Diagonal Grids with Members Having Torsional and Shear Rigidity

by

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A thesis submitted for the Degree of Doctor of Philosophy in the Faculty of Engineering

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Abstract

Diagonal grids are one of the commonly used structural forms. Roof and floor systems, bridge and ship decks, elevated highways and raft foundations are examples of structures for which diagonal grids may provide suitable solutions.

Diagonal grids belong to a family of flat grids and their structural behaviour, like that of all flat grids, is mainly governed by bending. The members of a diagonal grid, therefore, essentially work in bending and their flexural rigidity is the main load resisting agent.

Another two types of rigidities affecting the behaviour of flat grids are torsional and shearing rigidities. The nature of the effects of these rigidities, however, have not, in the past, been clearly understood and their importance has been usually underestimated.

This thesis presents an attempt to study the effects of these rigidities on the magnitudes and distribution of internal forces and displacements in diagonal grids. The material of the thesis is arranged in the following manner:

The preliminary definitions and relations and the background to the subject are given in chapter one.

Chapter two contains a description of the work involved in the analysis and the results of the analysis of a large number of diagonal grids. The conclusions obtained from these analytical results are given in chapter five.

An account of the experimental investigations is given in chapter four.
In chapter three, the concept of vector and matrix norms is employed to develop an original technique which is used to estimate the changes in the internal forces and displacements of flat grids due to variations in the torsional or shearing rigidities. Furthermore, it is shown that the scope of this technique is not confined to the matter under consideration and the idea may be applied to many other problems of structural analysis. In particular, subject to the conditions described in the text, the technique will provide a new and powerful means for structural optimization processes and it may even find uses in disciplines other than structural analysis.
To the memories of my father, mother and sister
Acknowledgements

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I am also greatly indebted to all my colleagues and friends in the civil engineering department of the University of Surrey and elsewhere and to the staff of the computing unit of the University of Surrey for their invaluable help.

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CHAPTER ONE

PRELIMINARIES
Section 1-1. Introduction

A diagonal grid may be defined as a structure consisting of two sets of parallel beams intersecting at (usually) right angle and making oblique angles (usually 45°) with the boundary lines (or boundary beams). The sketches in fig. 1-1-1 show the plan views of a number of possible configurations for a diagonal grid.

A diagonal grid under a general system of external loads is a rigidly jointed structure with all the six possible components of internal force* and all the six possible components of displacement present throughout the structure. However, "the analysis of grids" is normally understood to mean "the analysis of grids under loads perpendicular to the plane of the grid and/or external moments the axes of which lie in the plane of the grid". Under such a system of external loads, provided that the small deflection theory holds, the only internal forces present in the members are bending moments, torques and shearing forces and the number of degrees of freedom at each joint is limited to three (one translation perpendicular to the plane of the grid and two components of rotation the axes of which lie in the plane of the grid).

The above mentioned type of external loading and the validity of the small deflection theory will be presumed throughout the thesis.

The analysis of diagonal grids in the pre-computer days was considered to be highly complicated, the main difficulty being due to the necessity of dealing with large systems of simultaneous equations. The exact number of equations involved in the analysis

*Throughout the thesis, the word "force" is used to mean a force or a moment and the word "displacement" is used to mean a translation or a rotation.
depends on the grid configuration, boundary conditions and the technique of analysis. However, for most diagonal grids with a reasonably dense layout, the number of equations for both flexibility and stiffness methods of analysis may be taken to be about three times the number of joints and hence the analysis may easily involve the solution of hundreds or even thousands of simultaneous equations. It is not surprising, therefore, that the hand analysis of diagonal grids was regarded to be extremely difficult.

An interesting feature in the analysis of diagonal grids is that if the members have no torsional rigidity then the number of unknowns in certain versions of both flexibility and stiffness methods of analysis may be reduced to about a third. In such cases, the difficulties in the hand analysis will be greatly reduced and, therefore, in the pre-computer days it was common practice to ignore the torsional rigidities of the members regardless of the shapes of the cross-sections. Another simplifying assumption employed in the analysis was that the members have infinitely large shearing rigidities. A number of techniques of analysis incorporating the above two assumptions are described in [2], [18], [20], [21] and [22].

It must be mentioned that the hand analysis of diagonal grids, even when the members were assumed to have no torsional and infinite shearing rigidities, remained a difficult task and very few structural analysts had ever tried it. In fact, the design of diagonal grids was usually based on some sort of analogy (see [7], [26] and [30]).

*Numbers in square brackets refer to the numbers in the list of references given at the end of the thesis.
Nowadays, the situation is quite different. The laborious task of lengthy numerical computations may be done by an electronic digital computer and the structural analysts are able to tackle problems which are far beyond the scope of hand computation. Furthermore, many techniques of structural analysis have been reformulated to become suitable for automatic computation.

When analysing a grid by a computer, the most efficient technique to be used is the "standard stiffness method". In this case, there will be little point in assuming that the members have zero torsional and infinite shearing rigidities because these assumptions will hardly have any effect on the required programming efforts and the computation time and therefore one may as well consider the actual values of torsional and shearing rigidities.

An important question in the pre-computer days was:

If a diagonal grid is analysed with the assumptions that the members have zero torsional and infinite shearing rigidities, how accurate are the calculated values of internal forces and displacements?

The answer to this question may still be of some interest but a more relevant question to be asked today is:

How do the changes in the torsional and shearing properties of the members affect the internal forces and displacements and hence, for a given diagonal grid, what are the best ratios between different rigidities?

---

*The phrase "standard stiffness method" is used to refer to the modern formulation of the "stiffness methods" as, for instance, described in [17].
This thesis presents an attempt to find some answers to the above questions.

It is to be noted that the discussion of this section is more or less applicable to any type of flat grid and this will also be found to be true for many other parts of the thesis.

Section 1-2. Past Research

The study of the effects of torsional and shearing properties on the stress distribution and displacements of flat grids is a problem which in the past has received little attention. The reason may be explained as follows:

A thorough study of the problem involves the analysis of a large number of cases and this requires a tremendous amount of numerical computation. It is understandable, therefore, that in the pre-computer days the problem under consideration was not a popular research subject. Even with a computer, the problem remains to be comparatively difficult and unrewarding. Consequently, during the short period of availability of electronic computers the problem has not yet received much attention.

There are, nevertheless, a few investigators who have challenged the problem and, either bare handed or computer aided, have tried to shed some light on the matter (see [2],[11],[19],[24] and [28]). The results of these investigations contain some valuable information but they are usually based on comparison of the results of no more than two or three solved cases and hence are limited in scope.

Section 1-3. Basic Definitions and Relations

The standard stiffness method of structural analysis is used either as a means for analysis or as the background for discussion
in many parts of the thesis. In this section, a brief description of the standard stiffness method is used as a medium to introduce the basic relations, nomenclature and notation used in the thesis. This brief description of the standard stiffness method is not intended to be a comprehensive introduction to the subject, since such introductions are included in many textbooks (for instance, see [17]). The symbols are defined as they first appear and, for ease of reference, a list of all the symbols with their descriptions is given at the end of the thesis. A knowledge of the fundamentals of matrix algebra is presumed.

Consider a linear skeletal structure under a general system of loading. Let the terminals of a typical member be at joints i and j (j>i). To each member of the structure we allocate a Cartesian coordinate system which is referred to as a "member coordinate system" (o-x-y-z in fig. 1-3-1). For the structure as a whole we define a single Cartesian coordinate system which is referred to as the "frame coordinate system" (o'-x'-y'-z' in fig. 1-3-1).

The actual external loads are replaced by a system of "equivalent external loads" which are applied only at joints and produce joint displacements identical to those produced by the actual loading system. The structure is analysed for the equivalent external loading system but the results of the analysis may then be easily modified to correspond to the actual loading system.

The components of the internal force at the ends i and j of member b, relative to the coordinate system of the member, form the elements of column vectors \( p_{ij} \) and \( p_{ji} \) which are referred to as "member end force vectors". The components of displacement at the ends i and j of member b, relative to the coordinate system of the member, form the elements of column vectors \( d_{ij} \) and \( d_{ji} \) which are referred to as "member end displacement vectors".
FIG. 1-3-1

Contribution of a typical member \( b \) to the stiffness matrix of the structure.

\[
\begin{bmatrix}
(K_{11})_b & (K_{12})_b \\
(K_{21})_b & (K_{22})_b
\end{bmatrix}
\]

FIG. 1-3-2
The force-displacement relations for member $b$ are given by:

\[ p_{ij} = (K_{11})_b d_{ij} + (K_{12})_b d_{ji} \quad \ldots \ldots \ldots \quad 1-3-1a \]

\[ p_{ji} = (K_{21})_b d_{ij} + (K_{22})_b d_{ji} \quad \ldots \ldots \ldots \quad 1-3-1b \]

where \((K_{11})_b, (K_{12})_b, (K_{21})_b\), and \((K_{22})_b\) are the "stiffness matrices" of the member.

It can be shown that (see [17]),

\[ (K_{11})_b = H_b K_b H_b^T \quad \ldots \ldots \ldots \quad 1-3-2a \]

\[ (K_{12})_b = H_b K_b \quad \ldots \ldots \ldots \quad 1-3-2b \]

\[ (K_{21})_b = -K_b H_b^T \quad \ldots \ldots \ldots \quad 1-3-2c \]

\[ (K_{22})_b = K_b \quad \ldots \ldots \ldots \quad 1-3-2d \]

where $H_b$ is the "equilibrium matrix" of the member and $K_b$ which is equal to $(K_{22})_b$ is referred to as the "basic stiffness matrix" of the member.

Relations 1-3-1, relative to the frame coordinate system, may be written in the following form:

\[ p_{ij} = (K'_{11})_b d_{ij} + (K'_{12})_b d_{ji} \quad \ldots \ldots \ldots \quad 1-3-3a \]

\[ p_{ji} = (K'_{21})_b d_{ij} + (K'_{22})_b d_{ji} \quad \ldots \ldots \ldots \quad 1-3-3b \]

where

\[ p_{ij} = T_b p_{ij}, \quad d_{ij} = T_b d_{ij}, \ldots \text{etc.} \quad \ldots \ldots \quad 1-3-4a \]

\[ (K'_{11})_b = T_b (K_{11})_b T_b^T, \quad (K'_{12})_b = T_b (K_{12})_b T_b^T, \ldots \text{etc.} \quad 1-3-4b \]

$T_b$ is the "transformation matrix" by which vectors relative to the coordinate system of member $b$ are transformed to the frame coordinate system.

The conditions of compatibility, in general, require that all the member ends connected to a joint have the same components of
displacement. Relations 1-3-3, considering the conditions of compatibility, will become:

\[ p_{1j} = (K'_{11})_b d_i + (K'_{12})_b d_j \] .................................. 1-3-3a

\[ p_{1i} = (K'_{21})_b d_i + (K'_{22})_b d_j \] .................................. 1-3-3b

where \( d_i \) and \( d_j \) are column vectors the elements of which are the components of displacement of joints \( i \) and \( j \) relative to the frame coordinate system. \( d_i \) and \( d_j \) are referred to as "joint displacement vectors".

The components of equivalent external loads at a joint \( i \), relative to the frame coordinate system, form the elements of a column vector \( w_i \), which is referred to as a "joint external load vector".

The conditions of equilibrium at a joint \( i \) require that

\[ \sum p_{ij} = w_i \] .................................. 1-3-6

where the summation extends over all the member ends connected to the joint.

Relations similar to 1-3-6 may be written for all the joints of the structure giving rise to a system of simultaneous equations. This system, after being modified to take account of the constrained degrees of freedom (see \([17]\) and \([23]\)), may be represented by the following matrix relation:

\[ Kd = w \] .................................. 1-5-7

where \( d \) is the "displacement vector of the structure" containing all the joint displacement vectors, \( w \) is the "external load vector of the structure" containing all the joint external load vectors and \( K \) is the "stiffness matrix of the structure" which is composed of the stiffness matrices of all the members. The contribution of a typical member \( b \) to the stiffness matrix of the structure is
The system of equations represented by relation 1-1-7 may be solved to find the joint displacements. Knowing the joint displacements, the internal forces at the member ends can be obtained from relations similar to 1-1-5.

The above brief description of the standard stiffness method is in the most general form and applicable to the analysis of any linear skeletal structure. The dimensions of the vectors and matrices and the elements of these vectors and matrices, however, are different for different structures.

In the case of flat grids, the vectors associated with members are of dimension three and the matrices associated with members are of order three by three. For a typical member in a flat grid, the choice of member coordinate system and the notation for different components of force and displacement, as used in this thesis, are shown in fig. 1-3-3.

An important step in the analysis of any skeletal structure is the derivation of member stiffness matrices and techniques for obtaining these matrices are described in different textbooks.

In the case of flat grids, for a straight uniform member, the basic stiffness matrix is as shown in fig. 1-3-3 (see [9]). It is to be noted, however, that in the derivation of this basic stiffness matrix, the effects of restraint of torsional warping are not taken into account (this matter will be discussed in section 7-1). The other three member stiffness matrices may be obtained from the basic stiffness matrix and the equilibrium matrix of the member (see relations 7-2-3; the equilibrium matrix for a member of a flat grid is given in fig. 1-3-3. The symbols appearing in fig. 1-3-3 which have not been already defined are described in table 1-3-1.
\[ K_b = \begin{bmatrix}
    \frac{GJ}{L} = (t_0 \frac{EI}{L}) & 0 & 0 \\
    0 & \left(\frac{1 + 0.5k_o}{1 + 2k_o}\right) \frac{4EI}{L} & \left(\frac{1}{1 + 2k_o}\right) \frac{6EI}{L^2} \\
    0 & \left(\frac{1}{1 + 2k_o}\right) \frac{5EI}{L^2} & \left(\frac{1}{1 + 2k_o}\right) \frac{12EI}{L^3}
\end{bmatrix} \]

\[ H_b = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & -L \\
    0 & 0 & 1
\end{bmatrix} \]

FIG. 1-3-3
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>q &amp; ( \dot{q} )</td>
<td>Torques at ends 1 and 2</td>
</tr>
<tr>
<td>v ( \dot{v} )</td>
<td>Shearing forces at ends 1 and 2</td>
</tr>
<tr>
<td>x ( \dot{x} )</td>
<td>Twists at ends 1 and 2</td>
</tr>
<tr>
<td>\theta</td>
<td>Rotations at ends 1 and 2</td>
</tr>
<tr>
<td>\phi</td>
<td>Reflections at ends 1 and 2</td>
</tr>
<tr>
<td>E</td>
<td>Modulus of elasticity of the material in tension and compression</td>
</tr>
<tr>
<td>G</td>
<td>Modulus of elasticity of the material in shear</td>
</tr>
<tr>
<td>L</td>
<td>Length of the member</td>
</tr>
<tr>
<td>A</td>
<td>Cross-sectional area of the member</td>
</tr>
<tr>
<td>J</td>
<td>Second moment of area of the cross-section about member coordinate axis ( z )</td>
</tr>
<tr>
<td>J</td>
<td>Torsion constant of the cross-section</td>
</tr>
<tr>
<td>s &amp; ( \dot{s} )</td>
<td>Shear constant of the cross-section</td>
</tr>
<tr>
<td>c &amp; ( \dot{c} )</td>
<td>Torsion coefficient of the cross-section</td>
</tr>
<tr>
<td>k &amp; ( \dot{k} )</td>
<td>Shear coefficient of the cross-section</td>
</tr>
</tbody>
</table>
CHAPTER TWO

ANALYTICAL WORK
SECTION 3.1. Introduction

This chapter contains the results of the analysis of a large number of diagonal grids. The analysis was carried out by a digital computer and a programme was developed for this purpose. The reasons for analysing the grids were:

a. To study the effects of torsional and shear variations on the behaviour of diagonal grids;

b. To provide a basis for checking the predictability of the relations developed in the next chapter.

The results of analysis can also be used as a collection of solved cases for design purposes or can serve as a basis for some future research projects.

The grids were chosen to have commonly used dimensions and the torsional and shear properties of the members being within the critical range.

SECTION 3.2. The Programme

This section contains a description of the computer programme by which all the grids considered in this chapter were analysed.

The main structure of the programme is similar to the work of the Numerical Methods Laboratory at the University of California at Berkeley, except for a slight deviation of the IBM 704 onto an IBM 7090. Therefore, the programme can be divided into two main parts. The first is written in "IBM FORTRAN", while the second is written in "IBM ABOOK". In order to meet the requirements of the paper, a rather small but also efficient storage capacity of 14,000 words was used, for an extended version of the programme some 10,000 words were added. The programme is designed to analyse all common types of flat
The member stiffness matrices are given in the form of Eq. (1) as described in section 1.9 and include the term corresponding to torsional and shearing resistances.

The technique by which the stiffness matrix is developed to include the member stiffness matrices and to form the matrix corresponding to the contributions of the columns into the stiffness matrix of the structure is described. Therefore, when the stiffness matrix of the structure is obtained, necessary modifications for the constraints in the structure must be made. The components of the external load are treated in the usual manner and the load vector of the structure is obtained in its reduced form. The reason for adopting the above technique, rather than modifying the stiffness matrix of the structure after it has been formed, is to economize in the storage space.

Two types of external loading can be considered:

a. Vertical point loads at joints,
b. Uniformly distributed load covering the whole or parts of the surface.

In the case of uniformly distributed load, it is assumed that the share of each member is the load on the area formed by removing the length of the member from the centroid to the centroid of the area of the member. The load is then assumed to be uniformly distributed over the length of the member.
Fig. 2-26 illustrates the plan of a part of a grid, wherein uniformly distributed load and the assumed shape of beams for some of the members are indicated by the shaded areas.

If a grid has a symmetrical layout and if the members are also symmetric, the following technique can be used to simplify the analysis.

For each plane of symmetry, the grid is cut along the plane and the notation of each cut joint against the plane is constructed (in the case when the plane of symmetry passes through the middle of a member, a fictitious joint is defined there). Then, the concentrated load (if any) applied to each cut joint is taken and the procedure is repeated for all the planes of symmetry.

Knowledge of the forces and displacements may then be obtained by analysing a portion containing \( \frac{1}{n} \) of the grid, where \( n \) is the number of the planes of symmetry (see fig. 2-26). This technique is of general applicability (see \( \frac{27}{27} \)), but for the purpose under consideration it can only be used when the cross-sections are parallel to the frame coordinate axes.

The data required for the programme consists of the following:

a) Topological properties of the structure, i.e., the order in which the members are combined to constitute the structure,

b) Geometrical properties of the members, i.e., cross-sectional properties, location and orientations of the members,

c) Kinematic characteristics of the joints, i.e., degrees of freedom and constraints at the joints,

d) Properties of the material, of which the structure is built,

e) External loads.

A brief description of the contents of the programme is shown in a flow chart in Fig. 2-26 and a copy of the programme is given in the appendix at the end of the thesis.
plane of symmetry

PORTION TO BE ANALYSED

frame coordinate system

constrained components of rotation

FIG. 2-2-1

FIG. 2-2-2
Read information about topological properties of the network, geometrical properties of the members, kinematical data concerning the joints, the external loads and properties of the material, clear space for the stiffness matrix and the load vector, allocate no space for the rows and the columns corresponding to constraints.

Form the member stiffness matrixes relevant to the member ends connected to the ith joint.

Model the member stiffness matrixes for the constraints at plant then in the stiffness matrix, form the elements of the load vector corresponding to the ith joint.

Is it larger than the total No. of joints? Yes

Solve for the joint displacements. Enlarge the displacement vector by inserting zeros for the constraints.

Print the displacements of the ith joint.

Do the member stiffness matrices and the joint displacements to find the forces at the ends of the members connected to the ith joint and print them.

Print the displacements of the ith joint.

Is it larger than the total No. of joints? Yes

Stop
To demonstrate the application of the programme the following example is considered.

**Example.** The grid shown in Fig. 2-2-4a is to be analysed for a uniformly distributed load covering the area indicated in Fig. 2-2-4b. All the members of the structure are of the same cross-section having a bending rigidity equal to EI. The torsion coefficient of the members is \( t = \frac{1}{4} \) and the shear coefficient for diagonal members is \( k = \frac{1}{2} \).

The programme was used to analyse the grid and the results are shown in Table 2-2-1. The signs of the entries in this table are according to the following conventions:

a) Hogging bending moment is positive.

b) Torque is positive if it causes a right handed screw to move towards the other end of the member.

c) An element of a beam under positive shear force is shown in the following sketch.

\[ \text{Diagram of beam element} \]

d) Downward deflection is positive.

---

**Section 2-4. Torsional and Shearing Properties of Commonly Used Types of Members**

This section is concerned with estimating the values of torsion and shear coefficients of commonly used types of members in flat grids. It is recalled that:

- Torsion coefficient \( t = \frac{1}{4} \)
- Shear coefficient \( k = \frac{1}{2} \)
FIG. 2-2-4a  PLAN

U.D.L. of intensity $W_{\text{unit area}}$ over the dotted part.

FIG. 2-2-4b
<table>
<thead>
<tr>
<th>MEMBER</th>
<th>BENDING MUMENT WL&lt;sup&gt;3&lt;/sup&gt;</th>
<th>TORSION WL&lt;sup&gt;3&lt;/sup&gt;</th>
<th>SHEAR WL&lt;sup&gt;2&lt;/sup&gt;</th>
<th>DEFLECT. WL&lt;sup&gt;5&lt;/sup&gt;/EI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.512</td>
<td>-4.14</td>
<td>1.33</td>
<td>-1.314</td>
</tr>
<tr>
<td>2</td>
<td>7.901</td>
<td>-2.69</td>
<td>9.37</td>
<td>-1.311</td>
</tr>
<tr>
<td>3</td>
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<td>2.21</td>
<td>2.21</td>
<td>-1.15</td>
</tr>
<tr>
<td>4</td>
<td>-1.567</td>
<td>5.33</td>
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where the notation is defined in Table 1–1. These particular
properties depend on the material used, i.e., $k$ and $E$.

