Livšic Theorems and the Stable Ergodicity of Compact Group Extensions for Systems with Some Hyperbolicity

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Abstract

We consider Livšic regularity for Lie group valued cocycles over: a class of piecewise expanding maps of the interval, namely Lasota–Yorke maps; uniformly hyperbolic toral maps with singularities and a class of nonuniformly expanding interval maps.

As applications of the results we prove stable ergodicity theorems for compact Lie group extension of Lasota–Yorke maps and uniformly hyperbolic toral maps with singularities. Additionally we consider conditions for the ergodicity and weak-mixing of finite group extensions of hyperbolic basic sets given in terms of periodic data and cohomological equations. We also consider stable ergodicity results for a class of nonconnected compact Lie group extensions of hyperbolic basic sets.
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# Contents

Abstract i

Acknowledgements ii

List of Figures vi

Introduction 1

I Livšic Regularity Results 6

1 Livšic Theorems 7
  1.1 Cocycle Equations and Regularity ............................................ 7
  1.2 Hyperbolic Systems ................................................................. 8
  1.3 The Livšic Regularity and Periodic Point Theorems ....................... 10
  1.4 A Simple Example ........................................................................ 11
  1.5 Further Livšic Regularity Results .............................................. 12

2 Lasota—Yorke Maps 14
  2.1 Lasota—Yorke Maps .................................................................... 14
  2.2 The Main Theorem ...................................................................... 16
  2.3 Proof of The Main Theorem .......................................................... 17
  2.4 The β-transformation .................................................................. 23
  2.5 Markov Maps .............................................................................. 25

3 Uniformly Hyperbolic Toral Maps with Singularities 27
  3.1 Uniformly Hyperbolic Toral Maps with Singularities .................... 27
  3.2 Toral Linked Twist Maps ............................................................ 28
  3.3 Statement of Results .................................................................... 29
  3.4 Approximate Stable and Unstable Leaves ..................................... 30
  3.5 The Sinai Method ....................................................................... 34
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Nonuniformly Expanding Maps of the Interval</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>The Axiomatized System</td>
<td>39</td>
</tr>
<tr>
<td>4.2</td>
<td>Unimodal Maps</td>
<td>40</td>
</tr>
<tr>
<td>4.3</td>
<td>The Regularity Result</td>
<td>41</td>
</tr>
<tr>
<td>4.4</td>
<td>A Lebesgue Density Argument</td>
<td>41</td>
</tr>
<tr>
<td>4.5</td>
<td>Iterating the Cocycle</td>
<td>42</td>
</tr>
<tr>
<td>4.6</td>
<td>A Hölder Version on A</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>Further Work I</td>
<td>45</td>
</tr>
</tbody>
</table>

| II      | Stable Ergodicity of Compact Group Extensions | 47 |
| 5       | Compact Group Extensions                    | 48 |
| 5.1     | Definitions                                | 48 |
| 5.2     | Ergodicity and Mixing                       | 49 |
| 5.3     | Transitivity, Periodic Data and Cohomology  | 52 |
| 5.4     | Extension of Hyperbolic Bases               | 54 |
| 6       | Finite Group Extensions of Hyperbolic Basic Sets | 56 |
| 6.1     | Preliminaries                               | 56 |
| 6.2     | Abelian Extensions                          | 60 |
| 6.3     | General Finite Extensions                   | 62 |
| 6.4     | Finitely Many Connected Components          | 67 |
| 6.5     | Mixing                                      | 69 |
| 7       | Nonconnected Extensions of Hyperbolic Basic Sets | 72 |
| 7.1     | Setup                                       | 72 |
| 7.2     | $\Gamma_0$ Abelian                         | 73 |
| 7.3     | $\Gamma_0$ Semisimple                      | 77 |
| 7.4     | $\Gamma_0$ a General Compact Lie Group     | 80 |
| 8       | Extensions of Hyperbolic Systems with Singularities | 81 |
| 8.1     | Main Results                                | 81 |
| 8.2     | The Keynes–Newton Conditions                | 82 |
| 8.3     | Abelian and Semisimple Extensions of Lasota–Yorke Maps | 83 |
| 8.4     | Abelian and Semisimple Extensions of Uniformly Hyperbolic Toral Maps with Singularities | 86 |
| 8.5     | General Lie Group Extensions                | 88 |
List of Figures

2.1 A Lasota-Yorke Map ................................................................. 15
2.2 A $\beta$-transformation with $1 < \beta < 2$ ............................... 24
2.3 A Markov Map ........................................................................... 25

3.1 Singularity lines for the toral linked twist map $f := f_2 \circ f_1$, $f_1(x, y) = (x + \frac{1}{2} y, y)$ (mod 1), $f_2(x, y) = (x, y + \frac{1}{2} x)$ (mod 1) ........................................ 29
3.2 Describing the sector $\mathcal{K}_x$ .................................................. 35

6.1 $f_h$ invariant sets ..................................................................... 68
Livšic theorems are concerned with cohomological equations of the form

$$g_1(x) = h(fx)g_2(x)h(x)^{-1}$$  \hspace{1cm} (0.0.1)$$

where $f: X \to X$ is a dynamical system and $g_1, g_2, h: X \to \Gamma$ are maps into a Lie group $\Gamma$. Livšic regularity theorems generally state that given a solution $h: X \to \Gamma$ to the equation (0.0.1) with a low degree of regularity (measurable, $L^2$ etc.) there exists a solution $h': X \to \Gamma$ equal to $h$ almost everywhere with a higher degree of regularity ($C^r$ etc.). Typically such results are proved in the case where $g_1, g_2$ have the 'higher degree of regularity' and the dynamical system $f: X \to X$ has some degree of hyperbolicity.

The main uses of such results arise due to the fact that in many instances, an obstruction to a dynamical system possessing a property such as ergodicity or mixing is the existence of a solution to an equation of the form (0.0.1). It is often then possible to use dynamical data such as information on periodic orbits to show that a regular solution cannot exist.

Livšic' classical result states that if $f: M \to M$ is a transitive Anosov diffeomorphism and $g: M \to \mathbb{R}$ is Hölder, then for any measurable map $h: M \to \mathbb{R}$ satisfying

$$g(x) = h(fx) - h(x)$$

there exists a Hölder map $h'$ equal to $h$ almost everywhere.

Livšic regularity has been studied in a variety of contexts. The results obtained extend Livšic’ result in the following main directions:

- Considering a wider class of group;
- Generalizing the base dynamics $f: X \to X$;
- Looking at higher regularity.

For instance Parry and Pollicott consider the equation (0.0.1), where $g_1, g_2$ take values in a general (i.e. not necessarily abelian) compact Lie group. Results for noncompact groups have also
been considered for example by Nicol and Pollicott [46] who consider linear groups and Pollicott and Walkden [58] who consider cocycles which take values in arbitrary groups and satisfy certain spectral conditions.

Nitica and Torok consider diffeomorphism groups [48] as well as higher regularity [48],[49] where they show that if $g_1, g_2$ are $C^r$ then a Hölder solution $h$ can be taken to be $C^{k-1}$ with Hölder $(k-1)$-th derivative.

Livsic results for a wider class of base dynamics has also been well studied. Pollicott and Yuri [57] for example consider maps $f: X \to X$ which are not necessarily hyperbolic or have singularities. Nicol and Pollicott [46] consider maps $f: X \to X$ which are not necessarily hyperbolic but instead are either locally eventually onto (for every open set $U \subset X$ there exists $n \in \mathbb{N}$ such that $f^n(U) = X$) or minimal (every point in $X$ has a dense orbit under $f$).

The classical Livsic result is discussed in more detail in our opening chapter. We also state a related result: the so called Livsic periodic point theorem. We also formally state some of the recent generalisations of Livsic's result already mentioned.

The first part of this work is concerned with generalizing the Livsic result to more general dynamical systems such as those which have singularities and in which the hyperbolicity is not uniform. Specifically, we consider

(i) Piecewise $C^2$ expanding maps on the interval;

(ii) Uniformly hyperbolic toral maps with singularities;

(iii) Nonuniformly expanding maps of the interval.

The main problems arise due to: non-invertibility ((i) and (iii)); nonuniformity of hyperbolicity ((iii)) and the presence of singularities ((i) and (ii)). These are difficult obstacles to overcome and novel methods are required to cope.

The first results we prove are for a class of piecewise $C^2$ expanding interval maps known as Lasota–Yorke maps. For such maps $f: I \to I$ we show that if $h: I \to \Gamma$ is a measurable solution to the equation $g(x) = h(fx)h(x)^{-1}$, where $g: I \to \Gamma$ is a Lipschitz (Hölder) map into the compact Lie group $\Gamma$ satisfying certain spectral conditions, then there exists a map $h': I \to \Gamma$, which is Lipschitz (Hölder) on the neighbourhood of almost every point and such that $h = h'$ almost everywhere.

In order to deal with the noninvertibility of such piecewise expanding maps we work with the natural extension, which is invertible. Thus we consider separate inverse branches of points $(x_i)_{i=0}^\infty, f(x_i) = x_{i-1}$. Using the Borel–Cantelli argument we show that for almost all such branches $(x_i)_{i=0}^\infty$ there exists a family of intervals $\{J_i\}_{i=0}^\infty$ satisfying $x_i \in J_i$ and $f^i J_i = J_0$ for all $i$. Using a Lebesgue density argument we show that on $J_0$ we can find a Lipschitz (resp. Hölder version for $h$).

As well as the result for general Lasota–Yorke maps we consider two special subclasses, $\beta$-
transformation and Markov maps. For $\beta > 1$ the $\beta$-transformation $f(x) = \beta x \pmod{1}$ is an expanding map of the circle $[0, 1) \pmod{1}$ which, for $\beta \not\in \mathbb{Z}$ has a discontinuity at 0. Markov maps are Lasota–Yorke maps which have the additional property that the image of an interval on which $f$ is $C^2$ is the union of intervals on which $f$ is $C^2$. This is the so-called Markov property. Due to the simpler structure of these maps we are able to extend our result for Lasota–Yorke maps so that the Lipschitz (or Hölder) version $h'$ of $h$ can be taken to be Lipschitz (or Hölder) on the whole of the interval.

Following the work on Lasota–Yorke maps we proceed to consider two-dimensional analogues of the results. That is we consider a class of piecewise $C^2$ expanding, uniformly hyperbolic maps of the torus $\mathbb{T}^2$. Similarly to the result for Lasota–Yorke maps we show that a measurable solution $h$ to $g(x) = h(fx)h(x)^{-1}$, where $g: \mathbb{T}^2 \to \mathbb{T}$ is a Lipschitz map into the connected Lie group $\Gamma$ satisfying certain conditions, has a version $h'$ which is Lipschitz on an open neighbourhood of almost every point.

Our proof exploits the ‘Sinai Method’ as used by Liverani and Wojkowski [40] to establish ergodicity of the systems we are considering. We show that through almost every point there exist local stable and unstable leaves along which the map uniformly contracts and expands, respectively. We show that along almost all such leaves we can extend $h$ to a Lipschitz version. Utilizing the Sinai Theorem we then show that, by a Hopf chain argument, on a square neighbourhood of almost every point we have a uniform Lipschitz version $h'$.

A particularly simple class of uniformly hyperbolic maps of the torus with singularities is the class of toral linked twist maps. These maps have a simple linearization and as for $\beta$-transformations and Markov maps we obtain a map $h'$ which is Lipschitz on the whole of $\mathbb{T}^2$.

Moving on from considering uniformly hyperbolic systems, in Chapter 3 we consider a class of interval maps which expand nonuniformly. For such maps we exchange the problem of singularities for the fact that the derivative becomes arbitrarily near zero near to a point at which it vanishes. To deal with this we utilize the Tower construction of Young [66] and our results hold for systems for which a constructions holds. Although we do not explicitly use the dynamics of the tower, the salient properties of the tower are exploited to good effect.

Explicitly, we show that if $f: I \to I$ is an interval map which admits a Young Tower construction then there exists a subinterval $\Lambda$ such that any measurable solution to $g(x) = h(fx)h(x)^{-1}$, where $g: \mathbb{T}^2 \to \mathbb{T}$ is a Lipschitz map into the connected Lie group $\Gamma$ satisfying certain conditions, has a version $h'$ which is Lipschitz on $\Lambda$. The proof of this result uses a similar Lebesgue density argument we use for Lasota–Yorke maps.

One major application of Livsic regularity theorems in recent years has been in the study of compact group extension particularly with regards to stable ergodicity. A compact group extension is a map $f_h: X \times \Gamma \to X \times \Gamma$ defined by $f_h(x, \gamma) = (fx, h(x)\gamma)$ where $f: X \to X$ is a dynamical
system and \( h: X \to \Gamma \) is a measurable map into the compact Lie group \( \Gamma \). Regularity results are useful in the study of such maps as criteria exist for the ergodicity and weak-mixing of such extensions in terms of solutions to equations of a form similar to (0.0.1).

One major area of the theory of compact groups extension to which Livšic results have been effectively applied is in the study of stable ergodicity. It is this application to which the second part of this thesis is dedicated. Much work has been done for the stable ergodicity of compact connected group extension of Anosov diffeomorphism. Adler, Kitchens and Shub [3], for instance, show that if \( f: T^r \to T^r \) is an Anosov diffeomorphism then there exists an open and dense set of smooth maps \( \phi \) such that the circle extension is ergodic. Parry and Pollicott [54] proved similar results for circle extensions where \( f \) is a subshift of finite type or a hyperbolic system satisfying certain conditions. These results were further extended by Field and Parry [21] who considered general compact Lie group extension over hyperbolic systems.

We consider similar stable ergodicity results as those described above for nonconnected extensions of hyperbolic systems as well as compact connected extensions of discontinuous hyperbolic systems, namely the Lasota–Yorke maps and uniformly hyperbolic maps of the torus with singularities considered in Part II.

In Chapter 5 we give a brief review of compact group extension. As well as basic definitions we also state and prove the important results of Keynes and Newton [33] which give criteria for the ergodicity and weak-mixing of extensions in terms of solutions to dynamical cohomological equations.

Before considering stable ergodicity Chapter 6 considers straight ergodicity and mixing of finite group extensions over hyperbolic systems. We prove criteria for weak-mixing and ergodicity of such extensions in terms of periodic data and solutions to cohomological equations such as (0.0.1). Fundamental to the proofs of our results are the Livšic Regularity and Periodic Point Theorems. Initially the results are proved for subshifts of finite type and then ‘lifted’ to the general setting.

Chapter 7 is concerned with analogues of Field and Parry’s results for certain nonconnected compact Lie group extensions of hyperbolic systems. For hyperbolic maps \( f: \Lambda \to \Lambda \) we consider extensions by Hölder maps \( \Phi: \Lambda \to \Gamma \) where \( \Gamma \) is a compact Lie group which can be expressed as a semidirect product \( G = G \rtimes \Gamma_0 \), where \( G \) is a finite group and \( \Gamma_0 \) is the connected component of the identity in \( \Gamma \).

Due to the structure of \( \Gamma \) we are able to rewrite the \( \Gamma \)-extension \( f_\Phi \) as a \( \Gamma_0 \)-extension of a \( G \)-extension, that is a connected group extension of a finite group extension. Thus we apply the methods of Field and Parry with the finite group extension as the base. We assume that the finite extension is ergodic and show that if either

- \( \Gamma_0 \) is semisimple or
- \( \Lambda \) has finitely many connected components and is an attractor
then there is an open and dense set of $\Phi: \Lambda \to \Gamma$ (amongst the space of such $\Phi$ with the finite extension ergodic) such that $f_\Phi$ is ergodic.

Our proof of this results follows the same lines as in [21]. We consider separately the cases $\Gamma_0$ semisimple and $\Gamma_0$ abelian. In the semisimple case we exploit a result of [36] which states that the set of pairs which generates a compact, connected, semisimple Lie group is open and dense. In the abelian case we use the fact that the Keynes–Newton criteria for ergodicity can be recast in a form easier to work with. In both cases we rely on Livšic regularity to use data on periodic and homoclinic orbits. The results in the two cases are then combined to give the general result using a structural result for compact connected Lie groups.

In the final chapter we come full circle to consider applications of the results in Chapters 2 and 3. Using similar methods to Chapter 8 we prove stable ergodicity results for connected compact group extensions of Lasota–Yorke maps and uniformly hyperbolic maps of the torus with singularities. The proofs of these results utilize the Livšic regularity theorems proved in Chapters 2 and 3 in order to argue using data on periodic and homoclinic orbits. The main work of this chapter is thus taken up with establishing the density of such periodic and homoclinic orbits.

Throughout this thesis only a knowledge of basic concepts from ergodic theory is assumed and we refer the reader to any standard text such as [58] or [64].
Part I

Livšic Regularity Results
Chapter 1

Livšic Theorems

We begin gently with a brief précis of Livšic regularity theory. First we discuss cocycle regularity (or rigidity) in fairly general terms as well as its implications. Then after a swift review of hyperbolic dynamics we discuss the classical Livšic Theorem and look at a simple example to illustrate the method behind the proof of the result. To finish we look at more recent results which generalise the Livšic regularity Theorem particularly those concerning more general groups and base dynamics.

1.1 Cocycle Equations and Regularity

In general terms regularity or rigidity theorems assert that objects with a low degree of regularity, in fact have a higher degree of regularity. Livšic regularity results are concerned with solutions to certain dynamical cohomological equations. Specifically we consider solutions $h: X \to \Gamma$ to

\[ g_1(x) = h(f(x))g_2(x)h(x)^{-1} \tag{1.1.1} \]

where $f: X \to X$ is a measurable dynamical system and $g_1, g_2: X \to \Gamma$ are maps into the Lie group $\Gamma$. If such a solution exists we say the $g_1$ and $g_2$ are cohomologous via the cobounding function $h$. If $g_2 \equiv 1$ then $g_1$ is called a coboundary and (1.1.1) is a cocycle equation.

Livšic regularity theorems state that if $h$ has some degree of regularity then it in fact has some higher degree of regularity. Such results are useful in studying mixing properties of dynamical systems as in certain cases an obstruction to the system having a mixing property is the existence of a solution to an equation of the form (1.1.1). Clearly the higher the regularity such a solution must posses, the easier it is to prove it cannot exist.

One obvious application along these lines is the following. Let $f: X \to X$ be a measure preserving transformation of the measure space $(X, \mathcal{B}, \mu)$. We know that $f$ is ergodic with respect to $\mu$ if and only if any $\mu$-measurable solution $h: X \to \mathbb{R}$ to the equation $h(f(x)) - h(x) = 0$ is
constant \( \mu \) almost everywhere. Now suppose that \( X \) also admits a topology, then \( f \) is topologically transitive if and only if any continuous solution \( h: X \to \mathbb{R} \) to the equation \( h(fx) - h(x) = 0 \) is constant. Clearly, since any continuous function is measurable, transitivity of \( f \) implies ergodicity of \( f \) with respect to \( \mu \). In general, the converse is not true (see, for example [43, Theorem 7.1]). Suppose though that we have a regularity result which states that any measurable solution to the equation \( h(fx) - h(x) = 0 \) \( \mu \) a.e. is necessarily continuous. In this case we see that ergodicity with respect to \( \mu \) and transitivity are, in fact, equivalent.

1.2 Hyperbolic Systems

Livšic original cocycle regularity result, as well as recent generalisations, concern dynamical systems with at least some hyperbolic behaviour. Here we briefly review such systems. We concentrate on uniformly hyperbolic systems, though in later chapters we will come to look at systems with weaker hyperbolicity such as nonuniformly and partially hyperbolic dynamical systems. Our review is brief and for more details the reader is directed to [32].

Let \( M \) be a smooth compact Riemannian manifold, \( f: M \to M \) a diffeomorphism. A subset \( \Lambda \subset M \) is a hyperbolic basic set if

1. There is a continuous \( Df \)-invariant splitting of the tangent bundle \( T\Lambda M = E^s \oplus E^u \) and constants \( C > 0, \lambda > 1 \) such that for all \( n \in \mathbb{N} \)
   - \( ||Df^n v|| \leq C \lambda^n ||v||, v \in E^s \)
   - \( ||Df^{-n} v|| \leq C \lambda^n ||v||, v \in E^u \)

2. \( \Lambda \) is maximal and isolated, that is there exists an open neighbourhood \( U \) of \( \Lambda \) such that every compact \( f \)-invariant subset \( Y \subset U \) is contained in \( \Lambda \);

3. \( f: \Lambda \to \Lambda \) is transitive;

4. \( \Lambda \) is not a periodic orbit of \( f \).

We identify \( f \) with its restriction \( f|\Lambda \) and call \( f: \Lambda \to \Lambda \) a hyperbolic map. If \( \Lambda = M \) then \( f: M \to M \) is an Anosov diffeomorphism.

For \( x \in \Lambda \) and \( \epsilon > 0 \) we define the local unstable and stable manifolds through \( x \)

\[ W^u_\epsilon(x) = \{ y \in \Lambda \mid d(f^{-n}x, f^{-n}y) \leq \epsilon \text{ for all } n \geq 0 \} \]

and

\[ W^s_\epsilon(x) = \{ y \in \Lambda \mid d(f^n x, f^n y) \leq \epsilon \text{ for all } n \geq 0 \} \]

respectively.
1.2.1 Theorem (Stable/Unstable Manifolds Theorem). Let \( f: \Lambda \to \Lambda \) be a hyperbolic map with splitting \( T_xM = E_x^u \oplus E_x^s \). Then for each \( x \in \Lambda \), \( \epsilon > 0 \) small we have

(i) \( W^u_\epsilon(x) \) and \( W^s_\epsilon(x) \) are smooth embedded disks, varying smoothly with \( x \);

(ii) \( T_xW^s_\epsilon(x) = E_x^s \) and \( T_xW^u_\epsilon(x) = E_x^u \);

(iii) \( d(f^n x, f^n y) \leq \lambda^n d(x, y) \) for \( y \in W^s_\epsilon(x) \), \( n \geq 0 \);

(iv) \( d(f^{-n} x, f^{-n} y) \leq \lambda^n d(x, y) \) for \( y \in W^u_\epsilon(x) \), \( n \geq 0 \).

We define the \textit{global unstable and stable manifolds}

\[
W^u(x) = \{ y \in \Lambda \mid d(f^{-n} x, f^{-n} y) \to 0 \text{ as } n \to \infty \}
\]

and

\[
W^s(x) = \{ y \in \Lambda \mid d(f^n x, f^n y) \to 0 \text{ as } n \to \infty \}
\]

respectively.

The unstable and stable leaves of hyperbolic basic sets satisfy two important properties namely absolute continuity and local product structure. These are defined as follows:

**Local Product Structure** If for each \( x, y \in \Lambda \) given small \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( d(x, y) < \delta \) then \( W^u_\epsilon(x) \cap W^s_\epsilon(y) \) consists of a single point in \( \Lambda \) we say \( f: \Lambda \to \Lambda \) has \textit{local product structure}.

For \( x \) and \( y \) sufficiently close we define the holonomy map \( \phi_{x,y}: W^u(x) \to W^u(y) \) by \( \phi(x') = W^s(x') \cap W^u(y) \), where \( x' \in W^u(x) \).

**Absolutely Continuity** The unstable and stable leaves \( W^s(x), W^u(x) \) are said to form an \textit{absolutely continuous foliation} if for each \( x \) and \( y \) sufficiently close the holonomy map \( \phi_{x,y} \) is absolutely continuous.

Let \( f: \Lambda \to \Lambda \) be hyperbolic, \( v: \Lambda \to \mathbb{R} \) a Hölder map. We define the \textit{pressure of} \( v \) to be the quantity

\[
P(v) := \sup_{\mu} \left\{ h_\mu + \int v d\mu \right\}
\]

where \( \mu \) ranges over all \( f \)-invariant probability measures. It can be shown \([7]\) that the sup is obtained by a unique measure which we call the \textit{Hölder equilibrium state} or \textit{Gibbs measure} of \( f \) with respect to \( v \).

If \( f: \Lambda \to \Lambda \) is a hyperbolic basic set then periodic points are dense in \( \Lambda \) and \( f: \Lambda \to \Lambda \) is ergodic with respect to any Hölder equilibrium measure. Further if \( f: \Lambda \to \Lambda \) is topologically mixing, then it is weak-mixing with respect to any Hölder equilibrium measure.
Chapter 1. Livšic Theorems

As with many dynamical systems it often proves useful to take a symbolic dynamics approach to the study of hyperbolic basic sets. This is done by relating them to subshifts of finite type which we now describe. Let

\[ \Sigma_A := \{ \{ x_i \}_{i \in \mathbb{Z}} \in \{1, 2, \ldots, k\}^\mathbb{Z} \mid A[x_i, x_{i+1}] = 1 \text{ for all } i \in \mathbb{Z} \}, \]

where \( A \) is a \( k \times k \) matrix with entries \( A[i, j] \in \{0, 1\} \) and \( \sigma \) is the ‘left shift map’ defined by \( \sigma(\{x_i\}) = \{y_i\}, y_i = x_{i+1} \). The system \( \sigma: \Sigma_A \to \Sigma_A \) is called a subshift of finite type.

On \( \Sigma_A \) we define a metric \( d_\theta(\cdot, \cdot), 0 < \theta < 1 \) by

\[ d_\theta(\{x_i\}, \{y_i\}) = \theta^N \]

where \( N \) is the largest natural number such that \( x_i = y_i \) for all \( |i| \leq N \) with \( N = 0 \) if \( x_i \neq y_i \) for all \( i \in \mathbb{Z} \).

Given a metric space \( X \) we let \( F_\theta(\Sigma_A, X) = \{ h: \Sigma_A \to X \mid d_\theta(h(x), h(y)) < Cd(x, y) \text{ for some constant } C \} \) denote the space of functions from \( \Sigma_A \) to \( X \) which are Lipschitz with respect to the metric \( d(\cdot, \cdot) \).

With a slight abuse of terminology we shall say that a function \( h: \Sigma_A \to X \) is Hölder of exponent \( \theta \) if \( h \in F_\theta(\Sigma_A, X) \).

The relationship between hyperbolic basic sets and subshifts of finite type is given by the following.

1.2.2 Theorem ([7]). Let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( \mu \) a Hölder equilibrium state. There exists a subshift of finite type together with a Hölder equilibrium state \( \tilde{\mu} \) and a Hölder map \( \rho: \Sigma_A \to \Lambda \) such that

(i) \( \rho \circ \sigma = f \circ \rho \).

(ii) \( \rho \) is onto, finite-to-one and invertible \( \mu \) almost everywhere.

(iii) \( \mu = \rho_\ast \tilde{\mu} \).

Moreover if \( \Lambda \) is totally disconnected then \( f: \Lambda \to \Lambda \) is conjugate to a subshift of finite type.

1.3 The Livšic Regularity and Periodic Point Theorems

1.3.1 Theorem (The Livšic Regularity Theorem). ([41]) Let \( f: M \to M \) be a transitive Anosov diffeomorphism of the compact Riemannian manifold \( M \) with Hölder equilibrium state \( \mu \). Let \( g: M \to \mathbb{R} \) be a real valued Hölder map. Then given any measurable function \( h: M \to \mathbb{R} \) satisfying

\[ g(x) = h(fx) - h(x)\mu \text{ a.e.} \]

there exists a Hölder map \( h': M \to \mathbb{R} \) such that \( h = h', \mu \) almost everywhere.
We refer to the function $h'$ as a Hölder version of $h$. Note that this theorem says nothing about the existence of solutions to the cocycle equation. One obvious obstruction to a solution $h: M \to \Gamma$ to $g(x) = h(fx) - h(x)$ existing is given by ‘periodic data’, that is the quantities $g_n(p) = g(f_n p) + \cdots + g(fp) + g(p)$, where $f^n p = p$. It is clear that for such a solution to exist we must necessarily have that $g_n(p) = 0$ for all $p$ such that $f^n p = p$. The following result states that for Anosov systems this condition is also sufficient for a solution to exist.

1.3.2 Theorem (Livšic Periodic Point Theorem). [32] Let $f: M \to M$ be a transitive Anosov diffeomorphism of the compact Riemannian manifold $M$. Suppose that for every $x \in M$ such that $f^n x = x$ we have

$$g_n(p) = \sum_{i=0}^{n-1} g(f^ix) = 0.$$  

Then there exists a Hölder continuous $h: M \to \mathbb{R}$ such that $g(x) = h(fx) - h(x)$ for all $x \in M$.

1.4 A Simple Example

Let us consider a small example in which to demonstrate the Livšic argument. The aim is to display the heuristics of the argument and so details will be kept to a minimum. Let $f: T^2 \to T^2$ be the map of the two torus defined by the matrix

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$  

Note that $A$ has eigenvalues $\lambda = \frac{3-\sqrt{5}}{2} < 1$ and $\lambda^{-1} = \frac{3+\sqrt{5}}{2} > 1$ and $f: T^2 \to T^2$ is an Anosov diffeomorphism. Let $g: T^2 \to \mathbb{R}$ be a Lipschitz cocycle. We claim that if $h: T^2 \to \mathbb{R}$ is a measurable solution to

$$g(x) = h(fx) - h(x) \quad \text{i.e.}$$  

($l$ denotes Lebesgue measure) then $h$ has a Lipschitz version.

We first note that the set of points $x \in T^2$ such that $g(f^{i-1}x) = h(f^ix) - h(f^{i-1}x)$, for all $i \in \mathbb{N}$ is of full measure and we consider only such points. Thus given $x \in T^2$ we may iterate the cocycle equation as follows:

$$g(x) = h(fx) - h(x)$$  

$$\implies g(fx) = h(f^2x) - h(fx)$$  

$$= h(f^2x) - h(x) - g(x)$$  

$$\implies g(f^2x) = h(f^3x) - h(fx) - g(fx)$$  

$$= h(f^3x) - h(x) - g(x) - g(fx)$$  

$$\vdots$$  

$$\implies g(f^{n-1}x) = h(f^nx) - h(x) - \sum_{i=0}^{n-2} g(f^ix).$$
For \( p \in T^2 \) let \( W^s(p) \) and \( W^u(p) \) denote the lines through \( p \) having the directions of the eigenvectors corresponding to \( \lambda \) and \( \lambda^{-1} \) respectively. Fix \( x \) and take \( y \in W^s(x) \). For any \( n \in \mathbb{N} \), by the above and the triangle inequality, we have

\[
|h(x) - h(y)| \leq \sum_{i=0}^{n-1} |g(f^ix) - g(f^iy)| + |h(f^nx) - h(f^ny)|
\]

where \( K \) is a constant, \( | \cdot | \) is the standard norm on \( \mathbb{R} \) and \( d(\cdot, \cdot) \) the standard metric on \( T^2 \).

Now by a Theorem of Lusin's (which we will discuss later in Chapter 2) we may find a set \( H \), of measure at least \( \frac{1}{2} \) such that \( h \) restricted to \( H \) is uniformly continuous. Let \( \mathcal{A} \) denote the set of points on \( T^2 \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{0}^{N-1} \chi_H(f^nx) = l(H).
\]

By ergodicity \( l(\mathcal{A}) = 1 \). By absolute continuity we have that for \( l \) almost all \( x \in T^2 \) and \( l_x \) almost all \( y \in W^s(x) \), \( x, y \in \mathcal{A} \) (\( l_x \) denotes normalised one-dimensional Lebesgue measure on \( W^s(x) \)). Since \( l(\mathcal{A}) > \frac{1}{2} \) we can find a sequence \( n_k \to \infty \) such that \( f^{n_k}x, f^{n_k}y \in L \).

