GENERALIZED PLANE STRESS IN AN ELASTIC WEDGE
(UNDER DISTRIBUTED AND ISOLATED LOADS).

BY.
D.E.R. GODFREY M.Sc.

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ABSTRACT OF THESIS
ON
GENERALIZED PLANE STRESS IN AN ELASTIC WEDGE (UNDER DISTRIBUTED AND ISOLATED LOADS).

The thesis establishes general methods of solution for the stress and displacement in a wedge with distributed flank loadings or force or couple nuclei acting at any point. Full use has been made of the compactness of two-dimensional elasticity theory as expressed in terms of complex potentials and the stress combinations $\Phi$ and $\Psi$.

Complete theory of the Mellin transform in its application to the determination of stresses and displacements in two-dimensional elasticity, is provided. This is adapted for use in close connection with the theory mentioned above. The main part of the thesis deals with the application of the Mellin transform method to problems of a wedge under the action of various force and couple nuclei, with stress free or rigid boundaries.

Solutions are also given for certain problems involving body forces.
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In this thesis solutions are set out for various types of external loading of a plate in the shape of an infinite wedge. All the problems considered are two-dimensional in character and the state of generalised plane stress as defined by Filon, with its well known conditions, applies throughout. The general approach is that developed by A.C. Stevenson in terms of complex potentials. Details of this approach have been published (refs. 1 & 2) but a summary of the results required in this thesis is included in Chapter II. The elegance and compactness which is found in the use of complex potentials and the stress combinations $\Phi$ and $\Psi$ as compared with the Airy stress function characterises the problems considered here. In this connection Chapters III and V illustrate known solutions which are here solved by the use of suitable complex potentials.

The main part of this thesis is the consideration of force and couple nuclei at points in the material of the wedge other than at the vertex. It does not appear possible to find single complex potentials which will provide the nucleus and at the same time keep the flanks stress free. Well known complex potentials are however available which give a force or couple acting at any point (see ref 2 p 154) and a method has been devised in this thesis whereby the flank stresses may be nullified without affecting the nucleus itself.
It is appropriate at this point to refer to the work of C.J. Tranter who has used the Mellin transform in the solution of certain boundary problems on a wedge (ref. 3). It will be seen in Chapter IIb that by expressing this work in complex form and at the same time taking full advantage of the stress combinations $\Theta$ and $\Phi$, that the complexity of the algebraic work involved is considerably reduced, in fact it is never necessary at any stage to refer to any individual stress component. This thesis also contains details of the displacements which do not appear in the other publications to which reference is made. It is worth noting also that whereas a rather tentative approach is required in finding Airy stress functions suitable for certain problems, it is possible here to put forward a definite machinery for the solution of any force or couple nucleus at any point of the wedge. Such complete generality is not however attempted because it is in general preferable to keep to problems which have symmetry or anti-symmetry about the axis of the wedge and develop unsymmetrical problems by superposition.

Chapters VI to IX give details of four nucleus problems, which fall into two groups according to whether the nucleus is inside the material or on the flanks. This leads to particular difficulties in the case of a force at a point on the axis due to the presence in the complex potentials of the elastic constant $K$.

The solutions for the stresses are ultimately presented
formally in terms of integrals with an infinite range, which cannot be integrated except by numerical means. This feature was observed by W.M. Shepherd (ref. 4) in his work on a similar problem. It has however, been possible in the cases when the nuclei are inside the wedge to check the results in a special case when the wedge becomes a semi-infinite plate. The integrals can then be integrated and it is shown that the stresses obtained agree with the known results, as for example those given by Stevenson.
CHAPTER I

SURVEY OF PREVIOUSLY PUBLISHED WORK

Very little published work appears to be available on this subject. All the papers to which reference is made here are concerned with loadings of a distributed or concentrated nature on the flanks. No attention appears to have been given to nuclei in the material of the wedge. The work of Shepherd, Tranter and Timoshenko considers problems for which solutions are obtained satisfying the complete boundary conditions. There is also available a series of writings in Aircraft Engineering, mainly by Atkin in which solutions useful to the engineer are found.

Distributed Loads

(a) C.J. Tranter (ref.3)

A formal solution is obtained for any flank loading consisting of a normal traction and shear which are functions of the distance from the vertex. The Mellin transform is applied to the boundary problem expressed in polar co-ordinates so that it becomes the satisfaction of an ordinary differential equation with constant coefficients. It is interesting to note that in presenting the final stresses (equation 22) the formidable nature of the algebra forces the author to go part of the way towards the use of the stress combinations $\Theta$ and $\Phi$. 
(b) E. Atkin (ref. 7)

The classical solutions for a force and couple nucleus at the vertex of the wedge (see Ch V) are used to obtain a value of the shear stress on the axis of the wedge for various flank loadings. In effect a force and couple at the vertex which are statically equivalent to the actual loading are used. This will not in general give the same stress system, and in fact one stress component is overlooked (see Ch. V) in the special problem considered. However, for the wedge of small angle which is employed by Atkin the shear stress on the axis is numerically the correct value and it is for this reason, in spite of his method that Atkin obtains results which are fairly good. Wedge shaped beams of I-section are treated in his later paper (ref. 8).

(c) Pugsley and Weatherhead (ref. 9)

The problem is approached from the 'engineers' point of view and certain practical conclusions are arrived at which do not bear on the work of this thesis.

CONCENTRATED LOADS

W.M. Shepherd (ref. 4)

By the use of Airy stress functions in polar co-ordinates it has been found possible to build up solutions for loads perpendicular to the flanks. The tentative nature of the Airy stress function approach is much in evidence.
in this work, particularly if one considers how to choose \( \chi \) so that the loads are not necessarily perpendicular to the flanks. The author also envisages that the solution for a load at an internal point could be deduced from the solutions he gives, but it is not made clear how this would be done.

**LIST OF REFERENCES**

6. S.Timoshenko *Theory of Elasticity* p 121
8. E.H.Atkin " " Sept.1942
9. Pugsley & Weatherhead *Aircraft Eng.* Sept.1942
(a) Two Dimensional Elasticity in Complex Co-ordinates

The following is a brief summary of results relevant to this thesis taken from the work of A.C. Stevenson (refs 1&2)

If the body force/unit mass is given in terms of a potential $V$ by

$$ F = -\nabla V $$

the body stress equations become

$$ \begin{align*}
\frac{\partial (\sigma_{xx} - \rho V)}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial (\sigma_{yy} - \rho V)}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial (\sigma_{zz} - \rho V)}{\partial z} &= 0
\end{align*} \tag{2.1} $$

Make a change of co-ordinates

$$ z = x + iy \quad \bar{z} = x - iy \quad \tag{2.2} $$

and use $Z$ as the cartesian co-ordinate, but retain $z$ in the stress notation e.g. $\sigma_{zz}$ where no confusion will arise.

Also $V(xyZ)$ becomes $U(z\bar{z})$

Introducing the stress combinations

$$ \begin{align*}
\vartheta &= \sigma_{xx} + \sigma_{yy} \\
\Phi &= \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} \\
\Psi &= \sigma_{xz} + i\sigma_{yz}
\end{align*} \tag{2.3} $$

Equations 2.1 now become

$$ \begin{align*}
\frac{\partial \Phi}{\partial \bar{z}} + \frac{\partial \vartheta}{\partial z} - 2\rho \frac{\partial U}{\partial z} + \frac{\partial \Psi}{\partial \bar{z}} &= 0 \\
\frac{\partial \Psi}{\partial \bar{z}} + \frac{\partial \vartheta}{\partial z} + \frac{\partial (\sigma_{zz} - \rho U)}{\partial \bar{z}} &= 0
\end{align*} \tag{2.4} $$
Further the stress strain relations

\[ \widehat{pq} = \lambda \delta_{pq} + 2\mu \varepsilon_{pq} \]

where \( \varepsilon_{pq} \) are components of strain, \( \delta_{pq} = 1 \) if \( p=q \)

and \( \delta = \) dilatation \( p,q = x,y \) or \( z \)

take the form

\[ \Phi = 4\mu \frac{\partial D}{\partial Z} \]

\[ (1-2\gamma)\Theta = 2\mu \left\{ \frac{\partial D}{\partial Z} + \frac{\partial \Theta}{\partial Z} \right\} + 2\gamma \frac{\partial W}{\partial Z} \]

\[ \Psi = 2\mu \left\{ \frac{\partial W}{\partial Z} + \mu \frac{\partial D}{\partial Z} \right\} \]

\[ (1-2\gamma) \frac{\partial z^2}{\partial Z} = 2\mu \left\{ \frac{\partial D}{\partial Z} + \frac{\partial \Phi}{\partial Z} \right\} + (1-\gamma) \frac{\partial W}{\partial Z} \]

where \( u,v,w \) are components of displacement and \( D = u+iv \)

also \( \gamma = \frac{\lambda}{2(\lambda + \mu)} \) is Poisson's Ratio

**Plane Strain**

This is defined as given by \( \frac{\partial D}{\partial Z} = 0 \), \( w = 0 \) and \( \frac{\partial U}{\partial Z} = 0 \)

Using these conditions and eliminating the stresses

from 2.4 & 2.5 the displacement \( D \) is given by

\[ \frac{\partial}{\partial Z} \left\{ \frac{\partial D}{\partial Z} + \frac{\partial \Phi}{\partial Z} - \frac{1}{2} \frac{\partial}{\partial Z} (k-1)U \right\} = 0 \quad k = 3-4\gamma \]

In this case

\[ 8\mu D = k\chi(z) - \omega'(z) + b\rho w \]

where \( \chi = \frac{4(k-1)}{k+1} \) and \( U(z) = \frac{\partial W}{\partial Z} \) and \( \chi(z), \omega(z) \)

are complex potentials

from 2.5 the stresses now become

\[ 2 \widehat{\Theta} = \chi(z) + \omega'(z) + (4-b)\rho \frac{\partial W}{\partial Z} \]

\[ -2 \widehat{\Phi} = z \chi''(z) + \omega''(z) - 2b \rho \frac{\partial W}{\partial Z} \]
The solution is thus obtained formally in terms of two complex potentials \( \mathcal{J}(z) \) and \( \omega(z) \) which will be chosen so that the boundary conditions of any particular problem are satisfied.

**Generalized Plane Stress**

Consider now a plate-like material bounded by the planes \( Z = \pm c \) and define a state of generalized plane stress:

\[
\frac{\partial U}{\partial Z} = 0; \quad \bar{\varepsilon} = 0 \quad \text{everywhere}; \quad \bar{\psi} = 0 \quad \text{over} \quad Z = \pm c
\]

Filon has shown that if the stresses and displacements are averaged over the thickness \( 2c \), this state of stress is mathematically analogous to plane strain and the equations 2.8 and 2.9 will hold with reference to stresses averaged across the thickness of the plate. It is further necessary to replace Poisson's Ratio \( \gamma \) by a modified ratio \( \sigma \) where

\[
(1-\sigma)(1+\gamma) = 1
\]

**Transformation of Stress components**

Let the axes \( Ox, Oy \) be rotated through an angle \( \alpha \) to new positions \( O\xi, O\eta \). It is required to refer a state of stress to the new axes.

Using new stress combinations

\[
\tau' = \bar{\tau} + \bar{\sigma} \sigma; \quad \phi' = \bar{\phi} - \bar{\tau} \sigma + 2i\bar{\tau} \eta
\]

where \( \bar{\tau}, \bar{\sigma} \) etc are the averaged stresses, it can be shown that

\[
\tau' = \tau, \quad \phi' = \phi e^{-2i\alpha}
\]

In the particular case of polar co-ordinates when \( \alpha = \theta \) and \( \tau, \sigma \equiv r, \theta \).
\( \Omega' = \Omega, \quad \Phi' = e^{-2i\Theta} \Phi = \frac{\pi}{2} \Phi \)

so that with body force
\[
2 \Omega' = \mathcal{N}'(z) + \mathcal{N}'(\bar{z}) + (4 - \delta) \rho \frac{\partial W}{\partial z}
- 2i \Phi' = i \mathcal{N}''(z) + \frac{\pi}{z} \left\{ \mathcal{N}''(\bar{z}) - \delta \rho \frac{\partial W}{\partial \bar{z}} \right\}
\]

**Boundary conditions**

Suppose the boundary loading to be given on curve \( C \) i.e. \( \vec{m} + i \vec{s} \) is known say \( \vec{m} + i \vec{s} = -(N + iT) \)

Then on the boundary
\[
4 \left\{ \vec{m} + i \vec{s} - \frac{\partial W}{\partial z} \right\} \frac{\partial z}{\partial s} = \frac{\partial}{\partial s} \left\{ \mathcal{N}(z) + z \mathcal{N}'(\bar{z}) + \mathcal{N}'(\bar{z}) - \delta \rho \frac{\partial W}{\partial \bar{z}} \right\}
\]

Thus for a boundary which is to be stress free, under no body forces
\[
\mathcal{N}(z) + z \mathcal{N}'(\bar{z}) + \mathcal{N}'(\bar{z}) = \text{constant round } C
\]

**Force and Couple Resultant of stresses along an arc.**

Let \( X \) and \( Y \) be the components of the resultant force and \( N \) the resultant couple of the stresses defined by the complex potentials, acting over the arc \( C \).

Then
\[
X + iY = -\frac{1}{4}i \left[ \mathcal{N}(z) + z \mathcal{N}'(\bar{z}) + \mathcal{N}'(\bar{z}) \right]_A^B
\]
\[
N = \text{real part of } -\frac{1}{4} \left[ z \bar{z} \mathcal{N}'(\bar{z}) + \bar{z} \mathcal{N}'(\bar{z}) - \bar{\mathcal{N}(\bar{z})} \right]_A^B
\]
In the case of a closed curve \( C \) these results may be simplified so that

\[
X + iY = -i(1 - \sigma)C_y \mathcal{N}(z)
\]

\[
N = \text{real part of } \frac{1}{4} i \left\{ \omega(z) - z \omega'(z) \right\}
\]

Effect on the complex potentials of a change of origin

Let the origin be moved to \( z_c \) so that

\[
z = z_c + z_i
\]

The stresses cannot alter by this translation of axes and so from 2.9

\( \mathcal{N}'(z) \) and \( \bar{z} \mathcal{N}''(z) + \omega''(z) \) cannot alter

Denote the new complex potentials by a subscript 1

\[
\therefore \mathcal{N}_1(z) = \mathcal{N}(z)
\]

\[
\omega_1(z) = \omega(z) + \bar{z}_c \mathcal{N}(z)
\]
(b) The Mellin Transform

The Mellin transform \( \tilde{f}(p) \) of the function \( f(r) \) is defined by

\[
\tilde{f}(p) = \int_0^\infty r^{p-1} f(r) \, dr
\]  

(1)

where \( p \) must be such that the integral converges. Also the integral will be real if \( p \) is real.

The Mellin Inversion Theorem

If the integral \( \int_0^\infty r^{k-1} |f(r)| \, dr \) is bounded for some \( k > 0 \), and if \( \tilde{f}(p) \) is given by (1), then

\[
f(r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(p) r^{-p} \, dp
\]  

(2)

where \( c > k > 0 \) and here \( p \) is complex.

Mellin Transform of derivatives

Integrate (1) by parts

\[
\tilde{f}(p) = \left[ f(r) \frac{r^p}{p} - \int f'(r) \frac{r^p}{p} \, dr \right]_0^\infty
\]

so that if we assume that

\[
r^p f(r) \to 0 \text{ as } r \to 0 \text{ and as } r \to \infty
\]  

(3)

we have

\[
\int_0^\infty r^p f'(r) \, dr = -p\tilde{f}(p)
\]  

(4)

Integrating again by parts

\[
\left[ f'(r) \frac{r^{p+1}}{p+1} - \int f''(r) \frac{r^{p+1}}{p+1} \, dr \right]_0^\infty = -p\tilde{f}(p)
\]

so that if \( r^{p+1} f'(r) \to 0 \) at the upper and lower limits

\[
\int_0^\infty r^{p+1} f''(r) \, dr = p(p+1) \tilde{f}(p)
\]  

(5)

and generally

\[
\int_0^\infty f^{(n)}(r) r^{p+n-1} \, dr = (-1)^n \frac{\Gamma(p+n) \tilde{f}(p)}{f^{(n)}(p)}
\]  

(6)
Generalized Plane stress in polar coordinates

The mean stresses $\overline{\sigma_r}$, $\overline{\sigma_\theta}$, $\overline{\tau_\theta}$ are given in terms of Airy's stress function $\chi$, assuming no body force, by

$$\overline{\sigma_r} = \frac{1}{r^3} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \chi}{\partial r}, \quad \overline{\sigma_\theta} = \frac{\partial \chi}{\partial r}, \quad \overline{\tau_\theta} = -\frac{1}{r^2} \left( \frac{1}{r} \frac{\partial \chi}{\partial \theta} \right)$$

(7)

where

$$\nabla^4 \chi = 0$$

(8)

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

(9)

and the mean displacements $U_r$, $U_\theta$ are given in terms of $\chi$ and a displacement function $\psi$ satisfying

$$\nabla^2 \psi = 0 \quad \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial \theta} \right) = \nabla^2 \chi$$

(10)

as

$$2\mu U_r = -\frac{\partial \psi}{\partial r} + (1-\sigma) r \frac{\partial^2 \psi}{\partial \theta^2}$$

$$2\mu U_\theta = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} + (1-\sigma) r^2 \frac{\partial^2 \psi}{\partial r^2}$$

(11)

expanding the biharmonic equation (8)

$$\frac{1}{r^3} \frac{\partial}{\partial \theta} \nabla^2 \chi = \frac{1}{r^3} \left\{ \frac{\partial^4 \chi}{\partial \theta^4} + \frac{1}{r} \frac{\partial^3 \chi}{\partial \theta^3} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r^3} \frac{\partial \chi}{\partial \theta} \right\}$$

writing $D$ for $\frac{\partial}{\partial \theta}$

$$\frac{1}{r^3} \frac{\partial}{\partial \theta} \nabla^2 \chi = \frac{1}{r^3} \left\{ \frac{\partial^4 \chi}{\partial \theta^4} + \frac{1}{r} \frac{\partial^3 \chi}{\partial \theta^3} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r^3} \frac{\partial \chi}{\partial \theta} \right\}$$

so

$$\frac{1}{r^3} \frac{\partial}{\partial \theta} \nabla^2 \chi = \frac{1}{r^3} \left\{ \frac{\partial^4 \chi}{\partial \theta^4} + \frac{1}{r} \frac{\partial^3 \chi}{\partial \theta^3} - \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r^3} \frac{\partial \chi}{\partial \theta} \right\}$$

$$\frac{\partial^2}{\partial r^2} \nabla^2 \chi = \frac{\partial^4 \chi}{\partial r^4} + \frac{1}{r} \frac{\partial^3 \chi}{\partial r^3} - \frac{2}{r^2} \frac{\partial^2 \chi}{\partial r^2} + \frac{2}{r^3} \frac{\partial \chi}{\partial r} + \frac{1}{r^4} \frac{\partial^2 \chi}{\partial \theta^2}$$

$$- \frac{4}{r^5} \frac{\partial \chi}{\partial \theta} + \frac{6}{r^4} \frac{\partial \chi}{\partial \theta}$$

$$= \frac{1}{r^4} \left\{ \frac{\partial^4 \chi}{\partial \theta^4} + \frac{1}{r} \frac{\partial^3 \chi}{\partial \theta^3} - 2 \frac{2}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + 2 \frac{\partial \chi}{\partial \theta} + \frac{2}{r^3} \frac{\partial^2 \chi}{\partial \theta^2} \right\}$$

$$- \frac{4}{r^4} \frac{\partial \chi}{\partial \theta} + \frac{6}{r^4} \frac{\partial \chi}{\partial \theta}$$
whence \( r^4 \nabla^4 \chi = r^4 \frac{\partial^4 \chi}{\partial r^4} + 2 r^3 \frac{\partial^3 \chi}{\partial r^3} - r^2 \frac{\partial^2 \chi}{\partial r^2} + 2 r \frac{\partial \chi}{\partial r} + 2 \frac{\partial \chi}{\partial r} \chi + D^4 \chi \)  

(12)

Use of Mellin Transform in the Elastic Wedge problem

We first find the differential equation satisfied by the transform \( \tilde{\chi} \) of the Airy stress function \( \chi \), which is obtained by multiplying (12) by \( r^{p-1} \) and integrating from 0 to \( \infty \) with respect to \( r \), using

\[
\int_0^{\infty} r^{p-1} r \frac{\partial \chi}{\partial r} \, dr = -p \tilde{\chi}(p)
\]

\[
\int_0^{\infty} r^{p-1} r^2 \frac{\partial^2 \chi}{\partial r^2} \, dr = p(p+1) \tilde{\chi}
\]

\[
\int_0^{\infty} r^{p-1} r^3 \frac{\partial^3 \chi}{\partial r^3} \, dr = -p(p+1)(p+2) \tilde{\chi}
\]

\[
\int_0^{\infty} r^{p-1} r^4 \frac{\partial^4 \chi}{\partial r^4} \, dr = p(p+1)(p+2)(p+3) \tilde{\chi}
\]

and we have

\[
D^4 \tilde{\chi} + 2 \left( p(p+1) + p+2 \right) D^3 \tilde{\chi} + \left( p(p+1)(p+2)(p+3) - 2p(p+1)(p+2) - p(p+1) - p \right) \tilde{\chi} = 0
\]

which simplifies to

\[
D^4 \tilde{\chi} + \left( p^2 + (p+2)^2 \right) D^2 \tilde{\chi} + p^2(p+2)^2 \tilde{\chi} = 0
\]

or

\[
(D^4 + p^2) \left( D^2 + (p+2)^2 \right) \tilde{\chi} = 0
\]

(14)

of which the solution is

\[
\tilde{\chi} = Ae^{ip\Theta} + Ae^{-ip\Theta} + Be^{i(p+2)\Theta} + Be^{-i(p+2)\Theta}
\]

(15)

since it follows from the definition of \( \chi \) that it is real if \( p \) is real.
The stress combinations

From (7) the stress combinations

\[ \tilde{\omega}' = \tilde{rr} + \tilde{\theta} = \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \]

\[ \tilde{\Phi}' = \tilde{rr} - \tilde{\theta} + 2i \tilde{r} \tilde{\theta} = \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial r^2} - \frac{3}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} - 2i \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \chi}{\partial \theta} \right) \]

or \[ r^2 \tilde{\omega}' = \left\{ \frac{r^2 \frac{\partial^2 \chi}{\partial r^2}}{r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} + D \right\} \]

\[ r^2 \tilde{\Phi}' = \left\{ \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \chi}{\partial r} - \frac{3}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \right\} \chi - 2i \left( \frac{1}{r} \frac{\partial \chi}{\partial \theta} - 1 \right) D \chi \]

Multiply by \( r^{p-1} \) and integrate from 0 to \( \infty \) with respect to \( r \),

\[ \int_0^\infty r^2 \tilde{\omega}' r^{p-1} dr = p(p+1) \tilde{\chi} - p \tilde{\chi} + \tilde{\chi} \] using (13)

or \[ \int_0^\infty r^2 \tilde{\omega}' r^{p-1} dr = \{ D^2 + p^2 \} \tilde{\chi} \]

(16)

where from (3) it has been assumed that

\( r^{p+2} \tilde{\omega}' \to 0 \) for \( r \to 0 \) and \( r \to \infty \)

Since \( \tilde{\omega}' \) is finite at \( r=0 \), this means

\[ p+2 > 0 \quad \text{or} \quad -2 < p \]

If we further suppose that \( \tilde{\omega}' = 0(r^{-n}) \) as \( r \to \infty \)

\[ r^{p+2} \tilde{\omega}' = 0(r^{p+n-2}) \]

and this will \( \to 0 \) if \( p-n+2 < 0 \), or \( p < n-2 \)

hence \[ -2 < p < n-2 \]

(17)

Again

\[ \int_0^\infty r^2 \tilde{\Phi}' r^{p-1} dr = \{ D^2 - p - p(p+1) \} \tilde{\chi} + 2i(p+1)D \tilde{\chi} \]

\[ = \{ D^2 - p(p+2) + 2i(p+1)D \} \tilde{\chi} \]

\[ = (D + ip) \{ D + i(p+2) \} \tilde{\chi} \]

(18)

Where we assume \( \tilde{\Phi}' \) to be of order \( r^{-n} \) at \( \infty \), and (17) to be satisfied.
The Displacements

Rewriting equns. (11) as

\[ 2 \mu r U_r = -r \frac{\partial \chi}{\partial r} + (1 - \sigma) \frac{\partial^2 \psi}{\partial r^2} \]
\[ 2 \mu r U_\theta = -r \frac{\partial \chi}{\partial \theta} + (1 - \sigma) r \frac{\partial^2 \psi}{\partial r \partial \theta} \]

or on writing \[ r \frac{\partial \psi}{\partial \theta} = \psi_1 \]
these become

\[ 2 \mu r U_r = -r \frac{\partial \chi}{\partial r} + (1 - \sigma) \Delta \psi, \]
\[ 2 \mu r U_\theta = -\Delta \chi + (1 - \sigma) (r \frac{\partial}{\partial r} - 2) \psi, \]

where \[ \psi, \] satisfies equations obtained from (10),

\[ \nabla^2 \psi_1 = 0, \quad \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi_1}{\partial \theta} \right) = \nabla^2 \chi \]