Theoretically, the values of $k$ and $E$ are at infinity.
In practice, however, they are certain to be reduced
proportions and power laws usually apply to these values.
It is therefore proposed to cut down some of these practical
restrictions on the depth of which the values of the properties
could be estimated.

The following assumptions are believed to hold in most
practical cases:

1. The members are steel or aluminium I-sections or H-sections
   with rectangular cross-section.
2. The ratio $K/E$ is larger than 1, where $E$ is the depth of the
   cross-section and $K$ is the length of the member.
3. The ratio $K/E$ for a rectangular cross-section is larger
   than unity, where $E$ and $K$ are the depth and the width of the cross-
   section respectively.
4. In the case of R.C. beams, the contributions to the values
   of $k$ and $E$ are mainly due to concrete and the effects of reinforce-
   ments can be ignored.

Depending on the validity of the above assumptions, the limiting
values of $k$ and $E$ can now be closely estimated.

The ratio $E/G$

This ratio is given by:

$$E/G = 3(1 + \nu)$$

where $\nu$ is the Poisson's ratio. The values of $f_r$ and $E/G$ for
mild steel, aluminium and concrete are given in Table 1–1.
TABLE 2.4.1

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$k_3$ for Rectangular Cross-sections

For a solid rectangular cross-section, $k_3$ is defined and, assuming that the shear warping is restrained, $e_{10}$ and therefore, $k_3$ is given by

$$k_3 = \frac{e_{10}}{e_{15}} = 1.1$$

Introducing the restriction $e_{10}/e_{15} > 1$,

$$k_3 = \frac{e_{10}}{e_{15}} < 1$$

where the ratio $e_{10}/e_{15}$ is considered to be that for concrete. It may be concluded, therefore, that $k_3$ for an I-section beam with rectangular cross-section is unlikely to be ever more than 1.1.

$k_4$ for I-Sections

For an I-section, the value of $k_4$ is approximately given by $k_4 A_1$, where $A_1$ is the area of the web. The values of $k_4$ for the ratios $A_1$ equal to 1, 2 and 3 for all commonly used I-sections are given in Table 2.4.2. This table was obtained by a computer programme designed to evaluate $k_4$ for any I-section.

The abbreviations used in this table are:

B.S. for British Standard Beam
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<td>U1 18.0 X 9.0 3.3 1.3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>U1 18.0 X 9.0 5.0 1.3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>U1 18.0 X 9.0 7.0 1.3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>U1 18.0 X 9.0 10 1.3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>
### TABLE 2.3-2 cont.

<table>
<thead>
<tr>
<th>I-SECTION</th>
<th>( k_0 )</th>
<th>( L_{10}=3 )</th>
<th>( L_{10}=7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BB 10.001230.00</td>
<td>1.126</td>
<td>4.704</td>
<td>6.42</td>
</tr>
<tr>
<td>BB 10.001210.00</td>
<td>1.092</td>
<td>4.610</td>
<td>6.23</td>
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<tr>
<td>BB 10.001290.00</td>
<td>1.217</td>
<td>4.707</td>
<td>6.65</td>
</tr>
<tr>
<td>BB 10.001274.00</td>
<td>1.473</td>
<td>4.415</td>
<td>6.28</td>
</tr>
<tr>
<td>BB 10.001251.00</td>
<td>1.141</td>
<td>4.603</td>
<td>6.01</td>
</tr>
<tr>
<td>BB 10.001191.00</td>
<td>1.043</td>
<td>4.400</td>
<td>5.90</td>
</tr>
<tr>
<td>BB 10.001164.00</td>
<td>1.190</td>
<td>4.536</td>
<td>6.11</td>
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<td>BB 10.001133.00</td>
<td>1.252</td>
<td>4.631</td>
<td>6.14</td>
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<td>1.318</td>
<td>4.680</td>
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<tr>
<td>BB 10.001064.00</td>
<td>1.336</td>
<td>4.696</td>
<td>6.20</td>
</tr>
<tr>
<td>BB 10.001048.00</td>
<td>1.345</td>
<td>4.704</td>
<td>6.21</td>
</tr>
<tr>
<td>BB 10.001030.00</td>
<td>1.355</td>
<td>4.710</td>
<td>6.21</td>
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<tr>
<td>BB 10.001010.00</td>
<td>1.360</td>
<td>4.710</td>
<td>6.21</td>
</tr>
</tbody>
</table>

- The table lists various I-sections along with their corresponding values for \( k_0 \), \( L_{10}=3 \), and \( L_{10}=7 \).
for Universal Beam

W3 for I-shaped Beam

MANS for British Standard I-beam

It is seen from the table that the lowest value of \( t_0 \) is equal to 1.04 which corresponds to 12\% of \( h \) for \( t_0 \) for Rectangular Beam-Sections

For a solid rectangular cross-section, when \( t \) is the approximate value of \( \frac{h}{2} \) is given by:

\[
J = \frac{h^2}{12} \left( \frac{1}{4} + \frac{1}{12} \right) - \frac{h^3}{64}
\]

where the error involved is believed to be never more than \( \frac{1}{2} \% \) see [250], we have, therefore,

\[
J/l = \frac{\left( \frac{h^2}{12} \left( \frac{1}{4} + \frac{1}{12} \right) - \frac{h^3}{64} \right)}{bh^2}
\]

or,

\[
J/l = \frac{1}{8} \left( \frac{h}{b} - \frac{h}{2} \right) \left( 1 + \frac{h}{2} \right)
\]

For a number of practical values of \( h/b \), the ratios \( J/l \) are evaluated and the results are shown in table 1-1-1. The entries in the last column of this table are the corresponding values of \( t \), for which the material is assumed to be concrete. Furthermore, a curve showing the variation of \( t \) with respect to \( h/b \) is given in fig. 1-1-1.

From the values given in table 1-1-1, it may be concluded that in most practical cases for \( h/b \) beams with rectangular cross-section \( t \) will not exceed 0.2 \( h \).

\( t_0 \) for I-Sections

The values of torsion constant \( J \) for commonly used sections are given in the second column of table 1-1-1. These are obtained
\[ J = \left\{ \frac{2}{3}B t_f^3 + \frac{1}{3}(H-2t_f)t_w^3 + 2(0.094 + 0.07\frac{r}{t_f}) \left[ \frac{t_f^2 + t_w^2}{2} + r(2t_f + t_w + r) \right] \right\}^4 - 0.42t_f^4 \]
<table>
<thead>
<tr>
<th>I-SECTION</th>
<th>J (IN^4)</th>
<th>GJ/EI (LB-3)</th>
<th>GJ'/EI (LB-3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000 x 7.5x9.00</td>
<td>8.020</td>
<td>1.70</td>
<td>-</td>
</tr>
<tr>
<td>6.000 x 7.069.00</td>
<td>4.820</td>
<td>1.40</td>
<td>-</td>
</tr>
<tr>
<td>6.000 x 7.5x9.00</td>
<td>6.010</td>
<td>1.20</td>
<td>-</td>
</tr>
<tr>
<td>4.000 x 7.069.00</td>
<td>4.010</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>4.000 x 7.5x9.00</td>
<td>3.900</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>2.000 x 7.069.00</td>
<td>2.000</td>
<td>1.00</td>
<td>-</td>
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<tr>
<td>2.000 x 7.5x9.00</td>
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<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>1.500 x 7.069.00</td>
<td>1.500</td>
<td>1.00</td>
<td>-</td>
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<tr>
<td>1.500 x 7.5x9.00</td>
<td>1.500</td>
<td>1.00</td>
<td>-</td>
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<tr>
<td>1.000 x 7.069.00</td>
<td>1.000</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>1.000 x 7.5x9.00</td>
<td>1.000</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>0.500 x 7.069.00</td>
<td>0.500</td>
<td>1.00</td>
<td>-</td>
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<tr>
<td>0.500 x 7.5x9.00</td>
<td>0.500</td>
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<tr>
<td>0.000 x 7.069.00</td>
<td>0.000</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>0.000 x 7.5x9.00</td>
<td>0.000</td>
<td>1.00</td>
<td>-</td>
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<tr>
<td>8.000 x 7.5x9.00</td>
<td>8.020</td>
<td>1.70</td>
<td>-</td>
</tr>
<tr>
<td>6.000 x 7.069.00</td>
<td>4.820</td>
<td>1.40</td>
<td>-</td>
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<tr>
<td>6.000 x 7.5x9.00</td>
<td>6.010</td>
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<tr>
<td>4.000 x 7.069.00</td>
<td>4.010</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>4.000 x 7.5x9.00</td>
<td>3.900</td>
<td>1.00</td>
<td>-</td>
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<tr>
<td>2.000 x 7.069.00</td>
<td>2.000</td>
<td>1.00</td>
<td>-</td>
</tr>
<tr>
<td>2.000 x 7.5x9.00</td>
<td>1.900</td>
<td>1.00</td>
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<tr>
<td>1.500 x 7.069.00</td>
<td>1.500</td>
<td>1.00</td>
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<tr>
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<td>0.500 x 7.5x9.00</td>
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<td>-</td>
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</tr>
<tr>
<td>0.000 x 7.5x9.00</td>
<td>0.000</td>
<td>1.00</td>
<td>-</td>
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</tbody>
</table>

**TABLE 2.3.4**
### TABLE 2.3.4 cont.

<table>
<thead>
<tr>
<th>I-SECTION</th>
<th>J (IN.²)</th>
<th>GJ/E1</th>
<th>GJ/E1 (URR-3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>US13.0211</td>
<td>1</td>
<td>0.144</td>
<td>1.101</td>
</tr>
<tr>
<td>US12.0211</td>
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<td>0.137</td>
<td>1.097</td>
</tr>
<tr>
<td>US10.0211</td>
<td>0</td>
<td>0.131</td>
<td>1.094</td>
</tr>
<tr>
<td>US9.0211</td>
<td>0</td>
<td>0.125</td>
<td>1.091</td>
</tr>
<tr>
<td>US8.0211</td>
<td>0</td>
<td>0.120</td>
<td>1.088</td>
</tr>
<tr>
<td>US7.0211</td>
<td>0</td>
<td>0.115</td>
<td>1.085</td>
</tr>
<tr>
<td>US6.0211</td>
<td>0</td>
<td>0.110</td>
<td>1.082</td>
</tr>
<tr>
<td>US5.0211</td>
<td>0</td>
<td>0.105</td>
<td>1.079</td>
</tr>
<tr>
<td>US4.0211</td>
<td>0</td>
<td>0.100</td>
<td>1.076</td>
</tr>
<tr>
<td>US3.0211</td>
<td>0</td>
<td>0.095</td>
<td>1.073</td>
</tr>
<tr>
<td>US2.0211</td>
<td>0</td>
<td>0.090</td>
<td>1.070</td>
</tr>
<tr>
<td>US1.0211</td>
<td>0</td>
<td>0.085</td>
<td>1.067</td>
</tr>
<tr>
<td>US0.0211</td>
<td>0</td>
<td>0.080</td>
<td>1.064</td>
</tr>
</tbody>
</table>

Note: The table continues with similar entries for other sections.
<table>
<thead>
<tr>
<th>I-SECTION</th>
<th>J  (IN.(^4))</th>
<th>GJ/(E_1)</th>
<th>GJ/(E_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H B 1 0.00 X 12 X 110.0</td>
<td>9.710 0</td>
<td>2.044 0</td>
<td>4.020 2</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 100.0</td>
<td>7.705 0</td>
<td>2.044 0</td>
<td>4.020 2</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 90.0</td>
<td>5.510 0</td>
<td>2.044 0</td>
<td>4.020 2</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 80.0</td>
<td>4.437 0</td>
<td>1.822 0</td>
<td>3.644 2</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 70.0</td>
<td>3.341 0</td>
<td>1.624 0</td>
<td>3.248 2</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 60.0</td>
<td>2.888 0</td>
<td>1.532 0</td>
<td>3.064 2</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 50.0</td>
<td>2.002 0</td>
<td>1.093 0</td>
<td>2.186 1</td>
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<tr>
<td>H B 1 4.00 X 12 X 40.0</td>
<td>1.397 0</td>
<td>1.081 0</td>
<td>2.162 1</td>
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<tr>
<td>H B 1 4.00 X 12 X 30.0</td>
<td>1.074 0</td>
<td>1.061 0</td>
<td>2.122 1</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 20.0</td>
<td>0.845 0</td>
<td>1.048 0</td>
<td>2.084 1</td>
</tr>
<tr>
<td>H B 1 4.00 X 12 X 10.0</td>
<td>0.710 0</td>
<td>1.040 0</td>
<td>2.040 1</td>
</tr>
</tbody>
</table>
from the following sources:

a) Torsion constants for British Standard Beams have been obtained by Cassie and Dobie and the results are given in reference [6].

b) Torsion constants of all British Standard Aluminium Beams have been found by Cullimore and Fugaley and are given in reference [3].

c) Universal Beams are similar to American I-sections (Bethlehem wide Flange Sections) for which the values of J can be found in different handbooks (for instance, see [1]).

d) For Broad Flange Beams no information regarding the torsional properties could be found and the values of J were evaluated. The method used was that suggested by Lyse and Johnston (see [4]), which is based on an empirical formula given in fig. 2-3-2. This formula was used in a computer programme by which torsion constants of all I.F. Beams were calculated. The formula is believed to give reliable results provided that the flanges are parallel sided and the ratio of web to flange thickness is not greater than 0.6. For I.F. Beams, the first condition is always true and the second is almost always satisfied. It is, therefore, hoped that the values of J for I.F. Beams given in table 2-3-4 are reliable.

The third column of table 2-3-4 contains the ratios of GJ/EI, where the values of I are obtained from standard tables and E/2 for steel and aluminium are taken from table 2-4-1.

On examining the values of GJ/EI, it becomes clear that, in general, the torsion coefficients of I-sections are much smaller than those of solid rectangular sections. In fact "5/3 x 3/6" has the maximum value of GJ/EI equal to 0.1, which is 50 times smaller than that for a square solid cross-section.

Effects of Restraint of Torsional Waving

The evaluation of torsion constants for both rectangular and
I-sections in the above discussion was based on the assumption
that the end cross-sections can warp freely. Thus, however, is
hardly ever the case. The members are connected together at joints
and naturally the end cross-sections cannot warp freely. The warping
is not, of course, completely constrained out the degree of
restraint cannot be easily estimated.

For a solid rectangular section the change in the torsional
rigidity due to warping restraint is not appreciable and may be
ignored altogether (see [22]). The effects of warping restraint
on I-sections, however, may be important and cannot be disregarded.
For an I-section in the most critical case (i.e., when both ends of
the member are completely restrained from warping) the angle of
twist, from zero to \( L \), is approximately given by:

\[
\psi = \frac{Tl}{GJ} (1 - a \tanh a^{-1})
\]

where

\[
a = h \left( \frac{2KI}{GJ} \right)^{1/2}
\]

\( T \) is the torque, \( h \) is the distance
between the centre lines of the flanges, \( I_f \) is the second moment of
area of one flange about the vertical centre line of the cross-
section and the rest of the terms have their usual meaning (see [11]).

Comparing the above relation with the corresponding one for
pure torsion (i.e., \( \psi = TL/GJ \)), it appears that as though the torsion
constant of the section is

\[
J_1 = J(1 - a \tanh a^{-1})^{-1}
\]

That is, if the torsion constant is taken to be \( J_1 \), rather than \( J \),
the effects of warping restraint will automatically be taken into
account.

The values of \( GJ_1/\bar{J} \) corresponding to \( L/H = 1 \) for all commonly
used I-sections were evaluated by a computer programme and are given
in the last column of Table 3. It can be seen that the torsional
rigidity of an I-section, when the warping of the end cross-
sections are restrained, is much greater than that when the warping is free to take place. Nevertheless, even when the warping is considered to be completely constrained and \( k \) has the unusually low value of \( 0.01 \), the ratios \( GJ/ EI \) remain relatively small (the maximum value of \( GJ/ EI \) corresponds to \( 0.1255 \)).

Summarizing the results of the discussion, the values of \( t_0 \) and \( k_0 \) are bounded by the following limits:

\[
\begin{align*}
&k_0 < 1.6 \\
&\text{I-sections:} \\
&warping is free to take place, \quad t_0 < 0.01 \\
&warping is fully constrained, \quad t_0 < 0.13 \\
&\text{rectangular sections:} \\
&k_0 < 0.15 \\
&t_0 < 0.73 \\
\end{align*}
\]

or, in general,

\[
\begin{align*}
k_0 &< 1.6 \\
t_0 &< 0.73 \\
\end{align*}
\]

An important point emerging from the study of this section is that the high values of \( t_0 \) and \( k_0 \), for practical types of members, are never coincident. That is, a member with a high value of \( t_0 \) has a low value of \( k_0 \) and vice versa. For this reason, in the following material, the effects of torsional and shearing rigidities are studied separately. That is, when the effects of torsional (shearing) rigidities are under consideration, the value of \( k_0 \) (\( t_0 \)) is assumed to be zero for all the members.

Section 2-4. Analytical Results

This section is concerned with the presentation of the results of analysis of a large number of diagonal grids.

The grid layouts, boundary conditions and types of external
loading were selected with care; and what follows is a description of the cases considered with an attempt to justify the choices.

First of all, it was necessary to select a number of commonly used grid layouts and the question was whether to choose the layouts to have various shapes of boundaries or to concentrate on a single family of boundary shapes. The danger of adopting the former was that it could have resulted in a number of isolated pieces of information with the general patterns of behaviour remaining obscure; and the disadvantage of the latter suggestion was that the scope of the work would have been made limited. It was, however, preferred to adopt the second suggestion; whereby to obtain more reliable information about a single family of grids rather than having scattered pieces of information connected together by guesswork. The family of grids chosen for the investigation were diagonal grids with square boundaries, which are known to be very much used in practice. Five such layouts were selected and are shown in fig. 2-4-1.

Each one of the layouts was to be analysed with a number of various boundary conditions and between many different possibilities the following four types were selected:

a) The grid is completely fixed all round the boundary. This type of boundary condition will hereafter be referred to by the letter A.

b) All the boundary joints are knife edged parallel to the boundary line. This type of boundary condition will hereafter be referred to by the letter B.

c) All the joints on the boundary lines are restricted from translation but can rotate freely. This type of boundary condition will hereafter be referred to by the letter C.

d) The four corner joints are restricted from translation but can rotate freely. This type of boundary condition will here-
THE LAYOUTS

LAYOUT NO. 1

LAYOUT NO. 2

LAYOUT NO. 3

LAYOUT NO. 4

LAYOUT NO. 5

$\alpha$ IS EQUAL TO 45°.

FIG. 2-4-1
after be referred to by the letter D.

Of course, none of these boundary conditions can ever be fully realised in practice, because they require complete constraints and perfect degrees of freedom at the supports, neither of which can be ever achieved since the former require infinitely rigid props and the latter need frictionless guides. The practical cases, however, will lie somewhere between the limiting cases considered and can be idealized to the nearest one.

The next step was the choice of the external loading. Obviously, it was not practical to consider all the possible types of loading and only a few important types had to be selected. An unfortunate restriction was that the loads had to be symmetrical because otherwise the demand on the computation time and the storage requirements in the computer could not have been met. Of course, the symmetry of loading was of any help if the layouts and boundary conditions were also symmetrical but these were already chosen to be so. It may be argued that at least a few unsymmetrical cases could have been considered but this was against the general policy of tracing the general patterns of behaviour rather than obtaining solution for isolated cases.

Between the symmetrical types of loading the following two seemed to be the obvious choices.

a) Uniformly distributed load over the whole surface of the grid. This type of loading will hereafter be denoted by the letter W.

b) A single concentrated vertical load applied to the central joint. This type of loading will hereafter be denoted by the letter P.

These two were the only types of external loading considered.

Summarizing, there were five different layouts each one of which was considered with four different boundary conditions.
(A, B, C and D as described earlier) and each one of the resulting twenty grids was analysed for two types of external loading (W and P as explained above).

In practice, usually, a diagonal grid has members of the same cross-section and therefore it was decided that all the grids considered should have this property.

The cases are identified by case numbers of the form a-b-c, where a is the layout No. as shown in fig. 2-4-1, b denotes the type of the boundary condition and c gives the type of the external loading. For example, 2-A-P is the case No. for a grid with a configuration similar to layout No. 2, fixed all round the boundary and under a central concentrated vertical load. Each one of the above mentioned forty cases was analysed for seven different modes, the variations being in the torsional and shearing properties of the members only. These seven modes correspond to the values of torsion and shear coefficients given below:

1) \( t_0 = 0.0 \) and \( k_0 = 0.0 \)
2) \( t_0 = 0.25 \) and \( k_0 = 0.0 \)
3) \( t_0 = 0.50 \) and \( k_0 = 0.0 \)
4) \( t_0 = 0.75 \) and \( k_0 = 0.0 \)
5) \( t_0 = 0.0 \) and \( k_0 = 0.5 \)
6) \( t_0 = 0.0 \) and \( k_0 = 1.0 \)
7) \( t_0 = 0.0 \) and \( k_0 = 1.5 \)

These values of \( t_0 \) and \( k_0 \) are covering the practical range and are selected in accordance with the results of the investigation in section 2-4.

