Modelling dolphin hydrodynamics:
The numerical analysis & hydrodynamic
stability of flow past compliant surfaces

Leanne Allen
B.Sc. (Hons)

A thesis submitted for the degree of Doctor of Philosophy

Department of Mathematics & Statistics
University of Surrey
Guildford, Surrey, GU2 7XH

September 2001
Contents

List of figures vii
List of tables x
Acknowledgements xii
Summary xiii

Part I Introduction 1

1 General Overview 2
  1.1 Areas of commercial interest ........................................ 4
  1.2 Motivation ................................................................. 6
    1.2.1 The effects of wall compliance on boundary layer stability 7
  1.3 The secret of the Dolphin ............................................. 9
    1.3.1 Discussion of Gray’s Paradox ................................. 9
    1.3.2 Classification of instabilities ............................... 18
    1.3.3 Energy classification ........................................... 20
  1.4 Outline of contents ................................................... 21
Part III  Modelling compliant surfaces 110

10 Overview 111

10.1 Historic background for the plate-spring model .......... 111
10.1.1 Kramer's experimental investigation ................. 112
10.1.2 Review of stability analyses for boundary layers over compliant sur-
faces ................................................................. 114

11 Two-dimensional model for the compliant wall 118

11.1 Boundary Conditions for the two-dimensional Kramer-type compliant Surface 119
11.1.1 Derivation of Boundary Condition 11.7 ............... 122
11.1.2 Derivation of boundary condition 11.8 ............... 124

12 Three-dimensional model for the compliant surface 126

12.1 Boundary conditions for the three-dimensional Kramer-type compliant sur-
face - Ekman layer problem .................................... 126
12.1.1 Kinematic boundary conditions ......................... 126
12.1.2 Dynamic boundary condition .......................... 130
12.2 Attachment-line boundary layer over a compliant surface ........ 134
12.2.1 Kinematic boundary conditions ......................... 134
12.2.2 Dynamic boundary condition .......................... 136

13 Numerical values for the wall parameters 141

Part IV  Two-dimensional boundary layers interacting with a com-
pliant surface 143

14 Overview 144

14.1 Boundary conditions on $A^2(C^4)$ ......................... 145
14.2 Calculating the dimensionless parameters required for the boundary condi-
tions ................................................................. 146

15 Computed neutral curves 148
<table>
<thead>
<tr>
<th>Part V</th>
<th>Three-dimensional problem: Rotating flows and the Ekman layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>Overview</td>
</tr>
<tr>
<td>16.1</td>
<td>Ekman Spiral</td>
</tr>
<tr>
<td>16.2</td>
<td>Outline of Part V</td>
</tr>
<tr>
<td>16.3</td>
<td>Ekman layer solution</td>
</tr>
<tr>
<td>16.3.1</td>
<td>Ekman solution</td>
</tr>
<tr>
<td>16.4</td>
<td>Three dimensional instability</td>
</tr>
</tbody>
</table>

| 17    | Rotating boundary layer                                    |
| 17.1  | Non-inertial coordinate systems                            |
| 17.2  | Governing stability equations- Non-dimensionalizing the Ekman problem |
| 17.3  | Classical Ekman non-dimensionalization (Lilly, 1966)        |
| 17.4  | Non-dimensionalizing the N-S equations with emphasis on the Rossby and Ekman numbers |
| 17.5  | Comparison of the two forms of the equations               |

| 18    | A Numerical linear system and the Compound matrix method   |
| 18.1  | Compound matrix for restriction to $\bigwedge^3(\mathcal{C}^6)$ |
| 18.2  | Boundary conditions at infinity                            |
| 18.2.1| Starting values at $z = L_\infty$                         |

| 19    | Curves of neutral stability-rigid wall                     |

| 20    | Boundary conditions for the compliant surface              |
| 20.1  | Code Validation                                           |
| 20.1.1| Well-posed boundary conditions                            |
| 20.2  | Boundary conditions on $\bigwedge^3(\mathcal{C}^6)$       |
| 20.3  | Numerical values for parameters                           |
| 20.4  | Curves of neutral stability- compliant wall               |

| Part VI | Swept-wing boundary layers: stability of the attachment line |
flow past a compliant surface

21 Overview

22 Attachment line: basic flow and disturbance
22.1 Basic flow .............................................................................................................. 204
22.2 Linear stability problem ...................................................................................... 206
22.3 Derivation of the sixth order eigenrelation ........................................................ 207
22.4 Boundary conditions at infinity and the initial vector ...................................... 211
   22.4.1 Derivation of the initial vector ...................................................................... 211
22.5 Numerical comparison with Wilson and Gladwell ............................................ 213
   22.5.1 Accuracy check for $\bar{v}$ ........................................................................ 215
   22.5.2 Chebyshev accuracy results ...................................................................... 217
   22.5.3 Computed neutral curve - rigid wing ..................................................... 218

23 Boundary conditions for the compliant surface .................................................. 220
23.1 Well posed boundary conditions ..................................................................... 221
23.2 Boundary conditions on $\Lambda^3(C^d)$ ............................................................... 222
23.3 Computed neutral curves-compliant wall ....................................................... 223

Part VII Discussions and Conclusion

24 Development of numerical methods

25 Interaction between two-dimensional and three-dimensional viscous flow and the compliant surface
25.1 Stability features of rotating Ekman flow ......................................................... 229
25.2 Stability features of the attachment line flow past a swept wing .................... 230

26 Open problems

26.1 Global analysis ..................................................................................................... 231
26.2 The Compliant wall ............................................................................................ 231
26.3 Oscillating boundary layers ............................................................................. 232

References

vi
A  Compound matrix conversion code  245

B  Graph of the Eigenfunctions for plane Poiseuille flow  251
## List of Figures

1.1 Visualization of pressure drag ................................................. 8
1.2 Diagramatic classification of types of stability .......................... 19
2.1 Sketch of boundary layer on a flat plate in parallel flow at zero incidence 28
3.1 Schematic of plane Poiseuille flow ........................................... 42
5.1 Integration on a manifold ....................................................... 62
7.1 Neutral curve for plane Poiseuille flow ................................... 93
8.1 Linear method ................................................................. 96
8.2 Swapping fixed parameter at turning points ............................. 96
8.3 Continuation along a branch with limit points .......................... 98
8.4 Correction in the hyperplane orthogonal to the tangent .......... 98
9.1 Neutral Curve for the Blasius boundary layer ........................... 109
10.1 Cross section of Kramer’s coating. All dimensions in mm. (Drawings are based on those given by Kramer [71].) ......................... 112
10.2 Kramer’s model. All dimensions in mm. Shaded regions were coated. 113
10.3 The effect of surface compliance on the neutral curve for a non-dissipative flexible wall according to Benjamin’s theory. $\alpha$ is wavenumber, $\delta^*$ is boundary layer displacement thickness and $Re$ is the Reynolds number. 115
11.1 A schematic illustration of the theoretical model for a compliant coating 119
15.1 Effect of $E$ on the neutral curves, plotted in the $\alpha - R_e$ plane: values inside a curve correspond to instability $\text{Im}(c) > 0$. .................................................... 149
15.2 Blow up of the nose of the neutral curve in Figure 15.1 near the critical value of $E$. ................................................................. 149
15.3 Effect of compliant surface damping $C_D$ on the neutral curve ............. 150
15.4 Effect of tension $C_T$ variation on the neutral curve ............................ 150

16.1 Ekman spiral illustrates geostrophic oceanic flows induced by wind stress.
   Northern hemisphere flow is shown here. (Diagram reproduced following J.
   Vanyo [132].) ................................................................. 153
16.2 Mean velocity profiles for the Ekman layer flow ..................................... 161
16.3 Mean velocity profiles plotted as an Ekman profile ............................... 161

19.1 Curves of Neutral stability for the Ekman-layer with various Reynolds num-
   bers ......................................................................................... 186
19.2 Curves of Neutral stability for the Ekman layer in the $R_e - \gamma$ plane for rotation angles: $-30^\circ$, $-14^\circ$ and $13^\circ$ .............................. 187

20.1 Neutral curve for the Ekman layer at $R_e = 60$ with varying compliance on
   the surface. .................................................................................. 197
20.2 Blow up of figure showing the small effect of wall compliance at $R_e = 60$. 198
20.3 Neutral curve for the Ekman layer for $\epsilon = -30$ showing the effects of wall
   compliance. .................................................................................. 198

21.1 The swept back fins of the dolphin ..................................................... 200
21.2 Sketch of the attachment-line flow (from Wentz, Ahmed, and Nyenhuis 1985). 201
21.3 Sketch of attachment line region of swept Hiemenz flow (from Wentz, Ahmed,
   and Nyenhuis 1985). ......................................................... 202

22.1 Neutral curve for the attachment line boundary layer over a rigid surface 219

23.1 Curves of neutral stability for the attachment line boundary layer over a
   compliant surface with varying spring stiffness ................................. 224
23.2 Blow up of the nose of the neutral curve for the attachment-line boundary layer over a compliant surface ............................................. 224

26.1 Dolphin movements during acceleration .................................................. 232

B.1 The real and imaginary parts of the eigenfunction $\phi = \phi_r + i\phi_i$ when $\alpha = 1$
and $R_e = 10^6$ reproduced from Davey [35] ............................................. 251
# List of Tables

4.1 Table (a): Numerical errors for the well-conditioned problem, Table (b): Numerical errors for the ill-conditioned problem ................................................. 48

5.1 Comparison of the accuracy of results for various degrees of ill-conditioned problems with and without orthogonalization .................................................... 59

5.2 Comparison of the discrete Gram-Schmidt orthogonalization with the continuous orthogonalization technique .............................................................. 67

7.1 Comparison of current numerical results with results by Ng & Reid for the OS equation using GL-RK algorithm ............................................................... 92

9.1 The values of the parameters associated with the minimum critical Reynolds number for the Blasius boundary layer profile (based on the displacement thickness $\delta^*$) .................................................................................. 101

19.1 Summary of theoretical results for the Ekman instability .................................. 187

19.2 Accuracy of the critical Reynolds number .......................................................... 188

22.1 Comparison of numerical values for the wave number, $\alpha$ and corresponding growth rate $\lambda$ with those obtained by Wilson and Gladwell .......... 215

22.2 Accuracy of Chebyshev coefficients ................................................................. 218
Acknowledgements

The dedication of this thesis is two-fold. Foremost to my Mum and Dad whose care and support has always been, and continues to be, of immense encouragement to me, and is greatly appreciated. Secondly, to Sam Di Lieto who was a dear friend and shall never be forgotten.

I would like to thank my supervisor, Professor Tom J. Bridges, for his excellent supervision, organisation and guidance throughout my three years of research. I would also like to thank Professor Peter W. Carpenter for his time, views and useful discussions regarding the complaint surface analysis, Professor Frederic Dias for his supervision whilst carrying out my research at l'ecole normale superieure de Cachan, Emily Hudson for her excellent Russian translations and Mike Stonebank for all his computing help.

On the non-mathematical side, I would like to thank all the staff and postgraduate students in the Mathematics and Statistics department at the University of Surrey. Special thanks goes to Fiona Laine-Pearson for her support and for just being ‘Fi’! and to Richard Hillary and Alex Glaser for being my companions in the ‘other’ post-grad room! Finally, a huge thank you goes to my fantastic friends and house mates, Rachel Gartshore and Melike Palalioglu for everything from making me laugh to comforting me through tough times.

This degree programme was funded by the Engineering and Physical Sciences Research Council and also by a special supplementary research scholarship awarded by the University of Surrey.
Summary

Historical numerical methods for solving stiff ordinary differential equations are investigated and a new numerical framework developed and applied to a variety of hydrodynamic stability problems where the effects of passive wall compliance are investigated.

The compound matrix method is set in a general coordinate free framework using exterior algebra, and is considered to be the most accurate and easy to implement method for complex systems.

The effect of passive wall compliance on the Blasius boundary layer flow is studied. The linear stability of the mean flow state is considered using the new numerical framework and shooting technique. A Newton algorithm is implemented to converge the eigenvalue such that the boundary condition at the surface is satisfied. Curves of neutral stability are produced in the $R_e - \alpha$ plane for various degrees of compliance, damping and tension parameters.

Three dimensionality is incorporated first by a simple introduction of rotation in the flow leading to the investigation of the Ekman boundary layer problem where a Coriolis force instability mechanism (type-2) produces streamwise rolls at modest Reynolds numbers. The linear stability of the Ekman mean flow state is considered using an extension of the new numerical framework for use on the vector space $\Lambda^{(3)}(\mathbb{C})$. Curves of neutral stability are produced in the $\epsilon - \gamma$ plane for constant values of the Reynolds number and in the $R_e - \gamma$ plane for a selection of constant angle of orientation, $\epsilon$. This work is extended to consider the effects of wall compliance on the type-2 viscous instability mechanism, with the type-1 mode of instability briefly discussed.

Three-dimensionality is then used for a direct application to the dolphin. The stability
of the attachment-line boundary layer is investigated on flows past swept wings, relating
directly to the dolphins swept-back fins. Wall compliance, modelling the dolphins skin, is
included for analysis of its effect on the attachment-line instability.
PART I

Introduction
Since the originating work by Osborne Reynolds over a century ago, hydrodynamic stability has remained one of the central problems of fluid mechanics. It is concerned with whether a given laminar flow is stable and if so, when and how it breaks down, the subsequent development and eventual transition to turbulence or some other laminar flow. This unstable tendency leading to a turbulent state was first demonstrated by Reynolds [38] from his famous pipe flow experiments in 1883. This phenomenon continues to supply a great deal of interest due to its many applications in real situations and crosses over many scientific fields such as: engineering, meteorology, oceanography, astrophysics and geophysics. Most early studies were experimentally based, since theoretical studies were held back by the non-linear nature of the governing stability equations. Theoretical stability analyses had to rely heavily on simplified problems and the use of linearized theory since few laminar flows correspond to known solutions of these non-linear equations of motion.

The way in which instability occurs is dependent on the flow configuration and external forces such as buoyancy arising due to a fluid of variable density, surface tension and magneto-hydrodynamic forces. Centrifugal and Coriolis effects are also apparent in the consideration of system rotation. Laminar-turbulent transition is initiated by some disturbance of the equilibrium between external forces, inertia and internal stresses of pressure (viscous stresses) of the fluid, altering the flow structure and thus leading to the amplification of the disturbance and, hence, instability. In two-dimensional boundary layer flows, for example, instability appears when travelling waves called Tollmien-Schlichting (TS) waves, become amplified at some critical values of dependent parameters of the flow (usually Reynolds number and wave number). That is, if the Reynolds number is raised
above a critical value, the pressure forces are no longer balanced by the inertia and viscous forces, thus bringing about the amplification of the disturbances in the flow. Viscosity has an obvious stabilizing effect on the flow since it dissipates the energy of any disturbance, thus, the equilibrium forces will only become unbalanced if the viscosity is low. However, viscosity can also have a more complicated effect of diffusing momentum, rendering certain flows (e.g. parallel shear flows) unstable even though the same flow with an inviscid fluid may be stable. For example, a bounded flow, which usually constrains the development of a disturbance, can give rise to strong shear in the boundary layers which in turn gets diffused outwards by the viscosity thus leading to a breakdown of the stability of the flow. Generally, we consider the stability of primarily steady flows, however, useful information can be gained from unsteady flows. Laminar flow acceleration plays a large part in its stability. It has emerged from analysis that, in general, the acceleration of a laminar flow has a stabilizing effect and deceleration a destabilizing effect on the flow.

The forces required to maintain laminar flow can be of a completely different nature for different flow configurations, so that when laminar flow breaks down, the instability observed takes on a different form to that of the propagating TS waves. For example, the centrifugal instability observed in cylindrical Couette flow is caused by an imbalance between the centrifugal force on the fluid particles and the local pressure gradient. Taylor vortices are generated and then as the rotation speed is increased, this vortex structure undergoes a series of secondary-type instabilities, thus eventually leading to a turbulent state. The appearance of these stationary roll cells can result from the departure of the laminar state when a fluid is acted upon by a Coriolis force mechanism.

A general criteria for instability is that the basic velocity profile of the associated flow has an inflexion point. The importance of this fact and its bearing on the stability or instability of the flow was first shown in the late nineteenth century by Rayleigh [108] to give rise to an inviscid instability mechanism. This in turn generally leads to powerful instabilities. An example of this is brought about by the introduction of a negative pressure gradient which generates points of inflexion in the accompanying velocity profile and therefore has a destabilizing effect on the flow.
1.1 Areas of commercial interest

In this modern era of global mechanisation and world wide trade, the high density and use of motion vehicles and shipping, and the ever increasing demand for fast and frequent air transportation to move across large distances has brought about monetary and resource concerns. However, the scope for savings in costs and of natural resources through decreases in pressure drag and skin friction drag of these vehicles is tremendous. For land, air or sea vehicles, reduced drag means longer range, reduced fuel cost/volume, higher payload, or increased speed.

Laminar flow control (LFC) research in the US began in the 1930s and flourished through the early 1960s until it was de-emphasized because of a change in their national priorities. During the 1970s when the oil embargo by the organisation of petroleum exporting countries (OPEC) led to a fuel shortage and high-cost fuel, LFC research became important again because of the aerodynamic performance benefits it could potentially produce for commercial aircraft. Antonatos [2] presented a review of the concepts and applications of LFC, beginning with the realization that skin friction drag could amount to approximately 75 percent of the total drag for an aircraft. However, around twenty five years later, Thibert, Reneaux, and Schmitt [127] attributed friction drag to approximately 45 percent of the total drag, nevertheless, this is still a substantial amount. Because laminar skin friction can be as much as 90 percent less than turbulent skin friction at the same Reynolds number, laminar flow would obviously be more desirable than turbulent flow for reducing the drag of aerodynamic vehicles. In other words, a vehicle with laminar flow would have much lower skin friction drag than a vehicle with turbulent flow.

Unfortunately, achieving laminar flow over the entire configuration is impractical because of the sensitivity of the laminar flow to external disturbances. However, drag reduction due to laminar flow over select portions of a vehicle is feasible. For aircraft, the wings, engine nacelles, fuselage nose, and horizontal & vertical tail are areas for achieving laminar flow. LFC could yield reductions in take off gross weight, operating empty weight and block fuel for a given mission and significant improvement in cruise lift-to-drag ratio. Associated benefits may include reductions in both emissions (pollution) and noise and also smaller engine requirements.

In the early 1960s, Lachmann [75] discussed the design and operational economies of
Introduction

low-drag aircraft. Lachmann noted that the benefits of laminar flow obtained by LFC increased with the size of the aircraft, with benefits maximised for an all-wing aircraft. Also, if 39 percent of the aircraft fuselage could be laminarised for a typical trans-Atlantic airline, Lachmann predicted a 10 percent increase in lift-to-drag ratio. Thus, aerodynamic performance benefits from skin-friction drag reduction can translate into reduced operating costs of an aircraft.

The critical times in industry when fuel costs grew were in the late 1970s to early 1980s and also briefly in the 1990s, this caused a rapid increase in fuel costs in the 1970s inspiring researchers to study drag reduction [67]. However, in the last decade, the cost of fuel has become a small fraction of the operating cost for the industry and so the demand for LFC has diminished. However, the industry must be poised to cope with future uncertainty in fuel cost, since we must remember that the rise in fuel price in the early 1990s was spawned by the Iraq invasion of Kuwait.

In summary, LFC can lead to reduced skin friction drag and thereby reduced fuel consumption. This benefit can lead to either an extension in range for the same aircraft or to reduce aircraft weight for a fixed range. For the latter case less engine power is required and reduced emissions, noise and operating costs can be expected from the LFC aircraft. Noise and emission reductions have become more important and global pollution has become an important variable in the design concepts of the future. Note that although fuel costs have decreased in recent years, the total volume of fuel consumption has increased and potential fuel savings remain a significant cost saving to the industry [67].

Two examples follow to support and reinforce the above benefits of LFC. The first considers the effect on US commercial airlines, where the current annual fuel bill for all commercial airlines in the United States is approximately $10 billion. At sub-sonic cruising speeds, approximately half of the total drag of conventional take off and landing aircraft is due to skin friction. Hence, a conceivable reduction in skin-friction drag of 20 percent translates into an annual fuel saving of $1 billion. This alone represents a substantial cost saving in monetary terms and the associated saving in fuel would help cut the demands on global oil reserves [118].

Lets consider the military sector for our second example. The amount of propulsive power available for an underwater vehicle is limited by the volume allocated to its power plant and the efficiency of the various propulsive components. For these vehicles, about 90
percent of the total drag is due to skin friction. Accordingly, a reduction in skin-friction drag of 20 percent translates into an increase in speed of 6.8 percent. Although modest, this extra speed may be vital for the survival of a submarine being chased by another underwater vehicle [118].

1.2 Motivation

This research is motivated by the desire to control laminar-turbulent transition through the use of wall compliance. The report is divided into seven clearly defined parts. The second of these seven discusses and develops numerical techniques used for hydrodynamic stability analyses. The already well documented two-dimensional boundary layer flow, with its onset of TS instability, is used as a test case for a new concept for the analysis of the boundary layer problem, namely, a new representation of the compound matrix method. This new representation uses exterior algebra to produce a new numerical framework. Part III produces the model for the compliant surface to be incorporated throughout the latter parts, using Carpenter & Garrad's [23] two-dimensional plate-spring model as a starting point. Appropriate boundary conditions are derived for latter derivations for appropriate initial starting vectors in both the two and three-dimensional problems, and boundary conditions at the wall in the required form for use with our numerical framework. Part IV confirms the results of Carpenter and Garrad [23] for the two-dimensional Blasius problem with a flexible surface using the previously discussed numerical technique, producing, with great confidence, the most accurate results to date. Parts V and VI of this report focus on the more realistic three-dimensional boundary layer problems where part V moves on to discuss the introduction of three-dimensionality through rotating flows and the Ekman layer, initially producing the most accurate results confirming those obtained by Lilly [83] and Melander [90] for the rigid wall case and then investigating the effects on stability for this type of flow interacting with a flexible surface of the form previously discussed. Part VI investigates the stability of flow past a compliant swept wing with direct application to the swept fins of the dolphin. Finally, part VII gives an overall discussion and conclusions on the present research displayed throughout this report.
1.2.1 The effects of wall compliance on boundary layer stability

The reason laminar flow is usually more desirable than turbulent flow for external aero-dynamic vehicles lies with the reduction of the viscous drag penalty. So we need to ask ourselves, do we have a sufficient understanding of the fundamental flow physics for the problem to design an optimal, reliable, cost-effective system to control the flow? The answer to this question is encouraging if we consider the historic research on the subject at hand.

Nowadays, it has been well established, both theoretically and experimentally, that the implementation of wall compliance could substantially postpone laminar-turbulent transition. Such a response reaps great benefits particularly in marine applications. An appropriate beginning is to define and describe the various forms of drag affecting both natural and man-made objects.

The two types of drag

There are two types of drag associated with flow past any object. Potentially the largest drag component is pressure drag which is particularly troublesome when flow separation occurs. Basically, this type of drag is caused by a low pressure region created behind the object, for example, a blunt ended vehicle, figure (1.1). The difference in pressure fore and aft creates a force which pulls the vehicle backwards. The physics of this drag component involves the viscous influence upon the inviscid-flow pressure field.

Some pressure drag, at a relatively low level, occurs even if the flow is attached, simply because of the uncambering of the surface by viscosity-induced flow displacement. However, once flow separation occurs, this drag component increases tremendously. Therefore, the foremost consideration for drag control is probably the avoidance of flow separation. Streamlining the object eliminates most of the pressure drag.

The remaining drag component is skin friction drag which is generally much smaller than the pressure drag component. This type of drag is the result of the no-slip boundary condition on the surface created by viscous shear stress at the surface, and can either be laminar or turbulent.

If the boundary layer formed on the surface of the object is laminar (Blasius velocity profile, for example) then we have an acceptable level of drag. However, as the flow speed increases, the fluid flow past the object eventually becomes turbulent. In fact, it is the
Figure 1.1: Visualization of pressure drag

preservation of laminar flow to higher Reynolds number which is an obvious technique for obtaining skin friction reduction.

Most of the current research efforts are directed towards reducing the skin-friction drag associated with the onset of laminar-turbulent boundary layer transition. The skin friction drag reducing capabilities of compliant walls suggests possible increased vessel speeds and greater energy efficiency. Also, the maintenance of laminar flow over a body is known to greatly reduce noise emissions, thus giving submerged bodies a greater concealment from detection by others. Compliant walls can therefore be useful as an acoustic application.

The first major theoretical contributions to the study of hydrodynamic stability transition were made by Helmholtz [62], Kelvin [69], Reynolds [110], and Rayleigh [107, 108, 109]. Although these early investigations neglected the effects of viscosity, Rayleigh's inviscid inflexion theorem shows that the second derivative with respect to $z$ of the mean velocity proves to be a key issue in the explanation of hydrodynamic instabilities. Viscous effects were added in the early part of the 20th century, whereby Prandtl made a ground breaking discovery by introducing the concept of boundary layers. Orr [96] and Sommerfeld [122] went on to develop an ordinary differential equation (Orr-Sommerfeld equation) that governs the linear stability of two-dimensional disturbances in incompressible boundary-layer flow over a flat plate. Tollmien [131] and Schlichting [114] discovered convective travelling wave instabilities now termed Tollmien-Schlichting (TS) instabilities in boundary layer transition, and Liepmann [81] and Schubauer & Skramstad [117] experimentally confirmed the existence and amplification of these TS instabilities in the boundary layer. To set an understanding of this type of instability, we can visualize this disturbance by remembering the image of water waves created by dropping a pebble into a still lake or
puddle. In this image, the waves which are generated decay as they travel from the source. This is the case in boundary layer flow, except that the waves will grow in strength when certain critical flow parameters (e.g. Reynolds number) are reached, which can lead to turbulent flow.

1.3 The secret of the Dolphin

With relation to the numerous dynamical vehicles, the ongoing desire to improve their efficiency by means of drag reduction has driven researchers to study the perfections of nature. The presumption is that drag-reduction adaptations have evolved for improved efficiency and/or speed, thereby aiding species survival in the Darwinian sense.