As \( h \) is uniformly continuous on \( H \) along the subsequence \( n_k \) we have that \( d(f^{n_k}x, f^{n_k}y) \to 0 \). Thus \( h \) has a uniformly Lipschitz version on almost all stable leaves \( W^s(x) \). A similar argument shows that \( h \) has a uniformly Lipschitz version on almost all unstable leaves. Thus by the local product structure \( h \) has a Lipschitz version on \( T^2 \).

### 1.5 Further Livšč Regularity Results

The Livšč Regularity Theorem can be generalized in many ways. This can be done, for instance by relaxing the strict hyperbolicity conditions on the map \( f \) or by considering cocycles taking values in more general Lie groups. In the chapters following this we will consider Livšč regularity over more general maps namely those with singularities and whose hyperbolicity is nonuniform. For now let us consider results concerning more general Lie groups, that is non-abelian and noncompact.

In [53] Parry and Pollicott consider the nonabelian case as well as the more general cocycle equation. Specifically they prove:

**1.5.1 Theorem.** Let \( f : M \to M \) be a transitive Anosov diffeomorphism of the compact Riemannian manifold \( M \) with Hölder equilibrium state \( \mu \). Let \( g_1, g_2 : M \to \Gamma \) be Hölder maps taking values in the compact Lie group \( \Gamma \). Then given any measurable function \( h : M \to \mathbb{R} \) satisfying

\[
g_1(x) = h(fx)g_2(x)h(x)^{-1} \mu \text{ a.e.,}
\]
there exists a Hölder map $h': M \to \Gamma$ such that $h = h'$, $\mu$ almost everywhere.

For subshifts of finite type, Parry and Pollicott [53] prove the following result.

1.5.2 Theorem. Let $\sigma: \Sigma_A \to \Sigma_A$ be a subshift of finite type, $\mu$ a Hölder equilibrium state. Let $\Gamma$ be a compact Lie group and $h: \Sigma_A \to \Gamma$ a measurable solution to the equation

$$g_1(x) = h(fx)g_2(x)h(x)^{-1} \text{ a.e.}$$

where $g_1, g_2 \in F_0(\Sigma_A, \Gamma)$. Then there exists $h' \in F_0(\Sigma_A, \Gamma)$ such that $h = h'$, $\mu$ almost everywhere.

For linear cocycles Nicol and Pollicott [46] prove the following.

1.5.3 Theorem. Let $f: M \to M$ be a transitive Anosov diffeomorphism of the compact Riemannian manifold $M$. Let $\mu$ be an invariant, Lebesgue equivalent measure. Let $g: M \to GL(n, \mathbb{C})$ be Hölder. Then if $h: M \to GL(n, \mathbb{C})$ is a measurable solution to

$$g(x) = h(fx)h(x)^{-1}$$

such that both $h$ and $h^{-1}$ are essentially bounded then $h$ has a continuous version on $M$.

This result has also been generalised to the case of noncompact, connected Lie groups extensions of hyperbolic basic sets, for example by Pollicott and Walkden [58] who prove the following theorem.

1.5.4 Theorem. Let $f: \Lambda \to \Lambda$ be a hyperbolic map, $\mu$ a Hölder equilibrium state. Let $\Gamma$ be a connected Lie group which is either compact, abelian or nilpotent. Then if $g: \Lambda \to GL(n, \mathbb{C})$ is Hölder and $h: \Lambda \to \Gamma$ is a measurable solution to the equation

$$g(x) = h(fx)h(x)^{-1}$$

then $h$ has a Hölder version.

Pollicott and Walkden [58] also extend the Livsic Periodic Point Theorem to the same setting.

1.5.5 Theorem. Let $f: \Lambda \to \Lambda$ be a hyperbolic map, $\mu$ a Hölder equilibrium state. Let $\Gamma$ be a connected Lie group which is either compact, abelian or nilpotent. If $g: \Lambda \to \Gamma$ is Hölder and

$$h_n(p) = h(f^{n-1}p) \ldots h(fp)h(p) = e,$$ the identity in $\Gamma$

whenever $f^np = p$, then there exists a Hölder solution $h: \Lambda \to \Gamma$ to the equation

$$g(x) = h(fx)h(x)^{-1}.$$
Chapter 2

Lasota–Yorke Maps

In this chapter we establish a local Livsic regularity result for a class of piecewise expanding maps known as Lasota–Yorke maps. We consider the cocycle equation \( g(x) = h(fx)h(x)^{-1} \) where \( f: [0,1] \rightarrow [0,1] \) is Lasota–Yorke map and \( g: [0,1] \rightarrow \Gamma \) is a Lipschitz or Hölder cocycle into a connected Lie group satisfying certain spectral conditions. We show that any measurable solution \( h: [0,1] \rightarrow \Gamma \) to this equation has a version which is Lipschitz or Hölder on a neighbourhood of almost every point.

To finish we will consider two important subclasses of Lasota–Yorke maps, namely Markov maps and \( \beta \)-transformations. For these special cases we obtain full Livsic regularity. That is we show that measurable solutions to the cocycle equation have versions which are Lipschitz or Hölder on the whole base space.

2.1 Lasota–Yorke Maps

Let \( f: I \rightarrow I \) be a transformation of the interval \( I = [0,1] \). Let \( S = \{0 = s_0 < s_1, \ldots < s_d = 1\} \) be a finite set of points in \( X \) and suppose that \( f \) restricted to each open interval \( (s_i, s_{i+1}) \) is \( C^2 \), with \( C^2 \) extension to \( [s_i, s_{i+1}] \) having both left and right derivatives. Further suppose that for all \( x \in X \setminus S \), the derivative \( |f'(x)| \geq \lambda > 1 \), for some constant \( \lambda \). We call \( f \) a Lasota–Yorke map. A typical example of such a map is depicted in Figure 2.1.

Such piecewise expanding maps arise in many areas of mathematical study including:

- **Physiological Models**: The modelling of certain biological systems involves a measurement which increases (steeply) linearly until a particular constant value is attained where the measurement instantly drops to zero. Among the areas in which such an 'integrate and fire model' has been useful are neurobiology, cardiac electrophysiology and cell biology.

- **Rotary Drilling**: Lasota and Rusek’s study of the motion of a rotary drill [37] gave rise to
a piecewise expanding transformation of an interval.

- **Random Number Generators**: Certain Hewlett Packard calculators use piecewise expanding maps of the form \( f(x) = (n + x)^n \), \( n \geq 2 \) to simulate random number generation.

### Bounded Distortion

For our work we need to be able to compare the relative sizes of intervals after the map \( f \) is applied. That is given intervals \( I \) and \( J \) we would like some way of relating \( \frac{l(I)}{l(J)} \) to \( \frac{l(f(I))}{l(f(J))} \), where \( l(\cdot) \) denotes length. This is provided by the following.

#### 2.1.1 Lemma (Bounded Distortion). Let \( f : I \rightarrow I \) be a Lasota–Yorke map with singularity set \( S \). Then there exists \( R' \) such that for all \( n \in \mathbb{N} \) and \( x, y \in X \) such that \( f^j(x) \) and \( f^j(y) \) both belong to the same connected component of \( I \setminus S \) for \( j = 0, \ldots, n \)

\[
\frac{1}{R'} \leq \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leq R'.
\]

We can also interpret this result as a bound on the nonlinearity of \( f \). Note that if \( f \) is piecewise linear then \( R = 1 \). Thus given intervals \( I \) and \( J \) such that \( f^j I \) and \( f^j J \) are both wholly contained in the same connected component of \( I \setminus S \) for \( j = 0, \ldots, n \) then there exists \( R \) such that

\[
\frac{1}{R} l(f^n J) \leq l(f^n I) \leq R l(f^n J).
\]

### Invariant Measures

The existence of absolutely continuous, invariant probability measures for Lasota–Yorke maps was established (not coincidentally) by Lasota and Yorke [38]. Here we briefly mention the more salient properties of such measures. Let \( f : X \rightarrow X \) be a Lasota–Yorke map, we have the following:

- There are a finite number of distinct ergodic absolutely continuous measures.
Chapter 2. Lasota–Yorke Maps

• Such a measure is supported on a finite union intervals.

• On its support the density of such a measure, \( m \) satisfies

\[
\frac{1}{M} \leq \frac{dm}{dt} \leq M
\]

for some constant \( M \).

We note that if we relax the condition that \( f \) be a piecewise \( C^2 \) map to requiring that \( f \) be only piecewise \( C^1 \) then the second fails. In this case the support of an acim (absolutely continuous invariant measure) may be a Cantor like set (see [8, Example 8.2.1]). Further if the condition that \( |f'(x)| \) is strictly greater than \( 1 \) at all points is relaxed then there may be no finite absolutely continuous measures at all. For example Lasota and Yorke show that the map

\[
f(x) = \begin{cases} 
\frac{2x}{1-x} & 0 \leq x < \frac{1}{2} \\
2x - 1 & \frac{1}{2} \leq x \leq 1 
\end{cases}
\]

which satisfies \( f'(x) > 1 \) at all points except at 0 where \( f'(0) = 1 \), admits no finite absolutely continuous invariant measure.

To simplify the exposition we will assume that \( f \) is weak-mixing with respect to the acim \( m \), supported on a set \( X \subset [0,1] \). In particular \( f \) will be exact with respect to \( m \).

2.2 The Main Theorem

We now move on to formulate the result which forms the basis of this chapter.

A Pinching Condition on the Cocycle

Let \( \Gamma \) be a connected Lie group with Lie algebra \( L(\Gamma) \) which we identify with the tangent space at the identity \( T_e \Gamma \). Let \( g : X \to \Gamma \) be a Lipschitz or Hölder cocycle, \( f : X \to X \) a Lasota–Yorke map. In order to apply our result to noncompact groups it is necessary to impose certain spectral conditions on the cocycle.

Let \( r_g \) denote right multiplication by \( g \in \Gamma \), then given a norm \( \| \cdot \|_e \) on \( T_e \Gamma \) we define a norm \( \| \cdot \|_g \) on \( T_g \Gamma \) by \( \| v \|_g = \| r_g^{-1} v \|_e \). This norm induces a right invariant metric \( d(\cdot, \cdot) \) on \( \Gamma \), so that \( d(gk, hk) = d(g, h) \) for all \( g, h, k \in \Gamma \) [58].

We define the adjoint map \( \text{Ad} : \Gamma \to \text{Aut}(L(\Gamma)) \) by

\[
\text{Ad}(g)v = \frac{d}{dt} (g \exp(tv)g^{-1})
\]

where \( g \in \Gamma, v \in T_e \Gamma \). A calculation [58] shows that

\[
d(gh, hk) \leq \| \text{Ad}(g) \| d(h, k)
\]

(2.2.1)
Chapter 2. Lasota-Yorke Maps

for all \(g, h, k \in \Gamma\). Let
\[
\mu := \lim_{n \to \infty} \left( \sup_{x \in X} \| \text{Ad}(g_n(x)) \| \right)^{\frac{1}{n}}
\]
where \(g_n(x) = g(f^{n-1}x) \cdots g(fx)g(x)\). We assume the following condition holds:
\[
\text{(PH)} \quad 1 \leq \mu < \lambda.
\]
We further define
\[
\Theta_{\text{PH}} := \frac{\log \mu}{\log \lambda}.
\]

We remark that if \(\Gamma\) is compact, abelian or nilpotent then \(\mu = 1\) and (PH) is automatically satisfied.

**Statement of the Main Theorem**

We are now in a position to state the main result.

**2.2.1 Theorem.** Let \(f: X \to X\) be a weakly mixing Lasota-Yorke map with respect to an absolutely continuous invariant measure, \(m\) with support \(X\). Let \(g: X \to \Gamma\) be a Lipschitz (Hölder) cocycle satisfying condition (PH). If \(g: X \to \Gamma\) is Hölder of exponent \(\alpha\), assume that \(\alpha > \Theta_{\text{PH}}\). Then if \(h: X \to \Gamma\) is a measurable solution to the cocycle equation
\[
g(x) = h(fx)h(x)^{-1} \quad m \text{ a.e.}
\]
there exists \(h': X \to \Gamma\) such that \(h = h', m\) almost everywhere and for \(m\) almost every \(x \in X\) there exists an interval \(U(x)\) such that \(h'|_{U(x)}\) is Lipschitz (Hölder), uniformly over such intervals.

**2.3 Proof of The Main Theorem**

Over the following sections we shall work towards proving Theorem 2.2.1.

**The Natural Extension**

It is the backwards contraction property of \(f\) we wish to exploit here. As Lasota–Yorke maps are necessarily non–invertible, this leads us to work with the natural extension of \(f\). We recall its construction.

Let
\[
\Omega := \{(x_0, x_1, x_2, \ldots) \in X \times X \times X \times \cdots \mid x_n = f(x_{n+1}) \text{ for all } n \geq 0\}.
\]

We can think of \(\Omega\) as the set of all inverse branches of points in \(X\) under the mapping \(f\). We define a shift map, \(\sigma: \Omega \to \Omega\) on \(\Omega\), by
\[
\sigma((x_0, x_1, x_2, \ldots)) = (f(x_0), x_1, x_2, \ldots).
\]
This map is known as the natural extension of \( f: X \to X \). Clearly \( \sigma: \Omega \to \Omega \) is invertible.

On \( \Omega \) we define a measure \( \tilde{m} \), firstly on cylinder sets \( C(A_0, A_1, \ldots, A_k) \),

\[
C(A_0, A_1, \ldots, A_k) = \{(x_0, x_1, \ldots) \in \Omega \mid x_i \in A_i \text{ for all } 0 \leq i \leq k\}
\]

by

\[
\tilde{m}(C(A_0, A_1, \ldots, A_k)) = m(f^{-k}(A_0) \cap f^{-k+1}(A_1) \cap \ldots \cap f^{-k+i}(A_1) \cap \ldots \cap A_k),
\]

then extending to \( \Omega \) using Kolmogorov’s extension theorem.

It can be shown [8, Section 8.3] that \( \sigma: \Omega \to \Omega \) is Bernoulli. We will, though, only use the fact that \( \sigma \) and \( \sigma^{-1} \) are both ergodic with respect to \( \tilde{m} \).

**Inverse Branches of Intervals**

We begin the proof in earnest by constructing inverse branches of intervals, that is sequences of intervals \( \{I_n\} \) such that \( f(I_{n+1}) = I_n \). In the case \( S = \emptyset \) this is elementary since the preimage of an interval \( I \) will consist of a finite number of intervals \( I_1, \ldots, I_l \) with \( f(I_{(j)}) = I \), for all \( j = 1, \ldots, l \). The main problem in the general case is that the inverse image of an interval \( I \) may contain intervals which do not map back onto the whole of \( I \).

Let \( B \) denote the image of the singularity set \( S \) under \( f \); strictly speaking we take images of the points \( s_i \) under the appropriate \( C^2 \) extension of \( f \). That is we define

\[
B = \left\{ x = \left( x_0, x_1, \ldots, x_d \right) \in \Omega \mid x_0 < f(S), x_i \notin S, \text{ for all } i \geq 0 \right\}.
\]

Note that \( B \) may contain twice as many points as \( S \). The set of points \( x = (x_0, x_1, \ldots) \in \Omega \) such that \( x_0 \notin S \) and \( x_i \notin B \), for all \( i \geq 0 \) forms a full measure set. Thus without loss of generality we shall deal only with such points.

Now take a point \( x_0 \in X \) together with an open interval \( I \), disjoint from \( S \) which contains \( x_0 \). Consider the preimage, \( f^{-1}(I) \), which will consist of a finite number of intervals. Divide this partition further by deleting points in \( S \). Thus we have

\[
f^{-1}(I) \setminus S = I_{(1)} \cup I_{(2)} \cup \ldots \cup I_{(l)}.
\]

where all of the \( I_{(j)} \) are intervals which may or may not map onto \( I \). Note that if \( I \cap B = \emptyset \) then \( f(I_{(j)}) = I \) for all \( j \).

Next we take one of these intervals, \( I_{(j_1)} \) and carry out the same procedure giving us a partition into open intervals

\[
f^{-1}(I_{(j_1)}) \setminus S = I_{(j_1,1)} \cup I_{(j_1,2)} \cup \ldots \cup I_{(j_1,l_1)}
\]

where each of the intervals will map onto \( I_{(j_1)} \) if \( I_{(j_1)} \cap B = \emptyset \).
We can repeat this process inductively to give us partitions of the general form

\[ I((j_1, j_2, \ldots, j_n-1)) = \bigcup_{k=1}^{j_n} I((j_1, j_2, \ldots, j_{n-1}, k)) \cup \bigcup_{k=1}^{j_n} I((j_1, j_2, \ldots, j_{n-1}, j_n)). \]

Again if \( I((j_1, j_2, \ldots, j_n-1)) \cap B = \emptyset \) then all of the intervals \( I((j_1, j_2, \ldots, j_{n-1}, k)) \) will map onto \( I((j_1, j_2, \ldots, j_n-1)) \).

This process enables us to define modified inverse branches of the interval \( I \). For ease of notation we write \( I_n \) for a branch chosen in such a way (that is we define \( I_n := I((j_1, j_2, \ldots, j_n) \) and \( I_0 = I \)). For any point \((x_0, x_1, x_2, \ldots) \in \Omega \) we can define a corresponding 'modified inverse branch' of intervals \( I_0, I_1, I_2, \ldots \) with the properties that \( x_n \in I_n \) and \( f(I_{n+1}) \subseteq I_n \) for all \( n \geq 0 \). This branch is uniquely determined by the choice of initial interval \( I_0 \). Further, the branch will be 'ideal', i.e. will satisfy \( f(I_{n+1}) = I_n \) for all \( n \geq 0 \), if \( I_n \cap B = \emptyset \) for all \( n \geq 0 \). It turns out that \( \tilde{m} \) almost every \( x \in \Omega \) has a corresponding inverse branch of intervals which is ideal. To show this we use the following lemma.

2.3.1 Lemma. For \( \tilde{m} \) almost every \( x \in \Omega \) there exists \( N = N(x) \in \mathbb{N} \) such that for an inverse branch of intervals, \( \{I_n\} \) corresponding to \( x \), \( f(I_{n+1}) = I_n \) for all \( n \geq N \).

Proof. For \( n = 0, 1, 2, \ldots \) we define cylinder sets

\[ E_n := C(X, \ldots, X, N(B, \lambda^{-n}), X, \ldots) \]

where \( N(B, \lambda^{-n}) := \{x \in X \mid |x_n - B| < \lambda^{-n}\} \). Suppose that \( x = (x_0, x_1, \ldots) \notin E_n \), then \( |x_n - B| \geq \lambda^{-n} \). Given that the length of \( I_n \) is bounded above by \( \lambda^{-n} \) we can deduce that \( I_n \cap B = \emptyset \), so that \( f(I_{n+1}) = I_n \). Thus to prove the lemma it suffices to show that \( \tilde{m} \) almost all \( x \in \Omega \) lie in only a finite number of the \( E_n \). We achieve this via the Borel–Cantelli Lemma. We recall its statement.

2.3.2 Theorem (Borel–Cantelli Lemma). Let \((X, \mathcal{B}, m)\) be a probability space. Let \( A_1, A_2, \ldots \) be a sequence of measurable sets such that

\[ \sum_{i=1}^{\infty} m(A_i) < \infty. \]

Then for \( m \) almost every \( x \in X \), \( m(\{x \in X \mid x \in A_i \text{ for infinitely many } i\}) = 0 \).

To use this result we note that \( \tilde{m}(E_n) \leq M\lambda^{-n} \) for each \( n \), where \( M \) is a constant. Thus

\[ \sum_{n=0}^{\infty} \tilde{m}(E_n) \leq \frac{M}{1 - \lambda^{-1}} < \infty. \]

Thus we can use the Borel–Cantelli Lemma to conclude that for a full \( \tilde{m} \) measure set of \( x \in \Omega \), \( x \in E_n \) for only a finite number of \( n \). This proves the lemma.

Let \( \tilde{m} \) be a point satisfying Lemma 2.3.1. Let \( \{I_n\} \) be any modified inverse branch associated with \( \tilde{m} \). Then there exists some \( N \in \mathbb{N} \) such that for all \( n > N \), \( f(I_{n+1}) = I_n \). Define \( J_0 = f^N I_N \).
and let \( \{ J_n \} \) be the modified inverse branch of intervals with initial interval \( J_0 \) corresponding to \( x \). By construction for each \( n \) we have that \( x_n \in J_n \) and \( f(J_{n+1}) = J_n \). We shall proceed to show that \( h \) has a Hölder version on \( J_0 \).

From now on given a point \( y_0 \in J_0 \) we automatically associate the inverse branch \( \{ y_i \} \) satisfying \( y_n \in J_n \) for all \( n \geq 0 \). We first claim that

\[
m \left( \{ y \in J_0 \mid g(y_{i-1}) = h(y_i)h(y_{i-1}) \text{ for all } i \in \mathbb{N} \} \right) = m(J_0).
\]

To see this define

\[
A := \{ y_0 \in \Omega \mid g(y_i) = h(y_0)h(y_{i-1}) \}
\]

and consider the set

\[
\bigcup_{i=0}^{\infty} f^i(J_i \cap A^c) = \{ y_0 \in J_0 \mid y_i \notin A \text{ for some } i \in \mathbb{N} \}.
\]

Now since \( A \) has full Lebesgue measure, \( l(J_i \cap A^c) = 0 \) for all \( i \in \mathbb{N} \). Therefore as for each \( i \in \mathbb{N} \) \( f^i \) maps sets of zero Lebesgue measure to sets of zero Lebesgue measure \( l(f^i(J_i \cap A^c)) = 0 \) for all \( i \in \mathbb{N} \). Hence as \( m \) is equivalent to Lebesgue measure we have \( l(f^i(J_i \cap A^c)) = 0 \) for all \( i \in \mathbb{N} \) which gives

\[
\bigcup_{i=0}^{\infty} f^i(J_i \cap A^c) = 0.
\]

This proves the claim.

Fix a ‘reference point’ \( t_0 \in J_0 \) (the choice of \( t_0 \) will be made more explicit later). For \( n \in \mathbb{N} \) we define a function \( \Phi_n : J_0 \to \Gamma \) by

\[
\Phi_n(y_0) = g_n(y_0)g_n(t_0)^{-1}
\]

where \( g_n(y_0) = g(y_1) \ldots g(y_n) \). First note that the sequence \( \Phi_n \) converges. To see this we calculate

\[
d(\Phi_{n+1}(y_0), \Phi_n(y_0))
\]

\[
d = d(g_{n+1}(y_0)g_{n+1}(x_0)^{-1}, g_n(y_0)g_n(x_0)^{-1})
\]

\[
d = d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}, g_n(y_0)g_n(x_0)^{-1})
\]

\[
d = d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}, g_n(y_0)g(x_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1})
\]

\[
d = d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}, g_n(y_0)g(x_{n+1})) \quad \text{(by right invariance)}
\]

\[
\leq \| Ad(g_n(y_0))\| d(g(y_{n+1}), g(x_{n+1})) \quad \text{(by (2.2.1))}
\]

\[
\leq (\mu^\lambda - \alpha)^n + 1
\]

\[
\leq C \kappa^n,
\]
where \(|\kappa| < 1\) (since \(a > \Theta_{PH}\)). Thus the sequence \(\Phi_n(y_0)\) is Cauchy, hence converges. Next we show that each \(\Phi_n\) is Hölder with uniform Hölder constant. For \(y_0, z_0 \in J_0\) we have

\[
d(\Phi_n(y_0), \Phi_n(z_0)) = \sum_{i=0}^{n-1} d(g_i(y_0)g(y_{i+1})h_{n-i-1}(z_{i+1}), h_i(y_0)h(z_{i+1})h_{n-i-1}z_{i+1})
\]

\[
\leq \|A(d(g_i(y_0)), d(g(y_{i+1}), g(z_{i+1}))
\]

\[
\leq \sum_{i=0}^{n-1} C\mu^{i+1} \lambda^{-(t+1)\alpha} d(y_0, z_0)^\alpha
\]

\[
\leq K d(y_0, z_0)^\alpha
\]

where \(K = C \sum_{i=0}^{\infty} \mu^{i+1} \lambda^{-(t+1)\alpha}\). Thus \(\Phi_n\) is indeed Hölder (also \(K\) does not depend on \(n\)).

Next we recall a classical result of Lusin.

2.3.3 Theorem (Lusin's Theorem). Let \(\phi\) be a function defined and finite on the Lebesgue measurable set \(E \subset \mathbb{R}^n\). Then \(\phi\) is measurable if and only if given \(\epsilon > 0\) there exists a closed set \(F \subset E\) such that \(l(E \setminus F) < \epsilon\) \((l\) denotes Lebesgue measure) and the restriction of \(\phi\) to \(F\) is uniformly continuous.

Using this theorem we may take a set \(H\), of positive measure such that \(h\) restricted to \(H\) is uniformly continuous. We claim the following:

2.3.4 Lemma. Given \(\eta > 0\) there exists \(N \in \mathbb{N}\) such that for infinitely many \(n > N\)

\[
m \{y_0 \in J_0 \mid y_n \in J_n \cap H\} > (1 - \eta)m(J_0).
\]

The main tool in proving this lemma will be the following:

2.3.5 Theorem (Lebesgue Density Theorem). Let \(E \subset \mathbb{R}\) be a Lebesgue measurable set, \(l(E) > 0\). There exists a set \(F \subset E\) such that \(l(E \setminus F) = 0\) and for all \(x \in F\)

\[
\lim_{\epsilon \to 0^+} \frac{\mu(X \cap [x - \epsilon, x + \epsilon])}{2\epsilon} = 1.
\]

We invoke this result to find a sequence \(\rho_k\) of positive numbers, together with sets

\[
H_k := \left\{ p \in H \mid \frac{l((p - \rho, p + \rho) \cap H)}{l(p - \rho, p + \rho)} > 1 - \frac{1}{2^k} \text{ for all } 0 < p < \rho \right\}
\]

such that \(l(H_k) > (1 - \frac{1}{2^k})\). Note that \(m\) almost all \(y \in \Omega\) have the property that for all \(k, y_n \in H_k\) for infinitely many \(n\). To see this, for \(m \in \mathbb{N}\) let \(G_m\) denote the set of \(y \in \Omega\) such that

\[
\sigma^{-n} y_n \in H_m \times X \times X \times \cdots
\]

for infinitely many \(n\). As \(\sigma^{-1} : \Omega \to \Omega\) is ergodic each \(G_m\) is a full \(m\) measure set. Thus the countable intersection \(\bigcap_{m \in \mathbb{N}} G_m\) is also a full measure set. This is precisely the set of points \(y\) such that for each \(m, y_n \in H_m\) for infinitely many \(n\).
Now fix \( \eta > 0 \) and take \( k \) large enough so that \( \frac{M^2 R^2}{2k-1} < \eta \). Choose \( n \) large enough so that 
\[ \lambda^{-n} < \rho_k \] and \( x_n \in H_k \). This guarantees that \( J_n \subset (x_n - \rho_k, x_n + \rho_k) \) and consequently
\[ l(J_n \cap H) \leq l(J_n \cap H_k) < \frac{1}{2k-1} l(J_n). \]

Hence by the bounded distortion property (Lemma 2.1.1) we have
\[ l(f^n(J_n \cap H)) \leq \frac{R^2}{2k-1} l(J_0). \]

Now from the inequality (2.1.1)
\[ m(f^n(J_n \cap H)) \leq \frac{M^2 R^2}{2k-1} m(J_0). \]

Thus, recalling that \( k \) was chosen so that \( \frac{M^2 R^2}{2k-1} \), this proves the lemma.

**A Hölder version on \( J_0 \)**

Given \( 0 < \delta < 1 \) we may choose a subsequence \( n_i \) such that
\[ m(f^n(J_n \cap H)) < \frac{\delta}{2} m(J_0). \]

and hence
\[ m\{ y_0 \in J_0 \mid y_{n_i} \in J_{n_i} \cap H\} > (1 - \frac{\delta}{2}) m(J_0). \]

Since \( \sum_{i=1}^{\infty} \frac{1}{2^i} < 1 \) we may choose the reference point \( t_0 \in J_0 \) so that \( t_{n_i} \in J_{n_i} \cap H \) for all \( n_i \).

Define \( \Psi^{u,n}(y_0) = g_n(y_0) g_n(t_0)^{-1} \). Then \( \lim_{n \to \infty} \Psi^{u,n}(y_0) = \Phi^u(y_0) \Phi^u(t_0)^{-1} \) and further
\[ h(y_0) = \Psi^{u,n}(y_0)^{-1} \Psi^{u,n}(t_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n). \]

It may seem at first a cumbersome way of expressing \( h(y_0) \) but it simplifies the following calculations. Using the triangle inequality we estimate
\[ d(h(y_0), h(z_0)) \leq d(\Psi^{u,n}(y_0)^{-1} h(t_0) h(t_n)^{-1} h(y_n), \Psi^{u,n}(y_0) \Psi^{u,n}(t_0)^{-1} h(t_0)) \]
\[ + d(\Psi^{u,n}(y_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n), \Psi^{u,n}(y_0) \Psi^{u,n}(t_0)^{-1} h(t_0)) \]
\[ + d(\Psi^{u,n}(y_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n), \Psi^{u,n}(z_0) \Psi^{u,n}(t_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n)). \]

By right-invariance of the metric,
\[ d(\Psi^{u,n}(y_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n), \Psi^{u,n}(z_0) \Psi^{u,n}(t_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n)) \]
\[ = d(\Psi^{u,n}(y_0)^{-1} h(t_0), \Psi^{u,n}(z_0) \Psi^{u,n}(t_0)^{-1} h(t_0)) \]

The group valued function of \( y_0, \Psi^{u,n}(y_0)^{-1} h(t_0), \) is uniformly bounded and converges to the Hölder function of \( y_0, \Phi^u(y_0)^{-1} h(t_0) \) as \( n \to \infty \). Similarly the function
Chapter 2. Lasota-Yorke Maps

\[ \Psi^{u, n}(x_0) \Psi^{u, n}(t_0)^{-1} h(t_0) \] converges to \( \Phi^u(y_0) \Phi^u(t_0)^{-1} h(t_0) \). Given \( \eta > 0 \) there exists \( N \) such that if \( n_i \geq N \) and \( x_{n_i}, y_{n_i} \in H \) then

\[ d(\Psi^{u, n}(y_0) \Psi^{u, n}(t_0)^{-1} h(t_0), \Psi^{u, n}(y_0) \Psi^{u, n}(t_0)^{-1} h(t_0)) < \frac{\eta}{2} \]

and

\[ d(\Psi^{u, n}(y_0) \Psi^{u, n}(t_0)^{-1} h(t_0), \Psi^{u, n}(y_0) \Psi^{u, n}(t_0)^{-1} h(t_0) h(t_n)^{-1} h(z_n)) < \frac{\eta}{2} \]

Recall

\[ m(\{y_0 \in J_0 \mid y_{n_i} \in J_{n_i} \cap H\}) > (1 - \frac{\delta}{\eta}) m(J_0) \]

so that for any \( \eta > 0 \)

\[ (m \times m)(\{(y_0, z_0) \in J_0 \times J_0 \mid d(h(y_0), h(z_0)) \leq K' d(y_0, z_0)^{\gamma} + \eta\}) \geq (1 - 2\eta)(m \times m)(J_0 \times J_0) \]

Since \( \eta \) was arbitrary, this implies

\[ (m \times m)(\{(y_0, z_0) \in J_0 \times J_0 \mid d(h(y_0), h(z_0)) \leq K' d(y_0, z_0)^{\gamma}\}) = (m \times m)(J_0 \times J_0) \]

and hence \( h \mid J_0 \) has a Hölder version. Note that the interval \( J_0 \) is defined independently of the function \( h \). Thus for \( m \) a.e. \( x_0 \) there exists an interval \( J_0 := U(x_0) \) on which the restriction of any coboundary \( h \) has a Hölder version. This proves Theorem 2.2.1.