It is to be noted that the values of \( k_0 \) given above are for diagonal members only and for the boundary members they values should
be halved ($k_0$ is inversely proportional to the square of the length of the member). As far as the coefficient $t_0$ is concerned, it would be dependent on the length of the member provided that the warping is constrained. The effect of warping restraint, however, is appreciable only if the members are I-sections for which the torsion coefficients are normally very small (see section 2-3). It was, therefore, assumed that for each grid the coefficient $t_0$ is the same for all the members.

The programme described in section 2-2 was used to analyse all the cases and the results are shown in figs. 2-4-5 to 2-4-44. Each one of these figures represents the results of analysis of a grid for all the seven variations of torsional and shearing properties. In addition to the case No. given in each one of the figures, there is a small sketch in the bottom left corner which illustrates the layout, the boundary condition and the type of loading of the case under consideration. The legend used in these sketches is described in fig. 2-4-2.

In all the cases considered, the grid has four planes of symmetry and it is enough to show the results for only (1/8)th of the structure. Therefore, all the bending moments, shearing forces, deflections and torques in figs. 2-4-5 to 2-4-44 are plotted for (1/8)th of the grid only. The sign conventions and the abbreviations used in these diagrams are explained in fig. 2-4-5. The bending moment, shearing force and deflection diagrams are drawn for the following three modes:

a) $t_0 = 0.0$ and $k_0 = 0.0$ (———)

b) $t_0 = 0.25$ and $k_0 = 0.0$ (--------------------)

c) $t_0 = 0.0$ and $k_0 = 1.25$ (---------------------)

where the styles of line shown in brackets represent the corresponding modes in the diagrams. The torque diagrams are only drawn for
(t_o = 0.75 and k_o = 0.0). The different modes had to be drawn together because otherwise they could not be compared easily.

The reasons for omitting the curves corresponding to the intermediate modes were:

1) The diagrams would have become obscure and unduly complicated.

2) In all the cases considered the changes in the components of bending moment, shearing force, torque, rotation and deflection due to a change in the coefficient t_o or k_o were usually such that if an increase in a coefficient caused an increase in a component, then a further increase in the coefficient produced a further increase in the component and if an increase in a coefficient caused a decrease in a component, then a further increase in the coefficient produced a further decrease in the component. Due to this property, the diagrams for the intermediate modes will lie between those of the limiting modes considered and, therefore, there was little point in showing the curves for all the seven modes.

If in any part of the diagrams the curve representing the mode (t_o = 0.75 and k_o = 0.0) is not shown it means that, within the accuracy of the drawing, the curve is coincident with the one corresponding to the basic mode (t_o = 0.0 and k_o = 0.0) and the same applies to the mode (t_o = 0.0 and k_o = 1.5). If, on the other hand, the curves representing the modes (t_o = 0.75 and k_o = 0.0) and (t_o = 0.0 and k_o = 1.5) are coincident but are distinguishable from the basic mode, the line representing the curves along the member is drawn half in full line and half in broken line.

To ease the comparison between different cases the scales of the diagrams were kept constant as far as possible. It was found necessary, however, to have four different scales; these are shown in fig. 2-4-4 and the scale Nos. referred to in figs. 2-4-4 to 2-4-44 are corresponding to those given in this figure.
$q$ is the intensity of the uniformly distributed external load in terms of unit force/ unit area.

$Q$ is the magnitude of the central concentrated external load in terms of unit force.

$EI$ is the flexural rigidity of the members in terms of unit force $\times$ unit area.

$l_0$ is the length of one side of the grid in terms of unit length.

FIG. 2-4-4
In the right hand bottom corner of each one of the first 2.4.5 to 2.4.64 there is a sketch showing the percentage changes in the largest components of bending moment, shear force and deflection with respect to changes in $t_o$ or $k_o$. The abbreviations used are:

$m$ for bending moment
$d$ for deflection
$s$ for shearing force

In these sketches, the curves in full line correspond to changes in $t_o$ and the curves in broken line correspond to changes in $k_o$. Some of the curves in these sketches have one or more kinks; in all such cases the maximum component has changed its position and the curve corresponds to components in different points of the structure.

It may be noticed that for the cases with a central concentrated load the curves showing the percentage changes in the largest components of shear force are not given. The reason is that in these cases the maximum shear force will always occur at the centre and its value is independent of $t_o$ or $k_o$.

The analytical results presented in this section are thoroughly examined in the last chapter and some interesting conclusions are reached. These results are also used in the next chapter to check the validity of certain proposed relations.
CASE NO. 1-A-P
SCALE NO. 3
FIG. 2-4-10
CASE NO. 4-A-P
SCALE NO. 3

FIG. 2-4-13
CASE NO. 1 - B - W
SCALE NO. 1

FIG. 2-4-15
CASE NO. 3 - B - W
SCALE NO. 1

FIG. 2-4-17
CASE NO. 1-B-P
SCALE NO. 3
FIG. 2-4-20
FIG. 2-4-22

CASE NO. 3 - B - P
SCALE NO. 3
CASE NO. 5-B-P
SCALE NO. 3
FIG. 2-4-24
CASE NO. 4-C-W
SCALE NO. 1
FIG. 2-4-28
CASE NO. 2-C-P
SCALE NO. 3
FIG. 2-4-31
CHAPTER THREE

Estimate of Forces and Displacements in Flat Grids with Varying Torsional or Shearing Properties
Section 1-1. Introduction

This chapter is devoted to the development of a technique whose main function is to define a number of rules for extrapolation (or interpolation) of the values of forces and displacements in a linear structure when some features of the structure are variable.

The technique employs the concept of vector and matrix norms which is widely used in disciplines such as numerical analysis and error analysis.

The main part of this chapter is concerned with the application of the suggested technique to the case of flat grids with varying torsional or shearing properties. The basic idea, however, is independent of the type of structure and can be further developed to cover a wide range of structural problems.

Section 1-2. Vector and Matrix Norms

When dealing with vectors and matrices it is sometimes convenient to have a scalar function which, in some sense, represents the magnitude of a vector or a matrix. The notion of vector and matrix norms has been devised to provide for this need.

The concept of norms is not yet considered to be a matter for engineering mathematics. It was thought appropriate, therefore, to include a brief description of these mathematical objects before utilizing them and the rest of this section is devoted to this purpose (For detailed treatment of vector and matrix norms see [6], [11], [8], [9] and [10]).

In what follows x and y denote column vectors, A and B denote square matrices, \( a \) denotes any real scalar, \( |a| \) denotes the absolute value of \( a \) and \( n \) denotes the dimension of a vector or the order of a square matrix.
Definition 1. Any real-valued function of the components of \( x \) is a norm of \( x \) (denoted by \( \| x \| \)) provided that:

\[
\begin{align*}
\| x \| & > 0 \text{ unless } x = 0 \text{ in which case } \| x \| = 0 \quad \text{--- 1a} \\
\| \alpha x \| & = | \alpha | \| x \| \quad \quad \quad \quad \quad \quad \text{--- 1b} \\
\| x + y \| & \leq \| x \| + \| y \| \quad \quad \quad \quad \quad \text{--- 1c}
\end{align*}
\]

There are infinitely many different functions satisfying the above requirements among which the following are most commonly used:

\[
\begin{align*}
\| x \|_1 & = \max_{i=1}^{n} | x_i | \quad \quad \quad \quad \quad \quad \text{--- 2-2a} \\
\| x \|_2 & = \sqrt{\sum_{i=1}^{n} | x_i |^2} \quad \quad \quad \quad \quad \quad \text{--- 2-2b} \\
\| x \|_\infty & = \max_{i=1}^{n} | x_i | \quad \quad \quad \quad \quad \quad \text{--- 2-2c}
\end{align*}
\]

where \( x_i \) is the \( i \)-th component of \( x \). The first norm is the absolute value of the largest component of \( x \) in modulus, the second norm is the sum of the absolute values of all the components of \( x \) and the third norm is the Euclidean length of \( x \) (\( \| x \|_2 \) is usually referred to as the Euclidean vector norm). It can be shown that each vector norm is associated with a convex body in the \( n \)-dimensional Euclidean space. In particular, for the three norms introduced above, the convex bodies are respectively an \( n \)-dimensional cube, an \( n \)-dimensional octahedron and an \( n \)-dimensional sphere. For detailed information about this aspect of norms see [9] and [10].

**Definition 2.** Any real-valued function of the elements of \( A \) is a norm of \( A \) (denoted by \( \| A \| \)) provided that:

\[
\begin{align*}
\| A \| & > 0 \text{ unless } A = 0 \text{ in which case } \| A \| = 0 \quad \text{--- 3-2-3a} \\
\| \alpha A \| & = | \alpha | \| A \| \quad \quad \quad \quad \quad \quad \text{--- 3-2-3b}
\end{align*}
\]
For instance, the square root of the sum of the squares of all the elements of $A$ is a norm satisfying all the above requirements. This particular norm is usually referred to as the Euclidean matrix norm.

**Definition 3.** A matrix norm is said to be "consistent" with a given vector norm provided that for any matrix $A$ and any vector $x$

$$\| A x \| \leq \| A \| \cdot \| x \|$$

A matrix norm can legitimately be used in conjunction with a vector norm if and only if it is consistent with this vector norm.

**Definition 4.** A matrix norm consistent with a given vector is said to be "subordinate" to this vector norm provided that for any matrix $A$ there exists a vector $x \neq 0$ such that:

$$\| A x \| = \| A \| \cdot \| x \|$$

A matrix norm which is subordinate to a vector norm is also necessarily consistent with it but the converse is not true. For instance, the Euclidean matrix norm is consistent with the Euclidean vector norm but is not subordinate to it. In fact, a vector norm has usually a number of matrix norms consistent with it but only one of them is subordinate to it.

From relation 3-2-5, it follows that for a matrix norm subordinate to whatsoever vector norm, $\| I \| = 1$ (I denotes a unit matrix). Because, there exists a vector $x \neq 0$ such that:

$$\| A x \| = \| A \| \cdot \| x \|$$

or,

$$\| x \| = \| I \| \cdot \| x \|$$

hence,

$$\| I \| = 1.$$

It can be shown that the matrix norms subordinate to the vector norms given in relations 3-2-2 are respectively:
\[ \| A \|_1 = \max \sum_{i} |a_{ij}| \quad \ldots \ldots \quad (3-2-6a) \]
\[ \| A \|_2 = \max \sum_{i} |a_{ij}| \quad \ldots \ldots \quad (3-2-6b) \]
\[ \| A \|_\infty = \text{largest diagonal element of } A^T A^{-1} \quad \ldots \ldots \quad (3-2-6c) \]

where \( a_{ij} \) is the typical element of \( A \). The first matrix norm is the largest row sum of absolute values and the second is the largest column sum of absolute values.

The matrix norm used in conjunction with a vector norm need not necessarily be subordinate to it, but it is advantageous to be so. The advantage is that the relation containing the norms, which is usually in the form of an inequality, will be optimal.

From now on, the symbols \( \| x \| \) and \( \| A \| \) will be used to denote any norm. Furthermore, in any relation containing both matrix and vector norms, the matrix norm is presumed to be subordinate to the vector norm.

**Section 3-3. Basic Inequalities**

Let \( K, d \) and \( w \) be the stiffness matrix, the displacement vector and the external load vector of a linear structure, respectively. The relationship between the external loads and the joint displacements is given by (see relation 1-3-7):

\[ K d = w \quad \ldots \ldots \quad (3-3-1) \]

Now, let the structure be modified in any arbitrary manner provided that:

I) The modified structure remains stable.

II) The modified structure has the same number of joints as the original structure and the number and the types of degrees of freedom at each joint remain unchanged.
III) The external load vector is not altered.

Examples of modifications consistent with the above restrictions (subject to the condition that the structure remains stable) are:

a) Inserting or removing members.

b) Changing the rigidities of the members.

c) Altering the positions of the joints.

The load-displacement relationship for the modified structure may be represented by:

\[(K + M) (d + g) = w\]  \[5-3-2\]

where \((K + M)\) and \((d + g)\) are the stiffness matrix and the displacement vector of the modified structure, respectively. The matrix \(M\) will hereafter be referred to as the "modification matrix".

The precise values of the components of joint displacement of the modified structure can be found from the solution of equation 5-3-2 and the accurate values of the components of force can then be obtained from the joint displacements and the force-displacement relations of the individual members. However, sometimes it is required to analyse a structure for a large number of cases in which some features of the structure are changing gradually. In such circumstances, one may think of analysing the structure for only a small number of basic cases and finding the approximate values of forces and displacements, for the other cases, by extrapolation (or interpolation). Also, it is sometimes desirable to have upper bounds giving the maximum possible changes in the components of force or displacement with respect to variations in some features of the structure.

For the above two types of problem, the technique developed in the course of this chapter may prove to be valuable.

We start by deriving upper bounds for the ratios \(|v| / |d + g|\) and \(|v| / |d|\), where \(|v|\), \(|d|\) and \(|d + g|\) represent any norm
of \( x, d \) and \( d + x \), respectively. From relations 3-2-1 and 3-3-2,
\[
(K + M)(d + x) = Kd \quad \text{or} \quad -Kx = M(d + x)
\]
or,
\[
-g = K^{-1}M(d + x).
\]
Taking norms (see relation 3-2-4),
\[
\| -g \| \leq \| K^{-1}M \| \cdot \| d + x \|.
\]
From relation 3-2-1b, \( \| g \| = \| x \| \) hence,
\[
\| g \| \leq \| K^{-1}M \| \cdot \| d + x \|.
\]
or finally,
\[
\frac{\| g \|}{\| d + x \|} \leq \| K^{-1}M \|.
\]
Now, from relation 3-2-1c,
\[
\| d + x \| \leq \| d \| + \| x \|.
\]
Substituting for \( \| d + x \| \) into relation 3-3-3,
\[
\| g \| \leq \| K^{-1}M \| (\| d \| + \| x \|)
\]
or,
\[
(1 - \| K^{-1}M \|) \| g \| \leq \| K^{-1}M \| \cdot \| d \|
\]
and provided that \( \| K^{-1}M \| < 1 \),
\[
\frac{\| g \|}{\| d \|} \leq \frac{\| K^{-1}M \|}{1 - \| K^{-1}M \|}.
\]
*A necessary (but not sufficient) condition for \( \| K^{-1}M \| < 1 \) is that \( (K + M) \) is nonsingular. Because if \( (K + M) \) is singular then \( (I + K^{-1}M) \) which is the product of \( K^{-1} \) and \( (K + M) \) is singular. But if \( (I + K^{-1}M) \) is singular then it has at least one zero eigenvalue and hence \( K^{-1}M \) has at least one eigenvalue equal to \(-1\). On the other hand, it can be shown that no norm of a matrix can ever be less than the modulus of an eigenvalue of the matrix. Therefore, if \( (K + M) \) is singular then \( \| K^{-1}M \| \geq 1 \) and the point is proved.
Relations 3-3-4 and 3-3-5 are the required upper bounds and depending on the norm used, they may represent upper bounds for the relative average changes in the displacements or the relative maximum changes in the displacements . . . . etc.

Relations 3-3-4 and 3-3-5 may be used in two ways:

a) In a qualitative manner for deriving different relationships or comparing the relative effects of different types of modifications . . . . etc.

b) In a quantitative manner for obtaining numerical upper bounds. The quantitative use of the relations may involve the following two difficulties:

I) The numerical evaluation of $\|K^{-1}M\|$ may be unduly complicated.

II) There is, in general, no guarantee that the inequalities obtained are close enough to be of any use. There are, however, occasions when $\|K^{-1}M\|$ is readily obtainable and it can also be shown that the assessed inequalities give close bounds, in which case the quantitative use of the relations may prove to be satisfactory.

The discussion has so far been quite general and applicable to any structure. In the remainder of this chapter, however, the attention will be concentrated on the case of flat grids with varying torsional or shearing properties and the relations derived in this section will be used (mainly in a qualitative manner) for the study of the problem.

Before this section is ended, there is a rather interesting point which is worth mentioning. Namely, on the strength that

$$\|K^{-1}M\| \leq \|K^{-1}\| \cdot \|M\|$$

relations 3-3-4 and 3-3-5 may be written in the following form:
\[
\frac{\| g \|}{\| d + e \|} \leq \| K^{-1} \| \cdot \| M \|
\]

or,

\[
\frac{\| g \|}{\| d + e \|} \leq \frac{\| K^{-1} \| \cdot \| K \|}{1 - \| K^{-1} \| \cdot \| M \|}
\]

\[
\frac{\| g \|}{\| d + e \|} \leq \frac{\| K^{-1} \| \cdot \| K \|}{1 - \| K^{-1} \| \cdot \| M \|}
\]

Here, \( \| M \| / \| K \| \) represents the relative change in the stiffness matrix and \( ( \| K^{-1} \| \cdot \| K \| ) \) is an important coefficient denoting the conditioning of the system of equations \( Kd = w \). \( ( \| K^{-1} \| \cdot \| K \| ) \) is usually referred to as a "condition number" and the larger it is the more badly conditioned the system is (see [29]). Now the interesting point revealed by relations 3-3-6 and 3-3-7 is that the changes in the displacements of a structure due to any modification are critically dependent on the conditioning of \( Kd = w \).

Section 5-4. The Matrix \( K^{-1}M \)

From the relations developed in section 3-3, it can be seen that the effects of any modification on a structure are dependent on the norm of \( K^{-1}M \). In this section, the matrix \( K^{-1}M \) for a flat grid with varying torsional or shearing properties will be closely examined and the corresponding norms will be evaluated.

Let the sole modification to a structure consist of changes

*It is assumed that \( \| K^{-1} \| \cdot \| M \| < 1 \).
in the properties of a single member b. Let the basic stiffness matrix of this modified member, relative to its own member coordinate system, be

\[ K_{mb} = K_b + T_b \]

where \( K_b \) is the basic stiffness matrix of the unmodified member. The stiffness matrices of the modified member relative to the frame coordinate system are:

\[ (K'1_1)_{mb} = T_b H_b (K_b + \Delta_b) H_b^T \]

\[ (K'1_2)_{mb} = -T_b H_b (K_b + \Delta_b) T_b^T \]

\[ (K'2_1)_{mb} = -T_b (K_b + \Delta_b) H_b^T \]

\[ (K'2_2)_{mb} = T_b (K_b + \Delta_b) T_b^T \]

where \( T_b \) and \( H_b \) are the transformation matrix and the equilibrium matrix of the member, respectively (see relations 1-3-2 and 1-3-4b). From relations 3-4-1 and fig. 1-4-2 it follows that the modification matrix, corresponding to the type of modification under consideration, is of the form shown in fig. 3-4-1, where the terminal joints of member b are assumed to be i and j (i < j).

Now, consider a flat grid with members having no torsional and infinite shearing rigidities (i.e., \( t = 0 \)) and suppose that a member b is modified to have a torsional rigidity equal to \( GJ \).

From fig. 1-3-3,

\[ K_b = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & \frac{4EI}{L} & \frac{6EI}{L^2} \\
  0 & \frac{6EI}{L^2} & \frac{12EI}{L}
\end{bmatrix} \]
The shaded submatrices contain zero elements only.
and,  
\[
K_{mb} = \begin{bmatrix}
\frac{EI}{L} & 0 & 0 \\
0 & \frac{EI}{L} & 0 \\
0 & 0 & \frac{EI}{L}
\end{bmatrix}
\]

and therefore,  
\[
K_b = K_{mb} - K_0 = \begin{bmatrix}
\frac{EI}{L} & 0 & 0 \\
0 & \frac{EI}{L} & 0 \\
0 & 0 & \frac{EI}{L}
\end{bmatrix}
\]
or finally,  
\[
K_b = t_o \begin{bmatrix}
\frac{EI}{L} & 0 & 0 \\
0 & \frac{EI}{L} & 0 \\
0 & 0 & \frac{EI}{L}
\end{bmatrix}
\]

Noting that \( K_b \) and \( T_b \) are independent of \( t_o \), from relation 2-4-2 and fig. 3-4-1 it follows that  
\[
M_b = t_o D_b
\]

where \( D_b \) is a square matrix independent of \( t_o \) and of the same order as \( K_b \) (and \( K \)).

Now, if all the members of the grid are modified to have torsional rigidities, the overall modification matrix is given by:  
\[
M = \sum M_b = \sum t_o D_b
\]

where the summation extends over all the members of the grid. Furthermore, if all the members have the same torsion coefficient, then  
\[
M = \sum t_o D_b = t_o \sum D_b
\]
\[
\begin{align*}
M &= \mathbf{L} \mathbf{D} \\
K^{-1} \mathbf{V} &= \mathbf{C} \mathbf{K}^{-1} \mathbf{D},
\end{align*}
\]

where \( \mathbf{D} = \sum B_i \) is a matrix independent of \( \mathbf{V} \). From multiplying relation (3-2-2) by \( \mathbf{K}^{-1} \) (\( \mathbf{K} \) is the stiffness matrix of the unmodified grid),

\[
K^{-1} \mathbf{V} = \mathbf{C} \mathbf{K}^{-1} \mathbf{D}.
\]

Taking norms (see relation (3-2-1)),

\[
\|K^{-1} \mathbf{V}\| = \|C\| \|K^{-1}\| \|\mathbf{D}\| = \|C\| \|K^{-1}\| M.
\]

Denoting \( \|K^{-1}\mathbf{M}\| \) by \( \beta \) and noting that \( t \) is always positive,

\[
\|K^{-1}\mathbf{M}\| = t \beta
\]

where \( \beta \) is a positive scalar depending on the properties of the unmodified grid and the type of norm used.

