Blood flow in twisted arteries.

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Submitted for the degree of Doctor of Philosophy at The University of Surrey, December 1998.
To my mother
and Lise.
Abstract

The motivation for this research into flow in pipes with non-uniform geometry comes from physiological flows. It is now widely believed that haemodynamics plays an important role in the initiation and development of atherosclerosis. Experiments have shown that the preferred sites for atherogenesis are regions of low wall shear stress. The build-up of atherosclerotic plaques in the coronary arteries can lead to arterial blockage and coronary failure.

Previous studies have examined uniformly curved pipes and, more recently, uniformly curved and twisted pipes. However, it is well known that the arterial system displays non-uniform, time-dependent geometry.

The main objective of this thesis is to describe flow in various pipes with weakly non-uniform curvature and torsion, with a view to understanding the resulting wall shear stress distribution and velocity profiles.

The work herein models the flow of an incompressible Newtonian fluid through a pipe whose curvature and torsion vary along the pipe. The governing equations are first derived, then solved for both steady and oscillatory pressure gradients. The solution of these equations involves asymptotic and numerical techniques. The effects due to the non-uniform geometry and possible applications to physiology are discussed.

Finally, the effects of torsion upon fluid motion are studied from the Lagrangian viewpoint, using numerical particle tracking.
Firstly, thanks to my supervisor Dr. Peter Hydon, without whom this would not have been possible. Thanks also to Dr. Peter Ashwin and to Dr. Philip Aston for their help with the numerical work. I am also grateful for the helpful discussions I have had with Professor Ron Shail, Dr. Jonathan Mestel and, on the physiological side, Professor Colin Caro.

On the non-mathematical side, thanks to my mother and Lise who have had to put up with too much, to Susan for always being there, to Paul for being a star and to Mike, David, Qing and Kyriakos for being excellent and supportive "cell mates". Finally, thanks to Ron for inspiring me in the first place.

I would also like to acknowledge the support of the EPSRC.
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Chapter 1

Introduction.

1.1 Physiology.

1.1.1 The Heart.

The main purpose of the heart is to deliver all the nutrients the body's tissues require. The heart (Figure 1.1) is a muscle that pumps blood around the body. The right ventricle pumps blood through the lungs (pulmonary circulation), and the left ventricle pumps blood through the rest of the body (systemic circulation).

Blood enters each ventricle via an atrium. The atrium principally acts as a blood reservoir, but also weakly pumps the blood to help it enter the ventricle. The blood is emptied into the ventricle via a non-return atrioventricular (A-V) valve. The right and left A-V valves differ in structure; the right (tricuspid) valve has three cusps, the left (mitral) has two. The A-V valve stops any back-flow from the ventricle to the atrium. The ventricle then supplies the force that pushes the blood through the pulmonary or systemic circulation. The semilunar valve stops any back-flow into the ventricle.

The right ventricle propels the blood through the lungs via the pulmonary artery and circulation. Blood oxygenated by the lungs flows into the left atrium. The left ventricle then propels the blood through the aorta and the arterial tree, through which the oxygenated blood is delivered to the tissues of the body. Blood then returns to the heart via a converging network of increasingly larger veins. The blood enters the right atrium, and the cycle repeats.
Figure 1.1: Schematic diagram of the heart as a pump. (Guyton, 1991, Fig. 9–1 p. 99.)

The arteries transport blood under high pressure to the body’s tissues. Therefore the arteries have strong vascular walls which can withstand the high pressure and rapidly flowing blood. The arterioles are the last branches of the arterial system. The strong muscular wall of the arterioles is capable of closing completely, if necessary, or dilating. This allows the arterioles to regulate the flow of blood to the capillaries in response to the requirement of the body’s tissues. The capillaries are the smallest blood vessels and are the sites at which the fluids, nutrients and other substances are exchanged between the blood and the interstitial fluid. For this purpose, the capillaries have thin walls which are selectively permeable to small molecular substances. The permeability varies for different capillaries. Blood is collected from the capillaries by the venules which then merge into larger veins. The veins are the conduits for the transport of blood back to the heart. Venules and veins have thin walls. However, the walls are muscular and can contract or expand to hold varying amounts of blood. Therefore, the veins can also act as reservoirs for blood, depending on the requirements of the body.

The various portions of the circulation experience different levels of pressure (Fig-
Figure 1.2: Changes in the pulse pressure contour as the pulse wave travels toward the smaller vessels. (Guyton, 1992, Fig. 12–3, p.118.)

The aorta receives blood directly from the heart and hence is under the highest pressure in the vasculature. The arterial pressure oscillates, due to the pulsatile nature of the heart. The highest pressure in the arteries occurs during systole (the period of ventricular contraction), and the lowest pressure occurs during diastole (the period of ventricular relaxation). As the blood flows through the circulation, the pressure falls and becomes less pulsatile. The pressure is approximately equal to that of atmospheric pressure by the time it reaches the right atrium. The pulmonary circulation is similar to the systemic circulation. However, the pressure levels in the pulmonary circulation are considerably less.

The distensibility of blood vessels plays an important role in the vascular system. For example, when the pressure in the arterioles is increased, they dilate. Therefore the arterioles’ resistance is decreased, resulting in an increase in blood flow that is greater than one would expect (Guyton, 1986). The distensibility of the arteries allows them to average out the pulsatile form of the pressure. Therefore when the blood reaches the respiring tissues the pressure gradient is approximately steady.
and hence the flow is continuous (Figure 1.2).

1.1.2 Blood.

Blood is made up of a suspension of blood cells and particles in plasma. The cells account for 46% of the blood (volume of concentration): 45% erythrocytes (red cells), and 1% leucocytes (white cells) and platelets. Red blood cells are biconcave flexible discs (~8 μm in diameter), the white cells are spherical (7 to 20 μm in diameter) and the platelets are rounded/oval (2 to 4 μm in diameter). Blood transports matter (oxygen, carbon dioxide, nutrients, etc) and heat (for thermoregulation) and acts as a buffer for the body's fluids.

Blood contains haemoglobin (a globular protein) which gives the blood its capacity for transporting oxygen around the body. Each molecule of haemoglobin can transport up to four molecules of oxygen from the lungs to respiring tissues, this is due to oxygen and haemoglobin being able to combine reversibly. Haemoglobin also acts as a buffer to keep blood at a constant pH level by preventing the accumulation of hydrogen ions in the blood.

The main function of the red blood cells is to carry haemoglobin. Any haemoglobin that is not carried in the red blood cells diffuses easily through the capillary walls. Red blood cells are very flexible 'bag' like sacs that are deformable to virtually any shape. This allows them to pass through the capillaries, whose diameters are comparable with that of the undeformed red blood cells. Red blood cells are produced in bone marrow and their formation is controlled by a hormone secreted in the kidney.

White blood cells, although larger than the red cells, have little effect on the mechanical properties of blood because they are present in such low concentration. They can be split into three main groups; lymphocytes, monocytes, and granulocytes. White cells play an important role in protecting the body against infection. Lymphocytes can form antibodies, while monocytes and granulocytes combat infections by phagocytosis (engulfing foreign materials).

Platelets are more numerous than the white cells and are involved in haemostasis (clotting) which involves interaction between the platelets and the vessel wall.
Blood is therefore a mixture of large cells in plasma and so is not a homogeneous fluid. However, in the large arteries (0.8 to 2.6 cm in diameter) that we are considering, the scale of the microstructure is much smaller than that of the vessel geometry. Therefore, we can regard blood as homogeneous.

1.1.3 Atherosclerosis and Haemodynamics.

There are many diseases of the blood and circulation. Here we give a brief outline of atherosclerosis, which is the disease that motivates this thesis. Diseases of the arteries are responsible for 50% of all deaths in the western world. Atherosclerosis is the most common of the arterial diseases. Fatty lesions (atheromatous plaques) develop inside the arterial wall (Figure 1.3). This can lead to total occlusion of the artery which can lead to death. Vessels most commonly affected are the abdominal aorta, femoral, carotid, and coronary arteries (see Caro et al., 1971 and Pedley, 1980).

The walls of arteries have three layers (Figure 1.3 a); the outer layer or 'adventitia' is composed of fibres, the middle layer or 'media' is composed of muscle and the inner lining or 'intima' consists of a single layer of endothelial cells (which is called the endothelial layer). The first visible (microscopic) signs of atherosclerosis are fatty streaks within the intima. Next, the fibrous plaques form; these are separated
Figure 1.4: Arterial thrombosis: a. plaque in intima; b. plaque ruptures and thrombus formed; c. blockage. Double arrow indicates direction of flow.

from the bloodstream by a thin fibrous cap (Figure 1.4 a). The plaques gradually build up and narrow the artery. Calcification of the plaques cause the artery to harden making it stiff and unyielding (and in extreme cases completely rigid). This can lead to rupturing of the wall, which cracks the fibrous cap covering the plaque (Figures 1.3 b and 1.4 b), resulting in haemorrhaging into the plaque, and thrombosis (clotting). The clot may then break off and migrate. These 'moving' clots are called embolisms and, as they are swept through the arterial tree, can block smaller vessels. Alternatively, the thrombus can grow until it blocks the lumen (the cavity through which blood flows) totally (Figure 1.4 c) and the muscle beyond dies.

The bio-chemical development of atherosclerosis is not yet fully understood. However, three main 'life-style' factors are associated with the development of arterial disease; these are unbalanced diet, smoking and high blood pressure (Julian and Marley, 1991). Also, the relationship between high blood cholesterol levels and

In the atheromatous lesions there is an accumulation of lipids. The major component of these lipids is cholesterol. How these lipids enter the lesions is uncertain. Damage to the endothelial layer, pinocytosis (cells actively engulfing the lipids) and entry via intercellular junctions have all been suggested as possible factors in lipid build up (Adams, 1973). The highest concentration of cholesterol seems to occur in the low density lipoproteins (LDL) and hence it has been suggested that LDL plays an important role in the development of atherosclerosis. However, high density lipoproteins are thought to protect against atherosclerosis, but the mechanism is not yet understood (Guyton, 1991). The consumption of saturated fats and (to a lesser extent) salt and sugar, may contribute to arterial disease by increasing LDL concentration. Vitamins C and E probably protect against the development of coronary disease (Julian and Marley, 1991). Smoking and high blood pressure can be factors in increasing the concentration of LDLs and also in producing local changes in vascular tissue, which can then be predisposed to cholesterol deposition.

Caro et al. (1971) proposed that regions of low wall shear stress were susceptible to the build up of early atheroma. This hypothesis was based on observations which showed that regions of athromatous lesions were coincident with regions experiencing relatively low wall shear. It should be noted that these observations are for early athromatous lesions; once the lesions significantly effect the shape of the vessel the flow and shear will be altered (see Cavalcanti, 1995). Experiments of steady flow in models of the carotid artery bifurcation (Zarins et al., 1983, Ku and Giddens, 1987, Ravensbergen et al., 1998) showed that regions of low wall shear stress were coincident with observed sites of athromatous plaques. In addition, the in vivo studies of White et al. (1996), using duplex scanning, and of Gnasso et al. (1997), using echo-Doppler examination, have also confirmed the low wall shear hypothesis, Ku et al. (1985) also showed that regions where the wall shear stress changes direction during the cardiac cycle are susceptible to atherosclerosis.

When parts of the arterial tree are diseased it is sometimes necessary to use use prostheses to ensure the continuing flow of blood. Researchers have modelled prosthetic arterial bifurcations (Rieu and Pelissier, 1991), aortic heart valves (Lim
et al., 1998) and vascular grafts (Loth et al., 1997). Rie and Pelissier (1991) studied two types of prosthetic bifurcation (Figure 1.5). They conclude that type A (Figure 1.5 a.) was the better configuration and that the length of the mother branch \( (l_m) \) should be as short as possible: type B (Figure 1.5 b.) was shown to increase vortex formation and cell damage. Lim et al. (1998) tested four types of prosthetic aortic heart valves. They conclude that the St. Vincent Meditech valve was the most efficient due to having the lowest pressure drop and Reynolds stresses. For vascular grafts, Loth et al. (1997) show that the diameter of the graft should not be greater than that of the host artery; this is because this causes low-inertia flow (compared with those of equal diameter) which is associated with relatively low wall shear stress. The results of Caro et al. (1998) show how sensitive flow is to slight changes in geometry. They demonstrate that the addition of torsion reduces low wall stress regions in bypass-grafts (see §1.2.3).

Figure 1.5: Sketch of prosthetic arterial bifurcations examined by Rieu and Pelissier (1991).
1.2 Previous theoretical work.

1.2.1 Uniform geometries.

Flow driven by a steady pressure gradient in a straight pipe, with circular cross-section, is unidirectional and has a parabolic velocity profile (Figure 1.6). Eustice (1910, 1911) showed, by injecting dye into fluid flowing through a coiled pipe, that curvature gives rise to a secondary velocity.

![Figure 1.6: Poiseuille flow.](image)

Dean (1927,1928) theoretically studied the effect of curvature on steady flow through a rigid pipe. Dean assumed that the pipe was loosely coiled so that he could ignore higher powers of the curvature parameter, $\delta$ (the ratio of the radius to the radius of curvature of the pipe). The flow was shown to depended on one dimensionless parameter; the product of the Reynolds number, $Re$ and the square-root of the curvature parameter; known as the Dean number, $D$. Dean solved the Navier-Stokes equations by expanding the velocities and pressure of the flow field in powers of $D$. The leading order term is Poiseuille flow (Figure 1.6) and the effect of curvature is to create a secondary velocity motion which carries the flow from the inside to the outside of the curve, along the centerline of the tube, and back round via the walls (Figure 1.7 a). White (1929) extended Eustice's results by examining oil, as well as water, flowing through a curved pipe. He showed that Dean's results were in agreement with the experimental results. Van Dyke (1978) used computational methods to extend the series solution of Dean to 24 terms, allowing larger values of the Dean number. He assumed that the curvature ratio was small ($\delta < 0.002$) and showed that the series expansion converged for $D < 585$. The series solution was extended by using a Euler transformation and Van Dyke concluded that this made the solution valid for all Dean numbers.

As $D$ is increased further it is necessary to calculate the flow using numerical tech-
niques. The work of McConalogue and Srivastava (1968), Greenspan (1973) and Collins and Dennis (1975) numerically extended Dean’s work to $D = 600$ (McConalogue and Srivastava) and $D = 5000$ (Greenspan, Collins and Dennis). All results are qualitatively similar, but quantitatively Greenspan’s are different. Collins and Dennis suggest that this discrepancy is due to Greenspan not altering the grid coarseness of the numerical method as $D$ was increased. Observations of the numerical solutions showed a trend toward a flow consisting of an inviscid core with a boundary layer near the wall. When Van Dyke presented his results, he noted that there were discrepancies between his analytical work and previous numerical results. In recent years, Dennis (1980), Dennis and Ng (1982) and Nandakuma and Masliyah (1982) have given more accurate numerical results. Dennis and Riley (1991) suggested that increasing the number of terms in Van Dyke’s series solution would produce agreement between the high Dean number analytic and numerical results. Dennis and Riley (1991) again provided evidence for the boundary layer / inviscid core structure, however the computation breaks down near the inside wall. The reason for this breakdown is not yet fully understood. Ito (1968) assumed that in the inviscid core the transverse velocity components were small compared to the axial velocity, whereas in the boundary they were of comparable size. This gave two sets of equations; one for the core and one for the boundary layer. The problem was then solved using the Pohlhausen momentum-integral method. However Ito, like Dennis and Riley, found that the solution broke down at the inside bend. Further work is necessary to explain this phenomenon and to give a full description of the flow at large Dean number.
For $D > 956$, Dennis and Ng found that Dean's 2-vortex solution bifurcates to a 4-vortex solution (Figure 1.7 c.), which has since been confirmed by Yanase et al. (1988, 1989) and Daskopoulos and Lenhoff (1989). Daskopoulos and Lenhoff found two more 4-vortex solutions, but only one was stable to symmetric disturbances. Yanase et al. (1988, 1989) also found that the two-vortex solution was stable and that the 4-vortex solution was unstable to asymmetric disturbances. The stagnation point, that exists in the 4-vortex solution case, is located on the center plane of the pipe (Nandakumo and Masliyah).

Dean's work and that reviewed above relies on the curvature being small $\delta \ll 1$. However, this is not always the case in the vascular system (§1.2.2), or in other applications (e.g. plumbing systems). In general, therefore, the flow in curved pipes depends on the Dean number, $D$, and the curvature parameter, $\delta$. Topukoglu (1967) extended Dean's work by using higher powers of the curvature parameter and showed that this method would be valid for different pipe cross-sections. As the curvature is increased Topukoglu showed that the peak axial velocity is pushed towards the outer wall. (A similar effect is seen if $D$ is increased and $\delta$ is held constant (Berger et al. and Figure 1.7 b).) Larrain and Bonilla (1970) used a similar method to that of Topukoglu and showed that it compared favourably with experimental results. These results have been extended numerically by Austin and Seader (1973) for $5 < \delta < 100$ and $1 < D < 100$.

Truesdell and Adler (1970) numerically studied the effect of torsion (out-of-plane curvature) on flow in a curved pipe. For $D < 200$ they showed that the main effect of torsion was to distort the symmetry of the Dean velocity profiles. The first analytic solutions for flow in a helical pipe were by Germano (1982, 1988) and Wang (1981). Germano and Wang solved the Navier-Stokes equations by perturbing the pressure and velocity field about Poiseuille flow, however they had contradictory conclusions about the effect of torsion. Germano used an orthogonal coordinate system (see §2.2.2) which showed torsion to have an effect at second-order. Wang, however, using a non-orthogonal coordinate (§2.2.2), system argued that torsion was a first-order effect. The main issue was that Wang's solutions showed that the 2-vortex Dean solution could, at certain parameter values, break down into a single vortex. Kao (1987) tried to explain this difference. Kao concluded that Germano's results were indeed correct and went on to extend them numerically. However, Kao
did not examine his results from the perspective of the non-orthogonal coordinate system; and hence did not satisfactorily address the problem. Tuttle (1990) finally resolved the contradiction by examining both coordinate systems. Tuttle concluded that the results of Germano and Wang are the same but appear different due to the different frames of reference. This has since been confirmed by Liu and Masliyah (1993). The stability of perturbation solutions in helical pipe flow, has been discussed by Yamamoto et al. (1998).

For helical pipe flow Yanase et al. (1994 b) used numerical techniques to solve the full Navier-Stokes equations for a wider range of parameters. They noted that the secondary velocity does tend to a single-vortex when $\beta_0 = 1.4$; $\beta_0$ is the ratio of torsion to the square-root of the curvature (this is in agreement with Liu and Masliyah). Yanase et al. could not find a stable 4-vortex solution, suggesting this is due to the asymmetric nature of the flow in a helical pipe. Zabielski and Mestel (1998 a) have numerically studied flow in a helical pipe, by imposing helical symmetry on the velocity field. Using this method allows them to vary the curvature and torsion over a wide range of parameter values, e.g. Reynolds number = $50^3$. In concurrence with Yanase et al., Zabielski and Mestel do not find multiple solutions.

Various authors have studied time-dependent laminar flow in a straight pipe (Chatwin, 1975, Watson, 1983). However, we know that in many applications, including those relating to physiology, the pipes are curved or branched. The first theoretical study of unsteady flow in a curved pipe was carried out by Lyne (1970). Lyne examined flow driven by a purely oscillatory pressure gradient on the flow in a curved pipe. He showed the effects of two important parameters; $R_s$, the secondary streaming Reynolds number based on the secondary velocity scale, and $\beta$, which is a fre-
quency parameter \((\beta^2 = 2/\alpha^2, \alpha = \text{Womersley parameter})\). When \(R_s \gg O(1)\) the secondary flow consists of two vortices and a boundary layer. The boundary layer extends from the pipe walls along the diameter of the pipe (see Figure 1.8 a). For sufficiently small values of \(\beta\) (large frequency), Lyne finds that the secondary velocity flows in the opposite direction to that predicted by steady flow analysis. This is in agreement with Lyne's own experimental observations and, in the \(R_s \ll 1\) limit, with the experimental investigation of Bertelsen (1975). Smith (1975) extended Lyne's analysis to incorporate a pressure gradient that had non-zero mean, i.e. \(p_s = -G - \cos t\). This model introduces a third parameter, the Dean number, \(D\). Smith examined the interaction between the steady and oscillatory parts of the flow. This interaction of the two secondary flows was found to be complicated; due to the directions of the secondary velocity in steady- and unsteady-flow. However in the quasi-steady \((\alpha \ll 1)\) and large frequency limits Smith's results are in agreement with Dean and Lyne respectively. Zalosh and Nelson (1973) analytically examined the limits of large- and small- \(\alpha\), and numerically computed solutions for arbitrary values of \(\alpha\). As noted by Mullin and Greated (1980 b) there are errors in Zalosh and Nelson's paper. However, in the analytic results, for the quasi-steady case, there is agreement between Zalosh & Nelson and Mullin & Greated. Mullin and Greated's numerical method has been favourably compared with their own experimental results (Mullin and Greated 1980 a). The method reproduces the 4-vortex structure predicted by Lyne (see Figure 1.8 b). Though the solution is valid for arbitrary \(\alpha\), slow convergence restricts the solutions to \(\alpha \leq 11\). Lin and Tarbel (1980) extended Smith's analysis numerically \((D \leq 100)\) and experimentally \((D \sim 300)\). The limiting case, \(\alpha \ll 1\), compares well with Smith. For intermediate values of \(\alpha\) they described a 'new' phenomenon - resonance between the axial and secondary flow. Hamakiotes and Berger (1988, 1990) studied the effect of the Dean number on the fluid flow and the wall shear stress (see §1.2.3). In their paper Costas and Berger (1990) observed a period tripling bifurcation in the secondary flow at \(\alpha = 15\); creating a much more complicated vortex structure. Experimentally this bifurcation has not been found (Swanson et al. 1993). The experimental set-up of Swanson et al. involved a pipe curved around 180° in the form of a helix with small pitch. The incorporation of pitch however may destroy the period tripling bifurcation. Swanson et al. suggested that the narrow window in parameter space at which the bifurcation occurs may
mean that closer examination of the $\alpha \sim 15$ is required. The other possibility is that the numerical solution may be unstable. Physiological pulsatile flow in curved pipes has been studied numerically by Chandran et al. (1979), Chandran and Yearwood (1981) and Perktold et al. (1991).