These may be written,

\[
\begin{align*}
\left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{3}{r^2} \frac{\partial}{\partial r} \right) \left( \frac{1}{r^2} \frac{\partial \psi_1}{\partial r} \right) + \frac{1}{r^2} \Delta \psi_1 &= 0 \\
r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi_1}{\partial r} - \frac{2}{r^2} \psi_1 \right) + \Delta \psi_1 &= 0 \\
r \frac{\partial^2 \psi_1}{\partial r^2} - 3r \frac{\partial \psi_1}{\partial r} + 4\psi_1 + \Delta \psi_1 &= 0
\end{align*}
\]

(20)

and

\[
\begin{align*}
2 \frac{\partial}{\partial r} \left( \frac{1 - \sigma}{r} \frac{\partial \psi_1}{\partial r} \right) &= r \frac{\partial^2 \chi}{\partial r^2} + r \frac{\partial \chi}{\partial r} + \Delta \chi \\
r \frac{\partial \psi_1}{\partial r} - \Delta \psi_1 &= 2 \frac{\partial^2 \chi}{\partial r^2} + r \frac{\partial \chi}{\partial r} + \Delta \chi
\end{align*}
\]

(21)

hence from (19)

\[ 2 \mu \int_0^r r U_r r^{p-1} \, dr = p \tilde{\chi} + (1 - \sigma) \tilde{\psi}, \]

and

\[ 2 \mu \int_0^r r U_\theta r^{p-1} \, dr = -\tilde{\Delta} \chi - (1 - \sigma) (p+2) \tilde{\psi}, \]

These can be expressed entirely in terms of the Mellin transform \( \tilde{\chi} \) of the Airy stress function, since we find from (20) that

\[
\{ p(p+1) + 3p + 4 + \Delta^2 \} \tilde{\psi} = 0
\]
or \( \tilde{\psi} \) satisfies the equation
\[
\{ D^r + (p+2)^r \} \tilde{\psi} = 0 \tag{22}
\]
and is related to \( \tilde{\chi} \) from (21) as
\[
-(p+1)D\tilde{\psi} = \{ p(p+1) - p + D^r \} \tilde{\chi}
\]
or
\[
-(r+1)D\tilde{\psi} = (D^r + p^r) \tilde{\chi}
\tag{23}
\]
from the latter
\[
D\tilde{\psi} = - \frac{(D^r + p^r)}{p+1} \tilde{\chi}
\]
and then from (22)
\[
(p+2)\tilde{\psi} = -D\tilde{\psi} = \frac{(D^r + p^r)D\tilde{\chi}}{p+1}
\]

hence
\[
2\mu \int_0^\infty rU_0 r^{p-1} dr = p \tilde{\chi} - (1 - \sigma) \frac{(D^r + p^r)\tilde{\chi}}{p+1}
\]
\[
2\mu \int_0^\infty rU_0 r^{p-1} dr = -D\tilde{\chi} - (1 - \sigma) \frac{(D^r + p^r)D\tilde{\chi}}{(p+1)(p+2)}
\tag{24}
\]

The Inversion formulae for the stress combinations

Inverting (16) using (2)
\[
r^\gamma \Theta' = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (D^r + p^r) \tilde{\chi} r^{-p} dp
\]
\[
\Theta' = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (D^r + p^r) \tilde{\chi} r^{-p-2} dp
\tag{25}
\]
so that \( c + 2 > k > 0 \) or \( c > -2 \)

again inverting (18) using (2)
\[
r^\gamma \tilde{\Phi}' = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (D+ip) \{ D+ip+2 \} \tilde{\chi} r^{-p} dp
\]
or
\[
\tilde{\Phi}' = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (D+ip) \{ D+ip+2 \} \tilde{\chi} r^{-p-2} dp
\tag{26}
The Inversion formulae for the displacements

Inverting (24) using (2) we have

\[ 2 \mu r U_r = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \left\{ \frac{(1-\sigma)(D^2 + p^2)}{p+1} \right\} \tilde{\chi} r^{-p} dp \]  

(27)

\[ 2 \mu r U_\theta = -\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \left\{ 1 + \frac{(1-\sigma)(D^2 + p^2)}{(p+1)(p+2)} \right\} \tilde{D} \tilde{\chi} r^{-p} dp \]  

(28)

or

\[ U_r + iU_\theta = \frac{1}{4\pi i \mu} \int_{C-i\infty}^{C+i\infty} (p-iD) \tilde{\chi} r^{-p-1} \left\{ \frac{(1-\sigma)(D^2 + p^2)}{(p+1)(p+2)} \right\} \tilde{\chi} r^{-p-1} dp \]

\[ = \frac{1}{4\pi i \mu} \int_{C-i\infty}^{C+i\infty} \frac{(D+i(p+2))\tilde{\chi} r^{-p-1} dp}{(p+1)(p+2)} \]

\[ = \frac{1}{4\pi i \mu} \int_{C-i\infty}^{C+i\infty} \left\{ 1 + \frac{(1-\sigma)(D+i(p+2))\tilde{D} \tilde{\chi}}{(p+1)(p+2)} \right\} \tilde{\chi} r^{-p-1} dp \]  

(29)

gives the complex displacement inversion formula.

The boundary conditions

These may be conditions of displacement or of stresses. The complex boundary stress is

\[ \tilde{\sigma} + i\tilde{\sigma} \]

now \[ r^2(\tilde{\sigma} + i\tilde{\sigma}) = -r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \tilde{\chi}}{\partial \theta} \right) + ir^2 \frac{\partial \tilde{\chi}}{\partial r} \]

hence

\[ \int_0^\infty r^2(\tilde{\sigma} + i\tilde{\sigma}) r^{p-1} dr = \int_0^\infty \left\{ \frac{\partial \tilde{\chi}}{\partial \theta} - i r^2 \frac{\partial \tilde{\chi}}{\partial r} + \frac{\partial \tilde{\chi}}{\partial \theta} \right\} r^p dr \]

\[ = ip(p+1)\tilde{\chi} + (p+1)D\tilde{\chi} \]

or

\[ \int_0^\infty r^2(\tilde{\sigma} + i\tilde{\sigma}) r^{p-1} dr = (p+1)(D + ip)\tilde{\chi} \]  

(30)

Thus for example if the stresses were given on the wedge boundaries \( \theta = \pm \alpha \), the boundary conditions are, writing

\[ \int_0^\infty (\tilde{\sigma} + i\tilde{\sigma}) r^p dr = t(p) \]
\[(p+1)(D + ip)\tilde{\chi} = t(p) \quad \text{over } \theta = \alpha \]

\[= t_\alpha(p) \quad \text{over } \theta = -\alpha \]

where \(\tilde{\chi}\) is given by (15)

If the displacements are given and

\[\int_0^\infty (U_\rho + iU_\phi)r^2dr = d_\rho(p) \quad \text{over } \theta = \alpha \]

\[= d_\alpha(p) \quad \text{over } \theta = -\alpha \]

then the displacement boundary conditions are

\[-\frac{1}{2\mu} \left[ (D + ip) \left\{ 1 + \frac{(1-\sigma)(D - ip)(D - i(p+2))}{(p+1)(p+2)} \right\} \right] \tilde{\chi} = d_\rho(p) \quad \text{over } \theta = \alpha \]

\[= d_\alpha(p) \quad \text{over } \theta = -\alpha \]

(32)

We may of course have displacements given on one boundary and stresses on the other.
CHAPTER III

Continuous Load Problems

Solutions are obtained here for polynomial types of loading on the flanks, by the use of suitable complex potentials. Similar solutions are given in polar form by Timoshenko (ref. 6)

Constant normal traction and shear on \( \Theta = \alpha \)

referring to 2.17 the boundary condition may be written \( \vec{m} + \vec{n}s = p - iq \) on \( \Theta = \alpha \)

or, in terms of the complex potentials

\[ \mathcal{N}(z) + z\mathcal{N}'(z) + \mathcal{W}'(z) = 4(p - iq)z \] on \( z = ze^{-2i\alpha} \)

Consider the complex potentials

\[ \mathcal{N}(z) = Az + iBz\log z \quad \omega(z) = (C_i + iC_2)z^2 \]

substituting in 3.1

\[ 2Az - iBz(1-2i\alpha) + 2(C_i - iC_2)ze^{-2i\alpha} = 4(p - iq)z \]

\[ 2Az - iBz(1+2i\alpha) + 2(C_i - iC_2)ze^{2i\alpha} = 0 \]

giving

\[ 2A - 2B\alpha + 2(C_i \cos 2\alpha - C_2 \sin 2\alpha) = 4p \]

\[ B + 2(C_i \sin 2\alpha + C_2 \cos 2\alpha) = 4q \]

\[ 2A + 2B\alpha + 2(C_i \cos 2\alpha + C_2 \sin 2\alpha) = 0 \]

\[ B - 2(C_i \sin 2\alpha - C_2 \cos 2\alpha) = 0 \]

leading to

\[ A = p - q \cot 2\alpha \]

\[ C_i = \frac{q}{\sin 2\alpha} \]

\[ B = \frac{-2(p \cos 2\alpha + q \sin 2\alpha)}{2\alpha \cos 2\alpha - \sin 2\alpha} \]

\[ C_2 = \frac{p + 2aq}{2\alpha \cos 2\alpha - \sin 2\alpha} \]
Stresses

from $2, 15$

$$4(\theta + i \Phi) = 2A + i B \log \frac{r}{2} + i B + 2\Phi(C_i + i C)$$

$$= 2A - 2B \Phi + i B + 2(\cos 2\Theta + i \sin 2\Theta)(C_i + i C)$$

so that

$$\Phi = \frac{1}{2}(A - B \Phi + C, \cos 2 \Theta - C, \sin 2 \Theta)$$

$$\Theta = \frac{1}{4}(B + 2C, \sin 2 \Theta + 2C, \cos 2 \Theta)$$

$$\Theta' = \frac{1}{2}\{ \Theta'(z) + \Theta'(\bar{z}) \} = A - B \Phi$$

$$\Theta_0 = \frac{1}{2}(A - B \Theta - C, \cos 2 \Theta + C, \sin 2 \Theta)$$

3.3

3.4

A case of special interest is that of a uniform vertical load

Taking $p = -w \cos \alpha$ and $q = -w \sin \alpha$

$$A = -\frac{w}{2 \cos \alpha}$$

$$B = -\frac{2w \cos \alpha}{2a \cos 2 \alpha - \sin 2 \alpha}$$

3.5

$$C_i = -\frac{w}{2 \cos \alpha}$$

$$C = -\frac{w(\cos \alpha + 2a \sin \alpha)}{2a \cos 2 \alpha - \sin 2 \alpha}$$

Loading $r^n$ perpendicular to the axis of the wedge

we require

$$\bar{n} = -r^n \cos \alpha$$

$$\bar{n} = +r^n \sin \alpha$$

$$\bar{n} + \bar{n}_s = -r^n(\cos \alpha - i \sin \alpha)$$

$$= -(z \bar{z})^n e^{-ia}$$

and since $z = ze^{-2ia}$

$$\bar{n} + \bar{n}_s = -z^n e^{-(n+1)ia}$$

Using the complex potentials

$$\lambda(z) = (A_i + iA_s)z^{n+1}$$

$$\omega(z) = (B_i + iB_s)z^{n+2}$$

and boundary condition

$$\lambda(z) + z \lambda'(\bar{z}) + \omega'(\bar{z}) = -\frac{4}{n+1} z^{n+1} e^{-(n+1)ia} \quad \text{over } z = ze^{-2ia}$$

$$= 0 \quad \text{over } \bar{z} = ze^{2ia}$$
which gives
\[(A_e + iA_o) + (n+1)(A_e - iA_o)e^{-2n\alpha} + (n+2)(B_e - iB_o)e^{-(2n+2)i\alpha}\]
\[= -\frac{4}{n+1} e^{-(n+1)i\alpha}\]

\[(A_e + iA_o) + (n+1)(A_e - iA_o)e^{2n\alpha} + (n+2)(B_e - iB_o)e^{(2n+2)i\alpha} = 0\]

Thus
\[A_+ (n+1) (A_e \cos 2\alpha - A_o \sin 2\alpha) + (n+2)(B_e \cos (2n+2)\alpha - B_o \sin (2n+2)\alpha)\]
\[= -\frac{4}{n+1} \cos (n+1)\alpha\]

\[A_+ (n+1) (A_o \cos 2\alpha + A_e \sin 2\alpha) - (n+2)(B_e \sin (2n+2)\alpha + B_o \cos (2n+2)\alpha)\]
\[= \frac{4}{n+1} \sin (n+1)\alpha\]

\[A_+ (n+1) (A_e \cos 2\alpha + A_o \sin 2\alpha) + (n+2)(B_e \cos (2n+2)\alpha + B_o \sin (2n+2)\alpha) = 0\]

\[A_+ (n+1) (A_o \cos 2\alpha - A_e \sin 2\alpha) + (n+2)(B_e \sin (2n+2)\alpha - B_o \cos (2n+2)\alpha) = 0\]

leading to
\[A_e = \frac{2}{n+1} \frac{\sin (n+1)\alpha}{(n+1) \sin 2\alpha + \sin (2n+2)\alpha}\]
\[B_e = \frac{2}{(n+1)(n+2)} \frac{(n+1) \sin (n-1)\alpha - \sin (n+1)\alpha}{(n+1) \sin 2\alpha + \sin (2n+2)\alpha}\]
\[A_o = \frac{2}{n+1} \frac{\cos (n+1)\alpha}{\sin (2n+2)\alpha - (n+1) \sin 2\alpha}\]
\[B_o = -\frac{2}{(n+1)(n+2)} \frac{(n+1) \cos (n-1)\alpha - \cos (n+1)\alpha}{\sin (2n+2)\alpha - (n+1) \sin 2\alpha}\]

The stresses on \(\theta = 0\) now become
\[\bar{\theta} = \frac{i}{2} x^n \frac{(n+1) \sin (n-1)\alpha - (n+3) \sin (n+1)\alpha}{(n+1) \sin 2\alpha + \sin (2n+2)\alpha}\]
\[\bar{\theta} = \frac{i}{2} x^n \frac{\sin n\alpha \sin \alpha}{\sin (2n+2)\alpha - (n+1) \sin 2\alpha}\]
\[\bar{r} = \frac{i}{2} x^n \frac{(n-1) \sin (n+1)\alpha - (n+1) \sin (n-1)\alpha}{(n+1) \sin 2\alpha + \sin (2n+2)\alpha}\]

3.7
Approximate solution for an elliptic loading

A polynomial approximation to the ellipse is obtained on a least squares basis. The stresses are then found by superimposing solutions for the various powers of x.

The loading assumed is \( y = \sqrt{2x - x^2} \) and a polynomial approximation

\[
y = a_0 + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + \ldots
\]

is found. Here, \( P_r(x) \) is an orthogonal polynomial in x of degree r.

If the range of x is divided into \((n-1)\) equal parts giving discrete values of x as being \( x_i \), \( i = 0, 1, 2, \ldots, (n-1) \) or n ordinates, the polynomials then have the property

\[
\sum_{i=0}^{n-1} P_r(x_i) P_s(x_i) = 0 \quad \text{for all } r \neq s
\]

It is shown in Aitkin's "Statistical Mathematics" that the values of the coefficients \( a_r \) may be found so that the expression 3.8 for y will be a 'best' polynomial through the n ordinates, interpreted on a least squares basis.

The \( a_r \)'s are given by

\[
a_r = \frac{\sum_{i=0}^{n-1} y_i P_r(x_i)}{\sum_{i=0}^{n-1} \left\{ P_r(x_i) \right\}^2}
\]

Further, the method gives the 'best' polynomial of any desired degree simply by stopping the series 3.8 at the appropriate term. Tables suitable for this work are Fisher and Yates "Statistical Tables for Medical and Agricultural Research".
In the following numerical work \( n = 11 \) giving eleven ordinates. Also these ordinates are taken as for a circle of radius 10 units and finally reduced by a factor \( \lambda \).

The following table gives values of \( P_r(x) \) \( r = 1, 2, 3, 4, 5 \) compiled from Fisher and Yates, which also gives \( \sum \{P_r(x)\}^2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( P_1(x) )</th>
<th>( P_2(x) )</th>
<th>( P_3(x) )</th>
<th>( P_4(x) )</th>
<th>( P_5(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-5</td>
<td>15</td>
<td>-30</td>
<td>6</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>4.36</td>
<td>-4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6.00</td>
<td>-3</td>
<td>-1</td>
<td>22</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>7.14</td>
<td>-2</td>
<td>-6</td>
<td>23</td>
<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>4</td>
<td>8.00</td>
<td>-1</td>
<td>-9</td>
<td>14</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>5</td>
<td>8.66</td>
<td>0</td>
<td>-10</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>9.17</td>
<td>1</td>
<td>-9</td>
<td>-14</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>9.54</td>
<td>2</td>
<td>-6</td>
<td>-23</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>9.80</td>
<td>3</td>
<td>-1</td>
<td>-22</td>
<td>-6</td>
<td>-1</td>
</tr>
<tr>
<td>9</td>
<td>9.95</td>
<td>4</td>
<td>6</td>
<td>-6</td>
<td>-6</td>
<td>-6</td>
</tr>
<tr>
<td>10</td>
<td>10.0</td>
<td>5</td>
<td>15</td>
<td>30</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Values of the \( a \)'s corresponding are

\[
\begin{align*}
a_0 &= 7.5109 \\
a_1 &= 0.8157 \\
a_2 &= -0.1412 \\
a_3 &= 0.0259 \\
a_4 &= -0.0584 \\
a_5 &= 0.0445
\end{align*}
\]

The actual polynomials may be obtained by using Newtons central difference formula

\[
\begin{align*}
P_0' &= 1 \\
P_1' &= 0.5 \\
P_2' &= 0.14 \\
P_3' &= 0.059 \\
P_4' &= 0.035 \\
P_5' &= 0.018
\end{align*}
\]

\[
e.g., P_5(x) = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}
\]
so that \( P_5(x) = -3 + 9^\alpha C_1 x + 14^\alpha C_2 x + 14^\alpha C_3 x - 9^\alpha C_4 + 3^\alpha C_5 \)
and the polynomials become

- \( P_1(x) = x - 5 \)
- \( P_2(x) = x^2 - 10x + 15 \)
- \( P_3(x) = \frac{1}{6}(5x^3 - 75x^2 + 286x - 180) \)
- \( P_4(x) = \frac{1}{12}(x^4 - 20x^3 + 125x^2 - 250x + 72) \)
- \( P_5(x) = \frac{1}{40} x^5 - \frac{5}{8} x^4 + \frac{511}{24} x^3 - \frac{193}{8} x^2 + \frac{2331}{60} x - 3 \)

the 'best' cubic is then

\[
y = \frac{\lambda}{10} \left( 0.0216 x^3 - 0.4654 x^2 + 3.4641 x + 0.5362 \right) \quad 0 \leq x \leq 10
\]

the 'best' quartic is then

\[
y = \frac{\lambda}{10} \left( -0.0049 x^4 + 1.189 x^3 - 1.0736 x^2 + 4.6806 x + 0.1859 \right) \quad 0 \leq x \leq 10
\]

Numerical computation of stresses for a wedge of small angle, such as an aircraft wing girder is simplified by approximation to the first order in \( \alpha \).

Thus the stress \( \bar{\Theta}_\theta \) on \( \theta = 0 \) due to a loading \( \omega x^n \) on the upper flank is obtained from 3.7 as

\[
\bar{\Theta}_\theta = \frac{1}{2} \omega x^n \cos^n \alpha \frac{(n+1)\sin(n-1)\alpha - (n+3)\sin(n+1)\alpha}{(n+1)\sin2\alpha + \sin(2n+2)\alpha}
\]

\[
= \frac{1}{2} \omega x^n \frac{(n^2-1)\alpha - (n+3)(n+1)\alpha}{2(2n+2)\alpha} \text{ (approx)}
\]

\[
= \frac{1}{2} \omega x^n \cdot \frac{4(n+1)\alpha}{4(n+1)\alpha}
\]

\[
= -\frac{1}{2} \omega x^n \quad \text{which holds for } n = 1, 2, 3, 4,
\]
For the constant term in the polynomial approximations we have from 3.3
\[
\overline{\Theta}_0 = \frac{1}{2}(A + C_i)
\]
where from 3.5 \[ A = C_i = -\frac{w}{2c\sin a} \]
thus for small \( a \) \[ \overline{\Theta}_0 = -\frac{1}{2}w \] which fits in with 3.11.
Finally, the stresses corresponding to 3.9 and 3.10 will read
\[
\overline{\Theta}_0 = -\frac{wA}{20} \left(0.021x^2 - 0.4654x + 3.4641x + 0.5362\right)
\]
and
\[
\overline{\Theta}_0 = -\frac{wA}{20} \left(-0.0049x^4 + 0.1189x^3 - 1.0736x^2 + 4.6806x + 0.1859\right)
\]
For a truncated wedge, solutions having the same force and couple resultants over the end arcs are effectively the same at a distance from these ends, by Saint Venant's Principle. In calculating from these equations the end-values have not therefore been included.

The 'best' cubic gives the following values of the stresses at various positions along the wedge.

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{20wA\overline{\Theta}_0}{wA\overline{\Theta}_0} )</td>
<td>5.78</td>
<td>8.32</td>
<td>9.23</td>
<td>9.52</td>
</tr>
</tbody>
</table>
CHAPTER IV

Body Force Problems

(a) Wedge under its own weight

The components of body-force/unit mass are

\[ \mathbf{F}_x = -g \cos \beta \]
\[ \mathbf{F}_y = -g \sin \beta \]

and since \( \mathbf{F} = -\nabla V \)

we have

\[ V = g(x \cos \beta + y \sin \beta) \]

and

\[ U = \frac{1}{2}g(ze^{-i\beta} + \bar{z}e^{i\beta}) \]

Thus putting \( c = \frac{1}{2} ge^{-i\beta} \)

\[ U = cz + \bar{c}z = \frac{1}{2}W \]

\[ W = \frac{1}{2}cz^2 + \bar{c}z \overline{z} \quad \text{and} \quad \frac{\partial W}{\partial z} = \bar{c}z \]

Using 2.8 and 2.9 and introducing

\[ \chi = \frac{4(\kappa - 1)}{\kappa + 1} \quad \text{where} \quad \kappa = 3 - 4\sigma \quad \text{so that} \]

\[ \delta = 2(1 - \gamma) \quad \text{since} \quad (1 + \gamma)(1 - \sigma) = 1 \]

Consider now the complex potentials

\[ \mathcal{A}(z) = Az^2 \quad \omega(z) = Bz^3 \]

These make

\[ 2(\mathcal{O}) = 2Az + 2\bar{A}z + 2(1 + \gamma)\rho (cz + \bar{c}z) \]

or

\[ \mathcal{O} + \mathcal{O}' = \{A + \rho (1 + \gamma)c\}z + \{\bar{A} + \rho (1 + \gamma)\bar{c}\} \bar{z} \]

and

\[ -2\Phi = 2zA + 6\bar{B}z - 2(1 - \gamma)\rho z\bar{c} \]

but

\[ \Phi' = e^{-2i\sigma} \bar{\Phi} = \frac{\bar{z}}{z} \Phi \]

therefore

\[ \bar{z} \Phi - \bar{c} \Phi + 2i\bar{c} = \bar{\Phi}' = -\bar{z} \{A - (1 - \gamma)\rho \bar{c}\} - \frac{3B\bar{z}^2}{z} \]

or, combining 4.4 and 4.5

\[ 2(\mathcal{O} - i\mathcal{O}') = \{A + \rho (1 + \gamma)c\}z + 2\{\bar{A} + \gamma \rho \bar{c}\} \bar{z} + \frac{3\bar{B}z^2}{z} \]
For zero boundary stresses over both flanks we require the right hand side of 4.6 to be zero when $z = z \pm \text{i} \alpha$
or $\left\{ A + \rho(1+\gamma)c \right\} + 2\{ A + \gamma \rho \bar{c} \} e^{2\text{i} \alpha} + 3\rho e^{4\text{i} \alpha} = 0$
and $\left\{ A + \rho(1+\gamma)c \right\} + 2\{ A + \gamma \rho \bar{c} \} e^{-2\text{i} \alpha} + 3\rho e^{-4\text{i} \alpha} = 0$
subtracting we have

$$3B \cos 2\alpha = -(A + \gamma \rho \bar{c})$$

eliminating $B$ leads to

$$A \cos 2\alpha + A = -\gamma \rho \bar{c} - \rho(1+\gamma)c \cos 2\alpha$$
and

$$A \cos 2\alpha + A = -\gamma \rho \bar{c} - \rho(1+\gamma)c \cos 2\alpha$$
hence

$$A = -\gamma \rho \bar{c} - \rho(1+\gamma)c \cos 2\alpha$$
or using 4.1

$$A = -\frac{1}{4} \rho g \cos \beta \left\{ 2(1+\gamma) - \sec^2 \alpha \right\} + \frac{1}{4} \rho g \sin \beta \left\{ 2(1+\gamma) - \cosec^2 \alpha \right\}$$
and from 4.7

$$B = \frac{1}{12} \rho g \left\{ \cos \beta \sec^2 \alpha + i \sin \beta \cosec^2 \alpha \right\}$$

**STRESSES**

These may now be obtained from 4.4 and 4.6 but are here evaluated on the axis of symmetry.

thus

$$\frac{2}{\chi}(\Theta_0 - i\Theta_0) = A + 2A + 3B + \rho \left\{ (1+\gamma)c + 2\gamma \bar{c} \right\}$$
giving

$$\Theta_0 = \frac{1}{2} \rho gx \cos \beta \tan^2 \alpha$$
$$\Theta_0 = \frac{1}{2} \rho gx \sin \beta$$

Then from 4.4

$$\frac{1}{\chi}(\bar{r}_r + \bar{\Theta}_r) = A + \bar{A} + \rho(1+\gamma)(c + \bar{c})$$
giving

$$\bar{r}_r = \frac{1}{2} \rho gx \cos \beta$$
In the case when $\beta = 90^\circ$ it will be of some interest to obtain the stresses at any point of the wedge.