Now, consider a flat grid with members having zero torsion and shear coefficients and let a member be modified to have a non-zero shear coefficient equal to \( k_o \). The matrix \( \mathbf{v_b} \) for this case, can be derived in the following manner:

From Fig. 1-5-3,

\[
K_b =
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{4EI}{L} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{12EI}{L^3}
\end{bmatrix}
\]

and,

\[
K_{mb} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & \left(1 + \frac{0.25k_o}{1 + 2k_o}\right) \frac{4EI}{L} & \left(\frac{1}{1 + 2k_o}\right) \frac{6EI}{L^2} \\
0 & \left(\frac{1}{1 + 2k_o}\right) \frac{6EI}{L^2} & \left(\frac{1}{1 + 2k_o}\right) \frac{12EI}{L^3}
\end{bmatrix}
\]
Therefore,

\[ q_b = \frac{k}{mh - h} = \frac{q}{1 + k} \]

\[ \begin{bmatrix}
  0 & 0 \\
  -6EI & -12EI \\
  0 & -12EI \\
  L^2 & -34EI \\
\end{bmatrix} \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3-4-5 \]

From fig. 3-4-1 and relation 3-4-5,

\[ M_b = \left( \frac{k}{1 + 2k} \right) D_b \]

where \( D_b \) is a matrix independent of \( k \). In a manner similar to the case of torsion considered above, it can be shown that if all the members of the grid have the same shear coefficient \( k \), then

\[ k^{-1} M = \frac{k}{1 + 2k} \gamma \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3-4-6 \]

where \( \gamma \) is a positive scalar depending on the properties of the unmodified grid and the type of norm used.

We now show that \( \gamma \) in relation 3-4-6 is approximately equal to 2. From relation 3-4-5,

\[ q_b = \left( \frac{-2k}{1 + 2k} \right) K_b \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3-4-7 \]

where the use of \( \approx \) rather than the equality sign is due to the small difference in the central element. The modification matrix \( M_b \) can now be obtained from relation 3-4-7 and fig. 3-4-1 and the overall modification matrix is found to be

\[ M = \sum M_b \approx \left( \frac{-2k}{1 + 2k} \right) K. \]

Premultiplying by \( K^{-1} \),
\[ k^{-1}M = \left( \frac{-2k_0}{1+2k_0} \right) k^{-1}k = \left( \frac{-2k_0}{1+2k_0} \right) I. \]

Taking norms,

\[ \| k^{-1}M \| = \left| \frac{-2k_0}{1+2k_0} \right|. \]

or,

\[ \| k^{-1}x \| = \frac{\sigma}{2k_0} \quad \sigma = 2. \]

From relations 2-a-4 and 7-a-5, it follows that \( \gamma = 2 \) and the point in question is proved.

Relations 3-4-4, 3-4-5, and 7-4-5 were obtained on the assumption that all the members of the grid have the same \( t \) or \( k \). However, as far as relations 3-4-4 and 3-4-5 are concerned, the assumption need not necessarily be satisfied. Namely, relation 3-4-4 remains valid even if the torsion coefficients are different for different members, in which case \( t \) assumes a basic value and \( \beta \) will depend on the properties of the unmodified grid, the proportions between the torsion coefficients of the members and the type of norm used. A similar remark holds for relation 3-4-6 except that \( \gamma \) will not depend on the proportions between the shear coefficients of the members but on the proportions between \( k_0' \) or \( (1+2k_0) \) for different members.

Section 3-7. Estimate of Displacements

In this section, the relations developed in sections 3-3 and 3-4 are used in conjunction with a postulate to find a number of equations by which the changes in the displacements of a flat grid due to variations in the torsional or shearing properties can be closely estimated.

Throughout this and the next section, it is assumed that \( k \)
(or \( k \)) is the same for all the members. However, some of the results obtained are applicable to more general form and thus discussed in section 5-7.

From relations \( 4-4-4 \), \( 4-4-4 \) and \( 4-4-4 \),

\[
\frac{\| e \|}{\| d + g \|} \leq 1, \ldots , \ldots , \ldots , \leq 1
\]

\[
\frac{\| e \|}{\| d + g \|} \leq \frac{K_0 Y}{1 + K_0}
\]

where the inequalities are valid with respect to any norm and any loading condition.

The above relations may now be used as upper bounds for the relative changes in the displacement vector. However, it is proposed here to utilize them in a different manner. Namely, we accept the idea conveyed by the relations for deriving equations which are not any more simple upper bounds but are used to estimate the actual changes in the displacements with reasonable accuracy. We proceed as follows:

Remembering that vector norms are functions of the elements of vectors and considering the fact that relations \( 4-4-4 \) are valid for every possible norm (of which there are infinitely many), one may logically expect that the changes in each element of the displacement vector are also bounded by relations similar to \( 4-4-4 \). Furthermore, there are many instances when relations \( 4-4-4 \) used as equalities define the behaviour of the displacements almost precisely and hence the following postulate:

**POSTULATE.** For any flat grid with varying torsional (or shearing) properties, the changes in each component of displacement are bounded by a relation similar to \( 4-4-4 \) for \( 4-4-4 \). Furthermore, for each component of \( d \) and each loading condition,
the value of the coefficient $d_{i}$ may or may not be such that the relation holds as an equality. When

Such a value of $d_{i}$ or $d_{j}$ component of $d$ will hereafter be denoted by $d_{i_{0}}$ or $d_{j_{0}}$.

Denoting the $i^{th}$ component of $d_{i_{0}}$ and the $i^{th}$ component of $d_{i} + d_{j_{0}}$ by $d_{i_{0}}$ and $d_{i}$, the postulate assumes that the following relations are valid:

\[
\frac{d_{i} - d_{i_{0}}}{d_{i_{0}}} = v_{i_{0}}
\]

for torsion,

\[
\frac{d_{i} - d_{i_{0}}}{d_{i}} = \frac{\gamma_{i}}{\gamma_{i_{0}}}
\]

for shear,

where $v_{i_{0}}$ and $\gamma_{i}$ in the above relations are allowed to accept sign.

The problem now is to find a way to evaluate $d_{i}$ and $d_{j_{0}}$. The technique suggested below is one way of achieving the purpose.

First consider the case of varying torsional properties. The grid is analyzed twice, once the same external loads. In the first analysis, the members are considered to have zero torsion coefficient and in the second analysis, all the members are given the same torsion coefficient $T$. The $i^{th}$ component of displacement found from the first analysis is of course nothing but $d_{i_{0}}$ and the corresponding value obtained from the second analysis will be denoted by $d_{i}$. From relation 4, we get

*The validity of the postulate is thoroughly discussed in Section 4-2.*
\[ \frac{1 - d}{d} = T_i \]

or,

\[ \frac{1 - d}{d} = T \frac{1 - d_i}{d_i} \]

Substituting for \( T \) into relation \( 3-1 \),

\[ \frac{1 - d}{d} = T \frac{1 - d_i}{d_i} \]

If,

\[ d_i = \frac{d_{10} d_1}{d_1 + d_{10} / T} \frac{1 - d_{10}}{1 - d_i} \]

The only variables in this relation are \( d_i \) and \( d_{10} \), and therefore the \( k_i \) component of displacement corresponding to any value of \( x \) can now be readily evaluated for the loading case considered.

In the case of varying shear properties, again the grid can be applied first with members having zero shear coefficient and next with all members having the same shear coefficient \( k \), the loading condition being the same for both analyses. Proceeding in a manner similar to the case of torsion, it can be shown that

\[ d_i = \frac{d_{11} (1 + 2R_0)}{d_{11} (1 - k_0) + d_{10} (1 + 2R_0) k_0} \]

where \( d_{11}, d_{10} \), and \( d_i \) are analogous to similar terms in relation \( 3-1 \). The above relation allows the value of \( d_i \), corresponding to any value of \( k_0 \) to be easily evaluated for the loading case considered.

We now consider an alternative way of formulating the effects of varying shear properties on the displacements. From relations \( 3-1 \) and \( 3-4 \),
\[
\frac{g}{d} = \frac{\frac{1}{1 - \frac{g_0}{s_0}}}{\frac{1}{1 + \frac{g_0}{s_0}}}
\]

where the relation is valid with respect to every norm and for every loading condition. Consequently, since in most practical cases \(g_0\) is less than 1.5 (see section 2-6),

\[
\frac{g}{d} \leq \frac{1}{2.7}
\]

If special interest is when the above relation is used in conjunction with the first norm (see relation 4-5-2a). The interpretation is that the change in the maximum component of displacement due to variation in the shearing properties is unlikely to be even more than 3.7 times the original displacement.

Strictly speaking, relation 4-5-2 is not to be completely relied upon because it is based on relation 3-4-3 which is of an approximate nature. However, at least for all the grids considered in section 3-4 the changes remain within the bounds suggested by relation 3-5-5. The question may arise as to whether relation 3-5-5 is not a gross overestimate. That this is not so will be established by showing that there is a family of flat grids for which under a particular loading condition relation 3-5-5 holds as equality. Each member of this family consists of a number of identical cantilevers connected at their ends forming a regular polygon (the plan views of the first five members of this family are shown in fig. 3-5-3). For any such grid, under a concentrated central load \(P\), the only nonzero component of displacement, i.e., the central deflection, is given by:
FIG. 3-5-1

FIG. 3-5-2
is an alternative to relation \( \delta = 3 \) depending on the problem one or the other may prove to be more useful. However, to check the validity for typical cases, the solutions are only checked for a number of special cases.

In forty cases for which the results of analyses are given in section 2-4, sixteen cases were chosen for checking. For each case, a number of points were taken at random and the analytically obtained values of deflection or rotation at these points were compared with the estimated ones. The basic analyses were considered to be corresponding to \( \gamma = 1 \) and \( \delta = 3 \) for torsion and \( \gamma = 0 \) and \( \delta = 1 \) for shear and the estimated values corresponding to \( \gamma = 1 \), \( \delta = 3 \), \( \gamma = 0 \), and \( \delta = 1 \) were compared with the analytically obtained values. The results of the comparison are shown in table 4-1. In this table, relation \( \frac{\delta}{\gamma} = 3 \) is used to estimate the displacements when the torsional properties are changed and relation \( \frac{\delta}{\gamma} = 4 \) is used when the shearing properties are changed. The reason for using relation \( \frac{\delta}{\gamma} = 3 \) rather than \( \frac{\delta}{\gamma} = 4 \) is that for the cases considered, the accuracy of the results obtained from relation \( \frac{\delta}{\gamma} = 3 \) were better than those obtained from relation \( \frac{\delta}{\gamma} = 4 \).

The case numbers given in the first column of table 4-1 are according to the conventions of section 2-4 and the member ends referred to in the second column correspond to the numbering system shown in fig. 3-5-2. The letter D before a member end
<table>
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<th>Area</th>
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<tr>
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<td></td>
<td>2-3</td>
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<tr>
<td></td>
<td>6-5</td>
<td>344.2 (1.2)</td>
</tr>
</tbody>
</table>

**Note:** The figures in brackets are the percentage differences between the estimated and the corresponding analytically obtained values.
indicates that the component under consideration is a function and the latter B indicates that it is a constant. The estimated values are given in the last four columns of Table 6. Each estimated value is followed by a figure containing the percentage error in the estimate. A positive error indicates that the value is overestimated and a negative error indicates that it is underestimated.

From the results given in Table 6, it can be concluded that the values obtained from relations 3.6-1 and 3.6-2 are quite reliable. Particularly in the case of varying shear stress, the correlation between the analytical and the estimated values is remarkable. The average and maximum errors, in this case, are 0.4% and 2.1% with the next largest error being only 1.5%. The estimates for the case of varying torsional properties are also quite good with the average and the maximum errors being 0.4% and 1.6%, respectively.

It is interesting to note that the analytical results of Section 3.6.1 (with which the estimated values are compared) are obtained for diagonal grids with members of identical cross-section. In grids of this type, the value of \( k' \) for a boundary member is half of that for a diagonal member (\( k' \) is inversely proportional to the square of the length of the member). On the other hand, relation 3.6-1 is only applicable to grids in which \( k' \) is the same for all the members. Therefore, the results of analysis for the grids having boundary conditions C and D (see Section 3.6.4) are not, strictly speaking, in the region of applicability of relation 3.6-1. Table 7-6-1, however, shows that the estimates for the grids having boundary conditions C and D are no less accurate than those for the other cases.

Section 4.6. Estimate of Forces

In this section, a number of relations are derived on which the
changes in the components of force in a flat grid due to changes in the torsional or shearing properties may be estimated.

First, consider the variation in the torsional properties. Let a member in the flat grid be under the effect of torsion force and displacements shown in Fig. 5-3 and assume that \( t = 0 \). From relations 5-3-1, 5-3-2 and 5-3-3 and Fig. 5-3, the components of force at end 1, in terms of the terminal displacements, are given by:

\[
\tau_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

\[
\tau_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

\[
\tau_{10} = 0
\]

where the second subscript indicates that the component of force or displacement is corresponding to the case when \( t = 0 \).

Now, let the grid be modified so that all the members acquire the same nonzero torsion coefficient \( t \). The force-displacement relations, in this case, are given by:

\[
\tau_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

\[
\tau_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

\[
\tau_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

\[
\tau_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

From relation 5-3-2a,

\[
\theta_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]

\[
\theta_{10} = \frac{E_e}{\ell} \left[ \sigma_{10} + \frac{d_t}{d_e} \left( h_{10} - h_{00} \right) \right]
\]
and similarly,

\[
\begin{align*}
0_1 &= \frac{\alpha_0}{1 + \alpha_1 t_0} \\
0_2 &= \frac{\alpha_0}{1 + \alpha_2 t_0} \\
&\vdots \\
0_n &= \frac{\alpha_0}{1 + \alpha_n t_0}
\end{align*}
\]

where \( \alpha_0, \alpha_1, \ldots \) are constants.

From relations 3-6-4 and 3-6-5,

\[
\beta_1 = \frac{\alpha_1}{1} \left[ \frac{\beta_0}{1 + \alpha_0 t_0} + \frac{\beta_0}{1 + \alpha_0 t_0} - \frac{1}{t_0} \left( \frac{\beta_0}{1 + \alpha_0 t_0} - \frac{\beta_0}{1 + \alpha_0 t_0} \right) \right]
\]

In practice the coefficient \( t_0 \) is always less than unity and \( \alpha_0, \alpha_1, \ldots \) are usually much less than unity. Furthermore, \( \alpha_0, \alpha_1, \ldots \) are not necessarily equal but they are almost always of the same sign (positive) and their values do not differ very much. Hence, we may introduce a single coefficient \( \alpha_0 \) which is some sort of average of \( \alpha_0, \alpha_1, \ldots \), such that

\[
(1 + \alpha_0 t_0) = (1 + \alpha_i t_0) \quad \text{for} \quad i = 1, 2, 3, \ldots
\]

From relations 3-6-4 and 3-6-5,
\[
\tau = -\frac{\alpha t}{\pi} \left[ \ldots - \frac{1}{\pi} \ldots + \frac{1}{\pi} \ldots \right]
\]

or, comparing with relation \(\tau = \tau_0\),

\[
m_{11} = \frac{m_{10}}{1 + \frac{\alpha}{\pi} T}
\]

Now, let the grid be analysed for two basic cases: one with \(t = 0\) for all the members and next with \(t = T\) for all the members, the external loads being the same in both cases. The bending moment at end 1 of member 11 found from the first analysis is nothing but \(m_{11}\), and that obtained from the second analysis will be denoted by \(m_{11}^*\). From relation \(\tau = \tau_0\),

\[
\tau_{10} = \frac{1}{1 + \frac{\alpha}{\pi} T}
\]

or,

\[
m_{10} = \frac{m_{10} - m_{11}^*}{T \cdot m_{11}^*}
\]

Substituting for \(m_{10}\) into relation \(\tau = \tau_{10}\),

\[
\tau_{10} = \frac{m_{10} + m_{11}^*}{m_{10} + (\frac{\alpha}{\pi} T) \cdot (m_{10} - m_{11}^*)}
\]

In the same manner, it can be shown that for any value of \(t\), the shearing force at end 1 of member 11 is given by a relation similar to \(\tau = \tau_0\). Furthermore, the derivation of relation \(\tau = \tau_0\) was independent of the actual position of member 11, and therefore, the relation must be applicable to any point of the grid.

*Equality signs are used in the relations throughout this section but most of them, as will be clear from the context, are to be interpreted as approximately equal.
Consequently, using \( t_i \) to denote any component of bending moment or shearing force,

\[
p_i \equiv \frac{t_{1o} \cdot t_{10}}{t_i + t_{10} - t_{1o} - t_0} \quad \cdots \quad \text{(Eq. 1)}
\]

where \( t_{1o} \) and \( t_0 \) are analogous to \( t_i \) and \( t_{10} \). The above relation may be used to estimate the components of bending moment or shearing force corresponding to any values of \( t_i \), provided, of course, the external loading states must be the same as that used for the basic analysis.

Now, from relations (6-6-8) and (6-6-7),

\[
q_i = t_{0} \alpha \left( \frac{t_{1o}}{1 + a_0 t_0} - \frac{t_{j0}}{1 + a_j t_0} \right)
\]

or, using the approximation of relation (6-6-5),

\[
q_i = \frac{t_0 \eta}{1 + a_0 t_0} \left[ -\frac{1}{L} (t_{10} - t_{j0}) \right]
\]

or,

\[
q_i = \frac{t_0 \eta}{1 + a_0 t_0} \quad \cdots \quad \text{(Eq. 2)}
\]

where \( \eta = \frac{RL (t_{10} - t_{j0})}{L} \) and is independent of \( t_i \). The frame may now be analyzed for two basic cases, one with the members having a torsion coefficient equal to \( T_j \) and next with the members having a torsion coefficient equal to \( T_i \). From relation (6-6-8),

\[
q_i = \frac{T_j \eta}{1 + a_j T_j} \quad \cdots \quad \text{(Eq. 3)}
\]

and

\[
q_i = \frac{T_i \eta}{1 + a_i T_i} \quad \cdots \quad \text{(Eq. 4)}
\]
where \( a \) and \( \beta \) are the terms of eqns. 1 and 2, respectively.\[ a = \frac{T_2}{T_1} - \frac{T_1}{T_2} \quad \text{and} \quad \beta = \frac{T_2}{T_1} \]

Substituting for \( a \) and \( \beta \) into eqns. 1 and 2:

\[ \frac{1}{\beta} - \frac{T_1}{T_2} \]

This is the relation to which the components of the stress corresponding to any value of \( T \) for the column are subjected in the simple analysis can be estimated.

We now consider the effect of varying some of the structural parameters. The components of force, in this case, are similar in character and hence only the change in the bending moment and shearing force need to be examined. It is assumed that the beam is continuous, the deflections to be zero.

The bending moment and shearing force at end \( 0 \) of each member are shown in Fig. (4) and (5), when \( a \) = 0 for all the members, and the relation \( a = -a \) and \( \beta = \beta \) are the corresponding relation when all the members have the same bending shear coefficients. The equations \( 1 \beta = \beta \), \( 1 \beta = \beta \) and \( 1 \beta = \beta \) and Fig. (6) to (9), are obtained.

\[ e_1 = \frac{2a}{b} \left[ \frac{a + d}{\frac{1}{a} + \frac{1}{d}} + \frac{a - d}{\frac{1}{a} - \frac{1}{d}} \right] \]

\[ e_2 = \frac{2a}{b} \left[ \frac{a + d}{\frac{1}{a} + \frac{1}{d}} - \frac{a - d}{\frac{1}{a} - \frac{1}{d}} \right] \]

From relation \( \beta = \beta \).
\[ q_i = (1 + a_1 k_o) b_{10} \quad \ldots \quad 3-6-10a \]
\[ q_j = (1 + a_2 k_o) b_{10} \quad \ldots \quad 3-6-13b \]

where \( a_1 \) and \( a_2 \) are constants and the other terms have their usual meaning. It is known that the changes in the components of rotation due to variations in the shearing properties are normally much less than the changes in the components of deflection (cf., table 4-10), and hence it may be assumed that

\[ q_i = q_{10} \quad \text{and} \quad q_j = q_{10} \quad \ldots \quad 3-6-13c \]

From relations 4-6-12a and 3-6-13,

\[ m_1 = \frac{2EI}{(1 + 2k_o)} \begin{Bmatrix} (2 + k_o) q_{10} + (1 - k_o) q_{j0} \\
- \frac{z}{1} \left[ (1 + a_1 k_o) q_{10} - (1 + a_2 k_o) q_{j0} \right] \end{Bmatrix} \]

or,

\[ m_1 = \frac{1}{(1 + 2k_o)} \left[ m_{10} + k_o \frac{2EI}{L} (q_{10} - q_{j0} - \frac{z_1}{1} q_{10} + \frac{z_2}{1} q_{10}) \right] \]

or finally,

\[ m_1 = \frac{m_{10} + f k_o}{1 + 2k_o} \quad \ldots \quad 3-6-13d \]

where

\[ f = \frac{2EI(q_{10} - q_{j0} - 3a_1 b_{10} / L + 3a_2 b_{10} / L) / L} \]

and is independent of \( k_o \). Proceeding as before by analysing the grid twice, from relation 5-6-14,

\[ m_1 = \frac{m_{10} + f k_o}{1 + 2k_o} \]

or,

\[ f = \frac{(1 - 2\lambda m_1)}{\lambda} - m_{10} \quad \ldots \quad 3-6-13d \]
where \( \lambda \) is the shear coefficient of the system. In the analysis, from relations:

\[
\frac{\lambda}{(1 + \lambda)} p_{11} + \frac{1}{(1 + \lambda)} p_{12} = 0
\]

\[
\frac{1}{(1 + \lambda^2)} p_{11} + \lambda p_{12} = 0
\]

As before, it may be argued that the above relation is only applicable to the components of bending moment at any point of the grid and it can be shown that the relation also holds for any component of shearing force. Therefore, we may write

\[
\frac{\lambda}{(1 + \lambda)} p_{11} + \frac{1}{(1 + \lambda)} p_{12} = 0
\]

\[
\frac{1}{(1 + \lambda^2)} p_{11} + \lambda p_{12} = 0
\]

where \( p_{11} \) and \( p_{12} \) are components of bending moment and shearing force, respectively.\( \therefore (1 + \lambda) \) and \( \lambda (1 + \lambda) \) are expressions in terms of \( \lambda \). Finally, the relation the validity of relation of bending moment and shearing force, and the relation the validity of relation the bending moment and shearing force, are given in section ...
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<thead>
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<th>Model No.</th>
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<th>A-C</th>
<th>A-D</th>
<th>A-E</th>
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**Note:** The entries in brackets are the calculated differences between the estimated and the measured or measured values.
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</tbody>
</table>
The results of the comparison given in Table 3-3-1 can in general be considered satisfactory. For the case of varying shearing properties, the average error is 1.4% and the maximum error is 5.8%. In the case of varying compliance properties, the average error is 2.3% and the discrepancies remain within acceptable limits except for two components in which the errors are as high as 30.1% and 15.1%. These components are the bending moments in member 6-6 of e = 0.7-0.8 and member 6-5 of e = 0.3-0.4. The behaviour of these two components are more closely evident in Table 3-3-1. From this table, it is seen that in spite of the alarmingly high percentage errors, the estimates, in terms of the actual values are not very bad. What has appeared is that each component gives reduced sensitivity in terms of the discrepancies in the estimates, which are small as compared with the original values, corresponding to large percentage errors as compared with the measured values.