Zabielski and Mestel (1998) analytically and numerically studied the effect of torsion on unsteady-flow in curved pipes. As with steady-flow, helicity was found to break the 2- and 4-vortex symmetry of flow in curved pipes. The WSS distribution is also discussed.

All the above papers assume that the flow is fully developed, i.e. the fluid velocities are independent of the distance along the pipe. Entry flow into a curved pipe has been theoretically studied by Singh (1974), Yao and Berger (1975), Smith (1976) and Talbot and Wong (1982) for a steady pressure gradient, and by Talbot and Gong (1983) for a pulsatile pressure gradient. Rowe (1970) studied the problem of steady entry flow experimentally and numerically. In steady flow the wall shear increases at the inner wall near the entrance, then, as the flow becomes fully developed moves to the outer wall. Rowe (1970) showed the presence of two pairs of vortices at the entry region which dies out to one pair as the flow becomes fully developed. This could explain the change in position of the WSS. For pulsatile entry flow Talbot and Gong (1983) performed two experiments; the first was deemed to be quasi-steady and the second much more complex. The quasi-steady experiment gave results that had the same characteristics as low Dean number steady entry flow. The second experiment showed a separation of the axial flow that propagated towards the entrance of the pipe during deceleration. Talbot and Gong’s results agree well with the previous experimental work of Chandran et al. (1979) and Chandran and Yearwood (1981) which were conducted with physiological pulsatile pressure gradients.

For reviews of flows in curved pipe flows, see Berger, Talbot and Yao (1983) and Pedley (1980) and for helical flows see Tuttle (1990).

Sharp (1987) first studied the effects of oscillatory flow and curvature on solute dispersion. Pedley and Kamm (1988) discussed how secondary motion affects axial transport. This was done in a model where the axial and secondary velocities were prescribed. Sharp et al.(1991) used numerical techniques to extend the work
of Pedley and Kamm. Eckmann and Grotberg (1988) analytically studied how oscillatory flow in a curved pipe affects the concentration field and mass transport. This was done by first solving the Navier–Stokes equations, and then solving the convection–diffusion equation. Hydon (1994) used Kolmogorov-Arnol’d-Moser (KAM) theory to show the formation of “islands” in the transverse velocity plane. In certain islands there is rapid axial transport of particles; due to resonance between axial and transverse flows. The interaction between chaotic advection and resonant advection and between chaotic advection and diffusion has been studied by Hydon (1995) and Jones (1995) respectively.

1.2.2 Complex geometries.

It is now well known that the geometry in the arterial system is non-uniform (Caro et al., 1996). The arteries can curve (strongly) and bifurcate in a three-dimensional way, for example the aortic arch and the bifurcation of the aorta (Caro et al., 1996) have non-planar geometry. The arteries have distensible walls, non-uniform cross-sections and move within the body. Therefore there are many types of complex geometries that have been examined. Here we give the reader an overview of the literature available.

Cuming (1952) extended Dean’s work to include pipes with different rigid cross-sections. Non-Newtonian fluid flows in curved pipes of circular and elliptic cross-section have been studied by Thomas and Walters (1963, 1965). In their paper, Chow and Soda (1972), propose a method for analysing flow in pipes with general axisymmetric cross-section. The paper has a major error in the derivation of the second-order equations (eqn.15). This error causes the analytic results to be wrong, however the numerical results for sinusoidal wall variation appear to be accurate. Manton (1971), Sobey (1976) and Wild and Pedley (1977) looked at Newtonian flows in straight pipes with varying elliptic cross-section. This work was then extended, by Srivastava (1980), Topakoglu and Ebadian (1985, 1987), Vassilief (1994) and Dong and Ebadian (1991), to examine the flow in curved pipes of elliptic cross-section. The 2-vortex structure of Dean is still present, but in a slightly distorted form which depends on the orientation of the major/minor axis. Murata et al. (1976) studied flow in three types of pipe where the centre-line varies with distance along the pipe. They show that for large Reynolds number
the position of peak axial velocity shifts towards the outer wall, whereas for very small Reynolds number it shifts towards the inner wall; this is due to the viscous terms dominating the inertial terms. Schilt et al. (1996) experimentally studied the effect of time-varying curvature on steady-flow in a rigid pipe, while Lynch et al. (1996) and Waters (1996) have analytically examined the effect on both steady and unsteady flows. Experimentalists have also examined models of bifurcations (Delouche, 1985), branches (Karino et al., 1979, Karino and Goldsmith, 1985), non-newtonian flows in distensible arteries (Liepsch and Moravec, 1984) and the bending and twisting of a coronary bifurcation (Pao et al., 1992).

For a further review of flow in complex geometries, see Pedley (1980, 1995) and references therein.

1.2.3 Wall Shear Stress.

In a loosely coiled pipe, with a steady pressure gradient, at low Reynolds number, it is relatively easy to understand the form of the wall shear stress (WSS). The curvature creates a secondary flow and the faster moving fluid is pushed towards the outside bend of the pipe. This results in a steepening of the velocity gradient at the outer wall, leading to an increase in the WSS on the outside wall and a decrease on the inside wall. As the curvature increases, the peak axial velocity is pushed nearer the outer wall (Figure 1.7 b) and this increases (decreases) the shear at the outer (inner) wall. For similar parameter values in helical pipes the wall shear is skewed in the direction of the torsion. At very small Reynolds number ($Re < 1$) the viscous driving terms dominate the flow and the peak axial velocity shifts towards the inner bend (Murata et al., 1976) moving the maximum shear to the inner wall. A stable 4-vortex solution is found at high Reynolds number (Dennis and Ng 1982). The existence of two stable solutions means that the flow structure is sensitive to the initial and boundary conditions and any changes in geometry. This implies that small changes in any of these parameters are likely to result in substantial changes in the WSS.

If the dimensionless frequency, $\alpha$, is small (i.e. the flow is quasi-steady) the unsteady flow does not differ significantly from steady flow in curved pipes. At large Dean number, $D$, and large frequency, Lyne (1971) showed the occurrence of flow
with a viscous boundary layer (called a Stokes layer) and an inviscid core, the WSS for which has been calculated by Pedley (1980).

The WSS in unsteady flow in helical pipes has not been widely studied. Zabielski and Mestel (1998) show that, in a helically symmetric flow, the downstream component of velocity is not significantly affected by torsion. However, the secondary flow is skewed, first in the direction of the torsion and then in the opposite direction (depending on when the flow is sampled during the period). This change in position of the secondary velocity alters the azimuthal WSS and can affect the overall WSS if it becomes large enough. Zabielski and Mestel conclude that helical curvature can increase (from zero) the WSS at the inner bend and suggest that this may improve the cleansing at the inner wall. This in itself would suggest that helical geometry may play a beneficial role in combating the build up of atherosclerotic plaques. The experimental results of Caro (1998) show that flow is very sensitive to changes in geometry. If it is not possible to clear an arterial blockage,

![Figure 1.9: Sketch of Caro et al.'s (1997) results: Bypass grafts using: a curved pipe (a) and a curved and twisted pipe (b)](image-url)

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an arterial bypass graft is used to ensure the continuing flow of blood. Caro et al. considered the situation where an occlusion is bypassed by a curved pipe. The parent pipe is rigid, straight and the flow is driven by a steady pressure gradient. A secondary velocity is formed as the fluid passes through the curved pipe. Caro et al. observed was that a stagnation region was present at the down-stream end of the occlusion (Figure 1.9 a). This indicated a region of low wall shear, which could lead to the development of atheromatous lesions. However, when the graft was curved and twisted (i.e. has non-zero torsion) Caro et al. observed that this stagnation region was greatly reduced (Figure 1.9 b). A paper by Sherwin et al. (1997) uses a computational model to discuss this effect.

Figure 1.10: Sketch of a widening channel.

Entry flows produce more complicated flow structures which affects the WSS (Talbot and Wong 1982). Consider steady, unidirectional, high Reynolds number flow entering a wider channel (Figure 1.10). The increase in width will result in a decrease in velocity. This causes flow separation and a recirculation region; therefore the velocities and WSS, will be low in these regions. Another example of entry flows is flow into a bifurcation. Figure 1.11 shows the velocity profiles that result when flow in a straight pipe encounters a symmetric bifurcation. The faster-moving fluid is forced to the inner wall of the bifurcation, causing the formation of a secondary flow structure. The boundary layers at the inner wall will cause an increase in WSS, due to the steep velocity gradients that result from the split in the velocity profile.

1.3 Aims.

This thesis is concerned with modelling the flow in twisted pipes. The motivation for this is the flow of blood in large arteries. We wish to understand the distribution of wall shear stress and the structure of the velocity field in regions susceptible
to atherosclerosis. The arteries have non-uniform, time-dependent geometry. The work herein concentrates on how non-uniform time-independent geometry affects the wall shear stress and velocity field.

The geometry consists of a rigid, circular boundary fitted around a prescribed centreline in 3-space. We assume the blood is homogeneous; and therefore we can neglect the effects of cell interaction. The flow of a Newtonian fluid in a pipe with non-uniform curvature and torsion has not previously been unstudied.

We consider flows driven by (i) a steady pressure gradient and (ii) an oscillatory pressure gradient with zero mean.

In chapter 2, we review the work of Germano (1982, 1989) by deriving the full non-linear Navier–Stokes equations for an incompressible viscous fluid. Terms that arise due to helicity and non-uniform geometry are highlighted. The non-dimensionalizations used in later chapters are discussed.

In chapter 3, the equations are solved for flow with a steady pressure gradient. We assume that curvature and torsion depend exponentially on the arc-length, $s$. The curvature, $\epsilon(s)$, and torsion, $\tau(s)$, are taken to be small and vary by small amounts ($\epsilon^{\eta s}$, $\epsilon^{(\beta-\eta)s}$ respectively); this gives rise to four previously unstudied pipe geometries. The equations are then solved in power series expansions involving $\epsilon$, $\tau$, $\eta$ and $\beta$. At certain parameter values the expansion is not valid and a numerical approach is taken. The distribution of WSS, and the structure of the velocity field are found and discussed.
In chapter 4, we solve the equations for flow with an oscillatory axial pressure gradient, of frequency $\alpha$. An analytic solution for this problem cannot be found. Therefore a numerical method is used that is valid for all $\alpha$. This is an extension of the work of Lyne and Mullin & Greated. A new phenomenon is observed: flow reversal at low frequencies.

In chapter 5, we again discuss the flow driven by an oscillatory pressure gradient. An analytic solution is found for the case when the geometry is uniform. The flow is assumed to be quasi-steady, and we extend Hydon’s analysis to incorporate torsion. We discuss the affect of torsion on the velocity field and wall shear stress, and how the additional terms affect the “islands” described by Hydon (in oscillatory curved pipe flow).

In chapter 6, we review our results and examine the physiological applications. General conclusions are made and future work is proposed.
Chapter 2

The coordinate system and governing equations.

2.1 Introduction.

We aim to model flows in pipes with non-uniform geometry. This chapter describes the coordinate system used and the derivation of the governing equations. The boundary conditions and non-dimensionalizations are discussed.

2.2 The Coordinate system.

The pipe geometry is constructed by fitting a rigid boundary around a line in 3-space. Here we introduce the Serret-Frenet equations which are used to define a curve in 3-space. Germano (1982) and Wang (1981) used this method to construct their coordinate systems. We discuss the construction of both coordinate systems and the reasons we follow and extend Germano’s work.

2.2.1 The Serret-Frenet equations.

First, let us consider a general curve in 3-space; the locus defined by the vector:

\[ X(t) = (x_1(t), x_2(t), x_3(t)) \]
The length of the arc of the curve, extending from \( t_0 \), is:

\[
s(t) = \int_{t_0}^{t} |\mathbf{X}(\sigma)| \, d\sigma = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, d\sigma
\]

The curve can now be parametrized with respect to the arc-length, \( s \), \( \mathbf{R} = \mathbf{X}(\phi(s)) \).

Consider two neighbouring points in parameter space on the curve, \( P = P(u_0) \) and \( Q = Q(u) \). Hence,

\[
\mathbf{R}(u) - \mathbf{R}(u_0) = (u - u_0)\mathbf{\dot{R}}(u_0) + o(u - u_0),
\]

and, as \( u \to u_0 \),

\[
\lim_{u \to u_0} \frac{\mathbf{R}(u) - \mathbf{R}(u_0)}{|\mathbf{R}(u) - \mathbf{R}(u_0)|} = \frac{\mathbf{\dot{R}}(u_0)}{|\mathbf{\dot{R}}(u_0)|},
\]

which is a unit vector at \( P \), called the tangent vector, \( \mathbf{T} \):

\[
\mathbf{T} = \frac{\mathbf{\dot{R}}}{\mathbf{\dot{s}}} = \frac{d\mathbf{R}}{ds}.
\]

(2.2.1)

The curvature (\( \kappa \)) is defined as the rate at which the tangent changes direction as a point \( P \) moves along a curve. Thus, \( |\kappa| = |\mathbf{T}| \). We note that \( \mathbf{T} = \mathbf{\dot{R}} \) is normal to \( \mathbf{T} \) and together they define the osculating plane and the convention is made that

\[
\mathbf{T} = \kappa \mathbf{N},
\]

(2.2.2)

where \( \mathbf{N} \) is the normal vector. The third vector in this orthogonal triad is found by,

\[
\mathbf{B} = \mathbf{T} \times \mathbf{N},
\]

(2.2.3)

where \( \mathbf{B} \) is called the binormal vector.

The rate at which the osculating plane turns about the tangent is called the torsion (\( \tau \)) of the curve. Consider \( |\mathbf{B}^2| = 1 \); differentiating gives \( \mathbf{B} \mathbf{\dot{B}} = 0 \) and hence \( \mathbf{\dot{B}} \) lies in the osculating plane. Now, we know \( \mathbf{B} \mathbf{\dot{T}} + \mathbf{\dot{B}} \mathbf{T} = 0 \) (as \( \mathbf{B} \mathbf{T} = 0 \)). Therefore, as \( \mathbf{B} \mathbf{N} = 0 \), \( \mathbf{\dot{B}} \) is orthogonal to \( \mathbf{T} \). So, as \( \mathbf{\dot{B}} \) is in the osculating plane it must be parallel to \( \mathbf{N} \). Therefore

\[
\mathbf{\dot{B}} = -\tau \mathbf{N},
\]

(2.2.4)

(The negative sign indicates that, as \( s \) increases, the curve takes the form of a right-handed screw in the direction of \( \mathbf{T} \).) Using the identity \( \mathbf{N} = \mathbf{B} \times \mathbf{T} \) gives

\[
\mathbf{\dot{N}} = \tau \mathbf{B} - \kappa \mathbf{T}
\]

(2.2.5)

Equations (2.2.2), (2.2.4) and (2.2.5) are known as the Serret-Frenet equations.
2.2.2 **Germano’s and Wang’s coordinate systems.**

Germano (1980) and Wang (1981) studied flow in uniform helices. Though the coordinate systems they use are different the overall idea is similar: a helical centre-line is described in 3-space and then a tube with constant radius is fitted around it (see Figure 2.1). First we consider Wang’s system (Figure 2.1 (i)). We note that any point in the cross-section can be described as:

\[ x = R + \hat{r} \cos \psi N + \hat{r} \sin \psi B, \]

and the metric of this coordinate system can be found as follows:

\[ x + dx = R(\hat{s} + d\hat{s}) + (\hat{r} + d\hat{r}) \cos(\psi + d\psi)N(\hat{s} + d\hat{s}) + (\hat{r} + d\hat{r}) \sin(\psi + d\psi)B(\hat{s} + d\hat{s}), \]

which can be simplified to

\[ x + dx = x + (\hat{R} + \hat{r} \cos \psi N + \hat{r} \sin \psi B)d\hat{s} + (\cos \psi N + \sin \psi B)d\hat{r} \]
\[ + \quad (\hat{r} \cos \psi B - \hat{r} \sin \psi N)d\psi, \]

giving the metric

\[ dx^2 = \left(1 - \kappa \hat{r} \cos \psi \right)^2 \left(\hat{s} \right)^2 + \left(\hat{R} \right)^2 + \left(\hat{r} \right)^2 \left(\hat{d} \psi \right)^2 + 2\hat{r} \hat{r} d\hat{s} d\hat{\psi}. \quad (2.2.6) \]

Wang’s metric (2.2.6) shows that his coordinate system is not orthogonal (underlined term).
Now we construct the metric for Germano’s coordinate system (Figure 2.1 (ii)). We note that Wang’s coordinates can be written in terms of Germano’s as follows:

\[ \psi = \theta + \phi(\hat{s}) + \phi_0, \hat{r} = \hat{r}, \hat{s} = \hat{s}, \]

where \( \phi(\hat{s}) = -\int_{s}^{s'} \tau(s')ds' \), and \( \phi_0 = \frac{\pi}{2} \); this angle is used purely as a tool to make the coordinate system orthogonal (see below). Germano includes \( N^* \) and \( B^* \) to emphasise the role of \( \phi(\hat{s}) + \phi_0 \); while \( N \) and \( B \) rotate when the centreline has torsion, \( N^* \) and \( B^* \) do not. However, \( N^* \) and \( B^* \) play no role in the derivation of the equations of motion.

We can now use the following formula (Amari 1990) to find the metric matrix \( \tilde{G} \) for Germano’s system:

\[ \tilde{G} = BGB^T, \quad (2.2.7) \]

where \( G \) is the metric matrix in Wang’s coordinate system, and \( B \) is defined as follows:

\[ B_{ij} = \frac{\partial \Theta_j}{\partial \Xi_i}, \]

where \( \Theta = [\hat{s} \; \hat{r} \; \psi] \) and \( \Xi = [\hat{s} \; \hat{r} \; \theta + \phi + \phi_0] \). Therefore we find that

\[
B = \begin{pmatrix}
1 & 0 & -\tau \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

hence

\[
\tilde{G} = \begin{pmatrix}
(1 - \kappa \hat{s} \cos(\theta + \phi + \phi_0))^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \hat{s}^2
\end{pmatrix}.
\quad (2.2.8)
\]

From (2.2.8) we observe that there are no off-diagonal elements. Therefore Germano’s coordinate system is orthogonal. To avoid singularities in the coordinate system we must take \( \alpha < \frac{1}{\kappa} \), this ensures that the curvature is not so tight as to make the pipe intersect with itself.
2.3 Governing equations and boundary conditions.

2.3.1 Governing equations.

We can now construct the Navier-Stokes equations in the either Germano's or Wang’s coordinate system. We have decided to use the orthogonal coordinate system (Germano’s) because the results are easier to visualise and interpret (Tuttle, 1990).