2.4 The \( \beta \)-transformation

For fixed \( \beta > 1 \) we define the \( \beta \)-transformation \( f: X \to X \) by \( f(x) = \beta x \pmod{1} \) where \( X = [0, 1) \).

Such maps are useful in number theory, for instance in the study of \( \beta \)-adic representations of numbers and \( \mathbb{Z} \) numbers [22].

Livšic regularity in the context of the \( \beta \)-transformation has also been considered by Pollicott and Yuri [57]. Specifically they show that if \( f: X \to X \) is a \( \beta \)-transformation and, \( g: X \to \mathbb{R} \) a Hölder map and \( \mu \) is a Hölder equilibrium state corresponding to \( g \), then given any essentially bounded solution \( h: X \to \mathbb{R} \) to \( g(x) = h(f(x)) - h(x) \) and any \( \epsilon \) there exists \( h': X \to \mathbb{R} \) such that \( h' = h \mu \) almost everywhere and \( h' \) is of bounded variation on \([0, 1 - \epsilon]\). Their method of proof is based upon the transfer operator method and does not generalise to the general compact Lie group case.

Note that for \( \beta \notin \mathbb{Z} \) the map \( f: X \to X \) has a single discontinuity at the origin which is fixed. Further \( f \) is smooth on \((0, 1)\) with \( f'(x) = \beta > 1 \) for all \( x \in (0, 1) \).

It is known that \( \beta \)-transformations are exact with respect to a unique absolutely continuous invariant measure, with support \([0, 1) \pmod{1}\) (see, for example [56, Section 6.4] and [8, Problem 8.2.1]). The simple form of the \( \beta \)-transformation allows us to extend Theorem 2.1 to prove:
Figure 2.2. A $\beta$-transformation with $1 < \beta < 2$

2.4.1 Theorem. Let $f: X \rightarrow X$ be a $\beta$-transformation, $\Gamma$ a connected Lie group and $g: X \rightarrow \Gamma$ a Lipschitz (Hölder) cocycle satisfying condition (PH). If $g$ is Hölder of exponent $\alpha$ assume that $\alpha > \Theta_{PH}$. Then if $h: X \rightarrow \Gamma$ is a measurable solution to the equation

$$g(x) = h(fx)h(x)^{-1}$$

then there exists a Lipschitz (Hölder) function $h: X \rightarrow \Gamma$ such that $h = h' \ m \ a.e.$

To prove Theorem 2.4.1 let $(\gamma_1, \gamma_2)$ be an interval on which $h$ has a Hölder version. Let $n$ be the smallest natural number such that $f^n(\gamma_1, \gamma_2) = (\beta^n\gamma_1, \beta^n\gamma_2)$ contains 0. By the iterated cocycle equation

$$h(f^i x) = g(f^{i-1} x) \cdots g(f x)g(x)h(x)$$

(2.4.1)

we may extend the Hölder version of $h$ to the intervals $(\beta^n\gamma_1, \beta^n\gamma_2)$ for each $i = 1, \ldots, n - 1$.

Now since $h$ has a Hölder version on $f^{-1}(0, \gamma_2^2) \cap (\gamma_1, \gamma_2)$ we can also use (2.4.1) to extend the Hölder version of $h$ to $(0, \beta^n\gamma_2)$. Taking $r$ to be the smallest natural such that $(0, 1) \subset f^r(0, \beta^n\gamma_2)$ we see that $h$ has a Hölder version on $(0, 1)$.

In particular $h$ has a Hölder version on a neighbourhood of the point $\beta^{-1}$, which maps onto 0 we can use the cocycle equation to extend $h$ as a Hölder function on $X$.

Remark

It is easy to construct nontrivial examples of Lipschitz $\mathbb{R}^d$ or group-valued coboundaries where the base transformation is the $\beta$-transformation and the cobounding function is also Lipschitz. For example, let $h(x)$ be an $\mathbb{R}^d$ valued Lipschitz function on $X$ with the additional property that $h(\beta) = h(0) = h(1)$ and define $g(x) = h(fx) - h(x)$. Since $h$ is Lipschitz on $X$ and $f$ has a discontinuity only at the point $0 = 1$ the composition $h \circ f$ is Lipschitz except possibly at the point
Figure 2.3. A Markov Map

0 = 1. By checking left and right limits it follows that \( h \circ f \) is Lipschitz on \( X \) and hence \( g(x) \) is Lipschitz on \( X \). Hölder and Lie group valued examples may be constructed in the same manner.

2.5 Markov Maps

Let \( f: I \rightarrow I \) be a Lasota-Yorke map of the interval \( I = [0,1] \) with singularity set \{\( s_0, \ldots, s_d \)\}. Suppose that \( f \) has the further property that for any \( i \), \( f(s_i, s_{i+1}) \) is the union of intervals of the form \( (s_j, s_{j+1}) \). We call \( f: I \rightarrow I \) a Markov map.

2.5.1 Theorem. Let \( f: X \rightarrow X \) be a \( C^2 \) weakly mixing Markov map with respect to an absolutely continuous invariant measure, \( m \) with support \( X \). Let \( \Gamma \) a connected Lie group and \( g: X \rightarrow \Gamma \) a cocycle which is Lipschitz or Hölder of exponent \( \alpha \). Assume \( g \) satisfies condition (PH) and if \( g \) is \( \alpha \)-Hölder that \( \alpha > \Theta_{PH} \). Then if \( g: X \rightarrow \Gamma \) is Lipschitz (Hölder) and \( h: X \rightarrow \Gamma \) is a measurable solution to the cocycle equation

\[ g(x) = h(fx)h(x)^{-1} \]

then there exists a Lipschitz (Hölder) function \( h': X \rightarrow \Gamma \) such that \( h' = h \) m.a.e.

First note that we may assume that \( |f'(x)| > 2 \) for all \( x \) (where we take left or right derivatives at endpoints of \( (s_i, s_{i+1}) \)). If this is not the case then we may take some power \( f^n \) of \( f \) for which \( |(f^n)'(x)| > 2 \) and a refinement \( \{ (s_i, s_{i+1}) \} \) of the partition \( \{ (s_i, s_{i+1}) \} \) which is Markov for \( f^n \).

If \( g(x) = h(fx)h(x)^{-1} \) then \( h(f^n x)h(x)^{-1} = g(f^{n-1}) \cdots g(x) \) where \( g(f^{n-1}x) \cdots g(x) \) is Hölder on the interior of each partition element \( (s_j, s_{j+1}) \). Hence if we obtain a Hölder version of \( h \) on \( X \) solving \( h(f^n x)h(x)^{-1} = g(f^{n-1}) \cdots g(x) \) (recall \( h \) also solves \( g(x) = h(fx)h(x)^{-1} \)) then we have the Hölder version we are seeking. Suppose \( |f'(x)| \geq 2 + \delta \) where \( \delta > 0 \). Take an interval \( J_0 \) such that \( f^n(J_0) = J_0 \) and on which \( h \) is Hölder. We may extend \( h \) as a Hölder solution to \( f(J_0) \) unless \( f(J_0) \cap S \neq \emptyset \). If \( f(J_0) \cap S = \emptyset \) then the length of \( J_0 \) has been increased by a factor of at least 2. However if \( f(J_0) \cap S \neq \emptyset \) then either
(i) $(s_j, s_{j+1}) \subset f(J_0)$ for some $j$ or

(ii) we may divide $f(J_0)$ into two intervals one of which, say $J(1)$, has length strictly greater than that of $J_0$ by a factor of $1 + \frac{\alpha}{2}$.

Continuing this process we produce an interval $J(i)$ such that $h$ is Hölder on $J(i)$ and $(s_j, s_{j+1}) \subset f(J(i))$. Thus we may extend $h$ as a Hölder function to $(s_j, s_{j+1})$ and then the Markov property allows us to extend $h$ as a Hölder version on $X$. 
Chapter 3

Uniformly Hyperbolic Toral Maps with Singularities

We now consider two-dimensional analogues of the previous chapter’s results. That is, we consider a class of toral maps which have uniformly hyperbolic behaviour and a set of singularities. We show that for such maps \( f : T^2 \to T^2 \), measurable solutions \( h : T^2 \to \Gamma \) to the cocycle equation \( g(x) = h(fx)h(x)^{-1} \) where \( g : T^2 \to \Gamma \) is a Lipschitz or cocycle into a connected Lie group satisfying certain spectral conditions have versions Lipschitz on a neighbourhood of Lebesgue almost every point.

For our proof we will concentrate on a simple class of toral linked twist maps (tltm’s) as studied by Liverani and Wojkowski [40]. For this class of maps, as for Markov and \( \beta \)-transformations in the previous chapter, we obtain full regularity results.

Our approach is based upon the Sinai Method (itself based on the Hopf method) as used by Liverani and Wojkowski to establish the ergodicity of uniformly hyperbolic maps of the torus with singularities. We first show that Lebesgue almost all points have short approximate stable and unstable leaves, that is line segments upon which \( f \) uniformly contracts and expands uniformly. We then show that on almost all such leaves \( h : T^2 \to \Gamma \) has a Lipschitz version. Using the Sinai Theorem we show that almost all points in \( T^2 \) have an open neighbourhood on which \( h \) has a Lipschitz version.

3.1 Uniformly Hyperbolic Toral Maps with Singularities

We begin with an axiomatic description of the class of toral maps for which our results hold. Let \( f : T^2 \to T^2 \) be a Lebesgue measure preserving map of the two torus. We suppose that the following hold

(I) \( f \) is piecewise \( C^2 \) in the following sense. There exist finite sets of \( C^2 \) one-dimensional submanifolds \( \{ S_t^+, \ldots, S_t^- \} \) and \( \{ S_t^+, \ldots, S_t^- \} \) such that
Chapter 3. Uniformly Hyperbolic Toral Maps with Singularities

- for $i \neq j$, $S_i^+ \cap S_j^+$ and $S_i^- \cap S_j^-$ are either empty or consist of a finite number of points,
- $f$ and $f^{-1}$ are $C^2$ on $T^2 \setminus \left( \bigcup_{i=0}^{r} S_i^+ \right)$ and $T^2 \setminus \left( \bigcup_{i=0}^{r} S_i^- \right)$ respectively
- $f$ and $f^{-1}$ have $C^2$ extension to the boundary of open components of $T^2 \setminus \left( \bigcup_{i=0}^{r} S_i^+ \right)$ and $T^2 \setminus \left( \bigcup_{i=0}^{r} S_i^- \right)$ respectively.

(II) There exist piecewise continuous $Df$ invariant cone fields $C(x)$ and $D(x)$ in the tangent space to $T^2$ such that
- $DfC(x) \subset C(fx)$ (strictly) and $Df^{-1}D(x) \subset D(f^{-1}x)$ (strictly)
- vectors inside the cones $C(x)$ and $D(x)$ are expanded in length by $Df$ and $Df^{-1}$ respectively. That is there exist numbers $\lambda_0 < 1 < \lambda_n$ such that for all $n \geq 0$
  - $\|Df^n v\| \leq C\lambda_0^n \|v\|, v \in C(x)$
  - $\|Df^{-n} v\| \leq C\lambda_0^{-n} \|v\|, v \in D(x)$
- the angle between $C(x)$ and $D(x)$ is strictly bounded away from zero.

(III) the submanifolds $S_i^+$ have no tangents contained in the cones $C(x)$ and the submanifolds $S_i^-$ have no tangents contained in the cones $D(x)$.

We will call this class of such $f: T^2 \to T^2$ uniformly hyperbolic toral maps with singularities.

With reference to results of Pesin [55] and Katok and Strelcyn [31] through Lebesgue almost every point $p \in T^2$ there exist local stable and unstable manifolds of dimension one and the foliations are absolutely continuous.

3.2 Toral Linked Twist Maps

In order to give a clearer exposition of our argument we will concentrate on a simple class of uniformly hyperbolic toral maps with singularities known as toral linked twist maps. Such maps were considered by Liverani and Wojkowski [40] as a simple setting in which to demonstrate the Sinai Method of establishing ergodicity of the type of maps we are considering here. Other references which consider toral linked twist maps are [44, 59, 39]. In this section we briefly describe toral linked twist maps and their properties.

Let $f_1: T^2 \to T^2$ and $f_2: T^2 \to T^2$ be toral maps defined by

$$f_1(x, y) = (x + ay, y) \pmod{1}$$
$$f_2(x, y) = (x, y + ax) \pmod{1}.$$ 

where $a > 1$ is constant. Let $f := f_2 \circ f_1$. This map is linearized by the matrix

$$A = \begin{pmatrix} 1 & a \\ a & 1 + a^2 \end{pmatrix}.$$
We call \( f \) a toral linked twist map. More generally we could consider those maps for which \( f_2(x, y) = (x, y + bx) \), where \( a 
eq b > 0 \), but in the tradition of Liverani and Wojkowski we will restrict ourselves to the simplest notational case. For \( a \) an integer, \( f \) is uniformly hyperbolic on \( T^2 \). It is the case \( a \in \mathbb{Z} \) which interests us here. In this case \( f \) and \( f^{-1} \) have a finite number of singularity lines at which we have jump discontinuities. Specifically the singularity lines for \( f \) and \( f^{-1} \) are

\[
R^+ := \{(x, y) \in T^2 | y = 0 \text{ or } x + ay = 0 \pmod{1}\}
\]

and

\[
R^- := \{(x, y) \in T^2 | x = 0 \text{ or } y + ax = 0 \pmod{1}\}
\]

respectively. The discontinuity lines for the case \( a = \frac{1}{2} \) are shown in Figure 3.2.

We note that \( f \) obeys conditions (I)-(III) given in the previous section. In particular the matrix \( A \) has eigenvalues

\[
\lambda_+ = 1 + \frac{a^2}{2} - \sqrt{a^4 + 4a} < 1
\]

and

\[
\lambda_- = 1 + \frac{a^2}{2} + \sqrt{a^4 + 4a} > 1
\]

and with corresponding eigenvectors with directions not equal to \( \pm a \) or \( 0 \).

### 3.3 Statement of Results

As in the previous chapter it is necessary to impose some spectral assumptions on our cocycle. Let \( \Gamma \) be a connected Lie group with Lie algebra \( L(\Gamma) \) which we identify with the tangent space at the identity \( T\gamma \Gamma \). Let \( g: T^2 \to \Gamma \) be a Lipschitz or Hölder cocycle. Let \( r_g \) denote right multiplication by \( g \in \Gamma \), then given a norm \( \| \cdot \|_e \) on \( T\gamma \Gamma \) we define a norm \( \| \cdot \|_g \) on \( T_g \Gamma \) by \( \|v\|_g = \|r_g^{-1}v\| \). This norm induces a right invariant metric \( d(\cdot, \cdot) \) on \( \Gamma \), so that \( d(gk, hk) = d(g, h) \) for all \( g, h, k \in \Gamma \) [58].
We define the adjoint map $\text{Ad}: \Gamma \to \text{Aut}(L(\Gamma))$ by

$$\text{Ad}(g)v = \frac{d}{dt} (g \exp(tv)g^{-1})$$

where $g \in \Gamma$, $v \in T_\gamma \Gamma$. Recall that

$$d(gh, gk) \leqslant \|\text{Ad}(g)\|d(h, k) \tag{3.3.1}$$

We define

$$\mu_s := \lim_{n \to \infty} \left( \sup_{x \in \mathcal{X}} \|\text{Ad}(g_n(x))\| \right)^{-\frac{1}{n}}$$

$$\mu_u := \lim_{n \to \infty} \left( \sup_{x \in \mathcal{X}} \|\text{Ad}(g_n(x))\| \right)^{\frac{1}{n}},$$

where $g_n(x) = g(f^{n-1}x) \ldots g(fx)g(x)$. We assume the following condition holds:

$$\text{PH} \quad \lambda_s < \mu_s \leqslant 1 \leqslant \mu_u < \lambda_u.$$

Note that if $\Gamma$ is compact, nilpotent or abelian then $\mu_s = \mu_u = 1$ in which case PH is automatically satisfied.

The Results

3.3.1 Theorem. Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be a toral linked twist map, $g: \mathbb{T}^2 \to \Gamma$ a Lipschitz map into the connected Lie group $\Gamma$ satisfying condition (PH). If $h: \mathbb{T}^2 \to \Gamma$ is a measurable solution to the cocycle equation

$$g(x) = h(fx)h(x)^{-1} \text{ a.e.}$$

then there exists a Lipschitz function $h': \mathbb{T}^2 \to \Gamma$ such that $h' = h$ a.e.

3.3.2 Theorem. Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be a uniformly hyperbolic map of the torus, $g: \mathbb{T}^2 \to \Gamma$ a Lipschitz map into the connected Lie group $\Gamma$ satisfying condition (PH). If $h: \mathbb{T}^2 \to \Gamma$ is a measurable solution to the cocycle equation

$$g(x) = h(fx)h(x)^{-1} \text{ a.e.}$$

then there exists a function $h': \mathbb{T}^2 \to \Gamma$ such that $h' = h$ a.e. and such that almost every $x \in \mathbb{T}^2$ has a neighbourhood $U(x)$ on which $h'$ is Lipschitz.

3.4 Approximate Stable and Unstable Leaves

The first step in the proof of Theorem 3.3.1 will be to construct, for almost every $x \in \mathbb{T}^2$, stable and unstable leaves $W^s(x)$ and $W^u(x)$. These will be line segments along which we will have exponential contraction (for $W^s(x)$) and expansion (for $W^u(x)$) under $f$. Additionally they will
be constructed so that all forward iterates of $W^s(x)$ and all backwards iterates of $W^u(x)$ avoid the singular set.

Take $x \in T^2$. Define $W^s_0(x)$ to be the line through $x$ having the same direction as the eigenvector corresponding to $\lambda$. Let $W^s_1(x)$ be the open segment of $W^s_0(x)$ not intersecting $R^+$ but with endpoints on it. Clearly $W^s_1(x)$ is well defined provided $x \not\in R^+$.

Next we take $W^s_2(x)$ to be the open segment of $W^s_1(x)$ containing $x$ such that its image has endpoints on $R^+ \cup f(R^+)$ but does not intersect it. That is

$$W^s_2(x) := f^{-1}(f(W^s_1(x)) \cap W^s_0(f(x))).$$

Provided $x \not\in f^{-1}(R^+)$ this is a well defined open line segment with endpoints on $R^+ \cup f^{-1}(R^+)$. In general if we have defined $W^s_n(x)$, for $n \in \mathbb{N}$, we define

$$W^s_{n+1}(x) := f^{-n}(f^n(W^s_n(x)) \cap W^s_0(f^n(x))).$$

If $x \not\in \bigcup_{i=0}^{\infty} f^{-i}R^+$ (in which case we say that $x$ is smooth in the future) we have a sequence of open segments

$$W^s_0(x) \supset W^s_1(x) \supset W^s_2(x) \supset \ldots$$

each containing $x$ with endpoints on $\bigcup_{i=0}^{\infty} f^{-i}R^+$. It may happen that all inclusions $W^s_n(x) \supset W^s_{n+1}(x)$ are strict so that $\bigcap_{i=0}^{\infty} W^s_i(x) = x$. However a standard application of the Borel–Cantelli Lemma guarantees that for almost every $x \in T^2$ the sequence of line segments $W^s_n(x)$ stabilizes. In detail:

3.4.1 Lemma. For almost every $x \in T^2$ there exists $N = N(x) \in \mathbb{N}$ such that

$$\bigcap_{i=0}^{\infty} W^s_i(x) = \bigcap_{i=0}^{N} W^s_i(x).$$

Proof. For $t > 0$ define

$$X_t := \{x \in B^+ \mid d(x, R^+) \leq t\}. \quad (3.4.1)$$

As $R^+$ is made up of a finite number of lines we have that $l(X_t) \leq Ct$ for some constant $C$. Define a sequence $t_n = \frac{1}{n^2}$ and note that

$$\sum_{n=1}^{\infty} l(X_{t_n}) \leq \sum_{n=1}^{\infty} \frac{C}{n^2} < \infty.$$

Thus we have

$$\sum_{n=1}^{\infty} l(f^{-n}X_{t_n}) < \infty.$$

This allows us to use the Borel–Cantelli Lemma (Theorem 2.3.2) in order to conclude that almost every $x \in T^2$ belongs to only a finite number of the sets $f^{-n}X_{t_n}$.

Let $x$ be such a point so that, except for a finite number of values of $n$, we have

$$d(f^n x, R^+) > \frac{1}{n^2}.$$
Thus we can choose \( c = c(x) > 0 \) sufficiently small so that
\[
d(f^n x, R^+) > \frac{c}{n^2}
\]
for all values of \( n \).

Now suppose that \( W^s_n(x) \) is strictly shorter than \( W^s_{n-1}(x) \) so that
\[
d(f^n x, R^+) < \lambda^s_n l_x(W^s_{n-1}(x))
\]
where \( l_x \) denotes one-dimensional Lebesgue measure on \( W_0(x) \). This then gives
\[
\frac{c}{n^2} < \lambda^s_n l_x(W^s_{n-1}(x)) \leq l_x(W^s_1(x)).
\]
But as \( c > 0 \) this can only hold for a finite number of \( n \). Thus \( W^s_n(x) \) can be strictly shorter than \( W^s_{n-1}(x) \) only a finite number of times proving the lemma. \( \square \)

So for almost every \( x \) we may define the stable leaf \( W^s(x) \) through \( x \) by
\[
W^s(x) = \bigcap_{i=0}^{\infty} W^s_i(x).
\]
It is clear that on these leaves we have uniform exponential contraction so that for \( y, z \in W^s(x) \)
\[
d(f^i y, f^i z) < K \lambda^s_i d(y, z)
\]
for all \( i \in \mathbb{N} \), \( K \) a constant.

A similar process to that described above gives unstable leaves \( W^u(x) \) for almost every \( x \in T^2 \). We will next show that \( h \) has a version whose restriction to almost every stable leaf \( W^s(x) \) and unstable leaf \( W^u(x) \) is Lipschitz.

Fix \( x \in T^2 \) and on \( W^s(x) \) we define a function \( \Phi^s_y : W^s(x) \to \Gamma \) by
\[
\Phi^s_y(y) = \lim_{n \to \infty} g_n(y)^{-1}g_n(x),
\]
where \( g_n(x) = g(f^{n-1}x) \ldots g(x) \). Note
\[
d(g_{n+1}(y)^{-1}g_{n+1}(x), g_n(y)^{-1}g_n(x))
\]
\[
= d(g_n(y)^{-1}g(f^n y)^{-1}g(f^n x)g_n(x), g_n(y)^{-1}g_n(x))
\]
\[
= d(g_n(y)^{-1}g(f^n y)^{-1}g(f^n x)g_n(x), g_n(y)^{-1}g(f^n y)^{-1}g(f^n y)g_n(x))
\]
\[
= d(g_n(y)^{-1}g(f^n y)^{-1}, g_n(y)^{-1}g(f^n x)^{-1}) \quad \text{(right invariance)}
\]
\[
\leq \| \text{Ad}(g_n(y)^{-1})\| d(g(f^n y)^{-1}, g(f^n x)^{-1}) \quad \text{(by (3.3.1))}
\]
\[
\leq C ((\mu_s)^{-1} \lambda_s)^n
\]
\[
\leq C \kappa^n,
\]
where $|k| < 1$. Thus the sequence $g_n(y)^{-1}g_n(x)$ is Cauchy and so converges. Next we show $\Phi^e_x$ is Lipschitz. Let $y, z \in W^s(x)$, then for large $n$.

\[
\begin{align*}
    d(g_n(y)^{-1}g_n(x), g_n(z)^{-1}g_n(x)) &= d(g_n(y)^{-1}, g_n(z)^{-1}) \\
    &\leq \sum_{i=0}^{n-1} d(g_i(y)^{-1}g(f^iy)^{-1}g_{i-1}(f^{i+1}z)^{-1}, g_i(y)^{-1}g(f^i z)^{-1}g_{i-1}(f^{i+1}z)^{-1}) \\
    &\leq \sum_{i=0}^{n-1} \|Ad(g_i)^{-1}\|d(g(f^iy)^{-1}, g(f^i z)^{-1}) \\
    &\leq \sum_{i=0}^{n-1} C\mu_i^e \lambda^e_i d(y, z).
\end{align*}
\]

Letting $n \to \infty$ gives $d(\Phi^e_x(y), \Phi^e_x(z)) \leq Kd(y, z)$, $K$ a constant.

Define $Y' := \{x \in \mathbb{T}^2 \mid g(x) = h(fx)h(x)^{-1}\}$, a set of full measure. Further let $Y = \bigcap_{n \in \mathbb{Z}} f^{-n}Y'$. Note that $l(Y') = 1$, $g(x) = h(fx)h(x)^{-1}$ for all $x \in Y$ and $Y$ is $f$–invariant. For $x, y \in Y$ we can iterate the cocycle equation to give

\[
g_n(y)^{-1}g_n(x) = h(y)h(f^nx)^{-1}h(f^n x)h(x)^{-1}.
\]

By Lusin’s Theorem (Theorem 2.3.3) we can find a set $H$, $l(H) > \frac{1}{2}$ such that both $h$ and $h^{-1}$ are uniformly continuous when restricted to $H$. For almost stable leaves $W^s(x)$ and any full measure set $S$, the intersection of $W^s(x)$ with $S$ has full $\mu_s$ measure, where $\mu_s$ is the induced conditional measure on $W^s(x)$ (i.e. scaled one-dimensional Lebesgue measure). In particular if we let $S$ be the set of points $x \in \mathbb{T}^2$ such that

\[
l(H) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_H(f^i x) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_H(f^{-i} x)
\]

(by the ergodicity of $f$ a set of full measure) then we can conclude that for almost every $W^s(x)$

\[l_x(S \cap W^s(x)) = l_x(W^s(x)).\]

Since $l(H) > \frac{1}{2}$ for each $x, y \in W^s(x) \cap S$ there exists a sequence $n_i \to \infty$ such that $f^{n_i}y, f^{n_i}z \in H$ for all $i$. Thus if we let $n \to \infty$ through the sequence $n_i$ we obtain

\[
h(y)h(x)^{-1} = \Phi^e_x(y).
\]

Hence for $l$–almost all stable leaves $W^s(x)$ and $l_x$–almost all $y \in W^s(x)$

\[
h(y) = \Phi^e_x(y)h(x).
\]

Thus there is a version of $h$ whose restriction to almost every stable leaf $W^s(x)$ is Lipschitz. A similar argument shows that there is a version of $h$ whose restriction to almost every unstable leaf $W^u(x)$ is Lipschitz.
3.5 The Sinai Method

Our next aim is to prove that almost every \( x \) has a neighbourhood on which \( h \) has a Lipschitz version. For this we recall the Sinai Method used in [40] to prove the local ergodicity of \( f \).

Take a point \( x \in \mathbb{T}^2 \) and let \( U(x) \) be any square neighbourhood of \( x \), centered at \( x \), with sides parallel to the stable and unstable directions. To aid visualisation we shall regard the stable and unstable directions as vertical and horizontal, respectively. Given \( n \in \mathbb{N} \) we construct an open covering \( C_n \) of \( U(x) \) by firstly dividing it into \( n^2 \) squares of equal size and then expanding them by a factor of two, keeping centers fixed. For squares in \( C_n \) we have the following definitions.

**Definitions.** Let \( S \) be a square in \( \Gamma_n \), \( y \) a point in \( S \) which has a stable leaf and an unstable leaf. If \( W^s(y)(W^u(y)) \) intersects both horizontal (vertical) edges of \( S \) then we say that \( W^s(y)(W^u(y)) \) is long or connecting in \( S \). Otherwise \( W^s(y)(W^u(y)) \) is called short in \( S \).

A square \( H \in \Gamma_n \) is called connecting if:

\[
\{y \in S \mid W^s(y) \text{ and } W^u(y) \text{ is long in } S\} > \frac{3}{4} l(S).
\]

Otherwise \( S \) is called nonconnecting.

A stable (unstable) leaf is said to completely intersect a row (column) of squares if it is connecting in one of the squares in the row (column).

Note that if \( H_1, H_2 \) are two adjacent connecting squares then

\[
l \{p \in H_1 \cap H_2 \mid W^u(p) \text{ and } W^s(p) \text{ are long in } H_1 \text{ and } H_2\} \geq \frac{1}{4} l(H_1).
\]

3.5.1 Theorem (Sinai Theorem). Let \( x \) be smooth in both past and future. Then there exists a square neighbourhood \( U(x) \) such that

\[
\lim_{n \to \infty} n \left\{ \mathcal{S} \subseteq C_n \mid S \text{ is nonconnecting} \right\} = 0
\]

**Proof.** We consider only the ‘stable part’ of the theorem. That is we show that there exists a square neighbourhood \( U_s(x) \) such that

\[
\lim_{n \to \infty} n \left\{ \mathcal{S} \subseteq C_n \right\} = 0
\]

where \( \mathcal{B} \) consists of all squares \( S \in C_n \) such that

\[
l \left\{ y \in S \mid W^s(y) \text{ has an endpoint on } \bigcup_{t=0}^{\infty} f^{-t}R^+ \right\} \geq \frac{1}{8} l(S).
\]

First split the collection of nonconnecting squares in \( \mathcal{B} \) into three classes \( \mathcal{B}_{N_s}, \mathcal{B}_{M_s}, \text{ and } \mathcal{B}_{\infty} \), of squares which are nonconnecting by virtue of lines in \( \bigcup_{t=0}^{N_s-1} f^{-t}R^+ \), \( \bigcup_{t=N_s}^{M_s} f^{-t}R^+ \) and \( \bigcup_{t=N_s}^{\infty} f^{-t}R^+ \).
Figure 3.2. Shaded area is less than \( \frac{1}{8} \) the area of the square \( A \).

respectively, where \( N_s \) and \( M_s \) are constants to be defined. To be precise a square, \( S \), is in \( \mathcal{B}_{N_s} \), \( \mathcal{B}_{M_s} \), or \( \mathcal{B}_\infty \) if the measure of \( y \) such that \( W^s(y) \) has an endpoint on \( S \cap \bigcup_{i=0}^{N_s-1} f^{-i}R^+ \), \( S \cap \bigcup_{i=N_s}^{M_s} f^{-i}R^+ \) or \( S \cap \bigcup_{i=0}^{\infty} f^{-i}R^+ \) is at least an eighth the area of \( S \).

Fix a sector \( \mathcal{K}_s \), of directions of lines in \( \mathbb{R}^2 \) containing the stable direction such that for any square \( A \), intersected by a line with direction in \( \mathcal{K}_s \), the area shown in figure 3.5 is no more than \( \frac{1}{8} \) the area of the square. Note that applications of \( f^{-1} \) push all lines towards the stable direction and keeps those lines with the stable direction fixed. We are thus able to fix \( N_s \in \mathbb{N} \) so that for all \( i \geq N_s \), \( f^{-i}R^+ \) has direction in \( \mathcal{K}_s \). For \( x \) smooth in the future we choose \( U_s(x) \) to be the largest open square not intersecting \( \bigcup_{i=0}^{N_s-1} f^{-i}R^+ \). Clearly for this neighbourhood \( \mathcal{B}_{N_s} \) is empty.