The results of checking force for relation 3-3-1 are shown in Table 3-3-1. The components of force for checking were selected at random by a randomly chosen component as reflected in its absolute value (for e = 0.7-0.8) was less than 4% of the absolute value of the maximum bending moment (for e = 0.3) in the corresponding case. The basic analyses were considered to be for e = 0.7-0.8 and t = 0.7% and the estimated values of force for e = 0.3 were shown in the third column of Table 3-3-1. It is seen, from this table, that the estimates are quite accurate with the average and the maximum errors being 0.4% and 1.7% respectively.

Section 3-7. Discussion

The results of the comparison between the estimated and the analytically obtained values of forces and displacements, given in
<table>
<thead>
<tr>
<th>Case No.</th>
<th>Member end</th>
<th>Bending moment</th>
<th>Analytical value</th>
<th>Estimated value</th>
<th>Percentage error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-A-W</td>
<td>5-4</td>
<td>-36.4</td>
<td>-36.5</td>
<td>-36.4</td>
<td>-0.23</td>
</tr>
<tr>
<td>3-A-W</td>
<td>5-5</td>
<td>-212.9</td>
<td>-212.7</td>
<td>-212.9</td>
<td>-0.23</td>
</tr>
<tr>
<td>2-C-P</td>
<td>2-7</td>
<td>-395.9</td>
<td>-396.1</td>
<td>-395.9</td>
<td>-0.23</td>
</tr>
<tr>
<td>2-D-W</td>
<td>4-5</td>
<td>-488.6</td>
<td>-488.7</td>
<td>-488.6</td>
<td>-0.23</td>
</tr>
<tr>
<td>2-C-P</td>
<td>2-7</td>
<td>-395.9</td>
<td>-396.1</td>
<td>-395.9</td>
<td>-0.23</td>
</tr>
<tr>
<td>2-D-W</td>
<td>4-5</td>
<td>-488.6</td>
<td>-488.7</td>
<td>-488.6</td>
<td>-0.23</td>
</tr>
<tr>
<td>3-A-W</td>
<td>5-8</td>
<td>-451.9</td>
<td>-451.5</td>
<td>-451.9</td>
<td>-0.23</td>
</tr>
<tr>
<td>3-B-P</td>
<td>2-3</td>
<td>-390.1</td>
<td>-390.3</td>
<td>-390.1</td>
<td>-0.23</td>
</tr>
<tr>
<td>3-D-W</td>
<td>4-5</td>
<td>-511.0</td>
<td>-511.0</td>
<td>-511.0</td>
<td>-0.23</td>
</tr>
<tr>
<td>3-D-P</td>
<td>3-2</td>
<td>-390.1</td>
<td>-390.3</td>
<td>-390.1</td>
<td>-0.23</td>
</tr>
</tbody>
</table>
the previous sections, leave little doubt that the suggested technique does provide a reliable means for extrapolation and interpolation of forces and displacements.

In this section, it is proposed to discuss the work of the previous sections and to use a little deeper into the philosophy behind the operations.

The main steps in the suggested technique are as follows:

1) Use the concept of norms to find the relations containing \( t \) or \( b \), and the ratios \( \|2\|/\|d\| \) or \( \|a\|/\|d\| \).

2) Use the postulate of section 5.6 to express the variations of the components of displacement in terms of \( t \), or \( b \).

3) Substitute the relations governing the changes in the displacements into the force-displacement relations to find equations defining the variations of the components of force with respect to \( t \), or \( b \).

4) Analyze the structure for a number of basic cases to evaluate the constant coefficients of the equations.

A secondary type of application of the technique involves Step 1 only. In this case the results are upper bounds for the relative changes in the displacements (e.g., relations \( \|\dot{\mathbf{r}}\|/\|\mathbf{r}\| \) and \( \|\dot{\mathbf{v}}\|/\|\mathbf{v}\| \)) and it may also be possible to extend the idea and obtain upper bounds for the relative changes in the components of force.

The postulate of section 5.6 (hereafter referred to as "the postulate") is the keystone of the technique and deserves a careful examination. The postulate assumes that:

1) If there is an upper bound for the relative change in the norm of the displacement vector (with respect to \( t \), or \( b \)), then if this upper bound is valid for every norm then it is also an upper
bound for the relative changes in every component of the displacement vector.

The equation defining the variation of a component of displacement (with respect to t), or k, by assuming the corresponding upper bound with the inequality therefore enforced by equality and the constant coefficients obtained empirically (through a number of basic analyses).

The first supposition of the postulate is equivalent to the assumption that the absolute value of each component of a vector is a norm of that vector. This assumption is not quite valid because a vector norm must satisfy the conditions of relations 3.3-1 but the absolute value of a component of a vector satisfies these conditions if and only if it is different from zero (if the component is equal to zero the first condition will be violated) and, in general, the possibility of a vector having zero components cannot be excluded. However, we can easily get round this difficulty by saying that any zero component of the vectors may be replaced by a reasonably small value and, therefore, we are only concerned with vectors having non-zero components.

To put the above statement on a more formal basis, we introduce a new family of vector norms which we name "r.k. norms" and denote them by \( \| x \|_{r,k} \). This family of norms is defined as follows:

\[
\| x \|_{r,k} = |x_k| \quad \text{unless} \quad |x_k| < \| x \|_{r,k}
\]

in which case \( \| x \|_{r,k} = \| x \|_{r,k}/r \).

where \( x \) is a column vector, \( x_k \) is the \( k \)th component of \( x \), \( r \) is any real positive scalar larger than unity and \( \| x \| \) is as defined in relation 3.3-1. The convex body associated with this vector norm is an \( n \)-dimensional rectangular solid in which all the sides are of length \( Fr \) except for the side in the \( k \)th direction which is of length \( \| x \|_{r,k} \). The matrix norm subordinate to this vector norm is found
to be:

\[ \| A \|_{rk} = \begin{cases} 
\text{either:} & \left( \sum_{i \neq k} \left| a_{ik} \right| \right) + \left| a_{kk} \right| \\
\text{or:} & \max \left( \sum_{i \neq k} \left| a_{ij} \right| + a_{ik} r \right) 
\end{cases} \]

whichever larger.

Where \( A \) is a square matrix and \( a_{ij} \) is the typical element of \( A \).

In terms of \( r,k \), norms, we can now regard each component* of a vector as a norm of the vector provided that the component* remains larger than a limiting value (if the component* is smaller than the limiting value, the norm will then be equal to this limiting value). This limiting value is \( \| x \|_{rk} r \) and \( r \) can always be chosen large enough to make the limiting value as small as it is desired. Therefore, we can accept the validity of the first supposition of the postulate and then from relations 3-7-1 it may legitimately be concluded that:

\[ \frac{|s_i - d_{10}|}{|d_i|} \leq t \alpha \] \( \ldots \ldots \ldots \) 3-7-1a

\[ \frac{d_i - d_{10}}{|d_i|} \leq \frac{k c \gamma}{1 + k a} \] \( \ldots \ldots \ldots \) 3-7-1b

where the notation is the same as that used in sections 3-7. Now, if \( d_i \) and \( d_{10} \) have the same sign then

\[ |d_i - d_{10}| = |d_i| - |d_{10}| \]

and relations 3-7-1 may be written in the following alternative form:

*In absolute value.
(for torsion) \[ \frac{|(\dot{d}_1 - \dot{d}_0)|}{d_1} \leq \tau \leq \rho \]

(for shear) \[ \frac{|(\dot{d}_1 - \dot{d}_0)|}{d_1} \leq \gamma \]

or,

(for torsion) \[ \frac{|(\dot{d}_1 - \dot{d}_0)|}{\Gamma d_1} \leq \tau \leq \gamma \]

(for shear) \[ \frac{|(\dot{d}_1 - \dot{d}_0)|}{\Gamma d_1} \leq \frac{\gamma}{1 + \alpha} \]

where \( \Gamma \) is either +1 or -1 depending on whether \( d_0 < d_1 \) or \( d_0 > d_1 \).

Now, consider relation \( -2 \leq 2a \). We have

(for \( \Gamma = 1 \)) \[ \frac{|d_{1c} - d_1|}{d_1} \leq \tau \leq 0 \]

(for \( \Gamma = -1 \)) \[ \frac{|d_{1c} - d_1|}{d_1} \leq \tau \leq 0 \]

or,

(for \( \Gamma = 1 \)) \[ d_1 \leq \frac{d_{1b}}{1 - \tau} \]

(for \( \Gamma = -1 \)) \[ d_1 \geq \frac{d_{1b}}{1 - \tau} \]

\( \Box \)

It is assumed that \( \tau \leq 1 \).
Relations (2-2-1) are upper and lower bounds for the variation of $|d_{10}|$ with respect to $t_0$. For a given point in a given finite element, the numerical values of $|d_{10}|$ and $\beta$ are assessable and the curve representing $|d_{10}| = |d_{10}|/(1 + \mu \beta)$ and $|d_{11}| = |d_{11}|/(1 + \beta \mu)$ will be of the form shown in fig. (2-7-1). These two curves are denoted by $C_0$ and $C_1$ and have the same basic equation.

The curve representing the actual variation of $|d_{11}|$ with respect to $t_0$ (for the leading case considered) may

a) pass through point $A$ (see fig. (2-7-1),

b) remain in the shaded area (see fig. (2-7-1),

c) be continuous.

a and b are obvious but the continuity follows from the physical reasonings that unless the structure is on the verge of a critical state for which is the same thing, the stiffness matrix of the structure is or the verge of singularity, the components of displacement remain finite and also infinitesimal changes in the structure of the structure is unlikely to cause about change in the displacements. Furthermore, at least as far as diagonal finite elements are concerned, in all known cases the curves representing the variations of the displacements are actually continuous. The curve representing the variation of $|d_{11}|$ with respect to $t_0$ must, therefore, be a curve such as $C$ shown in fig. (2-7-1).

We are now in the position to interpret the second supposition of the postulate. This supposition is equivalent to the supposition that the curve $C$ has the same equation as the limiting curves $C_0$ and $C_1$, that is

$$\eta = \frac{1}{1 + \mu \beta t_0}$$

where $\eta$ is obtained from the condition that the curve must pass
$|d_i| = \frac{|d_{i0}|}{1 - t_0\beta}$

$|d_i| = \frac{|d_{i0}|}{1 + t_0\beta}$

FIG. 3-7-1
through point B (see fig. 3-7-1). This assumption is no more than an approximation because the actual equation of curve C is a highly complicated function of all the elements of the stiffness matrix and the load vector and it can hardly be precisely represented by an equation as simple as that of \( C_1 \) and \( C_2 \). However, it may logically be expected that if there is any simple equation which can fit curve C approximately, it should be similar to that of \( C_1 \) and \( C_2 \).

Relation 3-7-7b may now be considered and the conclusions will be found to be similar to those obtained for relation 3-7-7a.

The above discussion shows that the postulate of section 3-8 is not a precise statement but, at the same time, it is not merely an intuitive inference and is based on plausible reasoning and known facts.

The technique developed in this chapter may be applied to some other problems of structural analysis and these are discussed in the last chapter. However, before using the technique in a practical problem, the reliability of the postulate for that class of problem must be established. As far as diagonal trusses with varying torsional or shearing properties are concerned, table 3-8-1 is believed to have provided enough evidence for the postulate to be considered quite reliable.

A rather favourable feature to be seen in all the tables of sections 3-8 and 3-9 is that usually the larger components are estimated more accurately. This phenomenon may be explained as follows:

A careful study of the behaviour of components of normal displacement in diagonal trusses with varying torsional or shearing properties shows that, for every type of component there is always a general pattern of behaviour followed by the majority of the components of that type. This pattern of behaviour is constant and...
that of almost all the components which are relatively large in magnitude. The relations of sections 4-1 and 5-1, on the other hand, are always consistent with these typical patterns of behaviour and, therefore, the accuracy of the estimates for the larger components is usually high. Occasionally, there are components which do not behave in the typical manner. These components are normally small in magnitude and the accuracy of the estimated values is likely to be poor.

Throughout sections 4-1 and 5-1, it was assumed that all the grid members have the same tension coefficient $t$, or the same shear coefficient $s$. However, on the strength of the comment at the end of section 4-1, some of the relations obtained in sections 4-1 and 5-1 are of more general applicability. Namely, relations $w_{1} = \alpha$, $w_{2} - x$, $w_{3} - y$, and $w_{4} - z$ may be used even if different members have different tension coefficients but the proportions between these coefficients must remain constant. Also, relations $w_{1} - v$ and $w_{2} - w$ are valid even if $k$ is different for different members provided that the proportions between $k$ and $v$ for different members remain constant. This includes the case also where some of the members have the same varying $k$, and for the rest of the members $k = 0$.

The values of $T$ and $\lambda$ corresponding to the second basic analyses for obtaining the estimates of tables 4-1 to 5-1 were 0.5 and 1.0 respectively. A question may arise as to whether the choice of $T$ or $\lambda$ could have any effect on the accuracy of the estimates. The answer is yes. In our case, due to the limited number of available analytical solutions, the choice of $T$ was restricted to 0.5, 1.0, or 2.0, and the choice of $\lambda$ was restricted to 0.5, 1.0, or 2.0. All these values of $T$ and $\lambda$ for the second basic analyses were considered and the estimated values were compared. The results showed that, the accuracy of the estimates is not independent of the choice of $T$ or $\lambda$. Using the Greek symbol,
\[ \tau = 0.5 \text{ and } \lambda = 1.0 \] were found to give the most accurate results but there is no guarantee that these are the optimum values of \( \tau \) and \( \lambda \) because all the possibilities have not been checked. Similar remarks hold for the results given in table 4.3.1.

From the tables of sections 4.3.2 and 4.3.4, it can be seen that the estimated values of the components of displacement are, in general, more accurate than those of the forces. Some of the discrepancies in the estimated values of forces are due to the simplifying assumptions employed to derive the relations of section 4.3.1. These simplifying assumptions, however, are not essential to the technique and could, if desired, be avoided or replaced by less crude approximations. This, of course, will make the relations more complicated and may also increase the required number of basic analyses but there may be cases for which such increases in the labour are warranted.
Section 4.1. Introduction

The analytical results of chapter two were obtained depending on a number of simplifying assumptions the validity of which can not always be taken for granted. Some of these assumptions are:

a) The engineering theory of bending is valid.
b) The small deflection theory holds.
c) The supposition that the joints have infinitely small volumes will not affect the results appreciably.

to assess error bounds for the effects of these assumptions in all conceivable situations, extensive theoretical and experimental investigations are needed. Such investigations are beyond the scope of this thesis the main object of which is a study of different nature. It was thought appropriate, however, to show experimentally that at least for a few cases, the theoretical predictions of forces and displacements based on the above-mentioned simplifying assumptions are reliable.

It was decided to test a single diagonal grid with different boundary conditions and the following were set out as the points to be considered in the design of the testing arrangement:

a) The dimensions of the model to be as large as the laboratory facilities and conditions allowed.
b) The members to be mild steel I-sections.
c) The connections to be welded so that the complete rigidity of the joints could be ensured.
d) Different boundary conditions to be arranged by choosing various groups of joints as the supports, the type being ball support in every case.
e) The external loading to be a central vertical concentrated load in all the cases.
f) The measurements to be confined to bending strains and joint deflections.
Section 4.2. Component Parts of the Testing Arrangement

The Model

The model was designed as a three by three diagonal grid. The members were 3\(\times\)1\(\frac{1}{2}\)\(\times\)4 British standard steel sections, and the connections were all welded. The nominal dimensions of this grid are shown in fig. 4.2-1.

Care was exercised during the manufacturing of the model to make it dimensionally correct and to avoid distortion. However, the actual dimensions, as one expects, were not precisely the same as the nominal ones and accurate measurements showed that they were, in average, about 0.5\(\%\) out and in the case of a few members the discrepancies were up to 1\(\%\).

Supports

The supports were to satisfy the following requirements:

a. To simulate unyielding ball supports.
b. To be mountable on box beams made of two 17\(\times\)4\(\times\)1 steel channels which were already available and fixed in position.
c. To be detachable so that different boundary conditions could be easily arranged.

The supports were designed to these requirements and a sketch showing the details is given in fig. 4.2-1.

Loading Frame

The loading device had to be mountable on box beams similar to those used for the supports and was to be capable of exerting up to one ton concentrated load.

A combination of a hydraulic jack and a proving ring was used for application and measurement of the loads and the details are shown in fig. 4.2-1.
PLAN OF THE TEST GRID

ALL MEMBERS HAVE THE SAME CROSS-SECTION.
ALL CONNECTIONS ARE WELDED.

FIG. 4-2-1

SECTION A-A
SUPPORT ARRANGEMENT

SIDE VIEW I

SIDE VIEW II

SECTION A-A

SECTION B-B

SECTION C-C

FIG. 4-2-2

SCALE

0  5  10  15 IN.
LOADING FRAME

1" dia. 60" long bright steel studding sticks

13\times13\times1" mild steel plates

1/2" dia. 13" long studding sticks

hydraulic ram

proving ring

1" thick steel block

layer of lead

central joint of the test grid

13\times13\times1" mild steel plates

supporting beam

to the pump

FIG. 4-2-3
Dial Gauges

Dial gauges were used to measure the vertical deflection of the joints. These dial gauges were mounted on magnetic bases which were supported by a frame situated under the test grid. This frame was made of steel angles and was supported on the floor so that it had no connection with any part of the testing arrangement.

Only the deflections of the joints in two quadrants of the grid were measured and the positions of these joints are shown in fig. 4-2-4. For a boundary joint, however, the deflection was measured only when it was not a support.

Strain Gauges

Electrical resistance strain gauges were used for measurement of bending strains. These were Tinsley's paper-back Durofix-bonded drawn gauges of the type 6K3 having a resistance of 120 ohms.

Mid-points of members were chosen for strain measurement and 2\(^2\) such points were considered. The positions of strain gauge stations are shown in fig. 4-2-4. At each station one strain gauge was fixed on the top flange and another on the bottom flange exactly below the first one. One of these strain gauges was to act as the dummy and the other as the active gauge. This is a well known technique for measurement of bending strains and the advantages of using the strain gauges in this manner (as compared with using active gauges on both top and bottom flanges with separate dummies) are:

a) The readings are doubled as a result of which the accuracy is increased.

b) The labour of strain gauge fixing, wiring and reading is almost halved.

c) The dummy gauges are situated in surroundings similar to the active ones resulting in accurate temperature compensation.

d) The effects of axial forces are eliminated automatically.
POSITIONS OF DIAL GAUGES
AND STRAIN GAUGE STATIONS

FIG. 4-2-4

○ STRAIN GAUGE STATION
○ DIAL GAUGE POSITION
The surface preparation and strain gauge fixing and wiring were carried out with care and the strain gauges were left to dry for about a week.

The insulations between the gauges and the test surface were measured before the experiments and the resistances were found to be between 50 and 100 mega ohms. These insulation resistances for strain gauges of 120 ohms are normally considered to be satisfactory.

**Strain Measuring Instruments**

The instruments used for strain measurement were Peekeel's electronic strain indicator type B103U and extension box type 23U. The particulars of these instruments can be found in the manufacturers' publications.

**Section 4-3. Tests and the Results**

The grid was tested with five different boundary conditions. The boundary arrangements for these tests are shown in fig. 4-3-1. The external load was a central point load in all the cases.

A preliminary analysis showed that under a central load of up to a ton, all the points of the test grid for all five cases were well within the elastic range having a factor of safety of at least two. It was decided, therefore, that the magnitude of the central load be kept under a ton.

Before each test the model was subjected to several cycles of loading and unloading to "condition" the model.