The N-S equations are the continuity equation,

\[ \nabla \cdot \mathbf{u} = 0, \quad (2.3.9) \]

and the momentum equations,

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla \mathbf{p} + \nu \nabla^2 \mathbf{u}, \quad (2.3.10) \]

where \( \mathbf{u} \), \( \mathbf{p} \) and \( t \) are the dimensional velocity vector, pressure and time respectively, \( \rho \) is the density and \( \nu \) is the kinematic viscosity. Therefore, using the fact that the system is orthogonal (see Batchelor, 1967), we know that the divergence is given by

\[ \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 v_1)}{\partial \xi_1} + \frac{\partial (h_1 h_3 v_2)}{\partial \xi_2} + \frac{\partial (h_1 h_2 v_3)}{\partial \xi_3} \right], \quad (2.3.11) \]

the components of the gradient are

\[ \left( \begin{array}{ccc} \frac{1}{h_1} & 1 & \frac{1}{h_3} \\ \frac{1}{h_2} & \frac{1}{h_2} & \frac{1}{h_3} \end{array} \right) \]

and the components of the curl (\( \omega = \nabla \times \mathbf{u} \)) are

\[ \omega_1 = \frac{1}{h_2 h_3} \left( \frac{\partial}{\partial \xi_2} (h_3 v_3) - \frac{\partial}{\partial \xi_3} (h_2 v_2) \right), \quad \omega_2 = \frac{1}{h_1 h_3} \left( \frac{\partial}{\partial \xi_3} (h_1 v_1) - \frac{\partial}{\partial \xi_1} (h_3 v_3) \right), \]

\[ \omega_3 = \frac{1}{h_1 h_2} \left( \frac{\partial}{\partial \xi_1} (h_2 v_2) - \frac{\partial}{\partial \xi_2} (h_1 v_1) \right). \quad (2.3.13) \]

In the above equations, \( \xi = (\xi_1, \xi_2, \xi_3) \) are the orthogonal coordinates, \( \mathbf{h} = (h_1, h_2, h_3) \) are the scale factors and \( \mathbf{v} = (v_1, v_2, v_3) \) are the components of velocity. In Germano's coordinate system we have

\[ \xi = (\mathbf{s}, \mathbf{r}, \theta), \]

\[ \mathbf{h} = (1 + \kappa \mathbf{r} \sin(\theta + \phi), 1, \mathbf{r}), \]

\[ \mathbf{v} = (\tilde{u}, \tilde{v}, \tilde{w}), \quad (2.3.14) \]
where $\hat{s}, \hat{r}, \theta$ are the dimensional arc-length, radius and cross-sectional angle respectively, $\phi$ is the ‘unwinding’ angle (see Figure 2.1), $\kappa = \kappa(\hat{s})$ is the curvature and $(\hat{u}, \hat{v}, \hat{w})$ are the axial, radial and azimuthal velocities respectively. We can now obtain the following dimensional expressions for: the continuity equation

$$
\omega \frac{\partial \hat{u}}{\partial \hat{s}} + \frac{\partial \hat{v}}{\partial \hat{r}} + \frac{1}{\hat{r}} \frac{\partial \hat{w}}{\partial \theta} + \frac{\hat{v}}{\hat{r}} + \kappa \omega [\hat{v} \sin(\theta + \phi) + \hat{w} \cos(\theta + \phi)] = 0; \quad (2.3.15)
$$

and the momentum equations

$$
\frac{\partial \hat{u}}{\partial t} + \nabla \hat{u} + \frac{\hat{u}^2}{\hat{r}} - \kappa \hat{u}^2 \sin(\theta + \phi) = -\frac{1}{\hat{r}} \frac{\partial \hat{\rho}}{\partial \hat{r}},
$$

$$
\frac{\partial \hat{v}}{\partial t} + \nabla \hat{v} - \frac{\hat{v}^2}{\hat{r}} - \kappa \hat{u}^2 \sin(\theta + \phi) = -\frac{1}{\hat{r}} \frac{\partial \hat{\rho}}{\partial \hat{r}},
$$

$$
\frac{\partial \hat{w}}{\partial t} + \nabla \hat{w} - \frac{\hat{w}^2}{\hat{r}} + \kappa \hat{u}^2 \cos(\theta + \phi) = -\frac{1}{\hat{r}} \frac{\partial \hat{\rho}}{\partial \hat{r}},
$$

where

$$
\omega = \frac{1}{1 + \kappa \hat{r} \sin(\theta + \phi)}, \quad \tilde{D} = \omega \frac{\hat{u}}{\partial \hat{s}} + \frac{\partial \hat{\rho}}{\partial \hat{r}} + \frac{\hat{v}}{\hat{r}} \frac{\partial \phi}{\partial \theta}.
$$

The equations above ((2.3.15)-(2.3.18)) are the unsteady version of the Germano equations.
We wish to examine the effect of non-uniform geometry; assuming \( \kappa \) and \( \tau \) depend on \( s \), we can define pipes of non-uniform curvature and torsion. The non-underlined terms in (2.3.15)-(2.3.18) are the equations for unsteady-flow in a uniformly curved pipe \( (\kappa = \text{constant}) \). The \( s \)-derivative terms arise from the non-fully-developed nature of the flow and the additional angle \( \phi \) is due to the addition of torsion. We note that these new terms appear in the continuity equation and in both the inertial \( (\vec{D}) \) and viscous terms of the momentum equations. For constant curvature and zero torsion the equations reduce to the dimensional form of flow in a torus. When \( \tau \) and \( \kappa \) are both assumed to be constant the pipe geometry is that of a uniform helix. Therefore any steady-flow calculations can be compared with Dean (1927, 1928) and Germano, while unsteady-flow can be compared with Mullin and Greated (1980) and Hydon (1994).

### 2.3.2 Boundary conditions.

As we are considering a viscous fluid, the velocity at the pipe wall must be zero, i.e. \( \hat{u} = \hat{v} = \hat{w} = 0 \) at \( \hat{r} = a \), where \( \hat{r} = a \) is the equation of the pipe wall. This is known as the no-slip condition. We also assume that the fluid velocity and pressure gradient is finite everywhere in the pipe. This allows the derivation of finite conditions at \( \hat{r} = 0 \) for \( (\hat{u}, \hat{v}, \hat{w}) \), which are shown in subsequent chapters and not reproduced here.

### 2.3.3 Non-dimensionalization.

For flow with a steady pressure gradient we follow Germano and non-dimensionalise as follows:

\[
\begin{align*}
(\hat{u}, \hat{v}, \hat{w}) & \rightarrow U_0(u, v, w), \\
(\hat{s}, \hat{r}, \theta) & \rightarrow (as, ar, \theta), \\
\hat{p} & \rightarrow \hat{\rho} U_0^2 p, \\
\kappa & \rightarrow a^{-1} \epsilon,
\end{align*}
\]

where \( U_0 \) is a suitable velocity scale and \( a \) is the pipe radius.

We then make the substitution \( \xi = \theta + \phi \) and the associated change of variables:

\[
(\hat{r}, \hat{s}, \theta) \rightarrow (\hat{r}, \hat{s}, \xi),
\]
so that

\[
\begin{align*}
\frac{\partial}{\partial r} & \to \frac{\partial}{\partial r}, \\
\frac{\partial}{\partial s} & \to \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi}, \\
\frac{\partial}{\partial \xi} & \to \frac{\partial}{\partial \xi},
\end{align*}
\]  

(2.3.21)

where \( \lambda \) is the ratio of torsion to curvature.

The non-dimensionalization (2.3.19) and the transformation (2.3.21) alters the governing equations as follows:

the continuity equation

\[
\frac{\omega(u_s - \epsilon \lambda u_\xi)}{u} + v_r + \frac{w_\xi}{r} + \frac{v}{r} + \epsilon \omega [v \sin \xi + w \cos \xi] = 0,
\]  

(2.3.22)

and the momentum equations

\[
\begin{align*}
\omega w_s & + \nu u_r + \left( \frac{w}{r} - \epsilon \lambda \omega \right) u_\xi + \epsilon \omega u \left[ v \sin \xi + w \cos \xi \right] = -\omega p_s + \epsilon \omega \omega_\xi \\
& + \frac{1}{Re} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left( u_r + \epsilon \omega u \sin \xi - \omega v_s + \epsilon \lambda \omega_\xi \right) \\
& + \frac{1}{r} \frac{\partial}{\partial \xi} \left( u_\xi - \omega w_s + \epsilon \omega u \cos \xi + \epsilon \lambda \omega_\xi \right) \right], \\
& (2.3.23)
\end{align*}
\]

\[
\begin{align*}
\omega w_\xi & + \nu v_r + \left( \frac{w}{r} - \epsilon \lambda \omega \right) v_\xi - \frac{w^2}{r} - \epsilon \omega u^2 \sin \xi = -p_r \\
& + \frac{1}{Re} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \xi} \right) \left( -w_r - \frac{w}{r} + \frac{v_\xi}{r} \right) \\
& + \omega \left( \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi} \right) \left( \omega v_s - u_r - \epsilon \omega u \sin \xi - \epsilon \lambda \omega v_s \right) \right], \\
& (2.3.24)
\end{align*}
\]

\[
\begin{align*}
\omega w_s & + \nu w_r + \left( \frac{w}{r} - \epsilon \lambda \omega \right) w_\xi + \frac{vw}{r} - \epsilon \omega u^2 \cos \xi = -\frac{p_\xi}{r} \\
& + \frac{1}{Re} \left[ \left( \frac{\partial}{\partial r} + \epsilon \omega \sin \xi \right) \left( w_r + \frac{w}{r} - \frac{v_\xi}{r} \right) \\
& - \omega \left( \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi} \right) \left( -\omega w_s + \frac{u_\xi}{r} + \epsilon \omega u \cos \xi + \epsilon \lambda \omega v_s \right) \right], \\
& (2.3.25)
\end{align*}
\]

where

\[
\omega = \frac{1}{1 + \epsilon r \sin \xi} = 1 - \epsilon r \sin \xi + \epsilon^2 r^2 \sin^2 \xi + O(\epsilon^3).
\]
We find that there are three important parameters: the Reynolds number $Re = \frac{\nu a}{\mu}$, $\epsilon(s)$ (the curvature) and $\lambda(s)$. The Reynolds number shows the balance between the inertial and viscous terms; when $Re = O(1)$ this allows the study of both inertial and viscous effects and their relative dominance. Dean (1927, 1928) showed that the convective inertia terms are responsible for the formation of the two-vortex secondary flow pattern. (Due to the steady pressure gradient, the time derivatives are ignored.)

For unsteady-flow, we have followed the work of Hydon (1994) and non-dimensionalized as follows:

$$
\begin{align*}
(\hat{u}, \hat{v}, \hat{w}) & \rightarrow (\frac{a}{\gamma} u, \nu a, \frac{a}{\alpha} w), \\
(\hat{s}, \hat{r}) & \rightarrow (\frac{a}{\gamma} s, \alpha r), \\
\hat{p} & \rightarrow A_p + \beta K \hat{s} \cos(\Omega \hat{t}) + \frac{\beta \nu^2}{\alpha^2} \hat{p}, \\
\hat{\kappa} & \rightarrow a^{-1} \epsilon, \\
\end{align*}
$$

(2.3.26)

where $K$ is the amplitude and $\Omega$ is the frequency of the pressure pulse, with $(\nu, a)$ as before. ($A_p$ can be thought of as the atmospheric pressure, which we assume to be constant.)

The non-dimensionalization (2.3.26) and the transformation (2.3.21) give the following equations for unsteady flow:

the continuity equation

$$
\omega (u_s - \epsilon \lambda u_\xi) + v_r + \frac{1}{r} w_\xi + \frac{1}{r} v + \epsilon \omega (v \sin \xi + w \cos \xi) = 0; \quad (2.3.27)
$$

\begin{align*}
\frac{\alpha^2}{2\pi} u_t & + \frac{\omega u(u_s - \epsilon \lambda u_\xi) + \nu u_r + \frac{1}{r} \nu u_\xi + \epsilon \omega (\nu \sin \xi + w \cos \xi)}{G_0} + \frac{\partial}{\partial \xi} \left( \frac{1}{r} u_\xi + \epsilon \omega u \cos \xi \right) \\
& \quad - \frac{1}{G_0} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\omega (u_s - \epsilon \lambda u_\xi)) + \frac{1}{r} \frac{\partial}{\partial \xi} (\omega (w_s - \epsilon \lambda w_\xi)), \quad (2.3.28)
\end{align*}

\begin{align*}
\frac{\alpha^2}{2\pi} v_t & + \frac{\omega v(u_s - \epsilon \lambda u_\xi) + \nu v_r + \frac{1}{r} \nu v_\xi - \frac{1}{r} w^2 - \epsilon \omega G_0 u^2 \sin \xi}{\sin \xi} \\
& \quad - \frac{1}{G_0} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\omega (v_s - \epsilon \lambda v_\xi)) - \frac{1}{\sin \xi} \frac{\partial}{\partial \xi} (\omega (w_s - \epsilon \lambda w_\xi)),
\end{align*}

\begin{align*}
\frac{\alpha^2}{2\pi} w_t & + \frac{\omega w(u_s - \epsilon \lambda u_\xi) + \nu w_r + \frac{1}{r} w w_\xi}{\cos \xi} \\
& \quad - \frac{1}{G_0} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\omega (w_s - \epsilon \lambda w_\xi)) + \frac{1}{\cos \xi} \frac{\partial}{\partial \xi} (\omega (u_s - \epsilon \lambda u_\xi)),
\end{align*}

\begin{align*}
\frac{\alpha^2}{2\pi} \tilde{u}_t & + \frac{\omega \tilde{u}(\tilde{u}_s - \epsilon \lambda \tilde{u}_\xi) + \nu \tilde{u}_r + \frac{1}{r} \nu \tilde{u}_\xi - \frac{1}{r} \tilde{w}^2 - \epsilon \omega G_0 \tilde{u}^2 \sin \xi}{\sin \xi} \\
& \quad - \frac{1}{G_0} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\omega (\tilde{u}_s - \epsilon \lambda \tilde{u}_\xi)) + \frac{1}{\sin \xi} \frac{\partial}{\partial \xi} (\omega (\tilde{w}_s - \epsilon \lambda \tilde{w}_\xi)),
\end{align*}

\begin{align*}
\frac{\alpha^2}{2\pi} \tilde{v}_t & + \frac{\omega \tilde{v}(\tilde{v}_s - \epsilon \lambda \tilde{v}_\xi) + \nu \tilde{v}_r + \frac{1}{r} \nu \tilde{v}_\xi - \frac{1}{r} \tilde{w}^2 - \epsilon \omega G_0 \tilde{v}^2 \sin \xi}{\cos \xi} \\
& \quad - \frac{1}{G_0} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\omega (\tilde{v}_s - \epsilon \lambda \tilde{v}_\xi)) - \frac{1}{\cos \xi} \frac{\partial}{\partial \xi} (\omega (\tilde{u}_s - \epsilon \lambda \tilde{u}_\xi)),
\end{align*}

\begin{align*}
\frac{\alpha^2}{2\pi} \tilde{w}_t & + \frac{\omega \tilde{w}(\tilde{w}_s - \epsilon \lambda \tilde{w}_\xi) + \nu \tilde{w}_r + \frac{1}{r} \nu \tilde{w}_\xi}{\cos \xi} \\
& \quad - \frac{1}{G_0} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (\omega (\tilde{w}_s - \epsilon \lambda \tilde{w}_\xi)) + \frac{1}{\cos \xi} \frac{\partial}{\partial \xi} (\omega (\tilde{u}_s - \epsilon \lambda \tilde{u}_\xi)),
\end{align*}
In the unsteady-flow model we find four parameters that govern the flow: the frequency (Womersly) parameter ($\alpha = \frac{\omega}{\nu}$), the curvature ($\epsilon(s)$), the ratio of torsion to curvature ($\lambda(s)$) and $G_0$, where $G_0 = \frac{k^2\omega^2}{\mu^2\nu^2}$. The parameter $G_0 = \frac{1}{2\epsilon_0}G_m$, where $G_m$ is amplitude parameter used by Hydon and Mullin & Greated and $\epsilon_0$ is the curvature in a uniformly curved pipe. In the low frequency limit, Mullin and Greated showed that $G_m$ is equivalent to the Dean parameter (a quasi-steady Dean number) i.e. the flow is entirely driven by the parameter $G_m$ (a combination of the non-dimensional amplitude of the pressure and the curvature).

After making the change of variable ($\tilde{r}$, $\tilde{s}$, $\tilde{\theta}$) → ($\tilde{r}$, $\tilde{s}$, $\tilde{\xi}$), Germano sets all $s$-derivatives to zero. This is because he assumes a uniform pipe geometry and a fully developed flow. As we wish to examine the effect of non-uniform geometry, we cannot set $s$-derivatives to zero.
Chapter 3

Flow in twisted pipes: Steady pressure gradient.

3.1 Introduction.

In this chapter we discuss steady-flow in twisted pipes. Following Germano we solve the equations for small values of the curvature parameter, $\epsilon$, highlighting solutions that arise from the non-uniformity of the curvature and torsion. Due to the nature of the solution we find that for some pipe shapes a numerical solution is required. The method of solution and results are discussed.

3.2 Steady flow in twisted pipes.

3.2.1 Governing equations.

In chapter 2 the non-dimensional form of the Germano equations were given, these are reproduced below.

The continuity equation is

$$ \omega (u_\varepsilon - \epsilon \nu \epsilon \xi) + v_r + \frac{u_r}{r} + \frac{v}{r} + \epsilon \omega [v \sin \xi + w \cos \xi] = 0, \quad (3.2.1) $$

and the equations of motion are
\[ \begin{align*}
\omega w_s + \nu w_r + \left( \frac{w}{r} - \epsilon \lambda w \right) u_\xi + \epsilon \omega u [v \sin \xi + w \cos \xi] &= -\omega p_s + \epsilon \lambda \omega p_{\xi} \\
&+ \frac{1}{\text{Re}} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (u_r + \epsilon \omega u \sin \xi - \omega v_s + \epsilon \lambda \omega v_{\xi}) \right] \\
&+ \frac{1}{r} \frac{\partial}{\partial \xi} \left( \frac{u_\xi}{r} - \omega w_s + \epsilon \omega u \cos \xi + \epsilon \lambda \omega w_{\xi} \right) , \tag{3.2.2} \end{align*} \]

\[ \begin{align*}
\omega v_s + \nu v_r + \left( \frac{w}{r} - \epsilon \lambda w \right) v_\xi - \frac{w^2}{r} - \epsilon \omega u^2 \sin \xi &= -p_r \\
&+ \frac{1}{\text{Re}} \left[ \left( \frac{1}{r} \frac{\partial}{\partial \xi} + \epsilon \omega \cos \xi \right) \left( -w_r - \frac{w}{r} + \frac{v_\xi}{r} \right) \right] \\
&+ \omega \left( \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi} \right) \left( \omega v_s - u_r - \epsilon \omega u \sin \xi - \epsilon \lambda \omega v_{\xi} \right) \right] , \tag{3.2.3} \end{align*} \]

\[ \begin{align*}
\omega w_s + \nu w_r + \left( \frac{w}{r} - \epsilon \lambda w \right) w_\xi + \frac{vw}{r} - \epsilon \omega u^2 \cos \xi &= -\frac{p_s}{r} \\
&+ \frac{1}{\text{Re}} \left[ \left( \frac{\partial}{\partial r} + \epsilon \omega \sin \xi \right) \left( w_r + \frac{w}{r} - \frac{v_\xi}{r} \right) \right] \\
&- \omega \left( \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi} \right) \left( -\omega w_s + \frac{u_\xi}{r} + \epsilon \omega u \cos \xi + \epsilon \lambda \omega w_{\xi} \right) \right] , \tag{3.2.4} \end{align*} \]

where

\[ \omega = \frac{1}{1 + \epsilon r \sin \xi} = 1 - \epsilon r \sin \xi + \epsilon^2 r^2 \sin^2 \xi + O(\epsilon^3), \quad \text{Re} = \frac{U_0 a}{\nu}. \]

The above equations can now be solved. We assume that the curvature is small \((\epsilon(s) \ll 1)\) and varies slowly with distance along the pipe \((s)\).

A steady pressure gradient is assumed, i.e. to leading order in curvature

\[ p_s = -\frac{G}{\text{Re}}. \tag{3.2.5} \]

As it is assumed that curvature and torsion are dependent on the arc-length, \(s\), all higher-orders terms in the pressure and velocity will be functions of \(s\), \(r\) and \(\xi\). Therefore, the flow described here is not fully-developed.
In the following analysis we pose a perturbation solution in $\epsilon$. As the curvature parameter depends on $s$ we have to be careful that $\epsilon(s) \ll 1$ for a range of $s$. This will also be discussed at the end of this chapter.

### 3.2.2 The solution.

We assume that curvature is small, and expand in powers of $\epsilon(s)$:

$$
\begin{align*}
  u &= u_0(r) + \epsilon(s)u_1(r, s) \sin(\xi) + \epsilon(s)^2 u_2(r, s, \xi) + \mathcal{O}(\epsilon(s)^3), \\
  v &= \epsilon(s)v_1(r, s) \sin(\xi) + \epsilon(s)^2 v_2(r, s, \xi) + \mathcal{O}(\epsilon(s)^3), \\
  w &= \epsilon(s)w_1(r, s) \cos(\xi) + \epsilon(s)^2 w_2(r, s, \xi) + \mathcal{O}(\epsilon(s)^3), \\
  p &= p_0(s) + \epsilon(s)p_1(r, s) \sin(\xi) + \epsilon(s)^2 p_2(r, s, \xi) + \mathcal{O}(\epsilon(s)^3).
\end{align*}
$$

The boundary conditions are $u = v = w = 0$ at $r = 1$. We also insist that the velocities are finite everywhere in the cross-section.

At leading order we have:

$$
\begin{align*}
  u_0 &= \frac{\dot{\theta}}{4} (1 - r^2), \\
  p_0 &= \frac{\dot{\theta}}{Re} s,
\end{align*}
$$

which is equivalent to Poiseuille flow.

The equations at first order in $\epsilon$ are:

$$
\begin{align*}
  u_{1,s} + \frac{\dot{\epsilon}}{\epsilon} u_1 + v_{1,r} - \frac{w_1}{r} + \frac{v_1}{r} &= 0; \\ \\
  u_0 u_{1,s} + \frac{\dot{\epsilon}}{\epsilon} u_0 u_1 + u_0' v_1 &= -p_{1,s} - \frac{\dot{\epsilon}}{\epsilon} p_1 - \frac{8r}{Re} \\
  &+ \frac{1}{Re} \left[ \frac{1}{r^2} \right] \left( u_{1,r} + u_0 - \frac{\dot{\epsilon}}{\epsilon} v_1 - v_{1,s} \right) \\
  &- \frac{u_1}{r^2} - \frac{u_0}{r} + \frac{\dot{\epsilon} w_1}{\epsilon r} + \frac{w_{1,s}}{r}, \\ \\
  u_0 v_{1,s} + \frac{\dot{\epsilon}}{\epsilon} u_0 v_1 - u_0^2 &= -p_{1,r} \\
  &+ \frac{1}{Re} \left[ \frac{1}{\epsilon s} \right] \left( \epsilon \left( -u_{1,r} - u_0 + \frac{\dot{\epsilon}}{\epsilon} v_1 + v_{1,s} \right) \right) \\
  &+ \frac{w_{1,r}}{r} + \frac{w_0}{r^2} - \frac{v_1}{r^2}.
\end{align*}
$$
\[
u_0 w_{1,s} + \frac{\dot{\epsilon}}{\epsilon} u_0 w_1 - \frac{p_1}{r} = - \frac{\dot{p}_1}{r} + \frac{1}{Re} \left[ \frac{1}{\epsilon} \frac{\partial}{\partial s} \left( \epsilon \left( - \frac{u_1}{r} - u_0 + \frac{\dot{\epsilon}}{\epsilon} w_1 + w_{1,s} \right) \right) \right] + \frac{\partial}{\partial r} \left( w_{1,r} + \frac{u_1}{r} - \frac{v_1}{r} \right), \quad (3.2.10)\]

where \( \dot{\epsilon} = \frac{d\epsilon}{ds} \).