Now let \( M_s > N_s \) and take \( S \in \mathcal{B}_{M_s} \). We claim that \( S \) must be intersected by at least two lines in \( \bigcup_{i=N_s}^{M_s} f^{-i}R^+ \). Indeed if \( S \) were to intersect just one line, then since the direction of this line is in \( \mathcal{K}_s \) (by choice of \( N_s \)) the measure of \( y \in S \) such that \( W^s(y) \) intersects at least one line cannot exceed \( \frac{1}{8} \) the area of the square (by choice of \( \mathcal{K}_s \)). This contradicts the fact that \( S \in \mathcal{B}_{M_s} \). Now for \( n \) large enough, we can ensure that the only way \( S \) can intersect two lines in \( \bigcup_{i=N_s}^{M_s} f^{-i}R^+ \) is if it contains an intersection point. As there is a fixed finite number of such points, we can conclude that the total measure of all \( S \in \mathcal{B}_{M_s} \) is, for \( n \) sufficiently large, \( \frac{c}{n^\alpha}, \) a constant.

To finish we consider \( S \in \mathcal{B}_\infty \). Let \( W^s(x) \) be a short leaf in \( S \) with an endpoint on \( f^{-i}R^+ \), \( i > M_s \). Let \( X_i \) be as defined in equation 3.4.1 in the proof of Lemma 3.4.1 and define \( t_i := \frac{2s\lambda_s^i}{n} \), where \( s = \sqrt{t(U_s(x))} \). As \( l_s(W^s(y)) = \lambda_s l_s(f(W^s(y))) \) and \( l_s(W^s(y)) < l_s(S) = \frac{2s}{n} \) we have that \( f^i(W^s(y)) \cap S \subset X_i \). Thus \( S \) has at least \( \frac{1}{8} \) of its area covered by \( \bigcup_{i=M_s+1}^{\infty} f^{-i}R^+ \). So as each \( y \in U_s(x) \) lies in at most four squares from any \( \Gamma_n \), the measure of the union of \( S \in \mathcal{B}_\infty \) is at most

\[
\frac{4 \times 8 \times d}{n} \sum_{i=M_s+1}^{\infty} \lambda_s^i,
\]

\( d \) a constant.
Thus for any $\epsilon > 0$ we can make a choice of $M_s$ large enough so that $I \left\{ \bigcup_{S \in \mathcal{G}_M} S \right\} < \frac{\epsilon}{2n}$. Then for $n$ large enough so that

$$\frac{\epsilon}{n^2} < \frac{\epsilon}{2n},$$

we have

$$nl \left\{ \bigcup_{S \in \mathcal{G}_M} S \right\} < \epsilon,$$

proving the claim.

A similar argument gives a square neighbourhood $U_n(x)$ defined as the largest square neighbourhood of $x$ not intersecting the lines $\bigcup_{i=0}^{N_u-1} f^i R^-$, where $N_u$ is defined similarly to $N_s$. Letting $U(x)$ be the largest square neighbourhood of $x$ not intersecting $\bigcup_{i=0}^{N_u-1} f^i R^- \cup \bigcup_{i=0}^{N_u-1} f^{-i} R^+$ proves the theorem.

Let $x$ be smooth in past and future. Let $y$ and $z$ be points which are smooth in past and future and such that $h$ is uniformly Lipschitz on their stable and unstable leaves. We will call such points typical. Also assume that $W^s(y)$ and $W^u(z)$ are such that $l_y$ and $l_z$ almost every point on them is typical and contained in $\mathcal{A}$. The set of such points is of full measure in $U(x)$.

Using the Sinai Theorem it is straightforward to show:

3.5.2 Lemma. There exists $n \in \mathbb{N}$ so that at least one row of connecting squares in $\Gamma_n$ is completely intersected by $W^s(z)$ and one column of connecting squares by $W^u(y)$.

Proof. Suppose that for any $n$ all rows which are completely intersected by $W^s(z)$ contain at least one connecting square. Since the number of such rows grows linearly with $n$ and the measure of one square is $\frac{sm(U(x))}{n^2}$ this implies that $\lim_{n \to \infty} \frac{nm(U(x) \cap S \text{ is nonconnecting})}{n} = O(1)$ contradicting the Sinai theorem. The statement regarding $W^u(y)$ is proven similarly.

We may choose a row and column of connecting squares which are intersected completely by $W^s(z)$ and $W^u(y)$ respectively. We are thus able to construct a $W^{u,s}$ path of stable and unstable leaves which links $W^s(z)$ and $W^u(y)$. By absolute continuity of the foliation the intersections of the leaves of the path may be chosen to be typical points $\{x_i\}, i = 1, \ldots, N$ say, which also lie in $\mathcal{A}$. More explicitly

$$x_1 \in W^s(z), \quad x_N \in W^u(y)$$

and

$$x_i \in (W^s(x_{i-1}) \cup W^u(x_{i-1})) \cap (W^s(x_{i+1}) \cup W^u(x_{i+1})), \quad i = 2, \ldots, N - 1.$$
Furthermore since $x_i \in A'$, $d(h(x_i), h(x_{i+1}) \leq Kd(x_i, x_{i+1})$ for each $i = 0, \ldots, N + 1$ (where for convenience we take $x_0 = z$ and $x_{N+1} = y$). Since $\sum_{i=0}^{N+1} d(x_i, x_{i+1}) \leq \sqrt{2}d(z, y)$, it follows that $h$ has a Lipschitz version on $U(x)$ and hence $h$ has a version which is Lipschitz on each connected open component of $T^2/L$, where

$$L := \bigcup_{i=0}^{N_u-1} f^iR^- \cup \bigcup_{i=0}^{N_s-1} f^{-i}R^+.$$

**Proof of Theorem 3.3.1**

We now show that $h$ is uniformly Lipschitz on $T^2$. Let $C_1$ and $C_2$ be two open components of $T^2 \setminus L$ separated by a line segment $l \in \bigcup_{i=0}^{M-1} f^iR^-$. Take a point $z$ on $l$, not lying on any other line in $L$ and a small neighbourhood $A$ of $z$.

The point $fz$ is contained in the interior of one of the components of $T^2/L$. For $A$ sufficiently small, $f(A)$ will lie wholly inside one of the components. This is because the only invariant lines have direction vectors equal to the eigenvectors of the linearization of $f$. As $g$ is Lipschitz on $A$ we can thus, in view of the equation $g(x) = h(fz)h(x)^{-1}$, conclude that $h$ is Lipschitz on $A$, allowing us to extend the Lipschitz version of $h$ to the union of $C_1$ and $C_2$. We then argue inductively to take care of all lines $l \in \bigcup_{i=0}^{M-1} f^iR^-$. In a similar way we can also inductively extend $h$ as a Lipschitz version across the boundaries of the components given by lines $l \in \bigcup_{i=0}^{N-1} f^{-i}R^+$ using the identity $h(x) = g(f^{-1}x)h(f^{-1}x)$. Thus $h$ has a Lipschitz extension to $T^2$. This concludes the proof of Theorem 3.3.1.

**Proof of Theorem 3.3.2**

In the set-up we have conditions (I)-(III) imply conditions A-F of [40]. The same argument as in the case of the toral linked twist map shows that there is a full measure set $A \subset T^2$ and a version of $h$ which has the same Lipschitz constant $C$ when restricted to $W^u_s(x)$ and $W^u_s(x)$ for $x \in A$ (note that $\epsilon$ does not indicate the induced length of local stable or unstable manifolds, over which there is no uniformity, merely that they are not global). Take a neighbourhood $U(x)$ of $x$ and two points $y, z \in A \cap U(x)$ for which there exists a $W^u_s$ path which links $W^s(y)$ and $W^s(z)$. The absolute continuity of the foliations into local stable and unstable leaves allows us to assume that if $x_i, i = 1, \ldots, N$ are the intersections of the leaves of the $W^u_s$ path then $x_i \in A$. Conditions (I) and (II) imply that $\sum_{i=1}^{N} d(x_i, x_{i+1}) \leq Kd(y, z)$ and hence

$$d(h(y), h(z)) \leq \sum_{i=1}^{N} d(h(x_i), h(x_{i+1}))$$

$$\leq \sum_{i=1}^{N} Cd(x_i, x_{i+1})$$

$$\leq KCd(y, z).$$
Thus $h$ has a version $h'$ which is Lipschitz on each neighbourhood $U(x)$ with the same Lipschitz constant.
Chapter 4

Nonuniformly Expanding Maps of the Interval

Following similar methods to those employed in Chapter 2 we here prove a Livsic regularity result for a large class of nonuniformly expanding interval maps. In particular we consider those maps for which a tower construction of the type studied by Young holds. Such a construction gives rise to a countable partition which has properties which can be exploited.

There are two main differences between the maps we consider in this chapter and those encountered in Chapter 2. Firstly here the maps have no singularities which simplifies many aspects of the proof. Secondly, where the main problems arise, the maps considered in this chapter do not expand uniformly in that there is a point at which the derivative vanishes.

4.1 The Axiomatized System

We will first axiomatically describe the class of maps for which our result holds.

Let \( f : I \to I \) be a transformation of the interval \( I \). Suppose that there exists some subinterval \( A \subset I \), the reference set, of \( I \) together with

- a countable collection of subintervals \( \{ A_i \subset A \} \);
- an unbounded function \( R : A \to \mathbb{N} \), constant on each \( A_i \) to the value \( R_i \);
- a function \( s : A \times A \to \mathbb{N} \cup \{ \infty \} \).

satisfying the following axioms.

(A1) The collection \( \{ A_i \mid R_i > n \} \) partitions \( A \) for all \( n \in \mathbb{N} \).

(A2) For \( x, y \in A_i \) \( s(x, y) = R_i + s(f^{R_i} x, f^{R_i} y) \).
(A3) For each $i$, $f^{R_i}$ maps $A_i$ injectively onto $A$.

(A4) There exists $C > 0$ and $0 < \beta < 1$ such that for almost all $x, y \in A$ and all $0 \leq j < s(x, y)$
\[ d(f^j(x), f^j(y)) < C \beta^{s(x,y)-j}. \]

(A5) There exists $R > 0$ such that for all $A_i$ and all $x, y \in A_i$
\[ \frac{1}{R} \leq \frac{(f^j)'(x)}{(f^j)'(y)} \leq R \]
for all $j = 0, \ldots, s(x, y)$.

(A6) There exists an ergodic $f$-invariant measure $\mu$, absolutely continuous with respect to Lebesgue measure $I$ and such that
\[ \frac{1}{M} \leq \frac{d\mu}{dt} \leq M. \]

4.2 Unimodal Maps

We now look at a large class of unimodal maps for which conditions (A1)–(A6) (and hence our result) holds. Let $f: [-1, 1] \to [-1, 1]$ be a $C^2$ map with
\[ f'(x) = \begin{cases} > 0 & \text{if } x < 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x > 0 \end{cases} \]
so $f$ is a unimodal map. Suppose that $f$ additionally satisfies
- $f''(0) \neq 0$.
- There exists $\alpha > 0$ and $\lambda > 0$ such that:
  - $|f^n'(f(0))| \geq \lambda^j$ for all $j \geq 1$;
  - $|f^j(0)| \geq e^{-\alpha j}$ for all $j \geq 1$.
- There exists $\delta > 0$, $\delta > 1$ and $M \in \mathbb{N}$ such that:
  - If $x, f(x), \ldots, f^{M-1}x \not\in (-\delta, \delta)$ then $|f^M'(x)| \geq \delta^M$;
  - For all $j$, if $x, f(x), \ldots, f^{j-1}(x) \not\in (-\delta, \delta)$ and $f^j \in (-\delta, \delta)$ then $(f^j)'(x) \geq \delta$.

Then $f$ satisfies the axioms (A1)–(A6) with respect to some well chosen reference set $A$ (see [5, Section 3.4]). For such maps the partition $\{A_i\}$ which satisfies (A1)–(A6) arises via a tower construction in the work of Young [66, 65]. Here, though, we will not make explicit use of the tower dynamics.

It can be shown ([5, Section 3.4]) that for a positive measure set of $a \in (0, 1)$ the logistic map $f_a(x) = 1 - ax^2$ falls in this class of unimodal maps.
Chapter 4. Nonuniformly Expanding Maps of the Interval

4.3 The Regularity Result

Let $G$ be a connected Lie group $G$ with Lie algebra $L(G)$ which we identify with the connected component of the identity $T_eG$. We let $r_g$ denote right multiplication by $g \in G$. Given a norm $\| \cdot \|_e$ on $T_eG$ we define a norm $\| \cdot \|_g$ on $T_gG$ by $\|v\|_g = \|r_g \cdot v\|_e$. This norm induces a right invariant metric $d(\cdot, \cdot)$ on $G$ so that $d(gk, hk) = d(g, h)$ for all $g, h, k \in G$ [53, Section 4].

We define the adjoint map $\text{Ad}: G \to \text{Aut}(L(G))$, for $g \in G$ and $X \in T_gG$ by

$$\text{Ad}(g)X = \frac{d}{dt}(g \exp(tv)g^{-1}).$$

A calculation [58, Section 4] shows that

$$d(gh, hk) \leq \| \text{Ad}(g)\|_e d(h, k)$$

for all $g, h, k \in G$.

Let $g: X \to G$ be Hölder of exponent $\gamma > 0$. We define

$$\lambda := \lim_{n \to \infty} \left( \sup_{x \in I} \| \text{Ad}(g_n(x))\| \right)^{\frac{1}{n}}$$

where $g_n(x) = g(f^{n-1}x) \cdots g(x)g(x)$. We will assume the following condition holds:

$$(\text{PH}) \quad 1 \leq \lambda < \beta^{-1}.$$ 

Define

$$\Theta_{\text{PH}} := \frac{\ln \lambda}{\ln \beta}$$

then condition (PH) implies that $\Theta_{\text{PH}} < 1$.

Note that if $G$ is compact, nilpotent or abelian, then $\lambda = 1$ and (PH) is satisfied.

4.3.1 Theorem. Let $f: I \to I$ be a $C^1$ interval map which satisfies axioms (A1)–(A6) with respect to the reference set $\Lambda$. Let $g: I \to G$ be a Lipschitz or Hölder cocycle satisfying hypothesis (PH). If $g$ is Hölder assume it has exponent $\gamma > \Theta_{\text{PH}}$. Then for any measurable solution $h: I \to G$ to the cocycle equation

$$g(x) = h(fx)h(x)^{-1} \quad a.e.$$ 

there exists an $h': I \to G$ which is Lipschitz (Hölder) such that $h = h'$ a.e. on $\Lambda$.

4.4 A Lebesgue Density Argument

As $h$ is measurable we can use Lusin's Theorem to find a positive measure set $H$ such that $h$ restricted to $H$ is uniformly continuous. Further by the Lebesgue Density Theorem we can find a sequence of positive numbers $\rho_n \to \infty$ together with sets

$$H_n := \left\{ y \in H \mid \frac{l((y - \rho, y + \rho) \cap H)}{l(y - \rho, y + \rho)} > 1 - \frac{1}{2^n} \text{ for all } 0 < \rho < \rho_n \right\}$$

such that \( l(H_n) > (1 - \frac{1}{2^n}) \).

Fix some sequence \( i_n \to \infty \) of natural numbers such that

\[
R_{i_n} > \frac{\ln \beta}{\ln 2 \rho_n} \tag{4.4.1}
\]

and

\[
\Lambda_{i_n} \cap H_n \neq \emptyset. \tag{4.4.2}
\]

Given a point \( y \in \Lambda \) we define a sequence \( y_{i_n}, n \in \mathbb{N} \), recursively setting \( y_{i_n} \in \Lambda_{i_n} \) to be such that \( f^{R_{i_n}}y_{i_n} = y_{i_n} \), with \( y_0 = y \). By (A2) each \( y_{i_n} \) is uniquely defined. Thus given \( n \in \mathbb{N} \) any \( y \in \Lambda \) can be uniquely represented as \( f^{S_n}y_{i_n} \) where

\[
S_n := \sum_{j=1}^{n} R_{i_j}.
\]

By (A1) \( s(y_{i_n}, z_{i_n}) > S_n \) for all \( y, z \in \Lambda \).

We claim the following:

**4.4.1 Lemma.** Given \( \eta > 0 \) there exists \( N = N(\eta) \in \mathbb{N} \) such that for all \( n > N \)

\[
\mu(\{y \in \Lambda \mid y_{i_n} \in H\}) > (1 - \eta)\mu(\Lambda).
\]

Fix \( \eta > 0 \) and take \( n \) large enough so that \( \frac{C^2 D^2}{2^{n-1}} < \eta \). By conditions (4.4.1) and (4.4.2) we can express \( \Lambda_{i_n} = (y - \rho, y + \rho) \), where \( y \in H_n \) and \( \rho < \rho_n \). We thus have

\[
l(\Lambda_{i_n} \cap H^c) \leq l(\Lambda_{i_n} \cap H_n^c) < \frac{1}{2^{n-1}} l(\Lambda_{i_n}).
\]

Then by (A5) and (A6)

\[
\mu(f^{R_{i_n}}(\Lambda_{i_n} \cap H^c)) \leq \frac{C^2 D^2}{2^{n-1}}.
\]

Given that \( n \) was chosen so that \( \frac{C^2 D^2}{2^{n-1}} < \eta \) and that \( f^{R_{i_n}}(\Lambda_{i_n}) = \Lambda \), this proves the lemma.

### 4.5 Iterating the Cocycle

Note that the set of \( y \in \Lambda \) such that for any \( n \in \mathbb{N} \), \( g(f^{j-1}y_{i_n}) = h(f^jy_{i_n})h(f^{j-1}y_{i_n}) \) for all \( 0 < j \leq S_n \), is of full measure. We will thus deal only with such points. Given such \( y \), then we can iterate the cocycle equation so that for any \( n \) we may write

\[
h(y) = h(f^{S_n}y_{i_n}) = gs_n(y_{i_n})h(y_{i_n})^{-1} \tag{4.5.1}
\]

where \( gs_n = g(f^{S_n-1}y_{i_n}) \cdots g(fy_{i_n})g(y_{i_n}) \).

Now fix \( 0 < \delta < 1 \). By Lemma 4.4.1 we can find a subsequence \( j_m \to \infty \) such that

\[
\mu(\{y \in \Lambda \mid y_{j_m} \in \Lambda_{j_m} \cap H\}) > \left( 1 - \frac{\delta}{2^k} \right) \mu(\Lambda).
\]
Since \( \sum_{m=0}^{\infty} \frac{1}{2m+1} < 1 \) we may choose a point \( x \in \Lambda \) such that \( x_{jn} \in \Lambda_{jn} \cap H \) for all \( j_n \).

For \( n \in \mathbb{N} \) we define a function \( \Phi_n : \Lambda \to \Gamma \) by

\[
\Phi_n(y) = g_{s_n}(y_n)g_s(x_{i_n})^{-1}.
\]

Then by (4.5.1), given any \( n \in \mathbb{N} \) we can write

\[
h(y) = \Phi_n(y)h(x_{i_n})^{-1}h(y_n).
\]  
(4.5.2)

We claim that each \( \Phi_n \) is Hölder with uniform Hölder constant. To see this take arbitrary \( y, z \in \Lambda \) and calculate

\[
d(\Phi_n^a(y), \Phi_n^a(z)) = d(g_{s_n}(y_n)g_s(x_{i_n})^{-1}, g_{s_n}(z_{i_n})g_s(x_{i_n})^{-1})
\]

\[
= d(g_{s_n}(y_n), g_{s_n}(z_{i_n}))
\]

\[
\leq \sum_{i=0}^{S_n-1} d(g_i(f_{s_n}^{-i}y_n), g_i(f_{s_n}^{-i}z_{i_n}))g_{s_n}(f_{s_n}^{-i}y_n)g_{s_n}(f_{s_n}^{-i}(z_{i_n}))g_{s_n}(y_{i_n})g_{s_n}(z_{i_n})g_{s_n}(x_{i_n})^{-1}(f_{s-n}z_{i_n})^{-1}
\]

\[
\leq \sum_{i=0}^{S_n-1} \| \text{Ad}_i(f_{s_n}^{-i}y_n) \| d(g(f_{s_n}^{-i-1}(y_n)), g(f_{s_n}^{-i-1}(z_{i_n})))
\]

\[
\leq c \sum_{i=0}^{S_n-1} \| \text{Ad}_i(f_{s_n}^{-i}y_n) \| d(f_{s_n}^{-i-1}(y_n), f_{s_n}^{-i-1}(z_{i_n}))^{-1},
\]

where \( c \) is the Hölder constant of \( g \). Now by the bounded distortion condition \((A5)\) there is a constant \( R' \) such that for all \( y_n, z_{i_n} \in \Lambda_{i_n} \)

\[
\frac{1}{R'} \frac{d(f_{s_n}y_n, f_{s_n}z_{i_n})}{d(f_{s_n}Y_n, f_{s_n}Z_{i_n})} \leq \frac{d(f_{s_n}y_n, f_{s_n}z_{i_n})}{d(f_{s_n}Y_n, f_{s_n}Z_{i_n})} \leq R' \frac{d(f_{s_n}y_n, f_{s_n}z_{i_n})}{d(f_{s_n}Y_n, f_{s_n}Z_{i_n})}
\]

for all \( j = 0, \ldots, s(x_{i_n}, y_n) \), where \( Y = f_{s_n}Y_n \) and \( Z = f_{s_n}Z_{i_n} \) are the endpoints of \( \Lambda \). Let \( K \) be any real constant large enough so that \( K \times t(\Lambda) \geq \beta \). We thus have

\[
d(\Phi_n^a(y), \Phi_n^a(z)) \leq cR' \sum_{i=0}^{S_n-1} \lambda_i^j d(f_{s_n}^{-i-1}(Y_n), f_{s_n}^{-i-1}(Z_{i_n}))^{-1} \times d(f_{s_n}(y_n), f_{s_n}(z_{i_n}))^{\gamma} \times d(f_{s_n}(Y_n), f_{s_n}(Z_{i_n}))^{\gamma}
\]

\[
\leq cR' \lambda \sum_{i=0}^{S_n-1} \lambda_i^j d(f_{s_n}^{-i-1}(Y_n), f_{s_n}^{-i-1}(Z_{i_n}))^{\gamma} \times d(f_{s_n}(y_n), f_{s_n}(z_{i_n}))^{\gamma}
\]

\[
\leq cR'K \sum_{i=0}^{S_n-1} \lambda_i^j d(f_{s_n}(Y_n), f_{s_n}(Z_{i_n}))^{\gamma} \times d(f_{s_n}(Y_n), f_{s_n}(Z_{i_n}))^{\gamma} \sum_{i=0}^{S_n-1} \lambda_i^j
\]

\[
\leq K' \times (\lambda \beta')^j
\]

where \( K' = cR'K \sum_{i=0}^{S_n-1} (\lambda \beta')^j \) is a constant. Note that \( (\lambda \beta')^j \) sums since, by \((PH)\) and the fact that \( \gamma > \Theta_{PH}, \lambda \beta' < 1 \). Thus each \( \Phi_n^a \) is \( \gamma \)-Hölder with Hölder constant independent of \( n \).
4.6 A Hölder Version on $\Lambda$

Take points $y, z \in \Lambda$. Then by (4.5.2) and the triangle inequality, for any $n \in \mathbb{N}$ we have

$$d(h(x), h(y)) \leq d(\Phi_n(y)h(x_{i_n})^{-1}h(y_{i_n}), \Phi_n(y))$$

$$+ d(\Phi_n(z)h(x_{i_n})^{-1}h(z_{i_n}), \Phi_n(z))$$

$$+ d(\Phi_n(y), \Phi_n(z)).$$

Recall that we have a sequence $j_m \to \infty$ of natural numbers such that $x_{j_m} \in H$ for all $j_m$. Thus given $\eta > 0$ there exists $M \in \mathbb{N}$ such that if $j_m > M$ and $y_{j_m}, z_{j_m} \in H$

$$d(\Phi_{j_m}(y)h(x_{j_m})^{-1}h(y_{j_m}), \Phi_{j_m}(y)) < \frac{\eta}{2}$$

and

$$d(\Phi_{j_m}(z)h(x_{j_m})^{-1}h(z_{j_m}), \Phi_{j_m}(z)) < \frac{\eta}{2}$$

Further by Lemma 4.4.1 there exists $N$ such that for all $j_m > N$

$$(\mu \times \mu)((y, z) \in \Lambda \times \Lambda \mid y_{j_m}, z_{j_m} \in H) > (1 - 2\eta)\mu \times \mu(\Lambda \times \Lambda).$$

Thus, recalling that $d(\Phi_n(y), \Phi_n(z)) < K'd(y, z)$ for all $n$, we have

$$(\mu \times \mu)((y, z) \in \Lambda \times \Lambda \mid d(h(y), h(z)) < K'd(y, z)\gamma + \eta)) \geq (1 - 2\eta)(\mu \times \mu)(\Lambda \times \Lambda)).$$

Since $\eta$ was arbitrary this implies that

$$(\mu \times \mu)((y, z) \in \Lambda \times \Lambda \mid d(h(y), h(z)) \leq K'd(y, z)\gamma)) = (\mu \times \mu)(\Lambda \times \Lambda).$$

Hence $h|_{\Lambda}$ has a Hölder version proving Theorem 4.3.1.
Further Work I

To close this part we discuss some possibilities for expanding the results of the previous chapters.

Lasota–Yorke Maps

Recall that for Lasota–Yorke maps we obtained 'local' Livšic regularity in that a measurable solution to the cocycle equation has a version which was Hölder or Lipschitz on a neighbourhood of almost every point. It would be interesting to see if, as for $\beta$-transformations and Markov maps, this could be extended to give regularity on the whole of the support of the absolutely continuous measure. It is not clear how this could be done in general as the methods for $\beta$-transformations and Markov maps depend on the geometry of the maps. As it stands though, local regularity is sufficient for applications as we will see later.

It would also be interesting to see whether the results of this chapter could be extended to give higher regularity results. That is given a $C^r$ cocycle do measurable solutions to the cocycle equation have $C^r$ versions. For this the natural extension would be inadequate and we would have to use some other way of extending to an invertible map, perhaps using the solenoidal lift of [62].

Uniformly Hyperbolic Toral Maps with Singularities

Similarly to the case of Lasota–Yorke maps, for general uniformly hyperbolic toral maps with singularities we obtained only local regularity. Whilst this is still useful for applications it would be interesting to see whether this could be extended to a full regularity result as we did in the simpler case of toral linked twist maps.

As we discussed for Lasota–Yorke maps we could consider higher, $C^r$ versions of our results for uniformly hyperbolic toral maps with singularities. It is possible this could be done based upon the work of Journé [30].
Further we could consider higher dimensional analogues of the results, i.e. for uniformly hyperbolic maps of $n$-dimensional manifolds with singularities.

**Nonuniformly Expanding Maps of the Interval**

For the class of nonuniformly expanding interval $f: I \to I$ maps considered in Chapter 4 we showed that measurable solutions to the cocycle equation have versions which are Hölder or Lipschitz on the so-called 'reference set' $\Lambda$. It would be interesting to see whether this version could be extended to the whole of the interval $I$. Using the cocycle equation $g(z) = h(fz)h(z)^{-1}$ we can see that a Hölder (Lipschitz) version can be extended to any image of $\Lambda$. In particular if there exists a finite $N \in \mathbb{N}$ such that $f^N \Lambda = I$ then we have a Hölder (Lipschitz) version on $I$.

Combining with the case in Chapter 2 we could look at interval maps which are piecewise $C^3$, nonuniformly expanding. We could also consider Livšic regularity for nonuniformly expanding maps of $n$ dimensional manifolds including those with singularities.
Part II

Stable Ergodicity of Compact Group Extensions
Chapter 5

Compact Group

Extensions

In this chapter we give a brief overview of the theory of compact group extensions. We concentrate on their ergodic properties, in particular we give the Keynes-Newton conditions for the ergodicity and weak-mixing of compact group extensions. Towards the end of this chapter we consider extensions over hyperbolic bases.

5.1 Definitions

Let \((X, \mathcal{B}, \mu)\) be a probability space, \(f: X \to X\) a measure preserving map. Given a compact Lie group \(\Gamma\) and a measurable map \(h: X \to \Gamma\) we define the compact group extension of \(f\) with respect to \(h\) (or skew product) to be the map \(f_h: X \times \Gamma \to X \times \Gamma\) defined by

\[
 f_h(x, \gamma) = (fx, h(x)\gamma).
\]

We call \(f: X \to X\) the base map and \(h: X \to \Gamma\) the cocycle. It is clear that \(f_h\) preserves the product measure \(\mu \times \nu\), where \(\nu\) denotes Haar measure on \(\Gamma\). We also note here that for each \(n \in \mathbb{N}\)

\[
 f_h^n(x, \gamma) = (f^n x, h_n(x)\gamma)
\]

where \(h_n(x) = h(f^{n-1}x) \ldots h(fx)h(x)\).

We could also define \(f_h\) by \(f_h(x, \gamma) = (fx, \gamma h(x))\) but we will stick to the above convention. All results we consider will hold either way.

There are ways by which the concept of a compact group extension can be generalised. For example we could take a measure preserving transformation \(f: X_1 \to X_1\) of the probability space \((X_1, \mathcal{B}_1, \mu_1)\) together with a family \(\{h_x \mid x \in X_1\}\) of measure preserving transformations of a probability space \((X_2, \mathcal{B}_2, \mu_2)\). Assuming that the map from \(X_1 \times X_2\) to \(X_2\) taking \((x, y) \to h_x(y)\) is measurable we may define a measure preserving (with respect to the product measure \(\mu_1 \times \mu_2\)) map \(F: X_1 \times X_2 \to X_1 \times X_2\) by \(F(x, y) = (fx, h_x(y))\).
Chapter 5. Compact Group Extensions

5.2 Ergodicity and Mixing

We here consider criteria under which ergodic properties of base transformations can be 'lifted' to their compact group extensions. Originally proven by Keynes and Newton [33] the following results show that the ergodicity and mixing of compact group extensions is equivalent to the non-existence of solutions to certain cohomological equations. The proofs we present follow the particularly clear exposition given by Noorani [50].

5.2.1 Theorem. [50] Let \( f: X \rightarrow X \) be a measure preserving map of the probability space \((X, \mathcal{B}, \mu)\). Let \( \Gamma \) be a compact Lie group with Haar measure \( \nu \), \( h: X \rightarrow \Gamma \) a measurable map. Suppose \( f: X \rightarrow X \) is ergodic with respect to \( \mu \). Then the compact group extension \( f_h: X \times \Gamma \rightarrow X \times \Gamma \) is ergodic with respect to \( \mu \times \nu \) if and only if there are no nontrivial measurable solutions \( v: X \rightarrow \mathbb{C}^d \) to the equation

\[
v(fx) = R(h(x))v(x) \quad \text{a.e.} \quad x \in X
\]

(5.2.1)

where \( R: \Gamma \rightarrow U(d) \) is a non-trivial irreducible unitary representation of \( \Gamma \).