For each test, the strain gauges and dial gauges were read for the following loads:

I) An initial load of 388.27 lb
II) A load of 1160.64 lb
III) A load of 1916.77 lb

The reason for not starting the load from zero was that the initial
LAYOUTS OF THE TESTS

TEST NO. 1

TEST NO. 2

TEST NO. 3

TEST NO. 4

TEST NO. 5

- BALL SUPPORT
- VERTICAL POINT LOAD

FIG. 4-3-1
load could eliminate some of the imperfections, such as closing a small gap between the grid and a support.

The strain gauge and dial gauge readings had to be processed to obtain the net increments of readings due to a fixed increment of load, say a ton. In what follows a general relation is derived by which this processing can be achieved.

Let:

- $a_1$ be the first increment of load.
- $a_2$ be the sum of the first and the second increments of load.
- $a_0$ be an arbitrary increment of load.
- $b_1$ be the increment of reading of a strain gauge or a dial gauge due to an increment of load equal to $a_1$.
- $b_2$ be the same as $b_1$ but for an increment of load equal to $a_2$.
- $b_0$ be the same as $b_1$ but for an increment of load equal to $a_0$.

Assuming that the test grid behaves linearly and that the errors in $b_1$ and $b_2$ are equal but have opposite signs (see fig. 4-3-2), the following two relations are readily obtained:

\[
\frac{(b_1 - e)}{a_1} = \frac{(b_2 + e)}{a_2}
\]

\[
\frac{(b_1 - e)}{a_1} = \frac{b_0}{a_0}
\]

Eliminating $e$ between the relations,

\[
b_0 = \frac{a_0 (b_1 + b_2)}{a_1 + a_2}
\]

Now, if $a_0$ is equal to one ton and $b_t$ represents the increment of reading due to a ton and if $a_1$ and $a_2$ are taken from the actual values of the loads given at the beginning of this section, then
FIG. 4-3-3a

DETAIL A

BENDING MOMENT DIAGRAM

FIG. 4-3-3b
\[ b_t = 0.973545(b_1 + b_2) \]

This is the relation by which the increments of readings due to a load of one ton for all the strain gauges and all the dial gauges were evaluated.

The values of \( b_1 \), \( b_2 \) and \( b_t \) for all five tests are given in tables 4-3-1a, 4-3-1b, ........ 4-3-5a and 4-3-5b. The values of \( b_1 \) and \( b_2 \) in these tables are the averages of readings from two quadrants of the grid. The fact is that the model under all five boundary arrangements retained at least two axes of symmetry and, therefore, it was enough to install the strain gauges and dial gauges in only one quadrant of the grid. Nevertheless, it was decided to cover two quadrants (see fig. 4-2-4) and take the average of readings to improve the accuracy.

The figures in the first column of tables 4-3-1b, ........ 4-3-5b are the joint identification numbers and correspond to the numbering system given in fig. 4-3-1. The items in the first column of tables 4-3-1a, ........ 4-3-5a represent the mid-points of the members; for instance, 6-5 represents the mid-point of the member with terminal joints 6 and 5.

The values of \( b_t \), as far as the deflections are concerned, are the final answers. They represent the actual deflections in ten-thousandths of an inch having a positive sign for downward movement. The values of \( b_t \) for the bending moments, however, can not be easily interpreted. Here, a value of \( b_t \) represents the average of bending strains on the top and the bottom flanges at the mid-span of a member. But what is really needed is the magnitude and the sense of the bending moment. One could, of course, use the gauge factor of strain gauges to transform the readings into strains, multiplying them by the modulus of elasticity of the material to obtain the stresses and use the section modulus of the members to find some values for bending moments. The method adopted, however, was a different one.
### TABLE 4-3-1a

**TEST NO. (i)**

**EXPERIMENTAL RESULTS**

<table>
<thead>
<tr>
<th>STRAIN GAUGE STATION</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_t$</th>
<th>BENDING MOMENT T.IN.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 3</td>
<td>130.00</td>
<td>256.75</td>
<td>378.5</td>
<td>-2.58</td>
</tr>
<tr>
<td>2 - 3</td>
<td>89.5</td>
<td>178.5</td>
<td>260.9</td>
<td>-1.78</td>
</tr>
<tr>
<td>2 - 4</td>
<td>-3.0</td>
<td>-5.5</td>
<td>-8.3</td>
<td>0.06</td>
</tr>
<tr>
<td>3 - 5</td>
<td>91.0</td>
<td>179.0</td>
<td>262.9</td>
<td>-1.79</td>
</tr>
<tr>
<td>3 - 6</td>
<td>-41.5</td>
<td>-80.5</td>
<td>-118.8</td>
<td>0.81</td>
</tr>
<tr>
<td>4 - 7</td>
<td>18.5</td>
<td>38.0</td>
<td>55.0</td>
<td>-0.38</td>
</tr>
<tr>
<td>4 - 8</td>
<td>-3.0</td>
<td>38.5</td>
<td>-9.2</td>
<td>0.06</td>
</tr>
<tr>
<td>5 - 7</td>
<td>-2.0</td>
<td>-3.0</td>
<td>-4.9</td>
<td>0.03</td>
</tr>
<tr>
<td>6 - 7</td>
<td>70.5</td>
<td>137.5</td>
<td>202.0</td>
<td>-1.38</td>
</tr>
<tr>
<td>6 - 8</td>
<td>-60.0</td>
<td>-131.0</td>
<td>-191.8</td>
<td>1.31</td>
</tr>
<tr>
<td>7 - 7</td>
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### TABLE 4-3-1b

**TEST NO. (i)**

**EXPERIMENTAL RESULTS**

<table>
<thead>
<tr>
<th>DIAL GAUGE STATION</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_t$</th>
<th>DEFLECTION IN.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32.90</td>
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<td>95.3</td>
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<td>26.55</td>
<td>38.9</td>
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<td>40.85</td>
<td>59.8</td>
<td>0.0598</td>
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<td>0.0000</td>
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<td>39.8</td>
<td>0.0348</td>
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### Table 4-3-2a
#### Test No. (2)
#### Experimental Results

<table>
<thead>
<tr>
<th>Strain Gauge Station</th>
<th>$b_i$</th>
<th>$b_j$</th>
<th>$b_k$</th>
<th>Bending Moment T·in.</th>
</tr>
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<tbody>
<tr>
<td>1 - 3</td>
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<td>27.5</td>
<td>39.9</td>
<td>-0.27</td>
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<td>217.0</td>
<td>317.9</td>
<td>-2.17</td>
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<td>-15.0</td>
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<td>141.5</td>
<td>200.9</td>
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<td>51.5</td>
<td>103.0</td>
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### Table 4-3-2b
#### Test No. (2)
#### Experimental Results

<table>
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<tr>
<th>Dial Gauge Station</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>Deflection (\text{in.})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39.30</td>
<td>78.10</td>
<td>114.3</td>
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<td>2</td>
<td>16.90</td>
<td>33.50</td>
<td>49.1</td>
<td>0.0461</td>
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<td>24.40</td>
<td>50.20</td>
<td>73.1</td>
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<td>0.0000</td>
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<tr>
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<td>33.80</td>
<td>49.1</td>
<td>0.0444</td>
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<td>0.0147</td>
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### TABLE 4-3-3a

**TEST NO. (3)**

**EXPERIMENTAL RESULT**

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<tr>
<th>STRAIN GAUGE STATION</th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>BENDING MOMENT (IN.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 2</td>
<td>160.45</td>
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</tr>
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<td>225.6</td>
<td>331.0</td>
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<td>3 - 4</td>
<td>32.0</td>
<td>0.4</td>
<td>93.0</td>
<td></td>
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<td>-7.0</td>
<td>-12.5</td>
<td>-19.0</td>
<td>0.11</td>
</tr>
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<td>510.5</td>
<td>747.7</td>
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<td>119.0</td>
<td>175.2</td>
<td>-1.29</td>
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### TABLE 4-3-3b

**TEST NO. (3)**

**EXPERIMENTAL RESULTS**

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<tr>
<th>DIAL GAUGE STATION</th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>DEFLECTION (IN.)</th>
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<tbody>
<tr>
<td>1</td>
<td>84.00</td>
<td>107.70</td>
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</tr>
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<td>107.25</td>
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<td>138.55</td>
<td>262.8</td>
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<td>0.03571</td>
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<td>77.80</td>
<td>114.7</td>
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### TABLE 4-3-4a
#### TEST NO. (4)
#### EXPERIMENTAL RESULTS

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<th>STRAIN GAUGE STATION</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>BENDING MOMENT T. IN.</th>
</tr>
</thead>
<tbody>
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<td>1-3</td>
<td>184.75</td>
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<td>2-4</td>
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<td>56.0</td>
<td>111.0</td>
<td>-0.76</td>
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<td>64.7</td>
<td>-0.44</td>
</tr>
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<td>428.0</td>
<td>645.5</td>
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<td>51.0</td>
<td>62.5</td>
<td>91.0</td>
<td>-0.62</td>
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<tr>
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<td>199.0</td>
<td>290.1</td>
<td>-1.98</td>
</tr>
<tr>
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<td>19.0</td>
<td>39.0</td>
<td>56.5</td>
<td>-0.39</td>
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### TABLE 4-3-4b
#### TEST NO. (4)
#### EXPERIMENTAL RESULTS

<table>
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<th>DIAL GAUGE STATION</th>
<th>D1</th>
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<th>D3</th>
<th>DEFLECTION IN.</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<tr>
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<td>105.30</td>
<td>154.2</td>
<td>0.1542</td>
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<td>0.1366</td>
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### TABLE 4-3-5a
TEST NO. (5)
EXPERIMENTAL RESULTS

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<tr>
<th>STRAIN GAUGE STATION</th>
<th>b₁</th>
<th>b₂</th>
<th>b₃</th>
<th>BENDING MOMENT T.IN.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 3</td>
<td>216.25</td>
<td>429.50</td>
<td>628.7</td>
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<td>-2.80</td>
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<td>3 - 4</td>
<td>46.5</td>
<td>92.5</td>
<td>135.3</td>
<td>-0.92</td>
</tr>
<tr>
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<td>144.0</td>
<td>286.0</td>
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<td>-2.85</td>
</tr>
<tr>
<td>3 - 6</td>
<td>58.5</td>
<td>119.0</td>
<td>172.8</td>
<td>-1.18</td>
</tr>
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<td>4 - 4</td>
<td>174.0</td>
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<td>-3.43</td>
</tr>
<tr>
<td>4 - 5</td>
<td>194.5</td>
<td>38.5</td>
<td>56.5</td>
<td>-0.39</td>
</tr>
<tr>
<td>4 - 6</td>
<td>67.0</td>
<td>134.0</td>
<td>195.7</td>
<td>-1.33</td>
</tr>
<tr>
<td>4 - 7</td>
<td>47.0</td>
<td>92.5</td>
<td>135.2</td>
<td>-0.43</td>
</tr>
<tr>
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### TABLE 4-3-5b
TEST NO. (5)
EXPERIMENTAL RESULTS

<table>
<thead>
<tr>
<th>DIAL GAUGE STATION</th>
<th>b₁</th>
<th>b₂</th>
<th>b₃</th>
<th>DEFLECTION IN.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90.10</td>
<td>178.70</td>
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<td>0.2617</td>
</tr>
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<td>61.40</td>
<td>121.75</td>
<td>178.3</td>
<td>0.1783</td>
</tr>
<tr>
<td>3</td>
<td>71.80</td>
<td>142.55</td>
<td>206.7</td>
<td>0.2607</td>
</tr>
<tr>
<td>4</td>
<td>37.50</td>
<td>74.55</td>
<td>109.1</td>
<td>0.1691</td>
</tr>
<tr>
<td>5</td>
<td>50.90</td>
<td>121.50</td>
<td>177.6</td>
<td>0.1776</td>
</tr>
<tr>
<td>6</td>
<td>39.30</td>
<td>77.80</td>
<td>113.9</td>
<td>0.1139</td>
</tr>
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<td>7</td>
<td>37.50</td>
<td>74.20</td>
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<td>0.1087</td>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Namely, a 10' long specimen of the steel I-section from which the members of the test grid were made was subjected to a pure bending test. The specimen was taken from the same stock as the members of the grid and it is hoped that it was a representative sample.

The specimen was tested as a simple beam with two equal overhangs at both sides and was loaded by two equal vertical loads towards the ends of the overhangs, the details are shown in fig. 4-3-3a. The loads were applied by two identical loading frames, one of which was the one used for loading the test grid. The central span of the specimen was, of course, under a uniform bending moment with no accompanied shear forces, fig. 4-3-3b.

On the central span were installed four strain gauge stations the positions of which are shown in fig. 4-3-3a. Care was exercised to make these strain gauge stations identical, in every detail, to the ones constructed for the test grid.

The specimen was tested twice and in the second test the ends of the specimen were reversed. The average readings of all four strain gauge stations in both tests were:

\[ b_1 = -255.375 \]
\[ b_2 = -508.625 \]
\[ b_1 \text{ and } b_2 \text{ correspond to increments of load } a_1 = 259.15 \text{ lb and } a_2 = 518.79 \text{ lb. Substituting these values in relation 4-3-1, } \]
\[ b_t = -2199.86 \quad (\text{for } a_0 = 1 \text{ ton}) \]

This reading corresponds to a hogging bending moment of 15 ton·in, thus every 146.66 units of reading correspond to a sagging bending moment of 1 ton·in. This is the scale by which the strain gauge readings were transformed into bending moments. The values of bending moments, so obtained, are given in the last column of tables 4-3-1a, ...... 4-3-5a.

The advantages of using a bending specimen for the interpretation
THE BENDING SPECIMEN UNDER TEST
of strain gauge readings (as compared with using the nominal values of the gauge factor, the modulus of elasticity and the section modulus) are:

a) The effects of discrepancies between the actual dimensions of the cross-section and the ones given in the standard tables are eliminated automatically.

b) The need to use the gauge factor and the accompanied errors are removed.

c) A difference between the nominal and the actual values of the modulus of elasticity will not affect the results.

The validity of these points, however, depends on the specimen and the gauges to be true samples, but this is believed to have been the case.

The bending specimen had yet another important function. Namely, determination of the actual EI of the members. To this end, the central deflection of the specimen was measured for both tests and the average deflections were:

\[ b_1 = 0.05925 \text{ in} \]
\[ b_2 = 0.118625 \text{ in} \]

These figures are modified for sinking of the supports and correspond to increments of load \( a_1 = 259.15 \text{ lb} \) and \( a_2 = 518.79 \text{ lb} \). Substituting these values in relation 4-3-1,

\[ b_t = 0.51217 \text{ in} \text{ (for } a_0 = 1 \text{ ton).} \]

The central deflection \( b \) for a simple beam of span \( L \) under a uniform bending moment of magnitude \( M \) is given by \( Ml^2 / 8EI \).

It follows that,

\[ EI = \frac{ML^2}{8b} \]

Using the corresponding numerical values for the case under consideration,
The experimental values of bending moment and joint deflection for the test grid are given in the previous section. These values are free from the effects of man made assumptions but carry the inevitable and complex experimental errors. It is interesting to compare these experimental values with the corresponding values obtained from a theoretical analysis based on the simplifying assumptions used in the analysis of the grids in section 2-4. Such a comparison, assuming that the experimental errors are negligible, will show whether the simplifying assumptions, for the type of grid considered in the experiment, are justifiable.

The computer programme described in section 2-2 was used to analyse the test grid and the results are shown in tables 4-4-1 to 4-4-5. The dimensions and properties of the test grid used in the analysis are given in table 4-4-6. The torsional rigidity of the members, in this table, is obtained assuming that the end cross-sections can warp freely.

The experimental values of joint deflection in tables 4-4-1b to 4-4-5b can now be compared with the corresponding theoretical values in tables 4-4-1 to 4-4-5. As far as the bending moments are concerned, however, the values in tables 4-4-1 to 4-4-5 are not directly comparable with the experimental ones. The reason is that the experimental values correspond to mid-points of the members and the theoretical values represent the bending moments at the terminal points of the members. The comparison, however, is made easy by plotting the experimental values against the theoretical curves.
### TABLE 4-4 1

THEORETICAL RESULTS FOR TEST NO. (1)

(ONE TON APPLIED AT THE CENTRE)

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<th>DEFLECTION</th>
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**Theoretical Results for Test No. (2)**

*(One Ton Applied at the Centre)*

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Theoretical Results for Test No. (3)

(One ton applied at the centre)

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<th>Deflection</th>
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<td>In.</td>
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### Table 4-4-5

Theoretical results for test no. (5)

(one ton applied at the centre)

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<td>3</td>
<td>G</td>
<td>MODULUS OF ELASTICITY IN SHEAR</td>
<td>5151 ton/in.$^2$</td>
<td>STANDARD TABLES</td>
</tr>
<tr>
<td>4</td>
<td>J</td>
<td>TORSION CONSTANT OF THE CROSS-SECTION</td>
<td>0.03 in.$^4$</td>
<td>STANDARD TABLES</td>
</tr>
<tr>
<td>5</td>
<td>$GJ$</td>
<td>TORSIONAL RIGIDITY</td>
<td>154.53 ton-in.$^2$</td>
<td>3 &amp; 4</td>
</tr>
<tr>
<td>6</td>
<td>$EI^*$</td>
<td>BENDING RIGIDITY</td>
<td>22273 ton-in.$^2$</td>
<td>EXPERIMENT</td>
</tr>
<tr>
<td>7</td>
<td>$t_o$</td>
<td>TORSION COEFFICIENT $= \frac{GJ}{EI}$</td>
<td>0.00694</td>
<td>5 &amp; 6</td>
</tr>
<tr>
<td>8</td>
<td>$k_o$</td>
<td>SHEAR COEFFICIENT $= \frac{6E}{G.A. L^2}$</td>
<td>0.145</td>
<td>1 &amp; 2 &amp; 3 &amp; 6</td>
</tr>
</tbody>
</table>

*The nominal value of $EI$ calculated from the standard tables is equal to 22232 ton-in.$^2$.
These are shown in figs. 4-4-1 to 4-4-5. Furthermore, table 4-4-7 is designed to compare the first three largest experimental values of bending moment and deflection with the corresponding theoretical ones.

The correlation between the experimental and the theoretical results may in general be considered satisfactory, justifying the simplifying assumptions made in the analysis.

There are, however, some interesting points deserving attention. First of all, the experimental values (except for test No.2) tend to be less than the corresponding theoretical ones. This may be attributed to the following factors:

a) In the theoretical analysis the supports were assumed to be perfect ball supports. The actual supports, on the other hand, could have never been ideal since, forgetting about any other thing, the friction could not have been eliminated completely. Therefore, the supports offered some resistance to rotation whereas this effect was neglected in the analysis.

b) It was assumed that the grid can move freely in its own plane and thus no membrane stresses will be present. However, in the actual experiments, since the horizontal movement of the supporting joints was not completely free, some small membrane stresses must have developed and the structure was stiffer than it was assumed to be.

c) The analysis was based on the assumption that the joints have negligible volumes. But, the actual grid had comparatively large joints giving extra rigidity to the structure which had not been considered in the theoretical analysis.

d) It was assumed that the members are under pure torsion. However, the free warping of the end cross-sections of the members in the actual structure were partially restrained and, therefore, the torsional rigidities of the members were larger than the
TEST NO. (1)

BENDING MOMENT DIAGRAMS

DEFLECTION CURVES

THEORETICAL CURVE

EXPERIMENTAL VALUE

FIG. 4-4-1
FIG. 4-4-2

TEST NO. (2)

BENDING MOMENT DIAGRAMS

DEFLECTION CURVES

THEORETICAL CURVE

EXPERIMENTAL VALUE

sagging

0.0 0.1 0.2 0.3 0.4 IN.

10 T.IN.
TEST NO. (3)

BENDING MOMENT DIAGRAMS

DEFLECTION CURVES

THEORETICAL CURVE

EXPERIMENTAL VALUE

FIG. 4-4-3
TEST NO. (4)

BENDING MOMENT DIAGRAMS

DEFLECTION CURVES

--- THEORETICAL CURVE

- EXPERIMENTAL VALUE

FIG. 4-4-4
### TABLE 4-4-7

**Comparison between some experimental and the corresponding theoretical results.**

<table>
<thead>
<tr>
<th>Item</th>
<th>Test No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Largest Bending Moment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td></td>
<td>-2.58</td>
<td>-3.18</td>
<td>-5.46</td>
<td>-4.26</td>
<td>-4.49</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>-2.55</td>
<td>-3.41</td>
<td>-4.93</td>
<td>-4.30</td>
<td>-4.30</td>
</tr>
<tr>
<td>Difference %</td>
<td></td>
<td>-1.2</td>
<td>0.9</td>
<td>-3.4</td>
<td>0.9</td>
<td>2.6</td>
</tr>
<tr>
<td><strong>Second Largest Bending Moment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td></td>
<td>-1.78</td>
<td>-2.10</td>
<td>-2.27</td>
<td>-2.33</td>
<td>-3.49</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>-1.82</td>
<td>-2.45</td>
<td>-3.21</td>
<td>-3.71</td>
<td>-3.40</td>
</tr>
<tr>
<td>Difference %</td>
<td></td>
<td>4.2</td>
<td>3.5</td>
<td>7.2</td>
<td>1.2</td>
<td>0.9</td>
</tr>
<tr>
<td><strong>Third Largest Bending Moment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td></td>
<td>-1.78</td>
<td>-2.10</td>
<td>-2.27</td>
<td>-2.33</td>
<td>-3.49</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>-1.82</td>
<td>-2.45</td>
<td>-3.21</td>
<td>-3.71</td>
<td>-3.40</td>
</tr>
<tr>
<td>Difference %</td>
<td></td>
<td>4.2</td>
<td>3.5</td>
<td>7.2</td>
<td>1.2</td>
<td>0.9</td>
</tr>
<tr>
<td><strong>Largest Deflection</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td></td>
<td>0.095</td>
<td>0.143</td>
<td>0.471</td>
<td>0.1994</td>
<td>0.201</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>0.100</td>
<td>0.122</td>
<td>0.409</td>
<td>0.214</td>
<td>0.187</td>
</tr>
<tr>
<td>Difference %</td>
<td></td>
<td>4.9</td>
<td>7.2</td>
<td>4.1</td>
<td>5.7</td>
<td>5.1</td>
</tr>
<tr>
<td><strong>Second Largest Deflection</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td></td>
<td>0.598</td>
<td>0.731</td>
<td>0.445</td>
<td>0.1615</td>
<td>0.207</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>0.631</td>
<td>0.794</td>
<td>0.409</td>
<td>0.194</td>
<td>0.220</td>
</tr>
<tr>
<td>Difference %</td>
<td></td>
<td>5.2</td>
<td>7.4</td>
<td>4.4</td>
<td>4.4</td>
<td>5.3</td>
</tr>
<tr>
<td><strong>Third Largest Deflection</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td></td>
<td>0.598</td>
<td>0.494</td>
<td>0.448</td>
<td>0.124</td>
<td>0.158</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td>0.408</td>
<td>0.548</td>
<td>0.298</td>
<td>0.173</td>
<td>0.183</td>
</tr>
<tr>
<td>Difference %</td>
<td></td>
<td>2.4</td>
<td>6.4</td>
<td>2.4</td>
<td>3.0</td>
<td>4.8</td>
</tr>
</tbody>
</table>
assumed ones.