From 4.1

\[
\begin{align*}
c &= -\frac{1}{2}\rho g \\
A &= \frac{1}{4}\rho g \left\{ 2(1+\gamma) - \csc^2\alpha \right\} \\
B &= \frac{1}{2}\rho g \csc^2\alpha
\end{align*}
\]

also

\[
2(\bar{e} - \omega) = -\frac{1}{4}\rho g \csc^2\alpha z - \frac{1}{2}\rho g (2-\csc^2\alpha)z^2 - \frac{1}{4}\rho g \csc^2\alpha \frac{z^2}{2} = -\frac{1}{4}\rho g \left( z\csc^2\alpha + 2(2-\csc^2\alpha)z + \csc^2\alpha \frac{z^2}{2} \right)
\]

leading to

\[
\begin{align*}
\bar{e} &= \frac{1}{6}\rho g r \left[ \sin\theta(3\csc^2\alpha - 4) - \csc^2\alpha \sin\theta \right] \\
\bar{e} &= \frac{1}{6}\rho g r \left[ \cos\theta(4 - \csc^2\alpha) + \cos\theta \csc^2\alpha \right]
\end{align*}
\]

From 4.4

\[
\bar{r} + \bar{e} = -\frac{1}{4}\rho g \csc^2\alpha z + \frac{1}{4}\rho g \csc^2\alpha z
\]

\[
\bar{r} = \frac{1}{6}\rho g r \csc^2\alpha \sin\theta
\]

Leading to

\[
\bar{r} = \frac{1}{6}\rho g r \left[ \sin\theta(\csc^2\alpha + 4) + \csc^2\alpha \sin^2\theta \right]
\]

(b) Rotating wedge

Let the wedge rotate in its own plane about the vertex, with constant angular velocity $\omega$.

The body-force /unit mass = $\omega^2 r$

Thus

\[
\frac{\partial V}{\partial r} = -\omega^2 r \quad \text{and} \quad V = -\omega^2 r^2
\]

\[
U = \frac{1}{2} \omega^2 z \bar{z} = \frac{\partial W}{\partial z}
\]

\[
\therefore \ W = \frac{1}{4} \omega^2 z \bar{z}
\]

\[
\frac{\partial W}{\partial z} = \frac{1}{4} \omega^2 z \bar{z}
\]

4.16
from 2.15 the stresses are given by

\[ 2\theta' - 2\theta = 4(\theta + ir\theta) = \lambda'(z) + \lambda'(\bar{z}) + z\lambda''(z) + (4-\delta)\rho \frac{\partial \lambda}{\partial z} \]

\[ + \frac{1}{2} \frac{\partial \omega}{\partial z} - \frac{1}{2} \rho \frac{\partial \omega}{\partial z} \]

\[ 4(\theta + ir\theta) = \lambda'(z) + \lambda'(\bar{z}) + z\lambda''(z) + \frac{\partial \omega}{\partial z} \]

This function is required to be zero on \( \theta = \pm \alpha \), i.e. \( \bar{z} = ze^{\pm 2ia} \)

Use complex potentials

\[ \lambda(z) = Az^3 \quad \omega(z) = Bz^4 \quad (A \& B \text{ real}) \]

so that on \( \theta = \alpha \)

\[ 3Az^3 + 3A\bar{z}^3 + 6Az + 12Bz e^{2ia} - \frac{1}{2} \rho \omega(1 + 3\gamma)z\bar{z} = 0 \]

or

\[ 9A + 3Ae^{-4ia} + 12Be^{2ia} = \frac{1}{2} \rho \omega(1 + 3\gamma)e^{-2ia} \]

and

\[ 9A + 3Ae^{4ia} + 12Be^{-2ia} = \frac{1}{2} \rho \omega(1 + 3\gamma)e^{2ia} \]

so that

\[ 3Ae^{-2ia} + A e^{-6ia} + 4B = \frac{1}{6} \rho \omega(1 + 3\gamma)e^{-4ia} \]

\[ 3Ae^{2ia} + Ae^{6ia} + 4B = \frac{1}{6} \rho \omega(1 + 3\gamma)e^{4ia} \]

leading to

\[ A = \frac{1}{6} \frac{\rho \omega(1 + 3\gamma)\sin 4\alpha}{\sin 6\alpha + 3\sin 2\alpha} \]

\[ B = -\frac{1}{12} \frac{\rho \omega(1 + 3\gamma)\sin 2\alpha}{\sin 6\alpha + 3\sin 2\alpha} \]
Stresses

\[ 2\ddot{\theta} = \ddot{\theta}''(z) + \ddot{\theta}'(z) + (4 - \delta)\rho \frac{\partial^2 W}{\partial z^2} - 2(1 + \gamma)\rho \frac{\partial \dot{W}}{\partial z} \]

\[ = 3Az^2 + 3Az^2 - 2(1 + \gamma)\rho \frac{\partial \dot{W}}{\partial z} \quad \text{and} \quad z = re^{i\theta} \]

\[ 2(r\ddot{r} + \ddot{\theta}) = 6Ar^2 \cos^2 \theta - \rho \omega^2 (1 + \gamma) r^2 \]

\[ -2\ddot{\theta}' = \ddot{\theta}''(z) + e^{-2i\theta}\left\{ \ddot{\theta}''(z) - 6\rho \frac{\partial \dot{W}}{\partial z} \right\} \]

\[ = 6Az^2 + e^{-2i\theta}\left\{ 12Bz^2 + 2(1 - \gamma)\rho \frac{1}{4} \omega^2 \right\} \]

\[ = 6Ar^2 e^{-2i\theta} + 12Br^2 e^{-4i\theta} + \frac{1}{2} \rho \omega^2 (1 - \gamma) r^2 \]

so that

\[ -2(r\ddot{r} + \ddot{\theta}) = 6Ar^2 \cos^2 \theta + 12Br^2 \cos^4 \theta + \frac{1}{2} \rho \omega^2 (1 - \gamma) r^2 \]

\[ -4\ddot{\theta} = -6Ar^2 \sin^2 \theta - 12Br^2 \sin^4 \theta \]

leading to

\[ \ddot{\theta} = \frac{1}{8} \rho (1 + 3\gamma) \omega r^2 \left[ \frac{4\cos^2 \theta \cdot \sin 4\alpha - 2\cos^2 \theta \cdot \sin 2\alpha}{\sin 6\alpha + 3\sin 2\alpha} - 1 \right] \]

\[ \ddot{r} = \frac{1}{8} \rho \omega r^2 \left[ \frac{2(1 + 3\gamma) \cos 2\theta \cdot \sin 2\alpha}{\sin 6\alpha + 3\sin 2\alpha} - (3 + \gamma) \right] \]

\[ \ddot{\theta} = -\frac{1}{2} \rho \omega r^2 (1 + 3\gamma) \sin 2\theta \cdot \sin 2\alpha \left\{ \cos 2\theta - \cos 2\alpha \right\} \]
CHAPTER V
Isolated Force and Couple at the Vertex

(a) Force nucleus

Let the force be \( R \) at angle \( \beta \) to the axis of the wedge, so that we may write

\[
\mathbf{F} = R \cos \beta \mathbf{i} + R \sin \beta \mathbf{j} = R(\cos \beta + i \sin \beta) \mathbf{i} = R e^{i \beta} \mathbf{i}
\]

Consider the complex potentials

\[
\mathcal{A}(z) = A \log z \quad \mathcal{\omega}(z) = Bz \log z \quad (A \text{ and } B \text{ complex})
\]

with no body-force

\[
4(\bar{\mathcal{\Omega}} + i \mathcal{\Omega}) = \mathcal{A}'(z) + \bar{\mathcal{A}}'(\bar{z}) + z \mathcal{A}''(z) + \frac{z}{2} \mathcal{\omega}''(z)
\]

\[
= \frac{A}{z} + \frac{\bar{A}}{\bar{z}} - \frac{A}{z} + \frac{B}{z} \frac{\bar{B}}{\bar{z}}
\]

\[
= (A + B) \left( \frac{1}{z} \right)
\]

We may make \( \mathcal{\Omega} = r \mathcal{\Omega} = 0 \) everywhere by taking \( A = -B \)

Considering now the equilibrium of a portion OAB of material, the resultant couple must be zero since only a radial stress acts along AB. The resultant force may be found from 2.20

\[
-\text{Re}^{i \beta} = -\frac{1}{4} i \left[ A \log z + z \frac{A}{z} - A(1 + \log z) \right]^{B}_{A}
\]

\[
= -\frac{1}{4} i \left[ \frac{A}{z} - A + A \log z \right]^{B}_{A}
\]

\[
= \frac{1}{2} \left\{ A \sin 2\alpha + 2A\alpha \right\}
\]

or \( A \sin 2\alpha + 2A\alpha = -2\text{Re}^{i \beta} \)

and \( A \sin 2\alpha + 2\bar{A}\alpha = -2\text{Re}^{-i \beta} \)

leading to

\[
A = -2R \left\{ \frac{\cos \beta}{2\alpha + \sin 2\alpha} + i \frac{\sin \beta}{2\alpha - \sin 2\alpha} \right\}
\]
The remaining stress component
\[ \ddot{r}r = \Theta' = \frac{1}{2} \left\{ \mu'(z) + \mu'(\bar{z}) \right\} = \frac{1}{2r} \left\{ Ae^{-i\theta} + \overline{A}e^{i\theta} \right\} \]

or
\[ \ddot{r}r = \frac{2R}{r} \frac{\sin 2\alpha \cos (\beta + \phi) - 2\alpha \cos (\beta - \phi)}{4\alpha^2 - \sin^2 2\alpha} \]

(5.5)

(b) Couple nucleus

Consider \( \mathcal{J}(z) = \frac{A}{z} \) and \( \omega(z) = B\log z \)

\[ 4(\ddot{\Theta} + i\dot{\Theta}) = -\frac{A}{z^2} - \frac{\overline{A}}{z^2} + z^2 \frac{A}{z^3} + \frac{z}{2} \left( -\frac{B}{z^2} \right) \]

\[ = \frac{A}{z^2} - \frac{\overline{A}}{z^2} - \frac{B}{z^2} \]

\[ = \frac{1}{r} \left( Ae^{-2i\theta} - \overline{A}e^{2i\theta} - B \right) \]

which will vanish over \( \theta = \pm \alpha \) if

\[ Ae^{-2i\alpha} - \overline{A}e^{2i\alpha} - B = 0 \]

\[ Ae^{2i\alpha} - \overline{A}e^{-2i\alpha} - B = 0 \]

so that \( A + \overline{A} = 0 \) or \( A = iC \) where \( C \) is real

and \( \quad B = 2iC\cos 2\alpha \)

(5.6)

One further condition will be obtained by considering the equilibrium of the portion OAB of the wedge.

from 2.20 the resultant force is

\[ X + iY = -\frac{1}{4z} \left[ \frac{A}{z} + z(-\frac{\overline{A}}{z^2}) + \frac{B}{z} \right] A \]

or

\[ X - iY = \frac{1}{4z} \left[ \frac{A}{z} - \frac{\overline{A}}{z^2} + \frac{B}{z^2} \right] A \]

which is zero from 5.6

from 2.21 the resultant couple is

\[ N = \text{real part of} \left[ \frac{1}{4} \left[ z\mathcal{J}'(z) + z\omega'(z) - \omega(z) \right] \right] A \]

\[ = \left[ \text{"} \quad \text{"} \right] \frac{1}{4} \left[ -\frac{A\overline{z}}{z} + B - B\log z \right] A \]

\[ = \left[ \text{"} \quad \text{"} \right] \frac{1}{4} \left[ -Ae^{-2i\alpha} + Ae^{2i\alpha} - B(2i\alpha) \right] \]
or \[ G = \frac{1}{4} \{ 2A \sin 2\alpha - 2i\beta \} \]
\[ G = \frac{1}{2} \{-C\sin 2\alpha + a \cdot 2C\cos 2\alpha \} \]
\[ C = \frac{2G}{2\alpha \cos 2\alpha - \sin 2\alpha} \]

Stresses

from 5.6 \[ 4(\Theta + ir\Theta) = \frac{iC}{r^2} (2\cos 2\Theta - 2\cos 2\alpha) \]

so that \[ \Theta = 0 \]
\[ r\Theta = \frac{G}{r^2} \frac{\cos 2\Theta - \cos 2\alpha}{\cos 2\alpha - \sin 2\alpha} \]
\[ r\rho = \frac{i}{2} \{ \mathcal{L}(z) + \mathcal{L}^*(\bar{z}) \} = \frac{i}{2} \{ \frac{-A}{z} - \frac{\bar{A}}{\bar{z}} \} \]

or \[ r\rho = -\frac{2G}{r^2} \frac{\sin 2\Theta}{2\alpha \cos 2\alpha - \sin 2\alpha} \]

Criticism of paper on "Tapered Beams" by E.H. Atkin (ref. 7)

Using the solutions of this Chapter together with 3.3, 3.4 and 3.5 it is possible to find the extent of error in this work. The problem of a uniform flank loading is treated by considering the equivalent force and couple at the vertex, which will not in general give the same stress system.

Consider a uniform distribution of vertical load \( w/\text{unit flank length} \). The stresses on the axis distant \( d \) from the vertex will be found.

The equivalent apex force and couple are

\[ W = wd\sec \alpha \quad \text{and} \quad G = \frac{1}{2} wd^2\sec \alpha \]

True centre line stresses from 3.3, 4, 5

\[ \Theta = \frac{1}{2} (A + C) = -\frac{W}{2\alpha \sec \alpha} \]
\[ r\Theta = \frac{1}{4} (B + 2C) = -\frac{w\sin \alpha}{2\alpha \cos 2\alpha - \sin 2\alpha} \]
and \( \bar{r}r' = \frac{1}{2}(A - C,) = 0 \)

**Stresses on \( \theta = 0 \) obtained by superimposing \( W \) and \( G \)**

For \( W \)

\[
\bar{r}r = \bar{\theta} = \bar{r} = 0
\]

For \( G \)

\[
\bar{\theta} = \bar{r} = 0
\]

\[
\bar{r}_0 = -\frac{w}{2 \cos \alpha} \frac{1 - \cos 2\alpha}{2 \alpha \cos \alpha - \sin 2\alpha}
\]

It will be observed that when \( \alpha \) is small the stresses \( \bar{\theta}, \) and \( \bar{r}_0 \) are approximately equal to \( \frac{3w}{8\alpha} \) whilst \( \bar{\theta}, = -\frac{1}{2}w \).

**Three points should be mentioned**

1. The % error involved is

\[
\frac{\bar{r}_0 - \bar{r}_0}{\bar{r}_0} \times 100 = 100\left\{\frac{\tan \alpha}{\alpha} - 1\right\} = \frac{1}{3} \alpha^2 \times 100 \text{(approx)}
\]

so that Atkin does in fact over estimate, and the error is only 5% if \( 2\alpha = 44^\circ 18' \) (one of the values quoted), so that the approximation is quite good.

2. Atkin's method loses the stress \( \bar{\theta} \) entirely, and whilst this is negligible for one case which he uses (\( 2\alpha = 4^\circ 18' \)) it does reach 50% of the value of \( \bar{\theta} \) if \( 2\alpha = 44^\circ 18' \) so that the approximation whilst apparently good, would be somewhat in error for the principal stress.

3. The above comparison holds good only for the particular distribution of flank loading used and cannot be put forward, as Atkin does, as a method for any wedge loading, including force nuclei.
CHAPTER VI

Isolated Couple Nucleus at \( z = d \) (real) with stress free boundaries.

For an isolated couple nucleus \( G \) at \( z = d \) the complex potentials are known to be,

\[
\mathcal{J}_c(z_i) = 0, \quad \omega_c(z_i) = \frac{2iG}{\pi} \log z_i, \quad 6.1
\]

(see ref 2 pg 154 eqn 11.35)

on changing the origin to 0

\[
\mathcal{J}_c(z) = 0, \quad \omega(z) = \frac{2iG}{\pi} \log z, \quad \text{where } z_i = z - d
\]

now

\[
4(\hat{\theta} + ir\hat{\phi}) = \mathcal{J}_c'(z) + \mathcal{J}_c'(\bar{z}) + z\mathcal{J}_c(z) + \frac{z}{2}\omega''(z)
\]

\[
= \frac{z}{2} \frac{2iG}{\pi} \left( \frac{1}{z_i} \right) \quad \text{and on the flank, } z = re^{i\alpha}
\]

\[
\hat{\theta}' + i\hat{\phi}' = -\frac{G}{2\pi} \frac{e^{-2i\alpha}}{\left( re^{-i\alpha} - d \right)^2}
\]

\[
= -\frac{G}{2\pi} \frac{1}{(r - de^{i\alpha})^2} \quad 6.2
\]

We wish to wipe off this stress and so will apply the Mellin transform method described in Chapter II b.

Thus\[\tilde{t}(p) = \frac{G}{2\pi} \int_0^{\infty} \frac{r^{p+1}dr}{(r - de^{i\alpha})^2} = \frac{G}{2\pi} \: I(p, \alpha)\]

So the boundary problem which we have to solve is given by

\[
(p+1)(D+ip)\tilde{\chi} = \frac{G}{2\pi} \: I(p, \alpha) \quad \text{over } \theta = \alpha \quad 6.3
\]

\[
= \frac{G}{2\pi} \: I(p, -\alpha) \quad \text{over } \theta = -\alpha
\]
Since the stresses are of order $r^{-2}$ at infinity we have from Ch IIb (17) $n = 2$ and $-2 < p < 0$

**Evaluation of $I(p, \alpha)$**

This may be accomplished by means of a contour integral.

Use the contour shown and consider

$$\int_{C} \frac{z^{p+1}dz}{(z - de^{i\alpha})^2}$$

The zero of the denominator is at $z = de^{i\alpha}$

Thus

$$\int_{C} = \int_{C_{1}} + \int_{C_{2}} + \int_{C_{3}} + \int_{C_{4}} = 2\pi i \times \text{residue at } de^{i\alpha}$$

now on $C_{2}$, $z = x$ and on $C_{4}$, $z = xe^{2\pi i}$

$$\int_{C_{2}} = \int_{\infty}^{0} \frac{x^{p+1}dx}{(x - de^{i\alpha})^2} = I(p, \alpha)$$

$$\int_{C_{4}} = \int_{0}^{\infty} \frac{x^{p+1}e^{2(p+1)i\alpha}dx}{(x - de^{i\alpha})^2} = -e^{2\pi i} I(p, \alpha)$$

On $C_{1}$ and $C_{3}$, $z = re^{i\theta}$ and $|z - de^{i\alpha}| \geq (r-d)$

$$\int_{C_{1}} + \int_{C_{3}} \leq 2\pi r \frac{r^{p+1}}{(r-d)^2}$$

and since $-2 < p < 0$ this $\to 0$ as $r \to 0$ or as $r \to \infty$.

$$(1 - e^{2\pi i})I = 2\pi i x \times \text{residue of } \int \frac{z^{p+1}dz}{(z - de^{i\alpha})^2} \text{ at } z = de^{i\alpha}$$

to find the residue put $z - de^{i\alpha} = t$
residue = coefficient of \( \frac{1}{t} \) in \( \frac{1}{t^p} (t + de^{i\alpha})^{p+1} \)

\[ = (p+1)d^{p}e^{ipa} \]

Thus \( (1 - e^{2p\pi i})I(p,\alpha) = 2\pi i d^{p}e^{ipa}(p+1) \)

\[ e^{p\pi i(-2\sin p\pi)}I(p,\alpha) = 2\pi i d^{p}e^{ipa}(p+1) \]

\[ I(p,\alpha) = \frac{\pi d^{p}e^{-ip\beta}\theta(p+1)}{\sin p\pi} \text{ where } \beta = \pi - \alpha \]

and since \( I(p,-\alpha) = \overline{I(p,\alpha)} \)

\[ I(p,-\alpha) = \frac{\pi d^{p}e^{ip\beta}(p+1)}{\sin p\pi} \]

The boundary conditions for the partial (equilibrating) solution are accordingly obtained from

\[ \tilde{\chi} = Ae^{ip\theta} + \overline{A}e^{-ip\theta} + Be^{i(p+2)\theta} + \overline{B}e^{-i(p+2)\theta} \]

\[ (p+1)(D+ip)\tilde{\chi} = 2i(p+1)\left[ pAe^{ip\theta} + B(p+1)e^{i(p+2)\theta} - \overline{B}e^{-i(p+2)\theta} \right] \]

so that from Ch IIb (31)

over \( \theta = \alpha \)

\[ pAe^{ipa} + B(p+1)e^{i(p+2)\alpha} - \overline{B}e^{-i(p+2)\alpha} = \frac{iGd^{p}e^{-ip\beta}}{4 \sin p\pi} \]

over \( \theta = -\alpha \)

\[ pAe^{-ipa} + B(p+1)e^{-i(p+2)\alpha} - \overline{B}e^{i(p+2)\alpha} = \frac{iGd^{p}e^{ip\beta}}{4 \sin p\pi} \]

and the conjugates

\[ p\overline{A}e^{-ipa} + \overline{B}(p+1)e^{-i(p+2)\alpha} - Be^{i(p+2)\alpha} = \frac{-iGd^{p}e^{ip\beta}}{4 \sin p\pi} \]

\[ p\overline{A}e^{ipa} + \overline{B}(p+1)e^{i(p+2)\alpha} - Be^{-i(p+2)\alpha} = \frac{-iGd^{p}e^{-ip\beta}}{4 \sin p\pi} \]

These equations imply that \( \overline{A} = -A \) and \( \overline{B} = -B \) so that
\[ pA + B(p+1)e^{2i\alpha} + Be^{-2i(p+1)\alpha} = \frac{iGd^p e^{-ip\pi}}{4\sin p\pi} \]
\[ pA + B(p+1)e^{-2i\alpha} + Be^{2i(p+1)\alpha} = \frac{iGd^p e^{ip\pi}}{4\sin p\pi} \]

subtracting
\[ B \left\{ 2i\sin(2p+2)\alpha + (p+1)(-2i\sin 2\alpha) \right\} = \frac{iGd^p}{4\sin p\pi} \cdot 2i \sin p\pi \]
\[ B = -\frac{iGd^p}{4G(p,\alpha)} \]

where
\[ G(p,\alpha) = (p+1)\sin 2\alpha - \sin 2(p+1)\alpha \]
and
\[ H(p,\alpha) = (p+1)\sin 2\alpha + \sin 2(p+1)\alpha \]
\[ J(p,\alpha) = (p+1)\cos 2\alpha - \cos 2(p+1)\alpha \]
\[ K(p,\alpha) = (p+1)\cos 2\alpha + \cos 2(p+1)\alpha \]

which will be used in later problems.

solving for \( A \)
\[ pA = \frac{iGd^p}{4G(p,\alpha)} \left\{ e^{-i(2p+2)\alpha} + (p+1)e^{2i\alpha} \right\} = \frac{iGd^p e^{-ip\pi}}{4\sin p\pi} \]
\[ pA = \frac{iGd^p}{4G(p,\alpha)\sin p\pi} \left[ \sin p\pi \left\{ e^{-i(2p+2)\alpha} + (p+1)e^{2i\alpha} \right\} 
\[ + G(p,\alpha)e^{-ip\pi} \right] \]

leading to
\[ A = \frac{iGd^p}{4pG(p,\alpha)} \frac{K(p,\alpha)\sin p\pi + G(p,\alpha)\cos p\pi}{\sin p\pi} \]

The stresses
Consider first \( \Theta' \)
\[ (D^r + p^z)\tilde{\chi} = \left\{ -(p+2)^r + p^z \right\} \left[ B e^{i(p+2)\theta} + B e^{-i(p+2)\theta} \right] \]
but \( \bar{B} = -B \)
\[ = -4(p+1).B.2i\sin(p+2)\theta \]
\[ = -\frac{2Gd^p(p+1)\sin(p+2)\theta}{G(p,\alpha)} \]
whence from the inversion formula for \( i \cdot e \)

\[
\Theta' = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (D^x + p^x) \frac{\gamma}{\Gamma x} r^{-p-2} dp
\]

we have

\[
\Theta' = -\frac{G}{\pi i r} \int_{c-i\infty}^{c+i\infty} \frac{(d)^p}{(x)^2} \frac{(p+1)\sin(p+2)}{G(p,\alpha)} dp
\]

in which the complex \( p = c + i\gamma \) replaces the real \( p \) of the Mellin transform. Note that the restrictions on \( p \) given by eqn. (3) Ch IIb refer to the real part of \( p \), so that in the inversion formulae we must take

\[
-2 < c < 0
\]

The restriction of eqn. (2) Ch IIb is in accord with this and gives nothing new since,

\[
\Theta' = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ (D^x + (u-2)) \frac{\gamma}{\Gamma x} r^{-u} du \right\}
\]

so that from that eqn.