The dolphin has long been known for its superiority over man-made vessels with respect to its superb hydrodynamic ability. Observers aboard high-speed ocean craft have often seen dolphins travelling at apparent speeds of at least 21 knots (≈ 24.2 mph) and whales at apparent speeds of at least 30 knots (≈ 34.5 mph), indicating a good performance in water, especially when it is accomplished with the aid of the “notoriously weak muscle motor” [74]. Sir James Gray was the first to remark in the scientific literature on the abnormal high swimming speeds apparently achieved by the dolphin (the so called Gray’s paradox). An analysis of torpedoes and submarines, of roughly the same dimensions as dolphins and whales, indicates that the sea animals either are much more powerful than expected or possess some method of reducing hydrodynamic drag.

To consider the drag on a moving dolphin leads to a discussion of Gray’s paradox.

1.3.1 Discussion of Gray’s Paradox

The big discrepancy which Gray [51] suggested, based on energetics, to exist between the power apparently needed to overcome the associated drag at high speed, and the maximum power to be expected from the swimming muscles is incorporated by Gray’s paradox:

*The drag of various underwater creatures had to be inordinately low to correspond to speed claims.*

Gray used the following arguments. During vigorous exercise, a man is known to develop about 17 Watts of mechanical power per kg of muscle. If this factor is assumed to apply to whales then the power available in their muscles can be found as soon as the
weight is known. But, the power needed to drive a rigid streamlined body through water, at a speed $V$, is given by the expression $\frac{1}{2}\rho AV^3C_f$ where $C_f$ is the drag coefficient, $\rho$ is the density of water, $A$ is the surface area and $V$ is the speed [100]. Gray considered a dolphin swimming at 9 m/s (a generally accepted sustained swimming speed for the dolphin) and modelled the dolphin as a one-sided flat plate of length 2 m. The Reynolds number based on this length was approximately $20 \times 10^6$. Now, the Reynolds number for transition from laminar to turbulent flow does not exceed $2 \times 3 \times 10^6$ for flow over a flat plate. Thus Gray assumed that the flow would be mostly turbulent and so the dolphin body would experience a large drag force, and that for a speed of 9 m/s the dolphins muscles would have to deliver around seven times more power per unit mass than deemed possible. In fact, he found that the power met the demand only when the drag coefficient of laminar flow was used.

It should be noted that only small whales and dolphins have this power problem, in fact, Gray’s paradox doesn’t apply to whales over approx. 5 m in length. This result led Gray and others to argue that the dolphin must be capable of maintaining laminar flow by some extraordinary means. Unfortunately, nothing this good actually exists! Some explanations follow:

Gray’s first mistake was in approximating the dolphin by a flat plate and assuming the transition would be at the same place as that on the dolphin. Nowadays, we know that transition is delayed in favourable pressure gradients (accelerated flow) and promoted by adverse ones (decelerating flow) [116] thus, the transition part of the dolphin would actually occur near the point of minimum pressure—approximately half way along the body and so transition actually occurs at Reynolds number $\approx 10 \times 10^6$. Hence the drag is much less and the power output is not much more than two times that required [27].

There is also more recent evidence that dolphin muscle is capable of higher output than other mammalia muscle [44, 43]. With this in mind there is much less to explain. Nevertheless there is still lots of evidence for the use of passive artificial dolphin skins, compliant walls, to maintain laminar flow.

After taking into account the laminar body shape of the dolphin, a rigid surfaced airfoil, in comparison, still fails to compete with the extraordinary results of nature. So, how does the dolphin manage these high speeds using minimal energy?
Although the problem has not been resolved, evidence exists which indicates that the unusual performance is attributed to hydrodynamic rather than to physiological factors. A large body of evidence suggests that cetaceans and many types of fish possess unusually low drag because their boundary layer remains laminar at high speeds, and does not become predominantly turbulent as in rigid bodies [82]. If the compliance of the dolphin's skin is indeed the answer to their marvellous hydrodynamic ability, then the problem at its basic level is a question of stability.

The concept of using a compliant wall to postpone laminar-turbulent transition in marine applications originated with Kramer, who attempted to exploit the “dolphin’s secret” technologically.

When realizing how weak the propulsive power of the dolphin might be, for instance, in comparison with that of a man-made submarine, Kramer [70, 71, 74] became interested in investigating the hydrodynamics of the dolphin.

Some biologists suspected that the dolphin might establish a fully laminar flow at its skin during swimming which would reduce the drag of the dolphin to approximately one tenth of that encountered in a comparable man-made hull. This might explain the performance of the dolphin. However, biologists couldn’t explain how the dolphin might achieve this change in flow pattern and its corresponding reduction in frictional drag.

It was Kramer who undertook a study of the comparison of the hydrodynamic performance between the man-made submarine and the dolphin. Since no convincing explanation for a great hydrodynamic superiority of the dolphin had been found, he went on to thoroughly investigate the skin of a white-belly dolphin designing and producing various coatings to simulate the skin properties, then ran numerous tests to investigate their drag. His pioneering compliant coatings were based closely on the dolphin epidermis. As Carpenter [21] pointed out, nature does not yield up her secrets easily, and thus undoubtedly Kramer had an imperfect understanding of the structure and function of the dolphin’s epidermis. Nevertheless, Kramer obtained 1.6 feet of laminar flow on a four foot body of revolution at a model Reynolds number of $1.5 \times 10^7$. This indicates a laminar flow Reynolds number of $6 \times 10^8$, even with a slight adverse pressure gradient at the nose. The body was covered with a special fluid backed resilient rubber coating. The hypothesis is that tiny disturbances in laminar flow, which normally build up to cause turbulence, are damped out by the resilient coating, thereby maintaining laminar flow. After a series of
Introduction

runs involving the body being towed at high speed through sea water, the coated model was found to have only forty per cent of the drag of an equivalent uncoated rigid model.

Kramer was convinced that the drag reductions observed were a result of the transition delaying properties of his compliant coatings. His studies gave an impression that these soft, resilient coatings may stabilize the laminar boundary layer and thus, the astonishing hydrodynamic performance of the dolphin could be explained. In the early part of the 1960s, Kramer's experiments led to a flowering of theoretical studies concerned with hydrodynamic stability and transition in boundary layers over flexible surfaces. Several able theoreticians were attracted to the problem. The work of Benjamin [9, 10], Landahl [78], and Landahl & Kaplan [79] in particular have stood the test of time. Their research supported the case for achieving drag reduction through the use of compliant walls but also indicated the possibility of additional wall-based modes of instability absent in the rigid boundary case. Benjamin and Landahl laid the foundations for modifying the stability theory to account for the various boundary conditions at a compliant wall. In particular, Benjamin revealed the appearance of a least three different types of unstable waves, discussed in section 1.3.2. Several attempts were made to confirm Kramer's results by means of independent experiments. All such attempts, however, apparently ended in complete failure as far as verification of Kramer's large drag reductions was concerned. Despite Kramer's apparent success, this state of affairs, combined with some apparently major inconsistencies between Kramer's observations and theory, lead to general dismissal in the scientific and engineering community towards Kramer's pioneering work on compliant walls as a means of achieving transition delay. Consequently, little further work was undertaken on this topic until the 1980s, where a reassessment of Kramer's work, among other things, has led to a revival in the interest of the application of compliant walls as a means of controlling the transition process in water. Kramer's work still remains controversial but, in contradiction to the earlier prevailing view, Carpenter and Garrad [23, 24] have demonstrated that in theory, at least, substantial transition delays were indeed possible with Kramer's original coatings. It was also demonstrated that these original prototypes were close to the so called optimum walls, where the compliant walls optimum performance is realized through a critical choice of wall parameters. This brings about the delicate balance between reduction of flow based instability growth rates and the generation of wall-based instabilities of the type identified by Benjamin and Landahl.
In a very real sense, however, this is past history since recent theoretical & experimental work has amply confirmed that the use of wall compliance can lead to the substantial postponement of laminar-turbulent transition and therefore is a viable method for transition control. Drag reductions, or equivalent transition postponement, of magnitude reported by Kramer have yet to be confirmed but, on the basis of present knowledge, there is no reason to suppose that compliant coatings with even better performances will not ultimately be developed.

A substantial body of work followed and excellent theoretical contributions have been made by Carpenter & Morris [26] who used a plate-spring construction to represent an anisotropic compliant wall theoretically. Sen & Arora [119] and Yeo [137] modelled wall compliance with multiple layers of viscoelastic material.

In 1987, Gaster [47] made a breakthrough in carefully controlled experiments and Daniel et al. [32] assessed the effects of wall compliance on Tollmien-Schlichting (TS) waves in flat plate boundary layers, their results confirming that wall compliance could indeed have a significant stabilizing effect on the TS waves thus delaying the onset of transition.

Comprehensive reviews of the experimental work have been made by Riley et al. [111], Gad-el-Hak [45] and Carpenter [21] have reviewed the theoretical aspects of the work. The theory of Dixon et al. [37] have conservatively indicated six fold increases in the transition Reynolds number using specifically optimized viscoelastic compliant walls.

Compliant wall dynamics can be represented theoretically by using the plate-spring model of Carpenter and Garrad [23]. The observation that instability modes grow rapidly over rigid walls is perhaps the most significant difference between the transition process over a rigid flat plate and that over a compliant wall. The Tollmien-Schlichting wave is what leads to transition over a rigid surface. Accordingly, it is clear that this is fundamentally an instability of the boundary layer flow. This type of wave is modified by wall compliance but, for the most part, its basic character remains unchanged.

Any hydroelastic instabilities can severely impair the benefits achieved by reducing amplification rates of rigid wall instabilities, so that in order for wall compliance to be most effective, wall parameters should ideally be selected to give marginal stability with respect to wall-based instabilities.

At present, we must note that wall compliance only finds practical relevance in marine
applications. For the type of wall-flow interaction necessary in this technology, the densities of the two media, fluid and solid, need to be comparable. Water and rubber-type materials are well matched and permit the construction of a robust compliant coating. However, this does not limit the future capabilities for the use of compliant surfaces on aircraft, for example.

The majority of the preceding research has concentrated on the effects of wall compliance in relation to the two-dimensional flat plate (Blasius) boundary layer with zero external pressure gradient. In this case, the boundary-layer disturbances take the form of Tollmien-Schlichting (TS) waves which become destabilized by an essentially viscous mechanism. Cooper [29] states that wall compliance basically controls this type of instability by reducing the rate of production of disturbance kinetic energy by the Reynolds stress, by increasing the viscous dissipation and bringing in additional energy dissipation mechanisms. The effect of a compliant boundary therefore alters the ratio between energy production and dissipation, allowing the growth of boundary layer disturbances to be suppressed.

By reviewing the literature concerned with linear stability analysis of boundary layers over compliant surfaces, Carpenter and Garrad [23] developed a theoretical model for the compliant coating which resulted in an Orr-Sommerfeld equation with boundary conditions representative of the compliant surface. The analysis consisted of numerically integrating the Orr-Sommerfeld equation using Scott & Watts' numerical scheme, SUPORT, specially developed for stiff equations. The basis of SUPORT is a variable-step Runge-Kutta-Fehlberg integration scheme designed for the solution of two-point boundary-value problems coupled with the stabilizing scheme of orthogonalization. Each time the solutions started to lose their numerical independence, the solution vectors were orthonormalized again before the integration proceeded further. The desired solution was then obtained by piecing together intermediate solutions. They obtained results from the numerical analysis of the stability of the Blasius flow over a rigid flat plate, which compared favourably with previously obtained theoretical and experimental data. They then studied a wide range of compliant surfaces based on Kramer-type models with encouraging results obtained for the transition delay (see the review of Carpenter [21]).

Questions like "what limits the transition-delaying performance of a compliant wall?", "what is the greatest possible transition delay achievable?", and "what wall properties give
the best performance?” have already been answered for the plate-spring compliant wall interacting with Blasius type flow [28, 26]. Theoretical studies showed that as the wall compliance is increased, the growth of the TS instability is progressively suppressed. In fact, if the wall were to be made sufficiently compliant the TS waves would be completely stabilized, resulting in the maintenance of laminar flow for indefinitely high Reynolds numbers. Surprisingly, this ideal situation cannot occur in practice since the compliant wall itself is a wave bearing medium supporting two classes of essentially wall-based waves, travelling wave flutter (TWF) and divergence, in addition to the essentially flow-based TS instability. Thus, when the wall is sufficiently compliant, both wall-based instabilities can develop into hydro-elastic instabilities and it is actually these wall-based instabilities that limit the transition delaying performance of the compliant walls. Carpenter & Morris [26] state that the divergence would destroy any transition-delaying capabilities and must therefore be avoided.

Kramer, in his early pioneering work, considered energy dissipation in the wall (damping) to be an important and beneficial property. He carefully optimised the level of wall damping in his tests to obtain the greatest possible drag reductions and suggested that damping acted to dissipate the energy transferred from the TS wave to the wall. It was shown by Benjamin [9], and confirmed in the present research, that this rather plausible explanation on the control mechanism is, in fact, false. Damping actually leads to increased growth of the TS wave, that is, a destabilization of the TS instability. However, this does not mean that damping in the wall always has a deteriorating effect on the transition postponement. Damping in the original walls was probably successful in suppressing another mode of instability. This mode is a wall-based instability similar to the TWF instability in aeroelasticity. Damping reduces the growth of this type of instability and postpones its onset to a higher Reynolds number, rather than reducing the growth rate of the flow-based instability (TS) as first suggested by Kramer. Furthermore, the beneficial effect on the TWF type instability is much more pronounced than the adverse effect on the TS waves. In other words, the true role of damping in the Kramer coatings was to delay the onset of TWF, thereby allowing a more compliant wall to be used. The beneficial effects of increased compliance, in reducing the growth of the TS instability, more than offsets the damaging effects of damping [37]. Carpenter & Garrad [23] have studied the effect of a viscous fluid substrate on TS instabilities for Kramer-type surfaces. It appears
from their results that some sort of interaction occurs between TS and TWF modes. In their paper a second island of TS instability was found at a fairly high Reynolds number and low wave number. However, due to their numerical techniques, it was not possible to obtain complete sets of neutral curves. Carpenter, Gaster & Willis [25] overcame these difficulties by using a compound matrix method showing that this second island shrinks rapidly with an increase in the viscous ratio of wall substrate to fluid and disappears completely when the substrate viscosity is approximately two and a half times that of the fluid. Thus they have shown that perhaps viscous damping can indeed have a stabilizing effect on TS instability when mode coalescence occurs, thus possibly explaining how a viscous substrate can have a beneficial effect on hydrodynamic stability.

In real aerospace and marine applications, boundary layers are usually three-dimensional and/or develop in a non-zero pressure gradient, which is likely to be adverse over some part of the surface. Little work to date has focused on the effects of three-dimensionality of the boundary layer on compliant wall performance, with the most recent major works by Carpenter, Lingwood, and Cooper on the rotating disk. Cooper [29] gives a comprehensive review of works on the rotating disk problem. If wall compliance is to become a practical means of maintaining laminar flow, then its performance under three-dimensionality must be critically assessed.

The instabilities arising in boundary layers under these flow conditions are of a different nature to the TS instability. We see the development of velocity profiles with an inflexion point as a result of the effects of an adverse pressure gradient or of the three-dimensionality of the flow. The presence of an inflexion point in the two-dimensional velocity profile supports the more powerful inviscid instability mechanism identified by Rayleigh [108]. Lin [84] presented a physical reasoning for the occurrence of this type of instability. He considered the vorticity field associated with the mean flow and identified an instability mechanism qualitatively through momentum transfer arguments. The growth of disturbances by this mechanism continues to indefinitely high Reynolds number with amplification rates considerably higher than those of the TS instability. Therefore, it needs to be established whether passive wall compliance is capable of controlling disturbance growth to a sufficient extent when this inflexion instability dominates.

Three-dimensional flows of research interest include boundary layers which develop over swept-back wings and rotating boundary layers, as well as those specific to other
aerospace and marine applications.

Fully three-dimensional boundary layers are susceptible to an instability known as cross-flow which dominates the breakdown of laminar flow. This type of instability is only common to three-dimensional boundary layers. Previous workers have assumed that the instability is associated with a point of inflexion in the velocity profile which then promotes an instability mechanism dominated by inviscid effects resembling the Rayleigh instability of two-dimensional flows [108].

Discovery of the cross-flow instability is attributed to Gray [52], who observed, whilst studying the flow over swept wings, uniformly spaced vortices stationary with respect to the wing body prior to the onset of transition. This structure was not present in the two-dimensional flow. At the same time, these observations, now termed the cross-flow phenomena, were interpreted independently by the studies of Owen and Rendall [99].

Experimental work by Gregory et al. [54] and Faller [40] has revealed the existence and nature of this shearing instability in a boundary layer whose mean flow is produced by a balance of viscous, Coriolis and pressure forces. Gregory's mean flows were produced by a rotating disk and Faller's by withdrawing fluid from the centre of a rotating tank. Both these mean flows are characterised by a rotation of the flow direction vector with increasing distance from the bottom. The instability appears as vortex rolls in the form of spiral bands, with axes of symmetry lying approximately parallel to the mean flow outside the boundary layer.

Major theoretical analyses relevant to the stability problem were initially presented in the paper by Gregory, Stuart and Walker [54]. Stuart established general equations for the stability of three-dimensional flows in terms of curvilinear coordinates for analyses based on the inviscid equations for the rotating disk case.

Through the use of china-clay flow visualization techniques a stationary vortex structure similar to that observed over swept wings was demonstrated. In this geometry, the cross-flow instability manifests itself in the form of co-rotating spiral vortices which spiral out towards the edge of the disk at a constant angle. The flow visualization study revealed two critical radii with the first occurring at the edge of the central laminar flow region where the spiral vortices first set in. Moving further outwards, a second radius identifies the onset of the transition process.

Brown [17], Barcilon [5] and Faller [41] also contributed to the study of this stability
Introduction

problem. Brown's work was also confined to the rotating disk and to the related problem of flow over swept-back wings. This work consisted of numerical solutions of the Orr-Sommerfeld equation for given velocity profiles. Barcilon obtained analytic solutions of the perturbation equations with the mean velocity profile taken from the Ekman solution essentially appropriate to Faller's rotating tank experiments. Barcilon did not, however, obtain values for the critical Reynolds number, since his method of solution was not accurate enough for the relatively small critical Reynolds numbers observed.

Faller & Kaylor [42] and Lilly [83] used finite difference methods to solve the eigenvalue problem for the system of ordinary differential equations obtaining the parameter values at which the instabilities occur. Lilly's work indicated that the inclusion of Coriolis effects had a strongly stabilizing influence.

The following section gives a brief description of the classification schemes for instabilities in the coupled compliant wall/fluid boundary layer problem which are often used as a means of identifying modes of instability. Also presented are basic definitions of flow-based (hydro-elastic) instabilities which can arise as a direct result of the presence of a compliant boundary.

1.3.2 Classification of instabilities

The presence of a compliant wall in boundary layer problems generates the possibility of a number of instability modes. This section will attempt to provide the reader with a clear description of the classification schemes for instabilities in the coupled compliant wall/fluid boundary layer problem often used to identify modes of instability. The basic descriptions of flow-based (hydro-elastic) instabilities arising as a direct result of the presence of a compliant surface are also presented below. The presence of a compliant wall in fluid boundary layer problems generates the possibility of numerous modes of instability, hence the need for a clear classification scheme for instabilities. The features of the instabilities associated with each classification category provide a means of identification of unstable modes in the type of flow system presented in this report. Figure 1.2 shows a basic diagrammatic representation of the flow types.

There are essentially three main ways of categorising instabilities which arise in the coupled fluid boundary-layer/compliant wall flow problem.

1. Convective verses absolute instabilities:
This is the simplest form of classification and is applicable to any flow problem, since their definitions do not refer to wall compliance. A *convective* instability is defined as an amplification of a disturbance occurring as it moves downstream from its point of initiation. So the growth of an unstable mode arises with propagation distance as opposed to growth in time at a fixed spatial location which is the classic temporal view of unstable disturbances. *Absolute* instabilities are analogous to the temporal description and are usually associated with disturbances which have zero group velocity resulting in disturbance kinetic energy not being convected away from its point of origin. Disturbances of this type grow indefinitely at a given location. Absolute instabilities thus tend to dominate other modes of instability and significantly affect the process of transition. However, it must be noted that it is not always the case that zero group velocity ⇒ absolute instability.

2. **Wall-based verses flow-based instabilities:**

   Compliant walls themselves are wave-bearing media. If a compliant wall is subject to an impulsive line load in the absence of fluid flow, surface waves travel outward along the surface to the left and right of the point of impact. These are the free surface waves. In the presence of fluid flow, the free-surface waves can develop into instabilities. They can also interact with other waves to form instabilities. Thus compliant surfaces bring about the possible occurrence of the so-called *wall-
based or flow-induced-surface instabilities (FISI) above some threshold level of wall compliance. This type of instability occurs in the wall itself as a result of the action of the surrounding fluid on the flexible surface. Rigid walls do not have this type of instability.

Flow-based instabilities, however, are essentially modified flow instabilities which can arise in the presence of a rigid surface. Examples of flow-based modes are Tollmien-Schlichting instabilities (TSI) (the Rayleigh instability in two-dimensional boundary layers) and cross-flow instabilities (common to three-dimensional boundary layers).

Now there are two main types of wall-based (hydro-elastic) instabilities: travelling wave flutter (TWF) and divergence. The latter is basically induced by an imbalance between the walls structural forces and the conservative hydrodynamic pressure forces generated by fluid disturbances on the surface. It has been identified as an absolute instability and so its presence is damaging to the prospects of using wall compliance as an effective means of transition delay. TWF is a convective instability brought about by an essentially inviscid mechanism. It is characterised by high phase speeds close to the free-stream value and grows by the irreversible transfer of energy from the flow to the wall as a result of work done by the fluctuating pressure. Both divergence and TWF have their origins in the free wave modes of the compliant wall and the presence of wall-based instabilities introduces the possibility of coalescence with other modes to generate powerful new modes of instability.

1.3.3 Energy classification

The existence of flow-induced-surface instabilities promoted interest from Benjamin [9, 10] and Landahl [78], who talk about a classification of instability based on the effect of energy exchange between the wall and fluid and the response to damping in the wall. There are three types of instability within this classification scheme.

1. Class A (negative energy waves-NEW): Irreversible energy transfer to/from the compliant wall has a stabilizing/destabilizing effect on the instability. Damping tends to destabilize this type of mode.

2. Class B (positive energy waves-PEW): members of this class are more conventional in that they are stabilized by wall damping and the effect of irreversible energy
transfer in the wall/flow system is opposite to that of the Class A instability.

In their pure forms, the NEW and PEW can only become convective instabilities, i.e. grow exponentially until nonlinear effects intervene, as they propagate downstream, but do not grow with time at a fixed location. However, owing to their opposite energy requirements, NEW & PEW can combine to form a truly self sustained temporally growing instability, known as an *absolute* instability.

3. Class C: This type of instability is unaffected by both energy exchanges and wall damping. This type of wave can cause problems in the fabrication and maintenance of the coating.

The TS instability is an example of a Class A instability, the TWF belongs to Class B and divergence has been shown to belong to Class C.

The research described in this report is largely concerned with Tollmien Schlichting instabilities. In part IV, this is the only instability studied, although, the possibility of mode coalescence for the compliant surface case is noted, and results are discussed accordingly.

Part V focuses on the essentially viscous instability, type-2 (class A), occurring at low Reynolds numbers with the occurrence of the type-1 (class B) essentially inviscid instability briefly mentioned.

The final part of this report, part VI concerns the attachment line instability, namely a flow-based instability like the TSI.

1.4 Outline of contents

Following this brief introduction to the contents of this research, each major part will be preceded by a review of literature relevant to the particular problem in question.

Part II provides a review of the numerical methods used for hydrodynamic stability analyses, with regards to the two-dimensional stability analysis over a rigid flat plate. The Orr-Sommerfeld equation is initially derived and historic stabilizing techniques for its integration discussed. The stabilizing techniques considered are the discrete, Gram-Schmidt orthonormalization and a continuous orthonormalization technique and a discussion on their suitability and accuracy for the relevant boundary layer problems is given. A new
numerical framework for solving hydrodynamic stability problems is then developed. The new numerical framework uses exterior algebra as a starting point and compound matrices in a coordinate-free way, thus eliminating the requirement for additional stabilizing techniques. The method is easy to implement and can confidently be extended for use in any dimensions. Due to its formulation, this new framework provides us with confidence in its accuracy and applicability. Along with this numerical framework, another framework is also developed for obtaining asymptotic boundary conditions providing an effective numerical scheme for the generation of the starting vector required for shooting.

A commonly used numerical integration scheme - the fourth order Runge-Kutta algorithm - is assessed and found to not preserve a required quadratic equation exactly. A class of implicit integration schemes - using the geometric Gauss-Legendre algorithm - is shown to preserve this equation automatically to machine precision and so is used in replacement of the explicit scheme. The new exterior algebra based framework is then tested by applying it to the well studied cases of Poiseuille flow and the Blasius boundary-layer flow problems, producing with great confidence, the most accurate neutral curves to date.

The sophisticated numerical methods package, AUTO, used for efficiency in the continuation of neutral stability curves is also discussed.

Part III introduces the two-dimensional plate-spring model designed by Carpenter & Garrad [23] for the representation of the Kramer-type compliant surface. An adjusted model is then developed to incorporate three-dimensional effects for two separate studies. The first of these incorporates three-dimensionality in the most basic form using rotation effects in the system, that is, the effects of a compliant surface on the Ekman layer problem is to be studied as a prototype for this form of three-dimensional rotating flow. The second extension to a three-dimensional problem arises from the application to flow past a swept-wing, which is also directly applicable to the highly swept back fins of the dolphin. This motivation leads us naturally to the study of the effect of a compliant surface on the attachment line instability. The model for the three-dimensional compliant surface is further developed to enable the study of the effects of an anisotropic wall in the horizontal plane. The kinematic and dynamic boundary conditions at the compliant surfaces are derived in this part for use in the stability analyses in latter parts of this report. The numerical values for the compliant wall parameters are also discussed in this part, with values chosen relating directly to the measured parameters of Kramer’s best coating.
Part IV reproduces the two-dimensional stability analysis for the Blasius boundary layer interacting with the plate-spring compliant surface model discussed in part III. The analysis is carried out using, with confidence, the effective new numerical framework developed in part II. Curves of neutral stability are produced for a range of surface compliance and also a range of damping and tension coefficients. The results are discussed and comparisons made with corresponding results from analyses in the literature.