We see that the curvature parameter always appears as the ratio of \( \dot{\epsilon} \) to \( \epsilon \). Therefore we let \( \epsilon = \kappa_0 e^{\eta s} \) (so that \( \dot{\epsilon}/\epsilon = \eta \)) and look for solutions independent of \( s \). Using this form for the curvature allows us to compare our results with uniformly curved pipe flow \( (\eta = 0) \). However, the for \( \eta \neq 0 \) solutions are physically meaningful only for \( s \) small enough that epsilon remains small.

Using the substitutions
\[
u_1 = \tau f(\rho), \quad u_1 = g(\rho), \quad w_1 = h(\rho), \quad p_1 = \tau q(\rho), \quad \text{where} \quad \rho = r^2, \]
we obtain
\[
\eta f + 2g' - \frac{h}{\rho} + \frac{g}{\rho} = 0, \quad (3.2.11)
\]

\[
\eta u_0 f + \frac{u_0'}{r} g = -\eta q
\]

\[
- \frac{8}{Re} + \frac{1}{Re} \left[ 4\rho f'' + 8f' - 2\eta g' - \frac{\eta g}{\rho} + \frac{\eta h}{\rho} + \frac{u_0'}{r} \right], \quad (3.2.12)
\]

\[
\eta u_0 g - u_0^2 = -2\rho q' - q
\]

\[
+ \frac{1}{Re} \left[ -2\eta \rho f' - \eta f - \frac{g}{\rho} + \eta^2 g + 2h' + \frac{h}{\rho} - \eta u_0 \right], \quad (3.2.13)
\]

\[
\eta u_0 h - u_0^2 = -q
\]

\[
+ \frac{1}{Re} \left[ 4\rho h'' + 4h' - \frac{h}{\rho} + \eta^2 h - 2g' + \frac{g}{\rho} - \eta f - \eta u_0 \right], \quad (3.2.14)
\]

where the prime \( (') \) denotes the derivative with respect to \( \rho \).

If \( \eta \) was set to zero we would have the Dean equations for flow in a uniformly curved pipe. Therefore, if we take \( \eta \) to be small we can derive a small parameter

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solution to (3.2.11)-(3.2.14) in powers of $\eta$ i.e. we take

$$f = f_0 + \eta f_1 + O(\eta^2),$$

$$g = g_0 + \eta g_1 + O(\eta^2),$$

$$h = h_0 + \eta h_1 + O(\eta^2),$$

$$q = q_0 + \eta q_1 + O(\eta^2). \quad (3.2.15)$$

We assume a polynomial form of the solution and, to first order in $\eta$, this gives

the following solutions to (3.2.11)-(3.2.14):

$$f = \sum_{i=0}^{\infty} \eta^i f_i$$

$$= -\frac{3G}{16} (1 - \rho) + \frac{Re^2 G^3}{737280} (1 - \rho)(19 - 21\rho + 9\rho^2 - \rho^3)$$

$$+ \eta \left( -\frac{Re^2 G^4}{2477260800} (\rho - 1) \left( 5 \rho^5 - 72 \rho^4 + 348 \rho^3 - 877 \rho^2 + 1188 \rho - 807 \right) 
- \frac{Re^3 G^3}{928972800} (\rho - 1) \left( 817 \rho + 5 \rho^5 - 618 \rho^2 - 58 \rho^4 + 257 \rho^3 - 548 \right) 
- \frac{Re G^2}{9216} (\rho - 1) \left( -17 \rho - 17 + 7 \rho^2 \right) G^2 
+ \frac{Re G}{288} (\rho - 1) \left( \rho^2 - 3 \rho + 3 \right) \right) + O(\eta^2), \quad (3.2.16)$$

$$g = \sum_{i=0}^{\infty} \eta^i g_i$$

$$= \frac{Re G^2}{4608} (1 - \rho)^2 (4 - \rho)$$

$$+ \eta \left( \frac{Re^2 G^3}{4423680} (\rho - 1)^2 \left( -13 + 15\rho - 7\rho^2 + \rho^3 \right) - \frac{G}{24} (\rho - 1)^2 \right)$$

$$+ O(\eta^2), \quad (3.2.17)$$

$$h = \sum_{i=0}^{\infty} \eta^i h_i$$

$$= \frac{Re G^2}{4608} (1 - \rho)(4 - 23\rho + 7\rho^2)$$

$$+ \eta \left( \frac{Re^2 G^3}{4423680} (\rho - 1) \left( 13 - 224\rho + 266\rho^2 - 124\rho^3 + 17\rho^4 \right) 
+ \frac{G}{48} (\rho - 1) (2 - \rho) \right)$$

$$+ O(\eta^2), \quad (3.2.18)$$

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\[ q = \sum_{i=0}^{\infty} \eta^i q_i = \frac{G^2}{192} (9 - 6\rho + 2\rho^2) \\
+ \eta \left( \frac{Re G^2}{1105920} \left( -101 + 120\rho - 90\rho^2 + 30\rho^3 - 3\rho^4 \right) + \frac{G}{24Re} (1 - 3\rho) \right) \\
+ \mathcal{O}(\eta^3). \]  
(3.2.19)

The underlined terms are equivalent to Dean's solution for flow in a uniformly curved pipe.

The \( \mathcal{O}(\varepsilon^2) \)-equations are as follows:

\[
- \varepsilon r (\varepsilon u_1)_s sin \xi + (\varepsilon^2 u_2)_s - \varepsilon^2 \lambda u_1 cos \xi + \varepsilon^2 v_{2,r} + \frac{\varepsilon^2 u_{2,\xi}}{r} + \frac{\varepsilon^2 v_2}{r} \\
+ \varepsilon^2 \left[ v_1 sin^2 \xi + w_1 cos^2 \xi \right] = 0, \quad (3.2.20)
\]

\[
- \varepsilon ru_0 (\varepsilon u_1)_s sin \xi + \varepsilon u_1 (\varepsilon u_1)_s sin^2 \xi + \nu_0 (\varepsilon^2 u_2)_s + \varepsilon^2 v_{1,p,r} sin^2 \xi + \varepsilon^2 v_{2,u_0,r} \\
+ \varepsilon^2 \left( \frac{u_1}{r} \cos \xi - \lambda u_0 \right) u_1 cos \xi + \varepsilon^2 u_0 \left[ v_1 sin^2 \xi + w_1 cos^2 \xi \right] \\
= - \varepsilon^2 r^2 p_{0,s} sin^2 \xi + \varepsilon r (\varepsilon p_1)_s sin^2 \xi - (\varepsilon^2 p_2)_s + \varepsilon^2 \lambda p_1 cos \xi \\
+ \frac{1}{R} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left[ \varepsilon^2 u_{2,r} - \varepsilon^2 ru_0 sin^2 \xi + \varepsilon^2 u_1 sin^2 \xi + \varepsilon r (\varepsilon v_1)_s sin^2 \xi \\
- (\varepsilon^2 v_2)_s + \varepsilon^2 \lambda v_1 cos \xi \right] + \left( \frac{1}{r} \frac{\partial}{\partial \xi} \right) \left[ \frac{\varepsilon^2 u_{2,\xi}}{r} - \varepsilon^2 ru_0 sin \xi cos \xi + \varepsilon^2 u_1 sin \xi cos \xi \\
+ \varepsilon r (\varepsilon w_1)_s sin \xi cos \xi - (\varepsilon^2 w_2)_s - \varepsilon^2 \lambda w_1 sin \xi \right] \right], \quad (3.2.21)
\]

\[
- \varepsilon ru_0 (\varepsilon v_1)_s sin^2 \xi + \varepsilon u_1 (\varepsilon v_1)_s sin^2 \xi + \nu_0 (\varepsilon^2 v_2)_s + \varepsilon^2 v_{1,p,r} sin^2 \xi - \varepsilon^2 w_{1,r} \\
+ \varepsilon^2 \left( \frac{u_1 \cos \xi}{r} - \lambda u_0 \right) v_1 cos \xi + \varepsilon^2 ru^2 \sin^2 \xi - 2\varepsilon^2 ru_0 u_1 \sin^2 \xi \\
= - \varepsilon^2 p_{2,r} + \frac{1}{R} \left[ -\varepsilon \left( r sin \xi \frac{\partial}{\partial s} + \lambda \frac{\partial}{\partial \xi} \right) [(\varepsilon v_1)_s sin \xi - \varepsilon u_{1,r} sin \xi - \varepsilon u_0 sin \xi] \\
+ \frac{\varepsilon^2}{r} \left( -w_{2,r} - \frac{u_{2,\xi}}{r} + \frac{u_2 \xi}{r} \right) + \varepsilon^2 \cos \xi \left( -w_{p,r} - \frac{w_1}{r} + \frac{w_1 \xi}{r} \right) \\
+ \left( \frac{\partial}{\partial s} \right) \left[ -\varepsilon^2 u_{2,r} + \varepsilon^2 ru_0 sin^2 \xi - \varepsilon^2 u_1 sin^2 \xi - \varepsilon r (\varepsilon v_1)_s sin^2 \xi + (\varepsilon^2 v_2)_s \\
- \varepsilon^2 \lambda v_1 cos \xi \right] \right], \quad (3.2.22)
\]

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\[- \epsilon u_0 (\epsilon w_1)_s \sin \xi \cos \xi + \epsilon u_1 (\epsilon w_1)_s \sin \xi \cos \xi + u_0 (\epsilon^2 w_2)_s \]
\[+ \epsilon^2 v_1 u_{p,r} \sin \xi \cos \xi \frac{\epsilon^2 w_1 v_1}{r} \sin \xi \cos \xi - \epsilon^2 \left( \frac{w_1}{r} \cos \xi - \lambda u_0 \right) w_1 \sin \xi \]
\[- 2\epsilon^2 u_0 u_1 \sin \xi \cos \xi + \epsilon^2 r u_0^2 \sin \xi \cos \xi \]
\[= - \frac{\epsilon^2}{r} p_{2,\xi} \]
\[+ \frac{1}{R} \left[ \epsilon \left( r \sin \xi \frac{\partial}{\partial \xi} + \lambda \frac{\partial}{\partial \xi} \right) \left[ - (\epsilon w_1)_s \cos \xi + \frac{\epsilon}{r} u_1 \cos \xi + \epsilon u_0 \cos \xi \right] \right. \]
\[+ \epsilon^2 \left( w_{2,rr} + \frac{w_2}{r^2} - \frac{w_2}{r^2} + \frac{v_2,\xi}{r^2} + \frac{v_2,\xi}{r^2} \right) + \epsilon^2 \sin \xi \cos \xi \left( \frac{w_{p,r} + \frac{w_1}{r} - \frac{v_2,\xi}{r}}{r} \right) \]
\[- \left( \frac{\partial}{\partial \xi} \right) \left[ \epsilon^2 u_{2,\xi} - \epsilon^2 r u_0 \sin \xi \cos \xi + \epsilon^2 u_1 \sin \xi \cos \xi + \epsilon r (\epsilon w_1)_s \sin \xi \cos \xi \right. \]
\[- \left( \epsilon^2 w_2 \right)_s - \epsilon^2 \lambda w_1 \sin \xi \right], \quad (3.2.23) \]

where \( \lambda = \frac{r}{\epsilon} \).

We wish to discuss the effect of torsion. Therefore we want to include torsion in the model at low order, so we assume that \( \epsilon^{-1} \ll \lambda \gg 1 \) and

\[
w_2 = \lambda \left( w_2 + \mathcal{O}(\lambda^{-1}) \right), \\
v_2 = \lambda \left( v_2 + \mathcal{O}(\lambda^{-1}) \right), \\
w_2 = \lambda \left( w_2 + \mathcal{O}(\lambda^{-1}) \right), \\
p_2 = \lambda \left( p_2 + \mathcal{O}(\lambda^{-1}) \right), \quad (3.2.24) \]

and we consider the leading order terms. It follows that the next terms in the expansion are \( \mathcal{O}(\epsilon^2 \lambda) \). Therefore, the \( \mathcal{O}(\epsilon^2 \lambda) \) equations are as follows:

\[
\lambda^{-1} \epsilon^{-2} \left( \epsilon^2 \lambda w_2 \right)_s - u_1 \cos \xi + v_{20,r} \cos \xi + u_{20,r}^{\xi} \cos \xi + u_{20,\xi}^{\xi} = 0; \quad (3.2.25) \\
\lambda^{-1} \epsilon^{-2} u_0 \left( \epsilon^2 \lambda w_2 \right)_s - u_{0,r} v_{20} - u_0 u_1 \cos \xi = - \lambda^{-1} \epsilon^{-2} \left( \epsilon^2 \lambda p_2 \right)_s + p_1 \cos \xi \\
+ \frac{1}{Re} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left[ v_{20,r} - \lambda^{-1} \epsilon^{-2} \left( \epsilon^2 \lambda v_2 \right)_s + v_1 \cos \xi \right] \\
+ \frac{1}{r} \frac{\partial}{\partial \xi} \left[ \frac{w_{20,\xi}}{r} - w_{20} \sin \xi - \lambda^{-1} \epsilon^{-2} \left( \epsilon^2 \lambda w_2 \right)_s \right] \right], \quad (3.2.26) \\
\lambda^{-1} \epsilon^{-2} u_0 \left( \epsilon^2 \lambda v_2 \right)_s - u_0 v_1 \cos \xi = - p_{20,r} + \frac{1}{Re} \left[ \frac{1}{r} \left[ \frac{w_{20,\xi}}{r} - w_{20,\xi}^{\xi} + v_{20,\xi}^{\xi} \right] \\
+ \lambda^{-1} \epsilon^{-2} \frac{\partial}{\partial s} \left[ - \lambda \epsilon^2 u_{20,r} + \left( \epsilon^2 \lambda v_2 \right)_s - \lambda \epsilon^2 u_1 \cos \xi \right] \\
- \frac{1}{\epsilon} \left[ (\epsilon v_1)_s \cos \xi - \epsilon u_{1,r} \cos \xi - \epsilon u_0 \cos \xi \right] \right], \quad (3.2.27) 
\]
\[ \lambda^{-1} e^{-2} u_0 \left( e^{2\lambda w_{20}} \right)_s + u_0 w_1 \sin \xi = -\frac{p_{20,\xi}}{r} + \frac{1}{Re} \left( w_{20,r} + w_{20,rr} - \frac{w_{20}}{r^2} \right) \]

\[ \lambda^{-1} e^{-2} u_0 \left( e^{2\lambda w_{20}} \right)_s + u_0 w_1 \sin \xi = -\frac{1}{\epsilon} (\epsilon w_1)_s \sin \xi \left( \frac{u_1}{r} \sin \xi \right) \]

\[ \lambda^{-1} e^{-2} \frac{\partial}{\partial \beta} \left( e^{2\lambda w_{20}} \right) - e^{2\lambda w_1 \sin \xi} - \left( e^{2\lambda w_{20}} \right)_s \]

\[ \lambda^{-1} e^{-2} \frac{u_{20,r}}{r} + \frac{v_{20,\xi}}{r^2} - \frac{w_{20}}{r^2} \sin \xi \]  

(3.2.28)

We see that these terms explicitly depend on \( \xi \) via the factor \( \frac{e^{2\lambda}}{e^{2\lambda}} \). Therefore, taking \( e^{2\lambda} = k_0 \tau_0 e^{\beta s} \) (so that \( \frac{e^{2\lambda}}{e^{2\lambda}} = \beta \) and \( \tau = \tau_0 e^{(\beta-\eta)s} \)) we can look for solutions independent of \( s \) of the form:

\[ u_{20} = rF(\rho) \cos \xi, \quad v_{20} = rG(\rho) \cos \xi, \quad w_{20} = rH(\rho) \sin \xi, \quad p_{20} = rQ(\rho) \cos \xi. \]

This yields the following system of equations:

\[ \beta F' - f + 2G' + \frac{H + G}{\rho} = 0; \]  

\[ 2\beta F (1 - \rho) - 4G - 2(1 - \rho)f = -\beta Q + \frac{1}{Re} \left( 4pF'' + 8F' - 2\beta G' - \frac{\beta G}{\rho} \right) \]

\[ - \frac{\beta H}{\rho} + 2g' + \frac{g}{\rho} - \frac{h}{\rho}, \]  

(3.2.29)

\[ 2\beta G (1 - \rho) - 2(1 - \rho)g = -2\rho Q' - Q + \frac{1}{Re} \left[ -2\beta \rho F' - \beta F + \beta^2 G - \frac{G}{\rho} - 2H' \right] \]

\[ - \frac{H}{\rho} + 2\rho f' + f - \beta g - \eta g + 2(1 - \rho), \]  

(3.2.30)

\[ 2\beta H (1 - \rho) + 2(1 - \rho)h = Q + \frac{1}{Re} \left[ 2\beta F + 2G' - \frac{G}{\rho} + 4\rho H'' + 4H' + \beta^2 H \right] \]

\[ - \frac{H}{\rho} - f + \beta h + \eta h - 2(1 - \rho), \]  

(3.2.31)

These equations can be solved analytically by perturbing in the small parameter \( \beta \), i.e. \( F = F_0 + \beta F_1 + O(\beta^2) \), etc..
At this stage we must consider the validity of these expansions. Let us look at the structure of $u$,

\[ u = u_0 + \epsilon u_1 \sin(\xi) + \epsilon^2 \lambda u_{20} + \mathcal{O}(\epsilon^2) \]

\[ = u_0 + \epsilon \left( \tau f_0 + \eta \tau f_1 + \mathcal{O}(\eta^2) \right) \sin(\xi) \]

\[ + \epsilon^2 \lambda \left( \tau F_0 + \beta \tau F_1 + \mathcal{O}(\beta^2) \right) \cos(\xi) \]

\[ + \mathcal{O}(\epsilon^3). \]  

(3.2.33)

If $\eta$ is sufficiently small and $\epsilon^2 \lambda \gg \epsilon \eta$ then the expansion is valid. However, this assumption is very restrictive when considering non-uniform curvature and torsion. For this reason, in certain cases we solve the equations (3.2.11)-(3.2.14) and (3.2.29)-(3.2.32) numerically (see §3.1).