\( c + 2 > k > 0 \) or \( c > -2 \)

The restrictions 6.10 are the over-riding restrictions on the value of \( p \).

If we attempt to press the value of \( c \) to the limit zero, we must examine the integrand of \( \Theta' \) for possible poles on the imaginary axis, since the equivalence of the path BA to the path CD rests on Cauchy's theorem for the space enclosed between them and this must contain no singularity.

The zeros of the denominator \( G(p,\alpha) \) of 6.49 in the range occur at \( p = 0, -1, -2 \) but the numerator contains the factors
(p+1)\sin(p+2)\theta \) so that in fact the only pole is at \( p = 0 \)
and we must exclude this from the shaded region by a semi-
circular indentation of radius \( \xi \) which we allow to tend to zero.
writing \( p = i\gamma \)

\[
G(p,\alpha) = (1+i\gamma)\sin2\alpha - \sin(2+2i\gamma)\alpha = \sin2\alpha + i\gamma\sin2\alpha - \sin2\alpha\cosh2\gamma\alpha - i\cos2\alpha\sinh2\gamma\alpha = E_i + i0, \\
\]
where \( E_i = \sin2\alpha(1 - \cosh2\gamma\alpha) \)
\( O_i = \gamma\sin2\alpha - \cos2\alpha\sinh2\gamma\alpha \)

N.B. We are thinking here of even and odd parts rather than
real and imaginary parts, so as to reduce \( \int_{-\infty}^{0} \) to \( 2\int_{0}^{\infty} \)
at a later stage.

Also \( (\frac{d}{d\alpha})^{i\gamma} = e^{i\gamma}\log\frac{d}{\alpha} = e^{i\gamma}m \) where \( m = \log\frac{d}{\alpha} \)
and

\[
\begin{align*}
(1+i\gamma)\sin(2+2i\gamma)\theta &= (1+i\gamma)(\sin2\theta\cosh\gamma\theta + i\cos2\theta\sinh\gamma\theta) \\
&= E_2 + iO_2
\end{align*}
\]

where \( E_2 = \sin2\theta\cosh\gamma\theta - \gamma\cos2\theta\sinh\gamma\theta \)
\( O_2 = \gamma\sin2\theta\cosh\gamma\theta + \cos2\theta\sinh\gamma\theta \)

Thus we may now write
\[
\Theta' = -\frac{G}{\pi ir^2} \left( \int_{-\infty}^{0} e^{im\gamma\theta}(E_2 + iO_2)(E_i - i0) \right) \frac{d\gamma}{E_i^2 + O_i^2 - \pi R}
\]

where \( R \) is the residue of the integrand at \( p = 0 \) and

\[
\begin{align*}
E &= E_i E_2 + O_i O_2 \\
&= \sin2\alpha(1 - \cosh2\gamma\alpha)(\sin2\theta\cosh\gamma\theta - \gamma\cos2\theta\sinh\gamma\theta) \\
&+ (\gamma\sin2\alpha - \cos2\alpha\sinh2\gamma\alpha)(\gamma\sin2\theta\cosh\gamma\theta + \cos2\theta\sinh\gamma\theta)
\end{align*}
\]
\[ 0 = 0_2 E_1 - 0_1 E_2 \]
\[ = \sin 2\alpha (1 + \cosh 2\gamma \alpha) (\gamma \sin 2\theta \cosh \gamma \theta + \cos 2\theta \sinh \gamma \theta) \]
\[ - (\gamma \sin 2\alpha \cos 2\alpha \sinh 2\gamma \alpha) (\sin 2\theta \cosh \gamma \theta - \gamma \cos 2\theta \sinh \gamma \theta) \]

whence

\[ \omega' = - \frac{G}{\pi r} \left[ 2 \int_0^\infty \frac{\cos \eta m - \sin \eta m}{E_1'' + 0_1''} \, d\eta - \pi R \right] \]

Value of residue R

\[ E_1 + i0_1 = i\gamma (\sin 2\alpha - 2a \cos 2\alpha) + O(\gamma^2) \]
\[ R = \frac{i \sin 2\theta}{i(\sin 2\alpha - 2a \cos 2\alpha)} \]

so that

\[ \omega' = - \frac{G}{\pi r} \left[ 2 \int_0^\infty \frac{\cos \eta m - \sin \eta m}{E_1'' + 0_1''} \, d\eta - \frac{\pi \sin 2\theta}{\sin 2\alpha - 2a \cos 2\alpha} \right] \]

Consider now the stress combination \( \gamma' \)

\[ (D + i\rho)(D + i(p+2)) \tilde{X} = -4A(p+1)e^{ip\theta} - 4B(p+1)(p+2)e^{i(p+2)\theta} \]

and this becomes on putting in the values of A and B.

\[ = - \frac{iGd}{G(p,\alpha) \sin \pi \rho} \frac{e^{ip\theta}}{\rho \sin \pi \rho} Z(p,\theta) \]

where \( Z(p,\theta) = G(p,\alpha) \cos \pi \rho + K(p,\alpha) \sin \pi \rho - (p+2) e^{2i\theta} \sin \pi \rho \)

hence

\[ \gamma' = - \frac{G}{2\pi r^2} \int_{c-i\infty}^{c+i\infty} \frac{(d)^{1+\rho}}{G(p,\alpha) \sin \pi \rho} Z(p,\theta) \, dp \]

Although the zeros of \( G(p,\alpha) \sin \pi \rho \) in the relevant range are again \( p = -2, -1, 0 \), \( p = -1 \) will not be a pole because of the factor \( p+1 \) in the numerator, with \( p = -2 \) in \( Z(p,\theta) \) we have

\[ Z(-2,\theta) = G(-2,\alpha) = -\sin 2\alpha + \sin 2\alpha = 0 \]

and so, again, \( p = 0 \) is the only pole and we take \( c = 0 \), isolate the pole and put \( p = i\gamma \).
as before \( \left( \frac{d}{dr} \right)^p e^{ip\theta} = e^{ip\theta} e^{i\gamma m} = e^{i\gamma k} \) where \( k = m + i\theta \)

also \( G(p, \alpha) = E_3 + i0_3 \) and \( \sin \pi = \sinh \pi \gamma \)

\[
K(p, \alpha) = E_3 + i0_3
\]

where

\[
E_3 = \cos 2\alpha (1 + \cosh 2\gamma \alpha)
\]

\[
0_3 = \gamma \cos 2\alpha - \sin 2\alpha \sinh 2\gamma \alpha
\]

\[Z(p, \Theta) = (E_3 + i0_3) \sinh \pi \gamma + (E_1 + i0_1) \cosh \pi \gamma - (2+i\gamma)e^{2i\Theta} \sinh \pi \gamma
\]

\[= E_4 + i0_4
\]

where \( E_4 = -O_4 \sinh \pi \gamma + E_1 \cosh \pi \gamma + \gamma e^{2i\Theta} \sinh \pi \gamma \)

\[O_4 = E_3 \sinh \pi \gamma + 0_1 \cosh \pi \gamma - 2e^{2i\Theta} \sinh \pi \gamma
\]

thus

\[
\frac{d}{2\pi r^2} \left[ \int_{-\infty}^{\infty} \frac{e^{i\gamma k}(1+i\gamma)(E_4 + i0_4)id\gamma}{\sinh \pi \gamma (E_4 + i0_4)} - \pi iR \right]
\]

Residue R

\[
E_4 + i0_4 = i\gamma (\sin 2\alpha - 2\alpha \cos 2\alpha) + O(\gamma^2)
\]

\[
\sinh \pi \gamma = \pi \gamma + O(\gamma^2)
\]

\[
O_4 = i\gamma (2\pi \cos 2\alpha - 2\pi e^{2i\Theta} + \sin 2\alpha - 2\alpha \cos 2\alpha) + O(\gamma^2)
\]

\[
E_4 = O(\gamma^2)
\]

so that \( \pi R = 1 + \frac{2\pi(\cos 2\alpha - e^{2i\Theta})}{\sin 2\alpha - 2\alpha \cos 2\alpha} \)

and \( \frac{d}{2\pi r^2} \left[ \int_{-\infty}^{\infty} \frac{e^{i\gamma k}(E_4 + i0_4) + i(\gamma E_4 + 0_4) + (E_1 - i0_1) d\gamma}{\sinh \pi \gamma (E_4 + i0_4)} - \pi iR \right]
\]

the numerator \( = E_5 + i0_5 \) where

\[
E_5 = E_1 E_4 + 0_1 0_4 + \gamma (0_1 E_4 - E_1 0_4)
\]

\[
0_5 = E_1 0_4 - 0_1 E_4 + \gamma (E_4 E_4 + 0_1 0_4)
\]

... giving finally
giving finally
\[
\Phi' = -\frac{Gi}{2\pi \mu} \left[ 2 \int_0^{\infty} 0.5 \cos k \gamma + 0.5 \sin k \gamma \frac{d\gamma}{\sin \pi \gamma (\frac{E}{\pi} + 0^2)} - \pi R \right]
\]
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where \( R \) is given by 6.20. This completes the formal solution for the stresses as far as one can go without numerical calculation. The stresses arising from the original complex potentials as given by 6.2 make be added to those given by 6.15 and 6.21 to obtain the complete solution.

The Displacements

referring to Ch II (b) equn. (29)
\[
(D + ip)\tilde{\chi} = 2i [pAe^{ip\Theta} + B(p+1)e^{i(p+2)\Theta} - Be^{-i(p+2)\Theta}]
\]
\[
(D - ip)(D + ip)\tilde{\chi} = -4(p+1) [Be^{i(p+2)\Theta} + Be^{-i(p+2)\Theta}]
\]
\[
(D - ip)(D + ip)(D - i(p+2))\tilde{\chi} = 8i(p+1)(p+2) Be^{-i(p+2)\Theta}
\]
\[
U_i + iU_o = -\frac{1}{4\pi \mu} \int_{\infty}^{-\infty} 2i [pAe^{ip\Theta} + B(p+1)e^{i(p+2)\Theta} - Be^{-i(p+2)\Theta} + (1-\sigma)4Be^{-i(p+2)\Theta}]
\]
\[
\int_{\infty}^{\infty} dp
\]
or, inserting the values of \( A \) and \( B \) given by 6.6 and 6.8 we have
\[
U_i + iU_o = \frac{G}{8\pi \mu r} \int_{\infty}^{-\infty} \frac{Y(p,\Theta)}{\sin \pi \gamma G(p,\alpha)} \left( \frac{d\gamma}{\gamma} \right) dp
\]
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where
\[
Y(p,\Theta) = \sin \pi \gamma \left[ \frac{K(p,\alpha) - (p+1)e^{2i\Theta}}{2} + (3-4\sigma)e^{-2i(p+1)\Theta} \right] G(p,\alpha) \cos \pi \gamma
\]
to simplify 6.22 we again take \( c = 0 \) and put \( p = i \gamma \) and note that \( p = 0 \) is still the only pole of the integrand for
\[-2 < \Re p < 0 \]
as before \( e^{i\varphi}(\frac{d}{dr})^P = e^{ik}\gamma \) where \( k = \log_{r} + i\theta \)

\[ G(p,\alpha) = E_{i} + iO_{i} \quad \text{and} \quad K(p,\alpha) = E_{j} + iO_{j} \]

\[ Y(p,\Theta) = \text{isinh}^{-1}\left[ E_{i} + iO_{i} - (1+i\gamma)e^{2i\Theta} + (3-4\sigma)e^{-2i\Theta}(\cosh 2\gamma + \sinh 2\gamma) \right] \]

\[ + (E_{j} + iO_{j})\cosh 2\gamma \]

\[ = E_{i} + iO_{i} \quad \text{where} \]

\[ E_{i} = E_{i}\cosh \gamma + \sinh \gamma \left[ e^{2i\Theta} - 0_{2} + (3-4\sigma)e^{-2i\Theta}\sinh 2\gamma \right] \]

\[ 0_{i} = Q_{i}\cosh \gamma + \sinh \gamma \left[ E_{i} - e^{2i\Theta} + (3-4\sigma)e^{-2i\Theta}\cosh 2\gamma \right] \]

so that from 6.22

\[ U_{a} + iU_{\varphi} = \frac{G}{\frac{\partial}{\partial \mu} \pi} \left[ \int_{-\infty}^{\infty} \frac{e^{ik}\gamma(E_{i} + iO_{j})(E_{j} - iO_{j})}{\text{isinh}^{-1}(E_{i} + O_{i})} \, d\gamma - i\pi R \right] \]

where \( R \) is the residue at \( \gamma = 0 \)

\[ \therefore \pi R = \lim_{\gamma \to 0} \frac{e^{ik}\gamma(E_{i} + iO_{j})}{(E_{i} + iO_{j})} \]

\[ = \lim_{\gamma \to 0} \left[ \cosh \gamma + \sinh \gamma \left[ E_{i} + iO_{j} - (1+i\gamma)e^{2i\Theta} + (3-4\sigma)e^{-2i\Theta}(\cosh 2\gamma + \sinh 2\gamma) \right] \right] \]

\[ = 1 + \pi \left[ \frac{2\cos 2\alpha - e^{2i\Theta} + (3-4\sigma)e^{-2i\Theta}}{\sin 2\alpha - 2\cos 2\alpha} \right] \]

so that

\[ U_{a} + iU_{\varphi} = \frac{G}{\frac{\partial}{\partial \mu} \pi} \left[ \int_{-\infty}^{\infty} \frac{e^{ik}\gamma(E_{i} + iO_{j})}{\text{isinh}^{-1}(E_{i} + O_{i})} \, d\gamma - i\pi R \right] \]

where \( E_{i} = E_{i}, O_{i} = O_{j}, \) and \( O_{j} = O_{j}, E_{j} = E_{j}, 0_{j} \)

\[ U_{a} + iU_{\varphi} = \frac{G}{\frac{\partial}{\partial \mu} \pi} \left[ 2 \int_{-\infty}^{\infty} \frac{E_{i}\text{sink} + O_{i}\cos k\gamma}{\text{sinh}^{-1}(E_{i} + O_{i})} \, d\gamma - i\pi R \right] \]

Displacement due to original complex potentials must be added to this solution and is obtained from 2.8 and 6.1 as

\[ \delta_{\mu D} = \frac{e^{-i\Theta}}{(8\mu D)} \quad \therefore U_{a} + iU_{\varphi} = \frac{G}{4\pi \mu r} \frac{1}{\delta_{\mu D}} \]
Semi-infinite plate under couple nucleus.

The integrals obtained for the stresses and displacements are not integrable except by approximate methods, but an algebraic solution can be found for the case of a semi-infinite plate corresponding to the angle $\alpha = \pi/2$, since the integrals can then be evaluated. This reflects the fact that complex potentials cannot be found for the wedge under couple nucleus with stress free flanks when $\alpha \neq \pi/2$, but have been found by A.C. Stevenson (ref. 2 pg. 155) for the semi-infinite plate.

It is of interest to see how the above results reduce, and respond to direct integration in this case and also to show the agreement with the work of Stevenson.

Value of $\Theta'$ when $\alpha = \pi/2$

from eqns. 6.11 to 6.13

\[
E' = 0, \quad o' = \sinh \pi \gamma
\]

\[
E = \sinh \pi \gamma \left\{ \gamma \sin 2\theta \cosh \gamma + \cos 2\theta \sinh \gamma \right\}
\]

\[
o = -\sinh \pi \gamma \left\{ \sin 2\theta \cosh \gamma - \gamma \cos 2\theta \sinh \gamma \right\}
\]

from 6.15 we have for $\Theta'$

\[
-\frac{\pi}{2G} \Theta' + \frac{1}{2} \sin 2\theta = I_1 \sin 2\theta + I_2 \cos 2\theta
\]

\[6.27\]

where

\[
I_1 = \int_0^\infty \frac{\cosh \gamma \theta \sin \gamma \theta + \gamma \cosh \gamma \theta \cos \gamma \theta}{\sinh \gamma \theta} \, d\gamma
\]

\[
I_2 = \int_0^\infty \frac{\sinh \gamma \theta \cos \gamma \theta - \gamma \sinh \gamma \theta \sin \gamma \theta}{\sinh \gamma \theta} \, d\gamma
\]

These integrals have been evaluated in Edwards "Integral Calculus " Vol II Pg 277 and take the values;
\[
\int_0^\infty \frac{\cosh \theta \cdot \sin \gamma}{\sinh \pi \gamma} \, d\gamma = \frac{1}{2} \frac{\sinh m}{\cos \theta + \cosh m}
\]
\[
\int_0^\infty \frac{\sinh \theta \cdot \cos \gamma}{\sinh \pi \gamma} \, d\gamma = \frac{1}{2} \frac{\sin \theta}{\cos \theta + \cosh m}
\]
and by differentiating with respect to \( \theta \)
\[
\int_0^\infty \frac{\sinh \theta \cdot \sin \gamma}{\sinh \pi \gamma} \, d\gamma = \frac{1}{2} \frac{\sinh m \cdot \sin \theta}{(\cos \theta + \cosh m)^2}
\]
\[
\int_0^\infty \frac{\cosh \theta \cdot \cos \gamma}{\sinh \pi \gamma} \, d\gamma = \frac{1}{2} \frac{1 + \cos \theta \cdot \cosh m}{(\cos \theta + \cosh m)^2}
\]
so that since \( \cosh m = \frac{1}{2} (e^m + e^{-m}) = \frac{1}{2} (\frac{d}{r} + \frac{r}{d}) \)
\[
I_1 = \frac{4r^2 \partial^2 + d^4 - r^4 + 4rd^3 \cos \theta}{2 (r^2 + d^2 + 2r d \cos \theta)^2}
\]
\[
I_2 = \frac{2r^2 d \sin \theta (r + d \cos \theta)}{(r^2 + d^2 + 2r d \cos \theta)^2}
\]
from 6.27 we have on simplifying
\[
\theta' = \frac{4G \sin \theta (d + r \cos \theta)}{\pi (r^2 + d^2 + 2r d \cos \theta)^2}
\]
Value of \( \bar{\phi} \) when \( \alpha = \pi/2 \)
from 6.18 \( E_3 = -(1 + \cosh \gamma) \quad \gamma_3 = -\gamma \)
from 6.19
\[
E_4 = \gamma \sinh \gamma + \gamma e^{2i\theta} \sinh \gamma = \gamma (1 + e^{2i\theta}) \sinh \gamma
\]
\[
\bar{O}_4 = -(1 + \cosh \gamma) \sinh \gamma + \sinh \gamma \cdot \cosh \gamma \cdot \cosh \gamma = -2e^{2i\theta} \sinh \gamma
\]
\[
= -(1 + 2e^{2i\theta}) \sinh \gamma
\]
from 6.20
\[\pi R = 1 + \frac{2\pi(-1 - e^{2i\theta})}{\pi} = -1 - 2e^{2i\theta}\]

and
\[E_5 = -\sinh\pi\gamma(1 + 2e^{2i\theta})\sinh\pi\gamma + \gamma\sinh\pi\gamma(1 + e^{2i\theta})\]
\[O_5 = -\sinh^2\pi\gamma\gamma(1 + e^{2i\theta}) + \gamma\sinh\gamma(-1 - 2e^{2i\theta})\sinh\pi\gamma\]
\[= -\gamma\sinh^2\pi\gamma(2 + 3e^{2i\theta})\]

so that from 6.21
\[
\Phi' = -\frac{G_i}{2\pi R}\left[2\int_0^\infty \frac{\gamma(1+e^{2i\theta})-(1+2e^{2i\theta})^2}{\sinh\pi\gamma} \sin\gamma - \gamma(2+3e^{2i\theta})\cos\gamma d\gamma}{\sinh\pi\gamma} + 1 + 2e^{2i\theta}\right]
\]

Using the integrals
\[
\int_0^\infty \frac{\sin\gamma d\gamma}{\sinh\pi\gamma} = \frac{1}{2} \frac{\sinh k}{1 + \cosh k}
\]
\[
\int_0^\infty \frac{\cos\gamma d\gamma}{\sinh\pi\gamma} = \frac{1}{2} \frac{1}{1 + \cosh k}
\]
\[
\int_0^\infty \frac{\gamma\sin\gamma d\gamma}{\sinh\pi\gamma} = \frac{1}{2} \frac{\sinh k}{(1 + \cosh k)^2}
\]

equn. 6.29 may now be evaluated, and after considerable simplification, becomes
\[
\Phi' = \frac{G_i}{\pi} \frac{de^{-2i\theta} - r(e^{-3i\theta} + 2e^{-i\theta})}{(d + re^{-i\theta})^3}
\]

6.31
Value of $D' = U_x + iU_y$ when $\alpha = \pi/2$

referring to 6.26 and using $K = 3 - 4\sigma$, the modified bulk modulus

$$E_b = \sinh\pi\gamma \left[ \frac{\gamma e^{2i\theta} + \gamma + \kappa e^{-2i\theta} \sinh2\gamma\theta}{\sinh\gamma} \right]$$

$$O_b = \sinh\pi\gamma \cdot \cosh\pi\gamma + \sinh\pi\gamma \left[ -1 - \cos\pi\gamma - e^{2i\theta} + \kappa e^{-2i\theta} \cosh2\gamma\theta \right]$$

$$E_\gamma = \sinh\pi\gamma \cdot O_b \quad O_\gamma = -\sinh\pi\gamma \cdot E_b$$

$$\pi R = 1 + \left[ -2 - e^{2i\theta} + \kappa e^{-2i\theta} \right]$$

6.26 now reduces to

$$U_x + iU_y = \frac{G_i}{8\pi\mu r} \left[ 2 \int_0^\infty \frac{-\sin k\gamma - \gamma \cos k\gamma}{\sinh\gamma \gamma} d\gamma + 1 \right] \left( 1 + e^{2i\theta} \right)$$

$$+ \frac{G_i}{8\pi\mu r} \left[ 2 \int_0^\infty \frac{\cosh2\gamma\theta, \sin k - i\sinh2\gamma\theta, \cos k}{\sinh\gamma} \gamma d\gamma - 1 \right] \kappa e^{-2i\theta}$$

Using integrals previously quoted the first square bracket becomes

$$\frac{e^{-k}}{1 + \cosh k} = \frac{2r^2 e^{-2i\theta}}{(d + re^{-i\theta})^2}$$

and the second bracket to

$$\frac{-e^{-k} - e^{2i\theta}}{\cos2\theta + \cosh k} = -\frac{2r}{r + de^{-i\theta}}$$

giving

$$8\mu D' = \frac{2G_i}{\pi} \left[ \frac{r(1 + e^{-2i\theta})}{(d + re^{-i\theta})^2} - \frac{\kappa e^{2i\theta}}{(r + de^{-i\theta})} \right]$$

6.31a
Direct approach to the semi-infinite plate problem

Suitable complex potentials are to be found in the previous reference to the work of A.C. Stevenson Pg 156 equ.11.39. These involve the complex potentials 6.1 which correspond to the couple G and others required to clear the boundary of stress. These latter are,

\[ \mathcal{J}_b(z) = \frac{2iG}{\pi} \frac{1}{z_\tau} \quad \omega(z) = \frac{2iG}{\pi} \{ \log z_\tau + \frac{d}{z_\tau} \} \]

Using 2.15 the stresses corresponding are

\[ 2 \gamma' = -\frac{2iG}{\pi} \left\{ -\frac{1}{z_\tau} + \frac{1}{\bar{z}_\tau} \right\} \]

now \( z_\tau = z + d = re^{i\theta} + d \)

\[ z_\tau \bar{z}_\tau = r^\tau + d^\tau + 2rd\cos \theta \]

\[ -z_\tau + \bar{z}_\tau = -2i(r^\tau \sin 2\theta + 2r\sin \theta) \]

so that

\[ \delta = \frac{4G}{\pi} \frac{\tau \sin \theta (d + r\cos \theta)}{(r^\tau + d^\tau + 2r\cos \theta)^\tau} \]

\[ 2 \mathcal{F}' = -z \left( -\frac{2iG}{\pi} \right) \left( \frac{2}{z_\tau} \right) - \frac{2iG}{\pi} \left\{ -\frac{1}{z_\tau} + \frac{2d}{z_\tau} \right\} \]

\[ \mathcal{F}' = \frac{iG}{\pi} \frac{2z_\tau + z_\tau}{z_\tau} \]

and when \( z = re^{i\theta} \)

\[ \bar{z}_\tau = re^{-i\theta} - d \quad z_\tau = re^{i\theta} + d \]

\[ \mathcal{F}' = \frac{iG}{\pi} \frac{e^{-2i\theta} - r(2e^{-i\theta} + e^{-3i\theta})}{(re^{-i\theta} + d)^3} \]

It will be seen that these results agree with those obtained from the general solution 6.28 and 6.31.
Also \( 8\mu D = \kappa Jl(z) - zJl'(z) - \overline{\omega}'(z) \)

where \( D = u + iv \) is the complex displacement combination in cartesian coordinates.

\[ 8\mu D' = U_r + iU_\theta = e^{-i\theta}(8\mu D) \]

from the complex potentials 6.32

\[ 8\mu D = \frac{2iG}{n} \kappa \frac{2iG}{n} \left[ \frac{z - d}{z_\nu^2} + \frac{1}{z_\nu^2} \right] \]

\[ = \frac{2iG}{n} \kappa \frac{2iG}{n} \frac{z + \bar{z}}{\bar{z}_\nu^2} \]

\[ 8\mu D' = e^{-i\theta}.8\mu D = \frac{2iG}{n} \kappa e^{-i\theta} \frac{2iG}{n} \frac{ze^{-i\theta} + \bar{z}e^{-i\theta}}{\bar{z}_\nu^2} \]

\[ = \frac{2iG}{n} \kappa e^{-2i\theta} \frac{2iG}{n} \frac{r(1 + e^{-2i\theta})}{(d + re^{-i\theta})^2} \]

It will be seen that this result agrees with 6.31a
CHAPTER VII

Isolated Force Nucleus at \( z = d \) (real) with stress free Boundaries.