Proof. Suppose that there is some non-trivial measurable map \( v: X \rightarrow \mathbb{C}^d \) together with a non-trivial irreducible unitary representation \( R: \Gamma \rightarrow U(d) \) satisfying (5.2.1). We define a measurable function \( V: X \times \Gamma \rightarrow \mathbb{C}^d \) by

\[
V(x, \gamma) = R(\gamma^{-1})v(x) \quad (x, \gamma) \in X \times \Gamma
\]

which is, by assumption, nonconstant. Further we have

\[
V \circ f_h(x, \gamma) = v(fx, h(x)\gamma)
\]

(5.2.2)

\[
= R((h(x)\gamma)^{-1})v(fx)
\]

(5.2.3)

\[
= R(\gamma^{-1})R(h(x))^{-1}v(fx)
\]

(5.2.4)

\[
= R(\gamma^{-1})v(x)
\]

(5.2.5)

\[
= V(x, \gamma) \quad \text{a.e.} \quad (x, \gamma).
\]

Thus \( f_h \) cannot be ergodic.

Conversely suppose that \( f_h \) is not ergodic. We will construct a non-trivial irreducible representation \( R \) and a corresponding solution \( v \) to (5.2.1). Firstly note that, on \( L^2(X \times \Gamma) \) we have a continuous action by \( \Gamma \) given, for \( \gamma \in \Gamma \), \( \omega \in L^2(X \times \Gamma) \), by \( g \cdot f_h = U_{S_g}(\omega) = \omega \circ S_g \), where \( S_g: X \times \Gamma \) is the map given by \( S_g(x, \gamma) = (x, \gamma g) \). With regards to \( S_g \) we note that for \( (x, \gamma) \in X \times \Gamma \)

\[
(S_g \circ f_h)(x, \gamma) = S_g(fx, h(x)\gamma)
\]

(5.2.6)

\[
= (fx, h(x)\gamma g)
\]

(5.2.7)

\[
= f_h(x, \gamma g)
\]

(5.2.8)

\[
= (f_h \circ S_g)(x, \gamma).
\]
In other words the action of $\Gamma$ on $L^2(X \times \Gamma)$ commutes with $f_h$.

Consider the Hilbert space $\mathcal{H} := \{ \omega \in L^2(X \times \Gamma) \mid \omega \circ f_h = \omega \text{ a.e.} \}$, which by the assumption that $f_h$ is not ergodic is non-trivial. Now take $\omega \in \mathcal{H}$ and, for $g \in \Gamma$ calculate

\[(g \cdot \omega) \circ f_h = (\omega \circ S_g) \circ f_h = \omega \circ (f_h \circ S_g) = (\omega \circ f_h) \circ S_g = \omega \circ S_g = g \cdot \omega.\]

Thus $\mathcal{H}$ is a $\Gamma$-invariant subspace of $L^2(X \times \Gamma)$. Let $R' : \Gamma \to U(\mathcal{H})$ denote the unitary representation induced by the action of $\Gamma$ on $\mathcal{H}$. We require the following:

**5.2.2 Lemma.** [50] Let $R$ be a continuous unitary representation of a compact group $\Gamma$ with representation space $\mathcal{H}$. Then $\mathcal{H}$ decomposes into a direct sum of $R$-invariant closed finite dimensional subspaces $\{V_i\}_{i \in I}$ such that the restriction of $R$ on $V_i$ is irreducible for each $i$.

Using this lemma we can find a non-trivial finite-dimensional subspace $V$, of dimension $d$ say, of $\mathcal{H}$ such that $R|_V$ is irreducible. Let $\{\omega_1, \ldots, \omega_d\}$ be an orthonormal basis for $V$ so that

\[
\begin{pmatrix}
\omega_1 \\
\vdots \\
\omega_d
\end{pmatrix}
\circ S_g = A(g)
\begin{pmatrix}
\omega_1 \\
\vdots \\
\omega_d
\end{pmatrix}
\]

for some $d \times d$ matrix, dependent only on $g \in \Gamma$.

Since $\{\omega_1, \ldots, \omega_d\}$ is orthonormal the matrix $A(g)$ is unitary for all $g \in \Gamma$. We define a non-trivial representation $R$ of $\Gamma$ by $d \times d$ unitary matrices by $R(g) = A(g^{-1})$. As it is equivalent to $R'|_V$ we see that $R$ is also irreducible.

Define $\tilde{\omega} = (\omega_1, \ldots, \omega_d)^T$. Then by (5.2.2) we have

\[\tilde{\omega}(x, \gamma g) = R(g^{-1})\tilde{\omega}(x, \gamma)\]

and so

\[\tilde{\omega} \circ f_h(x, \gamma) = \tilde{\omega}(x, \gamma) \text{ a.e. } (x, \gamma) \in X \times \Gamma.\]

By Fubini's Theorem we can find a $\gamma_0$ such that

\[\tilde{\omega} \circ f_h(x, \gamma_0) = \tilde{\omega}(x, \gamma_0) \text{ a.e. } x.\]
Further we note that
\[
\tilde{\omega}(x, \gamma_0) = \tilde{\omega} \circ f_h(x, \gamma_0) \\
= \tilde{\omega}(fx, h(x)\gamma_0) \\
= \tilde{\omega}(fx, \gamma_0(\gamma_0^{-1}h(x)\gamma_0)) \\
= R(\gamma_0^{-1}h(x)^{-1}\gamma_0)\tilde{\omega}(fx, \gamma_0).
\]
Rearranging this gives
\[
R(\gamma_0)\tilde{\omega}(fx, \gamma_0) = R(h(x))R(\gamma_0)\tilde{\omega}(x, \gamma_0) \text{ a.e. } x.
\]
So setting \( v = R(\gamma_0)\tilde{\omega}(x, \gamma_0) \) we have
\[
v(fx) = R(h(x))v(x) \text{ a.e. } x.
\]
Thus we have our required representation \( R \) and function \( v \) completing the proof.  

5.2.3 Theorem. \([50]\) Let \( f: X \to X \) be a measure preserving map of the probability space \((X, \mathcal{B}, \mu)\). Let \( \Gamma \) be a compact Lie group with Haar measure \( \nu, h: X \to \Gamma \) a measurable map. Suppose \( f: X \to X \) is weakly-mixing with respect to \( \mu \) and that the compact group extension \( f_h: X \times \Gamma \to X \times \Gamma \) is ergodic with respect to \( \mu \times \nu \). Then \( f_h \) is weak-mixing if and only if for any \( e^{i\alpha} \neq 1 \) and any non-trivial one-dimensional irreducible representation \( R \) of \( \Gamma \), the equation
\[
v(fx) = e^{i\alpha}X(h(h))v(x) \text{ a.e. } x
\]
(5.2.3)
has no non-trivial solution \( v: X \to \mathbb{C} \).

Proof. First assume that \( f_h \) is weak-mixing and, for contradiction, that there exists a non-trivial solution \( v \) to (5.2.3) for some non-trivial one-dimensional representation \( R \) of \( \Gamma \) and constant \( e^{i\alpha} \neq 1 \).

Let \( \bar{v}: X \times \Gamma \to \mathbb{C} \) be the map defined by \( \bar{v}(x, \gamma) = R(\gamma^{-1})v(x) \). Then by definition \( \bar{v} \) is measurable and constant. Furthermore we have
\[
\bar{v} \circ f_h = \bar{v}(fx, h(x)\gamma) \\
= R((h(x)\gamma)^{-1})v(fx) \\
= R(h(x)^{-1})R(\gamma^{-1})v(fx) \\
= e^{i\alpha}R(\gamma^{-1})v(x) \\
= e^{i\alpha}\bar{v}(x, \gamma) \text{ a.e. } (x, \gamma).
\]
(5.2.4)
This contradicts the fact that \( f_h \) is weak-mixing. Conversely suppose that \( f_h \) is not weak-mixing. Define
\[
\mathcal{H} := \{ \omega \in L^2(X \times \Gamma) \mid \omega \circ f_h = e^{i\alpha}\omega \text{ a.e.} \}
\]
which by the assumption that \( f_h \) is not weak-mixing is non-trivial. As in the proof of Theorem 5.2.1 we have a representation \( R' : \Gamma \rightarrow U(\mathcal{H}) \) defined by \( R'(g) = U_{S_g} \). Again we use Lemma 5.2.2 to find a \( d \)-dimensional subspace \( V \) of \( \mathcal{H} \) such that \( R'|_V \) is irreducible.

As before let \( \{\omega_1, \ldots, \omega_d\} \) be an orthonormal basis of \( V \). Since \( \omega_i \neq 0 \) a.e. for all \( i = 1, \ldots, d \) we can divide each \( \omega_i \) by \( \omega_1 \) in order to obtain

\[
\frac{\omega_i}{\omega_1} \circ f_h = \frac{\omega_i}{\omega_1} \text{ a.e.}
\]

and so by the ergodicity of \( f_h \), for each \( i \) we have \( \omega_i = \omega_1 \) or \( \omega_i = c_j \omega_1 \) for all non-zero constants \( c_j \). Thus on setting \( \mathbf{v} = (\omega_1, \ldots, \omega_d) \) we have

\[
\mathbf{v}(x, \gamma) = \omega_1(x, \gamma)\mathbf{c}
\]

(5.2.5)

where \( \mathbf{c} = (c_1, \ldots, c_d) \). As in the proof of Theorem 5.2.1 we have a representation \( R : \Gamma \rightarrow U(\mathbb{C}^d) \) arising from \( R'|_V \) with respect to the basis \( \{\omega_1, \ldots, \omega_d\} \) for which

\[
\mathbf{v}(x, \gamma g) = R(g^{-1})\mathbf{v}(x, \gamma)\mathbf{c} \text{ a.e. } (x, \gamma).
\]

So using (5.2.5) we have that

\[
\omega_1(x, \gamma g)\mathbf{c} = R(\gamma^{-1})\omega_1(x, \gamma)\mathbf{c} \text{ a.e. } (x, \gamma).
\]

(5.2.6)

Note since \( R \) is irreducible and leaves the one-dimensional subspace generated by \( \mathbf{c} \) invariant (by (5.2)) it is necessarily one-dimensional. In particular now \( \mathbf{v} : X \times \Gamma \rightarrow \mathbb{C} \). So from (5.2.4) and (5.2) we can deduce (as in the proof of Theorem 5.2.1) that

\[
\mathbf{v}(x, \gamma_0) = e^{i\alpha} R(h(x)^{-1})\mathbf{v}(f x, \gamma_0) \text{ a.e. } x
\]

for some \( \gamma_0 \in \Gamma \). Thus setting \( v(x) = \mathbf{v}(x, \gamma_0) \) we have our required function and representation proving the Theorem. \( \square \)

### 5.3 Transitivity, Periodic Data and Cohomology

In some situations measure theoretic or topological properties of compact group extensions can be reduced to conditions on periodic data or the existence of particular solutions to cohomological equations.

We here consider extensions of a topological dynamical system \( f : X \rightarrow X \). Let \( h : X \rightarrow \Gamma \) be a continuous map into the topological group \( \Gamma \). We define the subgroup generated by the periodic data

\[
\mathcal{G} := \{h_n p \mid f^n p = p\},
\]

where \( h_n p = h(f^{n-1} p) \ldots h(fp)h(p) \).
Chapter 5. Compact Group Extensions

Recall that \( h \) is said to be continuously cohomologous to the continuous function \( \tilde{h} : X \to \Gamma \) if there exists a continuous function \( \omega : X \to \Gamma \) such that

\[
h(x) = \omega(fx)\tilde{h}(x)\omega(x)^{-1}
\]

for all \( x \in X \).

The following two results indicate the relationship between the transitivity of \( f_h \), the existence of certain functions cohomologous to \( h \) and conditions on the periodic data.

5.3.1 Proposition. Let \( f : X \to X \) be a continuous map of the topological space \( X \). Let \( h : X \to \Gamma \) be a continuous map into the topological group \( \Gamma \). If \( h \) is cohomologous to some \( \tilde{h} : X \to K \), where \( K \) is some proper subgroup of \( \Gamma \) then \( f_h \) is not topologically transitive.

Proof. Suppose that \( h \) is cohomologous to some \( \tilde{h} : X \to K < G \). Then we may write

\[
h = (\omega \circ f)h\omega^{-1}
\]

for some \( \omega : X \to \Gamma \). Take an arbitrary \( x \in X \) and let \( n_k \to \infty \) be any sequence of natural numbers such that \( f^{n_k}x \to x \) as \( n_k \to \infty \). We have

\[
f_h^{n_k}(x,\gamma) = (f^{n_k}x, h_{n_k}(x)\gamma) = (f^{n_k}x, \omega(f^{n_k}x)\tilde{h}_{n_k}\omega(x)^{-1}\gamma).
\]

Now as \( n_k \to \infty \), \( \omega(f^{n_k}x) \to \omega(x) \). So since \( \tilde{h} \) takes values only in the proper subgroup \( K \), \( \omega(f^{n_k}x)\tilde{h}_{n_k}\omega(x)^{-1}\gamma \) can only converge to values in the proper coset \( \omega(x)K\omega(x)^{-1}\gamma \), of \( \Gamma \). Since \( x \) and the sequence \( n_k \) were arbitrary \( f_h \) cannot be topologically transitive. \( \square \)

5.3.2 Proposition. Let \( f : X \to X \) be a continuous map of the topological space \( X \). Let \( h : X \to \Gamma \) be a continuous map into the topological group \( \Gamma \). Suppose that \( h \) is cohomologous to some \( \tilde{h} \) which takes values only in the subgroup \( K \) of \( \Gamma \). Then \( \mathcal{P} \) is contained in \( N_{\Gamma}(K) \), the normaliser of \( K \) in \( \Gamma \). In particular if \( K \) is normal then \( \mathcal{P} \neq \Gamma \).

Proof. Suppose that \( h \) is cohomologous to such an \( \tilde{h} \). Then

\[
h = (\omega \circ f)h\omega^{-1}
\]

for some \( \omega : X \to \Gamma \). Let \( p \) be a periodic point of order \( n \), then

\[
h_{n}p = \omega(f^{n}p)\tilde{h}_{n}(p)\omega(p)^{-1}.
\]

The right hand side of the equation can take values only in \( N_{\Gamma}(K) \) proving the lemma. \( \square \)

Further results along these lines in the context of extensions of hyperbolic basic sets will be considered, in some detail, in the proceeding chapter.
5.4 Extension of Hyperbolic Bases

We now specialise to the case where the base map \( f: \Lambda \rightarrow \Lambda \) has hyperbolic behaviour. In this case Theorems 5.2.1 and 5.2.3 can be extended via Livšic regularity to the following.

5.4.1 Theorem. Let \( f: \Lambda \rightarrow \Lambda \) be hyperbolic, \( \mu \) a Hölder equilibrium state. Let \( h: \Lambda \rightarrow \Gamma \) be a Hölder map into the compact Lie group \( \Gamma \).

(i) The compact group extension \( \Lambda \times \Gamma \rightarrow \Lambda \times \Gamma \) is not ergodic if and only if there exists a nontrivial irreducible unitary representation \( R \) of \( \Gamma \) and a non-trivial Hölder \( \nu: \Lambda \rightarrow \mathbb{C}^d \) such that

\[
R(h(x))\nu(x) = \nu(fx). \tag{5.4.1}
\]

(ii) If \( f: \Lambda \rightarrow \Lambda \) is weak-mixing and \( \Lambda \times \Gamma \rightarrow \Lambda \times \Gamma \) is ergodic then \( \Lambda \times \Gamma \rightarrow \Lambda \times \Gamma \) is not weak-mixing if and only if there exists a nontrivial irreducible one-dimensional representation \( R \) of \( \Gamma \), a constant \( e^{ia} \neq 1 \) and a non-trivial Hölder \( \nu: \Lambda \rightarrow \mathbb{C}^d \) such that

\[
e^{ia}R(h(x))\nu(x) = \nu(fx). \]

For toral groups it can be useful to recast equation (5.4.1) to state the following.

5.4.2 Theorem. Let \( f: \Lambda \rightarrow \Lambda \) be hyperbolic, \( \mu \) a Hölder equilibrium state. Let \( h = (h_1, \ldots, h_r): \Lambda \rightarrow \mathbb{T}^r \) be a Hölder map into the toral group \( \mathbb{T}^r \), \( r \in \mathbb{N} \). The compact group extension \( f_\Lambda: \Lambda \times \mathbb{T}^r \rightarrow \Lambda \times \mathbb{T}^r \) is not ergodic if and only if there exist integers \( k_1, \ldots, k_r \) not all zero and a Hölder map \( \nu: \Lambda \rightarrow \mathbb{S}^1 \), such that

\[
u \circ f/\nu = h_1^{k_1} \ldots h_r^{k_r}.\]

Recall that for topological dynamical systems in which a Livšic regularity theorem holds transitivity and ergodicity are equivalent. For subshifts of finite type Parry [51] has shown that this extends to extensions:

5.4.3 Theorem. Let \( \sigma: \Sigma_A \rightarrow \Sigma_A \) be a subshift of finite type, \( h: \Sigma_A \rightarrow \Gamma \) a Hölder map into the compact Lie group. If the compact group extension \( \sigma: \Sigma_A \times \Gamma \rightarrow \Sigma_A \times \Gamma \) is topologically transitive then it is ergodic with respect to \( \mu \times \nu \), where \( \mu \) is a Hölder equilibrium state and \( \nu \) is Haar measure on \( \Gamma \).

Also for subshifts of finite type we have the following extension to the results of the previous section.

5.4.4 Theorem ([53]). Let \( \sigma: \Sigma_A \rightarrow \Sigma_A \) be a subshift of finite type, \( h: \Sigma_A \rightarrow \Gamma \) a Hölder map into the compact Lie group \( \Gamma \). The compact group extension \( \sigma: \Sigma_A \times \Gamma \rightarrow \Sigma_A \times \Gamma \) is ergodic if and only if there is no measurable function \( \omega: \Sigma_A \rightarrow \Gamma \) satisfying

\[
h(x) = \omega(fx)h(x)\omega(x)^{-1}.\]
where \( \tilde{h} : \Sigma_A \to \Gamma \) is a Hölder map taking values in a proper subgroup of \( \Gamma \).

Extensions of hyperbolic maps by compact abelian and finite groups will be considered in detail in the next chapter.
Chapter 6

Finite Group Extensions of Hyperbolic Basic Sets

In this chapter we consider extensions of hyperbolic basic sets by finite groups. We are interested in criterion for ergodicity and mixing given in terms of cohomological conditions and periodic data. We first consider abelian extensions where we prove results in the more general setting of compact group extensions. We then move on to look at general finite group extensions firstly of subshifts of finite type. These results are then lifted to the setting of general hyperbolic basic sets. After considering the special case of a base with finitely many connected components we finish by looking at a weak-mixing of finite group extensions.

6.1 Preliminaries

We first cover some notation regarding subshifts of finite type as well as some basic properties of finite group extensions of subshifts and hyperbolic basic sets.

Subshifts of Finite Type

Let $\sigma: \Sigma_A \rightarrow \Sigma_A$ be a subshift of finite type. That is

$$\Sigma_A := \{\{x_i\}_{i \in \mathbb{Z}} \in \{1, 2, \ldots, k\}^\mathbb{Z} \mid A[x_i, x_{i+1}] = 1 \text{ for all } i \in \mathbb{Z}\},$$

where $A$ is a $k \times k$ matrix with entries $A[i, j] \in \{0, 1\}$ and $\sigma$ is the 'left shift map' defined by $\sigma(\{x_i\}) = \{y_i\}$, $y_i = x_{i+1}$. We assume that the matrix is irreducible, so that for all $1 \leq i, j \leq k$ there exists $N = N(i, j)$ such that $A^N[i, j] > 0$. Note that this is equivalent to assuming $\sigma$ is topologically transitive.

A finite word of length $n$ is simply any finite sequence $a = a_1, \ldots, a_n \in \{1, \ldots, k\}^n$ (we also write $|a| = n$). Such a word is called admissible if $A[a_i, a_{i+1}] = 1$ for all $1 \leq i < n$ and periodic if it is admissible and $A[a_n, a_1] = 1$. Thus a finite word $p$ is periodic if it generates a periodic point.
Chapter 6. Finite Group Extensions of Hyperbolic Basic Sets

...ppp... of \( \sigma : \Sigma_A \to \Sigma_A \). Note that throughout we will often identify periodic words with the periodic points they generate.

For \( i, j \in \{1, 2, \ldots, k\} \) we define a bridging word to be any finite word \( U \) such that \( iUj \) is admissible. Note that the assumption that \( A \) is irreducible guarantees the existence of at least one bridging word for all pairs of symbols \( i, j \in \{1, 2, \ldots, k\} \).

Given finite admissible words \( a = a_1 \ldots a_m \) and \( b = b_1 \ldots b_n \) with \( A[a_m, b_1] = 1 \) we define the cylinder set \( C^b_a \) to be the subset of \( \Sigma_A \):

\[
C^b_a := \{ x \in \Sigma_A \mid x_{i-m} = a_i \text{ for all } 0 \leq i < m \text{ and } x_i = b_i \text{ for all } 0 \leq i \leq n \}.
\]

Similarly we define

\[
C_a := \{ x \in \Sigma_A \mid x_{i-m} = a_i \text{ for all } 0 \leq i < m \}
\]

and

\[
C^b := \{ x \in \Sigma_A \mid x_i = b_i \text{ for all } 0 \leq i \leq n \}.
\]

Now let \( f : \Lambda \to \Lambda \) be a hyperbolic map and \( \sigma : \Sigma_A \to \Sigma_A \) the corresponding subshift, given by Theorem 1.2.2 via the Hölder semiconjugacy \( \rho : \Sigma_A \to \Lambda \). Then \( f : \Lambda \to \Lambda \) is ergodic or weak-mixing if and only if \( \sigma : \Sigma_A \to \Sigma_A \) is. If \( h : \Lambda \to \Gamma \) is a Hölder map into the compact Lie group \( \Gamma \), we define the map \( \widetilde{h} = h \circ \rho : \Sigma_A \to \Gamma \). Thus given a compact group extension \( f_h : \Lambda \times \Gamma \to \Lambda \times \Gamma \) of \( f : \Lambda \to \Lambda \) we have a corresponding compact group extension \( \sigma_{\widetilde{h}} : \Sigma_A \times \Gamma \to \Sigma_A \times \Gamma \), of \( \sigma : \Sigma_A \to \Sigma_A \).

Further if we define a map \( \rho_h : \Sigma_A \times G \to \Lambda \times G \) by \( \rho_h(x, \gamma) = (\rho(x), \gamma) \) then the properties of \( \rho \) imply that \( \rho_h \) is Hölder, onto, finite-to-one and invertible almost everywhere. Furthermore we have the following commutative diagram

\[
\begin{array}{ccc}
\Sigma_A \times G & \xrightarrow{\sigma_{\widetilde{h}}} & \Sigma_A \times G \\
\rho_h \downarrow & & \downarrow \rho_h \\
\Lambda \times G & \xrightarrow{f_h} & \Lambda \times G
\end{array}
\]

In particular \( \rho_h \) is a measurable isomorphism between \( \sigma_{\widetilde{h}} \) and \( f_h \). Thus \( \sigma_{\widetilde{h}} \) is ergodic or weak-mixing if and only if \( f_h \) is.

**Finite Extensions**

Let \( G \) be a finite group. On \( G \) we have define the discrete metric \( d_G(\cdot, \cdot) \) by

\[
d_G(g_1, g_2) = \begin{cases} 
0 & \text{if } g_1 = g_2 \\
1 & \text{if } g_1 \neq g_2
\end{cases}.
\]

We further take the measure \( \nu \) on \( G \) to be the normalised counting measure. That is if \( G \) has order \( m \) then a subset \( A \) of \( G \) has measure \( \nu(A) = |A|/m \), where \( |A| \) denotes the cardinality of \( A \).
Let \( f: \Lambda \to \Lambda \) be a hyperbolic map. We consider the finite group extension \( f_h: \Lambda \times G \to \Lambda \times G \) of \( f \) by \( h \), where \( h: \Lambda \to G \) is a Hölder map into \( G \). Note that as \( G \) is finite, \( h \) will be locally constant. Indeed suppose that \( h \) is Hölder of exponent \( \alpha \). Then if \( x, y \in \Lambda \) are such that \( h(x) \neq h(y) \) then

\[
1 = d(h(x), h(y)) \leq C d(x, y)^\alpha \\
\iff -\frac{\ln C}{\alpha} \leq \ln d(x, y) \\
\iff e^{-\frac{\ln C}{\alpha}} \leq d(x, y).
\]

Thus for all \( x \in \Lambda \), \( h \) restricted to \( B_\epsilon(x) \) is constant where \( \epsilon = e^{-\frac{\ln C}{\alpha}} \) and for any \( \delta > 0 \) we define

\[
B_\delta(x) := \{ y \in \Lambda \mid d(x, y) < \delta \}.
\]

We further define the subgroups

\[
\mathcal{B}_\delta(x) := \langle \{ h_n p \mid f^n p = p \in B_\delta(x) \} \rangle
\]

and

\[
\mathcal{P} := \langle \{ h_n p \mid f^n p = p \} \rangle,
\]

where for any \( x \in \Lambda \), \( h_n x = h(f^{n-1} x) \ldots h(f x) h(x) \).

Suppose that \( \sigma: \Sigma_{\Lambda} \to \Sigma_{\Lambda} \) is a subshift of finite type and that \( h \in F_\delta(\Sigma_{\Lambda}, G) \). Note that \( h \) will depend only on finitely many past and future coordinates. As before we define

\[
\mathcal{P} := \langle \{ h_n p \mid \sigma^n p = p \} \rangle
\]

and for finite admissible words, \( a \) and \( b \) let

\[
\mathcal{P}_a := \langle \{ h_n p \mid \sigma^n p = p \in C^a_b \} \rangle.
\]

Note that if \( p \) is a point such that \( \sigma^n p = p \) and \( p \in C^a_b \) then it is of the form \( p = \ldots b \ldots a \cdot b \ldots a \ldots \).

In the case of finite group extensions we can slightly extend Theorem 5.2.1 to the following result.

6.1.1 Theorem. Let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( \mu \) a Hölder equilibrium measure. Suppose \( h: \Sigma_{\Lambda} \to G \) is a Hölder map into the finite group \( G \) which depends on only \( T^- \) past and \( T^+ \) future coordinates and let \( \epsilon > 0 \) be such that \( h \) restricted to the open ball \( B_\epsilon(x) \) is constant for all \( x \in \Lambda \). The finite group extension \( f_h: \Lambda \times G \to \Lambda \times G \) is ergodic with respect to \( \mu \times v \) if and only if the only solutions to

\[
R(h(x))v(x) = v(fx)
\]

(6.1.1)

where \( R: G \to U(d) \) is an irreducible unitary representation and \( v: \Lambda \to C^d \) depends on only \( T^- \) past and \( T^+ \) future coordinates, is \( R \) trivial and \( v \) identically zero.
We first consider finite extensions of subshifts of finite type.

6.1.2 Theorem. Let \( \sigma: \Sigma_A \to \Sigma_A \) be a subshift of finite type, \( h: \Sigma_A \to \Gamma \) a map into the finite group \( G \) which depends on only \( T^- \) past and \( T^+ \) future coordinates. The compact group extension \( \sigma_h: \Sigma_A \times G \to \Sigma_A \times G \) is ergodic with respect to \( \mu \times \nu \) if and only if the only solutions to

\[
R(h(x))v(x) = v(\sigma x)
\]  

(6.1.2)

where \( R: G \to U(d) \) is an irreducible unitary representation and \( v: \Sigma_A \to \mathbb{C}^d \) depends on only \( T^- \) past and \( T^+ \) future coordinates, is \( R \) trivial and \( v \) identically zero.

Proof. Clearly the 'only if' part of the Theorem follows directly from Theorem 5.2.1, we consider the 'if' part. Suppose then that \( \sigma_h: \Sigma_A \times G \to \Sigma_A \times G \) is not ergodic. Then, by Theorem 5.4.1, there exists a non-trivial irreducible unitary representation \( R: G \to U(d) \) and a non-zero, continuous \( v: \Sigma_A \to \mathbb{C}^d \) satisfying (6.1.2). We claim that \( v \) depends on only \( T^- \) past and \( T^+ \) future coordinates.

Let \( C_a^b \) be any cylinder set with \( |a| = T^- \) and \( |b| = T^+ \). Let \( p = \ldots ppp \cdot ppp \ldots \) and \( q = \ldots qqq \cdot qqq \ldots \) be any two periodic points in \( C_a^b \).

For any \( r \in \mathbb{N} \) we form the point \( q_r = q^{oo}p^k \cdot p^{k}q^{oo} \), where \( q^{oo} \) represents an infinite string of \( q \)'s and \( p^{k} = \underbrace{pp\ldots p}_{k \text{ times}} \) with \( k \) the order of the group element \( h_np \), where \( n \) denotes the order of \( p \). Substituting \( q_r \) into (6.1.2) and iterating forwards gives

\[
R(h_npq^{oo}p^{k}q^{oo})v(q_r) = v(q^{oo}p^{2k}q^{oo}).
\]

Since \( h \) is constant on \( C_a^b \) and \( (h_np)^k = 1 \) we have

\[
R(h_npq^{oo}p^{k}q^{oo}) = R(h_np)^{kr} = 1
\]

for any \( r \in \mathbb{N} \). Thus letting \( r \to \infty \) we deduce that \( v(p) = v(p^{oo} \cdot q^{oo}) \). Using this same argument, though iterating (6.1.2) backwards, yields that \( v(p) = v(q^{oo} \cdot p^{oo}) \).

Repeating the whole argument with the roles of \( p \) and \( q \) reversed, we conclude that \( v(q) = v(q^{oo} \cdot p^{oo}) = v(q^{oo} \cdot q^{oo}) \). Thus \( v(p) = v(q) \). Since \( p \) and \( q \) were arbitrary we see that \( v \) is constant on periodic points in \( C_a^b \). As periodic points are dense this proves the theorem. \( \square \)

To prove Theorem 6.1.1 suppose that \( f_h: \Lambda \times G \to \Lambda \times G \) is not ergodic. Then there exists a non-trivial irreducible unitary representation \( R: \Lambda \to U(d) \) and a non-zero \( v: \Lambda \to \mathbb{C}^d \) satisfying equation (6.1.1). Let \( \sigma: \Sigma_A \to \Sigma_A \) be the subshift of finite type semi-conjugate to \( f: \Lambda \to \Lambda \) via the Hölder semi-conjugacy \( \rho: \Sigma \to \Lambda \) given by Theorem 1.2.2. Define \( \tilde{h} = h \circ \rho \) and note that for each \( z \in \Sigma_A \)

\[
R(\tilde{h}(z))v(\rho(z)) = R(h(\rho(z))v(\rho z) = v(f(\rho z)) = v(\rho(\sigma z)). \]

Then by Theorem 6.1.2 \( v \circ \rho \) is constant on cylinder sets of the form \( C_a^b \), where \( \tilde{h} \) is constant on \( C_a^b \). Thus \( v \) is constant on \( \rho(C_a^b) \) an hence \( B_\epsilon(x) \).
6.2 Abelian Extensions

We first consider abelian group extensions. For compact abelian group extensions of hyperbolic basic sets we show:

6.2.1 Theorem. Let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( h: \Lambda \to \Gamma \) a Hölder map into the compact abelian Lie group \( \Gamma \). Then \( f_h: \Lambda \times \Gamma \to \Lambda \times \Gamma \) is ergodic if and only if \( \mathcal{P} = \Gamma \).