Part V carries out the analysis for the basic three-dimensional flow by means of introducing the effects of rotation. It is natural to use the Ekman velocity profile as a prototype for three-dimensional rotating flows. A non-inertial coordinate system is taken as a starting point and a new generalized non-dimensionalization scheme developed with emphasis on the Rossby and Ekman numbers. This is then implemented obtaining a new generalized set of coupled ODEs for the Ekman boundary layer problem.

The numerical framework developed in part II is effectively extended to three-dimensions and incorporated to produce a linear system in $A^3(C^6)$ for a stability analysis. The initial starting vector for the Ekman boundary layer problem is computed using the new framework along with the full correct asymptotic boundary conditions.

The most accurate curves of neutral stability to date are produced confidently using the sophisticated continuation package, AUTO, the continuation method of which is discussed in part II. The critical Reynolds number for the viscous type-2 mode of instability is computed and compared to the corresponding results in the literature.

The three-dimensional plate-spring model for the compliant surface developed in part III is incorporated and the effect of wall compliance on the type-2 mode of instability is assessed followed by a discussion of these new results.

Part VI studies the three-dimensional flow for application to flow past a swept wing. This analysis has great relevance to the analysis of a dolphin where all three types of the dolphins fins can be considered as swept back wings. This focus naturally leads to the stability analysis of the attachment line boundary layer flow interacting with a compliant surface. A non-dimensionalization scheme is implemented and the basic flow and coupled perturbation equations are derived for analysis.

The idea of spanning sets is used in this part to create the initial vector from the full correct asymptotic boundary conditions since the system is dependent on $z$ at infinity. The accuracy of the code is checked by making a comparison with stagnation flow
stability results from Wilson & Gladwell [136]. Curves of neutral stability are produced for the flow past a rigid swept wing. The model for the three-dimensional anisotropic compliant surface developed in part III is incorporated and the effect of wall compliance on the attachment-line instability is assessed by means of a linear stability analysis using the new numerical framework developed in part II, followed by a discussion of the new results.

Finally, Part VII draws general conclusions from this study and suggests possible areas of interest for future work.
PART II

Numerical methods for hydrodynamic stability
Overview

The aim of this section is to review the historic numerical methods for hydrodynamic stability issues. The most common historic method of orthonormalization is critically reassessed and found to not be completely satisfactory for problems that depend analytically on a parameter. A new numerical framework initiating from exterior algebra is introduced and tested using the well known Poiseuille and Blasius flows obtaining comparable results to corresponding results in the literature. The remainder of this part is as follows. A review of theoretical and experimental work is given to provide some historical background to this hydrodynamic stability problem. Chapter 3 introduces the governing Navier-Stokes equations which are used to derive the flow stability equation, the well-known Orr-Sommerfeld equation, using linear stability theory. The eigenvalue problem which results from this analysis is solved numerically, first using a well known historic shooting method. This shows the need for orthogonalization which is required in order to overcome certain numerical difficulties associated with the Orr-Sommerfeld equation, and is discussed in Chapter 4. This technique is then discussed in Chapter 5 and shown to not preserve analyticity. An introduction to a new numerical framework, not requiring orthogonalization, comprising a new coordinate free formulation of the compound matrix method, is then given. This new framework is tested by solving the associated eigenvalue problem numerically along with an appropriate Runge-Kutta integration scheme to determine neutral curves for the rigid flat plate for two chosen flow fields, namely, the simple plane Poiseuille pipe flow and the Blasius flow field. The results produced are compared with historic results for these two problems and the conclusion drawn that this exterior algebra based numerical framework is indeed a simple and accurate scheme for solving stiff linear ODEs.
2.1 Review of Theoretical and Experimental Works.

The object of this section is to describe qualitatively the position and significance of boundary layers in fluid flows.

An important achievement in viscous-flow theory came in 1904 when Ludwig Prandtl postulated that a flow with small viscosity about a solid body can be divided into two regions: a very thin layer in the neighbourhood of the body, the boundary layer, where friction plays an essential part, and the remaining region outside this layer, where the forces due to friction are small and may be neglected. To set the idea of a boundary layer, consider a solid plate placed in a uniform flow of a fluid of low viscosity. Figure 2.1 represents diagrammatically the velocity distribution in such a boundary-layer at the plate, with the dimensions across it considerably exaggerated. The fluid immediately adjacent to the plate surface is at rest relative to the plate, while in the flow field far from the plate, the influence of the viscosity is not felt and so the flow exhibits no shear deformation. The influence of viscosity is confined to a thin region close to the plate, where the transverse velocity gradient is large and though the viscosity is small, the shear stress becomes significant and the fluid velocity continuously changes from zero on the surface to the uniform flow velocity. Thus, the plate is covered with a thin layer called the boundary layer. In front of the leading edge of the plate the velocity distribution is uniform. With increasing distance from the leading edge in the downstream direction the thickness, $\delta$, of the retarded layer increases continuously, as increasing quantities of fluid become affected. The thickness of the boundary layer decreases with decreasing viscosity.

Boundary layer theory finds its application in the calculation of the skin-friction drag which acts on a body as fluid moves over it: for example the drag experienced by a flat plate at zero incidence, the drag of a ship, or of an aeroplane wing.

When a semi-infinite flat plate is placed parallel to the flow direction in a uniform flow of velocity $U$, a boundary layer develops along it. The thickness of the boundary layer, denoted $\delta$ gradually increases in the stream-wise direction and the flow within it usually remains laminar until the Reynolds number, $Re = \frac{UL}{\nu}$ reaches an order of $10^5$, where $L$ is some characterised length scale and $\nu$ is the kinematic viscosity of the fluid.

The equation of Orr and Sommerfeld owes its name to two scientists of the early years of last century; Orr was Irish and Sommerfeld was German. Independently, they derived
and studied the equation bearing their names. This equation arises during a normal mode analysis of a parallel flow solution $U = U(y)$ to the Navier-Stokes equations. That is, $\ddot{u}(y)e^{(\alpha x - \omega t)}$ is the stream function of a two-dimensional disturbance to $U$, resulting in the famous Orr-Sommerfeld (OS) equation. The parameters in this equation are: $Re$, representing the Reynolds number, $\alpha$ the wave number, and $c = i\omega$, the wave speed. The independent variable, $y$, is assumed to vary over some open interval $I$. In the original form of the equation, the interval $I$ is finite, say $-1 < y < 1$. With this, the only steady solutions to the Navier-Stokes equations satisfying no-slip on the boundaries are

1. Plane Couette flow: $U(y) = y$,
2. Plane Poiseuille flow: $U(y) = 1 - y^2$, or
3. Some linear combination of 1. and 2. , where $U = (U(y), 0, 0)^T$.

The OS equation has been used to predict instability of boundary layers, jets, shear layers and other unbounded flows, so that $I$ becomes an interval such as $[0, \infty)$ or $(-\infty, \infty)$. The first person to employ the OS equation for boundary layers obtaining results appears to have been Tietjens [129]. A few years later, Tollmien [131] explained these results and computed the critical Reynolds number for flow over a flat plate.

The simplest example of the application of the boundary-layer equations is given by the flow along a very thin flat plate. Historically, this was the first example illustrating the application of Prandtl's boundary-layer theory; it was discussed by Blasius in his doctoral thesis at Goettingen. Let the leading edge of the flat plate be at $x = 0$, the plate being parallel to the $x$-axis and infinitely long downstream. Let us consider steady flow with a free-stream velocity, $U_\infty$ parallel to the $x$-axis. The velocity of potential flow is constant.

---

**Figure 2.1:** Sketch of boundary layer on a flat plate in parallel flow at zero incidence
in this case, thus, \( \frac{\partial p}{\partial x} = 0 \). The boundary layer equations then become

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= \nu \frac{\partial ^2 u}{\partial y^2},
\end{align*}
\]

u = v = 0 for \( y = 0 \) and \( u = U_\infty \) for \( y = \infty \).

Since this system has no preferred length, it is reasonable to assume that the velocity profiles at varying distances from the leading edge are similar to each other, which means that the velocity curves \( u(y) \) for varying distances, \( x \), can be made identical by selecting suitable scales for \( u \) and \( y \). These scale factors are naturally the free-stream velocity, \( U_\infty \), and the boundary layer thickness, \( \delta(x) \) respectively. Noting that the boundary-layer thickness increases with the distance \( x \), the principle of similarity of velocity profiles in the boundary layer can be written as \( u/U_\infty = \phi(y/\delta) \), where the function \( \phi \) must be the same at all distances \( x \) from the leading edge. The boundary layer thickness can then be estimated from the exact solutions of the Navier-Stokes equations as \( \delta = \sqrt{\nu x/U_\infty} \). Now by introduction of a new dimensionless coordinate, \( \eta = y/\delta \Rightarrow \eta = y \sqrt{\frac{U_\infty}{\nu \delta}} \), the equation of continuity can be integrated by introducing a stream function \( \psi(x, y) \) such that

\[
\psi = \sqrt{\nu x U_\infty} f(\eta),
\]

where \( f(\eta) \) denotes the dimensionless stream function. Thus we can now find the velocity components:

\[
u = \frac{\partial \psi}{\partial \eta} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = U_\infty f'(\eta),
\]

the prime denoting differentiation with respect to \( \eta \). Similarly, the second velocity component is

\[
v = \frac{\partial \psi}{\partial x} = \frac{1}{2} \sqrt{\frac{\nu U_\infty}{x}} (\eta f'' - f).
\]

Calculating the derivatives of these with respect to both \( x \) and \( y \) and the second derivative with respect to \( y \), substituting these into the boundary layer equations and finally simplifying we obtain the following ordinary differential equation famously known as the Blasius equation.

\[
f f'' + 2f'' = 0.
\]
with the boundary conditions:

\[ f = 0, \quad f' = 0 \quad \text{for} \quad \eta = 0; \quad f' \to 1 \quad \text{as} \quad \eta \to \infty. \]

This resulting differential equation is non-linear and of third order, thus the three boundary conditions are sufficient to determine the solution completely [135].

The Blasius equation appears exquisitely simple, however, the analytic evaluation of the solution of the Blasius differential equation is quite tedious. Blasius obtained this solution in the form of a series expansion around \( \eta = 0 \) and an asymptotic expansion for very large \( \eta \), the two forms then being matched at some suitable value of \( \eta \). The resulting procedure was described in detail by L. Prandtl [104]. Some years earlier Toepfer [130] solved the Blasius equation numerically by the application of the Runge-Kutta method. The same equation was solved again, this time with increased accuracy, by Howarth [65].

I will now provide a review of some of the historic work on the numerical approach of the Orr-Sommerfeld equation.

Owing to the fact that the OS equation is a linear ordinary differential equation, a natural question arising is: \textit{Why can't it simply be solved numerically?} The answer to this question is that it has been solved: Thomas [128] solved the OS equation for Poiseuille flow, determining its eigenvalues by using the matrix method, Gaussian elimination. This settled the dispute of whether plane Poiseuille flow was unstable above some finite Reynolds number. Details on numerical schemes for unbounded intervals, however, were not widely available until the 1960s. One of the many reasons for this was that much of the appropriate research in the 1950s was classified. It was Brown [16] who was the first to solve the OS equation, on \((0, \infty)\) with the Blasius profile, numerically. A flood of papers followed and an excellent account of the early work is given by Betchov and Criminale [12]. Since then, Osborne [98], Jordinson [66], Orszag [97], Davey [33], Mack [88] and Grosch & Orszag [57] among others have stimulated much further research. Here attention will be given to the last paper.

Grosch and Orszag [57] used a numerical approach to the OS equation on finite intervals. What they examined in their paper is the effect of mapping the infinite interval into a bounded one by an algebraic transformation or by an exponential one. Both

\[ \zeta = 1 - e^{-\psi/L} \]
and

\[ \zeta = \frac{y}{y + L} \]

map \([0, \infty) \rightarrow [0, 1]\) so what effect will it have on the numerical solutions? They answer this question by confirming that an algebraic transformation is preferred for the equations of boundary layer theory since solutions approach their limiting values at an exponential rate. In particular, for the OS equation they employed

\[ \zeta = \frac{y - L}{y + L} \]

which maps \([0, \infty) \rightarrow [-1, 1]\). This different finite interval means Chebyshev polynomials can be used. The value of the Chebyshev method is such that it is widely used by many researchers on related equations in fluid mechanics. This use of mapping is employed in chapter 9 of this report, to solve the Blasius equation by Chebyshev polynomial methods and also in part VI to obtain Chebyshev coefficients for use in the attachment-line stability analysis. However, in this report, the matrix method was not used to solve the OS equation on a semi-infinite domain, since significant problems arise from mapping the infinite interval onto a finite interval resulting in approximations of the asymptotic boundary conditions being taken.
The main area of research of the subject of hydrodynamic stability is the Orr-Sommerfeld equation, a fourth order linear ordinary differential equation with complex non-constant coefficients. This equation governs the stability of laminar boundary layers in the parallel flow approximation. The Orr-Sommerfeld equation is obtained by linearizing the Navier-Stokes equations about a parallel shear flow, and using a Fourier-Laplace transformation in the homogeneous directions.

3.1 Derivation of the Orr-Sommerfeld equation

Consider the two dimensional, Navier-Stokes equations

\[ u_x^* + u_y^* = 0 \] (incompressibility condition), \hspace{1cm} (3.1)

\[ u_t^* + u^*u_x^* + v^*u_y^* + \frac{1}{\rho}p_x^* = \nu \Delta u^*, \] \hspace{1cm} (3.2)

\[ v_t^* + u^*v_x^* + v^*v_y^* + \frac{1}{\rho}p_y^* = \nu \Delta v^*, \] \hspace{1cm} (3.3)

where, \( \nu \) is the kinematic viscosity, \( \rho \) is the fluid density and \( \Delta u^* = \frac{\partial^2 u^*}{\partial x^* \partial y^*} \). Here, the * indicates that the variables are dimensional.

The equations of motion require mathematically tenable and physically realistic boundary conditions at the solid surface. Now since we are studying the flow of a liquid in contact with a solid surface, the molecules are so closely packed and the mean free path is so small that fluid particles contacting the wall must essentially be in equilibrium with the solid, and so the liquid will stick to the wall. Thus, we take as a hypothesis that the boundary
conditions are \( u^* = v^* = 0 \) at \( y^* = 0 \), this is known as the no-slip condition, and the fluid region is \( -\infty < x^* < +\infty \), \( y^* > 0 \).

Now, our basic flow equations are extremely difficult to analyse, thus we must concentrate on casting them in a more efficient form. This is accomplished by non-dimensionalizing the equations and boundary conditions, which not only yields the minimum number of flow parameters but also places them in the correct context. We make them dimensionless by dividing them by constant reference properties appropriate to the flow. To do this, we introduce the new variables, \( u = \frac{u^*}{U_\infty^*}, \ x = \frac{x^*}{L}, \ y = \frac{y^*}{L}, \ t = \frac{t^*}{U_\infty^*}, \ v = \frac{v^*}{U_\infty^*} \) and \( p = \frac{p^*}{\rho U_\infty^*} \). Note that steady flows have no characteristic time of their own, hence we choose to non-dimensionalize time using the characteristic reference, residence time. Substituting these into equations (3.1),(3.2) and (3.3), we obtain

\[
\frac{d}{dx} \left( \frac{dx}{dx^*}(uU_\infty) \right) + \frac{d}{dy} \left( \frac{dy}{dy^*}(vU_\infty) \right) = 0
\]

\[
\Rightarrow \frac{d}{dx} \left( \frac{uU_\infty}{L} \right) + \frac{d}{dy} \left( \frac{vU_\infty}{L} \right) = 0
\]

\[
\Rightarrow \frac{U_\infty}{L}(u_x + v_y) = 0
\]

\[
\Rightarrow u_x + v_y = 0, \quad (3.4)
\]

\[
\frac{d}{dt} \left( \frac{dt}{dt^*}(uU_\infty) \right) + uU_\infty \left( \frac{dx}{dx^*}(uU_\infty) \right) + vU_\infty \left( \frac{dy}{dy^*}(uU_\infty) \right) + \frac{1}{\rho} \frac{dx}{dx^*}(p\rho U_\infty^2) = \nu \Delta uU_\infty
\]

\[
\Rightarrow \frac{U_\infty^2}{L} \left( u_x + uu_x + vv_y + p_x \right)
\]

\[
= \left( \frac{\partial^2 u^*}{\partial x^*^2} + \frac{\partial^2 u^*}{\partial y^*^2} \right) \nu
\]

\[
= \left[ \left( \frac{\partial}{\partial x} \frac{\partial x}{\partial x^*} \right) \left( \frac{\partial}{\partial x} \frac{\partial x}{\partial x^*} \right) (uU_\infty) \right.
\]

\[
+ \left( \frac{\partial}{\partial y} \frac{\partial y}{\partial y^*} \right) \left( \frac{\partial}{\partial y} \frac{\partial y}{\partial y^*} \right) (vU_\infty) \right]
\]

\[
= \left[ \frac{U_\infty}{L^2} u_{xx} + \frac{U_\infty}{L^2} v_{yy} \right] \nu
\]

\[
\Rightarrow u_x + uu_x + vv_y + p_x = \frac{1}{R_e} \Delta u, \quad (3.5)
\]

and

\[
\frac{d}{dt} \left( \frac{dt}{dt^*}(vU_\infty) \right) + uU_\infty \left( \frac{dx}{dx^*}(vU_\infty) \right) + vU_\infty \left( \frac{dy}{dy^*}(vU_\infty) \right) + \frac{1}{\rho} \frac{dy}{dy^*}(p\rho U_\infty^2) = \nu \Delta vU_\infty
\]

\[
\Rightarrow \frac{U_\infty^2}{L} \left( v_t + uu_x + vv_y + p_y \right)
\]

\[
= \left( \frac{\partial^2 v^*}{\partial x^*^2} + \frac{\partial^2 v^*}{\partial y^*^2} \right) \nu
\]

\[
= \left[ \left( \frac{\partial}{\partial x} \frac{\partial x}{\partial x^*} \right) \left( \frac{\partial}{\partial x} \frac{\partial x}{\partial x^*} \right) (vU_\infty) \right.
\]
Part II: The Orr-Sommerfeld equation

\[ + \left( \frac{\partial}{\partial y} \frac{\partial}{\partial y^*} \right) \left( \frac{\partial}{\partial y} \frac{\partial}{\partial y^*} \right) [vU_{\infty}] \nu \]

\[ = \left[ \frac{U_{\infty}}{L^2} u_{x\infty} + \frac{U_{\infty}}{L^2} u_{y\infty} \right] \nu \]

\[ \Rightarrow \quad u_t + uu_x + vv_y + p_y = \frac{1}{Re} \Delta u. \]  

(3.6)

These are the respective dimensionless Navier-Stokes equations. Note that \( Re = \frac{U_{\infty}L}{\nu} \) is the Reynolds number.

### 3.2 Linear Stability Analysis

Working with equations (3.4) to (3.6) we shall assume the existence of an equilibrium state. That is, we shall assume that we have found a laminar flow solution to these equations of the form \((U(y), 0, P(x))\), where the first component, \(U(y)\), is the velocity in the \(x\)-direction and the second component represents the \(y\)-velocity and is zero. For example, Poiseuille flow is obtained by taking \(U(y) = y(1 - y), \quad V = 0\) and the pressure \(P(x) = c_1 x + c_2\). In order to establish some stability characterisation of the flow problem, a space and time-dependent perturbation field, \(\epsilon[u, \theta, \rho]\), is imposed on the basic flow field, where \(\epsilon\) is an arbitrary constant introduced to represent small perturbations or infinitesimal disturbances.

\[
\begin{align*}
u(x, y, t) & = U(y) + \epsilon \bar{u}(x, y, t) \quad (3.7) \\
v(x, y, t) & = 0 + \epsilon \bar{\theta}(x, y, t) \quad (3.8) \\
p(x, y, t) & = P(x) + \epsilon \bar{\rho}(x, y, t). \quad (3.9)
\end{align*}
\]

The linear stability of the basic flow field is determined by substituting the superimposed variables given by equations (3.7), (3.8) and (3.9) into the governing equations, i.e., the continuity and Navier-Stokes equations. We subtract out the original steady state equalities and linearize with respect to the perturbation quantities about the basic state, that is, we can neglect terms of \(O(\epsilon^2)\) and smaller (occurring in the nonlinear convective acceleration). Thus,

\[ u_x + v_y = 0 \quad \text{becomes} \quad \frac{\partial}{\partial x} (U(y)) + \bar{u}_x + \bar{v}_y = 0 \]

\[ \Rightarrow \quad \bar{u}_x + \bar{v}_y = 0. \]
Equation (3.5) becomes

$$\frac{\partial u}{\partial t} + (U(y) + \hat{u}) \frac{\partial u}{\partial x} + \frac{\partial (U_y + \hat{u}_y)}{\partial y} + (P_x + \hat{p}_x) = \frac{1}{R_e} \Delta (U + \hat{u}),$$

if we neglect the higher order terms, we are left with

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + \frac{\partial U_y}{\partial y} + P_x + \hat{p}_x = \frac{1}{R_e} \Delta U + \frac{1}{R_e} \Delta \hat{u}. \quad (3.10)$$

Note that, by substituting our equilibrium solution, \([U(y), 0, P(x)]\) into the Navier-Stokes equation we obtain

$$U_t + U U_x + V U_y + P_x = \frac{1}{R_e} \Delta U$$

$$\Rightarrow P_x = \frac{1}{R_e} \Delta U,$$

thus, we can cancel \(P_x\) with \(\frac{1}{R_e} \Delta U\) in equation (3.10) to obtain

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + \frac{\partial U_y}{\partial y} = \frac{1}{R_e} \Delta \hat{u}. \quad (3.11)$$

Similarly, equation (3.6) becomes

$$\frac{\partial v}{\partial t} + (U + \hat{u}) \frac{\partial v}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial \hat{p}_y}{\partial y} = \frac{1}{R_e} \Delta \hat{\theta}$$

and neglecting nonlinear terms, we obtain

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{1}{R_e} \Delta \hat{\theta}. \quad (3.12)$$

thus, we now have our linearized system of pdes for \(\hat{u}, \hat{\theta}\) and \(\hat{p}\), with \(U\) assumed to be given. That is, we have the following system of linear equations.

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + \frac{\partial U_y}{\partial y} - \frac{1}{R_e} \Delta \hat{u} = 0 \quad (3.13)$$

$$\frac{\partial \theta}{\partial t} + U \frac{\partial \theta}{\partial x} + \frac{\partial \theta_y}{\partial y} - \frac{1}{R_e} \Delta \hat{\theta} = 0 \quad (3.14)$$

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{\theta}}{\partial y} = 0 \quad (3.15)$$

where \(U_y = \frac{dU}{dy}\). With boundary conditions

$$\hat{u} = \hat{\theta} = 0 \quad \text{at} \quad y = \pm 1. \quad (3.16)$$

These linearized equations are only valid for infinitesimal perturbations and results should be interpreted with this in mind. Any stability results determined using this method do
not indicate stability of the basic flow with respect to finite disturbances for which the non-linear terms, assumed negligible here, must be included.

The perturbation quantities can be modelled either as spatially or temporally developing disturbances. Although the spatial theory, first proposed by Gaster [46], is more physically realistic in cases when the instability is described as convective, for neutrally stable disturbances, the two methods are equivalent and the temporal method is generally more easily implemented. Thus, since the main purpose of this linear stability analysis is to determine neutral boundaries, the perturbation field is assumed to be made up of an exponential time component, therefore developing temporally with amplitude functions dependent on the spatial variables. Note that we have specified our basic state, $U$, to vary in one direction only, say the $y$-direction, that is, we have assumed a locally parallel basic flow. Since $y$ is the coordinate normal to the wall, we assume that the component $V$ across the layer is negligibly small ($\approx 0$) and further assume that $U \approx U(y)$. The disturbances are also assumed to be parallel flows. Having done this, the most general form of disturbance is a set of one-dimensional travelling waves whose amplitudes vary with $y$ (Tollmien-Schlichting waves). We can thus, systematically reduce the disturbance equations to a single ordinary homogeneous differential equation for the variation of the disturbance. With all this in mind, we specify the similarity solution form for the general solution of the linear system of PDEs

$$
\dot{u}(x, y, t) = \text{Re} \left( \bar{u}(y)e^{i(\alpha x - \omega t)} \right), \quad (3.17)
$$

$$
\dot{\theta}(x, y, t) = \text{Re} \left( \bar{\theta}(y)e^{i(\alpha x - \omega t)} \right), \quad (3.18)
$$

$$
\dot{\bar{p}}(x, y, t) = \text{Re} \left( \bar{\bar{p}}(y)e^{i(\alpha x - \omega t)} \right). \quad (3.19)
$$

All disturbances have wave number $\alpha$, propagation speed $c$, frequency $\omega = \alpha c$ and are historically referred to as Tollmien-Schlichting waves which are the first (infinitesimal) indications of laminar flow instability. The component of the time exponent is $-i\omega = -i\alpha c = \lambda$, where $\lambda = \lambda_r + i\lambda_i$ is complex valued with the real part of $\lambda$ ($\lambda_r$), giving the temporal growth rate.