First, suppose that $\epsilon^2 \lambda \gg \epsilon \eta$. Then the underlined terms in (3.2.29)-(3.2.32) can be neglected, therefore the leading- and first-order solutions for the $\mathcal{O}(\epsilon^2 \lambda)$ terms are:

\[ F_0 = \text{Re} \left[ \frac{G^3}{1061683200} (\rho - 1) \left( 14 \rho^5 - 196 \rho^4 + 929 \rho^3 - 2271 \rho^2 \right. \right. \]

\[ + 2989 \rho - 1985) \]

\[ + \frac{G^2}{928972800} (\rho - 1) \left( 62 \rho^5 - 820 \rho^4 + 3905 \rho^3 - 10095 \rho^2 \right. \]

\[ + 14125 \rho - 9857) \right) \]

\[ + \text{Re} \left[ - \frac{G^2}{4608} (\rho - 1) \left( \rho^2 - 5 \rho + 22 \right) \right. \]

\[ + \frac{G}{1536} (\rho - 1) \left( \rho^2 - 3 \rho + 3 \right) - \frac{1}{576} (\rho - 1) \left( \rho^2 - 3 \rho + 3 \right) \right. \]

(3.2.34)

\[ F_1 = \text{Re} \left[ - \frac{G^3}{416179814400} (\rho - 1) \left( 33 \rho^7 - 637 \rho^6 + 4627 \rho^5 - 18893 \rho^4 \right. \right. \]

\[ + 48433 \rho^3 - 80535 \rho^2 + 82985 \rho - 47199 \right) \]

\[ - \frac{G^2}{31213486800} (\rho - 1) \left( 217 \rho^7 - 3935 \rho^6 + 28321 \rho^5 - 118679 \rho^4 \right. \]

\[ + 319969 \rho^3 - 582023 \rho^2 + 664789 \rho - 414107 \right) \]

\[ + \text{Re} \left[ \frac{G^3}{88473600} (\rho - 1) \left( 2 \rho^4 - 28 \rho^3 + 122 \rho^2 - 278 \rho + 347 \right) \right. \]

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\[ G_0 = Re^2 \left( \frac{G^2}{11059200} \right) \left( \rho - 1 \right) \left( 33\rho^4 - 312\rho^3 + 2013\rho^2 - 4527\rho + 4483 \right) \]

\[ - \frac{G}{153600} \left( \rho - 1 \right) \left( 9\rho^4 - 56\rho^3 + 144\rho^2 - 186\rho + 119 \right) \]

\[ + \frac{1}{57600} \left( \rho - 1 \right) \left( 2\rho^4 - 13\rho^3 + 37\rho^2 - 53\rho + 37 \right) \]

\[ + \frac{1}{128} \left( \rho - 1 \right) \left( 2\rho - 5 \right) G - \frac{1}{48} \left( \rho - 1 \right) \left( \rho - 9 \right) \tag{3.2.35} \]

\[ G_1 = -Re^2 \left( \frac{G^3}{104044953600} \right) \left( \rho - 1 \right)^2 \left( 92\rho^5 - 1412\rho^4 + 7227\rho^3 - 19659\rho^2 \right) \]

\[ + 31365\rho - 32173 \]

\[ + Re^3 \left( \frac{G^2}{1857945600} \right) \left( \rho - 1 \right)^2 \left( 31\rho^5 - 378\rho^4 + 1670\rho^3 - 3807\rho^2 \right) \]

\[ + 4716\rho - 2987 \]

\[ + \frac{G^2}{92160} \left( \rho - 1 \right)^2 \left( 2\rho^2 - 11\rho + 81 \right) \]

\[ - Re \left( \frac{G}{153600} \right) \left( \rho - 1 \right)^2 \left( 8\rho^2 - 29\rho + 44 \right) \]

\[ - \frac{1}{5760} \left( \rho - 1 \right)^2 \left( \rho^2 - 3\rho \left( 32, \right) - 37 \right) \]

\[ H_0 = -Re^2 \left( \frac{G^3}{88473600} \right) \left( \rho - 1 \right) \left( \rho^4 - 14\rho^3 + 61\rho^2 - 139\rho + 311 \right) \]

\[ + Re^2 \left( \frac{G^2}{11059200} \right) \left( \rho - 1 \right) \left( 341\rho^4 - 2494\rho^3 + 5381\rho^2 - 4619\rho + 571 \right) \]

\[ - \frac{G}{128} \left( \rho - 1 \right) \left( \rho - 5 \right) - \frac{1}{48} \left( 5\rho - 1 \right) \left( \rho - 1 \right) \tag{3.2.38} \]

\[ H_1 = Re^3 \left( \frac{G^3}{104044953600} \right) \left( \rho - 1 \right) \left( 8\rho^6 - 160\rho^5 + 1163\rho^4 - 4962\rho^3 + 24928\rho^2 \right. \]

\[ - 60430\rho + 32173 \]

\[ - \frac{G^2}{1857945600} \left( \rho - 1 \right) \left( 589\rho^6 - 6895\rho^5 + 29582\rho^4 - 66143\rho^3 + 80297\rho^2 \right. \]

\[ - 48797\rho + 2987 \)

\[ + Re \left( \frac{G^2}{92160} \right) \left( \rho - 1 \right) \left( 2\rho^3 - 13\rho^2 + 2\rho + 81 \right) \]
The wall shear stress on the pipe boundary is given by

\[ \text{axial WSS} = -\frac{u_r}{a}|_{r=1} = -\frac{U_0}{a}u_r|_{r=1}, \quad (3.2.42) \]

\[ \text{azimuthal WSS} = -\frac{w_r}{a}|_{r=1} = -\frac{U_0}{a}w_r|_{r=1}. \quad (3.2.43) \]

The WSS for this asymptotic solution can be expressed as

\[ \text{Axial WSS} = 4 + \epsilon \left( -3 + \frac{Re^2}{120} - \eta \left( \frac{31Re}{72} + \frac{79Re^3}{90720} \right) \right) \sin \xi + \mathcal{O}(\eta^2 Re^3) + \mathcal{O}(\epsilon^2), \]

\[ + \epsilon^2 \lambda \left( \frac{31Re}{72} + \frac{79Re^3}{90720} + \beta \left( \frac{1}{24} - \frac{323Re^2}{17280} - \frac{6163Re^4}{87091200} \right) \right) \cos \xi \]

\[ + \mathcal{O}(\epsilon^2 \lambda^2 Re) + \mathcal{O}(\epsilon^2), \quad (3.2.44) \]

\[ \text{Azimuthal WSS} = \epsilon \left( -\frac{Re}{3} - \eta \left( \frac{1}{3} - \frac{13Re^2}{1080} \right) \right) \cos \xi + \mathcal{O}(\eta^2 Re) \]
and this can be converted into a dimensional quantity by multiplying by the factor \( \left( \frac{U_0}{a} \right) \). (In the following results, when referring to the WSS, we use the non-dimensional quantities: from Waters (1996) \( \left( \frac{U_0}{a} \right) \approx 10^2 \) in the extramyocardial coronary artery).

In the following results we refer to the total WSS, by this we mean the root-mean-squared value i.e. total WSS = \( \frac{1}{2} \sqrt{\text{axial WSS}^2 + \text{azim. WSS}^2} \). The results presented above have not been given in terms of a Dean number due to the non-uniformity of the curvature parameter. In the present model \( \epsilon = \kappa_0 e^{\eta s} \), so, assuming that \( \epsilon \ll 1 \) for a suitable range of \( s \), we can define a slowly varying pseudo-Dean number \( D(s) = 2 \kappa_0 e^{\eta s} Re^2 \). For \( \eta = 0 \), \( D(s) \) is the Dean number (as defined by Dean 1928). Dean found that the expansion is valid if the Dean number is less than 576.

### 3.2.3 Pipe shapes.

The flow in pipes with the following centre-line geometry will be examined (see also Figure 3.1). (Unless otherwise stated the parameters are assumed to be non-zero.)

(i). Torus, \( (\tau_0 = \eta = 0) \).

(ii). Helix, \( (\eta = \beta = 0) \).

(iii). Spiral, \( (\tau_0 = 0) \).

(iv). Spiral with uniform torsion, \( (\beta = \eta \neq 0) \).

(v). ‘Stretched’ helix, \( (\eta = 0) \).

(vi). A pipe with curvature inversely proportional to torsion, \( (\beta = 0) \).

A numerical solution (§3.2.4) is sought for pipes (iv) and (vi). The numerical solution is needed because (iv) and (vi) have non-uniform curvature and non-zero torsion which, in the analytic solution, could lead to important terms being inadvertently omitted.
Figure 3.1: Pipe shapes i. Torus, ii. Helix, iii. Spiral, iv. Spiral with uniform torsion, v. 'Stretched helix', vi. Pipe with curvature inversely proportional to torsion: $S_0$ indicates the start of the pipe; the arc-length increases as progress is made along the pipe (also see Figure 3.2).

Figure 3.2: Sketch of a generic centreline; indicating the direction of increasing arc-length, $s$: $S_0$ is the initial point on the line, $S_f$ is the final point on the line.

### 3.2.4 Numerical solution.

For two of the pipe geometries (§3.2.3) we have chosen to solve (3.2.11)–(3.2.14) and (3.2.29)–(3.2.32) numerically, using the collocation method. The NAG library routine D02TGF was used. The routine requires subroutines that define the ODEs and the boundary conditions. The boundary conditions are the no-slip condition at the wall and that all velocities and stresses are finite at $\rho = 0$. The finite conditions are found using the Frobenius expansion about $\rho = 0$.

Therefore, the boundary conditions for (3.2.11)–(3.2.14) are:
the no-slip condition

\[ f(1) = g(1) = h(1) = 0, \]

and the finite conditions

\[ g(0) - h(0) = 0, \]

\[
\left( \frac{\alpha^2}{Re} - 2\alpha \right) f(0) + 4g(0) - \alpha q(0) + \frac{8}{Re} f'(0) = \frac{12}{Re},
\]

\[
\frac{2\alpha}{3Re} f(0) + \left( 2\alpha - \frac{\alpha^2}{Re} \right) g(0) + q(0) - \frac{8}{3Re} h'(0) = 4 - \frac{2\alpha}{Re}.
\]

The boundary conditions for (3.2.29)-(3.2.32) are:

the no-slip condition

\[ F(1) = G(1) = H(1) = 0, \]

and the finite conditions

\[ G(0) + H(0) = 0, \]

\[
\frac{2\beta}{3} F(0) + \left( 2\beta Re - \beta^2 \right) G(0) + Re Q(0) + \frac{8}{3} H'(0)
= \frac{2}{3} f(0) + (2Re - \beta) g(0) - \alpha g(0) + 2,
\]

\[
(2\beta Re - \beta^2) F(0) - 4Re G(0) + \beta Re Q(0) - 8F'(0)
= (2Re - \beta) f(0) - \alpha f(0) + Re q(0).
\]

We now discuss the methods used to check the numerical solution. First, let us again consider the structure of \( u \),

\[
u = u_0 + \epsilon u_1 \sin(\xi) + \epsilon^2 \lambda u_2 + O(\epsilon^2)
= u_0 + \epsilon r f \sin(\xi) + \epsilon^2 \lambda r F + O(\epsilon^2)
= u_0 + \epsilon \left( r f_0 + \eta f_1 + O(\eta^2) \right) \sin(\xi)
+ \epsilon^2 \lambda \left( r F_0 + \beta r F_1 + O(\beta^2) \right) \cos(\xi)
+ O(\epsilon^2).
\]  
(3.2.46)
Table 3.1: Convergence of the non-uniform solution to the uniform solution. As \( \eta \) decreases in magnitude the numerical solution (for a non-uniformly curved pipe) converges to the analytic solution (A.S.)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( \frac{df}{d\eta}(0.0) )</th>
<th>( \frac{df}{d\eta}(0.5) )</th>
<th>( \frac{df}{d\eta}(1.0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-6.698667</td>
<td>-1.963630</td>
<td>0.530094</td>
</tr>
<tr>
<td>0.01</td>
<td>-9.261315</td>
<td>-3.076200</td>
<td>-0.089704</td>
</tr>
<tr>
<td>0.001</td>
<td>-9.575448</td>
<td>-3.207388</td>
<td>-0.158887</td>
</tr>
<tr>
<td>0.0001</td>
<td>-9.607538</td>
<td>-3.220736</td>
<td>-0.165888</td>
</tr>
<tr>
<td>0.00001</td>
<td>-9.610754</td>
<td>-3.222074</td>
<td>-0.166589</td>
</tr>
<tr>
<td>0.0 (A.S.)</td>
<td>-9.611111</td>
<td>-3.222222</td>
<td>-0.166666</td>
</tr>
</tbody>
</table>

The numerical solution evaluates the functions \( f \) and \( F \) (and the equivalents for \( v, w \) and \( p \)). Therefore we have been able to check the numerical solution with the analytic solution in 4 ways:

1. Comparing with the uniform curvature solution \( (f_0) \).
2. Comparing with the uniform helix solution \( (f_0 \text{ and } F_0) \).
3. Comparing with the non-uniform solution \( (f_i \text{ and } F_i, i = 0, 1) \).
4. Showing convergence to uniform solutions.

The NAG routine D02TGF allows the user to increase the number of collocation points \( (Cp) \) and the order of the Chebychev polynomial \( (Mo) \) to be used. To look at solutions in the range \( 1 \leq Re \leq 100 \) we found that we needed \( Cp \approx 20 \) and \( Mo \approx 10 \) for the uniform cases (this gave an error between the analytic and numeric solutions of \( O(10^{-5}) \), which is adequate). When the non-uniformities were introduced we found that, to get the required range of \( Re \), \( Cp \) and \( Mo \) had to be increased: \( Cp \approx 30 \) and \( Mo \approx 50 \). Therefore for all results shown in §3.3 we have used \( Cp = 30 \) and \( Mo = 50 \). The fourth check (above) was to do with convergence to the uniform solution as \( \eta \rightarrow 0 \). Table ?? shows the numerically evaluated solutions for \( \frac{df}{d\eta} \) (evaluated at \( \rho = 0, 0.5, 1 \)), of the non-uniformly curved pipe case, as \( \eta \) is decreased from \( \eta = 0.1 \) to \( \eta = 0.00001 \). The last line of the table shows the analytic solution for \( \frac{df}{d\eta} \) and that the numerical solution is indeed converging to the uniform limit. (This test was also done for the case where both non-uniformities were non-zero).
3.3 Results and discussion

For convenience we set \( G = 8 \) in the solutions, and we note that

\[
F_0 = -f_1, \quad G_0 = -g_1, \quad H_0 = h_1, \quad Q_0 = -q_0. \tag{3.3.47}
\]

\[
F_0 = -f_1, \quad G_0 = -g_1, \quad H_0 = h_1, \quad Q_0 = -q_0. \tag{3.3.48}
\]

\[
H_0 = h_1, \tag{3.3.49}
\]

\[
Q_0 = -q_0. \tag{3.3.50}
\]

Note: where the numerical solution has been used, the Figures have "(Numerical)" in the caption. The following parameter ranges have been studied, \( 1 \leq Re \leq 100, \epsilon(s) \ll 1 \) and \( \tau(s) \ll 1 \). Parameters used in this section have been chosen to show the effects of non-uniform geometry that are present in the entire parameter range.

Figure 3.3 shows how uniform torsion breaks the secondary-flow symmetry of toroidal flow. It is not possible to define a stream function, and therefore all secondary velocity fields are displayed by vector field plots. Dean flow and helical flow are discussed by other authors (see the reviews by Berger et al. 1983 and Tuttle 1990).

We first consider a spiral pipe whose centre-line has zero torsion and exponentially increasing curvature (Figures 3.4 to 3.6). The secondary flow patterns for the spiral look exactly the same as Dean flow. This similarity arises because increasing the curvature merely increases the magnitude of the vector field. Figure 3.4 shows contour plots of the axial velocity at two positions along the pipe. As the fluid travels along the pipe the region of peak axial velocity is forced towards the outer wall by the increasing curvature. In Poiseuille flow the contours of axial velocity
Figure 3.4: The axial flow for a pipe with a spiral centre-line: $Re = 60$, $\kappa_0 = 0.01$, $\eta = 0.1$ and a) $s = 0.001$; b) $s = 15$.

Figure 3.5: The axial perturbation to: a) Poiseuille flow, b) Dean flow. (+ denotes a perturbation in the direction of the mean flow, - denotes a perturbation in the opposite direction.)

are concentric circles, and the perturbation forces these circles towards the outer wall. The axial perturbations to Poiseuille flow and Dean flow (Figure 3.5) consist of two cells, with opposing flows. As the curvature increases so does the magnitude of the cells and hence the peak axial velocity is forced to the outer wall; as the curvature becomes even larger we would expect to see a deformation of the circles. A similar result is seen in high Dean number toroidal flow, as shown by Berger et al. The amplitude of the sinusoidal perturbation to the wall shear stress (WSS) increases with increasing $s$, and the position of maximum (minimum) WSS occurs at the inner (outer) wall (Figure 3.6).

For a pipe with a spiral centre-line and uniform torsion the results are similar to those seen when torsion is introduced into toroidal flow. The addition of uniform torsion skews the secondary flow vortices (Figure 3.7), breaking the symmetry, and affects the position of maximum axial velocity (Figure 3.8). The sinusoidal perturbation to the azimuthal WSS has an increasing effect on the total WSS as the flow progresses along the pipe, causing a 'bump' which reduces the region of low WSS (see Figure 3.9). This is because the sinusoidal perturbation to the azimuthal WSS increases in magnitude more rapidly than the axial WSS, due to
the increasing magnitude of the secondary velocity vector field, which in turn is due to the increasing curvature.

For the ‘stretched’ helix (Figure 3.10) the effects seen in helical flow are exaggerated as the fluid moves further along the pipe. The secondary flow vortices become more skewed, and the mechanisms for this are shown in the secondary-flow perturbation plots (Figure 3.11). The perturbation to Dean flow breaks the symmetry, and the perturbation to helical flow increases the skew. The contour plots of axial velocity (Figure 3.12) demonstrate the same effect of torsion. The axial perturbations to Dean flow and helical flow (Figure 3.13) both consist of two cells, with opposing flows, and demonstrate the mechanism for the increased skew in the axial velocity. Figure 3.14 shows the effect on the wall shear stress: as torsion increases there is a phase shift in the direction of the torsion. This phase shift is more prominent in the axial component because the coefficient of the $O(\epsilon^2 \lambda \beta)$-term is larger than that for the azimuthal component.

Finally we consider the pipe where torsion is inversely proportional to curvature.
Here torsion is dominant at the beginning of the pipe, but as the arc-length increases, curvature has a more significant effect on the flow. This is shown in the secondary velocity plots (figure 3.15) where the two-vortex structure skews towards Dean flow as $s$ increases. Figure 3.16 shows that there is no noticeable effect on the peak axial velocity. The WSS produces a result that at first seems counterintuitive: the minimum WSS occurs at the outer wall (Figure 3.17). The same phenomenon is seen in Dean flow at very low Reynolds numbers (see Larrain & Bonilla 1970, Murata et al. 1976). It is due to the geometrical effect of curvature upon the basic Poiseuille flow, which is dominant if the convective inertia is sufficiently small i.e. the viscous driving terms are dominant.
Figure 3.10: The secondary flow for a helical pipe with increasing torsion: $Re = 25$, $\kappa_0 = 0.09$, $\eta = 0$, $\tau_0 = 0.1$, $\beta = 0.1$ and a) $s = 0.001$; b) $s = 15$.

Figure 3.11: The perturbations to the secondary flow for a helical pipe with increasing torsion: a) perturbation to Dean flow; b) perturbation to helical flow.

Figure 3.12: The axial flow for a helical pipe with increasing torsion: parameters as for Figure 3.10 with a) $s = 0.001$; b) $s = 15$.

Figure 3.13: The perturbations to the axial flow for a helical pipe with increasing torsion: a) perturbation to Dean flow; b) perturbation to helical flow (parameters as for figure 3.10).
Figure 3.14: The wall shear for a helical pipe with increasing torsion: \( Re = 30, \kappa_0 = 0.09, \eta = 0, \tau_0 = 0.1, \beta = 0.1 \) and (-) \( s = 0.001 \), (-) \( s = 8 \), (...) \( s = 15 \). a) axial and azimuthal wss; b) total wss.

Figure 3.15: (Numerical) The secondary velocity vector field for a pipe with torsion inversely proportional to curvature: \( Re = 10, \kappa_0 = 0.01, \eta = 0.3, \tau = 0.31, \beta = 0.0 \) and a) \( s = 0.001 \); b) \( s = 5 \).

Figure 3.16: (Numerical) The axial velocity for a pipe with parameters as figure 3.15: a) \( s = 0.001 \); b) \( s = 5 \).

Figure 3.17: (Numerical) The wall shear stress for a pipe with torsion inversely proportional to curvature: \( Re = 10, \kappa_0 = 0.01, \eta = 0.3, \tau = 0.31, \beta = 0.0 \) and (-) \( s = 0.001 \), (-) \( s = 3 \), (...) \( s = 5 \). a) axial and azimuthal wss; b) total wss.
Chapter 4

Flow in pipes with non-uniform geometry: sinusoidal pressure gradient.

4.1 Introduction.

In this chapter we discuss unsteady flow in pipes of non-uniform geometry. We find that this problem is considerably more difficult than the steady-flow case. A numerical solution is found for flow in a pipe with non-uniform curvature and the results are presented. For certain parameter values this solution breaks down. We discuss how this limits the solution.

4.2 Unsteady flow in twisted pipes.

4.2.1 Governing equations.

In chapter 2 the non-dimensionalized Germano equations were given; these are reproduced below;

\[ \omega(u_x - \epsilon \lambda u_\xi) + v_r + \frac{1}{r} w_x + \frac{1}{r} v + \epsilon w(v \sin \xi + w \cos \xi) = 0; \]  

(4.2.1)
\[
\frac{\alpha^2}{2\pi} u_t + \frac{\omega(u_x - \epsilon \lambda u_\xi)}{1 + \epsilon \sqrt{e}} + \frac{v}{r} + \frac{1}{r} w u_x + \epsilon \omega u(v \sin \xi + w \cos \xi)
\]
\[
= -\frac{\omega}{G_0} (p_x - \epsilon \lambda p_\xi) + \alpha^2 \omega \cos 2\pi t + \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) (u_r + \epsilon \omega u \sin \xi)
\]
\[
+ \frac{1}{r} \frac{\partial}{\partial \xi} \left( \frac{1}{r} u_x + \epsilon \omega u \cos \xi \right)
\]
\[
- \frac{1}{G_0} \left( \left( \frac{\partial}{\partial r} + \frac{1}{r} \right)(\omega(u_x - \epsilon \lambda u_\xi)) + \frac{1}{r} \frac{\partial}{\partial \xi}(\omega(u_x - \epsilon \lambda u_\xi)) \right),
\] (4.2.2)

\[
\frac{\alpha^2}{2\pi} v_t + \frac{\omega(u_x - \epsilon \lambda u_\xi)}{1 + \epsilon \sqrt{e}} + \frac{v}{r} + \frac{1}{r} w v_x - \frac{1}{r} w^2 - \epsilon \omega G_0 u^2 \sin \xi
\]
\[
= -p_r - \left( \frac{1}{r} \frac{\partial}{\partial \xi} + \epsilon \omega \cos \xi \right) (w_r + \frac{1}{r} w - \frac{1}{r} v_x)
\]
\[
+ \omega \left( \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi} \right) \left( \frac{\omega}{G_0} (u_x - \epsilon \lambda u_\xi) - u_r - \epsilon \omega u \sin \xi \right),
\] (4.2.3)

\[
\frac{\alpha^2}{2\pi} w_t + \frac{\omega(u_x - \epsilon \lambda u_\xi)}{1 + \epsilon \sqrt{e}} + \frac{w}{r} + \frac{1}{r} w w_x + \frac{1}{r} w w - \epsilon \omega G_0 u^2 \cos \xi
\]
\[
= -\frac{1}{r^2} p_\xi + \left( \frac{\partial}{\partial r} + \epsilon \omega \sin \xi \right) (w_r + \frac{1}{r} w - \frac{1}{r} v_x)
\]
\[
- \omega \left( \frac{\partial}{\partial s} - \epsilon \lambda \frac{\partial}{\partial \xi} \right) \left( \frac{1}{r} u_x + \epsilon \omega u \cos \xi - \frac{\omega}{G_0} (u_x - \epsilon \lambda u_\xi) \right).
\] (4.2.4)

Here
\[
\omega = \frac{1}{1 + \epsilon \sqrt{e} \sin \xi} = 1 - \epsilon \sqrt{e} \sin \xi + \epsilon^2 r^2 \sin^2 \xi + O(\epsilon^3),
\]

where \(\epsilon\) is the curvature parameter and \(\lambda\) is the ratio of torsion to curvature. The underlined terms arise from non-uniform geometry and torsion.