For an isolated force nucleus \( W \) at \( z = d \) the complex potentials are known to be

\[
\Phi(z) = -\frac{2W}{\pi(1+K)} \log z, \quad \text{and} \quad \gamma_1
\]

\[
\omega(z) = \frac{2WK}{\pi(1+K)} z \log z,
\]

where \( K = 3-4\sigma \),

N.B. If \( W \) is replaced by \( F = We^{i\alpha} \) it would be possible to obtain a solution for the wedge under the action of a force \( W \) at an angle \( \alpha \) to the axis. However, as was mentioned in the Introduction, such a case would best be treated by superimposing the solutions for \( \alpha = 0 \) and \( \alpha = \pi/2 \). These two problems follow very much the same pattern and the case when \( \alpha = 0 \) is treated here.

writing

\[
P = \frac{2W}{\pi(1+K)}
\]

and changing the origin to the vertex, the complex potentials become

\[
\Phi(z) = -P \log z, \quad \omega(z) = K Pz \log z + dP \log z,
\]

where \( z_i = z - d \)

now

\[
4(\phi + i\theta) = \Phi(z) + \Phi^*(\bar{z}) + z \Phi(z) + \frac{z}{\bar{z}} \omega(z)
\]

\[
= -\frac{P}{z_i} - \frac{P}{\bar{z}_i} + z \frac{P}{z_i} + \frac{z}{\bar{z}} \left\{ \frac{K P}{z_i} - \frac{dP}{\bar{z}_i} \right\}
\]
so that
\[
4(\dot{\Phi} + i\dot{\Theta}) = \frac{P}{z} - \frac{P}{z_i} + \frac{z}{z_i} \frac{P}{z} (z - d) + \frac{K P z}{z z_i}
\]

Thus the loading to be removed by Mellin transform is
\[
\dot{\Theta} + i \dot{\Theta} = \frac{1}{4} \left[ \frac{P}{z} + \frac{P}{z_i} - \frac{z}{z_i} \frac{P z_i}{z} - \frac{K P z}{z z_i} \right]
\]

The algebra involved in this process will be simplified if we consider each term in \(7.4\) separately, but we will first evaluate \(t_i(p)\) for each term in case any two of them could be combined with advantage.

**Case (1)**
\[
\dot{\Theta} + i \dot{\Theta} = \frac{i P}{4z_i} = \frac{i P}{4} \text{re}^{-i\alpha} - d
\]
\[
t_i(p) = \frac{i P e^{i\alpha}}{4} \int_0^\infty \frac{r^{p+1} dr}{r - d e^{i\alpha}} = \frac{i P e^{i\alpha}}{4} I_\alpha(p,\alpha)
\]

Integrating \(\int \frac{z^{p+1} dz}{z - d e^{i\alpha}}\) round the same contour as in Ch. VI we have
\[
(1 - e^{2(p+1)\pi i}) I_\alpha(p,\alpha) = 2\pi i \times \text{residue at } z = d e^{i\alpha}
\]

putting \(z - d e^{i\alpha} = t\) we require the coefficient of \(\frac{1}{t}\) in
\[
\frac{1}{t} (t + d e^{i\alpha})^{p+1}, \text{which is } d^{p+1} e^{i(p+1)\alpha}
\]

\[
\therefore e^{pi(-2isinp\alpha)} I_\alpha(p,\alpha) = 2\pi i d^{p+1} e^{i(p+1)\alpha}
\]
or \[ I_\nu(p, \alpha) = \frac{\pi d^{p+1} e^{-i(p+1)\beta}}{\sin p\pi} \] where \( \beta = \pi - \alpha \)

\[ t_1(p) = -\frac{i\pi e^{-i\beta}}{4} \frac{\pi d^{p+1} e^{-i(p+1)\beta}}{\sin p\pi} \]

\[ = -\frac{i\pi}{4} \frac{\pi d^{p+1} e^{-i(p+2)\beta}}{\sin p\pi} \] \(7.5\)

\[ t_2(p) = -\frac{i\pi}{4} \frac{\pi d^{p+1} e^{i(p+2)\beta}}{\sin p\pi} \] \(7.6\)

Also since the stresses are of \( O(r^{-1}) \) as \( r \to \infty \) we have from Ch IIb (17)

\[-2 < p < -1\]

Case (2) \( \tilde{\varrho} + i\tilde{\vartheta} = \frac{i\pi}{4} \frac{1}{z} \)

t_1(p) may be obtained from the previous case as

\[ t_1(p) = -\frac{i\pi}{4} \frac{\pi d^{p+1} e^{i(p+2)\beta}}{\sin p\pi} \] \(7.8\)

\[ t_2(p) = -\frac{i\pi}{4} \frac{\pi d^{p+1} e^{-i(p+2)\beta}}{\sin p\pi} \] \(7.9\)

Case (3) \( \tilde{\varrho} + i\tilde{\vartheta} = -\frac{i\pi}{4} \frac{\tilde{z} z}{z \tilde{z}} \)

\[ \quad = -\frac{i\pi e^{-2i\alpha}}{4} \frac{re^{i\alpha} - d}{(re^{-i\alpha} - d)^2} \text{ on } \varrho = \alpha \]

\[ = -\frac{i\pi e^{i\alpha}}{4} \frac{(r - de^{-i\alpha})}{(r - de^{i\alpha})^2} \]

\[ \therefore t_1(p) = -\frac{i\pi e^{i\alpha}}{4} \int_0^\infty \frac{r^{p+1}(r - de^{-i\alpha}) dr}{(r - de^{i\alpha})^2} \]
or \( t'_1(p) = -\frac{i\rho e^{i\alpha}}{4} \int_0^{\infty} \left\{ \frac{r^{p+2}}{(r - de^{i\alpha})^2} - de^{i\alpha} \frac{r^{p+1}}{(r - de^{i\alpha})^2} \right\} dr \)

These integrals have been evaluated before and their values are given by 6.4 as

\[
\int_0^{\infty} \frac{r^{p+1} dr}{(r - de^{i\alpha})^2} = -\frac{\pi d e^{-ip\theta}(p+1)}{\sin r\pi}
\]

so that

\[
\int_0^{\infty} \frac{r^{p+2} dr}{(r - de^{i\alpha})^2} = \frac{\pi d^{p+1} e^{-ip\theta}(p+1)\beta(p+2)}{\sin r\pi}
\]

\[
t'_1(p) = \frac{i\rho e^{-i\beta}}{4} \left[ \frac{\pi d^{p+1} e^{-i(p+1)\beta(p+2)}}{\sin r\pi} - de^{i\beta} \frac{\pi d e^{-ip\theta}(p+1)}{\sin r\pi} \right]
\]

If we now add the values of \( t'_1(p) \) corresponding to Cases (1) and (3) we obtain,

\[
t'_1(p) = \frac{i\pi d^{p+1}}{4\sin r\pi} \left[ (p+1)e^{-i(p+2)\beta} - (p+1)e^{-ip\beta} - e^{-i(p+2)\beta} \right]
\]

\[
= \frac{i\pi d^{p+1}}{4\sin r\pi} \cdot (p+1)e^{-i(p+1)\beta}(-2i\sin \beta)
\]

\[
\therefore \ t'_1(p) = \frac{\pi d^{p+1}}{2\sin r\pi} \cdot (p+1)e^{-i(p+1)\beta} \sin \alpha \quad 7.10
\]

\[
t_2(p) = -\frac{\pi d^{p+1}}{2\sin r\pi} \cdot (p+1)e^{i(p+1)\beta} \sin \alpha \quad 7.11
\]

It will therefore be profitable, algebraically, to combine cases (1) and (3).
Case (4) \( r \Theta + i g \Theta = -iKP \frac{z}{4} - \frac{1}{z} = -iKP \frac{e^{-2i\alpha}}{4} \) on \( \theta = \alpha \)

\[
t_1(p) = \frac{iKP e^{i\beta}}{4} \int_0^{\infty} \frac{r^{p+1}}{r - de^{i\alpha}} \, dr
\]

or

\[
t_1(p) = \frac{iKP d^{p+1} e^{-i(p+1)\beta}}{4 \sin p \pi}
\]

\[
t_4(p) = \frac{iKP d^{p+1} e^{i(p+1)\beta}}{4 \sin p \pi}
\]

In the evaluation of stresses and displacements the solution will be presented in three parts as follows:

(a) case (1) + case (3)
(b) case (2)
(c) case (4)

(a) case (1) + case (3)

Boundary condition from Ch IIB (31) and 7.10 and 7.11

over \( \theta = \alpha \)

\[ pA e^{i\alpha} + B(p+1)e^{i(p+2)\alpha} = -Be^{-i(p+2)\alpha} = \frac{iPd^{p+1} \sin \alpha e^{-i(p+1)\beta}}{4 \sin p \pi} \]

over \( \theta = -\alpha \)

\[ pA e^{-i\alpha} + B(p+1)e^{-i(p+2)\alpha} = Be^{i(p+2)\alpha} = \frac{iPd^{p+1} \sin \alpha e^{i(p+1)\beta}}{4 \sin p \pi} \]

and the conjugates

\[ pA e^{-i\alpha} + B(p+1)e^{-i(p+2)\alpha} = Be^{i(p+2)\alpha} = \frac{iPd^{p+1} \sin \alpha e^{i(p+1)\beta}}{4 \sin p \pi} \]
\[ \rho A e^{i\rho} + B(p+1)e^{i(p+2)} = B e^{-i(p+2)} = \frac{i\pi d^{p+1}\sin \alpha e^{-i(p+1)\beta}}{4\sin \rho \pi} \]

These imply that \( A = A \) and \( B = B \) so that

\[ pA + B(p+1)e^{2i\alpha} = \frac{i\pi d^{p+1}\sin \alpha e^{-i(p+\beta)}}{4\sin \rho \pi} \]

\[ = A' + iB' \]

\[ pA + B(p+1)e^{-2i\alpha} = \frac{i\pi d^{p+1}\sin \alpha e^{i(p\beta)}}{4\sin \rho \pi} \]

\[ = A' - iB' \]

where

\[ A' = -\frac{\rho d^{p+1}}{4\sin \rho \pi} \sin \alpha \sin (p\beta + \alpha) \]

\[ B' = -\frac{\rho d^{p+1}}{4\sin \rho \pi} \sin \alpha \cos (p\beta + \alpha) \]

Subtracting 7.14 and 7.15 we have

\[ B \cdot 2\sin(2p+2)\alpha + (p+1)B \cdot 2\sin 2\alpha = 2iB' \]

\[ \therefore B \cdot H(p,\alpha) = B' \quad \text{from 6.7} \]

or

\[ B = -\frac{\rho \sin \alpha \cdot d^{p+1} \cos (p\beta + \alpha)}{4 \cdot H(p,\alpha) \sin \rho \pi} \]

Also

\[ pA = B \{ e^{-2i(p+1)\alpha} - (p+1)e^{2i\alpha} \} + A' + iB' \]

\[ = \frac{B'}{H(p,\alpha)} \{ -j(p,\alpha) - iH(p,\alpha) \} + A' + iB' \]

\[ A = \frac{1}{\rho H(p,\alpha)} \{ A' \cdot H(p,\alpha) - B' \cdot J(p,\alpha) \} \]

or

\[ A = \frac{\rho \sin \alpha \cdot d^{p+1}}{4 \cdot \rho H(p,\alpha) \sin \rho \pi} \{ J(p,\alpha) \cos (p\beta + \alpha) - H(p,\alpha) \sin (p\beta + \alpha) \} \]
Stresses

\[(D^p + p^r) \tilde{\chi} = -4(p+1)\left[Be^i(p+2)\theta + Be^{-i(p+2)\theta}\right]\]

\[= -4(p+1)2\beta \cos(p+2)\theta\]

\[= 2\pi \sin \alpha (p+1)c^{p+1} \frac{\cos(p+2)\theta \cos(p\tau + \beta)}{H(p,\alpha) \sin \pi} \]

giving

\[\tilde{\omega} = \frac{\pi \sin \alpha}{ir} \int_{-\infty}^{\infty} \left(\frac{d}{r}\right)^{p+1} \frac{(p+1)\cos(p+2)\theta \cos(p\tau + \beta)}{H(p,\alpha) \sin \pi} \, dp \quad (7.19)\]

This integral will be simplified by putting \(p = -1 + i\gamma\)

and taking \(c = -1\)

\[\left(\frac{d}{r}\right)^{p+1} = e^{i\gamma m} \quad \text{where} \quad m = \log_2 \frac{d}{r}\]

\[H(p,\alpha) = i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)\]

\[\cos(p\tau + \beta) = \cos(-\pi + i\gamma \pi + \pi - \alpha) = \cos(i\gamma \pi - \alpha)\]

\[\sin \pi = -i \sinh \gamma \pi\]

\[\tilde{\omega} = \frac{\pi \sin \alpha}{ir} \left[\int_{-\infty}^{\infty} e^{i\gamma m} \frac{i \gamma \cos(l + i\gamma)\theta \cos(i\gamma \pi - \alpha) \, dp}{\sin \gamma \pi \cdot i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)} - \frac{\pi i R}{r}\right] \quad (7.20)\]

where \(R\) is the residue at \(p = -1\).

now \(\cos(1 + i\gamma)\theta \cos(i\gamma \pi - \alpha) = E_{\xi} + iO_{\xi}\)

where

\[E_{\xi} = \cosh \gamma \cdot \cos \theta \cdot \cosh \gamma \cdot \cos \theta + \sinh \gamma \cdot \sin \theta \cdot \sinh \gamma \cdot \theta\]

\[O_{\xi} = -\cosh \gamma \cdot \cos \theta \cdot \sin \theta \cdot \sinh \theta + \sinh \gamma \cdot \sin \theta \cdot \cosh \gamma \cdot \theta\]

\[\pi R = \lim_{\gamma \to 0} \frac{\gamma \cos(l + i\gamma)\theta \cos(i\gamma \pi - \alpha)}{\gamma \sin 2\alpha + \sinh 2\gamma \alpha}\]

\[= \frac{\cos \theta \cdot \cos \alpha}{\sin 2\alpha + 2\alpha}\]
\[
\theta' = -\frac{P\sin\alpha}{r} \left[ 2 \int_{-\infty}^{\infty} \frac{\gamma(\cos^2m + E\sin^2m)}{\sin\pi\eta(\sinh2\gamma\eta + \gamma\sin2\alpha)} \ d\eta - \frac{\cos\gamma\cos\alpha}{\sin2\alpha + 2\alpha} \right] 7.21
\]

Stress function \( \Phi' \)
\[
(D + ip)(D + i(p+2))\tilde{\chi} = -4Ap(p+1)e^{ip\theta} - 4B(p+1)(p+2)e^{i(p+2)\theta}
\]
\[
= -4(p+1)e^{ip\theta} \left[ A_p + B(p+2)e^{2i\theta} \right] = -4(p+1)e^{ip\theta} \frac{\pi \sin \alpha}{4} \frac{dB+1}{H(p,\alpha) \sin \pi \alpha} Z, (p, \theta)
\]

where
\[
Z, (p, \theta) = J(p, \alpha) \cos (\pi(\pi + \beta)) - H(p, \alpha) \sin (\pi(\pi + \beta)) - (p+2)e^{2i\theta} \cos(\pi(\pi + \beta))
\]
\[
= \left\{ \frac{J(p, \alpha) - (p+2)e^{2i\theta}}{4} \right\} \cos(\pi(\pi + \beta)) - H(p, \alpha) \sin(\pi(\pi + \beta))
\]
\[
\Phi' = -\frac{P\sin\alpha}{2ir} \int_{-\infty}^{\infty} \frac{(d^{p+1})e^{ip\theta}}{H(p,\alpha) \sin \pi \alpha} Z, (p, \theta) \ dp
\]

Again taking \( c = -i \) and putting \( p = -1 + i \eta \)
\[
\left\{ \frac{d^{p+1}}{r} \right\} e^{i(p+1)\theta} = e^{i\gamma m} e^{i\gamma \theta} = e^{i\gamma k} \quad k = m + i\theta
\]

\[
J(p, \alpha) = (p+1)\cos\alpha - \cos(2p+2)\alpha = i\gamma\cos\alpha - \cosh2\gamma\alpha
\]
\[
\cos(\pi\pi-\beta) = \cos(i\gamma \pi-\alpha) = \cosh\gamma\pi \cos\alpha + i\sinh\gamma\pi \sin\alpha
\]
\[
\sin(\pi\pi+\beta) = \sin(i\gamma \pi-\alpha) = i\sinh\gamma\pi \cos\alpha - \cosh\gamma\pi \sin\alpha
\]
\[
Z, (p, \theta) = \left\{ -(\cosh2\gamma\alpha + e^{2i\theta}) + i\gamma(\cos2\alpha - e^{2i\theta}) \right\} \cos(\pi\pi+\beta)
\]
\[
- H(p, \alpha) \sin(\pi\pi+\beta)
\]
\[
= E_q + i\Omega_q \quad \text{where}
\]
\[
E_q = -(\cosh2\gamma\alpha + e^{2i\theta})\cosh\gamma\pi \cos\alpha - \gamma(\cos2\alpha - e^{2i\theta})\sinh\gamma\pi \sin\alpha
\]
\[
+ (\beta\sin2\alpha + \sinh2\gamma\alpha)\sinh\gamma\pi \cos\alpha
\]
\[
\Omega_q = \gamma(\cos2\alpha - e^{2i\theta})\cosh\gamma\pi \cos\alpha - (\cosh2\gamma\alpha + e^{2i\theta})\sinh\gamma\pi \sin\alpha
\]
\[
+ (\beta\sin2\alpha + \sinh2\gamma\alpha)\sinh\gamma\pi \sin\alpha
\]
\[
\Phi' = P e^{-i\Theta \sin a} \left[ \sum_{n=0}^{\infty} \frac{e^{i\gamma k} i\gamma (E_0 + i\theta) i\gamma}{i(\gamma \sin 2a + \sinh 2\gamma a) i \sinh \gamma} \right] - \pi i R
\]

and \[
\pi R = \lim_{\gamma \to 0} \frac{E_0 + i\theta}{\sin 2a + \frac{\sinh 2\gamma a}{2\gamma} \cdot 2a}
\]

\[
= - \frac{(1 + e^{2i\Theta}) \cos a}{\sin 2a + 2a}
\]

\[
\Phi' = P e^{-i\Theta \sin a} \left[ \sum_{n=0}^{\infty} \frac{\gamma (E_0 \sin \gamma k + \cos \gamma k)}{(\gamma \sin 2a + \sinh 2\gamma a) \sinh \gamma} \right] \frac{d\gamma}{\sin 2a + 2a}
\]

**Displacements** from Ch IIb (29)

\[
U_\varphi + iU_\theta = - \frac{1}{2 \pi \mu} \left\{ \frac{e^{i\varphi}}{e^{-i\varphi}} \int_{e^{-i\varphi}}^{e^{i\varphi}} \left[ \frac{p A e^{i\varphi} + B(p+1)e^{i(p+2)\varphi} + \kappa \theta e^{-i(p+2)\varphi}}{4 H(p,\varphi) \sin \varphi} \right] dp \right\}
\]

or inserting the values of A and B given by 7.17 and 7.18

\[
e^{i\varphi} \left\{ \frac{p A + B(p+1)e^{2i\varphi} + \kappa \theta e^{-i(2p+2)\varphi}}{4 H(p,\varphi) \sin \varphi} \right\}
\]

\[
e^{i\varphi} \left[ \frac{p+1}{4 H(p,\varphi) \sin \varphi} \left( J(p,\varphi) \cos(p\pi+\beta) - H(p,\varphi) \sin(p\pi+\beta) \right) \right]
\]

\[
\left( (p+1)e^{2i\varphi} + \kappa e^{-i(2p+2)\varphi} \right) \cos(p\pi+\beta)
\]

\[
U_\varphi + iU_\theta = \frac{P i \sin a}{8 \mu} \left[ \int_{e^{-i\varphi}}^{e^{i\varphi}} \frac{p+1}{4 H(p,\varphi) \sin \varphi} \right. \left. \frac{e^{i\varphi} Y_{p+1}(p,\varphi)}{H(p,\varphi) \sin \varphi} \right] dp
\]

where

\[
Y_{p+1}(p,\varphi) = \cos(p\pi+\beta) \left[ -J(p,\varphi) + (p+1)e^{2i\varphi} + \kappa e^{-i(2p+2)\varphi} \right]
\]

\[
+ H(p,\varphi) \sin(p\pi+\beta)
\]

putting \( p = -1 + i\gamma \) and using expressions obtained following equn 7.23.
\[ Y_1(p, \Theta) = (\cosh \eta \cdot \cos \alpha + i \sinh \eta \cdot \sin \alpha) \left[ \cosh 2\eta \alpha - i \zeta \cos 2\alpha + i e^{2i\Theta} \right. \\
+ K(\cosh 2\eta \Theta + \sinh 2\eta \Theta) \left. \right] + i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(\sinh \gamma \alpha \cos \alpha \right. \\
- \cosh \eta \alpha \sin \alpha) \\
= E_{io} + iO_{io} \text{ where} \\
E_{io} = \cosh \eta \alpha \cos \alpha (\cosh 2\eta \alpha + \kappa \cosh 2\eta \Theta) + \sinh \eta \alpha \sin \alpha (\gamma \cos 2\alpha - \zeta e^{2i\Theta} \\
+ iK \sinh 2\eta \Theta) - (\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \sinh \eta \alpha \cos \alpha \\
7.26 \\
O_{io} = \sinh \eta \alpha \sin \alpha (\cosh 2\eta \alpha + \kappa \cosh 2\eta \Theta) - \cosh \eta \alpha \cos \alpha (\gamma \cos 2\alpha - \zeta e^{2i\Theta} \\
+ iK \sinh 2\eta \Theta) - (\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \cosh \eta \alpha \sin \alpha \\
eqn. 7.25 \text{ now becomes} \\
U_1 + iU_\Theta = -\frac{Pie^{-i\Theta} \sin \alpha}{8\mu} \left[ \int_{-\infty}^{\infty} \frac{e^{i\gamma k} (E_{io} + iO_{io}) i d\gamma}{i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(i \sinh \gamma \alpha)} \right] - \pi \text{R} \\
\text{Calculation of the residue R} \\
\text{This presents a little more difficulty than in previous cases} \\
\text{since } p = -1 \text{ is a pole of order two of the integrand, for} \\
\text{which the series expansion will have to be obtained.} \\
E_{io} + iO_{io} = (1 + \kappa) \cos \alpha + i \alpha \gamma + \alpha \gamma \\
\text{where } \alpha \gamma = \pi (1 + \kappa) \sin \alpha - \cos \alpha (\cos 2\alpha - e^{2i\Theta} + 2iK \Theta) - (\sin 2\alpha + 2\alpha) \sin \alpha \\
= \frac{\pi (1 + \kappa) - 2\alpha}{\sin \alpha} \cos \alpha (1 - e^{2i\Theta} + 2iK \Theta) \\
\gamma \sin 2\alpha + \sinh 2\gamma \alpha = \gamma (\sin 2\alpha + 2\alpha) + O(\gamma^3) \\
\sinh \gamma \alpha = \gamma + O(\gamma^3) \\
\therefore \text{integrand becomes} \\
\left[ \frac{1 + i\gamma k + O(\gamma^\pm)}{i \gamma \pi (\sin 2\alpha + 2\alpha) + O(\gamma^\pm)} \right] \\
\left[ (1 + \kappa) \cos \alpha + i \alpha \gamma + \ldots \right] \\
\text{so that} \quad R = \frac{k(1 + \kappa) \cos \alpha + \alpha \gamma}{\pi (\sin 2\alpha + 2\alpha)} \\
7.27
we have finally
\[ U_\alpha + iU_\beta = \frac{Pe^{-i\theta \sin \alpha}}{8\mu} \left[ 2 \int_0^\infty \frac{E_\alpha \cos \gamma k - \varnothing \sin \gamma k}{(\gamma \sin \alpha + \sinh \gamma \alpha) \sinh \gamma \pi} d\gamma + \pi R \right] \]