Substituting this form for the perturbation, namely, (3.17), (3.18) and (3.19), into our system of PDEs removes time and some space derivatives. Thus, equation (3.10) becomes

$$
i\alpha \bar{u}(y)e^{i(\alpha x - \omega t)} + \bar{\theta} e^{i(\alpha x - \omega t)} = 0 \quad (3.20)
$$

$$
\Rightarrow \quad i\alpha \bar{u} + \bar{\theta}_y = 0. \quad (3.21)
$$
Equation (3.11) becomes

\[ -i\omega \tilde{u}e^{i(\alpha x - \omega t)} + U \left( i\alpha \tilde{u}e^{i(\alpha x - \omega t)} \right) + U_y \left( \tilde{u}e^{i(\alpha x - \omega t)} \right) + i\alpha \tilde{p}e^{i(\alpha x - \omega t)} = \frac{1}{Re} \Delta \tilde{u}. \]  

(3.22)

Now, \( \Delta \tilde{u} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), thus, from (3.17),

\[ \frac{\partial^2 \tilde{u}}{\partial x^2} = i^2 \alpha^2 \tilde{u}e^{i(\alpha x - \omega t)} \]

and

\[ \frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{\partial^2}{\partial y^2} \tilde{u}e^{i(\alpha x - \omega t)} \]

and so, substituting these expressions into (3.22), we obtain

\[ -i\omega \tilde{u} + U i\alpha \tilde{u} + U_y \tilde{u} + i\alpha \tilde{p} = \frac{1}{Re} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{u}. \]  

(3.23)

Equation (3.12) becomes

\[ -i\omega \tilde{v}e^{i(\alpha x - \omega t)} + U \left( i\alpha \tilde{v}e^{i(\alpha x - \omega t)} \right) + \tilde{p_y}e^{i(\alpha x - \omega t)} = \frac{1}{Re} \Delta \tilde{v}, \]  

(3.24)

and since, \( \Delta \tilde{v} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), from (3.18) we have

\[ \Delta \tilde{v} = i^2 \alpha^2 \tilde{v}e^{i(\alpha x - \omega t)} + \frac{\partial^2}{\partial y^2} \tilde{v}e^{i(\alpha x - \omega t)} \]

\[ = \left( \frac{\partial^2}{\partial y^2} - \alpha^2 \right) \tilde{v}e^{i(\alpha x - \omega t)}. \]

Finally, substituting this expression into (3.24) gives

\[ -i\omega \tilde{v} + U i\alpha \tilde{v} + \tilde{p_y} = \frac{1}{Re} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}. \]  

(3.25)

We also have the boundary conditions, \( \tilde{u} = \tilde{v} = 0 \) at \( y = 0 \). Thus, equations (3.21), (3.23) and (3.25), form a system of linear ordinary differential equations involving three variables, \( \tilde{u}, \tilde{v} \) and \( \tilde{p} \) with complex coefficients. These ODEs are second order in \( \tilde{u}, \tilde{v} \) and first order in \( \tilde{p} \). We reduce this system of ODEs to just one equation in the following way.

If \( \alpha \neq 0 \), we can solve equation (3.23) for \( \tilde{p} \)

\[ i\alpha \tilde{p} = -i(U\alpha - \omega)\tilde{u} - U_y \tilde{v} + \frac{1}{Re} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{u}. \]  

(3.26)

Rearranging equation (3.21) we have

\[ \tilde{u} = \frac{\tilde{v}_y}{i\alpha} = \frac{i}{\alpha} \tilde{v}_y, \]  

(3.27)
and so, substituting (3.27) into (3.26) we get

\[ i \alpha \tilde{p} = \frac{i}{\alpha} (U \alpha - \omega) \tilde{v}_y - U_y \tilde{v} + \frac{i}{Re \alpha} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}. \]  

(3.28)

Now, differentiating equation (3.28) with respect to \( y \) gives

\[ i \alpha \tilde{p}_y = U \tilde{v}_{yy} + \tilde{v}_y U_y - \frac{\omega}{\alpha} \tilde{v}_{yy} - (U_y \tilde{v}_y + \tilde{v} U_{yy}) + \frac{i}{Re \alpha} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_{yy} \]

\[ = \frac{1}{\alpha} (\alpha U - \omega) \tilde{v}_{yy} - \tilde{v}_y U_y + \frac{i}{Re \alpha} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_{yy} \]

\[ \Rightarrow \tilde{p}_y = \frac{1}{i \alpha^2} (\alpha U - \omega) \tilde{v}_{yy} - \frac{i}{\alpha} U_{yy} \tilde{v} + \frac{1}{Re \alpha^2} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_{yy}, \]

that is

\[ \tilde{p}_y = \frac{1}{i \alpha^2} (\alpha U - \omega) \tilde{v}_{yy} - \frac{i}{\alpha} U_{yy} \tilde{v} + \frac{1}{Re \alpha^2} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_{yy}. \]  

(3.29)

Now, substituting (3.29) into (3.25) we obtain

\[ -i \omega \tilde{v} + U i \alpha \tilde{v} + \frac{1}{i \alpha^2} (\alpha U - \omega) \tilde{v}_{yy} + \frac{i}{\alpha} U_{yy} \tilde{v} + \frac{1}{Re \alpha^2} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_{yy} = \frac{1}{Re} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}, \]

that is

\[ i(\alpha U - \omega) \tilde{v} - i \frac{i}{\alpha^2} (\alpha U - \omega) \tilde{v}_{yy} + \frac{i}{\alpha} U_{yy} \tilde{v} + \frac{1}{Re \alpha^2} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_y - \frac{1}{Re \alpha^2} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \tilde{v}_{yy}. \]

Making the substitution, \( \tilde{v} = \phi \) (the dimensionless disturbance amplitude) and rearranging yields

\[ i(\alpha U - \omega) \left( \phi - \frac{1}{\alpha^2} \phi'' \right) + \frac{i}{\alpha} U'' \phi = \frac{1}{Re} \left( \frac{d^2}{dy^2} - \alpha^2 \right) \left( \phi - \frac{1}{\alpha^2} \phi'' \right) \]

rearranging further gives

\[ \frac{1}{Re} \left( \frac{d^2}{\alpha^2} - \alpha^2 \phi + \phi'' \right) - i(\alpha U - \omega) \left( \phi - \frac{\phi''}{\alpha^2} \right) = 0 \]

\[ \Rightarrow \frac{1}{Re} \left( \frac{d^2}{\alpha^2} - 2 \phi'' + \alpha^2 \phi \right) + i(\alpha U - \omega) \left( \phi - \frac{\phi''}{\alpha^2} \right) + \frac{i}{\alpha} U'' \phi = 0. \]

Finally, multiplying through by \( \alpha^2 \) yields the required Orr-Sommerfeld equation.

\[ \frac{1}{Re} \left( \phi'' - 2 \alpha^2 \phi'' + \alpha^2 \phi \right) - i(\alpha U - \omega) \left( \phi - \alpha^2 \phi \right) + i \alpha U'' \phi = 0. \]  

(3.30)

The secrets of infinitesimal laminar-flow instability lie within this fourth order linear homogeneous equation first derived independently by Orr (1907) and Sommerfeld (1908). It is the eigenvalues of this equation that determine the stability of the parallel basic flow.
The boundary conditions are that the disturbance, $\bar{u}, \bar{v}$ must vanish at infinity and at any walls (no-slip). Now the continuity relation shows that if $\bar{u} = 0 \Rightarrow \bar{v}' = 0$, hence the proper conditions on the OS equation for boundary layers are:

$$\bar{v}(0) = \bar{v}'(0) = 0, \quad \lim_{y \to \infty} \bar{v}'(y) = \lim_{y \to \infty} \bar{v}'(y) = 0.$$ 

In spite of the relative simplicity of the Orr-Sommerfeld (OS) equation, it cannot be solved explicitly except for rare cases. Therefore, the majority of results known, concerning the stability of fluid flows governed by the OS equation, have been determined numerically [38]. In theory, a solution to the OS equation with appropriate boundary conditions would correctly simulate the flow of a real Newtonian fluid over a body in two-dimensional flow.

During this research we wish to analyse the stability of various fluid flows past a compliant surface in comparison to the stability of corresponding flows past a rigid flat plate. This requires the use of appropriate numerical techniques. The following section illustrates a historical numerical method using the simple plane Poiseuille flow velocity profile.

### 3.3 Numerical methods—Shooting techniques

Parabolic Poiseuille channel flow is one of the simplest laminar flow solutions. Its main feature consists of a pressure gradient between fixed walls so that $U(y) = U_{\infty}(1 - y^2)$. A review by White [135] states that Thomas [128] analysed this profile as one of the first applications of the digital computer to the Orr-Sommerfeld equation. White also states that it was also the occasion for one of the last of the asymptotic analyses by Shen [120] who computed the value of the critical Reynolds number to be $Re_c = \frac{U_{\infty}L}{\nu} = 5,360$ and corresponding critical wave number $\alpha_c = 1.05$ with a five per cent accuracy level. A subsequent machine calculation by Nachsheim [91] gave a critical Reynolds number of $Re_c = 5,767$ again with corresponding wave number $\alpha_c = 1.02$.

The Orr-Sommerfeld equation has three parameters, $\alpha, Re$ and $c$. We assume that the disturbances grow temporally so that the wave number, $\alpha$, is taken to be real and positive, while the wave speed, $c$, is in general complex. Since the OS equation and its boundary conditions are both homogeneous, it follows that what we are talking about is an eigenvalue problem. For a given profile, $U(y)$ and $U''(y)$, only a certain continuous but
limited sequence of these parameters (the eigenvalues) will satisfy the OS equation and its boundary conditions. The mathematical problem is to find this sequence which has the following functional form for temporal growth of disturbances

$$f(R_e, \alpha, c_r, c_i) = 0,$$

where the subscripts \( r \) and \( i \) mean real and imaginary parts, respectively. Our interest lies in the case of neutral stability. In this case, for fixed values of \( R_e \) and \( \alpha \), the basic flow is deemed to be linearly unstable if the perturbations grow exponentially in time, i.e., the imaginary part of \( c \), \( (c_i) \) is positive, \( (c_i > 0) \), and hence stable if \( c_i < 0 \). Thus, the aim is to find those combinations of the wave number, \( \alpha \), and Reynolds number, \( R_e \), for which the fluid flow changes from being stable to unstable. These points are then plotted to form, what is known as The Curve of Neutral Stability, where the locus of these neutral points, \( c_i = 0 \), or in practice, \( c_i \approx 0 \) to some reasonable computational accuracy, forms the boundary between stability and instability.

If we rewrite the Orr-Sommerfeld equation in the following form

$$\frac{\phi''''}{a(y,c)} + \left(-2\alpha^2 - i\alpha R_e (U(y) - c)\right) \frac{\phi''}{b(y,c)} + \left(\alpha^4 + i\alpha^3 R_e (U(y) - c) + i\alpha R_e U''(y)\right) \phi = 0$$

and using the velocity profile for Poiseuille flow, \( U(y) = 1 - y^2 \) for \(-1 \leq y \leq 1\). Then for fixed values of \( \alpha \) and \( R_e \), these equations define an eigenvalue problem for the complex eigenvalue \( c \).

The Orr-Sommerfeld equation is a fourth order linear boundary value problem. A number of numerical techniques can be employed to solve this eigenvalue problem. But, because of the convenience and power of codes for the initial value problem, historically, a popular way to solve boundary value problems is by initial value methods. The approach is called a shooting method. The following subsection introduces this numerical technique and moves on to describe its application with respect to this specific problem.

3.3.1 Application of shooting method to Poiseuille flow problem.

Shooting techniques for linear equations are based on the replacement of a linear boundary value problem by two initial value problems. To do this, we consider the transformation
of the problem into a linear system of ODEs of the form

\[ u_y = A(y, \lambda)u, \quad u \in \mathbb{C}^4 \]  

(3.31)

where \( \lambda \in \omega \subset \mathbb{C} \) is a complex parameter, \( \lambda \) is a function of wave number and wave-speed, \( (\lambda = -i\alpha c) \) and \( A(y, \lambda) \) is a \( 4 \times 4 \) matrix depending analytically on \( \lambda \) and \(-1 \leq y \leq 1\).

Since \( u \) and \( u_y \) represent the vertical and horizontal velocity respectively, the system has two homogeneous boundary conditions imposed at \( y = -1 \) and two at \( y = 1 \) for the linearization about Poiseuille flow (see figure 1.1). These boundary conditions are given by zero normal velocity and no-slip.

\[ u(-1) = u'(-1) = 0, \]
\[ u(+1) = u'(+1) = 0. \]

Rewriting these using the basis consisting of the standard four vectors, \( e_1, e_2, e_3 \) and \( e_4 \) we have

\[ (e_i, u(\pm 1, \lambda)) = (e_2, u(\pm 1, \lambda)) = 0, \]

where \((.,.)\) is a standard inner product on \( \mathbb{C}^4 \). Now, with the above setup, the system (3.31) is an eigenvalue problem with eigenvalue parameter \( \lambda \).

Due to the symmetry in the problem, and eigenfunction issues discussed in section (7.4), the natural approach to integrating (3.31) would be to integrate the induced system

\[ u'' = A(y, \lambda)u, \quad u(0, \lambda) = U_0 \in \mathbb{C}^{4 \times 2} \]

(3.32)

from \( 0 \leq y \leq 1 \), where the initial conditions are chosen to be two linearly independent and orthogonal vectors. That is, the columns of \( U_0 \) span the two dimensional subspace which satisfies the boundary conditions at \( y = 0 \).

The induced system, (3.32) is then integrated with a numerical method of sufficient accuracy, such as the fourth order Runge-Kutta algorithm, from \( y = 0 \) to \( y = 1 \). Imposition of the boundary conditions at \( y = 1 \) then leads to a complex analytic function \( D(\lambda) \) whose zeros correspond to eigenvalues of (3.31) [18].

Let \( u = [u_1 \quad u_2] \), then numerically integrate the ODE 3.32 from \( y = 0 \) to \( y = +1 \) using an integration method such as the fourth-order Runge-Kutta to obtain \( u_1(y, \lambda) \) and \( u_2(y, \lambda) \). The general solution for our boundary value problem, \( u_y = A(y, \lambda)u \), satisfying the boundary condition at \( y = 0 \) is then given by the expression

\[ u(y, \lambda) = a_1 u_1(y, \lambda) + a_2 u_2(y, \lambda) \]

(3.33)
for some complex constants $a_1$ and $a_2$ and $\lambda = -i\alpha c$.

These constants are determined by applying the right hand boundary conditions, namely: $(e_1, u(1, \lambda)) = (e_2, u(1, \lambda)) = 0$ which leads to

\[
\begin{pmatrix}
  u_{11}(1, \lambda) & u_{12}(1, \lambda) \\
  u_{21}(1, \lambda) & u_{22}(1, \lambda)
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\]

The trivial solution of this is \( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \). In order for \( \lambda \in \mathbb{C} \) to be an eigenvalue we require \( |a_1|^2 + |a_2|^2 \neq 0 \), which is equivalent to the determinant, \( \Delta(\lambda) = \det(M(\lambda)) = 0 \) (note that \( \Delta(\lambda) \) is a complex function of a complex variable). Thus we calculate \( \Delta(\lambda) \) and use Newton’s algorithm to converge its value to zero. For the majority of this report, code is written in Fortran 77. Parameter values for the OS equation are required, positive, real values for the wavenumber, $\alpha$ and Reynolds number, $Re$ are assigned and a good guess for the wave-speed, $c \in \mathbb{C}$, is necessary for convergence to occur.

### 3.3.2 Evaluation of the shooting method

This shooting method strategy is not always effective, unless the Reynolds number is small. When the Reynolds number is large, the shooting method becomes unstable. This problem is illustrated in the next chapter using a model example. The resolution to this problem will be presented in chapter 5.
The numerical solution of the Orr-Sommerfeld equation is bound to present some problems, since the highest order term $\phi'''$ is multiplied by the small factor $\frac{1}{Re}$ and so, the solutions are non-regular as the Reynolds number becomes very large. That is, the equation (3.31) is stiff. Stiffness occurs when there are two or more very different scales governing the solutions. The two solutions $u_1$ and $u_2$ in (3.33) are initially linearly independent, however, this independence is lost as the steps of the numerical integration proceed. Further, asymptotic analysis shows that of the four independent solutions needed, at least one of these, $\phi_4$ say, grows at the rate $\exp[Re^{\frac{1}{5}}]$ [135]. Now, since $\phi_1$ & $\phi_2$ are of order unity, they must somehow be kept separate in the computation. This effect is a purely numerical problem due in part to computer roundoff/truncation error. In this case, $u_1$ and $u_2$ contain parts of the rapidly varying viscous solutions, $\phi_4$, which tend to dominate the slower varying components and become increasingly more dominant as the integration proceeds. If this loss of independence is not checked, the result is a rapid accumulation in the numerical values of the solutions and this parasitic growth then renders the results meaningless.

To illustrate these difficulties associated with solving the eigenvalue problem of the Orr-Sommerfeld equation, a fourth order ODE with constant coefficients is used as a test equation. This type of equation is useful since we know the general solution, and so, can easily check the code output.
4.1 Fourth order ODEs and Numerical problems encountered: a test case

Consider the following fourth order ODE with constant coefficients

\[ U^{iv} - 24U''' - 169U'' - 324U' - 180U = 0. \]  

(4.1)

To find the general solution, let us assume the solution takes the general form \( U(y) = A e^{\mu y} \).

Substituting this into equation (4.1) we obtain

\[ (\mu^4 - 24\mu^3 - 169\mu^2 - 324\mu - 180)A e^{\mu y} = 0, \]

which can be factorised to give

\[ (\mu + 1)(\mu + 2)(\mu + 3)(\mu - 30) = 0. \]  

(4.2)

Thus, our general solution is a linear combination of exponentials with the roots of (4.2) as its exponents, namely

\[ U(y) = a_1 e^{-y} + a_2 e^{-2y} + a_3 e^{-3y} + a_4 e^{30y}, \]  

(4.3)

where \( a_1, a_2, a_3, a_4 \) are arbitrary constants.

Numerical problems arise due to the inability of the computer to handle the huge value of \( e^{30y} \approx 1 \times 10^{13} \), when \( y = 1 \). However, suppose the following boundary conditions are imposed on the ODE for \( 0 \leq y \leq 1 \)

\[ U(0) = 0, \quad U(1) = e^{-1} - 2e^{-2} + e^{-3}, \]

\[ U'(0) = 0, \quad U'(1) = -e^{-1} + 4e^{-2} - 3e^{-3}. \]

Using these boundary conditions, expressions for the coefficients \( a_1 \ldots a_4 \) in equation (4.3) are given by

\[ U(0) = 0 \Rightarrow 0 = a_1 + a_2 + a_3 + a_4 \]

\[ U'(0) = 0 \Rightarrow 0 = -a_1 - 2a_2 - 3a_3 + 30a_4 \]

\[ U(1) = e^{-1} - 2e^{-2} + e^{-3} \Rightarrow \]

\[ e^{-1} - 2e^{-2} + e^{-3} = a_1 e^{-1} + a_2 e^{-2} + a_3 e^{-3} + a_4 e^{30} \]

\[ U'(1) = -e^{-1} + 4e^{-2} - 3e^{-3} \Rightarrow \]

\[ -e^{-1} + 4e^{-2} - 3e^{-3} = -a_1 e^{-1} - 2a_2 e^{-2} - 3a_3 e^{-3} + 30a_4 e^{30}. \]
Solving the above equations gives $a_1 = 1$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$. Thus, the general solution which satisfies the four boundary conditions is

$$U(y) = e^{-y} - 2e^{-2y} + e^{-3y}.$$  

Hence, mathematically, the imposed boundary conditions have eliminated the large problem term, $e^{30y}$.

By setting up the ODE as a linear vector-matrix system in standard form in $\mathbb{R}^4$ for coding and by letting $u_1 = U$, $u_2 = U'$, $u_3 = U''$ and $u_4 = U'''$, we obtain

$$u_y = Au \quad u \in \mathbb{R}^4,$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 180 & 324 & 169 & 24 \end{pmatrix}.$$  

Let us define four standard unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and introduce the general inner product as

$$(u, v) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4,$$

where $\bar{u}$ represents complex conjugation. That is, each component of the vectors $\bar{u}$ and $v$ multiplied together and then summed. Then we can rewrite the boundary conditions in the following vector form.

$$\begin{align*}
(e_1, u(0)) &= 0 \quad (4.4) \\
(e_2, u(0)) &= 0 \quad (4.5) \\
(e_1, u(1)) &= e^{-1} - 2e^{-2} + e^{-3} \quad (4.6) \\
(e_2, u(1)) &= -e^{-1} + 4e^{-2} - 3e^{-3}. \quad (4.7)
\end{align*}$$

For example, the boundary condition (4.4) corresponds to $U(0) = 0$ since (4.4) means

$$1 \times u_1(0) + 0 \times u_2(0) + 0 \times u_3(0) + 0 \times u_4(0) = 0 \quad \Rightarrow \quad u_1(0) = 0.$$

Now, $u_1$ in the system...
is equivalent to \( U \), thus \( U(0) = 0 \) is satisfied.

Even though we have shown that the solution from the boundary conditions does not involve the large term, \( e^{30y} \), numerically, the solution will run to this large value since it emphasises the errors introduced at each iteration. An explanation of what occurs numerically is given below.

### 4.1.1 Algorithmic aspects of the numerical integration using shooting methods

Integrate the linear system (3.31) twice using two different starting vectors. That is, integrate

\[
\begin{align*}
\mathbf{u}_1' &= \mathbf{A} \mathbf{u}_1 \quad \text{with} \quad \mathbf{u}_1(0) = e^3 . \tag{4.8} \\
\mathbf{u}_2' &= \mathbf{A} \mathbf{u}_2 \quad \text{with} \quad \mathbf{u}_2(0) = e^4 . \tag{4.9}
\end{align*}
\]

from \( y = 0 \) to \( y = 1 \), where \( \text{prime} \) denotes differentiation with respect to \( y \) and use an accurate numerical integrator such as the fourth order Runge Kutta method. The general solution satisfying the boundary conditions at \( y = 0 \) is then defined by \( \mathbf{u}(y) = c_1 \mathbf{u}_1(y) + c_2 \mathbf{u}_2(y) \). This solution, however, does not necessarily satisfy the right hand boundary conditions, thus, we determine the constants \( c_1 \) and \( c_2 \) by imposing the remaining boundary conditions. That is, (4.6) and (4.7) respectively become

\[
\begin{align*}
\langle e_1, c_1 \mathbf{u}_1(1) + c_2 \mathbf{u}_2(1) \rangle &= a_1 \tag{4.10} \\
\langle e_2, c_1 \mathbf{u}_1(1) + c_2 \mathbf{u}_2(1) \rangle &= a_2 \tag{4.11}
\end{align*}
\]

where, \( a_1 = e^{-1} - 2e^{-2} + e^{-3} \) and \( a_2 = -e^{-1} + 4e^{-2} - 3e^{-3} \).

Thus, equations (4.10) and (4.11) can be written as a matrix equation

\[
\begin{pmatrix}
\langle e_1, \mathbf{u}_1(1) \rangle \\
\langle e_2, \mathbf{u}_1(1) \rangle
\end{pmatrix}
\begin{pmatrix}
\langle e_1, \mathbf{u}_2(1) \rangle \\
\langle e_2, \mathbf{u}_2(1) \rangle
\end{pmatrix}
\begin{pmatrix}
c_1 \\
\ c_2
\end{pmatrix} =
\begin{pmatrix}
a_1 \\
\ a_2
\end{pmatrix} ,
\]

where \( \langle e_1, \mathbf{u}_1(1) \rangle \) is the first component of the vector \( \mathbf{u}_1 \) and \( \langle e_2, \mathbf{u}_1(1) \rangle \) is the second component of the vector \( \mathbf{u}_1 \), similarly for the vector \( \mathbf{u}_2 \). If we rewrite this as

\[
M \begin{pmatrix}
c_1 \\
\ c_2
\end{pmatrix} =
\begin{pmatrix}
a_1 \\
\ a_2
\end{pmatrix} ,
\]
then the constants $c_1$ and $c_2$ can be obtained easily by finding the inverse of the matrix $M$ and calculating
\[
\begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix} = M^{-1} \begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix}.
\] (4.12)

Mathematically, $M$ is not singular, so an inverse exists. Numerically, however, the computations may lead to an $M$ which is almost singular (determinant near zero). This is because, when integrating $u_1$ and $u_2$ they are both attracted to the vector associated with the $e^{30y}$ term, and so, may lose linear independence. The code, written in Fortran 77, was first checked using the following well-posed problem:
\[
U''' + 10U'' + 35U' + 50U + 24U = 0,
\] (4.13)

which has general solution
\[
U(y) = a_1e^{-y} + a_2e^{-2y} + a_3e^{-3y} + a_4e^{-4y}.
\]

The boundary conditions imposed are $U(0) = U'(0) = 0$, $U(1) = e^{-1} - 2e^{-2} + e^{-3}$ and $U'(1) = -e^{-1} + 4e^{-2} - 3e^{-3}$. Thus the solution is equivalent to that of equation (4.1), namely
\[
U(y) = e^{-y} - 2e^{-2y} + e^{-3y}.
\]

Results

A comparison of the results for the well-posed problem, (4.13) and ill-posed problem (4.1) are set out below.

The following results were obtained using a standard fourth-order Runge-Kutta integration procedure.
Table 4.1: Table (a): Numerical errors for the well-conditioned problem, Table (b): Numerical errors for the ill-conditioned problem

From Table 4.1 (a), it can be seen that the program solves the ODE with acceptable accuracy. However, when numerically integrating often occurring ill-conditioned problems, the errors grow exponentially. Table 4.1 (b) shows results obtained from numerically integrating the ill-conditioned problem

\[ U''' - 24U'' - 169U' - 324U - 180U = 0, \quad (4.14) \]

using the same numerical methods carried out on the well-posed ODE, (4.13).

The main numerical difficulties in integrating the Orr-Sommerfeld equation using shooting come about because it is highly unstable. This highly stiff, and also unstable, characteristic makes the use of conventional numerical schemes impossible.

One remedy is to avoid numerical integration altogether and set up the OS equation as
a finite-difference problem, with eigenvalues found by inverting a large matrix. However, this method is limited to a growth factor of about $10^8$ in the dangerously increasing solution, $\phi_4$, say.

A second remedy is to use double precision arithmetic which has the disadvantage that it does chew up much more computer time but offers the advantage that it increases the allowable growth factor to about $10^{18}$ and with todays computers, the computing time is really not effected too much, thus double precision will certainly be used throughout the numerics in this research. However, without some form of stabilization, we are still restricted to low Reynolds numbers.