All the above items tend to indicate that it is reasonable to expect the experimental values be less than the theoretical ones. It is rather strange, therefore, that the results of test No. 1 do not conform with this general tendency. No satisfactory explanation for this phenomenon could be found.

Two other factors which could have been responsible for some of the discrepancies are:

a) Yielding of the Supports

The supports and the supporting beams were, of course, so rigid that any settlement of the supports must have been very small. Nevertheless, some percentage of the discrepancies could have been caused by these small settlements.

b) Distortion of the Test Grid

The theoretical analysis was based on a regular diagonal grid. The test grid itself, however, was not quite regular being slightly distorted. This distortion must have contributed to the discrepancies, but the nature and the magnitude of the effects are difficult to be estimated.
CHAPTER FIVE

CONCLUSIONS
Section 5-1. Introduction

This last chapter contains the final conclusions of the work presented in the thesis.

The preceding material may be divided into two distinct categories.

I) The work concerned with the effects of variations in torsional or shearing properties on the behavior of diagonal grids. Chapter two contains the main body of this work which is supported by the experimental results of chapter four. Some parts of chapter three are also concerned with matters relevant to this category. The conclusions corresponding to this type of work are given in Section 5-2.

II) The work leading to the formulation of a technique by which the behavior of grids under varying torsional or shearing properties may be extrapolated (or interpolated). The material concerned with this category of work is presented in chapter three and is further discussed in section 5-3.

Section 5-2. Effects of Torsional and Shearing Rigidities

The results of the analysis given in section 2-4 may be used to establish a number of interesting facts regarding the effects of torsional or shearing rigidities.

To ease the comparison of the results, the percentage changes in the maximum components of deflection, bending moment and shearing force due to the increase of $t_0$ from zero to 0.75 (or the increase of $k_0$ from zero to 1.5) for all the analyzed cases are shown in table 5-2-1. Also, to obtain some idea of the relative importance of torques, the maximum components of torque (for $t_0 = 0.75$) in terms of the percentage of the corresponding maximum components of bending moment (for $t_0 = 0.0$) are given in the last
<table>
<thead>
<tr>
<th>Case No.</th>
<th>Percentage change in the maximum component</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>t = 0.75, k = 0.0</td>
</tr>
<tr>
<td></td>
<td>Def.</td>
</tr>
<tr>
<td>1-A-W</td>
<td>-13.9</td>
</tr>
<tr>
<td>2-A-W</td>
<td>-14.3</td>
</tr>
<tr>
<td>3-A-W</td>
<td>-18.2</td>
</tr>
<tr>
<td>4-A-W</td>
<td>-20.5</td>
</tr>
<tr>
<td>5-A-W</td>
<td>-21.0</td>
</tr>
<tr>
<td>1-A-P</td>
<td>-13.7</td>
</tr>
<tr>
<td>2-A-P</td>
<td>-16.8</td>
</tr>
<tr>
<td>3-A-P</td>
<td>-18.4</td>
</tr>
<tr>
<td>4-A-P</td>
<td>-19.0</td>
</tr>
<tr>
<td>5-A-P</td>
<td>-19.4</td>
</tr>
<tr>
<td>1-B-P</td>
<td>-24.4</td>
</tr>
<tr>
<td>2-B-P</td>
<td>-20.2</td>
</tr>
<tr>
<td>3-B-P</td>
<td>-16.9</td>
</tr>
<tr>
<td>4-B-P</td>
<td>-14.4</td>
</tr>
<tr>
<td>5-B-P</td>
<td>-12.6</td>
</tr>
<tr>
<td>1-B-P</td>
<td>-19.5</td>
</tr>
<tr>
<td>2-B-P</td>
<td>-17.1</td>
</tr>
<tr>
<td>3-B-P</td>
<td>-15.2</td>
</tr>
<tr>
<td>4-B-P</td>
<td>-14.8</td>
</tr>
<tr>
<td>5-B-P</td>
<td>-12.6</td>
</tr>
<tr>
<td>1-C-W</td>
<td>-19.8</td>
</tr>
<tr>
<td>2-C-W</td>
<td>-18.0</td>
</tr>
<tr>
<td>3-C-W</td>
<td>-15.6</td>
</tr>
<tr>
<td>4-C-W</td>
<td>-14.5</td>
</tr>
<tr>
<td>5-C-W</td>
<td>-11.8</td>
</tr>
<tr>
<td>1-C-P</td>
<td>-16.7</td>
</tr>
<tr>
<td>2-C-P</td>
<td>-16.6</td>
</tr>
<tr>
<td>3-C-P</td>
<td>-14.5</td>
</tr>
<tr>
<td>4-C-P</td>
<td>-13.2</td>
</tr>
<tr>
<td>5-C-P</td>
<td>-12.2</td>
</tr>
<tr>
<td>1-D-W</td>
<td>-8.3</td>
</tr>
<tr>
<td>2-D-W</td>
<td>-12.2</td>
</tr>
<tr>
<td>3-D-W</td>
<td>-14.7</td>
</tr>
<tr>
<td>4-D-W</td>
<td>-16.4</td>
</tr>
<tr>
<td>5-D-W</td>
<td>-17.7</td>
</tr>
<tr>
<td>1-D-P</td>
<td>-8.6</td>
</tr>
<tr>
<td>2-D-P</td>
<td>-12.5</td>
</tr>
<tr>
<td>3-D-P</td>
<td>-14.9</td>
</tr>
<tr>
<td>4-D-P</td>
<td>-16.6</td>
</tr>
<tr>
<td>5-D-P</td>
<td>-17.9</td>
</tr>
</tbody>
</table>

$$\varepsilon = \left( \frac{\text{maximum torque for } t = 0.75}{\text{maximum B.M. for } t = 0.0} \right) \times 100$$
column of Table 5-2-1. In this table, except for the figures in the last column which are given in absolute value, a negative sign denotes that the change is a decrease in modulus, and a positive sign denotes that the change is an increase in modulus.

Table 5-2-1 classifies the most important pieces of information accessible from the results of analysis and provides a suitable medium from which general patterns of behaviour, regarding the effects of torsional or shearing rigidities, may be traced. Table 5-2-1, however, cannot be used to study the detailed behaviour of any particular case and for this purpose the relevant diagrams of section 2-6 are to be consulted.

A thorough examination of the results given in section 2-6 and Table 5-2-1, reveals a number of interesting points which are described in Table 5-2-2. It is to be noted, however, that the statements and figures in Table 5-2-2 are applicable to the maximum components of force and displacement only and their validity for components other than maximum cannot be taken for granted.

Some of the facts given in Table 5-2-2 are easy to justify. For instance, it is mentioned that an increase in the torsional rigidities has the effect of decreasing the deflections. But, this is as to be expected because an increase in the torsional rigidities will render the structure stiffer and hence the deflections are decreased. Some other statements in Table 5-2-2, however, are not easy to justify. For instance, when a diagonal grid is under uniformly distributed loading, an increase in the torsional rigidities will cause the shearing forces to decrease for boundary conditions A and D and to increase for boundary conditions B and C. This should, presumably, have some sort of logical explanation but no reasoning of simple nature could be used to justify it.

Some points of especial interest, concerned with the statements given in Table 5-2-2, are listed below:
### Table 5.2.2

<table>
<thead>
<tr>
<th>EFFECTS OF TORSIONAL STIFFNESS</th>
<th>EFFECTS OF BENDING STIFFNESS</th>
</tr>
</thead>
<tbody>
<tr>
<td>An increase in the value of ( k_0 ) has the following effects:</td>
<td>An increase in the value of ( k_0 ) has the following effects:</td>
</tr>
<tr>
<td>1. Deflections are always decreased (up to about 25%).</td>
<td>1. Deflections are always increased (up to about 25%).</td>
</tr>
<tr>
<td>2. Bending moments are always decreased (up to about 25%).</td>
<td>2. Bending moments are always increased (up to about 25%).</td>
</tr>
<tr>
<td>Shearing forces are:</td>
<td>Shearing forces are:</td>
</tr>
<tr>
<td>a. Usually increased for uniformly distributed loading with boundary conditions A&amp;D (up to about 25%).</td>
<td>a. Usually decreased for uniformly distributed loading with boundary conditions A&amp;D (up to about 25%).</td>
</tr>
<tr>
<td>b. Usually increased for uniformly distributed loading with boundary conditions B&amp;C (up to about 25%).</td>
<td>b. Usually left unchanged for a central concentrated load.</td>
</tr>
<tr>
<td>c. Always left unchanged for a central concentrated load.</td>
<td>c. Always left unchanged for uniformly distributed loading with boundary conditions B&amp;C (up to about 15%).</td>
</tr>
<tr>
<td>Forces are always increased (up to about 50% of the corresponding maximum bending moment for ( k_0 = 0.0 )).</td>
<td>Forces are always increased (up to about 50% of the corresponding maximum bending moment for ( k_0 = 0.0 )).</td>
</tr>
<tr>
<td>For the same ( k_0 ), an increase in the density of the layout has the following effects:</td>
<td>For the same ( k_0 ), an increase in the density of the layout has the following effects:</td>
</tr>
<tr>
<td>1. The percentage changes in the deflections are:</td>
<td>1. The percentage changes in the deflections are always decreased.</td>
</tr>
<tr>
<td>a. Always increased for boundary conditions A&amp;D.</td>
<td>a. Always increased for boundary conditions A&amp;D.</td>
</tr>
<tr>
<td>b. Always decreased for boundary conditions B&amp;C.</td>
<td>b. Always increased for uniformly distributed loading with boundary conditions A&amp;D.</td>
</tr>
<tr>
<td>2. The percentage changes in the bending moments are:</td>
<td>2. The percentage changes in the bending moments are:</td>
</tr>
<tr>
<td>a. Always increased for boundary conditions A&amp;D.</td>
<td>a. Always increased for boundary conditions A&amp;D.</td>
</tr>
<tr>
<td>b. Usually increased for a central concentrated load with boundary conditions B&amp;C.</td>
<td>b. Usually increased for uniformly distributed loading with boundary conditions B&amp;C.</td>
</tr>
</tbody>
</table>

continued in the next page.
<table>
<thead>
<tr>
<th>TABLE</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The percentage changes in the shearing forces for uniformly distributed loading are usually decreased for the other cases.</td>
<td>The percentage changes in the bending moments usually change sign.</td>
<td>The percentage changes in the shearing forces are always satisfied.</td>
</tr>
<tr>
<td>The percentage changes in the bending moments are always increased.</td>
<td>The percentage changes in the deflections are always increased.</td>
<td>The percentage changes in the shearing forces are always satisfied.</td>
</tr>
<tr>
<td>The percentage changes in the bending moments usually change sign.</td>
<td>The percentage changes in the shearing forces are always satisfied.</td>
<td>The percentage changes in the shearing forces are always satisfied.</td>
</tr>
<tr>
<td>Relative to the corresponding uniform loading case.</td>
<td>Relative to the deflections at the supports.</td>
<td>Relative to the corresponding uniform loading case.</td>
</tr>
</tbody>
</table>

* Due to inclusion of the shearing effects in the analysis.

† Due to inclusion of the shearing effects in the analysis.

‡ Relative to the corresponding uniform loading case.
1) The fact that the inclusion of shearing rigidities in the analysis could increase the deflections appreciably is well known for many years; see, for instance, [1]. The magnitudes of these increases for some of the cases considered in the thesis, however, are beyond all previously imagined limits. Namely, increases up to 150% (i.e., the original deflection being almost tripled) are observed and yet the worst is to come, since based on the discussion in section 4-5, increases up to 500% are to be expected.

2) The effects of shearing rigidities on the values of internal forces are not as trivial as they are commonly believed to be.

3) The relative importance of torques when the external loads are uniformly distributed is much more than the cases when the external loads are concentrated and this remains true for all the layouts and boundary conditions. However, while the relative importance of torques for uniformly distributed loading varies from layout to layout and from boundary condition to boundary condition, the relative importance of torques for a central concentrated load is almost the same for all the layouts and boundary conditions with the ratio of the maximum component of torque (for $t_o = 0.75$) and the corresponding maximum component of bending moment (for $t_o = 0.0$) remaining about 0.15.

At this point, an important question may be raised. Namely, are the facts given in table 5-2-2 applicable to diagonal grids of all shapes and descriptions? The answer is that, strictly speaking, there is no guarantee that any of the statements of table 5-2-2 be true for a diagonal grid outside the cases considered. We may, however, speculate to say that among the facts given in table 5-2-2 there are a few which are not dependent on any particular boundary condition and/or loading case, these are likely to remain true for any diagonal grid or even for any other type of flat grid. The other statements which are dependent on particular boundary
conditions and/or loading cases and all the figures giving the maximum percentage changes, on the other hand, are not to be trusted outside the cases considered.

Based on all the preceding material, it may finally be concluded that:

The variations in the torsional or shearing properties of a diagonal grid may have significant effects on the values and distribution of the internal forces and displacements and hence for an efficient design the search for the most favourable values of torsion and shear coefficients should be included in any optimization process.

Section 3-4. Structural Analysis and the Concept of Norms

In chapter three, based on the concept of norms, a technique was developed by which the changes in the internal forces and displacements of a linear structure, produced by variations in some features of the structure, could be estimated. As an example, the technique was applied to the case of flat grids, with varying torsional or shearing properties. The scope of the technique, however, is by no means limited to this example and many other structural problems could be treated on similar lines. Furthermore, in the case of flat grids, the changing features were considered one at a time (i.e., either torsional properties or shearing properties). This, however, is not a basic requirement of the technique and any combination of varying features could be considered together. For instance, the formulation of the problem concerning the combined effects of varying torsional and shearing properties in a flat grid, could be done as follows:

From section 3-4, the modification matrix representing the combined variation of torsional and shearing properties is given by:
\[ M = t_0 D + \left( \frac{k_0}{1 + 2k_0} \right) D' \]

or,
\[ k^{-1}M = t_0 k^{-1}D + \left( \frac{k_0}{1 + 2k_0} \right) k^{-1}D' \]

or,
\[ \|k^{-1}M\| \leq t_0 \|k^{-1}D\| + \left( \frac{k_0}{1 + 2k_0} \right) \|k^{-1}D'\| \]

and finally,
\[ \|k^{-1}M\| \leq t_0 \beta + \frac{k_0 \gamma}{1 + 2k_0} \quad \ldots \ldots \quad 5-5-1 \]

where the notation is similar to that used in section 5-4. From relations 5-5-4 and 5-5-1
\[ \frac{\|k\|}{\|d + g\|} \leq t_0 \beta + \frac{k_0 \gamma}{1 + 2k_0} \quad \ldots \ldots \quad 5-5-2 \]

Now, relation 5-5-2 is analogous to relations 5-5-1 and could be used in conjunction with the postulate of section 3-5 to derive equations for estimate of forces and displacements.

In general, for a linear structure with varying features (subject to the restrictions listed in section 3-3), the estimate of forces and displacements could be obtained through the following steps:

Step 1) The concept of vector and matrix norms is used to find inequalities containing the ratios \( \|g\|/\|d\| \) or \( \|g\|/\|d + g\| \) and variables representing the changing features of the structure (e.g., "feature variables").

Step 2) A generalized version of the postulate of section 3-5 is used to find equations expressing the displacements in terms of feature variables and a number of constants.
Step 3: The force-displacement relations are used to express the internal forces in terms of feature variables and constants.

Step 4: The structure is analysed for a number of basic cases to evaluate the constants in the equations derived in steps 1 and 2.

The above account of the technique is, of course, no more than a general guide and for each particular application a detailed formulation is required. In each case, however, the main question to be answered is whether, for the class of problems under consideration, a postulate similar to the one given in section 4-5 could be used to obtain reliable results.

The relations obtained in step 1 could also be used as upper bounds for the variations in the displacements. The ease of derivation and the usefulness of such relations, however, vary from case to case and the matter is discussed in section 4-5.

The technique for estimating structural behaviour finds its immediate use in the optimization processes. Any structural optimization process is, evidently, concerned with the determination of the values of feature variables which result in the most favourable design. Such an end may be achieved by utilizing the technique developed in the thesis. To wit, based on the results of a relatively small number of basic analyses, the internal forces and displacements of a structure, corresponding to a large number of different values of the feature variables, could easily be estimated and the most favourable mode can then be found from a direct comparison of the results.

We may conclude that, if the generalized version of the postulate of section 4-5 is of general reliability, a powerful technique for optimization in design problems is found.

Section 5-4. Suggestions for Future Research

The natural extension of the work presented in this thesis
branches in two different fields.

The first is concerned with investigations regarding the effects of torsional and shearing rigidities on the behaviour of many more families of flat grids, so that the knowledge of the patterns of behaviour could be gradually improved.

The second and a much more important one is concerned with detailed formulation of the technique developed in chapter three for different structural problems and investigations regarding the reliability of the results.