Again, we assume that the curvature is small, which allows us to perturb about unsteady flow in a straight pipe. The following leading-order pressure gradient has been assumed:

\[
p_0 = -\alpha^2 \cos 2\pi t.
\] (4.2.5)

This implies that at leading order the flow is purely in the axial direction and is independent of \(\xi\) and \(s\).
4.2.2 The solution.

As with the steady flow case we assume that $\epsilon \ll 1$ and construct a small parameter perturbation, i.e.

\[
\begin{align*}
  u &= u_0(r, t) + \epsilon(s)u_1(r, s, t) \sin(\xi) + \epsilon(s)^2 u_2(r, s, \xi, t) + O(\epsilon(s)^3), \\
  v &= \epsilon(s)v_1(r, s, t) \sin(\xi) + \epsilon(s)^2 v_2(r, s, \xi, t) + O(\epsilon(s)^3), \\
  w &= \epsilon(s)w_1(r, s, t) \cos(\xi) + \epsilon(s)^2 w_2(r, s, \xi, t) + O(\epsilon(s)^3), \\
  p &= p_0(s, t) + \epsilon(s)p_1(r, s, t) \sin(\xi) + \epsilon(s)^2 p_2(r, s, \xi, t) + O(\epsilon(s)^3).
\end{align*}
\]

The boundary conditions are $u = v = w = 0$ at $r = 1$. We also insist that the velocities and stresses are finite everywhere in the cross-section.

The leading order (straight pipe) solution was first solved by Sexl (1934)

\[
  u_0 = u_{0R} \cos 2\pi t - u_{0I} \sin 2\pi t,
\]

where

\[
  u_{0R} = \Re \left( -i + \frac{1}{\sqrt{2}} J_0 \left( (1 - i) \frac{\alpha r}{\sqrt{2}} \right) \right),
\]

\[
  u_{0I} = \Im \left( -i + \frac{1}{\sqrt{2}} J_0 \left( (1 - i) \frac{\alpha r}{\sqrt{2}} \right) \right).
\]

The $O(\epsilon^1)$ terms are:

\[
  v_{1,r} + \frac{1}{r} (v_1 - w_1) = -\eta u_1,
\]

\[
  \frac{\alpha^2}{2\pi} v_{1,t} + \frac{1}{r^2} (v_1 - w_1) - \frac{1}{r} w_{1,t} - u_{0,r} v_1 = \alpha^2 r \cos 2\pi t - u_{0,r}
\]

\[
  + \eta u_0 u_1 + \frac{\eta}{G_0} \left( v_{1,r} + \frac{1}{r} (v_1 - w_1) + p_1 \right),
\]

\[
  \frac{\alpha^2}{2\pi} v_{1,t} + \frac{1}{r^2} (v_1 - w_1) - \frac{1}{r} w_{1,r} + p_{1,r} = G_0 u_0^2 + \eta (-u_0 + u_{1,r} - u_0 v_1)
\]

\[
  - \frac{\eta^2}{G_0} v_1,
\]

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To solve the above equations we could pose a small parameter perturbation in \( \eta \). At leading order this is the uniformly curved pipe solution and is obtained by using the finite Hankel transformation (Mullin and Greated 1980). In principle this gives an analytic solution, but numerical techniques are needed to invert the transformation. Therefore we have chosen to solve equations (4.2.8)–(4.2.11) using numerical techniques. The small-\( \alpha \) and uniform-curvature cases can be solved analytically (see Hydon 1994, Mullin and Greated 1980) in terms of polynomials in \( \rho \) multiplied by sine (cosine) functions of time. Therefore we have chosen to approximate the spatial coordinate by finite differences and the time coordinate as a sum of Fourier modes.

4.2.3 Numerical method.

The numerical method we have chosen is the spectral method with a basis of Fourier modes. For simplicity we start by changing the variables as follows

\[
\begin{align*}
    u_1 &= r f(\rho, t), \\
    v_1 &= g(\rho, t), \\
    w_1 &= h(\rho, t), \\
    p_1 &= r q(\rho, t),
\end{align*}
\]

where \( \rho = r^2 \), giving:

\[
\begin{align*}
    2g_\rho + \frac{1}{\rho} (g - h) &= -\eta f, \quad (4.2.12) \\
    4\rho f_{\rho\rho} + 8f_\rho &= -\frac{\alpha^2}{2\pi} f_t - 2u_{0,\rho} g = \alpha^2 \cos 2\pi t - 2u_{0,\rho} \\
    &+ \eta \left( u_0 - \frac{\eta}{G_0} \right) f + \frac{\eta}{G_0} q, \quad (4.2.13) \\
    2pq_\rho + q &= -\frac{\alpha^2}{2\pi} g_t + \frac{1}{\rho} (g - h) - 2h_\rho = G_0 u_{0,\rho}^2 \\
    &- \eta (u_0 + f + 2\rho f) - \eta \left( u_0 - \frac{\eta}{G_0} \right) g, \quad (4.2.14)
\end{align*}
\]
\[ 4\rho h_{\rho\rho} + 4h_{\rho} = \frac{\alpha^2}{2\pi} h_{t} + \frac{1}{\rho} (g - h) - 2g_{\rho} - q = -G_0 u_0^2 \]
\[ + \eta \left( u_0 + f \right) + \eta \left( u_0 - \frac{\eta}{G_0} \right) h , \quad (4.2.15) \]

with the boundary conditions:

\[ f = g = h = 0, \text{ at } \rho = 1; \]

we also require all physical quantities to be finite at \( \rho = 0 \).

The above equations are now rearranged into two equations for \( f \) and \( g \). First we eliminate the pressure terms, \( q \), in (4.2.13) and (4.2.14). Then the terms in \( h \) are eliminated using (4.2.12), giving

\[ \frac{\alpha^2}{2\pi} f_t = \frac{\alpha^2}{2\pi G_0} f_{\rho\rho} + \left( \frac{4\eta^2 \rho^2}{G_0} - 4\rho \right) f_{\rho\rho} + \left( \frac{12\eta^2 \rho}{G_0} - 8 \right) f_{\rho} \]
\[ - \eta \left( \frac{\eta^2 \rho}{G_0} - 1 \right) u_0 - \eta \left( \frac{\eta^2 \rho}{G_0} + 1 \right) f - \frac{\alpha^2}{2\pi G_0} g_t - \frac{\alpha^2}{2\pi G_0} g_{t\rho} \]
\[ + \frac{8\eta^2}{G_0} g_{\rho\rho\rho} + \frac{28\eta}{G_0} g_{\rho\rho} - \frac{2\eta^2 \rho}{G_0} u_0 - \frac{2\eta^2 \rho}{G_0} + \frac{8\eta}{G_0} g_{\rho} \]
\[ + \left( 2u_{0,\rho} - \frac{\eta^2}{G_0} u_0 + \frac{\eta^2}{G_0} \right) g - 2u_{0,\rho} + \eta u_0^2 - \frac{\eta^2}{G_0} u_0 \]
\[ + \alpha^2 \cos 2\pi t = 0, \quad (4.2.16) \]

\[ \frac{4\alpha^2}{\pi} g_{t\rho} + \frac{2\alpha^2}{\pi} g_{\rho\rho} - 61\rho^3 g_{\rho\rho\rho} - 96\rho g_{\rho\rho} + \left( 4\eta \rho u_0 - \frac{4\eta^2 \rho}{G_0} - 96 \right) g_{\rho\rho} \]
\[ + \left( 4\eta \rho u_{0,\rho} + 8\eta u_0 - \frac{8\eta^2}{G_0} \right) g_{t} + 2\eta u_{0,\rho} g + \frac{3\alpha^2}{2\pi} f_t + \frac{\alpha^2}{\pi} f_{t\rho} \]
\[ - 8\eta^2 f_{\rho\rho\rho} - 44\eta f_{\rho\rho} + \eta \left( 2\eta \rho u_0 - \frac{2\eta^2 \rho}{G_0} - 40 \right) f_{\rho} \]
\[ + \eta \left( 2\eta \rho u_{0,\rho} + 3\eta u_0 - \frac{3\eta^2}{G_0} \right) f + 4G_0 u_{0,\rho} u_0 + 2\eta u_0 = 0. \quad (4.2.17) \]

We seek solutions which are periodic in time and so we write \( f \) and \( g \) as Fourier series:

\[ f(\rho, t) = \frac{a_{f0}(\rho)}{2} + \sum_{n=1}^{\infty} \left( a_{fn}(\rho) \cos 2\pi nt + b_{fn}(\rho) \sin 2\pi nt \right), \]
\[ g(\rho, t) = \frac{a_{g0}(\rho)}{2} + \sum_{n=1}^{\infty} \left( a_{gn}(\rho) \cos 2\pi nt + b_{gn}(\rho) \sin 2\pi nt \right), \]

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where the period is obtained from the forcing terms in (4.2.16) and (4.2.17). The spectral approximation is obtained by truncating the summation to $M$ terms. The coefficients of the Fourier series are found by solving the $4(M + 1)$ linear ordinary differential equations that are obtained upon taking the inner products

\[
\begin{align*}
A_F(p) &= 2 < F_{sp}, \cos 2\pi nt >, \\
B_F(p) &= 2 < F_{sp}, \sin 2\pi nt >, \\
A_G(p) &= 2 < G_{sp}, \cos 2\pi nt >, \\
B_G(p) &= 2 < G_{sp}, \sin 2\pi nt >. \\
\end{align*}
\] (4.2.18)

Here, $F_{sp}$ and $G_{sp}$ are the spectrally decomposed versions of (4.2.16) and (4.2.17) respectively. The equations (4.2.18) can be re-written as:

\[
Q_x + R = 0,
\] (4.2.19)

where

\[
x = [b_0(p) ... b_M(p) b_0(p) ... b_M(p) a_0(p) ... a_M(p) a_0(p) ... a_M(p)]^T
\] (4.2.20)

is the vector of coefficients of the two Fourier series, $Q$ is a linear differential operator in $p$ and $R$ is a vector of driving terms.

To solve (4.2.19) we decompose the spatial variables using finite difference techniques and solve the resulting system of linear equations. We choose to use finite differences that have error bounds of $O(h^2)$, where $h$ is the step size. At $p = 0$ we generate boundary equations for (4.2.16) and (4.2.17) by assuming that all physical quantities are finite; as in chapter 3 we use a Frobenius expansion to determine these conditions.

We need to be careful with the choice of finite differences at the boundaries, as $Q$ is a fourth-order differential operator. Therefore, at $p_0 = 0$ we use forward-biased finite differences when decomposing $Q$, and at $p_1 = h$ we use centred-differences for first- and second-order differentials with forward biased differences for all higher orders. At $p_{N-1} = 1 - h$ we use finite differences that incorporate the boundary condition at $p_N = 1$ so the $p_N$ terms are not needed. (See Appendix A for a list of the finite differences used.)

After some manipulation the problem reduces to solving the following

\[
A_x = b,
\] (4.2.21)
where \( b \) denotes the forcing terms and

![Matrix A](image)

The size of matrix \( A \) is \( 4N(M + 1) \times 4N(M + 1) \) with boxed terms \( (4N \times 4N \text{ matrices}) \) indicating a block tri-diagonal structure. Here we need to discuss the structure of \( A \). The first \( 2N \) terms of \( x \) are \((b_{f0i}, b_{g0i} : i = 0..N - 1)\), where \( i \) is the spatial discretization; these are all zero by definition. Therefore, the sub-matrix \( M^* \) has the following form

\[
M^* = \begin{pmatrix}
I & 0 \\
0 & M
\end{pmatrix},
\]

(4.2.22)

where \( I \) is the \( 2N \times 2N \) identity matrix and the entries of \( M \) are the mode-0 coefficients of \((a_{f0i}, a_{g0i} : i = 0..N - 1)\). The first \( 2N \) elements of \( b \) are set to zero ensuring that \( b_{f0i} = b_{g0i} = 0, \forall i \).

The computer algebra package Maple was used to generate the form of each sub-matrix. Although the matrix has a sparse structure, the construction of \( A \) is lengthy and therefore not included. The solution is found by using the NAG routines F07ADF, F07AEF and S17DEF. The routine S17DEF is used to find the values of the complex Bessel functions in \( u_0 \), while F07ADF forms the \( LU \) factorization of \( A \) before F07AEF solves the resulting system of real linear equations.

There are two ways in which to increase the accuracy of the numerical scheme, the first is to increase the number of modes used in the spectral approximation (increase \( M \)), and the second is to decrease the spacial step size, \( h \). The numerical method was checked by increasing \( M \) and decreasing \( h \) until little difference was made to the output. We found that \( M \approx 25 \) and \( h \approx \frac{1}{50} \) produce reasonable results.
4.3 Results.

The wall shear stress (WSS) is found in a similar to that in Chapter 3 i.e.

\[
\text{axial WSS} = -\hat{u}_r|_{r=1} = \frac{K}{\Omega a} u_r|_{r=1},
\]

(4.3.23)

\[
\text{azimuthal WSS} = -\hat{w}_r|_{r=1} = -\frac{\nu}{a^2} w_r|_{r=1},
\]

(4.3.24)

and we evaluate at specific values of $t$; the instantaneous form of the WSS. Following Waters (1996) and Pedley (1980) to find the dimensional form of the WSS

\[
K \approx 2 \times 10^{-1} \text{ms}^{-1}, \quad \Omega \approx 1, \quad a \approx 2 \times 10^{-3} \text{m}, \quad \nu \approx 4 \times 10^{-6} \text{m}^2\text{s}^{-1},
\]

giving

\[
\hat{u}_r \approx 10^5 u_r, \quad \hat{w}_r \approx w_r.
\]

(4.3.25)

First we consider flow in a pipe of uniform curvature ($\eta = 0$) and zero torsion. When $\alpha \ll 1$ the flow is quasi-steady, and therefore the velocity profiles are qualitatively similar to those for steady flow. At $t = 0$ the axial component of velocity is at a maximum, the fluid then slows and by $t = 0.5$ is completely reversed. The secondary velocity has a two-vortex structure throughout this cycle, but the strength is almost zero at $t = 0.25$ (when the unsteady pressure gradient forcing the leading-order flow is zero i.e. $p_{0s} = 0$). The axial component dominates the total WSS. Maximum axial and total WSS occurs at the outside wall and the minimum WSS occurs at the inside wall. The azimuthal WSS has zeros at these points.

When the pressure gradient ($p_{0s}$) is zero there is negligible azimuthal WSS. Figure 4.1 shows the relationship between the instantaneous WSS and the parameter $G_0$. (The "magnitude" of the WSS (the y-axis of Figure 4.1) is rough guide to how $G_0$ effects the WSS and is purely the maximum instantaneous WSS minus the minimum instantaneous WSS.) As the parameter $G_0$ is increased the magnitude of the WSS increases linearly. However, this increase is very small, especially in the axial and total WSS. Therefore, the accuracy of the results presented in Figure 4.1 (a and b) could be considered questionable. However, Figure 4.1 is only meant as a rough indication of the relationship of the WSS to the parameter $G_0$, and as a comparison with the non-uniform case (Figure 4.5); as such it does help to
highlight the fact that the non-uniform curvature plays an important role in the form and magnitude of the WSS. As the Womersley number $\alpha$, is increased the boundary layer / inviscid core structure (first observed by Lyne) forms. This is shown in Figure 4.2.

Figure 4.1: Relationship between the "magnitude" of the wall shear stress (WSS) and the parameter $G_0$ at $t = 0.0$; a. axial; b. azimuthal; c. total (root mean squared): $\alpha = 0.1, \eta = 0.0$.

When $\eta \neq 0$ and $\alpha \ll 1$ the effect of non-uniform curvature is most evident in the secondary velocity and azimuthal wall shear stress (WSS) (Figure 4.3). Due to the non-uniformity of the curvature, the additional driving terms do not all go to zero at $t = 0.25$ (unlike the uniform ($\eta = 0$) case). Therefore the secondary velocity is no longer zero at $t = 0.25$. The most surprising effect of the non-uniformity is seen at $t = 0.5$ where we observe that the secondary velocity field is reversed and hence the azimuthal WSS is inverted (Figure 4.3). Lyne (1971) showed that a reversal of secondary motion occurs along the center-line when $\alpha$ and $R_s$ (the secondary streaming Reynolds number) were large. When $\alpha$ is large the axial velocity has a boundary layer / inviscid core structure (Figure 4.2) and the secondary flow has a 4-vortex structure (Figure 1.8). Therefore, although the secondary flow is reversed along the center-line, at the wall it is orientated in the same direction as for steady flow. Hence the azimuthal WSS is altered in magnitude but not direction. This is where our results differ significantly from previous research. When $\alpha \ll 1$ and $\eta \neq 0$, Figure 4.3 shows that, at $t = 0.5$, the secondary flow is reversed everywhere in the cross-section; this causes the azimuthal WSS to be inverted (with respect to the $\eta = 0$ case). The reason for this is an inversion of the perturbed pressure.
profile which can be seen in Figure 4.4 b. The pressure profile in a uniformly curved pipe remains approximately the same shape throughout a period (Figure 4.4 a.). Therefore, although the strength changes (zero at $t = 0.25$), the direction of the secondary velocity remains unaltered. However, when the pipe has non-uniform curvature the pressure profile is inverted at $t = 0.5$ (when reversal is apparent); this causes a change in the sign of the pressure gradient, and hence a change in direction of the secondary flow. As $G_0$ is increased, the strength of the secondary flow at $t = 0.5$ decreases, and likewise the azimuthal WSS. From Figure 4.5 we see that the amplitude of azimuthal WSS increases (decreases) linearly with $G_0$ at $t = 0$ ($t = 0.5$). However, the relationship between the axial WSS and $G_0$ is no longer linear (Figure 4.5). As $G_0$ increases the dominant driving terms change, from those which occur due to non-uniform geometry to those which occur due to uniform geometry. As $\alpha$ is increased the non-uniformity has a decreased effect on the flow. When $\alpha \sim 4$ the secondary flow is no longer reversed (at $t = 0.5$); however both it and the azimuthal WSS are weaker than in pipes with uniform geometry.
Once $\alpha \geq 4$ all shear stresses increase linearly with $G_0$. Again, this is due to the dominance of the $\alpha^2$ driving terms compared with driving terms occurring due to the non-uniformity. When the boundary layer/core flow structure develops the secondary flow has a greater effect on the WSS. Figure 4.6 shows that, as the strength of the secondary velocity (and azimuthal WSS) increases, the total WSS has two maxima at $t = 0.25$. This effect occurs in both uniform and non-uniform geometries at $\frac{s}{G_0} \approx 0.05$; increasing $G_0$ or $s$ (in the non-uniform case) pushes the two peaks away from the outer wall. Zabielski and Mestel (1998) showed a similar effect in uniform helical pipe flows.

![Figure 4.3: Secondary velocity (top) and azimuthal (azim.) wall shear stress (bottom) (I - inside wall, O - outside wall): $\alpha = 0.1; G_0 = 150; \eta = 0.5$; (WSS plot) $s = 1(\_\_\_), 5(\_\_\_), 15(\cdots)$.](image)

As the arc-length $s$ increases, the magnitudes of the secondary flow and all shear stresses increase; the peak axial velocity is pushed towards the outer wall. This is in agreement with predictions made from the steady-flow case (chapter 3).
Figure 4.4: Profiles of pressure, extending from the centre of the cross-section to the outer wall, against radial distance: a. Uniform curvature; b. Non-uniform curvature: \( t = 0.0(\_\_\_), 0.25(\_\_), 0.5(\_\_), \). Note: in a. the profile is virtually the same at \( t = 0.0 \) and 0.5: \( \alpha = 0.1, G_0 = 150, \eta = 0.5. \)

### 4.3.1 Comments on the numerical method.

The simplest analytical results to which we can compare the numerical method are those for Hydon’s (1994) quasi-steady model. We note that, from Figure 4.7, the numerical method agrees well with Hydon's (analytic) solution for a pipe with uniform curvature. As \( \eta \) (the ‘non-uniformity’ parameter) is decreased, the numerical solution converges to the uniform curvature solution.