A breakthrough in numerical accuracy was made by Kaplan [68], who devised a scheme for purifying the small solutions by subtracting from them at each integration step a quantity proportional to the faster growing solution, thus preserving their linear independence. Kaplan's procedure allows growth factors as large as $10^{40}$ corresponding to Reynolds numbers as high as $R_e = 10^4$ (Betchov & Crininale [12], Landahl & Kaplan [79]). Finally, Kaplan's scheme was generalized rigorously by Bellman and Kalaba [7] into a Gram-Schmidt orthonormalization procedure for ensuring the linear independence of the numerical solution. The Gram-Schmidt approach is one of the most well known approaches to addressing this linear dependence problem and has proved useful in providing a stable numerical solution of an ODE of sufficient accuracy by elimination of the growth problem arising from numerically integrating these ill-conditioned problems. This method was used in boundary layer computations by Wazzan, Okamura and Smith [133] where results allow for a growth factor of $10^{80}$ or Reynolds number up to $10^5$ which covers almost any practical case of boundary-layer instability. The Gram-Schmidt procedure along with other useful methods shall be addressed in the next chapter.
The numerical instability illustrated in Table (3.2) can be eliminated by forcing the columns of $U$ to remain linearly independent. Let $U = (u_1 \ u_2)$ then one way of ensuring numerical independence of the solution over the whole integration range is to keep the vectors $u_1$ and $u_2$ mutually orthogonal. This means that as the vectors start to lose independence, some form of normalization process is required. We start by defining orthogonality and go on to introduce techniques that have been formulated to deal with maintaining it.

5.1 Orthogonality

In a two dimensional space, vectors $u_1 \in \mathbb{R}^2$ and $u_2 \in \mathbb{R}^2$ are orthogonal if they are at right angles to each other. However, in higher dimensional spaces, it is their inner product that determines orthogonality.

Let

$$u_1 = \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{pmatrix} \in \mathbb{R}^4 \quad \text{and} \quad u_2 = \begin{pmatrix} u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \end{pmatrix} \in \mathbb{R}^4$$

Then $u_1$ and $u_2$ are said to be orthogonal if their inner product is zero, that is, if

$$(u_1, u_2) = 0.$$
Now, the initial vectors for the numerical integration (i.e. at \( y = 0 \)) in section 2.1.1 are chosen to be

\[
\begin{align*}
\mathbf{u}_1(y)|_{y=0} &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2(y)|_{y=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

These initial vectors are orthogonal and linearly independent. However, as the steps of the numerical integration proceed, orthogonality is lost and in fact, the two vectors become close to a multiple of one another, hence, almost linearly dependent.

In a well-posed problem, two vectors will not necessarily remain orthogonal but will maintain linear independence. For example, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are orthogonal and linearly independent. Whereas, \( \begin{pmatrix} 1 \\ a \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are not orthogonal but are still linearly independent for any finite \( a \in \mathbb{R} \). Hence, orthogonal \( \Rightarrow \) linear independent, but, linear independent \( \not\Rightarrow \) orthogonal. Thus, to ensure that two vectors remain linearly independent, we shall orthogonize the two vectors at certain iteration points of the numerical integration.

A key question is what constitutes a loss of numerical independence and how often should this re-orthonormalization take place? This question shall be addressed in the next section.

### 5.2 Gram-Schmidt Orthogonalization

One simple rule which is effective when in conjunction with fixed time step integrators is to perform re-orthonormalization each time the magnitude of one of the vectors exceeds some specified value.

Gram-Schmidt orthogonalization is a discrete orthogonalization technique where the Gram-Schmidt algorithm is applied to the columns of (3.32) every few time steps. This criterion proves easy to incorporate and efficient to compute. Once a suitable value for this tolerance on the magnitude, \( M \), say, is decided upon for the particular problem at hand then the orthonormalization process is as follows.

Let \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) be two linearly independent vectors. The aim of Gram-Schmidt orthog-
onalization is to produce two new vectors which are orthogonal, the procedure of which follows. Let the two linearly independent vectors be defined as

\[ \mathbf{u}_1 = \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \end{pmatrix}. \]

If \( \|\mathbf{u}_1\| > M \) then \( \mathbf{u}_1 \) is normalized to give a new vector \( \mathbf{v}_1 \). That is, we scale \( \mathbf{u}_1 \) so that it has unit length. The length of a vector is defined to be

\[ \|\mathbf{u}_1\| = \sqrt{u_{11}^2 + u_{21}^2 + u_{31}^2 + u_{41}^2}. \]

Thus, we define a new vector variable \( \mathbf{v}_1 \), such that

\[ \mathbf{v}_1 = \frac{1}{\|\mathbf{u}_1\|} \mathbf{u}_1. \]

so that \( \mathbf{v}_1 \) is the vector \( \mathbf{u}_1 \) scaled with unit length. The second vector is then made orthogonal to this new first vector using the Gram-Schmidt transformation. This is achieved by defining a new vector, \( \mathbf{v}_2 \), such that

\[ \mathbf{v}_2 = \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1. \]

then, checking orthogonality between the two vectors,

\[ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \langle \mathbf{u}_2, \mathbf{v}_1 \rangle - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 0, \]

since \( \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1 \). Thus, the new vector \( \mathbf{v}_2 \) is orthogonal to \( \mathbf{v}_1 \). The second vector can now be normalized, however, this is not essential. To normalize \( \mathbf{v}_2 \) so that it has unit length, we set

\[ \mathbf{v}_2 \rightarrow \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2. \]

It is desirable for efficiency of computation to have the minimum number of re-orthonormalizations whilst maintaining the required degree of accuracy in the eigenvalues. For a given number of integration steps a value of \( M = 1000 \) was found to be sufficient and this typically resulted in around 10 re-orthonormalization steps in the integration process for the integration of the Orr-Sommerfeld problem. As a rigorous test on this value of
Part II: Maintaining linear independence numerically

$M$, results were compared to eigenvalues obtained with re-orthonormalization performed at every integration step. The results of this test revealed no change in the eigenvalues indicating the suitability of the chosen value of $M$, for this class of equations.

Once we have obtained the values of the required solution constants, $\tilde{c}_1$ and $\tilde{c}_2$, from the program combining orthogonalization with numerical integration of an ODE, it is necessary to scale these constants back to obtain their true values. In matrix form, the transformation at each step can be represented as

$$[v_1 v_2] = [u_1 u_2]H,$$

where $[v_1 v_2]$ and $[u_1 u_2]$ are $4 \times 2$ matrices and $H$ is the $2 \times 2$ matrix

$$H = \begin{pmatrix}
\frac{1}{\|u_1\|} & -\frac{\langle u_1, u_2 \rangle}{\|v_2\|\|u_1\|^2} \\
0 & \frac{1}{\|v_2\|}
\end{pmatrix}. \quad (5.1)$$

The global transformation is achieved by calculating the matrix $H_j$ at each iteration, $j$, where orthogonalization takes place. These are then multiplied together and the true values of the required solution constants are calculated from

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \prod_{j=1}^{N} H_j \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix},$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are the constants obtained from the program using orthonormalization.

5.2.1 Proof of how orthogonalization works

Let us look in general at the problem, $u_y = A(y)u$ where $u \in \mathbb{R}^4$. We initially have two vectors, $u_1(y)$ and $u_2(y)$ and so, are required to solve the following two problems:

$$u_{1y} = A(y)u_1 \quad \text{with} \quad u_1(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$u_{2y} = A(y)u_2 \quad \text{with} \quad u_2(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$
Let us, thus, define a $4 \times 2$ matrix such that

$$U(y) = [u_1(y) \quad u_2(y)] = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \\ u_{41} & u_{42} \end{bmatrix}. $$

Then, $U(y)$ satisfies

$$\frac{d}{dy} U(y) = A(y) U \quad \text{with} \quad U(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

For illustration, we shall now apply Euler's method to this differential equation.

Let $U^i = U(y_i)$ where $y_i = ih$ and $h$ is the step size of each iteration. Euler's method leads to

$$U^{i+1} = U^i + hA(y_i) U^i = [I + hA(y_i)] U^i = L_i U^i $$

with $U^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $L_i = I + hA(y_i)$. Therefore, the discrete version of the problem can be written in the form $U^{i+1} = L_i U^i$. So,

$$U^1 = L_0 U^0$$

$$U^2 = L_1 U^1 = L_1 L_0 U^0$$

$$U^3 = L_2 U^2 = L_2 L_1 L_0 U^0$$

$$\vdots$$

$$U^n = L_{n-1} U^{n-1} = L_{n-1} L_{n-2} \ldots L_1 L_0 U^0$$

$$= \left( \prod_{j=0}^{n-1} L_j^{-1} \right) U^0$$

$$= L U^0,$$
assuming each $L_j$ is invertible, where $L = L_{n-1}L_{n-2} \ldots L_1L_0$.

This procedure will now be repeated applying orthogonalization at each integration step. At some $i$, suppose the $4 \times 2$ matrix $U^i$ has linearly independent columns. Applying the Gram-Schmidt procedure to the two columns of $U^i$ gives

$$V^i = U^iH^i,$$

so, $V^i$ is a matrix with orthonormal columns (orthogonal and of unit length). Now, initially we have

$$U^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After the first iteration of the Euler method we have, $U^1 = L_0U^0$. Applying the Gram-Schmidt procedure to $U^1$, we obtain our orthogonalized vector, $V^1 = U^1H^1$. It then follows that

$$U^2 = L_1V^1 = L_1U^1H^1 = L_1L_0U^0H^1.$$

Applying the Gram-Schmidt procedure to $U^2$ to orthogonalize the vectors produces $V^2 = U^2H^2$, and so

$$U^3 = L_2V^2 = L_2U^2H^2 = L_2L_1L_0U^0H^1H^2$$

$$\vdots$$

$$U^i_{ortho} = L_{i-1}L_{i-2} \ldots L_1L_0U^0H^1H^2 \ldots H^{i-1}$$

$$= L(U^iH),$$

where $H = \prod_{j=1}^{i-1} H^j$ and $L$ is defined previously.

Hence, in comparison with the result obtained without orthogonalization, the initial data has been adjusted by $H$. Using $LU^0 = U^a$ and $U^a_{ortho} = LU^0H$ we obtain

$$U^a = U^a_{ortho}H^{-1}. \quad (5.2)$$

Using orthogonalization, at the end of the iterative process we have $U^n_{ortho}$. However, we want to apply the boundary conditions to $U^n_{ortho}$. Now,
and $U_n = U_{ortho}^n H^{-1}$.

**Lemma**

If $c_1$ and $c_2$ are the values of the constants found using $U_{ortho}^n$, then the required values of the constants for the original problem are obtained by

$$
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = H \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix},
$$

the proof of which follows:

**Proof**

From equation (5.2) we have

$$U_{ortho}^n = U^n H.$$  

The boundary conditions are

$$\langle e_1, U^n c \rangle = \text{boundary condition 1}$$

$$\langle e_2, U^n c \rangle = \text{boundary condition 2},$$

that is

$$
\begin{pmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix}
\text{boundary condition 1} \\
\text{boundary condition 2}
\end{pmatrix}.
$$

Thus,

$$\langle e_1, U_{ortho}^n H^{-1} c \rangle = \text{boundary condition 1}$$

$$\langle e_2, U_{ortho}^n H^{-1} c \rangle = \text{boundary condition 2}$$

and, this can be rewritten as

$$
\begin{pmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{pmatrix} (H^{-1}) \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix}
\text{boundary condition 1} \\
\text{boundary condition 2}
\end{pmatrix}.
$$  

(5.3)

Now, let

$$M = \begin{pmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{pmatrix},$$

rearranging (5.3) we obtain the required solution constants

$$
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = HM^{-1} \begin{pmatrix}
\text{boundary condition 1} \\
\text{boundary condition 2}
\end{pmatrix}.$$
As proposed.

The Gram-Schmidt orthonormalization technique will now be put to the test and results compared with those obtained by Bellman & Kalaba [7].

5.2.2 Comparison with results in Bellman & Kalaba

Reconsidering the ill-conditioned ODE, (4.1), which has general solution of the form

$$U(y) = a_1e^{-y} + a_2e^{-2y} + a_3e^{-3y} + a_4e^{30y}. \quad (5.4)$$

Let us focus our attention on the particular solution

$$U(y) = e^{-y} - 2e^{-2y} + e^{-3y}, \quad (5.5)$$

which satisfies the initial conditions

$$U(0) = 0,$$
$$U'(0) = 0,$$
$$U''(0) = 2.0,$$
$$U'''(0) = -12.0.$$

In addition, it satisfies the following conditions, at \( y = 1 \),

$$U(1) = e^{-1} - 2e^{-2} + e^{-3} \approx 0.146996,$$
$$U'(1) = -e^{-1} + 4e^{-2} - 3e^{-3} \approx 0.241005 \times 10^{-1}.$$

We shall attempt to determine the function \( U(y) \) given in (5.5) as the solution of (4.1) subject to the boundary conditions \( U(0) = 0, U'(0) = 0, U(1) = 0.146996, U'(1) = 0.0241005 \).

By employing the method of complementary functions with the Runge-Kutta integration procedure, using an integration interval of 0.01, Bellman & Kalaba [7] obtained disappointing results for the values of the solution constants, \( c_1 = -0.609136 \times 10^{-2} \) and \( c_2 = 0.365481 \times 10^{-1} \), which, theoretically, should be 2.0 and -12.0 respectively.

Bellman & Kalaba state that the difficulty lies in the fact that the characteristic values associated with the differential operator in (5.4), namely, (-1,-2,-3,30) differ greatly
in their real parts. This means that one of the complementary functions will dominate
the others and make it difficult to determine what linear combination of complementary
functions satisfies the given boundary conditions. It is this problem that is referred to as
ill-conditioning.

To remedy the situation, orthogonalization was used on the same problem which pro­
duced values of the solution constants, namely, 1.9999997 and -12.000003, which are very
close to the exact results (For details see [7] pages 101 - 103).

When trying to reproduce these results, I found that the ill-conditioning of the problem
was not great enough to give poor results without orthogonalization. This is most probably
due to my use of double precision as opposed to single precision adopted by Bellman &
Kalaba. I found that adopting single precision for the iterative procedure was enough
to obtain disappointing results, and have shown that with orthogonalization added, the
problem is resolved and accurate results are obtained. Thus, it is clear that for more
extreme ill-posed problems, orthogonalization will need to be employed for an accurate
result to be obtained. In the next section, I discuss an example of an ill-posed problem
where the orthogonalization procedure is essential.

5.2.3 A general 4th order, ill-posed, ODE requiring orthogonalization

Consider the general 4th order ODE

\[ U^{(m)} + (6 - a)U^{(m-1)} + (11 - 6a)U^{(m-2)} + (5 - 11a)U^{(m-3)} - QaU = 0 , \tag{5.6} \]

which has a general solution of the form

\[ U(y) = A_1 e^{-y} + A_2 e^{-2y} + A_3 e^{-3y} + A_4 e^{ay} . \tag{5.7} \]

If boundary conditions are as those adopted for the ill-conditioned ODE 4.14 then the
solution is

\[ U(y) = e^{-y} - 2e^{-2y} + e^{-3y} . \]

Thus, this allows me to enter different values of a, for testing the efficiency of the
orthogonalization procedure for different degrees of ill-conditioning. That is, by increasing
a, the ODE takes the form of an ODE with a greater ill-conditioning factor, i.e. the
difference between one of the roots of equation (4.2) and the others, is increasing.
Part II: Maintaining linear independence numerically

Table 5.1: Comparison of the accuracy of results for various degrees of ill-conditioned problems with and without orthogonalization.

<table>
<thead>
<tr>
<th>Value of $a$</th>
<th>Constants</th>
<th>No orthogonalization</th>
<th>Orthogonalization</th>
<th>Exact value</th>
</tr>
</thead>
<tbody>
<tr>
<td>40.0</td>
<td>$c_1$</td>
<td>2.4185880478486</td>
<td>1.9936699314336</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>$c_2$</td>
<td>-14.511528287092</td>
<td>-11.999999924767</td>
<td>-12.0</td>
</tr>
<tr>
<td>50.0</td>
<td>$c_1$</td>
<td>1.2207403824221D-03</td>
<td>1.9916283630386</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>$c_2$</td>
<td>-7.3244422945328D-03</td>
<td>-11.999999928085</td>
<td>-12.0</td>
</tr>
<tr>
<td>60.0</td>
<td>$c_1$</td>
<td>1.4189431530464D-07</td>
<td>1.9895637320747</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>$c_2$</td>
<td>-8.5136589182782D-07</td>
<td>-11.99999930293</td>
<td>-12.0</td>
</tr>
<tr>
<td>100.0</td>
<td>$c_1$</td>
<td>-7.229932740303D-24</td>
<td>1.9809734801837</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>$c_2$</td>
<td>4.337956464182D-23</td>
<td>-11.99999934704</td>
<td>-12.0</td>
</tr>
</tbody>
</table>

Results

The code using Gram-Schmidt and an explicit fourth-order Runge-Kutta method, was tested against that written without the orthogonalization procedure. Both programs use single precision and the number of steps in the integration is 5000. That is, the step size $h = 0.0002$. The results in table 5.1 show the effectiveness of the Gram-Schmidt orthogonalization technique.

From table 5.1, it is obvious that orthogonalization is necessary when numerically integrating ill-conditioned problems.

Now that the basic description of the Gram-Schmidt orthogonalization has been covered, the following subsection will discuss the method of orthogonalization in a complex space, since my research will involve numerically integrating the complex Orr-Sommerfeld equation.

5.2.4 Orthogonalization in a complex space

Let us first look at the case of a two dimensional subspace of $\mathbb{C}^n$. Initially, we choose two linearly independent vectors, $u_1 \in \mathbb{C}^n$ and $u_2 \in \mathbb{C}^n$. Orthogonalization works by defining new vectors

$$v_1 = u_1,$$
Part II: Maintaining linear independence numerically

\[ v_2 = u_2 - \langle w_1, u_2 \rangle w_1, \]

where

\[ w_1 = \frac{v_1}{\| v_1 \|}, \]
\[ w_2 = \frac{v_2}{\| v_2 \|}, \]

with \( \| u \| = \sqrt{\langle u, u \rangle} \), and

\[ (a, b) = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i, \]

where \( \bar{a} \) denotes the complex conjugate of \( a \). Thus, \( u_1, u_2 \) are linearly independent, \( v_1, v_2 \) are orthogonal, and \( w_1, w_2 \) are orthonormal.

Verification that \( v_1 \) and \( v_2 \) are orthogonal

\[ \langle v_1, v_2 \rangle = \langle u_1, u_2 - \langle w_1, u_2 \rangle w_1 \rangle \]
\[ = \langle u_1, u_2 \rangle - \langle w_1, u_2 \rangle \langle u_1, w_1 \rangle \]
\[ = \langle u_1, u_2 \rangle - \frac{\langle u_1, u_2 \rangle}{\| u_1 \|} \frac{\langle u_1, u_1 \rangle}{\| u_1 \|} \]
\[ = \langle u_1, u_2 \rangle - \frac{\langle u_1, u_2 \rangle}{\| u_1 \|} \| u_1 \|^2 \]
\[ = \langle u_1, u_2 \rangle - \langle u_1, u_2 \rangle \]
\[ = 0, \]

Q.E.D.

Let us now consider the three dimensional case. That is, initially we choose three linearly independent vectors, \( u_1 \in \mathbb{C}^n, u_2 \in \mathbb{C}^n \) and \( u_3 \in \mathbb{C}^n \). Then orthogonalization is carried out by defining new variables

\[ v_1 = u_1, \]
\[ v_2 = u_2 - \langle w_1, u_2 \rangle w_1, \]
\[ v_3 = u_3 - \langle w_1, u_3 \rangle w_1 - \langle w_2, u_3 \rangle w_2, \]

where

\[ w_1 = \frac{v_1}{\| v_1 \|}, \]
\[ w_2 = \frac{v_2}{\| v_2 \|}, \]
\[ w_3 = \frac{v_3}{\| v_3 \|}. \]
Part II: Maintaining linear independence numerically

Now, we have already shown that \( v_1 \) is orthogonal to \( v_2 \). Thus, we need to show that \( v_1 \) is orthogonal to \( v_3 \) and \( v_2 \) is orthogonal to \( v_3 \).

**Verification that \( v_1 \) is orthogonal to \( v_3 \)**

\[
\langle v_1, v_3 \rangle = \langle v_1, u_3 \rangle - \langle w_1, u_3 \rangle \langle v_1, w_1 \rangle - \langle w_2, u_3 \rangle \langle v_1, w_2 \rangle.
\]

Note that \( \langle v_1, w_2 \rangle \) vanishes, since it only involves terms of \( v_1 \) and a multiple of \( v_2 \) (from \( w_2 \)), and we have previously shown that \( v_1 \) and \( v_2 \) are orthogonal. Thus, we have

\[
\langle v_1, v_3 \rangle = \langle v_1, u_3 \rangle - \frac{\langle v_1, u_3 \rangle}{\|u_1\|} \|u_1\| = \langle v_1, u_3 \rangle - \langle v_1, u_3 \rangle = 0,
\]

Q.E.D.

Similarly, it is easily that \( \langle v_2, v_3 \rangle = 0 \).

In general, given \( k \) initial linearly independent vectors in \( \mathbb{C}^n \) we orthogonalize by defining the following new vectors

\[
\begin{align*}
v_1 &= u_1, \\
v_2 &= u_2 - \langle w_1, u_2 \rangle w_1, \\
v_3 &= u_3 - \langle w_1, u_3 \rangle w_1 - \langle w_2, u_3 \rangle w_2, \\
&\vdots \\
v_k &= u_k - \sum_{j=1}^{k-1} \langle w_j, u_k \rangle w_j,
\end{align*}
\]

where

\[
\begin{align*}
w_1 &= \frac{v_1}{\|v_1\|}, \\
&\vdots \\
w_k &= \frac{v_k}{\|v_k\|}.
\end{align*}
\]

The algorithm above carries out discrete orthogonalization in a complex space. Although this discrete method is very effective, improvements in terms of accuracy and efficiency can still be made by using a very neat continuous method. The theory and application of which is stipulated in the next section.
5.3 Continuous Orthogonalization - Integration on a Manifold

In this section we shall consider the method of continuous orthogonalization as analogous to integrating on a manifold. Since this method of orthogonalization is continuous, it is thus more efficient than that of the previously described discrete Gram-Schmidt procedure. Let us explain the idea by applying the procedure to a three-dimensional case.

Consider the ordinary differential equation $u_y = Au$ where $u \in \mathbb{R}^3$. Let us take some starting value in $\mathbb{R}^3$. Integration yields some trajectory as shown by the solid line in Figure (5.1).

If we have a unit sphere and choose a starting point lying on its surface, the trajectory obtained from integrating the ODE will not stay on the surface of this sphere, but will wonder freely in $\mathbb{R}^3$. Integration on a manifold forces the trajectory to stay on the surface of the sphere. Thus, visually, we take a point on the trajectory, draw a line from this point to the origin thus intersecting a point on the surface of the sphere.

Mathematically, this means that, instead of solving the ODE, $u_y = Au$, we solve for $h(y)$ and $v(y)$, where

$$v(y) = u(y)h(y),$$  \hspace{1cm} (5.8)

with $v \in \mathbb{R}^3$ and $h \in \mathbb{R}$ and

$$\langle v(y), v(y) \rangle = v_1^2 + v_2^2 + v_3^2 = 1,$$
namely, the condition to stay on the surface of a unit sphere. Thus, from equation (5.8) we obtain the following ODE

\[ v_y = u_y h + uh_y \]
\[ = Au + uh_y \]
\[ = Av + vh^{-1}h_y. \]

Taking \( h^{-1}h_y = g(y) \), we therefore have that \( v(y) \) and \( h(y) \) satisfy the following coupled differential equations,

\[ v_y = Av + vg \quad (5.9) \]
\[ h_y = hg. \quad (5.10) \]

Hence, visually, if we take the value on the sphere, \( v(y) \), and multiply by \( h^{-1}(y) \) we obtain the corresponding position on the original trajectory. This step must therefore be applied after integration is complete in order to obtain the solution of our original ODE.

5.3.1 Determination of \( g \)

Before we can implement this procedure, we need to determine \( g \). We said that \( v \) has to lie on the surface of a unit sphere. Thus, differentiating the condition for the trajectory to stay on the surface of the sphere, namely, \( \langle v(y), v(y) \rangle \) gives

\[ 2\langle v(y), v_y \rangle = 2v_1v_1 + 2v_2v_2 + 2v_3v_3 = 0, \]

where \( v(y) = v_y \). Thus, from this we can find \( g \).

\[ 0 = \langle v, v_y \rangle = \langle v, Av + vg \rangle = \langle v, Av \rangle + \langle v, vg \rangle \]
\[ \Rightarrow g = -\frac{\langle v, Av \rangle}{\langle v, v \rangle} = -\langle v, Av \rangle \]

and since we know \( A \) and \( v \), we know \( g \). Hence, integrating equations (5.9) and (5.10) with \( g \) as above will mathematically keep \( v \) on the sphere.

5.3.2 Application of the continuous orthogonalization technique to an equation in four dimensions

Suppose we have the following differential equation

\[ u_y = Au \quad \text{with} \quad u \in \mathbb{R}^4, \]
with two homogeneous boundary conditions at \( y = 0 \).

Let \( U(y) = [u_1(y) | u_2(y)] \), a 4 \( \times \) 2 matrix with \( u_1(0) = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \) and \( u_2(0) = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \).

Then, if \( u_y = Au \) is a "well behaved" equation, we would integrate

\[
U_y = AU \quad \text{with} \quad U \in \mathbb{R}^{4 \times 2}
\]

from \( y = 0 \) to \( y = 1 \) and apply the two boundary conditions at \( y = 1 \).