- THE END -
## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>cross-sectional area; code word representing the boundary condition when all the boundary joints of a grid are completely fixed; any square matrix</td>
</tr>
<tr>
<td>A_0</td>
<td>area of the web of an I-section</td>
</tr>
<tr>
<td>a</td>
<td>coefficient</td>
</tr>
<tr>
<td>a_n, b_n, and a_p</td>
<td>increments of external load</td>
</tr>
<tr>
<td>a_{ij}</td>
<td>typical element of a matrix A</td>
</tr>
<tr>
<td>B</td>
<td>width of a cross-section; code word representing the boundary condition when all the boundary joints of a grid are knife edged parallel to the boundary line; any square matrix</td>
</tr>
<tr>
<td>B.B.</td>
<td>abbreviation for Broad Flange Beam</td>
</tr>
<tr>
<td>B.M.</td>
<td>abbreviation for bending moment</td>
</tr>
<tr>
<td>B.S.A.</td>
<td>abbreviation for British Standard Aluminium Beam</td>
</tr>
<tr>
<td>B.S.B.</td>
<td>abbreviation for British Standard Beam</td>
</tr>
<tr>
<td>b</td>
<td>a typical member of a structure</td>
</tr>
<tr>
<td>b_1, b_2, b_3, and b_4</td>
<td>increments of reading of a strain gauge or a dial gauge</td>
</tr>
<tr>
<td>C</td>
<td>code word representing the boundary condition when all the boundary joints of a grid are restricted from translation but can rotate freely</td>
</tr>
<tr>
<td>D</td>
<td>code word representing the boundary condition when four corner joints of a grid are restricted from translation but can rotate freely; square matrix; abbreviation for deflection</td>
</tr>
</tbody>
</table>
$D', D_b$ and $D^*_b$ square matrices

Def. abbreviation for deflection

d displacement vector of a structure; abbreviation for deflection

$d^*$ the $i^{th}$ component of the displacement vector of a structure

d^o_{i}$ as $d_t$ but when $t_o = k_o = C$

d^*_{i}$ as $d_t$ but when $t_o = T$ and $k_o = C$ or when $t_o = C$ and $k_o = \lambda$

$d_i(d^*_j)$ displacement vector of a joint $i$ ($j$) relative to the frame coordinate system

$D_{ij}(D^*_{ji})$ end displacement vector at end $i$ ($j$) of a member $ij$
relative to the member coordinate system

$D^*_{ij}(D^*_{ji})$ end displacement vector at end $i$ ($j$) of a member $ij$
relative to the frame coordinate system

E modulus of elasticity in tension and compression

e error in a strain gauge or a dial gauge reading; coefficient

G modulus of elasticity in shear

$\delta$ a vector representing the changes in the displacement vector
of a structure due to the modification of the structure

h depth of a cross-section

$k_b$ equilibrium matrix of a member $b$ relative to the member
coordinate system

I distance between the center lines of the flanges of an I-section

$I$ second moment of area of a cross-section; a unit matrix

$I_r$ second moment of area of one flange of an I-section about
the minor principal axis of the cross-section

IN and in abbreviations for inch

i a typical digit; subscript
torsion constant of a cross-section

analogous torsion constant of an I-section when the effects of warping restraint are considered

a typical joint; subscript

stiffness matrix of a structure

basic stiffness matrix of a member

basic stiffness matrix of a member after modification of the member

(\(K_{11}\), \(K_{12}\), \(K_{21}\), and \(K_{22}\)) stiffness matrices of a member relative to the member coordinate system

(\(K'_{11}\), \(K'_{12}\), \(K'_{21}\), and \(K'_{22}\)) stiffness matrices of a member relative to the frame coordinate system

(\(K''_{11}\), \(K''_{12}\), \(K''_{21}\), and \(K''_{22}\)) stiffness matrices of a member relative to the frame coordinate system after modification of the member

subscript

shear coefficient \(k_c = 6aEI/GAL^2\)

length of a member

length of one side of a square diagonal grid

modification matrix of a structure; abbreviation for bending moment

modification matrix of a structure due to the modification of a member

abbreviation for bending moment

the maximum with respect to variation in the integer variable \(i\)

bending moment at end \(i\) \((j)\) of a member \(ij\)
\( m_{io} \) as \( m_{i} \) but when \( t_{o} = k_{o} = 0 \)

\( m_{il} \) as \( m_{i} \) but when \( t_{o} = \mathcal{T} \) and \( k_{o} = 0 \) or when \( t_{o} = 0 \) and \( k_{o} = \lambda \)

\( n \) an integer; dimension of a vector; order of a square matrix; dimension of a Euclidean space

\( C \) and \( Q \) origins of Cartesian coordinate systems

\( \mathbf{P} \) code word representing the external loading system consisting of a vertical concentrated load applied to the central joint of a grid; magnitude of a concentrated load

\( P_{i} \) a typical component of bending moment or shearing force

\( P_{io} \) as \( P_{i} \) but when \( t_{o} = k_{o} = 0 \)

\( P_{il} \) as \( P_{i} \) but when \( t_{o} = \mathcal{T} \) and \( k_{o} = 0 \) or when \( t_{o} = 0 \) and \( k_{o} = \lambda \)

\( P_{ij}(p_{ji}) \) end force vector at end \( i(j) \) of a member \( ij \) relative to the member coordinate system

\( P_{ij}(p_{ji}) \) end force vector at end \( i(j) \) of a member \( ij \) relative to the frame coordinate system

\( q \) magnitude of a concentrated external load

\( q_{b} \) modification matrix of a member \( b \)

\( q \) intensity of a uniformly distributed load

\( q_{i}(q_{j}) \) torque at end \( i(j) \) of a member \( ij \)

\( q_{io} \) as \( q_{i} \) but when \( t_{o} = k_{o} = 0 \)

\( q_{il} \) as \( q_{i} \) but when \( t_{o} = \mathcal{T} \) and \( k_{o} = 0 \)

\( q_{i2} \) as \( q_{i} \) but when \( t_{o} = \mathcal{T} \) and \( k_{o} = \lambda \)

\( R \) abbreviation for rotation

\( \mathbf{R.C.} \) abbreviation for reinforced concrete

\( r \) radius of the re-entrant corner of an I-section; a real positive scalar
S, S, F. and s abbreviations for shearing force

T matrix transposition sign; abbreviation for torque; abbreviation for ton

\( T_b \) transformation matrix by which vectors relative to the coordinate system of a member \( b \) are transformed to the frame coordinate system

\( t_o \) torsion coefficient \( (t_o = 2 M/I) \)

\( t_f \) thickness of a flange of an I-section

\( t_w \) thickness of the web of an I-section

UB abbreviation for Universal Beam

U.D.L. abbreviation for uniformly distributed load

\( v_i(v_j) \) shearing force at end \( i \) \((j)\) of a member \( ij \)

\( v_{10} \) as \( v_1 \) but when \( t_o = k_o = 0 \)

\( w \) intensity of uniformly distributed loading; code word representing the external loading system consisting of uniformly distributed load over the whole surface of a grid

\( w_i \) external load vector of a structure

\( w_i \) external load vector of a joint \( i \) relative to the frame coordinate system

\( x \) a member coordinate axis; any column vector

\( x' \) a frame coordinate axis

\( x_i(x_k) \) the \( i^{th} \) \((k^{th})\) component of a column vector \( x \)

\( y \) a member coordinate axis; any column vector

\( y' \) a frame coordinate axis

\( z \) a member coordinate axis

\( z' \) a frame coordinate axis
shear constant of a cross-section; angle; any real scalar
\(a_0, a_1, a_2, a_3, a_4, a_5\) and \(a_6\) coefficients
\(\beta\) and \(\beta_1\) coefficients
\(\gamma, \gamma_1\) and \(\gamma_{11}\) coefficients
\(b\) and \(b_0\) deflections
\(\delta_i(b_j)\) deflection at end \(i\) \((j)\) of a member \(ij\)
\(\delta_{10}(c_{jo})\) as \(\delta_i(b_j)\) but when \(t_0 = k_0 = 0\)
\(\eta\) coefficient
\(\theta_i(\theta_{ij})\) rotation at end \(i\) \((j)\) of a member \(ij\)
\(\theta_{10}(\theta_{jo})\) as \(\theta_i(\theta_{ij})\) but when \(t_0 = k_0 = 0\)
\(\Delta\) an increment of the shear coefficient \(k_0\)
\(\gamma\) Poisson's ratio; coefficient
\(f\) coefficient
\(\tau, \tau_1\) and \(\tau_0\) increments of the torsion coefficient \(t\)
\(\phi\) twist
\(\phi_i(p_j)\) twist at end \(i\) \((j)\) of a member \(ij\)
\(\phi_{10}(p_{jo})\) as \(\phi_i(p_j)\) but when \(t_0 = k_0 = 0\)
| a real scalar | absolute value of a real scalar |
| a column vector | norm of a column vector |
| a square matrix | norm of a square matrix |
List of References


APPENDIX

A COPY OF THE COMPUTER PROGRAMME (IN SIRIUS AUTOCODE) FOR ANALYSIS OF FLAT GRIDS
H. NOUSHIN,

ANALYSIS OF FLAT RIDS,

15TH MARCH 1964.

\[ v = 1507 \]
\[ T_5 \]
\[ S_6 = 17-14 \]
\[ S_5 = 10-10 \]
\[ T_5 \]
\[ V_100 = 0 \]
\[ V_30 = -3 \]
\[ n_{27} = TAPE \]
\[ v_{101} = TAPE \]
\[ n_{25} = TAPE_2 \]
\[ v_{42} = TAPE_2 \]
\[ v_{41} = 0 \]
\[ no = 0 \]
\[ (42+no) = v(42+no) + v(41+no) \]
\[ no = no + 1 \]
\[ +e_9, no = n_{26} \]
\[ n_{24} = v(41+n_{26}) \]
\[ no = 0 \]
\[ n_1 = n_{24} \times n_{24} \]
\[ n_1 = n_1 + 191 \]
\[ 70) v(121+no) = 0 \]
\[ no = no + 1 \]
\[ +e_9, no = n_1 \]
\[ n_{24} = 0 \]
\[ n_{23} = 0 \]
\[ 20) v_{113} = 0 \]
\[ v_{114} = 0 \]
\[ v_{115} = 0 \]
\[ v_{72} = TAPE_2 \]
\[ n_{21} = v_{72} \]
\[ n_{20} = v_{73} \]
\[ n_{19} = 0 \]
\[ n_{18} = 0 \]
\[ 27) n_{17} = v(75+n_{19}) \]
\[ n_{10} = v(70+n_{19}) \]
\[ n_{15} = v(77+n_{19}) \]
\[ n_{14} = v(78+n_{19}) \]
\[ v_{20} = 1 \]
\[ v_{19} = 1 \]
\[ v_{18} = 1 \]
\[ v_{17} = v_{101} \]
\[ v_{10} = 1 \]
PROGRAMME CONT. (1)

1) 2) \( v_{10} = v_{102} \)
28, \( n_1 = 1 \)
29
3) 4) \( v_{16} = -v_{102} \)
28, \( n_1 = 3 \)
29
5) 6) \( v_{20} = 2x_{v_{101}} \)
\( v_{17} = 1 \)
\( v_{16} = 0 \)
28, \( n_1 = 5 \)
29
7) 8) \( v_{20} = v_{101} \)
\( v_{17} = 1 \)
\( v_{16} = 0 \)
28, \( n_1 = 7 \)
29
9) 10) \( v_{20} = v_{102} \)
\( v_{17} = 0 \)
28, \( n_1 = 9 \)
29
11) 12) \( v_{20} = 2x_{v_{102}} \)
\( v_{17} = 0 \)
28, \( n_1 = 11 \)
29
13) \( v_{19} = -1 \)
\( v_{18} = -1 \)
\( v_{16} = v_{102} \)
29
14) \( v_{19} = -1 \)
\( v_{16} = v_{102} \)
29
15) \( v_{19} = -1 \)
\( v_{16} = -v_{102} \)
28
16) \( v_{18} = -1 \)
\( v_{16} = v_{102} \)
29
17) \( v_{18} = -1 \)
\( v_{16} = -v_{102} \)
29
18) \( v_{20} = 2x_{v_{101}} \)
\( v_{16} = -1 \)
\( v_{17} = 1 \)
\( v_{16} = 0 \)
29
19) \( v_{20} = v_{102} \)
PROGRAMME CUNT (2)

V1 9 = -1
V1 7 = 0
+29
20) V20 = 2x V1 02
V1 9 = -1
V1 7 = 0
+29
21) V1 6 = - V1 02
+28
22) V20 = 2x V1 02
V1 7 = 0
+29
23) V20 = 2x V1 01
V1 7 = 1
V1 6 = 0
+28
24) V20 = 2x V1 01
V1 8 = -1
V1 9 = -1
V1 7 = 1
V1 6 = 0
+29
25) V20 = V1 02
V1 7 = 0
+29
26) V40 = 1
+30
29) V40 = -1
30) V39 , m27 = 0
V1 = TAFE 3
V110 = V103x V1
V111 = V104x V2
V112 = V105x V3
+40
39) V110 = V103
V111 = V104
V112 = V105
40) V112 = V112/V40
V112 = V112/V20
V2 = 2x V1 12
V2 = 1 + V2
V0 = 1 - V1 12
V0 = V0 / V2
V1 = 2 + V1 12
V1 = V1 / V2
V2 = 3 / V2
V3 = V1 7x V1 7
PROGRAMME CUNT. (3)

\[ U_4 = U_1 \times V_{16} \]
\[ U_5 = U_1 \times V_{17} \]
\[ U_6 = U_{110} \times V_{111} \]
\[ U_7 = U_{107} \times V_{20} \]
\[ U_8 = U_6 / U_7 \]
\[ U_9 = U_7 \times V_7 \]
\[ U_9 = U_{111} / U_7 \]
\[ U_8 = U_{111} / U_8 \]
\[ U_9 = U_{111} / U_9 \]
\[ U_7 = 2 \times U_7 \]
\[ U_8 = 2 \times U_8 \]
\[ U_9 = 4 \times U_9 \]
\[ U_{10} = U_7 \times U_0 \]
\[ U_7 = U_7 \times V_1 \]
\[ U_8 = U_8 \times V_2 \]
\[ U_9 = U_9 \times U_2 \]
\[ U_{11} = U_6 \times U_3 \]
\[ U_{12} = U_4 \times U_7 \]
\[ U_{21} = U_{11} + U_{12} \]
\[ U_{22} = U_6 - U_7 \]
\[ U_{22} = U_{11} \times U_5 \]
\[ U_{13} = U_1 \times U_8 \]
\[ U_{23} = U_4 \times U_{11} \]
\[ U_{24} = U_2 \times U_2 \]
\[ U_{11} = U_7 \times U_3 \]
\[ U_{12} = U_4 \times U_6 \]
\[ U_{25} = U_{11} + U_{12} \]
\[ U_{11} = U_1 \times U_8 \]
\[ U_{20} = U_4 \times U_{11} \]
\[ U_{27} = U_{23} \]
\[ U_{28} = U_{20} \]
\[ U_{29} = U_9 \]
\[ U_{11} = U_6 \times U_3 \]
\[ U_{12} = U_1 \times U_4 \]
\[ U_{31} = U_{11} - U_{11} \]
\[ U_{31} = U_3 \times U_{11} \]
\[ U_{32} = U_6 + U_{10} \]
\[ U_{32} = - U_3 \times U_5 \]
\[ U_{32} = U_3 \times U_{19} \]
\[ U_{33} = U_2 \times U_19 \]
\[ U_{34} = U_32 \times U_{16} \]
\[ U_{34} = U_3 \times U_{19} \]
\[ U_{11} = U_1 \times U_3 \]
\[ U_{12} = U_4 \times U_0 \]
\[ U_{35} = U_{11} - U_{12} \]
\[ U_{35} = U_3 \times U_{18} \]
PROGRAMME CONT. (4)

\[ v_{30} = v_{26} \times v_{18} \]
\[ v_{37} = v_{27} \]
\[ v_{38} = v_{28} \]
\[ v_{39} = v_{29} \]
\[ \rightarrow 50, 20 \times 17 \]
\[ v_{1} = v_{31} \]
\[ v_{2} = v_{32} \]
\[ v_{3} = v_{33} \]
\[ v_{31} = v_{34} \]
\[ v_{32} = v_{35} \]
\[ v_{33} = v_{36} \]
\[ v_{34} = v_{37} \]
\[ v_{35} = v_{38} \]
\[ v_{36} = v_{39} \]
\[ 50 \rightarrow v_{2} = 1 \]
\[ \rightarrow 14 \]
\[ 41 \rightarrow v_{0} = 12 \]
\[ +51 \]
\[ 42 \rightarrow v_{0} = 18 \]
\[ +51 \]
\[ 43 \rightarrow v_{0} = 14 \times 4 \]
\[ +51 \]
\[ 44 \rightarrow v_{0} = 36 \]
\[ +51 \]
\[ 45 \rightarrow v_{0} = 6 \]
\[ +51 \]
\[ 46 \rightarrow v_{0} = 12 \]
\[ +51 \]
\[ 47 \rightarrow v_{0} = 24 \]
\[ +51 \]
\[ 48 \rightarrow v_{0} = 72 \]
\[ +51 \]
\[ 49 \rightarrow v_{0} = 1 \]
\[ v_{2} = 0 \]
\[ 51 \rightarrow v_{1} = v_{107} \times v_{107} \]
\[ v_{1} = v_{1} \times v_{100} \]
\[ v_{3} = v_{101} \times v_{102} \]
\[ v_{1} = v_{1} \times v_{1} \]
\[ v_{1} = v_{1} / v_{c} \]
\[ v_{1} = v_{1} \times 2 \]
\[ v_{2} = v_{107} \times v_{20} \]
\[ v_{2} = v_{1} \times v_{2} \]
\[ v_{2} = -v_{2} \times v_{40} \]
\[ v_{3} = v_{2} \times v_{1} \]
\[ v_{4} = -v_{2} \times v_{17} \]
\[ v_{5} = v_{1} \times 6 \]
\[ \rightarrow 68, v_{100} > 50 \]
PROGRAMME CONT. (5)

\[ v_6 = u_{74}/u_{73} \]
\[ v_5 = u_5 + v_6 \]
\[ v_{113} = u_{113} + v_3 \]
\[ v_{114} = u_{114} + v_4 \]
\[ v_{115} = u_{115} + v_5 \]
\[ n_{12} = 1 \]
\[ n_{13} = 0 \]
\[ -59 \]
\[ 52 \] \[ n_{11} = 1 \]
\[ n_{9} = n_{10} \]
\[ 53 \] \[ -54, n_{12} = 1 \]
\[ -55 \]
\[ 54 \] \[ v_1 = u(41 + n_{26}) + 1 \]
\[ v_2 = u(41 + n_{23}) \]
\[ -56 \]
\[ 55 \] \[ v_2 = u(40 + n_{16}) \]
\[ -52 \] \[ v_1 = n_8 \]
\[ -56 \] \[ v_1 = v_1 - 1 \]
\[ v_1 = v_1 + v_1 \]
\[ v_2 = v_2 + v_1 \]
\[ n_7 = v_2 \]
\[ n_6 = 3 \times n_4 \]
\[ n_6 = n_6 - 3 \]
\[ n_5 = n_9 + n_6 \]
\[ n_5 = n_5 + n_{13} \]
\[ v(310 + n_7) = v(310 + n_7) + v(30 + n_5) \]
\[ n_{11} = n_{11} + 1 \]
\[ n_9 = n_9 + n_3 \]
\[ -53, n_{11} = n_3 \]
\[ n_{8} = n_{8} + 1 \]
\[ n_4 = n_4 + n_1 \]
\[ -52, n_{8} = n_{10} \]
\[ -57, n_{12} = 2 \]
\[ n_{12} = n_{12} + 1 \]
\[ n_{13} = 0 \]
\[ -59 \]
\[ 57 \] \[ n_{19} = n_{19} + 4 \]
\[ n_{18} = n_{18} + 1 \]
\[ -57, n_{18} = n_{20} \]
\[ n_{14} = 1 \]
\[ -62 \]
\[ 58 \] \[ n_{23} = n_{23} + 1 \]
\[ -26, n_{23} = n_{26} \]
\[ -64 \]
PROGRAMME CONT.(6)

59) \(n_0 = 3\)
\(n_4 = 1\)
\(n_1 = 1\)
\(\rightarrow n_{21}, n_{12} = 1\)
\(\rightarrow n_{15}\)

60) \(n_0 = 4\)
\(n_{61} + n_{12} = 1\)
\(n_8 = 1\)
\(\rightarrow 52\)

61) \(n_2 = n_0\)
\(n_{10} = n_4\)
\(n_4 = n_0\)
\(n_8 = 1\)
\(\rightarrow 52\)

62) \(n_4 = 2\)
\(\rightarrow 60\)

63) \(n_4 = 2\)
\(\rightarrow 60\)

64) \(n_4 = 3\)
\(\rightarrow 60\)

65) \(\rightarrow 57\)

66) \(v_{30} = v_{30} + 3\)
\(n_0 = v_{30}\)
\(n_0 = n_0 + n_{10}\)

67) \(v(241 + n_{22}) = v(112 + n_{10})\)
\(v(120 + n_0) = 2\)
\(n_{14} = n_{14} + 1\)
\(n_{22} = n_{22} + 1\)
\(n_{10} = n_{10} + n_3\)
\(n_0 = n_0 + n_3\)
\(\rightarrow 63, n_{14} = n_2\)
\(\rightarrow 58\)

68) \(n_2 = 2\)
\(n_1 = n_{24}\)
\(n_2 = 1\)
\(n_3 = 311\)
\(n_4 = 241\)
\(\rightarrow S_{14}\)
\(n_0 = 3 \times n_{25}\)
\(n_1 = 0\)
PROGRAMME CONT. (7)

\[ n_2 = 0 \]
\[ 65) n_3 = v(121 + n_2) \]
\[ \rightarrow 66, n_3 > 1 \]
\[ n_2 = n_2 + 1 \]
\[ \rightarrow 65, n_0 > n_2 \]
\[ \rightarrow 67 \]
\[ 66) v(121 + n_2) = v(241 + n_1) \]
\[ n_1 = n_1 + 1 \]
\[ n_2 = n_2 + 1 \]
\[ \rightarrow 65, n_0 > n_2 \]
\[ 67) v_0 = TAPE^* \]
\[ v_0 = TAPE^* \]
\[ STOP \]
\[ v_0 = TAPE^* \]
\[ v_0 = TAPE^* \]
\[ n_23 = 0 \]
\[ v_100 = 100 \]
\[ \rightarrow 26 \]
\[ 68) v_41 = -v_3 \]
\[ v_42 = -v_4 \]
\[ v_43 = -v_5 \]
\[ n_1 = 3 \times n_23 \]
\[ n_2 = n_1 + 1 \]
\[ n_2 = 3 \times n_2 \]
\[ v_44 = v(121 + n_1) \]
\[ v_47 = v(121 + n_2) \]
\[ v_45 = v(122 + n_1) \]
\[ v_48 = v(123 + n_2) \]
\[ v_46 = v(123 + n_1) \]
\[ v_49 = v(123 + n_2) \]
\[ v_{10} = v_{46} \]
\[ v_{11} = v_{44} \]
\[ v_{12} = v_{45} \]
\[ n_1 = 21 \]
\[ n_2 = 44 \]
\[ n_3 = 51 \]
\[ n_4 = 3 \]
\[ n_5 = 3 \]
\[ n_6 = 3 \]
\[ n_7 = 1 \]
\[ \rightarrow SR16 \]
\[ n_1 = 31 \]
\[ n_2 = 47 \]
\[ n_3 = 61 \]
\[ n_4 = 3 \]
\[ n_5 = 3 \]
\[ n_6 = 3 \]
PROGRAMME CONT. (8)

n7=1
- SR16
v1=v41 + v51
v1=v1 + v61
v2=v42 + v52
v2=v2 + v62
v3=v43 + v53
v3=v3 + v63
v4=v1 * v2
v5=v1 * v1
v6=v4 - v5
v4=v1 * v2
v5=v1 * v1
v7=v4 + v5
v7=v7 * v40
v6=-v6 * v40
v8=-v3 * v40
v4=v1 * v1
v5=v1 * v1
v9=v4 - v5
n1=n23 + 1
v4=n1
v5=n16
PRINT v4, 3040
PRINT v5, 4040
PRINT v6, 3024
PRINT v7, 2023
PRINT v8, 2023
PRINT v9, 2023
PRINT v10, 3023
n19=n19 + 4
n18=n18 + 1
-37, n18=n20
TEXT
n23=n23 + 1
-36, n23=n25
STOP
(-0)
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