For the special case of finite extensions of hyperbolic basic sets and compact extensions of subshifts we extend this result to the following.

6.2.2 Theorem. Let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( h: \Lambda \to \Gamma \) a Hölder map into the compact abelian Lie group \( \Gamma \). Suppose either

(A) \( \Lambda \) is completely disconnected (i.e. \( f: \Lambda \to \Lambda \) is a subshift of finite type) or

(B) \( \Gamma \) is finite.

Then the following are equivalent:

(i) \( f_h \) is ergodic;

(ii) \( \mathcal{P} = \Gamma \);

(iii) \( h \) is not cohomologous to a Hölder \( h': \Lambda \to K \), where \( K \) is a proper subgroup of \( \Gamma \).

These results are proved via a series of lemmas.

6.2.3 Lemma. Let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( h: \Lambda \to \Gamma \) a Hölder map into the compact abelian Lie group \( \Gamma \). Then \( f_h: \Lambda \times \Gamma \to \Lambda \times \Gamma \) is ergodic if \( \mathcal{P} = \Gamma \).

Proof. Suppose that \( \mathcal{P} = \Gamma \) and, for contradiction, \( f_h \) is not ergodic. By Keynes-Newton and Livšic there exists a nontrivial, irreducible unitary representation \( R \) (one-dimensional as \( \Gamma \) is abelian) and a Hölder map \( v: \Lambda \to \mathbb{C} \) such that

\[
R(h(x))v(x) = v(fx)
\]  

(6.2.1)

for all \( x \in \Lambda \). Let \( p_1, \ldots, p_r \) be points such that \( \{h_{n_i}p_i \mid f^{m_i}p_i = p_i \}_{i=1}^r \) = \( \Gamma \). Then by iterating equation (6.2.1)

\[
R(h_{n_i})v(p_i) = v(p_i)
\]

for all \( 1 \leq i \leq r \). That is \( R(h_{n_i}) = 1 \) for all \( 1 \leq i \leq r \). But since \( \{h_{n_i}p_i \mid f^{m_i}p_i = p_i \}_{i=1}^r \) = \( \Gamma \), we have \( R(\Gamma) = 1 \) giving the desired contradiction. \( \square \)

6.2.4 Lemma. Let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( h: \Lambda \to \Gamma \) a Hölder map into the compact abelian Lie group \( \Gamma \). Suppose either
(A) $\Lambda$ is completely disconnected (i.e., $f: \Lambda \to \Lambda$ is a subshift of finite type) or

(B) $\Gamma$ is finite.

If $\mathcal{P} = K$ for some proper subgroup $K$ of $\Gamma$ then $h$ is cohomologous to some Hölder $h': \Lambda \to K$.

Proof. Suppose that $P = K$, a proper subgroup of $\Gamma$. Let $\pi: \Gamma \to \Gamma/K$ denote the quotient homomorphism, $\pi(\gamma) = \gamma + K$. Define

$$\tilde{h} := \pi \circ H: \Lambda \to \Gamma/K.$$

Then for all $p \in \Lambda$ with $f^n p = p$, $h_n p = 1$. We then use the Livšic periodic point theorem to find a Hölder map $\tilde{\omega}: \Lambda \to \Gamma/K$ such that

$$\tilde{h} = \tilde{\omega} \circ f - \tilde{\omega}.$$

We consider only case (A), for case (B) see Theorem 5.4.4. Fix a set of coset representatives for $K$ in $\Gamma$. Define a map $\omega: \Lambda \to \Gamma$ by letting $\omega(x)$ equal the corresponding coset representative of $\tilde{\omega}$. Note that since $\tilde{\omega}$ is Hölder it is locally constant, hence so is $\omega$. Thus $\omega$ is Hölder. Further for all $x \in \omega$, $h(x)$ and $\omega(fx)$ and $\omega(x)$ belong to the same coset of $K$ in $\Gamma$. We can thus find a Hölder $h': \Lambda \to \Gamma$ such that

$$h(x) = \omega(fx) - \omega(x) + h'(x)$$

as required. $\square$

Combining Lemmas 6.2.3, 6.2.4 and Proposition 5.3.1 constitutes a proof of Theorem 6.2.2.

To prove Theorem 6.2.1 we use Lemma 6.2.3 together with the following.

6.2.5 Lemma. Let $f: \Lambda \to \Lambda$ be hyperbolic, $h: \Lambda \to \Gamma$ a Hölder map into the compact abelian Lie group $\Gamma$. If $f_h: \Lambda \times \Gamma \to \Lambda \times \Gamma$ is ergodic then $\mathcal{P} = \Gamma$.

Proof. Suppose $\mathcal{P} = K$, for some proper subgroup, $K$ of $\Gamma$. Let $\sigma: \Sigma_A \to \Sigma_A$ be the subshift of finite type, semiconjugate to $f_h$ via the map $\rho: \Sigma_A \to \Lambda$ given by Theorem 1.2.2. Define the Hölder cocycle $\tilde{h} := h \circ \rho: \Sigma_A \to \Gamma$ and the subgroup

$$\mathcal{F} := \{ h_{np} | \sigma^n p = p \}.$$

We claim that $\mathcal{F}$ is a subgroup of $K$. Indeed let $p$ be a periodic point of $\sigma$. Then for some $n \in \mathbb{N}$

$$\sigma^n p = p \implies \rho(\sigma^n p) = \rho(p) \implies f^n(\rho(p)) = \rho(p).$$

So $\rho(p)$ is a periodic point of $f$. We calculate

$$\tilde{h}_{np} = \sum_{i=0}^{n-1} h(\sigma^i p) = \sum_{i=0}^{n-1} h(\rho(\sigma^i p)) = \sum_{i=0}^{n-1} h(f^i(\rho(p))).$$
which lies in \( \rho \), proving the claim. Hence by Theorem 6.2.2, \( \sigma_h \) is not ergodic and so neither is \( f_h \), proving the lemma.

\[ \square \]

### 6.3 General Finite Extensions

We now look at general finite group extensions, firstly over subshifts of finite type. For nonabelian extensions the periodic data condition from the previous section is not strong enough to guarantee ergodicity. This is illustrated by the following example.

**Example.** Let \( \sigma: \Sigma_A \to \Sigma_A \) be the subshift of finite type on three symbols, defined by the transition matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\]

Consider the third dihedral group \( D_3 = \{ \kappa, \rho \mid \kappa^2 = 1, \rho^3 = 1, \kappa \rho \kappa = \rho^2 \} \) and define a cocycle \( h: \Sigma_A \to D_3 \), depending on the first coordinate only by

\[
h(1) = \kappa, \quad h(2) = \kappa \rho^{-1}, \quad h(3) = \rho.
\]

(Here we identify \( i \) with \( C^i, i = 1, 2, 3 \).) We note that the words 1 and 323 are periodic and that

\[
h_11 = \kappa, \quad h_3323 = \rho \kappa.
\]

Thus \( \mathcal{F} = G \). The finite group extension \( \sigma_h: \Sigma_A \times D_3 \to \Sigma_A \times D_3 \), though, is not ergodic. To show this we will construct a nontrivial, irreducible unitary representation \( R \) together with a map \( \nu: \Sigma_A \to \mathbb{C}^3 \), which will depend on only the first coordinate, satisfying

\[
R(h(x))\nu(x) = \nu(\sigma x) \text{ for all } x \in \Sigma_A.
\]

Then using Keynes–Newton this will imply that \( \sigma_h \) is not ergodic.

We define \( R: D_3 \to U(3) \) firstly on the generators \( \kappa \) and \( \rho \) by

\[
R(\kappa) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad R(\rho) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]

and then extend to the whole group.

Define

\[
\nu(1) = \nu(2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \nu(3) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]
To check that equation (6.3.1) is satisfied we need only show that for i = 1, 2, 3
\[ R(h(i))v(i) = v(j) \]
whenever \( A[i,j] = 1 \). We thus calculate
\[
\begin{align*}
R(h(1))v(1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v(1) = v(2) \\
R(h(2))v(2) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v(3) \\
R(h(3))v(3) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v(1) = v(2).
\end{align*}
\]

Thus (6.3.2) is satisfied and \( \sigma_h \) is not ergodic.

The main result of this section is the following:

6.3.1 Theorem. Let \( \sigma : \Sigma_A \to \Sigma_A \) be a subshift of finite type, \( h : \Sigma_A \to G \) a cocycle into the finite group \( G \) which depends on a finite number of past and future coordinates. Then \( \sigma_h \) is ergodic if and only if for all finite words \( a \) and \( b \) with \( ab \) admissible \( \{h^n p \mid f^n p = p, \ p \in C_0^b\} = G \).

We will prove the theorem via a series of lemmas.

6.3.2 Lemma. Let \( \sigma : \Sigma_A \to \Sigma_A \) be a subshift of finite type, \( h : \Sigma_A \to G \) a cocycle in to the finite group \( G \) which depends on only the first \( T^- \) past and \( T^+ \) future coordinates. If for some pair of words \( a \) and \( b \) of lengths \( T^- \) and \( T^+ \), respectively \( \mathcal{S}_a^b = G \) then \( \sigma_h \) is ergodic.

Let \( a \) and \( b \) be finite words of lengths \( T^- \) and \( T^+ \), \( p_1, \ldots, p_r \) points in \( C_0^b \) such that \( \{\gamma_i = h_n p_i \mid \sigma^{n_i} p_i = p_i\}_{i=1}^r \) generates \( G \). Note that since \( h \) depends only on \( T^- \) past and \( T^+ \) future coordinates if we take any \( p_{l_1}, p_{l_2}, p_{l_3} \in \{p_i\}_{i=1}^r \) then
\[
h_{l_1} p_{l_2} = h_{l_2} (C_0^{p_{l_2} p_{l_3}}) = h_{l_2} (C_0^{p_{l_1} h}).
\]

For some fixed \( L \in \mathbb{N} \) we can write
\[
g_i = \gamma_{r_{m+1}} \ldots \gamma_{r_{1}}^{-1} n_{1,1}^3 \ldots n_{r_{1}}^3 \ldots n_{1,2}^3 \ldots n_{r_{2}}^3 \ldots n_{1,3}^3 \ldots n_{r_{3}}^3 \ldots n_{1,L}^3 \ldots n_{r,L}^3
\]
for each \( g_i \in G = \{g_1, \ldots, g_m\} \). For each \( i = 1, \ldots, m \) define
\[
P_i := p_{l_1}^{n_{1,1}} \ldots p_{l_r}^{n_{1,r}} p_{l_1}^{n_{1,2}} \ldots p_{r}^{n_{1,r}} \ldots p_{l_1}^{n_{1,L}} \ldots p_{r}^{n_{1,r}}
\]
where

\[ p_{j}^{n_{l}(j)} := p_{j}p_{j} \ldots p_{j} \text{ \text{\text{\text{\times} times}}} \]

\[ j = 1, \ldots, r \text{ and } l = 1, \ldots, L. \text{ Set} \]

\[ \gamma_{1} = h|_{\delta_{1}}(C_{\delta_{1}}^{ab}) \]

\[ \gamma_{2} = h|_{\delta_{2}}(C_{\delta_{2}}^{c'd'}) \]

where \( \kappa_{1} \) and \( \kappa_{2} \) are any suitable bridging words. Let \( t \in \{1, \ldots, m\} \) be such that \( g_{t} = \gamma_{2}^{-1}g^{-1} \gamma_{1}^{-1} \).

Then for any point of the form

\[ x = \ldots \cdot d^{k_{1}}a^{P_{t}}b^{k_{2}}c'd' \ldots \]

we have \( x \in C_{d}^{T} \) and

\[ \sigma_{h}^{N}(x, g) = (\sigma_{h}^{N}x, h_{N}(x)g) \]

\[ = (\sigma_{h}^{N}x, \gamma_{1}^{N}g_{t}^{N}g) \]

\[ = (\sigma_{h}^{N}x, g') \in C_{d}^{T} \times \{g'\} \]

where \( N = |\delta_{1}^{P_{t}}| \) as required.

As a converse to this lemma we have:

**6.3.3 Lemma.** Let \( \sigma: \Sigma_{A} \to \Sigma_{A} \) be a subshift of finite type, \( h: \Sigma_{A} \to G \) a cocycle into the finite group \( G \) which depends on only the first \( T^- \) past and \( T^+ \) future coordinates. If \( \sigma_{h} \) is ergodic then for some pair of words \( a \) and \( b \) of lengths \( T^- \) and \( T^+ \) respectively \( P_{ab}^{b} = G \).

**Proof.** To obtain a contradiction let \( a \) and \( b \) be finite words of lengths \( T^- \) and \( T^+ \) respectively, with \( ab \) admissible such that \( P_{ab}^{b} \) is a proper subgroup of \( G \). We will construct an irreducible unitary representation \( R \) of \( G \) and a function \( v: \Sigma_{A} \to C_{d}^{T} \) (which will depend on the \( T^- \) past and \( T^+ \) future coordinates) such that

\[ R(h)v = v \circ \sigma. \]  \[ (6.3.4) \]

Then via the Keynes–Newton result and Livšic regularity \( \sigma_{h} \) will not be ergodic.

Let \( R \) be any irreducible representation of \( G \) such that

\[ \text{Fix}(P_{ab}^{b}) = \{x \in C_{d}^{T} \mid R(\gamma)x = x \ \forall \gamma \in P_{ab}^{b}) \}

is nontrivial. For convenience of notation we will identify \( R \circ h \) with \( h \) and for finite words \( \alpha \) and \( \beta \) write \( \alpha \cdot \beta \) for \( C_{\delta}^{\alpha} \). We also fix a set of bridging words so that given words \( \alpha \) and \( \alpha' \) we have a uniquely defined word \( U_{\alpha}^{\alpha'} \) such that \( \alpha U_{\alpha}^{\alpha'} \alpha' \) is admissible. Further set \( N_{\alpha}^{\alpha'} = |\alpha U_{\alpha}^{\alpha'} \alpha'| \).

Define \( v(\cdot \cdot - \cdot b) \) to be any unit vector in \( \text{Fix}(P_{ab}^{b}) \). For arbitrary finite words \( c, d \) with \( |c| = T^- \), \( |d| = T^+ \) and \( cd \) admissible we define

\[ v(c \cdot d) := h|_{\delta_{1}}(a \cdot bU_{c}^{d}a)d)v(a \cdot b). \]
We note that
\[ h_{N_a}^s(c \cdot dU_a^s ab)v(c \cdot d) = h_{N_a}^s(c \cdot dU_a^s ab)h_{N_a}v(a \cdot b) \]
\[ = h_{N_a}^s(a \cdot bU_a^s cdU_a^s ad)v(a \cdot b) \]
\[ = v(a \cdot b) \text{ as } bU_a^s cdU_a^s a \in P_a^b. \]

To show that \( v: \Sigma_A \to C^d \) indeed satisfies equation (6.3.4) we need to show that for all \( c \) and \( d \) as above and all symbols \( s \in \{1, 2, \ldots, k\} \) such that \( A[dT^+, s] = 1 \) we have
\[ h(c \cdot d)h(c \cdot d)v(c \cdot d) = v(c_2 \cdot c_{T^-} \cdot d_2 \cdot d_{T^+} + sU_a^s ab)^{-1}v(a \cdot b). \]

We calculate
\[ v(c_2 \cdot c_{T^-} \cdot d_2 \cdot d_{T^+} + s) = h_{N_a}^s(c_2 \cdot c_{T^-} \cdot d_2 \cdot d_{T^+} + sU_a^s ab)^{-1}v(a \cdot b) \]
\[ = h(c \cdot d)h_{N_a}^s(c \cdot dU_a^s a)^{-1}v(a \cdot b) \]
\[ = h(c \cdot d)h_{N_a}^s(a \cdot bU_a^s cd)h_{N_a}^s(a \cdot bU_a^s dsU_a^s ab)^{-1}v(a \cdot b) \]
\[ = h(c \cdot d)h_{N_a}^s(a \cdot bU_a^s cd)v(a \cdot b) \]
\[ = h(c \cdot d)v(c \cdot d), \]
which completes the proof. □

To complete the proof of Theorem 6.3.1 we have:

6.3.4 Lemma. Let \( \sigma: \Sigma_A \to \Sigma_A \) be a subshift of finite type, \( h: \Sigma_A \to G \) a cocycle in to the finite group \( G \) which depends on only the first \( T^- \) past and \( T^+ \) future coordinates. Suppose that for some pair of finite words \( a \) and \( b \) of lengths \( T^- \) and \( T^+ \) respectively \( \mathcal{B}_a^b = G \). Then for any pair of finite words \( c \) and \( d \) with \( cd \) admissible
\[ \{h_n p | f^n p = p, p \in C^d \} = G. \]

Proof. Let \( a \) and \( b \) be finite words of lengths \( T^- \) and \( T^+ \), respectively such that \( \mathcal{B}_a^b = G \). Let \( c \) and \( d \) be any other finite words with \( cd \) admissible. Let \( g \) be an arbitrary group element. Define
\[ \gamma_1 := h_{N_1}(C_{a \kappa_1^1}^d ab) \]
and
\[ \gamma_2 := h_{N_2}(C_{b \kappa_2^2}^d cd), \]
where \( N_1 = |d\kappa_1^1 a| \) and \( N_2 = |a\kappa_2^2 c| \) and \( \kappa_1, \kappa_2 \) are any suitable bridging words. Let \( g_t \in G = \{g_1, \ldots, g_m\} \) be such that
\[ g_t = \gamma_2^{-1}g_1^{-1} \]
and define \( P_t \) as in equation (6.3.3). Let \( p \) be the periodic point generated by the finite word \( d\kappa_1 aP_t b\kappa_2 c \). Then \( p \in C^d \), \( f^n p = p \) and \( h_n p = g \), where \( n = |d\kappa_1 aP_t b\kappa_2 c| \). In other words \( g \in \{h_n p | f^n p = p, p \in C^d \} \). As \( g \) was arbitrary this proves the lemma. □
6.3.5 Theorem. Let \( f: \Lambda \to \Lambda \) be hyperbolic, \( h: \Lambda \to G \) a Hölder map into the finite group \( G \). Then \( f_h: \Lambda \times G \to \Lambda \times G \) is ergodic if and only if \( \mathcal{B}_\delta(x) = G \) for all \( x \in \Lambda \) and \( \delta > 0 \).

Proof. First suppose that \( f_h \) is ergodic and fix arbitrary \( \delta > 0 \) and \( x \in \Lambda \). Let \( y = \{y_i\} \in \Sigma_A \) be such that \( \rho(y) = x \) an set \( a = y_{-T} \cdots y_{-1}, \ b = y_1 \cdots y_T \) where \( T \) is any natural number larger than \( \frac{\ln (\delta/C')}{\ln \delta} \). Then \( \rho(C^b_a) \subseteq B_\delta(x) \). As \( \sigma_h \) is also necessarily ergodic we can use Theorem 6.3.1 to find periodic points \( \tilde{p}_1, \ldots, \tilde{p}_r \), in \( C_a^b \) of orders \( \tilde{n}_1, \ldots, \tilde{n}_r \) such that \( \{\gamma_i := \tilde{h}_{\tilde{n}_i, \tilde{p}_i}\}_{i=1}^r \) generates \( G \).

For each \( i = 1, \ldots, r \) set \( p_i = \rho(\tilde{p}_i) \), we then have

\[
\begin{align*}
  f^{\tilde{n}_i} p_i &= f^{\tilde{n}_i}(\rho(\tilde{p}_i)) \\
 &= \rho(\sigma^{\tilde{n}_i} \tilde{p}_i) \\
 &= \rho(\tilde{p}_i) \\
 &= p_i,
\end{align*}
\]

so that \( p_i \) is a periodic point for \( f \) of order \( \tilde{n}_i \). Further

\[
\begin{align*}
  h_{\tilde{n}_i, p_i} &= h(f^{\tilde{n}_i-1} p_i \ldots f p_i h(p_i)) \\
 &= h(f^{\tilde{n}_i-1} \rho(\tilde{p}_i) \ldots f \rho(\tilde{p}_i)) h(\rho(\tilde{p}_i)) \\
 &= h(\rho(\sigma^{\tilde{n}_i-1} \tilde{p}_i)) \ldots h(\rho(\tilde{p}_i)) h(\rho(\tilde{p}_i)) \\
 &= \tilde{h}(\sigma^{\tilde{n}_i-1} \tilde{p}_i) \cdots \tilde{h}(\tilde{p}_i) \tilde{h}(\tilde{p}_i)) \\
 &= \tilde{h}_{\tilde{n}_i, \tilde{p}_i}.
\end{align*}
\]

Thus \( h_{\tilde{n}_i, p_i} = \gamma_i \) and so \( \{h_{\tilde{n}_i, p_i} = \gamma_i\}_{i=1}^r \) generates \( G \). Moreover since \( \rho(C^b_a) \subseteq B_\delta(x) \) we have that \( \mathcal{B}_\delta(x) = G \) as required.

Conversely suppose that \( \mathcal{B}_\delta(x) = G \) for all \( x \in \Lambda \) and \( \delta > 0 \). Assume, for contradiction, that \( f_h \) is not ergodic. Then by Keynes–Newton and Livšic regularity we have a nontrivial, irreducible, unitary representation \( R: G \to U(d) \) and a Hölder map \( v: \Lambda \to G \) such that

\[ R(h(x)) v(x) = v(f(x)) \quad (6.3.5) \]

for all \( x \in \Lambda \). Fix any \( x \in \Lambda \). By Theorem 6.1.1 we can pick a \( \delta > 0 \) small enough so that \( v \) is constant on the open ball \( B_\delta(x) \). Let \( p_1, \ldots, p_r \) be periodic points, of orders \( n_1, \ldots, n_r \), in \( B_\delta(x) \) such that \( \{h_{n_i, p_i}\}_{i=1}^r \) generates \( G \). By iterating equation (6.3.5) we have that

\[ R(h^{n_i} p_i) v(p_i) = v(f^{n_i} p_i) \]

\[ = v(p_i). \]

But since \( v \) is constant on \( B_\delta(x) \) we conclude

\[ R(h_{n_i, p_i}) v(x) = v(x) \]

for all \( i = 1, \ldots, r \). Thus as \( \{h_{n_i, p_i}\}_{i=1}^r \) generates \( G \), \( v(x) \) is fixed by the whole of \( G \). This gives the desired contradiction and concludes the proof of the theorem. \( \square \)
Chapter 6. Finite Group Extensions of Hyperbolic Basic Sets

6.4 Finitely Many Connected Components

Let \( f : \Lambda \to \Lambda \) be a hyperbolic map, \( h : \Lambda \to G \) a Hölder cocycle into the finite group \( G = \{ g_0 = 1, \ldots, g_m \} \). In this section we specialise to the case where \( \Lambda \) has a finite number of connected components \( \Lambda_1, \ldots, \Lambda_n \). Note that, in this case, \( h \) will be constant on each of the \( \Lambda_i \).

For \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) let \( \Lambda_i \times \{ g \} \) denote the \( m \times n \) connected components of \( \Lambda \times G \). We further define \( h_j := h(\Lambda_{j-1})h(\Lambda_{j-2}) \ldots h(\Lambda_1), \) with \( h_0 = 1 \) and \( H = \langle h_n \rangle, k = \vert H \vert \).

\( f_h \)-Invariant Sets

We consider \( f_h \) invariant sets by studying orbits, under \( f_h \) of the connected components of \( \Lambda \times G \).

In particular we show:

6.4.1 Proposition. Let \( \{ \gamma_1, \ldots, \gamma_r \} \) be a left transversal for \( H \) in \( G \). Then we have \( r \) disjoint cyclic orbits \( \{ C_i \}_{i=1}^r \), where

\[
C_i := \bigcup_{j=1}^n \Lambda_j \times \{ H\gamma_i \}.
\]

In particular \( \Lambda \times G \) is a single orbit if and only if \( H = G \).

First take the component \( \Lambda_1 \times \{ 1 \} \) and consider the \( f_h \)-orbit

\[
\Lambda_1 \times \{ 1 \} \to f_h(\Lambda_1 \times \{ 1 \}) \to f_h^2(\Lambda_1 \times \{ 1 \}) \to \ldots \to f_h^k(\Lambda_1 \times \{ 1 \}) \ldots
\]

Since \( f \) is smooth and ergodic we can assume that the \( \Lambda_i \) are ordered so that \( f(\Lambda_i) = \Lambda_{i+1} \) for \( i = 1, \ldots, n - 1 \) and \( \Lambda_n = \Lambda_1 \). Thus this orbit becomes

\[
\begin{align*}
\Lambda_1 \times \{ 1 \} &\to \Lambda_2 \times \{ h_1 \} \to \ldots \to \Lambda_n \times \{ h_{n-1} \} \\
\Lambda_1 \times \{ h_n \} &\to \Lambda_2 \times \{ h_1 h_n \} \to \ldots \to \Lambda_n \times \{ h_{n-1} h_n \} \\
\vdots &\to \Lambda_1 \times \{ (h_n)^{k-1} \} \to \Lambda_2 \times \{ h_1 (h_n)^{k-1} \} \to \ldots \to \Lambda_n \times \{ h_{n-1} (h_n)^{k-1} \} \\
\Lambda_1 \times \{ 1 \} &\to \Lambda_2 \times \{ h_1 \} \to \ldots
\end{align*}
\]

This gives a cyclic orbit of order \( kn \). Next consider the \( f_h \)-orbit of \( \Lambda_1 \times \{ \gamma_2 \} \):

\[
\begin{align*}
\Lambda_1 \times \{ \gamma_2 \} &\to \Lambda_2 \times \{ h_1 \gamma_2 \} \to \ldots \to \Lambda_n \times \{ h_{n-1} \gamma_2 \} \\
\Lambda_1 \times \{ h_n \gamma_2 \} &\to \Lambda_2 \times \{ h_1 h_n \gamma_2 \} \to \ldots \to \Lambda_n \times \{ h_{n-1} h_n \gamma_2 \} \\
\vdots &\to \Lambda_1 \times \{ (h_n)^{k-1} \gamma_2 \} \to \Lambda_2 \times \{ h_1 (h_n)^{k-1} \gamma_2 \} \to \ldots \to \Lambda_n \times \{ h_{n-1} (h_n)^{k-1} \gamma_2 \} \\
\Lambda_1 \times \{ \gamma_2 \} &\to \Lambda_2 \times \{ h_2 \gamma_2 \} \to \ldots
\end{align*}
\]
In general we have \( r \) cyclic orbits of the form
\[
\begin{align*}
\Lambda_1 \times \{\gamma_i\} & \rightarrow \Lambda_2 \times \{h_1\gamma_i\} & \rightarrow \cdots & \rightarrow \Lambda_n \times \{h_{n-1}\gamma_i\} \\
\Lambda_1 \times \{h_{n}\gamma_i\} & \rightarrow \Lambda_2 \times \{h_1h_{n}\gamma_i\} & \rightarrow \cdots & \rightarrow \Lambda_n \times \{h_{n-1}h_{n}\gamma_i\} \\
\vdots & & & \\
\Lambda_1 \times \{(h_{n})^{k-1}\gamma_i\} & \rightarrow \Lambda_2 \times \{h_1(h_{n})^{k-1}\gamma_i\} & \rightarrow \cdots & \rightarrow \Lambda_n \times \{h_{n-1}(h_{n-1})^{k-1}\gamma_i\} \\
\Lambda_1 \times \{\gamma_i\} & \rightarrow \Lambda_2 \times \{h_1\gamma_i\} & & \\
\end{align*}
\]
\( i = 1, \ldots, r \) as claimed.

**Example.** To illustrate Proposition 6.4.1 we consider a basic example. Let \( G = \mathbb{Z}_4 = \langle \kappa \rangle \) and suppose that \( \Lambda \) has 5 connected components. Define \( h: \Lambda \rightarrow G \) as follows
\[
h(\Lambda_1) = \kappa \quad h(\Lambda_2) = 1 \quad h(\Lambda_3) = \kappa^2 \quad h(\Lambda_4) = \kappa \quad h(\Lambda_5) = \kappa^2.
\]
Then we obtain two \( f_h \)-invariant sets as depicted in figure 6.4

**Ergodic Components**

From Proposition 6.4.1 it is clear that for \( f_h \) to be ergodic we must necessarily have that \( H = G \).

We now show that this condition is in fact sufficient for ergodicity. Moreover we show:

**6.4.2 Theorem.** The orbits \( C_i \) are precisely the ergodic components of \( f_h \).

We consider only the case of a single orbit, the general case is proved similarly.

**6.4.3 Lemma.** Suppose that \( H = G \). Then \( h \) is not cohomologous to some Hölder \( \tilde{h}: \Lambda \rightarrow K \) where \( K \) is some proper subgroup of \( G \).
Chapter 6. Finite Group Extensions of Hyperbolic Basic Sets

Proof. Suppose that \( h \) is cohomologous to such a \( \tilde{h} \). Then for some \( \omega: \Lambda \to G \) we can write

\[
h(x) = \omega(fx)h(x)\omega(x)^{-1}.
\]

Thus \( h(A_i) = \omega(A_{i+1})\tilde{h}(A_i)\omega(A_i)^{-1} \) for \( i = 1, \ldots, n \) (with \( A_{n+1} = A_1 \)). Hence

\[
h_n = \omega(A_1)\tilde{h}_n\omega(A_1)^{-1} = \tilde{h}_n.
\]

Now since \( \tilde{h} \) takes values only in \( K \), \( \tilde{h}_n \in K \). This, though contradicts the fact that \( h_n \) generates \( G \).

Thus with reference to Section 6.2, in particular Theorem 6.2.2, Theorem 6.4.2 follows.

6.5 Mixing

We finish with a discussion of weak-mixing for finite group extensions over hyperbolic basic sets. As in previous sections let \( f: \Lambda \to \Lambda \) be a hyperbolic map, \( f_h: \Lambda \times G \to \Lambda \times G \), the corresponding group extension via the H"older cocycle \( h: \Lambda \to G \) into the finite group \( G \).

In this section we apply the extra assumption that \( f: \Lambda \to \Lambda \) is weak-mixing. (So in particular \( \Lambda \) is either connected or has infinitely many connected components.) If \( f_h \) is ergodic then by Theorem 5.4.1 \( f_h \) is not weak-mixing if and only if there exists a non-trivial, one-dimensional representation \( R \), a non-trivial H"older map \( v: \Lambda \to \mathbb{C} \) and a constant \( \epsilon_{\alpha} \neq 1 \) such that

\[
v(fx) = \epsilon_{\alpha} R(h(x))v(x).
\]

In particular we note that if \( G \) has no nontrivial one-dimensional representations then \( f_h \) will be automatically weak-mixing. This motivates:

6.5.1 Proposition. Let \( f: \Lambda \to \Lambda \) be a weak-mixing hyperbolic map, \( h: \Lambda \to G \) a H"older map into the finite group \( G \). Assume that \( G \) has no cyclic factors, that is for all proper, normal subgroups \( N < G \), the factor group \( G/N \) is noncyclic. Then if \( f_h \) is ergodic it is also weak-mixing.

Proof. Suppose that \( f_h \) is ergodic. If \( f_h \) is not weak-mixing \( G \) has a non-trivial irreducible one-dimensional representation, \( R \) say. Let \( K = \text{Ker} R \) denote the kernel of \( R \). Then \( K \) is a normal subgroup of \( G \) such that \( G/K \) is cyclic (see, for example [23]). This proves the proposition.