(b) Case (2)
Boundary condition from Ch IIb (31) and 7.8 and 7.9

over \( \theta = \alpha \)
\[ pAe^{i\alpha} + B(p+1)e^{i(p+2)\alpha} - Be^{-i(p+2)\alpha} = -\frac{P\pi d^{p+1}e^{i(p+2)\beta}}{8(p+1)\sin \pi} \]

over \( \theta = -\alpha \)
\[ pAe^{-i\alpha} + B(p+1)e^{-i(p+2)\alpha} - Be^{i(p+2)\alpha} = -\frac{P\pi d^{p+1}e^{-i(p+2)\beta}}{8(p+1)\sin \pi} \]

and the conjugates
\[ \bar{p}Ae^{-i\alpha} + \bar{B}(p+1)e^{-i(p+2)\alpha} - \bar{B}e^{i(p+2)\alpha} = -\frac{P\pi d^{p+1}e^{-i(p+2)\beta}}{8(p+1)\sin \pi} \]
\[ \bar{p}Ae^{i\alpha} + \bar{B}(p+1)e^{i(p+2)\alpha} - \bar{B}e^{-i(p+2)\alpha} = -\frac{P\pi d^{p+1}e^{i(p+2)\beta}}{8(p+1)\sin \pi} \]

These again imply \( \bar{A} = A \) and \( \bar{B} = B \) so that

\[ pA + B(p+1)e^{2i\alpha} - Be^{-(2p+2)i\alpha} = -\frac{P\pi d^{p+1}e^{i\delta}}{8(p+1)\sin \pi} = A' + iB' \]
\[ pA + B(p+1)e^{-2i\alpha} - Be^{i(2p+2)\alpha} = -\frac{P\pi d^{p+1}e^{-i\delta}}{8(p+1)\sin \pi} = A' - iB' \]

where
\[ A' = -\frac{P\pi d^{p+1}\cos \delta}{8(p+1)\sin \pi} \]
\[ B' = -\frac{P\pi d^{p+1}\sin \delta}{8(p+1)\sin \pi} \]
and \( \delta = (p+2)\beta - p\alpha \)
as in section (a),

\[
B = \frac{B}{H(p, a)} = -\frac{\pi d^{p+1} \sin \delta}{8(p+1)H(p, a) \sin \pi}
\]  \hfill 7.31

\[
A = \frac{1}{pH(p, a)} \left\{ A'H(p, a) - B'J(p, a) \right\}
\]

\[
A = -\frac{\pi}{p(p+1)H(p, a) \sin \pi} \left\{ H(p, a) \cos \delta - J(p, a) \sin \delta \right\}
\]  \hfill 7.32

**Stresses**

\[
(D + p^2) \chi = -3(p+1)B \cos(p+2)\theta
\]

\[
= \frac{\pi d^{p+1} \sin \delta, \cos(p+2)\theta}{H(p, a) \sin \pi}
\]

giving

\[
\Theta' = \frac{P}{2\pi r} \int_{C-i\infty}^{C+i\infty} \left( \frac{d}{r} \right)^{p+1} \frac{\sin \delta \cos(p+2)\theta}{H(p, a) \sin \pi} \, dp
\]  \hfill 7.33

again put \( p = -1 + i\gamma \) and take \( c = -1 \) so that

\[
\left( \frac{d}{r} \right)^{p+1} = e^{i\gamma m} \quad \text{and} \quad H(p, a) = i(\gamma \sin 2\alpha + \sinh 2\gamma a)
\]

\[
\delta = (p+2)\theta - pa = (1 + i\gamma)(\pi - \alpha) - (-1 + i\gamma)\alpha
\]

\[
= \pi + i\gamma(\pi - 2\alpha) = \pi + i\gamma \psi \quad \text{(say)}
\]

\[
\sin \delta = -\sin \gamma \psi = -i \sinh \gamma \psi
\]

\[
\cos(p+2)\theta = \cos(1 + i\gamma)\theta = \cos \theta \cosh \gamma \theta - i \sin \theta \sinh \gamma \theta
\]

so that 7.33 becomes

\[
\Theta' = \frac{P}{2\pi r} \left[ \int_{-\infty}^{\infty} e^{i\gamma m} \left( -i \sinh \gamma \psi \right) \left( \cos \theta \cosh \gamma \theta - i \sin \theta \sinh \gamma \theta \right) \, id\gamma - \pi i \right]
\]
\[ \pi R = \lim_{\gamma \to 0} \frac{\sinh \gamma \left( \cos \theta \cdot \cosh \gamma - i \sin \theta \cdot \sinh \gamma \right)}{\gamma \sin 2\alpha + \sinh 2\gamma} \]
\[ = \lim_{\gamma \to 0} \frac{\sinh \gamma \omega \cos \theta \cdot \cosh \gamma - i \sin \theta \cdot \sinh \gamma \omega}{\sin 2\alpha + \sinh 2\gamma} \cdot \omega \frac{2\alpha}{2\gamma} \]
\[ = \frac{\omega \cos \theta}{\sin 2\alpha + 2\alpha} \]

and finally
\[ (\omega') = \frac{p}{2r} \left[ 2 \int_0^\infty \frac{\sinh \gamma \omega \left( \cos \theta \cdot \cosh \gamma - i \sin \theta \cdot \sinh \gamma \right) \sin m}{(\gamma \sin 2\alpha + \sinh 2\gamma) \sin m} \, d\gamma \right] \cdot \frac{\omega \cos \theta}{\sin 2\alpha + 2\alpha} \]

Stress function \( \Phi' \)
\[ (D + ip)(D + i(p+2)) \tilde{\chi} = -4(p+1)e^{ip\theta} [A_0 + B(p+2)e^{2i\theta}] \]
\[ = \frac{p}{2r} \frac{e^{ip\theta} e^{p+1}}{H(p,\alpha) \sin \pi} Z_2(p,\theta) \]
where
\[ Z_2(p,\theta) = H(p,\alpha) \cos \delta - \left\{ J(p,\alpha) - (p+2)e^{2i\theta} \right\} \sin \delta \]
and
\[ \Phi' = \frac{p}{4ir} \int_{c-i\infty}^{c+i\infty} \left( \frac{d}{1} \right)^{p+1} e^{i(p+1)\theta} Z_2(p,\theta) \, dp \]

again putting \( p = -1 + i\gamma \)
\[ \left( \frac{d}{1} \right)^{p+1} e^{i(p+1)\theta} = e^{i\frac{1}{2}k} \quad \text{where} \ k = m + i\theta \]
\[ \cos \delta = -\cosh \gamma \omega \]
\[ Z_2(p,\theta) = -i(\gamma \sin 2\alpha + \sinh 2\gamma) \cosh \gamma \omega + i \sinh \gamma \omega \left\{ i \gamma \cos 2\alpha - \cosh 2\gamma \right\} (1 + i\gamma) e^{2i\theta} \]
\[ = E_{uu} + 10_z \]
\[ E_{uu} = -\gamma(\cos 2\alpha - e^{2i\theta}) \sinh \gamma \omega \]
\[ 0_u = -\gamma(\sin 2\alpha + \sinh 12\gamma) \cosh \gamma \omega - (\cosh 2\gamma + e^{2i\theta}) \sinh \gamma \omega \]
\[
\Phi' = -\frac{Pe^{-i\theta}}{4ir} \left[ \int_{-\infty}^{\infty} \frac{e^{i\gamma k}(E_{\mu} + i0_{\mu}) i d\gamma}{i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \sinh \gamma \pi} \right] = \pi i R
\]

and \[
\pi R = \lim_{\gamma \to 0} \frac{E_{\mu} + i0_{\mu}}{i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)}
\]

\[
= \frac{1}{i(\sin 2\alpha + 2\alpha)} \lim_{\gamma \to 0} \frac{E_{\mu} + i0_{\mu}}{\gamma}
\]

\[
= \frac{1}{i(\sin 2\alpha + 2\alpha)} \left\{ -i(\sin 2\alpha + 2\alpha) + i(-1 - e^{2i\theta})\tilde{\omega} \right\}
\]

\[
\pi R = -1 - \frac{(1 + e^{2i\theta})\tilde{\omega}}{\sin 2\alpha + 2\alpha}
\]

and finally
\[
\Phi' = \frac{Pe^{-i\theta}}{4r} \left[ \int_{0}^{\infty} \frac{E_{\mu} \cosh \gamma - O_{\mu} \sinh \gamma}{(\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \sinh \gamma \pi} d\gamma \right] = 1 - \frac{(1 + e^{2i\theta})\tilde{\omega}}{\sin 2\alpha + 2\alpha}
\]

Displacements from Ch IIb (29)
\[
e^{ip\theta} [pA + B(p+1)e^{2i\theta} + KBe^{-i(2p+2)\theta}] \]

becomes on using 7.31 and 7.32
\[
e^{ip\theta} \frac{Fr}{8H(p,\alpha)(p+1)\sin \pi} Y_1(p,\theta)
\]

where
\[
Y_\gamma(p,\theta) = H(p,\alpha) \cos \delta - J(p,\alpha) \sin \delta + \{ (p+1)e^{2i\theta} + \}
\]
\[
\quad + K e^{-i(2p+2)\theta} \}
\]
\[
\sin \delta
\]
\[
U_{\gamma} + iU_{\tilde{\gamma}} = \frac{Pi}{10\mu} \left[ \int_{0}^{\infty} \left( \frac{d\gamma}{r} \right)^{p+1} e^{ip\theta} Y_2(p,\theta) \frac{dp}{H(p,\alpha)(p+1)\sin \pi} \right]
\]

putting \( p = \gamma + i \gamma \)
\[
Y_1(p,\theta) = \{ i e^{2i\theta} + K(\cosh 2\gamma \theta + \sinh 2\gamma \theta) - i \cos 2\alpha + \cosh 2\gamma \alpha \} \{- \sinh \tilde{\gamma} \tilde{\omega} \}
\]
\[
\quad + i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(-\cosh \tilde{\gamma} \tilde{\omega})
\]
\[
= E_{\gamma_1} + i0_{\tilde{\gamma}_1}
\]
\[
E_{\gamma_1} = \sinh \gamma \tilde{\omega} \{ e^{2i\theta} - K \sinh 2\gamma \theta - \gamma \cos 2\alpha \}
\]
\[
o_{\tilde{\gamma}_1} = -\sinh \gamma \tilde{\omega} \{ \cosh 2\gamma \alpha + K \cosh 2\gamma \theta \} - \cosh \tilde{\gamma} \tilde{\omega} (\gamma \sin 2\alpha + \sinh 2\gamma \alpha)
\[ U_t + iU_\theta = \frac{\text{Pe}^{-i\vartheta}}{16\mu} \left[ \int_{-\infty}^{\infty} \frac{e^{i\gamma k(E_{1/2} + i0_{1/2})} id\gamma}{i(\gamma\sin2\alpha + \sinh2\gamma\alpha)\gamma\sinh\gamma\pi} - nR \right] \]

**Calculation of \( R \)**

\[ E_{1/2} + i0_{1/2} = b_1 \gamma + b_\vartheta \gamma^2 \quad \text{where} \]

\[ b_1 = -i\left\{ \tilde{\omega}(1+\kappa) + \sin2\alpha + 2\alpha \right\} \]

\[ b_\vartheta = \tilde{\omega}_i e^{2i\vartheta} - 2\kappa i\vartheta - \cos2\alpha \]

\[ \sinh\gamma\pi(\gamma\sin2\alpha + \sinh2\gamma\alpha) = \gamma\pi(\sin2\alpha + 2\alpha) + O(\gamma^4) \]

so that on expansion of the integrand of 7.38

\[ \pi R = \frac{b_\vartheta - ikb_1}{\sin2\alpha + 2\alpha} \]

and then

\[ U_t + iU_\theta = -\frac{\text{Pe}^{-i\vartheta}}{16\mu} \left[ \frac{2}{\gamma(\gamma\sin2\alpha + \sinh2\gamma\alpha)\sinh\gamma\pi} \int_{0}^{\infty} \right] \]

**Case (4)**

Boundary condition from Ch IIb (31) and 7.12 and 7.13

over \( \theta = \alpha \)

\[ pAe^{i(p+2)}\alpha - \overline{B}e^{-i(p+2)}\alpha = \frac{KP_{1d}p+1e^{ip\beta}}{8(p+1)\sin\varpi} \]

over \( \theta = -\alpha \)

\[ pAe^{-i(p+2)\alpha} - \overline{B}e^{-i(p+2)\alpha} = \frac{KP_{1d}p+1e^{-ip\beta}}{8(p+1)\sin\varpi} \]

and the conjugates

\[ p\overline{A}e^{i(p+2)\alpha} - \overline{B}(p+1)e^{-i(p+2)\alpha} = \frac{KP_{1d}p+1e^{ip\beta}}{8(p+1)\sin\varpi} \]

\[ p\overline{A}e^{-i(p+2)\alpha} - \overline{B}(p+1)e^{-i(p+2)\alpha} = \frac{KP_{1d}p+1e^{-ip\beta}}{8(p+1)\sin\varpi} \]

which again imply that \( \overline{A} = A \) and \( \overline{B} = B \) so that
\[ pA + B(p+1)e^{2i\alpha} - Be^{-i(2p+2)\alpha} = \frac{KP\pi d^{p+1}e^{-ip\pi}}{8(p+1)\sin\pi} = A' + iB' \]

\[ pA + B(p+1)e^{-2i\alpha} - Be^{i(2p+2)\alpha} = \frac{KP\pi d^{p+1}e^{ip\pi}}{8(p+1)\sin\pi} = A' - iB' \]

so that

\[ B = \frac{B'}{H(p,\alpha)} = -\frac{\frac{KP}{8}}{(p+1)H(p,\alpha)} \]

and

\[ A = \frac{1}{pH(p,\alpha)} \left\{ A' H(p,\alpha) - B' J'(p,\alpha) \right\} \]

\[ = \frac{1}{pH(p,\alpha)} \frac{KP\pi d^{p+1}}{8(p+1)\sin\pi} \left\{ H(p,\alpha) \cos\pi + J(p,\alpha) \sin\pi \right\} \]

**Stresses**

\[ (D^m + p^n)\chi = -8(p+1)B \cdot \cos(p+2)\theta \]

\[ = \frac{KP\pi d^{p+1}\cos(p+2)\theta}{H(p,\alpha)} \]

\[ \Theta' = \frac{KP}{2ir} \left[ \int_{-\infty}^{\infty} \frac{(d)\pi^{p+1}\cos(p+2)\theta}{H(p,\alpha)} \right] \]

on putting \( p = -1 + i\gamma \) and using previously obtained expressions \( 7.44 \) becomes

\[ \Theta' = \frac{KP}{2ir} \left[ \int_{-\infty}^{\infty} e^{i\gamma m \cos(l+i\gamma)\theta} d\gamma - \pi R \right] \]

\[ \cos(i+i\gamma)\theta = \cos\theta \cdot \cosh\gamma\theta - i\sin\theta \cdot \sinh\gamma\theta \]

and \( R = \lim_{\gamma \to \infty} \frac{e^{i\gamma m \cos(l+i\gamma)\theta}}{\gamma \sin2\alpha + \sinh2\gamma\alpha} = \frac{\cos\theta}{\sin2\alpha + 2\alpha} \)

finally

\[ \Theta' = \frac{KP}{2ir} \left[ \int_{-\infty}^{\infty} \frac{\sin\gamma \cdot \cos\theta \cdot \cosh\gamma\theta - \cos\gamma \cdot \sin\theta \cdot \sinh\gamma\theta}{\gamma \sin2\alpha + \sinh2\gamma\alpha} d\gamma = \frac{\pi \cos\theta}{\sin2\alpha + 2\alpha} \right] \]
Stress function $\Phi'$

\[
(D + i p)\{D + i(p+2)\}x = -4(p+1)e^{ip\theta}[A_p + B(p+2)e^{2i\theta}]
\]

\[
= -\frac{K_p}{2} \frac{d^{p+1}e^{i\theta}}{H(p,\alpha)\sin\pi}\,
\]

where

\[
Z_3(p,\theta) = \{J(p,\alpha) - (p+2)e^{2i\theta}\} \sin\pi + H(p,\alpha)\cos\pi
\]

\[
\Phi' = -\frac{Kp}{4\pi r} \int_{C-i\infty}^{C+i\infty} \frac{d^{p+1}e^{i(p+1)\theta}Z_3(p,\theta)}{H(p,\alpha)\sin\pi} d\rho.
\]

Putting $p = -1 + i\gamma$

\[
Z_3(p,\theta) = -i\{i\gamma \cos 2\alpha - \cosh 2\gamma \alpha - (1 + i\gamma)e^{2i\theta}\} \sinh \gamma \pi
\]

\[
+i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(-\cosh \gamma \pi)
\]

\[
= E_3 + iO_3 \quad \text{where}
\]

\[
E_3 = \gamma (\cos 2\alpha - e^{2i\theta}) \sinh \gamma \pi
\]

\[
O_3 = (\cosh 2\gamma \alpha + e^{2i\theta}) \sinh \gamma \pi - (\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \cosh \gamma \pi
\]

equation 7.46 now becomes

\[
\Phi' = \frac{Kp}{4\pi r} \int_{-\infty}^{\infty} \frac{e^{i\gamma k}(E_3 + iO_3) id\gamma}{i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(i\sinh \gamma \pi)} - \pi i R
\]

where

\[
R = \lim_{\gamma \to 0} \frac{\gamma e^{i\gamma k}(E_3 + iO_3)}{(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(i\sinh \gamma \pi)}
\]

\[
\pi R = -1 + \frac{\pi (1 + e^{2i\theta})}{\sin 2\alpha + 2\alpha}
\]

giving finally

\[
\Phi' = -\frac{Kp}{4\pi r} \int_{0}^{\infty} \frac{E_3 \cos \gamma - O_3 \sin \gamma}{(\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \sinh \gamma \pi} d\gamma -1 + \frac{\pi (1 + e^{2i\theta})}{\sin 2\alpha + 2\alpha}
\]

7.47
Displacements

\[ e^{i\theta}[pA + B\{(p+1)e^{2i\theta} + \kappa e^{-i(2p+2)\theta}\}] \]

\[ = e^{i\theta} \kappa p \frac{d^{p+1}}{8H(p,\alpha)(p+1) \sin \pi} Y_3(p, \theta) \]

where

\[ Y_3(p, \theta) = \sin \pi \left[ J(p, \alpha) - (p+1)e^{2i\theta} - \kappa e^{-i(2p+2)\theta} \right] + H(p, \alpha) \cos p \pi \]

so that

\[ U_\theta + iU = - \frac{i\kappa e^{-i\theta}}{16\mu} \int_{-\infty}^{\infty} \left( \frac{d^{p+1}}{r} \right) \frac{e^{i(p+1)\theta} e^{-i(p+1)\phi}}{H(p,\alpha)(p+1) \sin \pi} \]

on putting \( p = -1 + i \gamma \)

\[ Y_3(p, \theta) = -i \sin \gamma \pi \left[ \Gamma \cos 2\alpha - \cosh 2\gamma \alpha - i\gamma e^{2i\theta} - \kappa (\cosh 2\gamma \theta + \sinh 2\gamma \theta) \right] \]

\[ + i(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(-i\cosh \gamma \pi) \]

\[ = \mathbb{E}_{\mu} + i \mathbb{O}_{\mu} \]

where

\[ \mathbb{E}_{\mu} = \sin \gamma \pi \left[ \Gamma \cos 2\alpha - e^{2i\theta} + i \kappa \sinh 2\gamma \theta \right] \]

\[ \mathbb{O}_{\mu} = \sinh \gamma \pi \left[ \cosh 2\gamma \alpha + \kappa \cosh 2\gamma \theta \right] - \cosh \gamma \pi (\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \]

\[ U_\theta + iU = - \left( \frac{i\kappa e^{-i\theta}}{16\mu} \right) \left[ \int_{-\infty}^{\infty} \left( \frac{e^{i\gamma (\mathbb{E}_{\mu} + i \mathbb{O}_{\mu})}}{\mathbb{E}_{\mu}(\gamma \sin 2\alpha + \sinh 2\gamma \alpha)(i\gamma)(-i\sinh \gamma \pi)} \right) \right] \]

Calculation of \( R \)

\[ \mathbb{E}_{\mu} + i \mathbb{O}_{\mu} = \mathcal{C}_1 \gamma + \mathcal{C}_2 \gamma \]

where

\[ \mathcal{C}_1 = \gamma (1 + \kappa) - (\sin 2\alpha + 2\alpha) \]

\[ \mathcal{C}_2 = \pi [\cos 2\alpha - e^{2i\theta} + 2i\kappa \theta] \]

and finally

\[ U_\theta + iU = \frac{\kappa e^{-i\theta}}{16\mu} \left[ \int_{0}^{\infty} 2 \left( \frac{\mathbb{E}_{\mu} \sin \gamma k + \mathbb{O}_{\mu} \cos \gamma k}{\gamma (\gamma \sin 2\alpha + \sinh 2\gamma \alpha) \sinh \gamma \pi} \right) \right] - \pi R \]
Stresses and Displacements due to original complex potentials

In order to complete the solution, the stresses and displacements due to the complex potentials $7,3$ which correspond to the force nucleus, must be added to those obtained in terms of integrals in the previous Cases (a), (b) and (c).

From 2.15 and 2.8 with no body force

$$2 \mathbf{\Omega}' = -P \left\{ \frac{1}{z_i} + \frac{1}{z'_i} \right\} \quad \text{and} \quad z_i = re^{i\theta} + d$$

or

$$\mathbf{\omega}' = \frac{P(d - \text{roo}se\theta)}{r^2 + d^2 - 2rd\cos\theta}$$

$$-2\Omega' = \frac{z}{z} \frac{Pz_i'}{z'} + \frac{Pz}{zz'_i}$$

$$= Pe^{2i\theta} \left\{ \frac{(re^{i\theta} - d)}{(re^{i\theta} - d)^2} + \frac{K}{(re^{i\theta} - d)} \right\}$$

$$\Phi' = -\frac{1}{2} Pe^{-2i\theta} \left\{ \frac{re^{i\theta} - d}{(re^{-i\theta} - d)^2} + \frac{K}{(re^{-i\theta} - d)} \right\}$$

$$8\mu D = -K P \log z_i + z_i \frac{1}{z_i} - P \left\{ K(1 + \log z_i) + \frac{d}{z_i} \right\}$$

$$= -KP(1 + \log z_i, z_i) + Pe^{z_i}$$

$$8\mu(U + i\Phi) = -KPe^{-i\theta} \left[ 1 + \log(r^2 + d^2 - 2rd\cos\theta) \right] + Pe^{-i\theta} \frac{re^{i\theta} - d}{r^2 + d^2 - 2rd\cos\theta}$$
Special case of a semi-infinite plate

We shall now make a partial check on the previous work by evaluating the stresses in the case when \(\alpha = \pi/2\). This has been solved by the use of complex potentials by Stevenson, Proc. Roy. Soc. Vol. 184 p 157.

Referring to equn. 11.46 of this paper we have when \(F = W\) the complex potentials suitable for the complete solution. It should be noted that the sign of the potentials quoted is incorrect.

\[
\begin{align*}
\mathcal{J}(z) &= -P\left\{ \log z + \kappa \log z - \frac{2d}{z^2} \right\} \quad \text{7.54} \\
\omega(z) &= P\left\{ \kappa z, \log z, + z, \log z, + d \log z, + \kappa d \log z - \frac{2d^2}{z^2} \right\} \quad \text{7.55}
\end{align*}
\]

These may be separated into three parts:

I \(\mathcal{J}(z) = -P \log z\) \hspace{1cm} \(\omega(z) = P\kappa z, \log z, + Pd \log z\) \hspace{1cm} \text{7.56}

II \(\mathcal{J}(z) = \frac{2Pd}{z}\) \hspace{1cm} \(\omega(z) = -\frac{2Pd}{z} + Pz, \log z\) \hspace{1cm} \text{7.57}

III \(\mathcal{J}(z) = -\kappa P \log z\) \hspace{1cm} \(\omega(z) = \kappa Pd \log z\) \hspace{1cm} \text{7.58}

where \(z_1 = z - d\) and \(z_2 = z + d\)

Group I consists of the original complex potentials which give rise to the force nucleus at \(z = d\), whilst Groups II and III are the complex potentials required to wipe off the boundary load included in Group I. It is to be expected then, that the stress combinations \(\Theta\) and \(\Phi\) given previously in terms of integrals should reduce to the stresses given by II and III.
The form of III which contains the constant \( K \) suggests that it will correspond to Case (4). Group II will, therefore, be checked against Cases (1), (2) and (3).