When \( \frac{\eta}{\alpha^2} \) becomes significant, the numerical solution breaks down at the origin, \( (\rho = 0) \). This can be explained by means of a crude analytic method. We take our current expansion:

\[
\begin{align*}
u &= u_0 + \epsilon u_1 + \ldots, \\
\end{align*}
\]

and at each order expand in terms of \( \alpha^2 \). At leading order in \( \epsilon \) this gives the quasi-steady solution for unsteady Poiseuille flow, ie.

\[
\begin{align*}
\frac{u_0}{\rho_0} &= -\frac{1}{4} \alpha^2 (r - 1)(r + 1) \cos 2\pi t + \frac{1}{64} \alpha^4 (r - 1)(r + 1)(r^2 - 3) \sin 2\pi t + \mathcal{O}(\alpha^5), \\

\end{align*}
\]

However, if we look at the first-order in \( \epsilon \) solutions we can see where the problems occur:

\[
\begin{align*}
u_1 &= -\alpha^2 \frac{1}{16} r(r - 1)(r + 1) \cos 2\pi t + \mathcal{O}(\alpha^2 \eta), \\

v_1 &= \alpha^2 \eta u_{10} + \mathcal{O}(\alpha^2 \eta^2),
\end{align*}
\]

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Figure 4.5: The "magnitude" of the WSS vs. $G_0$, in a non-uniformly curved pipe: $\alpha = 0.1; \eta = 0.5$.

$$w_1 = \alpha^2 \eta w_{10} + O(\alpha^2 \eta^2),$$

$$p_1 = \alpha^2 \eta p_{10} + O(\alpha^2 \eta^2),$$

where

$$v_{10} = \frac{1}{512} \left(3r^4 + 20r^2 \ln r - 16r^2 + 13\right) \cos 2\pi t,$$

$$w_{10} = \frac{1}{512} \left(-17r^4 + 60r^2 \ln r + 4r^2 + 13\right) \cos 2\pi t,$$

$$p_{10} = \frac{1}{64} r \left(13r^2 - 20 \ln r - 11\right) \cos 2\pi t.$$

From the logarithmic term in $p_1$ we can see that there will be a singularity at $\rho = 0$ in the radial pressure gradient. Therefore $\eta$ has to be taken sufficiently small so that the $G_0$ driving terms are dominant and $\frac{\eta}{G_0}$ is small. It turns out that when $\eta = 0.05$, the lowest value of $G_0$ that we can take before the singularity becomes dominant is $G_0 \sim 10$. Therefore, we cannot study the effects of very small $G_0$. To find out whether this restriction is important, we estimate $G_0$, using the relationship

$$2\epsilon G_0 \sim G_m = 2\alpha^2 R_s,$$

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where $G_m = 2\delta K^2 a^2$ is the Dean-like perturbation parameter used by Mullin and Greated and $R_s$ is the secondary streaming Reynolds number (see Pedley, 1980). Waters (1996) notes that $R_s \approx 650$ and $\alpha \approx 1$ in the extramyocardial coronary arteries, which means that $G_0 \approx O\left(\frac{a^2 R_s}{\varepsilon}\right)$. Therefore, as we have assumed that $\epsilon(s) < 1$, $G_0$ must be large and hence the breakdown of the numerical solution at certain parameter values is not a problem.

The current numerical method has taken a long time to complete due to the lengthy (and messy) computer algebra involved. Extending the numerical method to include non-uniform torsion would be (in principle) straightforward. At $O(\epsilon^2)$, the first order at which torsion appears, a spectral approximation for time would be used. However, as the spectral method was also used at $O(\epsilon)$ the resulting matrix would not be sparse. Although in principle this is not a problem, in practice this would produce very long programming code (i.e. at present the C-program is $\sim 5000$ lines long, and we would expect this to increase by an order of magnitude). Hence, due to time restrictions, helicity has not been incorporated in this model.
Figure 4.7: Comparison of numerical solution for $u_1$ (-) and the analytic equivalent (o) (from Hydon, 1994) for $\alpha = 0.01$, $G_0 = 50$ and $t = 0.0$. 
Chapter 5

Flow in a uniform helix: unsteady pressure gradient.

5.1 Introduction.

In this chapter we discuss unsteady flow in a uniform helix. The Navier-Stokes equations are used to derive the flow as an expansion in powers of $\alpha^2$, as described in Hydon (1994). We discuss how torsion affects the velocity field and particle transport.

5.2 Unsteady flow in uniformly twisted pipes.

5.2.1 Extension of Hydon’s method.

Hydon (1994) derived a small-$\alpha$ expansion for the flow in a toroidal pipe of uniform curvature, and used this expansion to study the motion of fluid particles. Following Pedley (1980) and Hydon (1994) we non-dimensionalize the unsteady Navier-Stokes equations (2.3.15)-(2.3.18) as follows:

\[
\begin{align*}
(u, v, w) &\rightarrow (Ku, \frac{v}{a}, \frac{w}{a}), \\
(s, r) &\rightarrow (as, ar), \\
\dot{\rho} &\rightarrow A_p + \dot{\rho}Ks \cos(\Omega t) + \frac{\dot{\rho}}{2a^2} p, \\
\dot{t} &\rightarrow \frac{2x}{\Omega}, \\
\kappa &\rightarrow a^{-1} \epsilon,
\end{align*}
\]

(5.2.1)
and we define a parameter \( G = \frac{\alpha^2 R^2}{\nu^4 R^4} \sim \mathcal{O}(1) \). Ignoring all higher order terms in \( \epsilon \) gives the following equations:

\[
-GD_G \lambda u_\xi + \frac{1}{r} w_\xi + v_r - \frac{1}{r} w_\xi + v_r = 0, \tag{5.2.2}
\]

\[
\frac{\alpha^2}{2\pi} u_t - \frac{1}{r} Gd_G \lambda w_\xi + v_r + \frac{1}{r} w_\eta - \frac{1}{r} w^2 - Gu^2 \sin \xi = \alpha^2 \cos(2\pi t) + u_{rr} + \frac{1}{r} u_r + u_{\xi \xi}, \tag{5.2.3}
\]

\[
\frac{\alpha^2}{2\pi} v_t - \frac{1}{r} Gd_G \lambda w_\xi + v_r + \frac{1}{r} w_\eta - \frac{1}{r} w^2 - Gu^2 \sin \xi = -D_G p_r - \frac{1}{r^2} (r w_\xi + w_\xi - v_\xi), \tag{5.2.4}
\]

\[
\frac{\alpha^2}{2\pi} w_t - \frac{1}{r} Gd_G \lambda w_\xi + v_r + \frac{1}{r} w_\eta - \frac{1}{r} w^2 - Gu^2 \sin \xi = -D_G r p_\xi + w_{rr} + \frac{1}{r} w_r - \frac{1}{r^2} w + \frac{1}{r^2} v_r - \frac{1}{r} v_{\xi \xi}, \tag{5.2.5}
\]

where \( D_G = \frac{\nu^3}{4 K} \) and the underlined terms are due to helicity. Again, the boundary conditions are that the velocities are zero at the wall and all physical quantities are finite throughout the cross-section. We eliminate the pressure terms and assume that \( \alpha \ll 1 \), leading to the following power series expansion:

\[
u = \alpha^2 u_0 + \alpha^4 u_1 + \alpha^6 u_2 + \alpha^8 u_3 + \mathcal{O}(\alpha^{10}), \tag{5.2.6}
\]

\[
v = \alpha^4 v_1 + \alpha^6 v_2 + \alpha^8 v_3 + \mathcal{O}(\alpha^{10}), \tag{5.2.7}
\]

\[
w = \alpha^4 w_1 + \alpha^6 w_2 + \alpha^8 w_3 + \mathcal{O}(\alpha^{10}). \tag{5.2.8}
\]

We solve the equations at each order in \( \alpha^2 \) by the same method as that used in Chapter 3; we assume a polynomial solution in \( \rho \).

For simplicity we take \( \lambda_D = D_G \lambda \), and therefore the \( \mathcal{O}(\alpha^2) \) equation is

\[
u_{rr} + \frac{1}{r} \nu_{r} = -\alpha^2 \cos 2\pi t, \tag{5.2.9}
\]

which gives

\[
u_0 = \frac{1}{4} (r^2 - 1) \cos 2\pi t. \tag{5.2.10}
\]
By substitution, the $O(\alpha^4)$ equations are found to be

$$u_{1,rr} + \frac{1}{r} u_{1,r} + \frac{1}{r^2} u_{1,\xi\xi} = -\frac{1}{4} (1 - r^2) \sin 2\pi t,$$  \hspace{1cm} (5.2.11)

$$rw_{1,rrr} + 2w_{1,rr} - \frac{1}{r} w_{1,r} + \frac{1}{r^2} w_{1,\xi\xi} - 2v_{1,rr\xi} - \frac{1}{r} v_{1,r\xi} - \frac{1}{r^2} v_{1,\xi\xi\xi} - \frac{2}{r^2} v_{1,\xi}$$

$$+ G\lambda_D u_{1,\xi\xi} + \frac{2G\lambda_D}{r} u_{1,\xi\xi} = \frac{1}{4} r^2 (1 - r^2) \cos^2 2\pi t \cos \xi,$$  \hspace{1cm} (5.2.12)

$$v_{1,r} + \frac{1}{r} v_{1} + \frac{1}{r} w_{1,\xi} = G\lambda_D u_{1,\xi}.$$

(5.2.13)

On solving (5.2.11)–(5.2.13) we find:

$$u_1 = \frac{1}{64} \left( r^2 - 1 \right) \left( r^2 - 3 \right),$$

$$v_1 = -\frac{G}{4608} \left( r^2 - 1 \right)^2 \left( r^2 - 2 \right) \sin \xi \left( \cos 2\pi t \right)^2,$$

$$w_1 = -\frac{G}{4608} \left( r^2 - 1 \right) \left( 7r^4 - 23r^2 + 4 \right) \cos \xi \left( \cos(2\pi t) \right)^2.$$  \hspace{1cm} (5.2.14)

Substituting into the $O(\alpha^6)$ equations gives

$$u_{2,rr} + \frac{1}{r} u_{2,r} + \frac{1}{r^2} u_{2,\xi\xi} = \frac{G}{9216} r (r^2 - 1)^2 (r^2 - 2) \cos^3 2\pi t \sin \xi$$

$$+ \frac{1}{64} (r^2 - 1)(r^2 - 3) \cos 2\pi t,$$  \hspace{1cm} (5.2.15)

$$rw_{2,rrr} + 2w_{2,rr} - \frac{1}{r} w_{2,r} + \frac{1}{r^2} w_{2,\xi\xi} - 2v_{2,rr\xi} - \frac{1}{r} v_{2,r\xi} - \frac{1}{r^2} v_{2,\xi\xi\xi} - \frac{2}{r^2} v_{2,\xi}$$

$$+ G\lambda_D u_{2,\xi\xi} + \frac{2G\lambda_D}{r} u_{2,\xi\xi} = \frac{G^2\lambda_D}{9216} r^2 (r^2 - 1) (31r^4 - 95r^2 + 40) \cos^3 2\pi t \sin \xi$$

$$+ \frac{G}{192} r^2 (13r^4 - 42r^2 + 27) \sin 2\pi t \cos \xi,$$  \hspace{1cm} (5.2.16)

$$v_{2,r} + \frac{1}{r} v_{2} + \frac{1}{r} w_{2,\xi} = G\lambda_D u_{2,\xi},$$

(5.2.17)

and we find that

$$u_2 = \frac{1}{2304} (r^2 - 1) \left( r^4 - 8r^2 + 19 \right) \cos 2\pi t$$

$$+ \frac{G}{737280} r (r^2 - 1) \left( r^6 - 9r^4 + 21r^2 - 19 \right) \sin \xi \left( \cos 2\pi t \right)^3,$$

$$v_2 = \frac{G}{737280} (r^2 - 1)^2 \left( 13r^4 - 114r^2 + 299 \right) \sin \xi \sin 2\pi t \cos 2\pi t$$

$$- \frac{G^2\lambda_D}{4423680} (r^2 - 1)^2 \left( r^6 - 7r^4 + 15r^2 - 13 \right) \cos \xi \left( \cos 2\pi t \right)^3,$$

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\[ w_2 = \frac{G}{737280} (r^2 - 1) \left( 117r^6 - 863r^4 + 1837r^2 - 299 \right) \cos \xi \sin 2\pi t \cos 2\pi t + \frac{G^2 \lambda_D}{4423680} \left( r^2 - 1 \right) \left( 17r^8 - 124r^6 + 266r^4 \right) - \frac{224r^2 + 13}{\sin \xi (\cos 2\pi wt)^3}. \]  

(5.2.18)

Finally the \( \mathcal{O}(\alpha^3) \) equations are

\[ u_{3,rr} + \frac{1}{r}u_{3,r} + \frac{1}{r^2}u_3, \xi \xi = \frac{G^2 \lambda_D}{4423680} \left( r^2 - 1 \right)^2 \left( 2r^6 - 17r^4 + 39r^2 - 35 \right) \cos^4 2\pi t \cos \xi \]

\[ - \frac{G}{491520} r^2 \left( 13r^6 - 107r^4 + 273r^2 - 191 \right) \sin 2\pi t \sin \xi - \frac{1}{2304} \left( r^2 - 1 \right) \sin 2\pi t, \]  

(5.2.19)

\[ ru_{3,rrr} + 2u_{3,rr} - \frac{1}{r}u_{3,r} + \frac{1}{r^2}u_3 = 2u_{3,rr\xi} - \frac{1}{r}u_{3,r\xi} - \frac{1}{r^2}u_3, \xi \xi - \frac{1}{r^2}u_3, \xi \]

\[ + G \lambda_D u_{3,\xi \xi} + \frac{2G \lambda_D}{r} u_3, \xi \xi = \frac{G^3 \lambda_D}{4423680} r^2 \left( r^2 - 1 \right) \left( 55r^8 - 377r^6 + 808r^4 - 712r^2 + 174 \right) \cos^4 2\pi t \cos \xi \]

\[ - \frac{G^2 \lambda_D}{1474560} r^2 \left( 1203r^8 - 8680r^6 + 20940r^4 - 18624r^2 + 5057 \right) \cos^2 2\pi t \sin 2\pi t \sin \xi \]

\[ + \frac{G^2}{2211840} \left( r^2 - 1 \right) \left( 5r^6 - 67r^4 + 158r^2 - 162 \right) \cos^4 2\pi t \sin \xi \cos \xi \]

\[ + \frac{G}{7680} r^2 \left( 50r^6 - 330r^4 + 675r^2 - 362 \right) \cos^2 2\pi t \cos \xi \]

\[ - \frac{G}{46080} r^2 \left( 155r^6 - 960r^4 + 1800r^2 - 896 \right), \]  

(5.2.20)

\[ v_{3,r} + \frac{1}{r}v_3 + \frac{1}{r}w_{3,\xi} = G \lambda_D u_3, \xi \]  

(5.2.21)

on solving gives

\[ u_3 = - \frac{1}{147456} \left( r^2 - 1 \right) \left( r^6 - 15r^4 + 93r^2 - 211 \right) \sin 2\pi t \]

\[ - \frac{G}{58982400} r \left( r^2 - 1 \right) \left( 13r^8 - 167r^6 + 783r^4 - 1537r^2 + 1328 \right) \sin \xi \sin 2\pi t \cos \xi \]

\[ + \frac{G^2 \lambda_D}{7431782400} r \left( r^2 - 1 \right) \left( 20r^{10} - 274r^8 + 1301r^6 - 3249r^4 + 4381r^2 - 2969 \right) \cos \xi \cos 2\pi t \]

\[ \cos 2\pi t \]  

(5.2.22)
\[ v_3 = -\frac{G}{88473600} \left( r^2 - 1 \right)^2 \left( 31r^6 - 418r^4 + 2133r^2 - 4276 \right) \sin \xi \\
+ \frac{G}{2949120} \left( r^2 - 1 \right)^2 \left( 2r^6 - 29r^4 + 165r^2 - 365 \right) \sin \xi (\cos 2\pi t)^2 \\
- \frac{G}{59454259200} \left( r^2 - 1 \right)^2 \left( 5r^8 - 134r^6 + 777r^4 - 2792r^2 \right) \\
+ 4979 \cos 2\xi (\cos 2\pi t)^4 \\
+ \frac{G^2\lambda_D}{4954521600} \left( r^2 - 1 \right)^2 \left( 116r^8 - 1420r^6 + 6389r^4 \right) \\
- 11702r^2 + 8567 \cos \xi (\cos 2\pi t)^2 \sin 2\pi t \\
+ \frac{G^2\lambda_D^2}{24970788864000} \left( r^2 - 1 \right)^2 \left( 825r^{10} - 10446r^8 + 47961r^6 \right) \\
- 117072r^4 + 152035r^2 - 90418 \sin \xi (\cos 2\pi t)^4 , \tag{5.2.23} \]

\[ w_3 = -\frac{G}{88473600} \left( r^2 - 1 \right) \left( 341r^8 - 3979r^6 + 17021r^4 - 27779r^2 \right) \cos \xi \\
+ \frac{G}{2949120} \left( r^2 - 1 \right) \left( 22r^8 - 275r^6 + 1300r^4 - 2320r^2 \right) \cos \xi (\cos 2\pi t)^2 \\
+ 365 \cos \xi (\cos 2\pi t)^4 \\
+ \frac{G^2\lambda_D}{59454259200} \left( r^2 - 1 \right) \left( 35r^{10} - 829r^8 + 4421r^6 - 13499r^4 \right) \\
+ 20521r^2 - 4979 \sin 2\xi (\cos 2\pi t)^4 \\
- \frac{G^2\lambda_D^2}{4954521600} \left( r^2 - 1 \right) \left( 2600r^{10} - 30692r^8 + 133213r^6 - 242967r^4 \right) \\
+ 189493r^2 - 8567 \sin \xi (\cos 2\pi t)^2 \sin 2\pi t \\
+ \frac{G^3\lambda_D^2}{24970788864000} \left( r^2 - 1 \right) \left( 12375r^{12} - 144873r^{10} + 621585r^8 \right) \\
- 1389375r^6 + 1649605r^4 - 908195r^2 + 90418 \cos \xi (\cos 2\pi t)^4. \tag{5.2.24} \]
The above solution includes effects due to convective inertia at $O(\alpha^8)$, i.e. $G^2$ terms, and underlined terms are those due to torsion.

We can now state the asymptotic expansions for the (dimensionless) wall shear stress:

axial WSS = $-u_r|_{r=1}$

$$= \frac{1}{2} \alpha^2 \cos 2\pi t + \frac{1}{16} \alpha^4 \sin 2\pi t - \alpha^6 \left( \frac{1}{96} \cos 2\pi t - \frac{G}{61440} \cos^3 2\pi t \sin \xi \right)$$

$$- \alpha^8 \left( \frac{11}{6144} \sin 2\pi t - \frac{7G}{491520} \sin 2\pi t \cos^2 2\pi t \sin \xi \right)$$

$$- \frac{79G^2 \lambda_D}{371589120} \cos^4 2\pi t \cos \xi \right),$$

(5.2.25)

azimuthal WSS = $-\omega_r|_{r=1}$

$$= -\frac{G}{192} \alpha^4 \cos^2 2\pi t \cos \xi$$

$$- \alpha^6 \left( \frac{11G}{5120} \sin 2\pi t \cos 2\pi t \cos \xi - \frac{G^2 \lambda_D}{552960} \cos^3 2\pi t \sin \xi \right)$$

$$- \alpha^8 \left( \frac{253G}{1105920} \cos \xi - \frac{227G}{368640} \cos^2 2\pi t \cos \xi \right.$$

$$+ \frac{G^2}{5242880} \cos^4 2\pi t \sin 2\xi$$

$$+ \left(- \frac{359G^2 \lambda_D}{20643840} \sin 2\pi t \cos^2 2\pi t \sin \xi \right.$$

$$- \frac{163G^3 \lambda_D^2}{2972712960} \cos^4 2\pi t \cos \xi \right) \right).$$

(5.2.26)

To convert to dimensional parameters one would use the following:

$$\hat{u}_r = \frac{K}{\Omega a} u_r, \quad \hat{w}_r = \frac{\nu}{a^2} w_r,$$

(5.2.27)

where (from Waters (1996) and Pedley (1980)) we can estimate the parameters as in §4.3.

### 5.3 Results.

#### 5.3.1 General.

The range of parameters we have studied are $\alpha < 1$, $20 < G < 5000$ and $\lambda < 5$. The parameters used in this results section are qualitatively similar to results found
Figure 5.1: Secondary velocity vector field plots for a pipe with uniform curvature and torsion. $I$ and $O$ denote the innermost and outermost parts of the pipe wall respectively: $G = 50$, $\alpha = 0.9$ and $\lambda = 0.9$.

Figure 5.2: Wall shear stress in a uniform helix with oscillatory pressure gradient. $I$ and $O$ denote the innermost and outermost parts of the pipe wall respectively: $G = 50$, $\alpha = 0.9$ and $\lambda = 0.9$.

at lower values of $\alpha$ and $\lambda$.