Now, we devise an algorithm to restrict this problem to a manifold. Starting with

\[
U_y = AU \quad \text{with} \quad U \in \mathbb{R}^{4 \times 2},
\]

and

\[
U(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

we let

\[
V(y) = U(y)h(y),
\]

where \( V \in \mathbb{R}^{4 \times 2} \) and \( h \in \mathbb{R}^{2 \times 2} \). Now, the condition for orthogonality of the columns of \( V \) requires that \( V(y) \) satisfy

\[
V^T V = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

that is, if \( V = [v_1 | v_2] \) then \( V^T = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \) and

\[
V^T V = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{pmatrix}.
\]

Therefore, since \( \langle v_1, v_1 \rangle = 1 \) (unit length) and \( \langle v_1, v_2 \rangle = 0 \) (condition for orthogonality) \( V^T V = I_2 \Rightarrow v_1 \in \mathbb{R}^4 \) and \( v_2 \in \mathbb{R}^4 \) are orthonormal. Note that this generalization of the sphere is called a Stiefel Manifold

\[
V_k(\mathbb{R}^n) = \{ M \in \mathbb{R}^{n \times k} : M^T M = I_k \},
\]
Part II: Maintaining linear independence numerically

so, when \( k = 1 \) then \( M \) is a vector and so \( M^T M = 1 \) ⇒ a sphere. When, for example, \( k = 2 \) and \( n = 4 \), we have a five dimensional surface in \( \mathbb{R}^{4\times2} = \mathbb{R}^8 \). We can say this because each column of the matrix is an independent vector with four entries, Hence, we can vary each entry independently ⇒ \( V = [v_1, v_2] \in \mathbb{R}^8 \) (see T.J Bridges [14] for details on the OS equation on a manifold).

We now need to obtain an equation for \( V \) and an equation for \( h \). To obtain these equations, let us differentiate \( V(y) = U(y)h(y) \)

\[
V_y = U_y h + Uh_y = AUh + Vh^{-1}h_y .
\] (5.11)

If we let \( h^{-1}h_y = g(y) \) then we obtain the following coupled system

\[
\begin{align*}
V_y &= AV + Vg \quad (5.12) \\
h_y &= hg . \quad (5.13)
\end{align*}
\]

Following Bridges [14], we determine \( g \) by using the requirement that \( V \) must stay on the manifold, which yields (after some calculations)

\[
g(y) = -\text{sym}(V^T AV) ,
\] (5.14)

where, \( \text{sym}(m) = \frac{m^T + m}{2} \), the symmetric part of a square matrix \( m \).

Now, Davey [35] used the whole matrix \( V^T AV \) for orthogonalization instead of just the symmetric part as used in this report. It has since been shown that Davey’s choice of \( g(y) \) maintains linear independence but does not correspond precisely to orthogonalization.

5.3.3 Testing - Continuous v Discrete

This method of continuous orthogonalization, obtained by restricting the solution to a manifold, was tested using the model with constant coefficients stated in equation (5.15) below, and the results were compared to those obtained using the Gram-Schmidt method of orthogonalization.

\[
\phi'''' + (a^2 - b^2)\phi'' + a^2 b^2 \phi = 0 ,
\] (5.15)

with boundary conditions: \( \phi(0) = \phi'(0) = 0 \), \( \phi(1) = \cos(a) - \cosh(b) \) and \( \phi'(1) = -a \sin(a) - b \sinh(b) \). This problem has the exact solution

\[
\phi(y) = \cos(ay) - \cosh(by)
\]
The linear system for this problem was set up and the code written in FORTRAN 77, to solve the following coupled system

\[
V_y = AV + Vg, \\
h_y = hg,
\]

where

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a^2b^2 & 0 & b^2 - a^2 & 0
\end{pmatrix}
\]

\[
g = -\text{sym}(V^TAV) = -\frac{1}{2}V^TAV - \frac{1}{2}V^TATV
\]

and the initial conditions are

\[
V(0) = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad h(0) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

The above system was integrated with various values of $a$ and $b$, using a standard fourth-order Runge-Kutta algorithm. A selection of the results follow.
Results

<table>
<thead>
<tr>
<th>Value of a</th>
<th>Value of b</th>
<th>Constants</th>
<th>Gram-Schmidt</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2.0</td>
<td>$c_1$</td>
<td>-4.9999993539468</td>
<td>-4.9999993481748</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_2$</td>
<td>-9.0564725496733BD-07</td>
<td>-9.1515925504382D-07</td>
</tr>
<tr>
<td>2.0</td>
<td>3.0</td>
<td>$c_1$</td>
<td>-12.99998332133</td>
<td>-12.99998471853</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_2$</td>
<td>-1.7954490658667D-06</td>
<td>-1.3203442392751D-06</td>
</tr>
<tr>
<td>10.0</td>
<td>2.0</td>
<td>$c_1$</td>
<td>-103.99977396958</td>
<td>-104.000304179653</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_2$</td>
<td>-5.3301191321964D-05</td>
<td>5.522898992626D-04</td>
</tr>
</tbody>
</table>

Table 5.2: Comparison of the discrete Gram-Schmidt orthogonalization with the continuous orthogonalization technique

Conclusion/Discussion

It can be easily seen, from the results in table 5.2, that both the discrete and continuous orthogonalization techniques return similar values for the constants, $c_1$ and $c_2$, to a reasonable accuracy. Since the continuous orthogonalization method is a “built in” orthogonalization scheme, it is assumed more accurate. The following section discusses the numerical methods required to find the roots of the equation, that is, moves on to discuss the inclusion of the Newton’s algorithm for the eigenvalue search scheme.

5.4 Combining Newton’s method with orthogonalization

The method for continuous orthogonalization in the real case requires us to numerically integrate the adjusted ODE,

$$V'_\lambda = A(y, \lambda) V + V_g, \quad V \in \mathbb{R}^{4 \times 2}, \quad (5.16)$$

where $g$ is given by

$$g = -\frac{1}{2} V^T (A + A^T) V.$$

Now, given an initial guess for $\lambda$, Newton’s method requires the derivative of the function. Thus, we need to integrate the derivative of (5.16) with respect to $\lambda$.

Let $W = \frac{\partial}{\partial \lambda} V$, then

$$W'_\lambda = AW + \frac{\partial}{\partial \lambda} AV + W_g + V \frac{\partial}{\partial \lambda} g, \quad (5.17)$$
thus, we need to find $\frac{\partial}{\partial \lambda} g$ which is given by

$$\frac{\partial}{\partial \lambda} g = -\frac{1}{2} W^T (A + A^T)V - \frac{1}{2} V^T (A + A^T)W - \frac{1}{2} V^T (DA + DA^T)V,$$

where $DA = \frac{\partial}{\partial \lambda} A$.

For computational ease, we will group together matrices and their transpose, i.e.

1. $-\frac{1}{2} V^T DAV$ is the transpose of $-\frac{1}{2} V^T (DA)^T V$
2. $-\frac{1}{2} W^T AV$ is the transpose of $-\frac{1}{2} V^T A^T W$
3. $-\frac{1}{2} W^T A^T V$ is the transpose of $-\frac{1}{2} V^T AW$

Thus, only one from each of the three pairs listed above needs to be calculated, since their transpose can be easily found from the resulting matrices.

So, both equations (5.16) and (5.17) are integrated and Newton’s method used. The next section reveals a problem occurring from implementing either the discrete or continuous orthogonalization schemes with complex functions.

5.5 Considerations for combining complex operations with continuous orthogonalization and Newton’s method

Since discrete and continuous orthogonalization make the system non-analytic, one alternative approach to combining Newton’s method with orthogonalization is to split the complex number up into real and imaginary parts. So, we end up solving for two functions $f_1(\lambda_r, \lambda_i)$ and $f_2(\lambda_r, \lambda_i)$.

In this case, Newton’s method becomes

$$\begin{pmatrix} \lambda_r^{n+1} \\ \lambda_i^{n+1} \end{pmatrix} = \begin{pmatrix} \lambda_r^n \\ \lambda_i^n \end{pmatrix} - \left( \begin{array}{cc} \frac{\partial f_1}{\partial \lambda_r} & \frac{\partial f_1}{\partial \lambda_i} \\ \frac{\partial f_2}{\partial \lambda_r} & \frac{\partial f_2}{\partial \lambda_i} \end{array} \right)^{-1} \begin{pmatrix} f_1(\lambda_r^n, \lambda_i^n) \\ f_2(\lambda_r^n, \lambda_i^n) \end{pmatrix}.$$

Note that this is to be carried out if we require $f_1(\lambda_r, \lambda_i) = 0$ and $f_2(\lambda_r, \lambda_i) = 0$. The generalization of this is

$$X_{n+1} = X_n - f'(X_n)^{-1} f(X_n), \quad \text{for} \quad f: \mathbb{R}^m \rightarrow \mathbb{R}^m.$$
Part II: Maintaining linear independence numerically

Thus, we would start off with

\[ V_y = A(y, \lambda_r, \lambda_i) V + V g, \]

where \( V \) is now in \( \mathbb{R}^{8 \times 4} \) since everything is split up into its real and imaginary part. So, \( g \) takes the form

\[ g = -\text{sym}(V^T A V). \]

However, we must note that we will also need to calculate the following two derivatives

\[ W_r = \frac{\partial V}{\partial \lambda_r} \quad \text{and} \quad W_i = \frac{\partial V}{\partial \lambda_i}, \]

thus substantially increasing the computational time.

One of the difficulties with both discrete and continuous orthogonalization is that when the matrix, \( A \), is complex and depends analytically on a parameter, such as the eigenvalue, \( c \), in the Orr-Sommerfeld equation, the operation of complex conjugation in complex orthogonalization makes the differential equation non-analytic. Therefore, implementation of Newton's method is more cumbersome, and methods based on Cauchy's theorem can not be used. A second difficulty with orthogonalization is that the induced system is non-linear. An alternative to orthogonalization was proposed by Ng & Reid [92], where compound matrices are used as coordinates for integrating (3.32). This method is discussed in chapter 7.

Before we venture into the theory and implementation of the suggested alternative techniques of Ng & Reid, the Orr-Sommerfeld equation shall be rewritten to incorporate a more realistic setting. That is, to study the OS equation on a semi-infinite domain.
For the study of the stability of the boundary layer flow past a rigid or compliant surface, we wish to consider the Orr-Sommerfeld equation on a semi-infinite domain. Thus, the independent variable, \( y \), is assumed to vary over the open interval \([0, \infty)\). The intent of this chapter is to derive the boundary conditions for the OS equation on this semi-infinite interval by obtaining a good approximation of the boundary conditions at infinity with \( y = L \) where \( L \) is some acceptably large value.

For the rigid wall, we already know the boundary conditions at the surface, namely

\[
\phi(0) = \phi'(0) = 0.
\]

Thus, we now need only derive the boundary conditions at infinity. Before we derive the boundary conditions at infinity, a brief introduction on asymptotic boundary conditions and left eigenvector theory is given below.

### 6.1 Asymptotic boundary condition theory

In the numerical solution of eigenvalue problems on infinite intervals, a common method of proceeding is to replace the infinite interval \([0, \infty)\) by a finite one, \([0, y_\infty]\), say. The main problem then is to determine the appropriate boundary conditions to be imposed at \( y = y_\infty \), where \( y_\infty \) must be chosen sufficiently large so that the coefficients in the governing OS equation can be approximated for \( y \geq y_\infty \) by their limiting values as \( y \to \infty \). On this interval the limiting form of the governing equations can be solved exactly and the appropriate boundary conditions at \( y = y_\infty \) then follow by requiring continuity of the solution and it’s derivative at \( y = y_\infty \). Now, for boundary layer flows, \( U(y) \to 1 \) and \( U'' \to 0 \) as \( y \to \infty \), and in this limit the bounded solutions of the OS equation have the
asymptotic behaviour

\[ \phi_1(y) \sim \text{constant} \times e^{-\alpha y} \quad \text{and} \quad \phi_2(y) \sim \text{constant} \times e^{-\beta y}. \]

If we now choose \( y_\infty \) to be sufficiently large so that for \( y \geq y_\infty \), \( U(y) \) and \( U''(y) \) are numerically indistinguishable from their corresponding values at infinity, then \( \phi_1 \) and \( \phi_2 \) and their derivatives can be treated as continuous at \( y = y_\infty \).

### 6.1.1 Left Eigenvector theory

For the linear system, \( u_y = Au \quad u \in \mathbb{C}^n \), let \( \mu_1 \) be an eigenvalue of \( A \), where \( A \) is any \( n \times n \) complex matrix. The eigenvector, \( \xi_1 \) of \( A \) associated with \( \mu_1 \) satisfies \( A\xi_1 = \mu_1 \xi_1 \) and the left eigenvector, \( \eta_1 \), satisfies \( \eta_1^H A = \mu_1 \eta_1^H \), where \( \eta_1^H \) is the conjugate transpose of \( \eta_1 \).

Let \( \mu_1, \ldots, \mu_n \) be \( n \) distinct eigenvalues of \( A \) and suppose

\[
\text{Real}(\mu_j) < 0 \quad \text{for} \quad j = 1, \ldots, k
\]

\[
\text{Real}(\mu_j) > 0 \quad \text{for} \quad j = k + 1, \ldots, n.
\]

Then let \( \xi_1, \ldots, \xi_k \) be the right eigenvectors associated with \( \mu_1, \ldots, \mu_k \) and \( \eta_{k+1}, \ldots, \eta_n \) the left eigenvectors associated with \( \mu_{k+1}, \ldots, \mu_n \), then the left eigenvectors satisfy the following identity

\[ A^H \eta_j = \bar{\mu}_j \eta_j \quad j = k + 1, \ldots, n, \]

and the asymptotic boundary conditions at \( y = y_\infty \) are found to be

\[ \eta_j^H u(y_\infty) = 0 \implies \langle \eta_j, u(y_\infty) \rangle = 0 \quad \text{for} \quad j = k + 1, \ldots, n. \]

Here, the \( \eta_j \)'s are the left eigenvectors associated with the eigenvalues in the solution which we wish to eliminate. Thus, these terms vanish leaving the solution of the governing equation bounded as \( y \to \infty \).

### 6.2 Derivation of the boundary conditions at infinity

Writing the Orr-Sommerfeld equation as a linear system in the form

\[ u_y = A(y,p)u, \quad u \in \mathbb{C}^4, \]
where $p$ represents the parameters $\alpha$, $c$, and $Re$. $u$ is defined by

$$
\mathbf{u} = 
\begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    u_4 
\end{pmatrix} = 
\begin{pmatrix}
    \phi \\
    \phi' \\
    \phi'' - \alpha^2 \phi \\
    \phi''' - \alpha^2 \phi'
\end{pmatrix},
$$

and

$$
\mathbf{A}(y,p) = 
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    \alpha^2 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    -i\alpha ReU''(y) & 0 & \gamma(y) & 0
\end{pmatrix},
$$

where $\gamma(y) = \lambda Re + \alpha^2 + i\alpha ReU(y)$.

Let us denote the limit of $\mathbf{A}$ as $y$ tends to infinity to be

$$
\lim_{y \to \infty} \mathbf{A}(y,p) = \mathbf{A}_{\infty}(p) = 
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    \alpha^2 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & \beta^2 & 0
\end{pmatrix},
$$

where, $\beta^2 = \lambda Re + \alpha^2 + i\alpha Re$, since $U(y) \to 1$ and $U''(y) \to 0$ as $y \to \infty$. So, $\mathbf{A}_{\infty}$ is a constant coefficient matrix, with eigenvalues: $\mu_1 = -\alpha$, $\mu_2 = -\beta$, $\mu_3 = \alpha$, $\mu_4 = \beta$, with $\alpha^2 \neq \beta^2$, found from the characteristic polynomial equation, $\det(\mu I - \mathbf{A}_{\infty}(p)) = 0$. Since we obtain a constant matrix as $y \to \infty$, the theory for asymptotic boundary conditions can be applied. Now, the eigenvalues of $\mathbf{A}_{\infty}(p)$ are distinct, therefore, the general solution of the constant coefficient ODE, namely $\mathbf{u}_y = \mathbf{A}_{\infty}(p)\mathbf{u}$, depends on the eigenvalues in the following way

$$
\mathbf{u}(y) = a_1 \xi_1 e^{-\alpha y} + a_2 \xi_2 e^{-\beta y} + a_3 \xi_3 e^{\alpha y} + a_4 \xi_4 e^{\beta y},
$$

where $\xi_i$ are eigenvectors and the $a_i$'s are complex.

Looking at the system with $y$ very large, we wish to exclude $e^{\beta y}$ and $e^{\alpha y}$ in the solution, since they grow exponentially and cause the solution to be unbounded. Thus, we require asymptotic boundary conditions that will have the effect of removing these unwanted terms from the solution. To obtain these boundary conditions, we require the left eigenvectors associated with the positive eigenvalues, $\mu_3 = \alpha$ and $\mu_4 = \beta$. Thus, we need to solve the equation

$$
(\mathbf{A}_{\infty}(p)^H - \bar{\mu}_j I)\eta_j = 0 \quad \text{for } j = 3, 4,
$$

**Part II: The Orr-Sommerfeld equation on a semi-infinite domain**
Part II: The Orr-Sommerfeld equation on a semi-infinite domain 73

where the superscript $H$ denotes complex conjugate transpose. Now, solving equation (6.2) yields

$$
\begin{pmatrix}
-\bar{\mu}_j & \alpha^2 & 0 & 0 \\
1 & -\bar{\mu}_j & 0 & 0 \\
0 & 1 & -\bar{\mu}_j & \beta^2 \\
0 & 0 & 1 & -\bar{\mu}_j
\end{pmatrix} \eta = 0
$$

$$
\Rightarrow \eta_j = C_j \begin{pmatrix}
\alpha^2 (\bar{\mu}_j^2 - \beta^2) \\
\bar{\mu}_j (\bar{\mu}_j^2 - \beta^2) \\
\bar{\mu}_j^2 \\
\bar{\mu}_j
\end{pmatrix},
$$

where $C_j$ is an arbitrary complex multiple, and the bar notation represents complex conjugation. Substituting in the offending positive roots of the characteristic polynomial equation, $\mu_3 = \alpha$ and $\mu_4 = \beta$, we obtain

$$
\eta_3 = C_3 \begin{pmatrix}
\alpha (\alpha^2 - \beta^2) \\
(\alpha^2 - \beta^2) \\
\alpha \\
1
\end{pmatrix}, \quad \eta_4 = C_4 \begin{pmatrix}
0 \\
0 \\
\beta \\
1
\end{pmatrix},
$$

where $C_3$ and $C_4$ are some arbitrary complex numbers. The two asymptotic boundary conditions are then defined to be

$$
\eta^H_3 \cdot u = 0, \quad (6.3)
$$

$$
\eta^H_4 \cdot u = 0, \quad (6.4)
$$

From equations (6.3) and (6.4) we obtain

$$
\alpha (\alpha^2 - \beta^2) u_1 + (\alpha^2 - \beta^2) u_2 + \alpha u_3 + u_4 = 0,
$$

$$
\beta u_3 + u_4 = 0.
$$

respectively, then using the original transformation of variables, (6.1), and simplifying, we obtain the asymptotic boundary conditions at $y = L$ defined as

$$
\phi'''' + \alpha \phi'' - \beta^2 \phi' - \beta^2 \alpha \phi = 0 \quad (6.5)
$$

$$
\phi'''' + \beta \phi'' - \alpha^2 \phi' - \alpha^2 \beta \phi = 0 \quad (6.6)
$$

at $y = y_{\infty} = L$ for a sufficiently large value of $L$. 
6.3 Converting the four boundary conditions into a generalized form

Consider the ODE
\[ u_y = A(y, p)u, \quad u \in \mathbb{C}^4. \]

The two boundary conditions at the wall, \( y = 0 \), can be represented in the form,

\[
\langle e_1, u(0, \lambda) \rangle = 0 \quad , \quad \langle \eta_1, u(0, \lambda) \rangle = 0
\]

or in general

\[
\langle e_2, u(0, \lambda) \rangle = 0 \quad , \quad \langle \eta_2, u(0, \lambda) \rangle = 0
\]

where \( \eta_1 \) and \( \eta_2 \) are some vectors in \( \mathbb{C}^4 \), and are given by the standard unit vectors \( e_1 \) and \( e_2 \) in this case. (The more general form is required for compliant surface equations.)

The first boundary condition at infinity is

\[
\phi''' + \beta \phi'' - \alpha^2 \phi' - \alpha^2 \beta \phi = 0.
\]

Re-writing this in terms of it's \( u \) transformation, (6.1) we have

\[ u_4 + \beta u_3 = 0 . \tag{6.7} \]

If we let \( \eta_3 = \begin{pmatrix} 0 \\ 0 \\ \beta \\ 1 \end{pmatrix} \) then equation (6.7) \( \Rightarrow \langle \eta_3, u \rangle = 0 \). That is, the boundary condition can be characterised as a complex inner product being set equal to zero. Similarly, the second boundary condition can be written as \( \langle \eta_4, u \rangle = 0 \), where, \( \eta_4 = \begin{pmatrix} \alpha (\alpha^2 - \beta^2) \\ \alpha^2 - \beta^2 \\ \alpha \\ 1 \end{pmatrix} \).

6.3.1 Numerical integration, semi-infinite case

Let

\[ U(y, \lambda) = [u_1(y, \lambda) \quad u_2(y, \lambda)], \]

where \( u_1(y, \lambda) \) is a vector in \( \mathbb{C}^4 \) and \( u_2(y, \lambda) \) is a vector in \( \mathbb{C}^4 \) and so, \( U(y, \lambda) \) is a \( 4 \times 2 \) matrix.
Now, \( U(y, \lambda) \) is chosen so that when \( y = 0 \), each column of \( U(y, \lambda) \) satisfies the boundary conditions at \( y = 0 \). Thus, the general solution of \( U_y = A(y, \lambda)U \), \( U \in \mathbb{C}^{4 \times 2} \) satisfying the boundary conditions at \( y = 0 \) is:

\[
u(y, \lambda) = U(y, \lambda)c(\lambda),
\]

where, \( c(\lambda) = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \end{pmatrix} \). Thus, we integrate \( U_y = A(y, \lambda)U \) over to the right hand side and apply the boundary conditions at the right hand side to find \( c_1 \) and \( c_2 \).

Now, at \( y = L \), we have to satisfy the boundary conditions:

\[
\begin{align*}
\langle \eta_3, u(L, \lambda) \rangle &= 0, \\
\langle \eta_4, u(L, \lambda) \rangle &= 0,
\end{align*}
\]

for vectors \( \eta_3 \) and \( \eta_4 \) as described above. That is

\[
\begin{align*}
\langle \eta_3, U(L, \lambda)c(\lambda) \rangle &= 0, \\
\langle \eta_4, U(L, \lambda)c(\lambda) \rangle &= 0.
\end{align*}
\]

If we let \( \Sigma = [\eta_3 \ \eta_4] \) be a \( 4 \times 2 \) matrix, then, (6.9) and (6.10) are equivalent to

\[
\Sigma^H U(L, \lambda)c(\lambda) = 0.
\]

So, this equation has a non-trivial solution \( c(\lambda) \neq 0 \) if and only if

\[
det(\Sigma^H U(L, \lambda)) = 0,
\]

since if this doesn't hold, then \( c(\lambda) \) must be zero.

Let \( D(\lambda) = det(\Sigma^H U(L, \lambda)) \). We wish to apply Newton's Method to find the roots of \( D(\lambda) \). However, we must note that \( D \) is a complex valued function. The following section shows how Newton’s algorithm is combined with the numerical integration technique, shooting as an eigenvalue search scheme.

### 6.4 Combining Newton’s method with shooting

Suppose \( \lambda_n \) is a guess for the root of \( D(\lambda) \). Then an improved estimate is obtained by Newton's method:

\[
\lambda_{n+1} = \lambda_n - \frac{D(\lambda_n)}{D'(\lambda_n)} \text{ for } n = 1, 2, \cdots
\]
To apply Newton’s method, we require $D'(\lambda)$. Now, if $D(\lambda) = \det(M(\lambda))$, and $D'(\lambda) \neq 0$, then

$$D'(\lambda) = D(\lambda) Tr(M(\lambda)^{-1}M'(\lambda)),$$

where $Tr$ is the sum of the diagonal entries of $M(\lambda)$, that is, the Trace.

Since $M(\lambda) = \Sigma^H U(1, \lambda)$, $M$ is a $2 \times 2$ matrix. Thus, let

$$M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix},$$

then the determinant of $M$ is defined by $\det(M) = m_{11}m_{22} - m_{12}m_{21}$. Thus, Newton’s method can be written as

$$\lambda_{n+1} = \lambda_n - \frac{1}{t_n},$$

where $t_n = Trace(M(\lambda_n)^{-1}M'(\lambda_n))$, however, we don’t have $M'(\lambda_n)$. Now

$$M'(\lambda) = \Sigma^H \frac{d}{d\lambda} U(1, \lambda) + \left( \frac{d}{d\lambda} \Sigma \right)^H U(1, \lambda),$$

thus, we need to construct a differential equation to compute $\frac{\partial}{\partial \lambda} U(1, \lambda)$.

Let

$$W(y, \lambda) = \frac{\partial}{\partial \lambda} U(y, \lambda),$$

then

$$W_y = A(y, \lambda)W + \frac{\partial}{\partial \lambda} A(y, \lambda) U \quad \text{with} \quad W(0, \lambda) = 0. \quad (6.11)$$

The idea of the numerical algorithm is to integrate $U_y = A(y, \lambda)U$ and equation (6.11) together. This yields $M(\lambda_n)$ and $M'(\lambda_n)$ at $y = 1$, which is then used to obtain a new estimate for $\lambda$ using Newton’s method.
Eigenvalue problems for ordinary differential equations are most commonly treated by first defining a solution matrix which satisfies some prescribed initial conditions, then the required eigenvalues are obtained from the roots of some minor of the solution matrix. By using methods such as *shooting*, where the evaluation of the minor is attempted by computing its elements separately, then a serious loss of numerical accuracy occurs when the governing equation is stiff, and techniques such as the previously discussed orthonormalization method are generally implemented to overcome this problem.

However, in the late seventies, another new method was proposed along with continuous orthogonalization which again is substantially more robust that the Gram-Schmidt orthogonalization technique. This method involves considering the differential equation satisfied by certain compound matrices whose elements are the minors of the original solution matrix and thus, the required minor can be computed directly (Ng & Reid [92], Davey [34], Ng & Reid [94]).