Conversely we show:

6.5.2 Proposition. Let \( f: \Lambda \to \Lambda \) be a weak mixing hyperbolic map, \( G \) a finite group. Suppose that for each H"older map \( h: \Lambda \to G \) such that \( f_h \) is ergodic, \( f_h \) is weak-mixing. Then \( G \) has no cyclic factors.
Proof. Suppose that $G$ has a cyclic factor so that there exists some proper normal subgroup, $N$ of $G$ with $G/N$ cyclic. Let $\langle \kappa \rangle \in G$, be a generator of $G/N$. Then $\langle \kappa \rangle$ is a subgroup of $G$ with $G = \langle \kappa \rangle N$ and $\langle \kappa \rangle \cap N = \{e\}$. Thus $\langle \kappa \rangle$ is normal in $G$ and $G$ is isomorphic to the direct product $\langle \kappa \rangle \times N$. This is how we identify $G$ from now on and write any Hölder map $h: \Lambda \rightarrow G$ as $(h^1, h^2)$, where $h^1: \Lambda \rightarrow \langle \kappa \rangle$ and $h^2: \Lambda \rightarrow N$ are Hölder. Note that if $f_h: \Lambda \times G \rightarrow \Lambda \times G$ is weak-mixing then so are $f_h^1: \Lambda \times \langle \kappa \rangle \rightarrow \Lambda \times \langle \kappa \rangle$ and $f_h^2: \Lambda \times N \rightarrow \Lambda \times N$.

Define a Hölder cocycle $h: \Lambda \rightarrow G$ with $h(x) = (h^1(x), h^2(x))$, where $h^1 \equiv \kappa$ and $h^2: \Lambda \rightarrow N$ is such that $f_h^2: \Lambda \times N \rightarrow \Lambda \times N$ is weak-mixing. (If no such $h^2$ exists we are done.) We claim that $f_h$ is ergodic but not weak-mixing.

To see that $f_h$ is ergodic we first note that $f_h^1$ is ergodic. Indeed suppose the contrary, then there exists a nontrivial irreducible representation $R$ of $G$ and a nonzero, locally constant map $v: \Lambda \rightarrow \mathbb{C}$ such that

$$R(h^1(x))v(x) = v(fx)$$

(6.5.1)

for all $x \in \Lambda$. Fix a point $x$ and a neighbourhood $U(x)$, such that $v$ is constant on $U(x)$. As $f$ is weak-mixing there exists some $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $f^{-n}(U(x)) \cap U(x) \neq \emptyset$. In particular we can pick $n \equiv 1 \pmod{k}$ and choose $y \in f^{-n}(U(x)) \cap U(x) \neq \emptyset$. Iterating (6.5.1) and substituting in $y$ gives

$$R(h^1(x))v(y) = v(fy).$$

Since both $y$ and $fy$ lie in $U(x)$ and $v|_{U(x)}$ is constant we have that $R(h^1(y)) = 1$. Then since $n \equiv 1 \pmod{k}$ we conclude that

$$R(h^1(y)) = R(\kappa^n) = R(\kappa)$$

and hence $R(\langle \kappa \rangle)$ is equal to 1, a contradiction. Thus $f_h^1$ is ergodic as claimed.

Next, as $f_h^2$ is weak-mixing, $f_h^2$ is ergodic. As a consequence we can pick periodic points $p_1, \ldots, p_r$, all lying in some open set $U \subset \Lambda$ with $h^2|_U$ constant, such that the set

$$\{h^2_{n_k}p_i | f^{n_k}p_i = p_i\}$$

generates $N$. Then as for all $i = 1, \ldots, r$ $h_{n_k}p_i = 1$ the set $\{h_{n_k}p_i\}$ generates $\{1\} \times N$. Further as $f_h^1$ is ergodic we have periodic points $p_{r+1}, \ldots, p_s$ lying in $U$ such that $\{h^1_{n_k}p_i | f^{n_k}p_i = p_i\} \times_{i=r+1}^s$ and hence $\{h^1_{n_k}p_i | f^{n_k}p_i = p_i\} \times_{i=r+1}^s$ generates $\langle \kappa \rangle$. Thus we have periodic points $p_1, \ldots, p_s$, $f^{n_k}p_i = p_i$ such that the set $\{h_{n_k}p_i\}$ generates $\langle \kappa \rangle = G$. Hence $f_h: \Lambda \times G \rightarrow \Lambda \times G$ is indeed ergodic.

To finish we show that $f_h^1$ and hence $f_h$, is not weak-mixing. Let $R$ be the one-dimensional irreducible representation on $\langle \kappa \rangle$ defined by $R(\kappa^j) = e^{2\pi ij/k}$, $j = 1, \ldots, r$. Let $v: \Lambda \rightarrow \mathbb{C}$ be any constant function, $|v| = 1$. Then for all $x \in \Lambda$

$$v(fx) = v(x) = e^{2\pi i/k}R(h(x))v(x).$$
Hence by the Keynes–Newton result $f_h$ is not weak-mixing. □

To finish we prove:

6.5.3 Proposition. Let $f: \Lambda \to \Lambda$ be a weak-mixing, hyperbolic map and $h: \Lambda \to G$ a Hölder map into the finite group $G$. Then $f_h$ is weak-mixing if and only if for all $n \in \mathbb{N}$ $f^n_h$ is ergodic.

Clearly the ‘only if’ part of the proposition is true. The ‘if’ part follows from:

6.5.4 Lemma. Let $f: \Lambda \to \Lambda$ be a weak-mixing, hyperbolic map and $h: \Lambda \to G$ a Hölder map into the finite group $G$ such that $f_h$ is ergodic. If there exists $p \in \Lambda$, $f^n p = p$ such that $h_{n^2} p = 1$ and $f^n_h$ is ergodic, then $f_h$ is weak-mixing. In particular if $f$ has a fixed point, $p$, such that $h(p) = 1$, then $f_h$ is weak-mixing.

Proof. Suppose that $f_h$ is not weak-mixing, then there exists a one-dimensional, nontrivial, irreducible representation $R$ together with a Hölder function $v: \Lambda \to \mathbb{C}$ and a constant $e^{ia}$ such that

$$v(f(x)) = e^{ia} R(h(x)) v(x). \tag{6.5.2}$$

Let $p_1, \ldots, p_r$ be such that $f^{n_i} p_i = p_i$, with $n_i = m_i n$ for all $i = 1, \ldots, r$ and $\{h_{n_i} p_i\} = G$. Substituting the $p_i$ into (6.5.2) and iterating gives

$$e^{in_i a} R(h_{n_i} (p_i)) = 1$$

for all $n_i$. Given that $h_{n} p = 1$, substituting $p$ into (6.5.2) and iterating yields that $e^{in a} = 1$. Thus for all $i = 1, \ldots, n$

$$e^{in_i a} = (e^{in a})^{m_i} = 1$$

and so

$$R(h_{n_i} (p_i)) = 1.$$ Since the $h_{n_i} p_i$ generate the whole of $G$, $R$ is trivial giving the desired contradiction. □
Chapter 7

Nonconnected Compact Group Extensions of Hyperbolic Basic Sets

The main purpose of this chapter is to show that the methods of Field and Parry in proving stable ergodicity results for compact connected Lie group extensions of hyperbolic basic sets still hold when considering certain nonconnected compact Lie group extensions of hyperbolic basic sets. Particularly we consider groups which can be expressed in the form \( \Gamma = G \ltimes \Gamma_0 \), where \( G \) is a finite group and \( \Gamma_0 \) is the connected component of the identity in \( \Gamma \).

Our method closely follows that of Field and Parry. We consider separately the cases \( \Gamma_0 \) abelian and \( \Gamma_0 \) semisimple later combining the results obtained using a structural result for compact connected Lie groups.

7.1 Setup

Let \( \Gamma \) be a compact Lie group which can be expressed as a semidirect product

\[
\Gamma = G \ltimes \Gamma_0
\]

where \( G \) is a finite group and \( \Gamma_0 \), the connected component of the identity in \( \Gamma \). Thus elements of \( \Gamma \) are of the form \((g, \gamma)\) where \( g \in G \) and \( \gamma \in \Gamma_0 \). Given elements \((g_1, \gamma_1), (g_2, \gamma_2) \in \Gamma \) the group multiplication is defined by

\[
(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 g_2, \rho_{g_2}(\gamma_1) \gamma_2),
\]

where the map \( \psi : G \rightarrow \text{Aut} G \) taking \( g \mapsto \rho_g \) is a homomorphism. In particular we have \( \Gamma / \Gamma_0 \cong G \).

Let \( f : \Lambda \rightarrow \Lambda \) be the restriction of a diffeomorphism on a compact Riemannian manifold to a hyperbolic basic set \( \Lambda \). Let \( \Phi : \Lambda \rightarrow \Gamma \) be a Hölder map which we write as \( \Phi = (h, \phi) \), where
Chapter 7. Nonconnected Extensions of Hyperbolic Basic Sets

$h: \Lambda \rightarrow G$ and $\phi: \Lambda \rightarrow \Gamma_0$ are each Hölder. Let $f_\Phi: X \times G \rightarrow X \times G$ denote the compact group extension of $f$ by $\Phi$. Then for $x \in \Lambda$ and $(g, \gamma) \in \Gamma$ we have

$$f_\Phi(x, (g, \gamma)) = (f(x), \Phi(x)(g, \gamma))$$

$$= (f(x), (h(x), \phi(x))(g, \gamma))$$

$$= (f(x), (h(x), \rho_\phi(\phi(x)))\gamma)).$$

Thus we can rewrite $f_\Phi$ as $F_\Phi: (\Lambda \times G) \times \Gamma_0 \rightarrow (\Lambda \times G) \times \Gamma_0$ where

$$F_\Phi((x, g), \gamma) = (F(x, g), \Psi(x, g)\gamma) \quad (x, g, \gamma) \in \Lambda \times G \times \Gamma_0$$

and $F: \Lambda \times G \rightarrow \Lambda \times G$ is defined by $F(x, g) = f_h(x, g) = (f(x), h(x)g)$.

### 7.2 Abelian

First we consider the case where $\Gamma_0$ is abelian, whereby we write $\Gamma_0 = \mathbb{T}^m$ and $\Psi = (\Psi_1, \ldots, \Psi_n)$ for some $m \in \mathbb{N}$ and $\Psi_1: X \times \Gamma \rightarrow S^1$. By Theorem 5.4.2 we know that $F_\Psi$ is not ergodic if and only if there exist integers $k_1, \ldots, k_m$ not all zero and a Hölder map $v: X \times G \rightarrow S^1$ such that

$$v \circ F/v = \Psi_1^{k_1} \ldots \Psi_n^{k_m}.$$

For this case we will also assume that $\Lambda$ has a finite number of connected components.

### Homotopy Groups

For a connected topological space we let $H^1(X)$ denote the Brushtinsky group of homotopy classes of continuous functions from $X$ to the circle $S^1$. We consider

$$H^1(\Lambda \times G) \cong \prod_{i=1}^m H^1(\Lambda_i) \times \cdots \times H^1(\Lambda_n)$$

where

$$H^1(\Lambda) \cong H^1(\Lambda_1) \times \cdots \times H^1(\Lambda_n).$$

Let $C^0(\Lambda \times G, S^1)$ denote the group of continuous functions from $\Lambda \times G$ to $S^1$ and $N^0(\Lambda \times G, S^1)$ the (normal) subgroup of $C^0(\Lambda \times G, S^1)$ consisting of null homotopic maps. Then

$$H^1(\Lambda \times G) \cong C^0(\Lambda \times G, S^1)/N^0(\Lambda \times G, S^1).$$

Moreover since every map in $C(\Lambda \times G, S^1)$ is homotopic to a map in $C^0(\Lambda \times G, S^1)$,

$$H^1(\Lambda \times G) \cong C^\alpha(\Lambda \times G, S^1)/N^\alpha(\Lambda \times G, S^1).$$

Fix $(\lambda, g) \in \Lambda \times G$ and a subgroup $\Pi$ of $C^0(\Lambda \times G, S^1)$ containing functions which map $(\lambda, g)$ to the identity in $S^1$. In particular $\Pi \cong H^1(\Lambda \times G)$. 
Let \( F^* : H^1(\Lambda \times G) \to H^1(\Lambda \times G) \) denote the homomorphism defined by \( F^*(\xi) = \xi \circ F \). As \( \pi \cong H^1(\Lambda \times G) \) \( F^* \) induces a homomorphism \( F^*_\Pi : \Pi \to \Pi \). Following Field and Parry, for \( \xi \in \Pi \), we express

\[
F^*_\Pi(\xi) = F^*_\Pi(\xi) \exp(2\pi i (r_\xi + c_\xi))
\]

where the map \( \Pi \to C^0(\Lambda \times G, \mathbb{R}) \) taking \( \xi \to r_\xi \) is a uniquely determined homomorphism satisfying \( r_\xi(\lambda, g) = 0 \) and \( c_\xi \) is a constant in \([0, 1)\).

For our results we will require the condition that \( \dim(\ker(F^* - I)) < \infty \). In particular this holds if \( H^1(\Lambda \times G, \mathbb{Z}) \) has finite rank (see [21]). By the next theorem this also holds when \( \Lambda \) is a hyperbolic attractor.

**7.2.1 Theorem.** Suppose that \( f : \Lambda \to \Lambda \) is a hyperbolic attractor. Then \( \text{rank}(H^1(\Lambda \times G, \mathbb{Z})) < \infty \).

**Proof.** We follow [21]. Suppose that \( \Lambda \) is a hyperbolic attractor of some diffeomorphism \( \tilde{f} : M \to M \) and let \( \tilde{f}_h : M \times G \to M \times G \) denote the corresponding finite group extension of \( \tilde{f} \) by \( h \). Since \( \Lambda \) is an attractor we can find a closed neighbourhood \( Y \) of \( \Lambda \) such that \( \tilde{f}_h(Y \times G) \) is contained in the interior of \( Y \) and \( \bigcap_{n \geq 0} \tilde{f}_h(Y) = \Lambda \times G \). Further we can assume that \( Y \) is a smooth manifold with boundary.

Let \( \tau_1, \ldots, \tau_k \in \Pi \) be linearly independent. We extend the \( \tau_i \) as continuous maps on some open neighbourhood \( U \times G \) of \( \Lambda \times G \) of \( \Lambda \). Now for some \( N \geq 0 \) the \( \tau_i \) extend to \( \tilde{f}_h(Y \times G) \). As these extensions are integrally independent we have that

\[
k \leq \text{rank}(H^1(\tilde{f}_h(Y \times G), \mathbb{Z})) = \text{rank}(H^1(Y \times G, \mathbb{Z})).
\]

Hence \( \Pi \leq \text{rank}(H^1(Y \times G, \mathbb{Z})) \). Since \( Y \) is compact with boundary \( \text{rank}(H^1(Y \times G, \mathbb{Z})) \) is finite. Noting \( H^1(\Lambda \times G, \mathbb{Z}) \), the result follows. \( \square \)

**Stable Ergodicity**

Our main result for abelian extensions is the following:

**7.2.2 Theorem.** Suppose that \( f_h \) is ergodic and that \( \text{rank}(\ker(F^* - I)) < \infty \). Then there exists an open and dense set of maps \( \phi : \Lambda \to \Gamma_0 \) such that \( F_\psi \) is ergodic.

Our proof follows closely that of [21] with minor deviations.

**7.2.3 Lemma.** Suppose \( \Psi \) is homotopic to \( \varphi = (\varphi_1, \ldots, \varphi_n) \), \( \varphi_i \in \Pi \) with

\[
\langle \varphi_1, \ldots, \varphi_n \rangle \cap (F^*_\Pi - I)\Pi
\]

Then \( F_\Psi \) is ergodic for all \( \tilde{\Psi} : \Lambda \times G \to \mathbb{T}^m \) in \( C^0 \)-open neighbourhood of \( \Psi \).

**Proof.** Suppose that \( \Psi \) is homotopic to such a \( \varphi \). Clearly if \( \tilde{\Psi} \) is \( C^0 \) close to \( \Psi \), then it is homotopic to \( \Psi \) and hence to \( (\varphi_1, \ldots, \varphi_n) \). Thus it will be sufficient to prove that \( F_\Psi \) is ergodic. Suppose it is
not, then there are integers $k_1, \ldots, k_m$, not all zero together with a nontrivial continuous solution to

$$v \circ F/v = \Psi_1^{k_1} \cdots \Psi_m^{k_m}.$$ 

Then as $\Psi_i$ is homotopic to $\phi$ for each $i = 1, \ldots, m$, $\Psi_1^{k_1} \cdots \Psi_m^{k_m}$ is homotopic to $\phi_1^{k_1} \cdots \phi_m^{k_m}$. Thus $\phi_1^{k_1} \cdots \phi_m^{k_m} \in (F^n - I)\Pi$, a contradiction. \hfill \Box

Given maps $\eta_1, \ldots, \eta_m \in \Pi$ we define the vector subspace of $C^\alpha(\Lambda, \mathbb{R})$

$$\overline{V}_{\eta_1, \ldots, \eta_m} = \text{span}\{r_\xi \mid (F^n - I)\xi \in \langle \eta_1, \ldots, \eta_m \rangle\}.$$ 

Let $V_{\eta_1, \ldots, \eta_m}$ denote the sum of $\overline{V}_{\eta_1, \ldots, \eta_m}$ and the space of locally constant functions on $\Lambda$. Since $\Lambda$ has finitely many connected components we have that

$$\dim(V_{\eta_1, \ldots, \eta_m}) = n' + \dim(\overline{V}_{\eta_1, \ldots, \eta_m}) \leq n + \text{rank}(\ker(F^n - I))$$

for $n, n' \in \mathbb{N}$. Let $B^\alpha$ denote the space of $\alpha$–Hölder, $\mathbb{R}$ valued coboundaries, that is

$$B^\alpha := \{h \circ F - F \mid h \in C^\alpha(\Lambda, \mathbb{R})\}.$$ 

It follows from the Livšic Periodic Point Theorem (Theorem 1.3.2) that $B^\alpha$, and hence $B^\alpha + V_{\eta_1, \ldots, \eta_m}$ is $C^0$–closed on $C^\alpha(\Lambda, \mathbb{R})$.

**7.2.4 Lemma.** Write $\Psi = (\Psi_1, \ldots, \Psi_m) \in C^\alpha(\Lambda, \mathbb{T})$, where, for $i = 1, \ldots, m$ $\Psi_i = \eta_i \exp(2\pi i \nu_i)$, $\eta_i \in \Pi$, $\nu_i \in C^\alpha(\Lambda, \mathbb{R})$. Then $F_\Psi$ is ergodic if

$$\text{span}(\nu_1, \ldots, \nu_m) \cap (V_{\eta_1, \ldots, \eta_m} + B^\alpha) = \{0\}$$

**Proof.** Suppose $F_\Psi$ is not ergodic. Then there exists a nontrivial Hölder $v: \Lambda \to \mathbb{S}^1$ together with nonzero natural numbers $k_1, \ldots, k_m$ such that

$$\Psi_1^{k_1} \cdots \Psi_m^{k_m} = v \circ F/v.$$ 

Let $v = \xi \exp(2\pi i v)$ where $\xi \in \Pi$ and $v \in C^\alpha(\Lambda, \mathbb{R})$. Then

$$v \circ F/v = \frac{\xi \circ F \exp(2\pi i v \circ F)}{\exp(2\pi i v)}$$

$$= \frac{F^n(\xi)}{\xi} \exp(2\pi i (v \circ F - v))$$

$$= \frac{F^n(\xi) \exp(2\pi i (r_\xi + c_\xi))}{\xi} \exp(2\pi i (v \circ F - v)).$$ (7.2.1)

We can also write

$$\Psi_1^{k_1} \cdots \Psi_m^{k_m} = \eta_1^{k_1} \cdots \eta_m^{k_m} \exp(2\pi i (k_1\nu_1 + \cdots + k_m\nu_m)).$$ (7.2.2)
Thus equating (7.2.1) and (7.2.1) gives

\[(F_\Pi - I)\xi = \eta_1^{k_1} \cdots \eta_m^{k_m}\]  
\[n + r_\xi + c_\xi + \nu \circ F - \nu = k_1\nu_1 + \cdots + k_m\nu_m\]

where \(n: \Lambda \times G \to N\) is constant on each connected component of \(\Lambda \times G\). From (7.2.3) \(r_\xi\) is in \(\mathcal{V}_{\eta_1, \ldots, \eta_m}\). Further since \(c_\xi\) is constant and \(n\) is locally constant, \(n + r_\xi + c_\xi\), belongs to \(\mathcal{V}_{\eta_1, \ldots, \eta_m}\).

Clearly \(\nu \circ F - \nu \in B^\alpha\) so that by (7.2.4) \(k_1\nu_1 + \cdots + k_m\nu_m\) is a nonzero element of \(\mathcal{V}_{\eta_1, \ldots, \eta_m} + B^\alpha\). Thus if \(\text{span}(\nu_1, \ldots, \nu_m) \cap (\mathcal{V}_{\eta_1, \ldots, \eta_m} + B^\alpha) = \{0\}\), \(F_\Phi\) must be ergodic. \(\square\)

**7.2.5 Lemma.** The space \(C^\alpha(\Lambda \times G, \mathbb{R})/B^\alpha\) is infinite dimensional.

**Proof.** Suppose that \(C^\alpha(\Lambda \times G, \mathbb{R})/B^\alpha\) has finite dimension \(l\). Then there exist linearly independent \(\rho_1, \ldots, \rho_l \in C^\alpha(\Lambda \times G, \mathbb{R})\) and linear \(\tau_1, \ldots, \tau_r, \tau_i : C^\alpha(\Lambda \times G, \mathbb{R})\) such that given \(\Phi \in C^\alpha(\Lambda \times G, \mathbb{R})\) we may write

\[\Phi = \sum_{i=1}^l \tau_i(\Phi)\rho_i + \nu \circ F - \nu\]

for some \(\nu \in C^\alpha(\Lambda \times G, \mathbb{R})\). Let \(\mu\) be any \(F\)-invariant measure on \(\Lambda\). Then integrating both sides of (7.2.5) with respect to \(\mu\) gives

\[\int_\Lambda \Phi d\mu = \sum_{i=1}^l \tau_i(\Phi) \int_\Lambda \rho_i + \int_\Lambda \nu \circ F - \int_\Lambda \nu = \sum_{i=1}^l \tau_i(\Phi) \int_\Lambda \rho_i\]

Hence \(\int_\Lambda d\mu\) is a linear combination of the \(\tau_i\). This implies that the space of invariant measures on \(\Lambda\). This is a contradiction since, for example, the space of periodic point measures is infinite dimensional. \(\square\)

**7.2.6 Lemma.** The set

\[\{ (\eta_1, \ldots, \eta_m) \in C^\alpha(\Lambda, \mathbb{R})^m \mid \text{span}(\nu_1, \ldots, \nu_m) \cap \mathcal{V}_{\eta_1, \ldots, \eta_m} = \{0\} \}\]

is \(C^\alpha\)-open and \(C^r\)-dense \((r > 0)\) in \(C^\alpha(\Lambda, \mathbb{R})^m\).

**Proof.** As in [21] openness follows from the closedness of \(\mathcal{V}_{\eta_1, \ldots, \eta_m} + B^\alpha\) whilst density follows since \(\text{dim}(\mathcal{V}_{\eta_1, \ldots, \eta_m})\) is bounded above by \(n + \text{rank}(\ker(F^* - I))\) and from the previous lemma. \(\square\)

Combining Lemmas 7.2.3–7.2.6 proves Theorem 7.2.2.
Chapter 7. Nonconnected Extensions of Hyperbolic Basic Sets

7.3 $\Gamma_0$ Semisimple

We now consider the case where $\Gamma_0$ is semisimple (i.e. its center $Z(\Gamma_0)$ is finite). In this case we will make use of the following result of Kuranishi [36].

7.3.1 Theorem. Let $\Gamma$ be a compact, connected, semisimple Lie group. The set of pairs which generate $\Gamma$ is open and dense in $\Gamma^2$.

Stable Ergodicity

In this section we prove:

7.3.2 Theorem. Suppose that $f_h$ is ergodic. Then, given $r > 0$ there exists a $C^0$-open and $C^r$-dense set of Hölder maps $\phi: \Lambda \to \Gamma_0$ such that $F_\phi$ is ergodic.

Suppose that $F_\phi$ is not ergodic. Then by the Keynes-Newton condition and Livšic regularity there exists a nontrivial irreducible representation $R: \Gamma_0 \to U(d)$ and a nonzero Hölder function $v: \Lambda \times G \to \mathbb{C}^d$, such that

$$R(\Psi(x,g))v(z,g) = v(F(z,g))$$

for all $(z,g) \in \Lambda \times G$. Or simplicity we will identify $R \circ \Psi$ with $\Psi$ from now on.

Let $z$ be a periodic point for $f$. By taking a power of $F$ if necessary we may assume, without loss of generality, that $(z, e)$ is a fixed point of $F$. Let $y \in \Lambda$ be a homoclinic point to $x$ under $f$ such that $(y, e)$ is homoclinic to $(x, e)$ under $F$.

Iterating equation (7.3.1) we find that

$$\Psi_n(z,g)v(z,g) = v(F^n(z,g))$$

for all $n \in \mathbb{N}$, $z \in \Lambda$ and $g \in G$, where

$$\Psi_n(z,g) = \Psi(F^{n-1}(z,g)) \cdots \Psi(F(z,g)) \Psi(z,g)$$

$$\Psi_{-n}(z,g) = \Psi(F^{-1}(z,g)) \cdots \Psi(F^{-n+1}(z,g)) \Psi(F^{-n}(z,g)).$$

In particular we have that

$$\Psi^n(x,e)v(x,e) = v(x,e)$$

$$\Psi_{-n}(x,e) = v(y,e)$$

$$\Psi_n(y,e) = v(F^n(y,e)).$$

Define $\gamma_n^+ = \Psi(x,e)^{-n} \Psi_n(y,e)$. Then for any $n \in \mathbb{N}$

$$\gamma_n^+ v(y,e) = \Psi(x,e)^{-n} v(F^n(y,e)).$$
Furthermore let $\gamma^+ := \lim_{n \to \infty} \gamma^+_n$. To see that this is well defined we calculate

$$
d(\gamma^+_{n+1}, \gamma^+_n) = d(\Psi(z, e)^{(n+1)}\Psi_n(y, e), \Psi(x, e)^{-n}\Psi_n(y, e))
$$

$$
= d(\Psi(x, e)^{-1}\Psi(F^n(y, e)), I)
$$

$$
= d(\Psi(F^n(y, e)), \Psi(x, e))
$$

$$
\leq C d(F^n(y, e), (x, e))^\alpha
$$

$$
\leq C \lambda^{n \alpha}
$$

where $d(\cdot, \cdot)$ is a bi-invariant metric on $\Gamma_0$ and $0 < \lambda < 1$ estimates the contraction on the stable manifold through $x$. Thus for all $m > n \in \mathbb{N}$

$$
d(\gamma^+_m, \gamma^+_n) \leq \sum_{i=n}^{m-1} d(\gamma^+_{i+1}, \gamma^+_i) \leq c \lambda^\alpha \sum_{i=0}^{m-(n+1)} C \lambda^{i \alpha}.
$$

Letting $m \to \infty$ the left hand side of this inequality becomes $C \lambda^\alpha \frac{\lambda^n}{1-\lambda}$ which tends to zero as $n \to \infty$. In other words the sequence $\gamma^+_n$ is Cauchy and hence converges.

Furthermore we calculate

$$
d(\Psi(x)^n v(F^n(y, e)), v(x, e)) = d(v(F^n(y, e)), \Psi(x, e)^{-n}v(x, e))
$$

$$
= d(v(F^n(y, e)), v(x, e))
$$

$$
= C \lambda^n \to 0 \text{ as } n \to \infty.
$$

Thus

$$
\gamma^+ v(y, e) = v(x, e). \quad (7.3.2)
$$

Similarly we define $\gamma^- = \Psi_{-n}(y, e)\Psi(x, e)^{-n}$ for $n \in \mathbb{N}$ and set $\gamma^- := \lim_{n \to \infty} \gamma^-_n$, which can be shown to exist via a similar calculation to the $\gamma^+$ case. Note

$$
d(\gamma^-_n v(x, e), v(y, e)) = d(\Psi_{-n}(y, e)v(x, e), \Psi_{-n}(y, e)v(F^{-n}(y, e)))
$$

$$
= C \lambda^n \to 0 \text{ as } n \to \infty.
$$

Thus

$$
\gamma^- v(x, e) = v(y, e). \quad (7.3.3)
$$

Combining (7.3.2) with (7.3.3) gives $\gamma^+\gamma^- v(x, e) = v(x, y)$. In particular we have two elements, namely $\gamma^+\gamma^-$ and $v(x, e)$ which fix the vector $v(x, e)$. Thus if $\gamma^+\gamma^-$ and $v(x, e)$ were to together generate $\Gamma_0$, the whole of $\Gamma_0$ would fix $v(x, e)$ providing us with a contradiction, i.e. implying that $F_0$ is ergodic.
Chapter 7. Nonconnected Extensions of Hyperbolic Basic Sets

Openness

Suppose that $\Psi$ is such that $\gamma^+\gamma^-$ and $\Psi(x, e)$ generate $\Gamma_0$. Let $\Psi^*: \Lambda \to \gamma_0$ be Hölder and fix $\epsilon > 0$. For $n \in \mathbb{N}$ define:

$$
\gamma_n = \Psi^*(x, e)^{-n} \Psi^*_n(y, e) \Psi^*(y, e)^{-n} \\
\gamma = \lim_{n \to \infty} \gamma_n \\
\gamma_n = \gamma^+_n \gamma^-_n \\
\gamma = \gamma^+ \gamma^- .
$$

By the triangle inequality for any $n \in \mathbb{N}$ we have

$$
d(\gamma, \gamma^*) \leq d(\gamma, \gamma_n) + d(\gamma^*, \gamma^-_n) + d(\gamma_N, \gamma^-_N) .
$$

From the earlier calculations, by choosing $n$ large enough we can guarantee that $d(\gamma, \gamma_n)$ and $d(\gamma^*, \gamma^-_n)$ are each less than $\epsilon/3$. Note also that for a $C^0$ open set of $\Psi^*$ (containing $\Psi$) the same $n$ will work. Further assuming that $\Psi^*$ is close enough to $\Psi$, specifically that

$$
d(\Psi(f^i(y, e)), \Psi^*(f^i(y, e))) < \frac{\epsilon}{3N}
$$

for all $i \leq n$, we have that $d(\gamma, \gamma^*)$. Further suppose that $d(\Psi(x, e), \Psi^*(x, e)) < \epsilon$. Given that the set of generating pairs for $\Gamma_0$ is open we can take $\epsilon$ small enough so that $(\Psi^*(x, e), \gamma^*) = \Gamma_0$. This proves the openness part of the Theorem.

Density

To establish the density part of the theorem we first assume that $\Psi$ is such that $\gamma^+\gamma^-$ and $\Psi(x, e)$ do not generate $\Gamma_0$. We make a small perturbation of $\Psi$ to a Hölder function $\Psi^*: \Lambda \to \Gamma_0$ as follows.