The flow is quasi-steady (as $\alpha \ll 1$), and hence we would expect the velocity profiles to be qualitatively similar to those in chapter 3 (steady-flow). Figure 5.1 shows the secondary velocity vector field for different values of $t$. The effect of torsion is to break the symmetric structure of the secondary flow seen in pipes without torsion. As the fluid flows up the pipe (i.e. into the page) the 2-vortex structure is skewed in the direction of the torsion (Figure 5.1, $t = 0; 0.25$). The extreme skewing seen at $t = 0.25$ is not as significant as it looks. The strength of the vector field at $t = 0.25$ is very small; this is reflected in the negligible azimuthal wall shear stress at this moment in time (Figure 5.2). When the fluid flows in the opposite direction ($t = 0.5$) the 2-vortex structure is skewed in the opposite sense. The axial velocity is not significantly affected by the torsion. This is not surprising as torsion first has an effect at $O(\alpha^3)$ for the axial velocity and at $O(\alpha^6)$ for the secondary velocity.

The axial component of the WSS dominates the overall distribution (Figure 5.2).
As torsion has virtually no effect on the axial velocity, the total WSS is at a maximum at the outer wall (O) and at a minimum at the inner wall (I). In curved pipe flow the azimuthal WSS is zero at the inner and outer walls. The introduction of torsion markedly alters the distribution of the azimuthal component of the WSS, as seen in figure 5.3 which shows contours of equal azimuthal WSS over a single period. For the zero-torsion ($\lambda = 0$) case it is seen that the azimuthal WSS is always zero at the inside and outside bends. In addition, the azimuthal WSS is negligible over the whole circumference at $t \approx 0.35 + 0.5(n + 1)$. The addition of torsion ($\lambda \neq 0$) breaks the symmetry seen in curved pipe flow. The maximum (and minimum) azimuthal shear stresses now appear at different sites and times during a single period (in agreement with Zabielski and Mestel, 1998b). When $G_0$ is increased the effect of torsion becomes less significant. When $G_0 = 5000$ (the value used by Hydon) there is no noticeable difference between the velocity profiles in the uniform torus and the uniform helix. The effect on the flow of particles is discussed in §5.4.

Figure 5.3: Contours of azimuthal wall shear stress for $G_0 = 50$, $\alpha = 0.9$ and $\lambda = 0$ (left); 1.4 (right).
5.4 Particle tracking.

In this section, we examine how torsion affects the motion of a particle. In §5.4.1 we reproduce and extend the results of Hydon (1994) to include torsion.

5.4.1 Results.

![Image of trajectories](image)

Figure 5.4: A sample of trajectories for the uniform curvature model: A Poincaré section at $t = \frac{1}{2}k$, $k = 1, 2, ...25000$: $G = 2000$ and $\alpha = 1$.

In his paper, Hydon (1994) shows how the perturbation terms effect the integrable secondary flow ($v_1, w_1$) by examining the equations of motion of a fluid "particle", i.e. the Lagrangian equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t), \quad \mathbf{x}(0) \text{ given.} \quad (5.4.28)$$

To convert into cartesian coordinates $(x, y)$ and then into C-programming code we used the computer algebra package Maple. Here, we give the method of derivation of the Lagrangian equations. First, we note that we can define a psuedo-stream function,

$$\psi = \psi_0 + \alpha^2 \psi_1 + \mathcal{O}(\alpha^4), \quad (5.4.29)$$
that satisfies equation (5.2.2) (see Kao, 1987, Germano, 1989, Tuttle, 1990), such that:

\[ v = -\frac{1}{r} \frac{\partial \psi}{\partial \xi}, \quad w - G \lambda \partial r = \frac{\partial \psi}{\partial r}, \]

(5.4.30)

and that (see also Hydon 1994)

\[ \frac{dx}{dt} = \frac{2\pi}{\alpha^2} \frac{\partial \psi}{\partial y}, \]
\[ \frac{dy}{dt} = -\frac{2\pi}{\alpha^2} \frac{\partial \psi}{\partial x}, \]

(5.4.31, 5.4.32)

with the decoupled equation for longitudinal motion

\[ \frac{ds}{dt} = \frac{2\pi}{\alpha^2} \left( \frac{G}{\epsilon} \right)^{1/2} u(x(t), y(t)) . \]

(5.4.33)

For the present model, \((x, y) = (r \sin \xi, r \cos \xi)\) and hence the leading order term for the stream function is given by:

\[ \psi_0 = -\frac{1}{4608} \alpha^4 G y \left( x^2 - 4 + y^2 \right) \left( -1 + x^2 + y^2 \right)^2, \]

(5.4.34)

which is equivalent to that of Hydon; as one would expect as this is the first correction due to curvature alone.

For our flow we use the numerical integration package *DsTool* to solve (5.4.28). We have used a 4th-order Runge-Kutta method with time-step size \(10^{-6}\), which
is sufficiently small to avoid significant numerical dissipation. This was done by decreasing the step size until no significant difference in output was detected. We discuss the same parameter range as in §5.3, parameters in this section were chosen to demonstrate the effects of torsion as forcibly as possible without being qualitatively different to results at lower parameter values.

Hydon showed that, where the period of the unperturbed secondary orbit is commensurate with the period of the perturbation, the orbit breaks into a structure of “islands” of regular motion which are interwoven with chaotic regions. Figure 5.4 shows a Poincaré section displaying the islands which appear in place of the orbits of period \( \frac{1}{2}, 1, \frac{3}{2} \) and 2. The chaotic regions that surround the islands and the KAM tori within are not shown (for these see Hydon, 1994). Hydon’s most important result is that resonance can occur between the perturbed secondary motion and the longitudinal flow; in fact resonance occurs wherever the resulting island chain has an even number of islands. This can be seen in the time series plot (Figure 5.5) where the two- and four-island chain trajectories (green and yellow) show resonant transport.

Figure 5.6: A sample of trajectories for the model with torsion: a Poincaré section at \( t = k, k = 1, 2, ... 25000 \): \( G = 400, \alpha = 0.5 \) and \( \lambda = 5 \).

We now discuss the effects due to helicity of the pipe. The first significant detail to notice is that at \( O(\alpha^6) \) in the transverse velocities there is a \( (\cos 2\pi t)^3 \) term. This means that when we plot a Poincaré section for the helical pipe we must sample
every period rather than every $\frac{1}{2}$ period. The most striking difference between the Poincaré section for curved pipe flow (Figure 5.4) and that for flow in a helical pipe (Figure 5.6) is breaking of the symmetry due to helicity. Most islands (and KAM tori) in the top half of the cross-section are larger than their counterparts in the lower half. The other major difference is that island chains with an even or an odd number of islands give rise to longitudinal transport.

When $G \approx 500$ most of the particles are on chaotic trajectories (Figure 5.8). However, two new island regions are now apparent; as well as the two small regions at the top and lower part of the cross-section, there are now two more either side of the origin on the centre-line. Figures 5.9 and 5.10 show the trajectories and time series plots for two initial conditions; one leading to a chaotic trajectory (red) and the other a non-chaotic trajectory (blue). The non-chaotic trajectory behaves in the expected manner; it stays on the orbit and is transported longitudinally along the pipe. The chaotic trajectory moves on the outside of a island for some time (dense region of the Poincaré section; almost linear region of the time series) and then it moves quickly away from the island and then the motion becomes chaotic.

When $\alpha^2 = 1$ and $G = 500$ no islands are visible and all trajectories appear chaotic (Figure 5.11).
Figure 5.8: A sample of trajectories for the model with torsion: a Poincaré section at $t = k$, $k = 1, 2, \ldots 25000$: $G = 500$, $\alpha = 0.5$ and $\lambda = 5$.

Figure 5.9: Two trajectories for the model with torsion (red: chaotic, blue: non-chaotic): a Poincaré section at $t = k$, $k = 1, 2, \ldots 25000$: $G = 500$, $\alpha = 0.5$ and $\lambda = 5$. 
Figure 5.10: Time series showing longitudinal transport of particles whose secondary orbits are depicted in Figure 5.9: non-chaotic trajectory (left), chaotic trajectory (right).

Figure 5.11: Two trajectories for the model with torsion: A Poincaré section at $t = k$, $k = 1, 2, ..., 25000$: $G = 400$, $\alpha^2 = 1$ and $\lambda = 5$. 
Chapter 6

Discussion and physiological applications.

The aim of this thesis was to model flow in twisted pipes. The motivation for this work was a desire to see how non-uniform geometry affects the distribution of flow and wall shear stress (WSS) in arteries susceptible to atherosclerosis.

We have modelled the arteries as rigid-walled, twisted pipes whose curvature and torsion vary slowly with longitudinal position. The pipes are constructed by fitting a rigid, circular, boundary (of uniform radius) around a prescribed centreline. Blood is treated as an incompressible, Newtonian fluid.

Blood flow in the arteries is driven by a highly pulsatile pressure gradient. To understand the geometrical effects, we have modelled the flow with both steady and oscillatory pressure gradients.

In this thesis, we have extended the steady-flow calculations of Germano to incorporate non-uniform geometry, and then studied the oscillatory flow problem in the same geometry.

6.1 Parameter values.

From Waters (1996) and Pedley (1980) we have the following data for the extramyocardial coronary arteries
kinematic viscosity of blood \( \nu \approx 4 \times 10^{-5} \text{m}^2\text{s}^{-1} \),
period of the heart beat \( T \approx 1 \text{s} \),
diameter of vessel \( a \approx 2 \times 10^{-3} \text{m} \),
uniform curvature \( \kappa_0 \approx 0.07 \),
Reynolds number \( Re \approx 150 \),
steady pressure gradient \( G \approx 0.03 \),
Womersley parameter \( \alpha^2 < 15 \),
secondary streaming Reynolds number \( R_s \approx 650 \),
the amplitude parameter \( G_0 \approx \frac{a^2 R_s}{\kappa_0} \approx 9000 \),
frequency \( \Omega \approx 1 \).

Using computer-tomographically reconstructed images, Pao et al. (1992) constructed a wire frame model of the left coronary artery tree of a dog. They showed that the approximate range of the torsion-to-curvature ratio was \( \lambda \in (0, 2) \).

All the above estimations are from measurements of coronary arteries. In §6.2 and §6.3 we compare our results with the above parameters. Other parameter ranges may occur in other arteries, e.g. the brain, which has vessels that display highly non-uniform geometry. For smaller arteries, both \( \alpha^2 \) and \( Re \) will be smaller than the above values. However, there is insufficient data on other vessels in the arterial system to enable comparison with the current work.

### 6.2 Steady flow.

The Navier-Stokes equations for pipes with non-uniform geometry have been constructed in accordance with Germano's coordinate system for a helical pipe. These equations have been solved in the limit of small, slowly-varying curvature and torsion. The solution was found to depend on five parameters: the Reynolds number \( Re \) and the geometry parameters \( \kappa_0, \tau_0, \eta \) and \( \beta \).

From the numerical values given in §6.1, we note that the assumptions of small \( \kappa_0 \) and small \( \tau_0 \) are reasonably well satisfied, but the (necessary) additional assumption of small \( Re \) is not satisfied. The asymptotic expansion and the numerical solution in chapter 3 converge for \( Re < 100 \). However, in practice we have taken \( Re < 50 \) in the numerical solution to restrict computation time. We have also assumed that \( \eta \) and \( \beta \) are small; we have not found any data to disprove or prove this assumption.
6.2.1 The effect of non-uniformity on the velocity profiles and WSS. (See chapter 3 for velocity plots and WSS.)

The wall shear stress (WSS) in a weakly curved pipe is dominated by its axial component. When the pipe has weakly non-uniform curvature (with $\epsilon(s) = \kappa_0 e^{\omega s}$) the components of WSS increase in magnitude; the azimuthal component is slightly more affected by the non-uniformity than the axial component. As the arc-length increases, the region of peak axial velocity is forced closer to the outside wall; the secondary velocity increases in magnitude but has unaltered form. (When $Re \sim 1$ the viscous driving terms dominate the flow and the region of peak axial velocity is forced closer to the inside wall; and the wall shear stress is altered accordingly.

The azimuthal WSS component is dramatically increased by the introduction of torsion. For a uniform pipe, the introduction of torsion breaks the symmetric structure (of flow without torsion) and skews the velocity profiles in the direction of the torsion: the axial component of velocity is also deformed in shape. The azimuthal WSS has an increasing effect on the total WSS as the arc-length ($s$) increases, causing a "bump" in the total WSS distribution.

When a pipe has uniform curvature and non-uniform torsion the axial components of velocity and WSS are skewed more than the azimuthal components. The axial component of velocity is not only skewed, but is also increasingly deformed in shape. The azimuthal velocity becomes more skewed in the direction of torsion, but its strength is only slightly increased. The axial component of WSS is skewed more than the azimuthal component and this is highlighted in the altered distribution of the total WSS.

To conclude, we can say that as curvature increases, the axial velocity is forced towards the outer wall and the secondary velocity field is increased in strength. Increasing torsion skews the components of velocity in the direction of increasing torsion. From (3.2.44) and (3.2.45) we can see that as soon as torsion is introduced, for moderate values of $Re$, the azimuthal WSS is increased.
6.3 Unsteady flow.

The unsteady Navier-Stokes equations, as derived in chapter 2, are used to model oscillatory flow in non-uniform pipes. This problem proved to be much more difficult than the steady-flow model. We tackled this problem in two ways: (i) by studying the small curvature (e), general frequency (a) case, and (ii) by studying the small a limit. The main objective from (i) was to understand how the non-uniform geometry affects the velocity profiles and WSS, while in (ii) we wished to extend the particle tracking work of Hydon (1994) to incorporate the effects of torsion. The assumption of small curvature and small torsion are in accordance with the numerical values given in §6.1. The numerical code in chapter 4 is valid for $G_0 > 10$ and for all values of $a$ for which the assumed solution is stable; in practice we take $a < 12$ and $G_0 < 200$ due to restrictions on computation time. These additional assumptions are not at odds with the values given above. When extending Hydon’s particle tracking model we have used $G_0 < 500$; this is not in accordance with the numerical values in §6.1, but allows us to compare Hydon’s results with ours.

When the pipe is non-uniformly curved and $a < 2$, flow reversal occurs. This type of flow reversal is different to that seen by previous authors (e.g. Lyne, 1970) because it is due to an inversion of the perturbed pressure profile, whereas Lyne describes reversal along the centreline which is due to the formation of a boundary-layer structure. As $a$ is increased the effect of the non-uniformity weakens and the velocity profiles become approximately the same as for oscillatory flow in a uniformly curved pipe. When $a > 4$ the boundary-layer structure forms and when $a \approx 0.05$ the azimuthal WSS has a noticeable effect on the total WSS (at $t = 0.25$): namely that a double peak is seen in the total WSS (this is true for both uniform and non-uniform pipes). As $G_0$ becomes small ($\sim 10$) the numerical method breaks down for non-uniform geometry, due to the occurrence of a singularity at the origin. A somewhat crude analysis shows the occurrence of logarithmic terms in the pressure and transverse velocities at first order in $e$. To get around this we would suggest some form of matched asymptotic expansion. This seems straightforward, but there is no obvious occurrence of a boundary layer at $\rho = 0$. However, as $G_0$ decreases, the two vortices in the secondary flow move closer together and could cause a complicated flow structure when they interact. As this
problem only occurs when the curvature is non-uniform, we initially tried re-scaling \( s \), but this did not get rid of the singularity, it merely pushes it to a higher order. We suggest that a possible way forward would be to non-dimensionalize the secondary velocity differently close to the origin in order to match the two different solutions (see Hinch, 1991). Lack of time has not allowed us to follow this idea through.

Our work shows that, for small curvature, torsion and frequency, the azimuthal WSS rises (from zero) at the inside wall during a period. From the results from chapter 2, we postulate that larger curvature and torsion would cause the azimuthal WSS to have a greater effect on the total WSS and hence should converge to the results of Zabielski and Mestel.

We have extended Hydon's particle tracking model to incorporate torsion. The islands seen at low values of \( G_0 \) are larger than equivalent islands in curved-pipe flow. The symmetric structure of curved-pipe flow is broken, and the islands in the top half of the cross-section are larger than those in the lower half. As \( G \) is increased, the total area where islands occur is decreased and more of the trajectories become chaotic. When \( G = 500 \) and \( \alpha^2 = 1 \) no islands are seen and all trajectories appear chaotic. From these results it seems that for low \( G \) and \( \alpha \), the helical pipe produces increased longitudinal transport, but at high \( G \) and \( \alpha \) a curved pipe is more effective at generating longitudinal transport.

### 6.4 Possible applications to physiology.

It is now widely accepted that regions susceptible to atherosclerosis are correlated with regions of low wall shear stress and in regions where wall shear stress changes direction in the course of the cardiac cycle.

The steady flow calculations have shown that torsion skews and increases the strength of the secondary velocity field, which in turn alters the azimuthal wall shear stress similarly. For oscillating flows, we agree with other authors that, torsion raises (from zero) the wall shear stress in certain regions of the pipe. This has a genuine application to bypass graft surgery. It suggests that adding a twist to a graft will reduce the possibility of plaque build-up. The steady flow experiments
of Caro (1998) have already shown that torsion reduces the downstream stagnation region, our results (and those of Zabielski and Mestel) imply that torsion is also beneficial in oscillating flows.

For oscillating flows in non-uniformly curved pipes we have shown that, at low frequencies, the secondary velocity and azimuthal wall shear stress are reversed at a specific time during a period. We have not been able to study large curvature and large frequencies, but we conjecture that a similar reversal effect would be seen for higher frequencies if the curvature is also sufficiently increased. It would therefore be possible for a bypass graft to increase the chances of a build-up of atherosclerotic plaques (assuming the parameters are satisfied!).

### 6.5 Future work.

In this thesis we have studied steady and oscillatory flow in pipes with non-uniform curvature and torsion. The parameter regimes we have studied are not exhaustive. In particular solutions for medium and large values of curvature and torsion would be interesting, as this would give better approximations to highly curved areas of the arterial tree (e.g. the aortic arch). We have been quite restricted in the size of both $Re$ and $G_0$ due to the form of the asymptotic expansions and the numerical methods. However, the parameter ranges we have used would be important in testing any future numerical simulations.

The natural extension to the current work is to study oscillatory flow in non-uniformly twisted pipes. Of particular interest would be the effect of helicity on the flow reversal described in chapter 4. If the torsion increases the strength of the secondary vector field the azimuthal wall shear stress may have an important impact in the overall distribution of the wall shear stress.

We have studied the effects of steady and oscillatory pressure gradients. In the circulatory system, the pulsatile pressure gradient is made up of a steady and oscillatory part. Therefore combining the results for the two pressure gradients is important.

Various other authors have looked at other aspects of the arterial system, such as,
elasticity of the walls, shape of the cross-section, time-dependent curvature and the non-Newtonian behaviour of blood. It would be desirable to incorporate these factors into the present model.
Appendix A

Finite differences.

Below are the finite differences used in chapter 4; the method used to derive the finite differences can be found in Williams (1980). The differential operator $Q$ is of order 4 i.e. highest derivative in $\rho$ is $\frac{d^4}{d\rho^4}$, hence we show the finite differences used at each spatial point at each order. (We use general notation where $f = f(\rho)$ and $f^k = \frac{d^k}{d\rho^k}(\rho_i)$, $k = 1, 2, 3, 4$.

A.1 Finite difference at $\rho_i$, $2 \leq i \leq N - 2$.

\[ f^1 = -\frac{1}{h} (f_{i-1} - f_{i+1}) , \]
\[ f^2 = \frac{1}{h^2} (f_{i-1} - 2f_i + f_{i+1}) , \]
\[ f^3 = -\frac{1}{2h^3} (f_{i-2} - 2f_{i-1} + 2f_{i+1} - f_{i+2}) , \]
\[ f^4 = \frac{1}{h^4} (f_{i-2} - 4f_{i-1} + 6f_i + 4f_{i+1} - f_{i+2}) . \]

A.2 Finite difference at $\rho_0$.

\[ f^1 = -\frac{1}{2h} (f_0 - 4f_1 + f_2) , \]
\[ f^2 = \frac{1}{h^2} (2f_0 - 5f_1 + 4f_2 - f_3) , \]
\[ f^3 = -\frac{1}{2h^3} (5f_0 - 18f_1 + 24f_2 - 14f_3 + 3f_4) , \]
\[ f^4 = \frac{1}{h^4} (3f_0 - 14f_1 + 26f_2 - 24f_3 + 11f_4 - 2f_5) . \]
A.3 Finite difference at \( \rho_1 \).

\[
\begin{align*}
    f^1 &= -\frac{1}{2h} (f_0 - f_2), \\
    f^2 &= \frac{1}{h^2} (f_0 - 2f_1 + f_2), \\
    f^3 &= -\frac{1}{2h^3} (3f_0 - 10f_1 + 12f_2 - 6f_3 + f_4), \\
    f^4 &= \frac{1}{h^4} (2f_0 - 9f_1 + 16f_2 - 14f_3 + 6f_4 - f_5).
\end{align*}
\]

A.4 Finite difference at \( \rho_{N-1} \).

\[
\begin{align*}
    f^1 &= \frac{1}{2h} (f_{N-2} - 4f_{N-1} + 3f_N), \\
    f^2 &= -\frac{1}{h^2} (f_{N-3} - 4f_{N-2} + 5f_{N-1} - 2f_N), \\
    f^3 &= \frac{1}{2h^3} (3f_{N-4} - 14f_{N-3} + 24f_{N-2} - 18f_{N-1} + 5f_N), \\
    f^4 &= -\frac{1}{h^4} (2f_{N-5} - 11f_{N-4} + 24f_{N-3} - 26f_{N-2} + 14f_{N-1} - 3f_N).
\end{align*}
\]
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