Compound matrices have been used by Gilbert and Backus [48] in their discussion of elastic wave problems and also by Lakin *et al.* [76] to derive a uniform approximation to the eigenvalue relation for the Orr-Sommerfeld problem.

In 1979, Ng and Reid [92] showed how compound matrices could be used for eigenvalue problems for linear ordinary differential equations, and a few years later, they generalized the compound matrix method to deal with eigenvalue and boundary-value problems involving unstable systems of ordinary differential equations giving details for fourth and sixth order problems [94].

In 1999, Bridges [14] gave a new definition of orthogonality and showed that continuous orthogonalization does not in fact satisfy this definition, and suggested a new continuous
Part II: The compound matrix method

orthogonalization algorithm: restriction to a Steifel manifold. He then showed that continuous orthogonalization and the compound matrix method are actually dual formulations of the same idea: in order to ensure linear independence, integrate the differential equation on a Grassmanian manifold.

The compound matrix method is a numerical method for integrating stiff linear systems of differential equations, particularly the Orr-Sommerfeld equation. The main features of this method are the preservation of linear independence without requiring orthogonalization, as well as maintaining analyticity of the system of ODEs.

In this chapter, we introduce a new formulation of the compound matrix method in terms of exterior algebra producing a new numerical framework for solving stiff ODEs in hydrodynamic stability problems (cf. Allen and Bridges [1]) and apply it to the computation of the neutral curve for plane Poiseuille flow. In the following chapters, we apply this technique to the Blasius boundary layer flow past a rigid and compliant surface respectively. The compound matrix method corresponds to a restriction of the differential equation to a Grassmanian manifold using Plucker coordinates [14], and the exterior algebra formulation leads to a new efficient algorithm for constructing the induced matrices central to the implementation of the compound matrix method. In the section to follow we introduce briefly the idea of the compound matrix method and then in the following sections we discuss the required aspects of exterior algebra along with the construction of the induced system, the numerical implementation of such algorithms including efficiency problems and accuracy issues. Finally, as an illustrative example, the curve of neutral stability for plane Poiseuille flow is computed.

7.1 Compound matrix method

We shall work with the induced system for the Orr-Sommerfeld equation

$$U_y = A(y, \lambda)U, \quad U(y)|_{y=0} = U_0 \in \mathbb{C}^{4 \times 2},$$

where $a \leq y \leq b$ and $a$ or $b$ can be infinite with $\lambda \in \mathbb{C}$ as the eigenvalue representing a stability exponent.
Part II: The compound matrix method

Let

\[
U = \begin{bmatrix}
u_1 & v_1 \\
u_2 & v_2 \\
u_3 & v_3 \\
u_4 & v_4
\end{bmatrix} \in \mathbb{C}^{4 \times 2},
\]

(7.1)

and consider all possible 2 \times 2 sub-determinants of \( U \in \mathbb{C}^{4 \times 2} \) as coordinates:

\[
z_1 = u_1v_2 - u_2v_1, \quad z_2 = u_1v_3 - u_3v_1, \quad z_3 = u_1v_4 - u_4v_1,
\]

\[
z_4 = u_2v_3 - u_3v_2, \quad z_5 = u_2v_4 - u_4v_2, \quad z_6 = u_3v_4 - u_4v_3.
\]

Differentiating the compound matrix coordinates \( z_1, \ldots, z_6 \) and using the property that \( u \in \mathbb{C}^4 \) and \( v \in \mathbb{C}^4 \) satisfy the differential equation, it follows that the coordinates \( z \in \mathbb{C}^6 \) satisfy

\[
z_y = A^{(2)}(y, \lambda)z, \quad z \in \mathbb{C}^6,
\]

(7.3)

where \( A^{(2)}(y, \lambda) \) is a 6 \times 6 matrix whose entries depend linearly on the entries of \( A(y, \lambda) \). The compound matrix coordinates then lead to induced boundary conditions at \( y = 0 \) and \( y = 1 \) (see Ng & Reid [92] and Drazin & Reid [38] for full details of this derivation).

The advantage of integrating the induced system, (7.3), over the original system occurs in the integration of a line of two-dimensional subspaces for (7.3) and therefore eliminating the numerical linear independence problem. Moreover, when \( A(y, \lambda) \) depends analytically on \( \lambda \), \( A^{(2)}(y, \lambda) \) will also depend analytically on \( \lambda \).

Presented in the next section are the required aspects of exterior algebra along with the development of the new numerical framework for the construction of the induced system (7.3) in a coordinate-free way, thus providing an easily understandable method for implementation on, theoretically, an \( n \)-dimensional system.

7.2 Exterior algebra and compound matrices

The starting point is the system of linear differential equations for the Orr-Sommerfeld equation in the form

\[
u_y = A(y, \lambda)u, \quad u \in \mathbb{C}^4, \quad y \in \mathbb{R}, \quad \lambda \in \Lambda,
\]

(7.4)

where \( A(y, \lambda) \) is a continuously differentiable function of \( y \) and an analytic function of \( \lambda \) for all \( \lambda \in \Lambda \), and \( \Lambda \) is some specified subset of the complex plane. We consider the
restriction of (7.4) to two-dimensional subspaces of \( \mathbb{C}^4 \). The most obvious such restriction is to consider
\[
U_y = A(y, \lambda)U \quad U(y, \lambda)|_{y=0} = U_0(\lambda) \in \mathbb{C}^{4 \times 2},
\]
where the columns of \( U_0(\lambda) \) span some two-dimensional starting subspace at \( y = 0 \).

Alternatively, following Bridges [14], exterior algebra can be used to represent two-dimensional subspaces. If \( \xi_1, \xi_2 \) span a two-dimensional space, then \( \xi_1 \wedge \xi_2 \) where \( \wedge \) is the wedge product, is a 2-form which represents the two-dimensional subspace as a 'point'. The linear space of all 2-forms in \( \mathbb{C}^4 \) creates a vector space \( \Lambda^2(\mathbb{C}^4) \). Introducing a basis enables a straightforward method for approaching constructive aspects of these vector spaces.

Let \( e_1, \ldots, e_4 \) be an orthonormal basis for \( \mathbb{C}^4 \). Then the nonzero and distinct members of the set
\[
\{ e_{i1} \wedge e_{i2} : i1, i2 = 1, \ldots, 4 \}
\]
form a basis for the vector space \( \Lambda^2(\mathbb{C}^4) \) with exactly \( d = \frac{4!}{2!2!} = 6 \) (the dimension of \( \Lambda^2(\mathbb{C}^4) \)), distinct elements in this set.

By choosing an ordering such as standard lexical ordering and labelling the nonzero distinct elements in the set (7.5) by \( \omega_1, \ldots, \omega_6 \), any element \( W \in \Lambda^2(\mathbb{C}^4) \) can be represented as
\[
W = \sum_{j=1}^{6} W_j \omega_j.
\]
The compound matrix method can be interpreted as the restriction of (7.1) to \( \Lambda^2(\mathbb{C}^4) \).

The system (7.1) restricted to \( \Lambda^2(\mathbb{C}^4) \) is defined to be
\[
W_y = A^{(2)}(y, \lambda) W, \quad W \in \Lambda^2(\mathbb{C}^4) \approx \mathbb{C}^6,
\]
where \( A^{(2)}(y, \lambda) \) is a 6 \( \times \) 6 matrix. The key to constructing this induced system lies in an algorithm for constructing the matrix \( A^{(2)}(y, \lambda) \). Now, there is a natural way to construct the induced matrix \( A^{(2)} \), given \( A \in \mathbb{C}^{4 \times 4} \), in a coordinate-free way using the vector space structure of the spaces \( \Lambda^2(\mathbb{C}^4) \).

A complex inner product on \( \mathbb{C}^4 \) denoted by \( \langle \cdot, \cdot \rangle_\mathbb{C} \), with conjugation on the first element, induces an inner product on each vector space \( \Lambda^2(\mathbb{C}^4) \) as follows. Let
\[
W = w_1 \wedge w_2 \quad \text{and} \quad V = v_1 \wedge v_2, \quad w_i, v_j \in \mathbb{C}^4, \quad \forall i, j = 1, 2,
\]
be any decomposable 2-forms. A 2-form is decomposable if it can be written as a pure form: a wedge product between two linearly independent vectors in $\mathbb{C}^4$. The inner product of $W$ and $V$ is defined by

$$\langle W, V \rangle = \det \begin{bmatrix} \langle w_1, v_1 \rangle & \langle w_1, v_2 \rangle \\ \langle w_2, v_1 \rangle & \langle w_2, v_2 \rangle \end{bmatrix}, \quad W, V \in \bigwedge^2(\mathbb{C}^4).$$

This definition extends by (bi)-linearity to any 2-form; i.e. not necessarily decomposable, since every element in $\bigwedge^2(\mathbb{C}^4)$ is a sum of decomposable elements.

The induced matrix $A^{(2)}$ is then the $6 \times 6$ matrix with entries

$$\{A^{(2)}\}_{i,j} = [\omega_i, A\omega_j]_2, \quad i, j = 1, \ldots, 6,$$

where, for any $\omega = e_i \wedge e_j \in \bigwedge^2(\mathbb{C}^4)$,

$$A\omega \overset{\text{def}}{=} (A e_i \wedge e_j) + (e_i \wedge A e_j).$$

Note that with this definition, $A^{(2)}(y, \lambda)$ is an analytic function of $\lambda$ whenever $A(y, \lambda)$ is analytic. Furthermore, if the basis $e_1, \ldots, e_4$ is independent of $y$ and $\lambda$ then $A^{(2)}(y, \lambda)$ inherits exactly the differentiability properties of $A(y, \lambda)$. Another advantage of this definition of the induced matrix is that it is easily automated, in either FORTRAN, MAPLE, or MATLAB, which is essential when the order of the original system, $n$, is large (see Appendix A for an example code written in Maple). However, for small $n$, the induced matrices can be constructed explicitly.

Working with the Orr-Sommerfeld equation,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -i\alpha R_e U''(y) & 0 & \gamma(y) & 0 \end{pmatrix},$$

where $\gamma(y) = \lambda R_e + \alpha^2 + i\alpha R_e U(y)$.

Take $e_1, \ldots, e_4$ to be the standard basis for $\mathbb{C}^4$ and let $\omega_1, \ldots, \omega_6$ be a basis for $\bigwedge^2(\mathbb{C}^4)$. For example, using a standard lexical ordering,

$$\omega_1 = e_1 \wedge e_2, \quad \omega_2 = e_1 \wedge e_3, \quad \omega_3 = e_1 \wedge e_4,$$

$$\omega_4 = e_2 \wedge e_3, \quad \omega_5 = e_2 \wedge e_4, \quad \omega_6 = e_3 \wedge e_4.$$
The basis $\omega_1, \ldots, \omega_6$ in (7.10) is orthonormal with respect to the inner product $[,]_2$. Therefore

\[
\{A^{(2)}\}_{1,1} = [\omega_1, A\omega_1]_2 = [e_1 \wedge e_2, Ae_1 \wedge e_2 + e_1 \wedge Ae_2]_2
\]

\[
= [e_1 \wedge e_2, Ae_1 \wedge e_2]_2 + [e_1 \wedge e_2, e_1 \wedge Ae_2]_2
\]

\[
= \det \begin{bmatrix} \langle e_1, Ae_1 \rangle_c & \langle e_1, e_2 \rangle_c \\ \langle e_2, Ae_1 \rangle_c & \langle e_2, e_2 \rangle_c \end{bmatrix} + \det \begin{bmatrix} \langle e_1, e_1 \rangle_c & \langle e_1, Ae_2 \rangle_c \\ \langle e_2, e_1 \rangle_c & \langle e_2, Ae_2 \rangle_c \end{bmatrix}
\]

\[
= \langle e_1, Ae_1 \rangle_c + \langle e_2, Ae_2 \rangle_c = a_{11} + a_{22} = 0.
\]

Similarly

\[
\{A^{(2)}\}_{1,2} = [\omega_1, A\omega_2]_2 = \langle e_2, Ae_3 \rangle_c = a_{23} = 1.
\]

Continuing this way, we find

\[
A^{(2)} = \begin{bmatrix}
a_{11} + a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\
a_{31} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\
a_{42} & a_{43} & a_{11} + a_{44} & 0 & a_{12} & a_{13} \\
a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\
-a_{41} & 0 & a_{21} & a_{43} & a_{22} + a_{44} & a_{23} \\
0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33} + a_{44}
\end{bmatrix}
\]

The induced matrix in (7.11) is precisely the form obtained using the compound matrix method (see equation (2.11) in [94]). However, unlike the compound matrix method, this is achieved in a coordinate-free way. Also, with exterior algebra it is clear how to change basis, to automate the construction, and to generalize it to any dimensions, $\Lambda^k(\mathbb{C}^n)$. This generalized form along with a simple and illuminating example showing the effect of basis change are given in ALLEN & BRIDGES [1].

In the original system for the Orr-Sommerfeld equation, the standard boundary conditions for this system when the basic state is Poiseuille flow are:
1. boundary condition at the left hand side: \((e_1, u(0, \lambda)) = 0\) and \((e_2, u(0, \lambda)) = 0\)

2. boundary condition at the right hand side: \((e_1, u(l, \lambda)) = 0\) and \((e_2, u(l, \lambda)) = 0\).

In the previous shooting-orthogonalization scheme, we integrate two vectors with initial values \(e_3\) and \(e_4\). Now, \(\omega_5\) represents the \(e_3 \wedge e_4\) plane, and so it is only necessary to integrate with one starting vector, namely

\[
W(0, \lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

From the original problem, two boundary conditions are required at the right hand side

\[
(e_1, u(1, \lambda)) = 0 \quad \text{and} \quad (e_2, u(1, \lambda)) = 0
\]

Thus, for the revised problem using our new numerical framework with exterior algebra, we require only one boundary condition, since the first component of \(W\) represents the plane spanned by \(e_1 \wedge e_2\):

\[
\Delta(\lambda) = (\dot{e}_1, W(1, \lambda)) = 0, \quad \text{where} \quad \dot{e}_1 = (1, 0, 0, 0, 0, 0)^T.
\]

Since we wish to impose this condition, we will use Newton's algorithm to determine \(\lambda\) such that \(\Delta(\lambda) = 0\), given an initial guess for \(\lambda\). (See Allen & Bridges [1] for the method on the restriction of (7.1) to \(k\)-dimensional subspaces on \(C^n\)).

7.2.1 Newton's method and analyticity

Since we need to use Newton's method to satisfy the right-hand boundary condition when \(A^{(2)}\) depends on \(\lambda\), we will need the derivative of \(W\) with respect to \(\lambda\). The derivative of \(W(0, \lambda)\) can be computed by appending a differential equation for \(\partial_\lambda W(y, \lambda)\) to the basic ordinary differential equation. That is, the basic ODE becomes

\[
\begin{pmatrix} W \\ \partial_\lambda W \end{pmatrix}_y = \begin{bmatrix} A^{(2)}(y, \lambda) & 0 \\ A^{(2)}_\lambda(y, \lambda) & A^{(2)}(y, \lambda) \end{bmatrix} \begin{pmatrix} W \\ \partial_\lambda W \end{pmatrix}.
\]
The initial condition for (7.14) is formed from the initial condition for $W(y, \lambda)$ and the derivative of this initial condition. When the domain is finite, this approach is straightforward, since we can integrate from 0 to 1 with the initial starting vector

$$
\begin{pmatrix}
W \\
W_\lambda
\end{pmatrix}
|_{y=0} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix},
$$

and the "1" is in the 6th entry. However, if the domain is infinite or semi-infinite, this approach is still satisfactory although computing the starting vector is not so straightforward. This construction is required, for example, when computing the stability of three-dimensional rotating flows, as discussed in part V of this report. An algorithm for computing the starting vector for a semi-infinite domain is given in the next chapter for Blasius boundary layer flow problems.

Now, for the bounded Poiseuille flow case, the corresponding system can be written as

$$Z_y = \tilde{A}^{(2)}(y, \lambda)Z$$

$$Z = \begin{pmatrix}
W \\
W_\lambda
\end{pmatrix} \in \mathbb{C}^{12},$$

where $\tilde{A}^{(2)}(y, \lambda)$ is the $12 \times 12$ matrix defined in equation (7.14).

Once we have integrated from $y = 0$ to the right hand side, $y = 1$, we let

$$\Delta(\lambda) = \langle \hat{e}_1, Z(y, \lambda) \rangle,$$

that is, the first component of $Z$ (which is the first component of $W$) at $y = 1$. Now, the function $\Delta(\lambda)$, whose roots are eigenvalues, is a complex analytic function and so application of Newton's method to find these roots is straightforward. Since we want $\Delta(\lambda)$ to be zero, numerically we compare this value to a small value, $\varepsilon$, say. If the value is less than $\varepsilon$, then we have converged. If not, then we use Newton's method in the form

$$\lambda_{n+1} = \lambda_n - \frac{\Delta(\lambda_n)}{\Delta'(\lambda_n)},$$

where $\Delta'(\lambda) = \langle \hat{e}_1, W_\lambda(1, \lambda) \rangle = \langle \hat{e}_7, Z(1, \lambda) \rangle$, that is, the seventh component of $Z$. Then repeat the integration until convergence of the eigenvalue, $\lambda$, occurs.
7.3 Geometric numerical integration

In choosing a numerical method for integrating the induced systems on $\Lambda^2(\mathbb{C}^4)$, accuracy is an important factor. However, it is also important to preserve 2-dimensional subspaces.

The induced systems obtained by the compound matrix method have been integrated using explicit fourth order Runge-Kutta algorithms [18]. Even though this method is widely used and reasonably accurate, it could still be improved since it was found that the explicit algorithms will not necessarily preserve the two-dimensional subspaces accurately, especially over long range integration. Our main observation, discussed in the following sections, is that the natural family of integrators for these systems is the class of implicit Gauss-Legendre Runge-Kutta algorithms, because they possess the special property that strong quadratic invariants are preserved automatically to machine precision [28].

7.3.1 Gauss Legendre

The Gauss-Legendre procedure is a fourth order, two stage implicit Runge-Kutta algorithm. For the Orr-Sommerfeld equation, the induced system to be integrated on $\Lambda^2(\mathbb{C}^4)$ is

$$Z_y = B(y, \lambda)Z, \quad Z(y, \lambda)|_{y=a} = \xi(\lambda) \in \Lambda^2(\mathbb{C}^4). \quad (7.15)$$

Here, $B(y, \lambda) = A^{(2)}(y, \lambda)$, and $\xi(\lambda)$ is a decomposable element of $\Lambda^2(\mathbb{C}^4)$.

Implicit midpoint rule

The second order Gauss-Legendre method is the midpoint rule, and when applied to (7.15) takes the form

$$Z^{s+1} = Z^s + \Delta yB(y_{s+\frac{1}{2}}, \lambda)Z^{s+\frac{1}{2}}, \quad (7.16)$$

where $Z^{s+\frac{1}{2}} = \frac{1}{2}(Z^s + Z^{s+1})$ and so

$$Z^{s+1} = \left[I - \frac{1}{2}\Delta yB_{s+\frac{1}{2}}\right]^{-1}\left[I + \frac{1}{2}\Delta yB_{s+\frac{1}{2}}\right]Z^s,$$

where $B_{s+\frac{1}{2}} = B(y_{s+\frac{1}{2}}, \lambda)$. The implicit nature results in having to invert a $6 \times 6$ matrix at each step.
Fourth order Gauss-Legendre

There is a two-stage implicit Runge-Kutta method of order 4 (cf. LAMBERT [77], p.153), which when applied to (7.15) takes the form

\[ Z_{s+1} = Z_s + \frac{1}{2} \Delta y(K_1 + K_2), \quad (7.17) \]

where \( K_1 \) and \( K_2 \) are implicitly defined by

\[ K_1 = B(y_s + \frac{1}{2} + \frac{\sqrt{3}}{6}) \Delta y)(Z^s + \frac{1}{4} \Delta yK_1 + (\frac{1}{4} + \frac{\sqrt{3}}{6}) \Delta yK_2), \quad (7.18) \]

\[ K_2 = B(y_s + \frac{1}{2} - \frac{\sqrt{3}}{6}) \Delta y)(Z^s + \frac{1}{4} \Delta yK_2 + (\frac{1}{4} - \frac{\sqrt{3}}{6}) \Delta yK_1). \quad (7.19) \]

If we let

\[ \sigma_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad b_1 = 1 + \frac{2}{\sqrt{3}} \]

\[ \sigma_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad b_2 = 1 - \frac{2}{\sqrt{3}}, \]

then (7.18) becomes

\[ K_1 = B(y_s + \sigma_1 \Delta y)(Z^s + \frac{1}{4} \Delta yK_1 + \frac{1}{4} b_1 \Delta yK_2) \]

\[ = B(y_s + \sigma_1 \Delta y)Z^s + \frac{1}{4} \Delta yB(y_s + \sigma_1 \Delta y)K_1 + \frac{1}{4} b_1 \Delta yB(y_s + \sigma_1 \Delta y)K_2. \]

Thus,

\[ K_1 - \frac{1}{4} \Delta yB(y_s + \sigma_1 \Delta y)K_1 = B(y_s + \sigma_1 \Delta y)Z^s + \frac{1}{4} b_1 \Delta yB(y_s + \sigma_1 \Delta y)K_2. \]

If we let \( B(y_s + \sigma_1 \Delta y) = B_1 \) then we obtain

\[ (I - \frac{1}{4} \Delta yB_1)K_1 - \frac{1}{4} b_1 \Delta yB_1K_2 = B_1Z^s. \quad (7.20) \]

Similarly, by letting \( B_2 = B(y_s + \sigma_2 \Delta y) \), equation (7.19) yields

\[ (I - \frac{1}{4} \Delta yB_2)K_2 - \frac{1}{4} b_2 \Delta yB_2K_1 = B_2Z^s. \quad (7.21) \]

Since we are dealing with a linear system of equations, all we need to do is rearrange the implicit equation to form an explicit integration procedure. That is, we need to solve the following linear system

\[ \begin{bmatrix} I - \frac{1}{4} \Delta yB_1 & -\frac{1}{4} \Delta yb_1B_1 \\ -\frac{1}{4} \Delta yb_2B_2 & I - \frac{1}{4} \Delta yB_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} B_1Z^s \\ B_2Z^s \end{bmatrix}. \quad (7.22) \]
Note that the block 2 \times 2 matrix is well conditioned since \( B_1 \) and \( B_2 \) involve the time step, \( \Delta y \). The time step is generally small (\( \ll 1 \)) and so this matrix is just a perturbation of the identity.

In summary, at each time step we solve equation (7.22) for the vectors \( K_1 \) and \( K_2 \), which are then substituted into equation (7.17) to obtain \( Z^{s+1} \).

### 7.3.2 Reducing computational time

Inverting an \( n \times n \) matrix requires \( n^3 \) operations, so for large systems of equations, computational time becomes an issue. It is possible to reduce the computational time when using either of the implicit Gauss-Legendre Runge-Kutta procedures by splitting the systems to be inverted into smaller systems and computing their inverses independently.

For the implicit midpoint rule we have the system

\[
[I - \frac{1}{2} \Delta y \, B_{s+1/2}] Z^{s+1} = [I + \frac{1}{2} \Delta y \, B_{s+1/2}] Z^s, \tag{7.23}
\]

where \( B \) is a block lower triangular matrix of order \( n \times n \). The right hand side of (7.23) is known and for simplification shall be labelled \( F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \). To find \( Z^{s+1} \), the inverse of \([I - \frac{1}{2} \Delta y \, B_{s+1/2}]\) requires to be calculated. Since \( B \) is block lower triangular, (7.23) can be rewritten as

\[
\begin{pmatrix}
I - E & 0 \\
-G & I - E
\end{pmatrix} W^{s+1} = F,
\]

where each sub-matrix is of order \( \frac{n}{2} \times \frac{n}{2} \).

Let \( Z^{s+1} = \begin{pmatrix} W \\ v \end{pmatrix} \), then the system can be split in two and thus we solve

\[(I - E)w = f_1\]

for \( w \), then

\[(I - E)v = f_2 + Gw\]

for \( v \). For both systems of equations, the solution requires inverting \( I - E \) taking \( (\frac{n}{2})^3 \) operations thus the computational time is reduced.