First modify $\Psi$ to a function $\tilde{\Psi}: \Lambda \to \Gamma_0$ by setting $\tilde{\Psi}(x, e) = \Psi(x, e) \cdot \delta$ where $\delta \in \Gamma_0$ is chosen such that there exists a dense set of $\kappa \in \Gamma_0$ such that $\Psi(x), \delta$ and $\kappa$ generate $\Gamma_0$. The fact that this can be done follows from Theorem 7.3.1. Further $\delta$ can be chosen arbitrarily close to the identity.

Now define the map $\Psi^*: \Lambda \to \Gamma_0$ by setting $\Psi^*(x, g) = \tilde{\Psi}(x, g) \cdot \exp(\xi b(e))$ where $\xi \in L(\Gamma_0)$, the Lie algebra of $\Gamma_0$ and $b: \Lambda \to L(G)$ is a bump function centered on $y$. That is $b$ is a smooth function such that $b(y, e) = 1$ and $\{(x, e) \in \Lambda \times G \mid b(x) \neq 0\} \subset U$, where $U$ is an open neighbourhood of $(y, e)$. We take $U$ small enough so that $x \notin U$, i.e. $\Psi^* = \tilde{\Psi}$. We can choose the element $\xi$ such that $\Psi^*(x, e)$ and $\gamma^*$ generate $\Gamma_0$. Further we can choose $\xi$ arbitrarily close to zero so that $\exp(\xi b(x))$ will be arbitrarily close to the identity. This completes the proof of Theorem 7.3.2.
Chapter 7. Nonconnected Extensions of Hyperbolic Basic Sets

Weak-Mixing

Given that the only irreducible one-dimensional representations of semisimple groups are trivial, with reference to Theorem 5.2.1 we see that if $F_\phi$ is ergodic then it is also weak-mixing. Thus Theorem 7.3.2 can be extended to

7.3.3 Theorem. Suppose that both $f: \Lambda \to \Lambda$ and $f_h: \Lambda \times G \to \Lambda \times G$ are weak-mixing. Then, given $r > 0$ there exists a $C^0$-open and $C^r$-dense set of Hölder maps $\phi: \Lambda \to \Gamma^0$ such that $F_\phi$ is ergodic.

7.4 $\Gamma_0$ a General Compact Lie Group

7.4.1 Theorem. Suppose that $f_h$ is ergodic and that either

(a) $\Gamma_0$ is semisimple or

(b) $\Lambda$ has finitely many connected components and $\text{rank}(\ker(F^* - I)) < \infty$.

Then there exists an $C^0$-open and $C^r$-dense set of Hölder cocycles $\phi: \Lambda \to \Gamma_0$ such that $F_{\phi_k}$ is ergodic.

Following Field and Parry we will use a structural result for compact connected Lie groups to combine Theorems 7.2.2 and 7.4.1 in order to prove Theorem 7.4.1.

In detail there exists a finite covering homomorphism $\pi: \Gamma_0 \to \Gamma$, where $\Gamma$ is the direct product of a torus $T_r$ and a compact semisimple group $S$. We define $\Gamma_A = \pi(T_r \times I_S)$ and $\Gamma_S = \pi(I_T \times S)$. Clearly $\Gamma_A$ and $\Gamma_S$ are normal subgroups of $\Gamma_0$ with $\Gamma_A \cong T_r$ and $\Gamma_S$ semisimple. Define $\bar{A} = \Gamma/\Gamma_S$, $\bar{S} = \Gamma/\Gamma_A$ and let $\pi_A: \Gamma \to \Gamma_A$ and let $\pi_A: \Gamma \to \Gamma_S$ be the quotient homomorphisms.

Define $\Psi_A = \pi_A \circ \Psi: \Lambda \times G \to \bar{S}$ and $\Psi_S \circ \Psi = \pi_S: \Lambda \times G \to \bar{A}$. For an open and dense set $\mu$ of $\Psi: \Lambda \to \Gamma$ the extensions $F_{\psi_A}$ and $F_{\psi_S}$ will be ergodic.

Suppose $\phi \in \mathcal{U}$ and let $\Sigma$ denote the isotropy subgroup of an ergodic component of $F_\phi$. Then $\pi_A(\Sigma) = \bar{A}$ and $\bar{S}$ and so $\Sigma = \Gamma_0$. Hence $F_\phi$ is ergodic thus proving the result.
Chapter 8

Compact Group

Extensions of Hyperbolic Systems with Singularities

As we saw have seen in previous chapters, one use of Livšic regularity results is to establish stability or genericity of ergodicity for compact group extensions by strengthening the Keynes-Newton conditions. In this chapter we apply the results obtained in Chapters 2 and 3 to the same ends for compact group extensions of Lasota-Yorke maps and uniformly hyperbolic toral maps with singularities.

8.1 Main Results

The main results of this chapter are the following:

8.1.1 Theorem. Let $f: X \to X$ be a Lasota-Yorke map which is weak-mixing with respect to an absolutely continuous invariant measure with support $X$. Let $\Gamma$ be a compact Lie group, $F$ the space of $C^1$ cocycles $h: X \to \Gamma$.

(i) If $\Gamma$ is semisimple there exists a $C^1$ open and dense subset $F'$ of $F$ such that if $h \in F'$ then $f_h$ is ergodic.

(ii) If $\Gamma$ is a compact connected Lie group then there exists a $C^1$ prevalent ([38]) and residual subset $F'$ of $F$ such that for $h \in F'$, $f_h$ is Bernoulli.

8.1.2 Theorem. Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be a uniformly hyperbolic toral map with discontinuities. Let $F$ denote the space of $C^1$ cocycles $h: X \to \Gamma$, $\Gamma$ a compact Lie group.

If $\Gamma$ is semisimple there exists a $C^1$ open and dense subset $F'$ of $F$ such that if $h \in F'$ then $f_h$ is Bernoulli.

If $\Gamma$ is a compact connected Lie group then there exists a $C^1$ prevalent and residual subset $F'$ of $F$ such that for $h \in F'$, $f_h$ is Bernoulli.
8.2 The Keynes–Newton Conditions

Analogously to Theorems 5.4.1 and 5.4.3 we can extend the Keynes–Newton conditions using the Livšic regularity results obtained in chapters 2 and 3. We thus obtain:

8.2.1 Theorem. Let \( f : X \to X \) be a Lasota-Yorke map, weakly mixing with respect to the absolutely continuous invariant measure \( m \) with support \( X \). Let \( h : X \to \Gamma \) be a Lipschitz cocycle into the compact Lie group \( \Gamma \) with Haar measure \( \nu \).

(i) The compact group extension \( f_h : X \times \Gamma \to X \times \Gamma \) is ergodic with respect to \( m \times \nu \) if and only if the only solutions to
\[
\nu(fx) = R(h(x))\nu(x)
\]
where \( R \) is an irreducible unitary representation of degree \( d \) and \( \nu : X \to \mathbb{C}^d \) is Lipschitz on an open neighbourhood of \( m \) almost every point, are \( \nu \) constant and \( R \) trivial.

(ii) If \( f_h : X \times \Gamma \to X \times \Gamma \) is ergodic with respect to \( m \times \nu \) then it is weak-mixing with respect to \( m \times \nu \) if and only if the only solutions to
\[
e^{i\alpha}R(h(x))\nu(x) = \nu(fx)
\]
where \( R \) is an irreducible one-dimensional representation of degree \( d \), \( e^{i\alpha} \) is a constant and \( \nu : T^2 \to \mathbb{C}^d \) is Lipschitz on an open neighbourhood of \( m \) almost every point are \( \nu \) constant \( e^{i\alpha} = 1 \) and \( R \) trivial.

Similarly for uniformly hyperbolic toral maps with singularities we have:

8.2.2 Theorem. Let \( f : T^2 \to T^2 \) be a hyperbolic toral map with singularities. Let \( h : T^2 \to \Gamma \) be a Lipschitz cocycle into the compact Lie group \( \Gamma \) with Haar measure \( \nu \).

(i) The compact group extension \( f_h : T^2 \times \Gamma \to T^2 \times \Gamma \) is ergodic with respect to \( l \times \nu \) (\( l \) denotes Lebesgue measure) if and only if the only solutions to
\[
\nu(fx) = R(h(x))\nu(x)
\]
where \( R \) is an irreducible unitary representation of degree \( d \) and \( \nu : T^2 \to \mathbb{C}^d \) is Lipschitz on an open neighbourhood of \( l \) almost every point, are \( \nu \) constant and \( R \) trivial.

(ii) If \( f_h : T^2 \times \Gamma \to T^2 \times \Gamma \) is ergodic with respect to \( l \times \nu \) then it is weak-mixing with respect to \( m \times \nu \) if and only if the only solutions to
\[
e^{i\alpha}R(h(x))\nu(x) = \nu(fx)
\]
where \( R \) is an irreducible one-dimensional representation of degree \( d \), \( e^{i\alpha} \) is a constant and \( \nu : T^2 \to \mathbb{C}^d \) is Lipschitz on an open neighbourhood of \( m \) almost every point are \( \nu \) constant \( e^{i\alpha} = 1 \) and \( R \) trivial.
We sketch only the main points of Theorem 8.2.1, the details for Theorem 8.2.2. From Chapter 2 we know that for \( m \) almost every \( x_0 \in X \) there exists a sequence of intervals \( \{J_n\} \) and an inverse branch \( \{z_n\} \) satisfying \( x_n \in J_n \) and \( f^n(J_0) = J_0 \). Fix such a point \( x_0 \) and intervals \( \{J_0\} \) and take points \( y_0, z_0 \in J_0 \) together with inverse branches \( \{y_n\}, \{z_n\} \) satisfying \( y_n, z_n \in J_n \). We calculate

\[
\|v(y_0) - v(z_0)\| = \|R(h(z_1)) \cdots R(h(z_{n-1}))(v(z_n) - R(h(y_1)) \cdots R(h(y_{n-1}))v(y_n)\|
\]

\[
\leq \sum_{i=1}^{n-2} \|R(h(z_i)) \cdots R(h(y_{i+1}))(R(h(y_{i+1})) - h(y_{i+1})))R(h(y_{i+2})) \cdots R(h(y_{n-1}))v(z_n)\|
\]

\[
+ \|R(h(z_1)) \cdots R(h(z_{n-1}))(h(y_n) - h(z_n))\|.
\]

The sum

\[
\sum_{i=1}^{n-2} \|R(h(z_i)) \cdots R(h(y_{i+1}))(R(h(y_{i+1})) - h(y_{i+1})))R(h(y_{i+2})) \cdots R(h(y_{n-1}))v(z_n)\|
\]

is bounded above by \( Kd(y_0, z_0) \) for some \( K \) since \( R(h(y_{i+2})) \cdots R(h(y_{n-1}))v(z_n) \) is uniformly bounded in \( k \) and \( d(y_i, z_i) < \lambda^{-i}d(y, z) \). Since \( v \) is measurable we may use Lusin's Theorem (Theorem 2.3.3) to find a set \( H, m(H) > \frac{1}{2} \) such that \( h \) restricted to \( H \) is uniformly continuous (hence bounded). We can then use exactly the same argument as in Section 2.3, i.e. using the Lebesgue Density Theorem (Theorem 2.3.5) and the fact that \( f \) has bounded distortion to show that given \( \eta > 0 \)

\[
(m \times m)(\{(y_0, z_0) \mid \|R(h(y_1)) \cdots R(h(y_{n-1}))(v((z_n) - v(y_n))\| \leq \eta\}) \geq (1 - 2\eta)(m \times m)(J_0 \times J_0)
\]

and hence for \( m \) almost every \( x_0 \in X \) there is an interval \( J_0 \ni x_0 \) such that \( v \) restricted to \( J_0 \) is Lipschitz.

The same analysis shows that, in equation (8.2.1), we may assume that for \( l \) almost all \( x \in \mathbb{T}^2 \) there is a neighbourhood \( U(x) \) on which \( v \) is Lipschitz.

### 8.3 Abelian and Semisimple Extensions of Lasota–Yorke Maps

As in Chapter 7 we will first consider abelian and semisimple extensions and then, following Field and Parry [21] use a structural result for compact connected Lie groups to obtain the results for general compact connected Lie groups.

#### Abelian Extensions

Suppose that \( \Gamma = \mathbb{T}^r \) is a toral group. With reference to Theorem 5.4.2 the group extension \( f_h \) is ergodic if and only if there are no nontrivial Lipschitz solutions \( v: X \to \mathbb{T}^r \) to the equation

\[
v(f(x))v(x)^{-1} = h(x)^k \quad (8.3.1)
\]
for any nonzero \( k = (k_1, \ldots, k_r) \in \mathbb{Z}^r \) and non-trivial \( \nu: X \to T^r \).

We note that if \( p \) is a point such that \( f^n p = p \) and \( f^i p \not\in S \) for all \( i \) then equation (8.3.1) implies that

\[
(h(f^{n-1}p) \ldots h(fp)h(p))^k = 1.
\]

But this in turn implies that \( h(f^{n-1}p) \ldots h(fp)h(p) \) is not a generator of \( T^r \). Now Lasota–Yorke maps have a dense set of periodic points. We may, then choose a point \( p \in X, f^n p = p \) which avoids the singular set and whose iterates are contained in an open set on which \( h \) is Lipschitz. For a prevalent and residual set of maps \( h(f^{n-1}p) \ldots h(fp)h(p) \) will generate \( T^r \). Thus for a prevalent and residual set of maps \( h: X \to T^r \), the toral extension \( f_h: X \times T^r \to X \times T^r \).

To establish the prevalence and residualness of weak-mixing we note that by the result just proved, given \( h: X \to T^r \) we can assume, without loss of generality, that \( f_h \) is ergodic. Clearly we may take two points \( x \) and \( y \) such that \( f^m x = x \) and \( f^m y = y \) and such that their iterates are contained in neighbourhoods on which \( h \) is Lipschitz. If \( f_h \) is not weak-mixing then neither

\[
e^{i\alpha} h(f^{m-1}x) \ldots h(fx)h(x)
\]

nor

\[
e^{i\alpha} h(f^{n-1}y) \ldots h(fy)h(y)
\]

generate \( T^r \). This implies that \( h(f^{m-1}x) \ldots h(fx)h(x) \) and \( h(f^{n-1}y) \ldots h(fy)h(y) \) are rationally related but for a prevalent and residual set of \( h: X \to T^r \), \( h(f^{m-1}x) \ldots h(fx)h(x) \) will not be rationally related to \( h(f^{n-1}y) \ldots h(fy)h(y) \). Hence for a prevalent and residual set of extensions \( f_h \) will be weak–mixing. A result of Rudolph [61] states that a weak–mixing compact group extension of a Bernoulli dynamical system is Bernoulli and hence for a prevalent and residual set of maps \( h \) the extension \( f_h \) is Bernoulli.

**Semisimple Extensions**

Now suppose that \( \Gamma \) is semisimple. Recall (Theorem 7.3.1) that the set of pairs \((\gamma_1, \gamma_2) \in \Gamma^2 \) which generate \( \Gamma \) is open and dense.

We first claim that there exists a periodic point \( x_0 \) with inverse branch \((x_0, x_1, \ldots) \in \Omega \) together with a point \( y_0 \) with inverse branch \((y_0, y_1, \ldots) \in \Omega \) homoclinic to \((x_0, x_1, \ldots) \). Presumably it is well-known that such points can be found for \( C^2 \) piecewise expanding maps but we were unable to find a reference in the literature. We thus include a proof here. By taking a suitable power of \( f \) we may assume without loss of generality that \( |f'(x)| > 2 \). Now from the work in Section 2.5 we know that for each \( i \) and subinterval \( I \subset (s_i, s_{i+1}) \) there exists \( J_0 \subset I \) such that \( f^k(J_0) \supset (s_j, s_{j+1}) \), for some \( f, k \) and \( f^i(J_0) \cap S \neq \emptyset \) for \( i = 0, \ldots, k \). Hence we may extend \( v \) to be Hölder on \((s_j, s_{j+1}) \).

Furthermore for any periodic point \( x_0 \) and a sufficiently small interval \( I(x_0) \ni x_0 \) there exists an
inverse branch \( \{ I_n(x_0) \} \) of \( I(x_0) \) which contains an iterate of \( x_0 \) at each stage, that is \( f^j(x_0) \in I_n(x_0) \) for some \( j \), and which limits on \( x_0 \) in the sense that \( \lim_{j \to \infty} I(I_n(x_0)) = 0 \).

So we may take a subinterval \( J_0 \subset I(x_0) \subset (s_{j_0}, s_{j_0+1}) \), for some \( j_0, x_0 \not\in J_0 \) and an integer \( k_0 \) such that \( f^{k_0}(x_0) \in (s_{j_0}, s_{j_0+1}) \), for some \( j_0 \). First suppose that we can take \( j_0 = j_1 \). In this case there exists a point \( y_0 \in J_0 \) distinct from \( x_0 \) such that \( f^{k_0}(y_0) = x_0 \) and which has an inverse branch limiting on \( x_0 \), this is our homoclinic point. Suppose now that \( j_0 \neq j_1 \). We may then choose a periodic point \( z_1 \in (s_{j_1}, s_{j_1+1}) \) with corresponding interval \( I(z_1) \) and subinterval \( J_1 \subset I(z_1) \) such that \( f^{k_1}(z_1) \in (s_{j_2}, s_{j_2+1}) \), for some \( k_1, j_2 \). Continuing this process, since there are only a finite number of intervals \( (s_j, s_{j+1}) \) we find a cycle \((j_0 j_1 \ldots j_t \ldots)\) of \( (s_j, s_{j+1}) \) interval indices. By refining the interval \( I(t) \subset J_t \subset (s_t, s_{t+1}) \) we find an interval \( J_t \) such that \( f^{k_t}(J_t) \subset (s_t, s_{t+1}) \) and \( f^t(I(t)) \cap S = \emptyset \) for \( t = 0, \ldots, k_t \).

Thus we may find a periodic point \( x_0 \in (s_i, s_{i+1}) \) for some \( i \) on which \( h \) has a Hölder version and an interval \( I \subset (s_i, s_{i+1}) \) such that \( f^k(I) \supset (s_i, s_{i+1}) \) and an inverse branch of \( I \) limits on \( x_0 \).

We assume without loss of generality that \( x_0 \) is a fixed point (otherwise take an appropriate power of the mapping) and that \( \lim_{n \to \infty} f^n y_0 \) and \( \lim_{n \to \infty} y_n = x_0 \). We may also assume by construction that forward iterates of \( x_0 \) and \( y_0 \) avoid the singular set \( S \) and that \( x_i, y_i \not\in B \) for all \( k \geq 0 \) where

\[
B = \left\{ \lim_{z \in I_i} f(z), \lim_{z \not\in I_i} f(z) \mid i = 1, \ldots, d \right\}.
\]

Now consider the cohomology equation

\[
R(h(z))v(z) = v(fz) \quad (8.3.2)
\]

where \( R \) is an irreducible unitary representation and \( v \) is Hölder on a neighbourhood of \( m \) almost every point. To simplify notation we write \( R(h) \) as \( h \). Iterating (8.3.2) we obtain

\[
v(y_0) = h(f^{n-1}y_n) \cdots h(y_n)v(y_0)
\]

and

\[
h(f^{n-1}y_0) \cdots h(fy_0)v(y_0) = v(f^n y_0)
\]

for each positive integer \( n \). As shown in Section 7.3 the products \( h(f^{n-1}y_n) \cdots h(y_n)g(y_0)^n \) and \( h(f^{n-1}y_n) \cdots h(y_n)g(x)^{-n} \) converge, hence we may define

\[
\gamma^+ := \lim_{n \to \infty} h(f^{n-1}y_n) \cdots h(y_n)g(y_0)^n
\]

and

\[
\gamma^- := \lim_{n \to \infty} h(f^{n-1}y_n) \cdots h(y_n)g(x)^{-n}
\]

(cf. Chapter 7). Further \( \gamma^+ \) and \( \gamma^- \) satisfy

\[
v(x_0) = \gamma^+ v(y_0)
\]

\[
v(y_0) = \gamma^- v(x_0)
\]
and hence
\[ v(x_0) = \gamma^+ v(y_0) = \gamma^+ \gamma^- v(x_0). \]
Further as \( x_0 \) is fixed
\[ v(x_0) = h(x_0)v(x_0). \]
Thus if the pair \((h(x_0), \gamma^+ \gamma^-)\) generates the group \( \Gamma \), then the whole group would fix \( v(x_0) \) providing a contradiction to show that the extension \( f_h \) is ergodic. Arguing as in Section 7.3 we use Theorem 7.3.1 to find a \( C^1 \) open and dense set of extensions \( h \) such that the pair \((h(x_0), \gamma^+ \gamma^-)\) does generate \( \Gamma \). Hence for a \( C^1 \) open and dense set of cocycles \( h, f_h \) is ergodic.

As the only irreducible one-dimensional representations of semisimple groups are trivial ergodicity implies weak-mixing. Thus an open and dense set of extensions are weak mixing and hence (by Rudolph [61]) Bernoulli.

### 8.4 Abelian and Semisimple Extensions of Uniformly Hyperbolic Toral Maps with Singularities

We now consider the case of uniformly toral hyperbolic maps with singularities. The method of proof is virtually identical to that used for extensions of Lasota–Yorke maps. The main work to be done is to establish the existence of periodic and homoclinic points required for the proofs to work. This is what takes the lions share of this section. To show that the required points exist we rely on the work of Krüger and Troubetskoy [35] who have shown that for a class of non-uniformly hyperbolic systems with singularities (which includes the class of toral maps we are interested in) periodic points are dense and there exists a countable Markov partition for the system. In this section we show that the existence of a Markov partition with the properties Krüger and Troubetskoy establish is enough to prove our stable ergodicity results.

Define
\[ M := T^2 \setminus \bigcup_{i \geq 0} f^{-i}R^+ \cup f^iR^- \]
where \( R^- \) and \( R^+ \) are the singularity lines of \( f^{-1} \) and \( f \) respectively (see Chapter 3).

Krüger and Troubetskoy have shown that for almost every \( x \in M \) there exists a unique corresponding symbol sequence from an at most countable alphabet given by a countable Markov partition. Further they show that \( f: M \to M \) is conjugate to a topological Markov chain in the sense that there is a subset \( B \) of full measure in \( M \) so that the conjugating map \( \rho: (B, f) \to (\rho(B), \sigma) \) is such that \( \rho^{-1} \) is continuous. Define \( \overline{B} = \rho(B) \).
Chapter 8. Extensions of Hyperbolic Systems with Singularities

Abelian Extensions

Take a periodic point $\omega \in \Sigma$ (not necessarily in $\tilde{B}$) which has a neighbourhood $U \subset \Sigma$ such that the restriction of $\rho^{-1}$ to $\tilde{B} \cap U$ is continuous. Since $\tilde{B}$ is dense we may define $p = \rho^{-1}(\omega)$ by continuity if $\omega \notin \tilde{B}$. Suppose for the moment that $\omega \notin \tilde{B}$ and so $\rho^{-1}(\omega) \in M$. Suppose, without loss of generality that $\omega$ is a fixed point of $\sigma$. Since Lebesgue almost every point $x \in \rho^{-1}U$ satisfies

$$\rho(f^j(x)) = \sigma^j(\rho(x))$$

for all $j \in \mathbb{Z}$.

By using the continuity of $\rho^{-1}$ and the symbolic dynamics of $(B, \sigma)$ we can find a sequence of points $\omega_j \in \tilde{B}$ such that

- For all $i \in \mathbb{Z}$, $j \geq 0$, $\rho^{-1}\sigma^i\omega_j = f^i\rho^{-1}\omega_j$
- $\lim_{j \to \infty} f\rho^{-1}(\omega_j) = f\rho^{-1}(\omega)$
- $\lim_{j \to \infty} \rho^{-1}(\omega_j) = \rho^{-1}(\omega)$.

Further by the continuity of $\rho^{-1}$ and $f$ it follows that

$$\rho^{-1}(\omega) = \lim_{j \to \infty} \rho^{-1}(\omega_j) = \lim_{j \to \infty} f\rho^{-1}(\omega_j) = f\rho^{-1}(\omega).$$

Thus if $\rho^{-1}(\omega) \in M$ then $p = \rho^{-1}(\omega)$ is a periodic orbit under $f$. Hence if $R$ and $v$ are solutions to 8.2.3 we have

$$R(h(p))v(p) = v(p).$$

(8.4.1)

Suppose that $\rho^{-1}(\omega) \notin M$. In this case we define $p_j := \rho^{-1}(\omega_j)$, $f(p_j) := f\rho^{-1}(\omega_j)$ and note that

$$v(f(p_j)) = R(h(p_j))v(p_j).$$

Taking limits as $j \to \infty$ again yields (8.4.1). Thus we may argue as in Section 8.3 to establish the prevalence and residualness of abelian extensions.

Semisimple Extensions

Now we consider the semisimple case. Let $\tilde{\omega} \in U$ be a point homoclinic to $\omega$ under $\sigma$. Now $\rho^{-1}(\tilde{\omega})$ may not be homoclinic to $p = \rho^{-1}(\omega)$ as it may not be true that $\rho^{-1}\sigma^j\tilde{\omega} = f^j\rho^{-1}\tilde{\omega}$ for all $j$. However since for $l$ almost all $x \in \rho^{-1}U$ we have that

$$\rho f^j(x) = \sigma^j \rho(x)$$

for all $j$

we can find a sequence of points $\{\tilde{\omega}_k\}$ such that if we define $y'_i := f^i\rho^{-1}\tilde{\omega}_k$ we have that

(a) $\rho^{-1}\sigma^j\tilde{\omega}_k = f^j\rho^{-1}\tilde{\omega}_k$.

(b) For each $i$, $\{y'_i\}$ is Cauchy in $t$. 

Chapter 8. Extensions of Hyperbolic Systems with Singularities

(c) \( \lim_{t \to \infty} y^t_{\omega,x} = p \).

Now if 8.2.3 holds then the function \( v \) satisfies, for each \( t \), the equations

\[
v(y^t_0) = h(y^t_{t-1}) \cdots h(y^t_0)v(y^t_0)
\]

and

\[
v(y^t_0) = h(y^t_{t-1}) \cdots h(y^t_{t-t})v(y^t_{t-t}).
\]

We may choose the rate of convergence in (b) such that

\[
\gamma^+ := h(y^t_{t-1}) \cdots h(y^t_0)
\]

\[
\gamma^- := h(y^t_{t-1}) \cdots h(y^t_{t-t})
\]

both exist. So by continuity and (c) \( v(p) = \gamma^+ \gamma^- h(p) \). The same argument as in Section 8.3 shows that a \( C^1 \) open and dense set of semisimple extensions are Bernoulli.

8.5 General Lie Group Extensions

We now combine the results of the previous sections in order to complete the proofs of Theorems 8.1.1 and 8.1.2. To do this we use the same method of Field and Parry [21] as was used in Chapter 7. As the argument is much the same we will only sketch the details. Also we will only consider the case where \( f: X \to X \) is a Lasota–Yorke map, the argument in the case of uniformly hyperbolic maps of the torus is identical.

Let \( \Gamma \) be a compact connected Lie group. Then there exists a covering homeomorphism \( \pi: \widetilde{\Gamma} \to \Gamma \) where \( \widetilde{\Gamma} \) is the direct product of a torus \( \mathbb{T}^r \) and a compact semisimple Lie group \( S \). Define \( \Gamma_A = \pi(\mathbb{T}^r \times I_S) \) and \( \Gamma_S = \pi(I_T \times S) \). Note that \( \Gamma_A \) are normal subgroups of \( \Gamma \) with \( \Gamma_A \cong \mathbb{T}^r \) abelian and \( \Gamma_S \) semisimple. Let \( \pi_A: \Gamma/\Gamma_A \) and \( \pi_S: \Gamma/\Gamma_S \) denote the quotient homomorphisms. If we set \( \tilde{A} = G/S \) and \( \tilde{S} = G/A \) then \( \tilde{A} \cong \mathbb{T}^r \) and \( \tilde{S} \) is semisimple.

Let \( h: X \to \Gamma \) be a \( C^1 \) cocycle. Define \( h_A = \pi_A h: X \to \tilde{S} \) and \( h_S = \pi_S h: X \to \tilde{A} \). Note that \( h_A \) and \( h_S \) are \( C^1 \) cocycles which define \( \tilde{S} \) and \( \tilde{A} \) extensions respectively. For a prevalent set \( P \) of cocycles \( h: X \to \Gamma \) both extensions \( f_{h_A} \) and \( f_{h_S} \) are Bernoulli.

Now let \( \Sigma \) denote the isotropy subgroup of an ergodic component of the extension \( f_h \) and suppose that \( h \in P \). Then \( \pi_A(\Sigma) = \tilde{A} \) and \( \pi_S(\Sigma) = \tilde{S} \) which implies that \( \Sigma = \Gamma \). Hence \( f_h \) is ergodic. By Theorem 8.2.2 \( f_h \) is weak-mixing if and only if for any \( e^{it} \neq 1 \) and any non-trivial, one-dimensional representation \( R \) of \( \Gamma \) the equation

\[
v(fx) = e^{it} R(h(x))v(x)
\]

has no nontrivial solution \( v: X \to \mathbb{C} \) which is Lipschitz on an open neighbourhood of \( m \) almost every point. Since \( S \) is semisimple \( R \) restricted to \( S \) is trivial and hence determines a one-dimensional
irreducible representation $R_A : \bar{A} \to S^1$ by $R_A(\gamma B) = R(\gamma)$, where $\gamma \in A$. Since the extension $f_{h_A}$ is weak-mixing $R_A$ is trivial, and $e^{i\alpha} = 1$. Thus $f_h$ is weak-mixing and by Rudolph's Theorem [61] Bernoulli.
Further Work II

To finish we briefly discuss ways in which the results of the previous chapters may be extended.

Finite Group Extensions of Hyperbolic Basic Sets

For nonabelian finite extensions of hyperbolic basic sets we obtained conditions for ergodicity in terms of periodic data. It would also be of interest to obtain conditions in terms of cohomological equations as in the abelian case.

Also as the proofs in this chapter mainly rely on the cocycle being locally constant it seems simple to extend the results to locally constant cocycles taking values in arbitrary compact groups.

Further for weak-mixing hyperbolic maps we gave conditions on the group under which ergodicity and weak-mixing of extensions were equivalent. It may be possible to find conditions for weak-mixing similar to those for ergodicity, i.e. in terms of periodic data and cohomological equations.

Nonconnected Compact Group Extensions of Hyperbolic Basic Sets

There are a number of ways in which we may improve upon the results of Chapter 7. For instance we recall in the case where the connected component \( \Gamma_0 \) we require that the base space \( \Lambda \) consist of finitely many components and that \( \text{rank}(H^1(\Lambda \times G, \mathbb{Z})) \) be finite. It would be interesting to see whether these conditions could be removed.

It would also be valuable to extend the results to hold for arbitrary compact groups \( \Gamma \), not only ones which can be written as the semidirect product of a finite group and the connected component of the identity.
Further Work II

Compact Group Extensions of Hyperbolic Systems with Singularities

Generalisations of the results in this chapter could potentially follow in tandem with generalisations of the Livšic results in Part I. That is to obtain stable ergodicity results for compact group extensions of the systems discussed in the further work for Part I such as nonuniformly hyperbolic systems and higher dimensional systems with singularities.

We could also try to combine the results of this chapter with those of the previous chapter. In other words we would consider nonconnected compact group extensions of hyperbolic systems with singularities.
References


References