**Stresses corresponding to II**

\[
2 \Omega' = - \frac{2Pd}{z_2^2} - \frac{2Pd}{z_2^3} + 2\pi \theta + \frac{1}{z_2}
\]

but \( z_2 = \text{re}^{\text{i} \theta} + \text{d} \); \( z_2 = \text{re}^{\text{i} \theta} + 2\pi \theta + \text{d} \)

\[
z_2 \bar{z}_2 = \text{r}^* + 2\pi \text{d} \cos \theta + \text{d}^2
\]

so that

\[
\Omega' = \frac{-2Pd(r^* \cos 2\theta + 2\pi \text{d} \cos \theta + \text{d}^2)}{(r^* + 2\pi \text{d} \cos \theta + \text{d}^2)^2}
\]

7.59

\[
2 \phi' = z \left\{ -\frac{4Pd}{z_2^3} \right\} + e^{2\text{i} \theta} \left\{ \frac{4Pd}{z_2^3} - \frac{1}{z_2} - \frac{1}{z_2^2} \right\}
\]

\[
= \frac{-P}{z_2} \left[ 4dz + e^{2\text{i} \theta}(z^2 + 4dz - d^2) \right]
\]

or

\[
\phi' = \frac{-P}{2(\text{re} - \text{i} \theta + \text{d})^2} \left[ r^2 e^{-4 \text{i} \theta} + 4 \pi e^{-2 \text{i} \theta}(1 + e^{-2 \text{i} \theta} - d^2 e^{-2 \text{i} \theta}) \right]
\]

7.60

**Stresses corresponding to III**

\[
2 \Omega' = -K \left\{ \frac{1}{z_2} + \frac{1}{z_2^2} \right\}
\]

\[
\Omega' = -\frac{K \text{r} \cos \theta + \text{d}}{r^* + 2\pi \text{d} \cos \theta + \text{d}^2}
\]

7.61

\[
2 \phi' = z \left\{ \frac{K \text{r} \cos \theta}{z_2} \right\} + e^{2\text{i} \theta} \left( \frac{K \text{d}}{z_2^2} \right)
\]

\[
\phi' = \frac{-K \text{r}}{2(\text{re} - \text{i} \theta + \text{d})^2} \left( \text{re} - \text{i} \theta - d e^{-2 \text{i} \theta} \right)
\]

7.62
Evalutation of the integrals when $a = \pi/2$

Case (1) + Case (3)  ref 7.21 and 7.24

$$E_\theta = \sinh \gamma \pi \cdot \sin \theta \cdot \sinh \gamma \theta$$

$$O_\theta = \sinh \gamma \pi \cdot \cos \theta \cdot \cosh \gamma \theta$$

$$\theta' = -\frac{2P}{r} \int_0^\infty \frac{(\cos \theta \cdot \cosh \gamma \theta \cdot \cos m + \sin \theta \cdot \sinh \gamma \theta \cdot \sin m)}{\sinh \gamma \pi} \, d\gamma$$

The values of these integrals have been given in Ch VI so that

$$\theta' = \frac{2P}{r} \left[ \cos \theta \cdot \frac{1}{2} \frac{1 + \cos \theta \cdot \cosh m}{(\cos \theta + \cosh m)^2} + \sin \theta \cdot \frac{1}{2} \frac{\sinh m \cdot \sin \theta}{(\cos \theta + \cosh m)^2} \right]$$

$$= \frac{2P}{2r} \left[ \frac{2 \cos \theta + \cos^n (e^m + e^{-m}) + \sin^n (e^m - e^{-m})}{\cos \theta + \frac{1}{2} \frac{d}{r + \frac{r}{d}}} \right]$$

$$\Phi' = \frac{2Pd}{(r^2 + d^2 + 2r \cos \theta)^2} \left\{ r \cos 2\theta + 2r \cos \theta + d^2 \right\}$$

Also from 7.24

$$E_\theta = -\gamma (-1 - e^{2i \theta}) \sinh \gamma \pi = \gamma (1 + e^{2i \theta}) \sinh \gamma \pi$$

$$O_\theta = - (\cosh \gamma \pi + e^{2i \theta}) \sinh \gamma \pi + \sinh \gamma \pi \cdot \cosh \pi = -e^{2i \theta} \sinh \gamma \pi$$

$$\Phi' = \frac{Pe^{-i \theta}}{r} \int_0^\infty \frac{[\gamma (1 + e^{2i \theta}) \sin k - e^{2i \theta} \cos k]}{\sinh \gamma \pi} \, dk$$

$$= \frac{Pe^{-i \theta}}{2r} \left[ \frac{(1 + e^{2i \theta}) \sinh k}{(1 + \cosh k)^2} - \frac{e^{2i \theta}}{1 + \cosh k} \right]$$

but $k = m + i \theta$ so that

$$1 + \cosh k = \frac{e^{i \theta}}{2r} \left( d + re^{-i \theta} \right)^2$$

and

$$\sinh k = \frac{e^{i \theta}}{2r} \left( d^2 - r^2 e^{-2i \theta} \right)$$
so that, \[ \Phi' = \frac{2Pe^{-i\theta}rd^2}{(d + re^{-i\theta})^2} \left[ \frac{e^{-i\theta}}{2rd} \left( d^2 - r^2e^{-2i\theta} \right) - 1 - \frac{r e^{-i\theta}}{d} \right] \]
\[ \Phi' = \frac{Pde^{-i\theta}}{(d + re^{-i\theta})^3} \left[ de^{-i\theta} - r(2 + e^{-2i\theta}) \right] \]

**Case (2)**

from 7.33 \( \phi = 0 \) and \( \delta = \pi \) so that \( \sin \delta = 0 \)

\[ \phi' = 0 \]

from 7.36 \( E'' = 0 \quad 0'' = -\sinh \gamma \pi \)

\[ \phi' = \frac{Pe^{-i\theta}}{4r} \left[ 2 \sum_{n=0}^{\infty} \frac{\sin k}{\sinh \gamma \pi} d^2 - 1 \right] \]
\[ = \frac{Pe^{-i\theta}}{4r} \left[ -\sinh k \frac{\sinh k}{1 + \cosh k} - 1 \right] \]
\[ \Phi' = -\frac{Pe^{-2i\theta}}{2(d + re^{-i\theta})} \]

for Cases (1), (2) and (3) the value of \( \phi' \) is therefore given by 7.63 which it will be seen, agrees with 7.59 corresponding to Group II. Also the value of \( \Phi' \) is obtained as the sum of 7.64 and 7.65 giving

\[ \Phi' = \frac{Pe^{-i\theta}}{2(d + re^{-i\theta})^3} \left[ 2de^{-i\theta} - 2rd(2 + e^{-2i\theta}) - e^{-i\theta}(d^2 + 2rde^{-i\theta} + r^2e^{-2i\theta}) \right] \]

which simplifies to

\[ \Phi' = \frac{-P}{2(d + re^{-i\theta})^3} \left[ r^2e^{-4i\theta} + 4rde^{-i\theta}(1 + e^{-2i\theta}) - d^2e^{-2i\theta} \right] \]

and this result agrees with 7.60.
Case (4) from 7.45

\[ \mathbf{H}' = \frac{KP}{2r} \left[ 2 \int_{0}^{\infty} \left\{ \cos \theta \frac{\sin m \cdot \cosh \gamma}{\sinh \gamma} - \sin \theta \frac{\cosh m \cdot \sinh \gamma}{\sinh \gamma} \right\} d\gamma - \cos \theta \right] \]

\[ = \frac{KP}{2r} \left[ \frac{\cos \theta \cdot \sinh m}{\cos \theta + \cosh m} - \frac{\sin^{2} \theta}{\cos \theta + \cosh m} - \cos \theta \right] \]

\[ = \frac{KP}{2r} \frac{2rd}{r^{2} + d^{2} + 2rd \cos \theta} \left( -e^{-m \cdot \cos \theta} - 1 \right) \]

\[ \Phi' = -\frac{KP(r \cos \theta + d)}{r^{2} + d^{2} + 2rd \cos \theta} \]

and this agrees with 7.61 from 7.47

\[ E_{3} = -\gamma(1 + e^{2i\theta}) \sinh \gamma \pi \]

\[ O_{3} = (\cosh \gamma \pi + e^{2i\theta}) \sinh \gamma \pi - \sinh \gamma \pi \cdot \cosh \gamma \pi \]

\[ = e^{2i\theta} \sinh \gamma \pi \]

\[ \Phi' = -\frac{KPe^{-i\theta}}{4r} \left[ 2 \int_{0}^{\infty} \frac{-\gamma(1 + e^{2i\theta}) \cosh k - e^{2i\theta} \sinh k}{\sinh \gamma \pi} d\gamma + e^{2i\theta} \right] \]

\[ = \frac{KPe^{-i\theta}}{4r} \left[ \frac{(1 + e^{2i\theta}) + e^{2i\theta} \sinh k}{1 + \cosh k} - e^{2i\theta} \right] \]

\[ \Phi' = \frac{KP}{2} \frac{(de^{-2i\theta} - re^{-i\theta})}{(d + re^{-i\theta})^{2}} \]

and this agrees with 7.62
Isolated Force Nuclei at $z = e^{ia}$ with otherwise stress free boundaries.

Consider loads $W$ acting at $z_0 = e^{ia}$ and at $\bar{z}_0$, as shown.

For the load on the upper flank use the complex potentials

$$\lambda(z) = E \log z, \quad \omega(z) = -Ez_0 \log z,$$

where $z_1 = z - z_0$.

As was shown in 5.3 these give rise to a radial distribution of stress

$$\frac{\dd}{\dd r} = \frac{1}{3} \left( \frac{E}{z_1} + \frac{E}{\bar{z}_1} \right),$$

Consider the force and couple resultants on a small semi-circular piece of the wedge with centre at $z_0$. By symmetry the couple resultant is zero. No force acts along HK, whilst the resultant force over the semicircle is given by,

$$(X + iY)\mathbf{i} = \int_{z_0}^{\alpha} \dd \mathbf{R} \cdot \mathbf{R}_0 (\dd \phi)$$

where $\mathbf{R}_0$ = unit vector and $z_1 = \Re e^{i\phi}$ so that $\mathbf{R}_0 = \mathbf{i} e^{i\phi}$

$$(X + iY)\mathbf{i} = \frac{1}{2} \int_{z_0}^{\alpha} \left\{ \left( \frac{E}{R} e^{-i\phi} + \frac{E}{\bar{R}} e^{i\phi} \right) \dd \phi \right\} e^{i\phi} \dd \phi = \frac{1}{2} \mathbf{E} i$$

for equilibrium of the element

$$Wc\cos\phi - iW\sin\phi + \frac{TE}{2} = 0$$
\[ E = - \frac{2We^{-i\delta}}{\pi} \]

For the load on the lower flank

\[ \mathcal{N}(z) = F \log z, \quad \omega(z) = -\overline{F}z \log z \]

where \( z = z - \overline{z_0} \)

These again give rise to a radial distribution of stress

\[ \overline{r} = \frac{1}{2} \left( \frac{F}{\overline{z}} + \frac{\overline{F}}{z} \right) \]

In this case \( X + iY = \frac{TF}{2} \) and for equilibrium of the semi-circular element

\[ W \cos \delta + iW \sin \delta + \frac{TF}{2} = 0 \]

or

\[ F = - \frac{2W e^{i\delta}}{\pi} \]

The complex potentials 8.2 and 8.5 give no stress distribution on the flank to which they refer, but will stress the opposite flank. From the combination of the two nuclei, there will be a stress \( \overline{r} + i\overline{r} \) to be removed from both boundaries.

It is worth noting at this point that the problem of a load on one flank with the other entirely free of load could be solved. However, the advantage in having symmetrical equations for the boundary conditions which determine the constants \( A \) and \( B \) of the transform \( \tilde{\chi} \), is so great that it is better to obtain the case of a single load by superimposing

\[ \frac{W}{2} \quad \frac{W}{2} \]

and

\[ \frac{W}{2} \quad \frac{W}{2} \]

This follows the method adopted by Shepherd (ref.4).
Boundary Stresses

For the nucleus on the upper flank.

The complex potentials 8.2 become when referred to the vertex as origin

\[ \Phi(z) = E \log z, \quad \omega(z) = -\overline{E}z \log z - \overline{z} E \log z \]

so that

\[ 4(\Phi + i\omega) = \Phi(z) + \Phi'(\overline{z}) + z \Phi''(z) + \frac{\overline{z}}{z} \omega''(z) \]

\[ = \frac{E}{z} + \frac{\overline{E}}{\overline{z}} - z \frac{E}{\overline{z}} + \frac{z}{\overline{z}} \left\{ - \frac{E}{z} + \frac{\overline{E}}{\overline{z}} \right\} \]

\[ = \frac{E}{z} + \frac{\overline{E}}{\overline{z}} - \frac{z}{\overline{z}} \frac{E}{z} - \frac{z}{\overline{z}} \left( \frac{\overline{E}}{z} - \frac{E}{z} \right) \]

\[ = \left( \frac{E}{z} + \frac{\overline{E}}{\overline{z}} \right) \left( 1 - \frac{z \overline{z}}{\overline{z}} \right) \]

so that the distribution of stress to be removed from the lower boundary is

\[ \phi + i\omega = -\frac{i}{4} \left( \frac{E}{z} + \frac{\overline{E}}{\overline{z}} \right) \left( 1 - \frac{z \overline{z}}{\overline{z}} \right) \]

where \( z = re^{-i\alpha} - de^{i\alpha} \)

\[ 1 - \frac{z \overline{z}}{\overline{z}} = 1 - e^{2i\alpha} \frac{(re^{-i\alpha} - de^{i\alpha})}{re^{i\alpha} - de^{-i\alpha}} \]

\[ = 1 - \frac{r - de^{2i\alpha}}{r - de^{-2i\alpha}} \]

\[ = \frac{2id \sin 2\alpha}{r - de^{-2i\alpha}} \]
\[
\frac{E + \bar{E}}{z_i + \bar{z}_i} = -2W \left[ \frac{e^{-i\theta}}{\pi} \left( e^{-i\alpha(r - de2i\alpha)} + e^{i\alpha(r - de-2i\alpha)} \right) \right]
\]
\[
(r\theta + i\bar{\theta})_{\theta = \alpha} = -\frac{Wd \sin 2\alpha}{\pi} \left[ \frac{e^{-i(\delta - \alpha)}}{(r - de -2i\alpha)^2} + \frac{e^{i(\delta - \alpha)}}{(r - de -2i\alpha)(r - de -2i\alpha)} \right]
\]

For the nucleus on the lower flank

The distribution of stress to be removed from the upper flank is
\[
\frac{\bar{r}\theta + i\bar{\theta}}{z_j + \bar{z}_j} = -\frac{i}{4} \left( \frac{\bar{F}}{z_j} + \frac{\bar{F}}{\bar{z}_j} \right) \left( 1 - \frac{\bar{z}_j}{\bar{z}_j} \right)
\]

where
\[
z_j = r e^{i\alpha} - d e^{-i\alpha}
\]
\[
1 - \frac{\bar{z}_j}{\bar{z}_j} = 1 - e^{-2i\alpha} \frac{(r e^{i\alpha} - d e^{-i\alpha})}{r e^{-i\alpha} - d e^{i\alpha}}
\]
\[
= -\frac{2id \sin 2\alpha}{r - de2i\alpha}
\]
\[
\frac{\bar{F} + \bar{F}}{z_i + \bar{z}_i} = -2W \left[ \frac{e^{i\theta}}{\pi} \left( e^{i\alpha(r - de -2i\alpha)} + e^{-i\alpha(r - de -2i\alpha)} \right) \right]
\]
\[
(r\theta + i\bar{\theta})_{\theta = \alpha} = \frac{Wd \sin 2\alpha}{\pi} \left[ \frac{e^{-i(\delta - \alpha)}}{(r - de -2i\alpha)^2} + \frac{e^{i(\delta - \alpha)}}{(r - de -2i\alpha)(r - de -2i\alpha)} \right]
\]

referring now to Ch I1b (31)
\[
t_1(p) = \int_0^{p\alpha} \left( r\theta + i\bar{\theta} \right) r^{p+1} dr = -t_2(p)
\]
\[
= \frac{Wd \sin 2\alpha}{\pi} \left[ \int_0^{\infty} \left( \frac{r^{p+1} e^{-i(\delta - \alpha)}}{(r - de -2i\alpha)^2} + \frac{r^{p+1} e^{i(\delta - \alpha)}}{(r - de -2i\alpha)(r - de -2i\alpha)} \right) dr \right]
\]
\[
= \frac{Wd \sin 2\alpha}{\pi} \left[ I_3(p, \alpha) e^{-i(\delta - \alpha)} + I_4(p, \alpha) e^{i(\delta - \alpha)} \right]
\]

from 6.4
\[
I_3(p, \alpha) = \frac{\pi d e^{-ip\nu}(p+1)}{\sin \nu \pi} \text{ where } \nu = \pi - 2\alpha
\]

\[
I_4(p, \alpha) = \frac{\pi d e^{-ip\nu}(p+1)}{\sin \nu \pi}
\]
Evaluation of $I_p(p,\alpha)$ follows the method of contour integration detailed in Ch VI, so that

$$(1 - e^{2\pi i})I_p(p,\alpha) = 2\pi i \sum \text{residues of } \int \frac{z^{p+1}dz}{(z - d e^{2i\alpha})(z - de^{-2i\alpha})}$$

Residue at $de^{2i\alpha}$ put $z - de^{2i\alpha} = t$

Integrand becomes

$$\frac{(t + de^{2i\alpha})^{p+1}}{t(t + de^{2i\alpha} - de^{-2i\alpha})}$$

$$= \frac{dp+1e^{2i(p+1)\alpha}}{2id \sin 2\alpha} \left\{1 + \frac{t}{2id \sin 2\alpha}\right\}$$

residue = $\frac{dp+1e^{2i(p+1)\alpha}}{2id \sin 2\alpha}$ \hfill 8.14

Residue at $de^{i(2\pi - 2\alpha)}$, (so as to avoid crossing the cut.) put $z - de^{i(2\pi - 2\alpha)} = t$

Integrand becomes

$$\frac{(t + de^{i(2\pi - 2\alpha)})^{p+1}}{(t + de^{2i\alpha} - de^{-2i\alpha})t}$$

residue = $\frac{-dp+1e^{i(p+1)(2\pi-2\alpha)}}{2id \sin 2\alpha}$ \hfill 8.15

sum of residues = $\frac{dp}{2i \sin 2\alpha} \left[e^{2i(p+1)\alpha} - e^{i(p+1)(2\pi-2\alpha)}\right]$
\[ e^{ip\pi}(-2i\sin\pi)I_4(p, \alpha) = -2\pi i \frac{dp_{e^{i(p+1)\pi}}\sin(p+1)\alpha}{\sin2\alpha} \]

\[ I_4(p, \alpha) = -\frac{\pi dp\sin(p+1)\alpha}{\sin2\alpha \sin\pi} \]

and finally
\[ t_1(p) = -\frac{Wp^{p+1}}{\sin\pi} \left[ (p+1)\sin2\alpha \cdot e^{-ip\alpha} - e^{i(p+2)\alpha} + \sin(p+1)e^{i(p+2)\alpha} \right] \]

Also from Ch IIb (17) \[ n = -2 \text{ and so} \]

\[ -2 < p < 0 \]

**Boundary condition**
\[ (p+1)(D + ip)\tilde{\chi} = 2i(p+1)\left[ pAe^{ip\alpha} + B(p+1)e^{i(p+2)\alpha} - Be^{-i(p+2)\alpha} \right] \]

\[ = t_1(p) \text{ over } \Theta = \alpha \]

\[ = t_2(p) \text{ over } \Theta = -\alpha \]

over \( \Theta = \alpha \),

\[ pAe^{ip\alpha} + B(p+1)e^{i(p+2)\alpha} - Be^{-i(p+2)\alpha} = \frac{t_1(p)}{2i(p+1)} \]

over \( \Theta = -\alpha \),

\[ pAe^{-ip\alpha} + B(p+1)e^{-i(p+2)\alpha} - Be^{i(p+2)\alpha} = \frac{t_2(p)}{2i(p+1)} = -\frac{t_1(p)}{2i(p+1)} \]

and the conjugates
\[ p\overline{A}e^{ip\alpha} + B(p+1)e^{i(p+2)\alpha} - B\overline{e}^{i(p+2)\alpha} = -\frac{\overline{t_1}(p)}{2i(p+1)} \]

\[ p\overline{A}e^{ip\alpha} + B(p+1)e^{i(p+2)\alpha} - B\overline{e}^{i(p+2)\alpha} = \frac{t_2(p)}{2i(p+1)} \]

These imply \( \overline{A} = A \) and \( \overline{B} = B \) so that
\[ pA + B(p+1)e^{2i\alpha} - Be^{-i(2p+2)\alpha} = \frac{t_1(p)e^{-ip\alpha}}{2i(p+1)} = A' + iB' \]

\[ pA + B(p+1)e^{-2i\alpha} - Be^{i(2p+2)\alpha} = -\frac{t_1(p)e^{ip\alpha}}{2i(p+1)} = A' - iB' \]
\[ A' + iB' = \frac{W_d p + 1}{2(p+1)\sin \pi} \left[ (p+1) \sin 2\alpha, e^{-i(p+\theta)} e^{-i(\delta-\alpha)} \right. \\
\left. + \sin (p+1) \omega e^{i(\delta-\alpha)} e^{-ip \alpha} \right] \]

or writing

\[ p(\alpha+\omega) + \delta - \alpha = p \pi = (p+1) \alpha + \delta = \theta, \quad 8.21 \]

and

\[ (p+1) \alpha - \delta = \theta_2 \quad 8.22 \]

\[ A' + iB' = \frac{W_d p + 1}{2(p+1)\sin \pi} \left[ (p+1) \sin 2\alpha, e^{-i\theta}, + \sin (p+1) \omega e^{-i\theta_2} \right] \]

\[ A' = \frac{W_d p + 1}{2(p+1)\sin \pi} R_1(\rho, \alpha) \quad 8.23 \]

\[ B' = \frac{W_d p + 1}{2(p+1)\sin \pi} R_2(\rho, \alpha) \quad 8.24 \]

where

\[ R_1(\rho, \alpha) = (p+1) \sin 2\alpha, \sin \theta, + \sin (p+1) \omega, \sin \theta_2 \quad 8.25 \]

\[ R_2(\rho, \alpha) = (p+1) \sin 2\alpha, \cos \theta, + \sin (p+1) \omega, \cos \theta_2 \quad 8.26 \]

as in 7.17 and 7.18

\[ B = \frac{B'}{H(p, \alpha)} = \frac{W_d p + 1}{2(p+1)H(p, \alpha)\sin \pi} \quad 8.27 \]

\[ A = \frac{1}{pH(p, \alpha)} \left\{ A'(H(p, \alpha) - B', J(p, \alpha)) \right\} \]

\[ = \frac{W_d p + 1}{2p(p+1)H(p, \alpha)\sin \pi} \left\{ R_1(p, \alpha), H(p, \alpha) - R_2(p, \alpha), J(p, \alpha) \right\} \quad 8.28 \]

**Stresses**

\[ (D^2 + p^2) \chi = -8(p+1)B \cos (p+2) \theta \]

\[ = -\frac{4W_d p + 1}{H(p, \alpha)\sin \pi} \quad R_2(p, \alpha) \cos (p+2) \theta \]

\[ \theta' = \frac{2W_d}{\pi i r^2} \int_0^\infty \frac{d}{r} \left( \frac{1}{p} \right) \frac{R_2(p, \alpha) \cos (p+2) \theta}{H(p, \alpha)\sin \pi} \quad dp \quad 8.29 \]
Bearing in mind the restriction on the real part of $p$ given by 8.18, this integral will be simplified by putting $p = i \gamma$ and taking $c = 0$. The point $p = 0$ is however a pole of the integrand.

$$H(i \gamma, \alpha) = (1+i \gamma) \sin 2 \alpha + \sin(2+2i \gamma) \alpha$$

$$= E_{15} + i 0_{15} \quad \text{where}$$

$$E_{15} = \sin 2 \alpha (1 + \cosh 2 \gamma \alpha)$$

$$0_{15} = \gamma \sin 2 \alpha + \cos 2 \alpha \sinh 2 \gamma \alpha$$

$\Omega_1 = i \gamma (\pi - \alpha) - (\alpha - \delta) = i \gamma \beta - (\alpha - \delta)$

$\Omega_2 = i \gamma \alpha + (\alpha - \delta)$

$$R_2(i \gamma, \alpha) = (1+i \gamma) \sin 2 \alpha \cos \left\{ i \gamma \beta - (\alpha - \delta) \right\} + \sin(1+i \gamma) \cos(1+i \gamma) \cos \left\{ i \gamma \alpha + (\alpha - \delta) \right\}$$

$$= (1+i \gamma) \sin 2 \alpha \left[ \cosh \gamma \beta \cos(\alpha - \delta) + i \sinh \gamma \beta \sin(\alpha - \delta) \right]$$

$$+ \sin \omega \cosh \omega \beta \sinh \alpha \cos(\alpha - \delta)$$

$$Q_{1b} = \sin 2 \alpha \left[ \sinh \gamma \beta \sin(\alpha - \delta) + \gamma \cosh \gamma \beta \cos(\alpha - \delta) \right]$$

$$- \sin \omega \cosh \omega \beta \sinh \alpha \sin(\alpha - \delta) + \cosh \omega \sinh \omega \beta \cosh \alpha \cos(\alpha - \delta)$$

$$= \frac{2Wd}{\pi i r^2} \left[ \int_{-\infty}^{\infty} \frac{e^{i \gamma m(E_{15} + i 0_{15})} (\cos 2 \theta \cosh \gamma \theta - i \sin 2 \theta \sinh \gamma \theta)}{(E_{15} + i 0_{15}) i \sinh \gamma \alpha} \, d \gamma \right]$$

$$\pi R = \frac{\cos 2 \theta}{2 \sin 2 \alpha \left\{ \sin 2 \alpha \cos(\alpha - \delta) + \sin \omega \cos(\alpha - \delta) \right\}}$$

$$= \cos 2 \theta \cos(\alpha - \delta)$$

$$= E_{15} + i 0_{15} \quad \text{where}$$
\[ E_{17} = (E_{15}E_{16} + 0_{15}0_{16})\cos 2\theta \cdot \cosh \gamma + (0_{16}E_{15} - E_{16}0_{15})\sin 2\theta \cdot \sinh \gamma \]

\[ 0_{17} = (0_{16}E_{15} - E_{16}0_{15})\cos 2\theta \cdot \cosh \gamma - (E_{15}E_{16} + 0_{15}0_{16})\sin 2\theta \cdot \sinh \gamma \]

so that

\[
\Theta ' = -\frac{2Wd}{\pi r^2} \left[ 2 \int_{0}^{\infty} \frac{E_{17}\sin m + 0_{17}\cos m}{(E_{15}^{2} + 0_{15}^{2})\sinh \gamma} \, dm - \cos 2\theta \cdot \cos (5-\alpha) \right] \tag{8.31}
\]

**Stress Function** \( \Phi' \)

\[
(D + ip)\{D + i(p+2)\}{\chi} = -4(p+1)e^{ip\theta}\{A\theta + B(p+2)e^{2i\theta} \}
\]

inserting the values of \( A \) and \( B \) given by 8.27 and 8.28, this expression

\[
= -4(p+1)e^{ip\theta} \frac{Wd^{p+1}}{2(p+1)H(p,\alpha)\sinh \pi} Z_{4}(p,\theta) \tag{8.32}
\]

where

\[
Z_{4}(p,\theta) = R_{1}(p,\alpha)H(p,\alpha) - R_{2}(p,\alpha) \{ J(p,\alpha) - (p+2)e^{2i\theta} \} \tag{8.33}
\]

putting \( p = i\gamma \)

\[
R_{1}(i\gamma,\alpha) = (1+i\gamma)\sin 2\alpha \cdot \sin [i\gamma - (\alpha - \delta)] + \sin (1+i\gamma)\sin [i\gamma + (\alpha - \delta)]
\]

\[
= (1+i\gamma)\sin 2\alpha [i\sinh \gamma \cdot \cos (\alpha - \delta) - \cosh \gamma \cdot \sin (\alpha - \delta)] + [i\sinh \gamma \cdot \cosh \gamma + i\cosh \gamma \cdot \sinh \gamma] [i\sinh \gamma \cdot \cos (\alpha - \delta) + \cosh \gamma \cdot \sin (\alpha - \delta)]
\]

\[
= E_{18} + i_{018} \text{ where}
\]

\[
E_{18} = -\sin 2\alpha [i\sinh \gamma \cdot \cos (\alpha - \delta) + \cosh \gamma \cdot \sin (\alpha - \delta)]
\]

\[
+ \sin \gamma \cdot \cosh \gamma \cdot \cosh \gamma \cdot \sin (\alpha - \delta) - \cos \gamma \cdot \sinh \gamma \cdot \sinh \gamma \cdot \cos (\alpha - \delta)
\]

\[
0_{18} = \sin 2\alpha [i\sinh \gamma \cdot \cos (\alpha - \delta) - \gamma \cosh \gamma \cdot \sin (\alpha - \delta)]
\]

\[
+ \sin \gamma \cdot \cosh \gamma \cdot \sinh \gamma \cdot \cos (\alpha - \delta) + \cos \gamma \cdot \sinh \gamma \cdot \cosh \gamma \cdot \sin (\alpha - \delta)
\]

\[
J(i\gamma,\alpha) = (1+i\gamma)\cos 2\alpha - \cos 2(1+i\gamma)\alpha = E_{19} + i_{019} \text{ where}
\]

\[
E_{19} = \cos 2\alpha (1 - \cosh 2\gamma \alpha)
\]

\[
0_{19} = \gamma \cos 2\alpha + \sin 2\alpha \cdot \sinh 2\gamma \alpha
\]

\[
Z_{4}(p,\theta) = (E_{16} + i_{016})(E_{15} + i_{015}) - (E_{16} + i_{016})[E_{19} + i_{019} - (2+i\gamma)e^{2i\theta}]
\]

\[
= E_{0} + i_{020} \text{ where}
\]
\[ E_{20} = E_{15} E_{18} - O_{15} O_{18} - E_{16}(E_{17} - 2e^{2i\theta}) + O_{16}(0_{17} - \gamma e^{2i\theta}) \]

\[ O_{20} = O_{18} E_{15} + O_{15} E_{16} - E_{16}(O_{17} - \gamma e^{2i\theta}) - O_{16}(E_{17} - 2e^{2i\theta}) \]

\[
\Phi' = -\frac{Wd}{\pi r^2} \int_0^\infty \frac{e^{i\gamma k(E_{20} + iO_{20})i\gamma} - \pi R}{(E_{15} + iO_{15})i\sinh \gamma} \, d\gamma - \pi R, \quad k = m + i\theta
\]

\[ \pi R = \lim_{\gamma \to 0} \frac{E_{20} + iO_{20}}{E_{15} + iO_{15}} = 2e^{2i\theta} \cos(\alpha - \delta) \]

so that

\[
\Phi' = -\frac{Wd}{\pi r^2} \left[ 2 \int_0^\infty \frac{E_{20} \sin \gamma + O_{20} \cos \gamma}{(E_{15} + iO_{15})\sinh \gamma} \, d\gamma - \pi R \right]
\]

where

\[ E_{21} = E_{20} E_{18} + O_{20} O_{18} \quad \text{and} \quad C_{20} = O_{20} E_{15} - E_{20} O_{15} \]

Displacements

\[ e^{ip\theta} \frac{\beta pA + B}{(p+1)e^{2i\theta} + K e^{-i(2p+2)\theta}} \]

\[ = \frac{Wd}{2(p+1)H(p,a)\sin \pi} Y_\mu(p,\theta) \quad \text{where} \]

\[ Y_\mu(p,\theta) = R_h(p,a) H(p,a) + R_\mu(p,a) \left\{ (p+1)e^{2i\theta} + K e^{-i(2p+2)\theta} \right\} J(p,a) \]

\[ U_\mu + iU_\theta = -\frac{Wl}{4\pi \omega} \int_{c-i\omega}^{c+i\omega} \frac{(d\theta)^{p+1} e^{ip\theta} Y_\mu(p,\theta)}{(p+1)H(p,a)\sin \pi} \, dp \]

putting \( p = i\gamma \)

\[ (1 + i\gamma)(E_{15} + iO_{15}) = E_{22} + iO_{22} \quad \text{where} \]

\[ E_{22} = E_{15} - \gamma O_{15} \quad \text{and} \quad O_{22} = \gamma E_{15} + O_{15} \]

\[ Y_\mu(i\gamma,\theta) = (E_{18} + iO_{18})(E_{15} + iO_{15}) + (E_{16} + iO_{16}) \left\{ (1 + i\gamma)e^{2i\theta} + K e^{-2i\theta}(\cosh 2\gamma + \sinh 2\gamma)(E_{17} + iQ_{17}) \right\} \]

\[ = E_{23} + iO_{23} \quad \text{where} \]

\[ E_{23} = E_{18} E_{15} - O_{18} O_{15} + E_{16}(e^{2i\theta} + K e^{-2i\theta} \cosh 2\gamma - E_{19}) \]

\[ - O_{16}(e^{2i\theta} - K e^{-2i\theta} \sinh 2\gamma e^{2i\theta} - O_{19}) \]
\begin{align*}
o_{\phi} &= O_{\phi} E_{\phi} + E_{\phi} O_{\phi} + O_{\phi} (e^{2i\theta} + \kappa e^{-2i\theta} \cosh 2\gamma \theta - E_{\phi})
&= E_{\phi} (\gamma e^{2i\theta} - i \kappa e^{-2i\theta} \sinh 2\gamma \theta - O_{\phi})
U_{\phi} + i U_{\phi} &= \frac{\text{Wid}}{4\pi \mu r} \left[ \int_{-\infty}^{\infty} \frac{e^{i \gamma k (E_{2\phi} + i O_{2\phi})}}{(E_{2\phi} + i O_{2\phi}) \sinh \gamma \pi} \, d\gamma - \pi i R \right]
\pi R &= \frac{X(0, \theta)}{H(0, \alpha)} = \frac{2 \sin 2\alpha \cos (\alpha - \delta) \{e^{2i\theta} + \kappa e^{-2i\theta}\}}{2 \sin 2\alpha}
&= \cos (\alpha - \delta) \{e^{2i\theta} + \kappa e^{-2i\theta}\}
U_{\phi} + i U_{\phi} &= \frac{\text{Wid}}{4\pi \mu r} \left[ \int_{-\infty}^{\infty} \frac{e^{i \gamma k (E_{2\phi} + i O_{2\phi})}}{(E_{2\phi} + i O_{2\phi}) \sinh \gamma \pi} \, d\gamma - \pi i R \right]
\text{where } E_{2\phi} &= E_{2\phi} + O_{2\phi} \text{ and } O_{2\phi} = Q_{2\phi} E_{2\phi} - E_{2\phi} O_{2\phi}
U_{\phi} + i U_{\phi} &= \frac{\text{Wid}}{4\pi \mu r} \left[ 2 \int_{-\infty}^{\infty} \frac{E_{2\phi} \sin \gamma k + Q_{2\phi} \cos \gamma k}{(E_{2\phi}^* + O_{2\phi}^*) \sinh \gamma \pi} \, d\gamma - \pi R \right]
\end{align*}
Chapter IX

Isolated Couple Nucleus at \( z = d \) (real) with Rigid boundaries.

Using the complex potentials 5.1 which give a couple nucleus at \( z = d \), the displacement of the flanks is given from 6.27 as

\[
U_{\mu} + iU_{\nu} = \frac{iG}{4\pi\mu} \frac{1}{r - de^{i\theta}} \quad \text{on } \theta = \pm \alpha \quad 9.1
\]

We wish to remove this displacement and so from Ch IIb eqn. (32)

\[
d_i(p) = -\frac{iG}{4\pi\mu} \int_{0}^{\infty} \frac{r^p}{r - de^{i\alpha}} dr = -\frac{iG}{4\pi\mu} I_i(p, \alpha) \quad 9.2
\]

Integrating \( \int \frac{z^p}{z - de^{i\alpha}} dz \) round the same contour as before

\[(1 - e^{2\pi i}) I_i(p, \alpha) = 2\pi i \times \text{Residue at } de^{i\alpha}\]

putting \( z = de^{i\alpha} = t \) the integrand becomes

\[
\frac{(t + de^{i\alpha})^p}{t} \quad \text{so that the residue is } dp e^{i\alpha} \]

\[
I_i(p, \alpha) = 2\pi i d_p e^{i\alpha} \]

giving \( I_i(p, \alpha) = -\frac{\pi d_p e^{-ip\beta}}{\sin \pi} \quad \beta = \pi - \alpha \quad 9.3 \)

also \( \left| \int \frac{z^p}{z - de^{i\alpha}} dz \right| < 2\pi r. \frac{r^p}{r - d} \rightarrow 0 \text{ as } r \rightarrow 0 \text{ if } p > -1 \)

\( \rightarrow 0 \text{ as } r \rightarrow \infty \text{ if } p < 0 \)

\(-1 < p < 0 \quad 9.4\)
\[ d_1(p) = \frac{iG \, p e^{-ip\beta}}{4\mu \sin \mu} \]
\[ d_2(p) = -d_1(p) \]

**Boundary condition**

\[ (D + ip) \left[ 1 + \frac{(1-\sigma)(D-ip)\{D-i(p+2)\}}{(p+1)(p+2)} \right] \chi = 2\mu i \, d_1(p) \text{ on } \theta = \alpha \]
\[ = 2\mu i \, d_2(p) \text{ on } \theta = -\alpha \]

over \( \theta = \alpha \)

\[ 2i \left[ pAe^{ip\alpha} + B(p+1)e^{i(p+2)\alpha} - \overline{\alpha}e^{-i(p+2)\alpha} \right] + (1-\sigma)8i\overline{\alpha}e^{-i(p+2)\alpha} = 2\mu i \, d_1(p) \]

\[ pAe^{ip\alpha} + B(p+1)e^{i(p+2)\alpha} + \kappa \overline{\alpha}e^{-i(p+2)\alpha} = \mu d_1(p) \]

where \( \kappa = 3 - 4\sigma \)

using 9.5 we have

\[ pA + B(p+1)e^{2i\alpha} + \kappa \overline{\alpha}e^{-i(2p+2)\alpha} = \frac{iG \, p e^{-ip\alpha}}{4 \sin \mu} \]

over \( \theta = -\alpha \)

\[ pA + B(p+1)e^{-2i\alpha} + \kappa \overline{\alpha}e^{i(2p+2)\alpha} = \frac{iG \, p e^{ip\alpha}}{4 \sin \mu} \]

and the conjugates

\[ \overline{pA} + \overline{B(p+1)e^{-2i\alpha}} + \kappa \overline{\alpha}e^{i(2p+2)\alpha} = -\frac{iG \, p e^{-ip\alpha}}{4 \sin \mu} \]

\[ \overline{pA} + \overline{B(p+1)e^{2i\alpha}} + \kappa \overline{\alpha}e^{-i(2p+2)\alpha} = -\frac{iG \, p e^{ip\alpha}}{4 \sin \mu} \]

These imply \( \overline{A} = -A \) and \( \overline{B} = -B \)

\[ pA + B(p+1)e^{-2i\alpha} - \kappa \overline{\alpha}e^{i(2p+2)\alpha} = \frac{iG \, p e^{-ip\alpha}}{4 \sin \mu} \]

\[ pA + B(p+1)e^{2i\alpha} - \kappa \overline{\alpha}e^{-i(2p+2)\alpha} = \frac{iG \, p e^{ip\alpha}}{4 \sin \mu} \]
subtracting
\[ \mathbf{B}(p+1)(-2\sin 2\alpha) - \kappa \mathbf{B}_r 2\sin(2p+2)\alpha = \frac{iG \, dp}{4 \sin \pi} \]

or
\[ B = \frac{iG \, dp}{4 \, H(p, \alpha)} \]

where
\[ G(p, \alpha) = (p+1)\sin 2\alpha - \kappa \sin 2(p+1)\alpha \]
\[ H(p, \alpha) = (p+1)\sin 2\alpha + \kappa \sin 2(p+1)\alpha \]
\[ J(p, \alpha) = (p+1)\cos 2\alpha - \kappa \cos 2(p+1)\alpha \]
\[ K(p, \alpha) = (p+1)\cos 2\alpha + \kappa \cos 2(p+1)\alpha \]

solving for \( A \) from 9.8
\[ \frac{pA}{4 \sin \pi} = \frac{iG \, dp e^{i\pi}}{4 \, H(p, \alpha)} \left[ (p+1)e^{-2i\alpha} - \kappa e^{i(2p+2)\alpha} \right] \]
\[ = \frac{iG \, dp}{4 \, H(p, \alpha) \sin \pi} \left[ H(p, \alpha) \cos \pi + J(p, \alpha) \sin \pi \right] \]

Stresses
\[ (D^p + p^r) \tilde{\chi} = -8i \mathbf{B}(p+1)\sin(p+2)\Theta \]
\[ = -\frac{2G \, dp}{H(p, \alpha)} (p+1)\sin(p+2)\Theta \]
so that
\[ \Theta \cdot -\frac{G}{\pi ir^p} \left( \frac{d}{dr} \right)^p (p+1)\sin(p+2)\Theta \, dp \]

in simplifying this integral we note that whilst \( p = -1 \) is a zero of \( H(p, \alpha) \) it is not a pole of the integrand due to the presence of \( p + 1 \) in the numerator.

putting \( p = i \gamma \)
\[ \left( \frac{d}{dr} \right)^p = e^{i \gamma m} \quad \text{where} \quad m = \log \frac{d}{r} \]
\[ (1 + i \gamma)\sin(2 + i \gamma)\Theta = \mathbf{E}_r + i \mathbf{O}_r \] as in Ch VI
\[ H, (i \gamma, \alpha) = (1 + i \gamma) \sin 2\alpha + \kappa \sin(2i \gamma \alpha + 2\alpha) \]
\[ = (1 + i \gamma) \sin 2\alpha + \kappa \{ \sin 2\alpha \cosh 2\gamma \alpha + i \cos 2\alpha \sinh 2\gamma \alpha \} \]
\[ = X, + i Y, \quad \text{where} \]
\[ X, = \sin 2\alpha (1 + \kappa \cosh 2\gamma \alpha) \]
\[ Y, = \gamma \sin 2\alpha + \kappa \cos 2\alpha \sinh 2\gamma \alpha \]
\[ \phi = -\frac{G}{\pi \tau^{\gamma}} \int_{-\infty}^{\infty} \frac{e^{i \gamma m} (E_x + i 0) (X, - i Y,)}{X,^2 + Y,^2} \, d\gamma \]
\[ = \frac{E_x \cos \gamma m - Q_x \sin \gamma m}{X, + Y,^2} \quad \text{where} \quad E_x = E_2 X, + O_x Y, \quad \text{and} \quad Q_x = O_2 X, - E_2 Y, \]

**Stress function** \( \Phi' \)

\[ (D + ip) \{ D + i(p+2) \} \tilde{\chi} = -4A(p+1) e^{ip\Theta} - 4B(p+1)(p+2) e^{i(p+2)\Theta} \]
\[ = -i(p+1) e^{ip\Theta} \left[ A_{\gamma} + B(p+2) e^{2i\Theta} \right] \]
\[ = - \frac{ig_{p+1} e^{ip\Theta}}{H, (p, \alpha) \sin \pi \gamma} Z_5 (p, \Theta) \]

where \( Z_5 (p, \Theta) = H, (p, \alpha) \cos \pi \gamma + J, (p, \alpha) \sin \pi \gamma - (p+2) e^{2i\Theta} \sin \pi \gamma \)

so that \[ \Phi' = -\frac{G}{2\pi \tau^{\gamma}} \int_{-\infty}^{\infty} \frac{e^{ip\Theta}}{H, (p, \alpha) \sin \pi \gamma} Z_5 (p, \Theta) \, dp \]

and this integral does have a pole at \( p = 0 \)

putting \( p = i \gamma \) \[ \left( \frac{d}{r} \right)^p e^{ip\Theta} = e^{i\gamma k} \quad \text{where} \quad k = m + i \Theta \]

\[ J, (p, \alpha) = (1 + i \gamma) \cos 2\alpha + \kappa \{ \cos 2\alpha \cosh 2\gamma \alpha - i \sin 2\alpha \sinh 2\gamma \alpha \} \]
\[ = X, + i Y, \quad \text{where} \]
\[ X_2 = \cos 2\alpha (1 + \kappa \cosh 2\gamma \alpha) \]
\[ Y_2 = \gamma \cos 2\alpha + \kappa \sin 2\alpha \cdot \sinh 2\gamma \alpha \]
\[ Z_5(p, \Theta) = (X_2 + iY_2)i \sin \gamma \pi + (X_1 + iY_1) \cosh \gamma \pi \]
\[ = (2 + i\gamma)e^{2i\Theta}i \sin \gamma \pi \]
\[ X_3 = -Y_1 \sinh \gamma \pi + X_1 \cosh \gamma \pi + \gamma e^{2i\Theta} \sinh \gamma \pi \]
\[ Y_3 = X_2 \sinh \gamma \pi + Y_1 \cosh \gamma \pi - 2e^{2i\Theta} \sinh \gamma \pi \]
\[ \phi' = -\frac{G}{2\pi \tau} \left[ \int_{-\infty}^{\infty} \frac{e^{i\gamma k} (1 + i\gamma)(X_2 + iY_2)}{i \sinh \gamma \pi (X_1 + iY_1)} \mathrm{d}\gamma + \pi i \tau \right] \]

since \( \sinh \gamma \pi = \pi \gamma + O(\gamma^2) \)
\[ \pi \tau = \lim_{\gamma \to 0} e^{i\gamma k} (1 + i\gamma)(X_2 + iY_2) = 1 \]
so that
\[ \phi' = -\frac{G}{2\pi \tau} \left[ 2 \int_{0}^{\infty} \frac{O_{2\theta} \cos \gamma k + E_{2\theta} \sin \gamma k}{\sinh \gamma \pi (X_1 + iY_1)} \mathrm{d}\gamma - 1 \right] \]
where
\[ O_{2\theta} = X_1X_3 + Y_1Y_3 + \gamma (Y_1X_3 - X_1Y_3) \]
\[ E_{2\theta} = X_1X_3 + Y_1Y_3 + \gamma (X_1X_3 + Y_1Y_3) \]

**Displacements**

referring to the work leading up to 6.22
\[ U_{\phi} + iU_{p} = -\frac{1}{2\pi \mu \nu} \int_{c-i\infty}^{c+i\infty} \left\{ p \mathrm{e}^{i\Theta} + B(p+1) \mathrm{e}^{i(p+2)\Theta} + K \mathrm{Be}^{-i(p+2)\Theta} \right\} \tau^{p-1} \mathrm{d}p \]
or inserting the values of \( A \) and \( B \) given by 9.10 and 9.11
\[ U_{\phi} + iU_{p} = \frac{G}{8\pi \mu r} \int_{c-i\infty}^{c+i\infty} \frac{Y_{5}(p, \Theta) \mathrm{e}^{ip\Theta}}{H_{\gamma}(p, \alpha) \sin \pi \nu r} \left( \frac{d}{r} \right)^{p} \mathrm{d}p \]
where
\[ Y_{5}(p, \Theta) = \sin \pi \nu \left[ \left\{ J_{\gamma}(p, \alpha) - (p+1) \mathrm{e}^{2i\Theta} \right\} + K \mathrm{e}^{-2i(p+1)\Theta} \right] \]
\[ + H_{\gamma}(p, \alpha) \cos \pi \nu \]
again putting $p = i\gamma$ to simplify 9.22

$$Y_5(p, \theta) = X_4 + iY_4$$ where

$$X_4 = X_1 \cosh \gamma \pi + \sinh \gamma \pi \left[ \gamma e^{2i\theta} - Y_2 + \kappa e^{-2i\theta} \sinh 2\gamma \theta \right]$$

$$Y_4 = Y_1 \cosh \gamma \pi + \sinh \gamma \pi \left[ X_2 - e^{2i\theta} + \kappa e^{-2i\theta} \cosh 2\gamma \theta \right]$$

$$U_4 + iU_4 = \frac{G}{\xi \mu r \pi} \left[ \int_{-\infty}^{\infty} \frac{e^{i\eta k} (X_4 + iY_4) (X_4 - iY_4)}{i \sinh \gamma \pi (X_4^2 + Y_4^2)} \, i d\xi \right] \pi i R \quad 9.24$$

and

$$\pi R = \lim_{\gamma \to 0} \frac{e^{i\gamma k} (X_4 + iY_4)}{X_4 + iY_4} = 1$$

$$U_4 + iU_4 = \frac{G}{\xi \mu r \pi} \left[ \int_{-\infty}^{\infty} \frac{e^{i\eta k} (E_{2\gamma} + i0_{2\gamma})}{\sinh \gamma \pi (X_4^2 + Y_4^2)} \, d\gamma \right] \quad 9.25$$

$$U_4 + iU_4 = \frac{G}{\xi \mu r \pi} \left[ 2 \left( \int_{-\infty}^{\infty} \frac{E_{2\gamma} \sin k + O_{2\gamma} \cos k}{\sinh \gamma \pi (X_4^2 + Y_4^2)} \, d\gamma \right) - 1 \right]$$

where

$$E_{2\gamma} = X_4 X_4 + Y_4 Y_4 \quad \text{and} \quad O_{2\gamma} = Y_4 X_4 - Y_4 X_4$$