With the fourth-order Gauss-Legendre Runge-Kutta procedure, numerical solutions of systems of the form

\[
\begin{bmatrix}
I - \frac{1}{4} \Delta y B_1 & -\frac{1}{4} \Delta y b_1 B_1 \\
-\frac{1}{4} \Delta y b_2 B_2 & I - \frac{1}{4} \Delta y B_2
\end{bmatrix}
\begin{pmatrix}
K_1 \\
K_2
\end{pmatrix}
= \begin{pmatrix}
B_1 Z^s \\
B_2 Z^s
\end{pmatrix}, \tag{7.24}
\]
are required, where $\mathbf{B}_1$ and $\mathbf{B}_2$ are $n \times n$ matrices and $\mathbf{K}_1, \mathbf{K}_2, \mathbf{B}_1 \mathbf{Z}^s, \mathbf{B}_2 \mathbf{Z}^s$ are $n$-vectors. Let

$$
\begin{pmatrix}
\mathbf{K}_1 \\
\mathbf{K}_2
\end{pmatrix} =
\begin{pmatrix}
\mathbf{I} & \mathbf{M} \\
0 & \mathbf{I}
\end{pmatrix}
\begin{pmatrix}
\bar{\mathbf{K}}_1 \\
\bar{\mathbf{K}}_2
\end{pmatrix},
$$

(7.25)

for some $n \times n$ matrix, $\mathbf{M}$ to be determined. Using this, system (7.24) becomes

$$
\begin{bmatrix}
\mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_1 & -\frac{1}{4} \Delta y b_1 \mathbf{B}_1 \\
-\frac{1}{4} \Delta y b_2 \mathbf{B}_2 & \mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_2
\end{bmatrix}
\begin{bmatrix}
\mathbf{I} & \mathbf{M} \\
0 & \mathbf{I}
\end{bmatrix}
\begin{pmatrix}
\bar{\mathbf{K}}_1 \\
\bar{\mathbf{K}}_2
\end{pmatrix} =
\begin{pmatrix}
\mathbf{B}_1 \mathbf{Z}^s \\
\mathbf{B}_2 \mathbf{Z}^s
\end{pmatrix}.
$$

(7.26)

By choosing

$$
\mathbf{M} = \frac{1}{4} \Delta y b_1 (\mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_1)^{-1} \mathbf{B}_1
= -b_1 \mathbf{I} + b_1 (\mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_1)^{-1},
$$

(7.26) simplifies to the lower block triangular matrix

$$
\begin{bmatrix}
\mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_1 & 0 \\
-\frac{1}{4} \Delta y b_2 \mathbf{B}_2 & \mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_2 - \frac{1}{4} \Delta y b_2 \mathbf{B}_2 \mathbf{M}
\end{bmatrix}
\begin{pmatrix}
\bar{\mathbf{K}}_1 \\
\bar{\mathbf{K}}_2
\end{pmatrix} =
\begin{pmatrix}
\mathbf{B}_1 \mathbf{Z}^s \\
\mathbf{B}_2 \mathbf{Z}^s
\end{pmatrix},
$$

(7.27)

which can be reduced to two $n \times n$ systems. The first is solved for $\bar{\mathbf{K}}_1$, namely

$$
(\mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_1) \bar{\mathbf{K}}_1 = \mathbf{B}_1 \mathbf{Z}^s,
$$

(7.28)

then the second is solved for $\bar{\mathbf{K}}_2$,

$$
(\mathbf{I} - \frac{1}{4} \Delta y \mathbf{B}_2 - \frac{1}{4} \Delta y b_2 \mathbf{B}_2 \mathbf{M}) \bar{\mathbf{K}}_2 = \mathbf{B}_2 \mathbf{Z}^s + \frac{1}{4} \Delta y b_2 \mathbf{B}_2 \bar{\mathbf{K}}_1.
$$

(7.29)

Once $\bar{\mathbf{K}}_1$ and $\bar{\mathbf{K}}_2$ have been determined, the original $\mathbf{K}_1$ and $\mathbf{K}_2$ can be found by

$$
\mathbf{K}_1 = \bar{\mathbf{K}}_1 + \mathbf{M} \bar{\mathbf{K}}_2
$$

$$
\mathbf{K}_2 = \bar{\mathbf{K}}_2.
$$

The above construction reduces the number of numerical computations. However, (7.28) can be simplified further since both $\mathbf{B}_1$ and $\mathbf{B}_2$ have block lower triangular structure,

$$
\mathbf{B}_1 = \begin{pmatrix}
\mathbf{B}_{11} & 0 \\
\mathbf{B}_{12} & \mathbf{B}_{11}
\end{pmatrix},
\mathbf{B}_2 = \begin{pmatrix}
\mathbf{B}_{22} & 0 \\
\mathbf{B}_{21} & \mathbf{B}_{22}
\end{pmatrix},
$$

(7.30)
where each sub-matrix $B_{ij}$ is of order $\frac{n}{2} \times \frac{n}{2}$. From (7.28) and (7.30)

$$(I - \frac{1}{4} \Delta y B_1) = \begin{pmatrix} I - \frac{1}{4} \Delta y B_{11} & 0 \\ -\frac{1}{4} \Delta y B_{12} & I - \frac{1}{4} \Delta y B_{11} \end{pmatrix},$$

and a matrix of the form

$$\begin{pmatrix} E & 0 \\ G & E \end{pmatrix}$$

has inverse

$$\begin{pmatrix} E & 0 \\ G & E \end{pmatrix}^{-1} = \begin{pmatrix} E^{-1} & 0 \\ -E^{-1}GE^{-1} & E^{-1} \end{pmatrix},$$

so

$$(I - \frac{1}{4} \Delta y B_1)^{-1} = \begin{bmatrix} (I - \frac{1}{4} \Delta y B_{11})^{-1} & 0 \\ \frac{1}{4} \Delta y(I - \frac{1}{4} \Delta y B_{11})^{-1}B_{12}(I - \frac{1}{4} \Delta y B_{11})^{-1} & (I - \frac{1}{4} \Delta y B_{11})^{-1} \end{bmatrix}. (7.31)$$

Thus only one matrix of order $\frac{n}{2} \times \frac{n}{2}$ is to be inverted requiring yet again fewer operations.

From (7.29), $M$ is block lower triangular and so $(I - \frac{1}{4} \Delta y B_2 - \frac{1}{4} \Delta y b_2 B_2 M)$ also has block lower triangular structure, thus using the method above, the number of operations required to obtain the inverse can be reduced again. Therefore, inverting the separate systems requires substantially fewer operations as opposed to the number needed to invert the original fourth order GL-RK system.

### 7.4 Other accuracy issues to note

If we look at the graph of the eigenfunctions for plane Poiseuille flow given in appendix B, produced by Davey [35], we can see that the most rapid variation takes place in the vicinity of the so called critical layer near the wall at $y = \pm 1$. That is, large oscillations occur at -1 and 1. Thus, it is better to use only half the interval, either, from 1 to 0 or -1 to 0. This is because any errors get greatly magnified by the oscillations at the ends, so integrating from -1 to 1 leads to small errors, gradually built up from the start, being greatly magnified at the other end (just before 1).

Taking half the interval and integrating from the edge with an oscillation to a smooth well-behaved point, only very small starting errors are amplified. Then only the usual small step-size errors are added at each step until the right hand side has been reached. Thus, making the overall numerical integration more accurate.
Note that, splitting the interval into two parts requires the boundary conditions at 
\(y = 0\) and not \(y = 1\). Now at \(y = 0\), \(\phi' = \phi'' = 0\). So, \(u_2\) and \(u_4\) are zero \(\Rightarrow w_5 = 0\) at the 
right hand side.

### 7.5 Problems encountered

Code written in FORTRAN 77 with the function to produce a converged value of \(\lambda\) and, 
hence, a value for the wave speed \(c\) for the stability question of the OS equation for plane 
Poiseuille flow failed to produce any such convergence. Instead, the numerics looped into 
oscillations around some small values of \(\lambda\).

The attempt to debug the code involved checking numerous values at different time 
steps in the program. It was noticed that very small errors were being introduced during 
calculations, but these were much smaller than machine precision and so thought not to 
be the cause of the error. It turned out that the code would determine the direction of 
the solution vector very accurately, however it allowed the magnitude to get very large, 
causing unnecessary round-off error. In any case, the problem lead us to look at the theory 
in more detail.

#### 7.5.1 Check suggested by Ng and Reid [92]

We have already seen that in order to integrate our induced system \(W_y = A^{(2)}(y, \lambda)W\) 
along a path of two-dimensional subspaces, it has to be restricted to decomposable 2-
forms. Now, a 2-form \(W \in \Lambda^2(\mathbb{C}^4)\) is decomposable if \(W \wedge W = 0\) (cf. Griffiths & Harris 
[56]). Expanding \(W\) in terms of the standard basis (7.10) of \(\Lambda^2(\mathbb{C}^4)\), gives

\[
0 = W \wedge W = \sum_{i=1}^{6} \sum_{j=1}^{6} W_i W_j \omega_i \wedge \omega_j = (W_1 W_6 - W_2 W_5 + W_3 W_4) e_1 \wedge e_2 \wedge e_3 \wedge e_4.
\]

Define \(\mathcal{I} : \Lambda^2(\mathbb{C}^4) \to \mathbb{C}\) by

\[
\mathcal{I}(W) = W_1 W_6 - W_2 W_5 + W_3 W_4. \tag{7.32}
\]

For a path of the equation (7.6) to be a path of two-dimensional subspaces, the quadratic 
function (7.32) has to be preserved. Due to decomposability, the identity \(\mathcal{I}(W)\) is equal 
to zero for all \(y\) in the range of integration, therefore, if possible, the numerical method 
should be designed to preserve this constraint exactly. The surface defined by \(\mathcal{I}(W) = 0\)
Part II: The compound matrix method

is $G_2(\mathbb{C}^4)$, the Grassmannian manifold of two planes in $\mathbb{C}^4$ (cf. [56]).

Cooper [28] has proved that the implicit GL-RK method preserves the quadratic invariant constraint, $I(W)$ to machine precision by the discretization based on GL-RK, when $I$ is a strong invariant (Allen & Bridges [1]). Note that an explicit RK algorithm will not have this property. That is, for explicit RK algorithms

$$I^{s+1} = I^s + O(\Delta y^p)$$

for some $p$.

Since $I(W) = 0$ is a necessary constraint, its value was checked at set time points during the numerical integration. The results revealed an exponentially increasing invariant. The norm of the solution was then checked and found to be very large implying that our initial equations were very unstable. This lead us to conclude that the errors were caused by the inability of the computer system to cope with the huge numbers involved throughout the calculations within the code, even though the direction of the vector was being computed accurately. Hence the rounding errors noticed were in fact the cause of the error in the running of the program.

The eigenvalue problem was re-addressed. Our aim being to find the eigenvalues and corresponding eigenvectors of the fluid flow. The important factor in our problem is the direction of the eigenvector. Since a vector is not unique in magnitude, this appeared to be a good way of controlling the rounding errors introduced due to calculations involving huge numbers. Thus a few lines of code were added to scale the solution vector down to its unit vector in regular intervals during the iterative Runge-Kutta procedure. However, our solution vector obtains both the required eigenvector and its derivative. Thus, to scale the vector, the length was found from only the first six components of the solution vector, and then the complete $12 \times 1$ vector is scaled down by this value. Although this method is non-analytic, the same results would be obtained from an analytic scaling.

To test the effectiveness of the present method on this problem and to check the accuracy of my code, we have considered the unstable mode for $\alpha = 1$ and $Re = 10,000$, as this is a case for which a comparison can be made with existing results obtained by Ng & Reid [92].
7.6 Numerical results for the Orr-Sommerfeld equation using GL-RK with $\alpha = 1.0$ and $Re = 10,000$

<table>
<thead>
<tr>
<th>Number of steps, N</th>
<th>Value of wave speed, $c$</th>
<th>Ng &amp; Reid [92], $c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.2369 916 757 + 0.0037 979 897i</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>0.2374 934 492 + 0.0037 446 524i</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.2375 199 725 + 0.0037 407 017i</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>0.2375 244 279 + 0.0037 400 016i</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>0.2375 256 451 + 0.0037 398 072i</td>
<td>0.2375 221 + 0.0037 409i</td>
</tr>
<tr>
<td>600</td>
<td>0.2375 260 825 + 0.0037 397 360i</td>
<td>0.2375 243 + 0.0037 402i</td>
</tr>
<tr>
<td>700</td>
<td>0.2375 262 692 + 0.0037 397 064i</td>
<td>0.2375 253 + 0.0037 400i</td>
</tr>
<tr>
<td>800</td>
<td>0.2375 263 602 + 0.0037 396 915i</td>
<td>0.2375 258 + 0.0037 398i</td>
</tr>
<tr>
<td>900</td>
<td>0.2375 264 084 + 0.0037 396 838i</td>
<td>0.2375 260 + 0.0037 498i</td>
</tr>
<tr>
<td>1000</td>
<td>0.2375 264 361 + 0.0037 396 793i</td>
<td>0.2375 262 + 0.0037 397i</td>
</tr>
<tr>
<td>1100</td>
<td>0.2375 264 531 + 0.0037 396 758i</td>
<td>0.2375 263 + 0.0037 397i</td>
</tr>
<tr>
<td>1200</td>
<td>0.2375 264 635 + 0.0037 396 748i</td>
<td>0.2375 263 + 0.0037 397i</td>
</tr>
<tr>
<td>1300</td>
<td>0.2375 264 704 + 0.0037 396 735i</td>
<td></td>
</tr>
<tr>
<td>1400</td>
<td>0.2375 264 750 + 0.0037 396 729i</td>
<td></td>
</tr>
<tr>
<td>1500</td>
<td>0.2375 264 784 + 0.0037 396 723i</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>0.2375 264 856 + 0.0037 396 711i</td>
<td></td>
</tr>
<tr>
<td>2500</td>
<td>0.2375 264 875 + 0.0037 396 709i</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>0.2375 264 882 + 0.0037 396 707i</td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>0.2375 264 886 + 0.0037 396 707i</td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>0.2375 264 887 + 0.0037 396 707i</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.1: Comparison of current numerical results with results by Ng & Reid for the OS equation using GL-RK algorithm

Table 7.1 shows the effect of step-size on the eigenvalue, $c$ including a comparison with the values obtained by Ng & Reid [92].

In the calculations from Ng & Reid their aim was not to achieve great accuracy but rather to show that reasonable accuracy could be obtained without difficulty. Their cal-
Calculations were made by using the Runge-Kutta-Gill procedure with constant step size and were performed in single-precision arithmetic on a CDC-6600 computer, (equivalent to double precision arithmetic on our present computers).

From this comparison, we can clearly conclude that our code is producing accurate results. Thus, our FORTRAN 77 code designed to produce parameter values of the Reynolds number and the wave number for which the imaginary part of the wave speed is approximately zero, was executed with confidence using our new numerical framework and the following curve of neutral stability obtained for Poiseuille flow.

### 7.7 Neutral curve for plane Poiseuille flow

Calculations for the neutral curve in the temporal scheme were performed using a value for the number of steps in the integration, \( N = 500 \) along with the fourth order GL-RK integration scheme to ensure the most accurate results possible. This accuracy is not essential and the curve of neutral stability could adequately be produced with much fewer steps and a less accurate integration scheme. Figure 7.1 shows the temporal neutral curve for the Poiseuille flow velocity profile. It is formed by plotting the values of \( \alpha \) and \( Re \) which return the eigenvalue, \( c \), with zero imaginary part.

![Neutral Curve for plane Poiseuille flow](image-url)
The critical point, nose of the neutral curve occurs at

\[ c_l \approx 0, \quad c_r = 0.2640, \quad R_e \approx 5772.0, \quad \alpha \approx 1.02. \]

This critical value of the Reynolds number represents the ratio of magnitudes of destabilizing forces of shear and stabilizing forces, for which their effects may be said to balance.

The region inside the curve represents unstable flow, whereas outside the curve, the flow perturbations are stable. The parameter values found for the critical point reflect those produced by Davey (unpublished) [38] with values quoted as \( R_{ec} = 5772.2218 \), \( \alpha_c = 1.020547 \), \( c = 0.2640003 \) using a shooting method and the stabilizing technique of orthogonalization.
The research in this report is based on computing neutral curves to study the stability of fluid flow past a compliant surface, that is, we fix all but two of the parameters in the problem and plot $\text{Im}(c) = 0$. For illustrative purposes, let us assume we are studying the stability of the Ekman boundary layer flow (discussed in part V of this report). In this case we fix the Reynolds number, Rossby number and all other parameters except the modulus of the wave number, $\gamma$ and the angle of rotation orientation, $\varepsilon$ and plot $\text{Im}(c) = 0$. Let $\text{Im}(c) = I$, so

$$I(\gamma, \varepsilon) = 0 \quad (\gamma \in \mathbb{R}, \varepsilon \in \mathbb{R}, I \in \mathbb{R}),$$

and we wish to obtain a data set of values of $\gamma$ and $\varepsilon$ that satisfy (8.1). On the surface of neutral stability, i.e., the surface in the $(Re, \gamma, \varepsilon)$-space along which $c_i$ vanishes, contours of constant Reynolds numbers are found using a continuation procedure.

### 8.1 Numerical continuation

Numerical continuation is a technique to find consecutive points of a solution branch to (8.1). Suppose we have already found a solution point $(\gamma_0, \varepsilon_0)$. To find a new point, call it $(\gamma_1, \varepsilon_1)$, we need a starting point $(\hat{\gamma}, \hat{\varepsilon})$ and a strategy to determine $(\gamma_1, \varepsilon_1)$. A historical method is to choose $\varepsilon$ and to fix $\gamma_1 = \hat{\varepsilon}$, then $\gamma_1$ is determined by solving

$$I(\gamma, \varepsilon_1) = 0.$$ 

Solving this system by Newton's method we construct a sequence $\gamma^0, \gamma^1, \gamma^2, \ldots$ with $\gamma^0 = \hat{\gamma} = \gamma_0$ and

$$\Delta \gamma^k = -\frac{I(\gamma^k, \varepsilon_1)}{[I(\gamma^k, \varepsilon_1)]_\gamma} \quad (8.2)$$

$$\gamma^{k+1} = \gamma^k + \Delta \gamma^k \quad (8.3)$$
for $k = 0, 1, 2, \ldots$. Of course, any other method to solve nonlinear systems can be used instead of (8.2) and (8.3).

Geometrically, this amounts to approximating the curve first by a straight line (predictor step) and then correcting in a hyper-plane $\varepsilon = \varepsilon_1$ (corrector step) see figure (8.1). If $\varepsilon$ is increased at a fixed step size then it may overshoot the turning point of the curve and not converge. Thus a very small step size must be taken. A possible solution to this problem is to swap the fixed parameter when a turning point is close, that is, to fix $\gamma$ and vary $\varepsilon$ until convergence occurs and then increase or decrease $\gamma$ by a fixed amount, see figure 8.2. However, the step size has to be small and thus computing time is large. The

Figure 8.1: Linear method

Figure 8.2: Swapping fixed parameter at turning points

following section discusses a more advanced method for the continuation of a curve used
Part II: Numerical methods for continuation of the neutral stability curves

by the sophisticated numerical bifurcation and continuation package, AUTO.

8.2 Pseudo-Arc length continuation

The historical method discussed in the previous section has been used in most of the calculations in this report. However, the method has difficulties if a branch of equation (8.1) contains limit points; \( \varepsilon \) is not a good parameterization of the curve in the neighbourhood of such points. A method to overcome this problem is to use pseudo-arc length continuation. The following principles of this idea are explained by Willy Govaerts [50].

The idea is that an arc-length is introduced as a new parameter, and at fixed values of this parameter we search for a zero of the determinant along a line approximately orthogonal to the contour of neutral stability. Suppose that a point \((\gamma_1, \epsilon_1)\) and a previous point \((\gamma_0, \epsilon_0)\) with tangent vector \(t_0\) are known. To find a tangent vector \(t_1\) at \((\gamma_1, \epsilon_1)\) we remark that

\[
[I_\gamma, I_\epsilon]t_1 = 0. 
\]  

(8.4)

This equation determines \(t_1\) up to a scale factor. To preserve the orientation of the branch we require

\[
t_0^T t_1 = 1 . 
\]  

(8.5)

If we decompose \(t_1\) in a natural way as \(t_1 = \begin{pmatrix} t_1^{(\gamma)} \\ t_1^{(\epsilon)} \end{pmatrix}\) and do similarly for \(t_0\), then we can write (8.4) and (8.5) as

\[
\begin{pmatrix} I_\gamma & I_\epsilon \\ t_0^{(\gamma)} & t_0^{(\epsilon)} \end{pmatrix} \begin{pmatrix} t_1^{(\gamma)} \\ t_1^{(\epsilon)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\]

Suppose that a step length \(\Delta S\) is chosen (the choice of which is discussed in the following section). Then our predictor is

\[
\hat{\gamma} = \gamma_1 + \frac{\Delta S}{||t_1||} t_1^{(\gamma)}, \quad \hat{\epsilon} = \epsilon_1 + \frac{\Delta S}{||t_1||} t_1^{(\epsilon)} .
\]

This is called a pseudo-arclength predictor (figure 8.3) because \(\Delta S\) measures arc-length along the tangent line and, therefore, approximates the arc-length along the branch. In general there are several possibilities for the corrector step. However, only the method actually used in the research is discussed in this report, other methods can be found in [50].
A solution to (8.1) is sought in the hyper-plane orthogonal to \( t_1 \), figure 8.4. The system for the computation of the next point on the branch is therefore,

\[
I(\gamma, \varepsilon) = 0, \quad t_1^{(\gamma)} (\gamma - \dot{\gamma}) + t_1^{(\varepsilon)} (\varepsilon - \dot{\varepsilon}) = 0,
\]

with Jacobian

\[
\begin{bmatrix}
IMC_{\gamma} & IMC_{\varepsilon} \\
\dot{t}_1^{(\gamma)} & \dot{t}_1^{(\varepsilon)}
\end{bmatrix}
\]

This is called Keller's method [50] and references therein.

The next section considers the important aspects of step-length control, automatically carried out in the continuation and bifurcation package, AUTO.

### 8.3 Step-length control

Step-length control is an important part of a continuation method. If the step-length is too small, then a lot of unnecessary work is done. If it is too large, then the corrector algorithm may converge to a point on a different branch of (8.1) or in our problem not
converge at all. It is possible, however, to make an a priori estimate of a good step-length, although such an estimate is never completely reliable, see [50] for further details.

The package, AUTO, was used for the continuation of the neutral curves for the Ekman boundary layer stability analysis, where the speed of this method was greatly beneficial in the particular curves forming closed loops. However, this package was not used for the majority of neutral curves produced for the other hydrodynamic stability problems studied in this report since its methods did not appear to provide, in general, a significant decrease in computational time.
The stability of the boundary layer flow along a semi-infinite flat plate is of interest both theoretically and experimentally. The study of the stability of laminar boundary layers was originally undertaken in an attempt to account for the phenomenon of transition, and the early work by Tollmien and Schlichting provided detailed theoretical predictions for the growth or decay of small disturbances, known as Tollmien-Schlichting (TS) waves, in a Blasius boundary layer.

During the 1930's, no experimental support emerged for the theory, and the view amongst experimentalists of that time was that transition depended primarily on the magnitude of the disturbances in the boundary layer. It was thought that when the pressure gradients associated with the disturbances became large, transition would be caused by local separation of the boundary layer. However, the predictions of linear stability theory didn’t support this view. Instead the theory shows that the stability/instability of small disturbances depend only on the frequency or wavelength and on the Reynolds number. In 1947, Schubauer and Skramstad [117] experimentally confirmed this theory. They conducted their experiments in low-turbulent wind tunnels using a vibrating ribbon to produce a controlled disturbance of known amplitude and frequency, and the growth/decay of forced oscillations were detected by highly sensitive hot-wire anemometers. This technique has continued to be widely used for experimental work on TS waves and related phenomena.

The first theoretical calculations of the curve of marginal stability for the Blasius boundary layer profile was made by Tollmien [131] who, using an approximation, [38], obtained values of the parameters at the minimum critical Reynolds number given in table 9.1. A subsequent calculation by Schlichting [115] led to significantly different results (see
Part II: Linear stability of the Blasius boundary layer

Then Lin [84] tried to resolve the discrepancy between these two sets of results obtaining results close to those of Tollmien's except for the value of the wave number, $\alpha$. Nevertheless, exact numerical calculations by Jordinson [66] led to a substantial increase in the value of the critical Reynolds number. But, when these more accurate results were compared with the experimental data, the agreement became much less satisfactory, especially at lower Reynolds numbers. Drazin & Reid [38] suggest that agreement can be restored, however, by taking into account the non-parallel character of the basic flow, that is, the difference between theoretical and experimental results is thought to be due to the effects of boundary layer growth.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_{crit}$</th>
<th>$c$</th>
<th>$R_{crit}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tollmien (1929)</td>
<td>0.34</td>
<td>0.41</td>
<td>420</td>
</tr>
<tr>
<td>Schlichting (1933)</td>
<td>0.23</td>
<td>0.42</td>
<td>575</td>
</tr>
<tr>
<td>Lin (1945)</td>
<td>0.3718</td>
<td>0.411</td>
<td>421</td>
</tr>
<tr>
<td>'Exact' (Jordinson 1970)</td>
<td>0.3012</td>
<td>0.3961</td>
<td>520</td>
</tr>
<tr>
<td>'Exact' (Davey)</td>
<td>0.30377</td>
<td>0.39664</td>
<td>519.060</td>
</tr>
</tbody>
</table>

Table 9.1: The values of the parameters associated with the minimum critical Reynolds number for the Blasius boundary layer profile (based on the displacement thickness $\delta^*$)

9.1 Blasius solution

To solve the stability problem for the Blasius boundary layer equation, we are required to solve the Orr-Sommerfeld equation for the boundary layer flow past one fixed boundary, that is, we can choose the upper boundary at infinity, and so we use the theory of boundary layer flows on the semi-infinite domain $[0, \infty)$.

Starting with the Orr-Sommerfeld equation on $0 \leq y \leq y_\infty$, we use a large number, $y_\infty$, as the upper boundary to represent infinity. This large number is chosen so that the velocity profile of the flow at this value is approximately equal to the theoretical velocity profile at infinity.

The boundary conditions at the rigid wall are as before

$$\phi(0) = \phi'(0) = 0.$$
and the boundary conditions at the imaginary upper wall are

$$
\phi'' + \beta \phi' - \alpha^2 \phi = 0 \quad \text{at} \quad y = y_\infty \quad (9.1)
$$

$$
\phi'' + \alpha \phi' - \beta^2 \phi = 0 \quad \text{at} \quad y = y_\infty , \quad (9.2)
$$

where

$$
\beta^2 = \alpha^2 + i\alpha R_e (U_\infty - c) = \alpha^2 + iR_e \alpha U_\infty + \lambda R_e
$$

and $U_\infty = \lim_{y \to \infty} U(y)$.

The Reynolds number in the Orr-Sommerfeld equation is based on the free-stream velocity and the thickness of the boundary layer, and so it is usually treated as a nearly parallel flow. In this approximation, the basic velocity distribution is given by $U(y) = f'(y)$, where $f(y)$ is the Blasius function which satisfies the equation

$$
f'' + \frac{1}{2} f f'' = 0 , \quad (9.3)
$$

along with the boundary conditions

$$
f(0) = f'(0) = 0 , \quad \text{and} \quad f'(y) \to 1 \quad \text{as} \quad y \to \infty .
$$

The Blasius equation is non-linear, thus, for efficiency, we will need to use the GL-RK procedure for any numerical integration. We shall first create a vector, $w$, comprising of $w_1 = f$, $w_2 = f'$, $w_3 = f''$. Then, the non-linear system can be written as

$$
\frac{d}{dy} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_2 \\ w_3 \\ -\frac{1}{2} w_1 w_3 \end{pmatrix}
$$

with initial data, $w(0) = (0, 0, a)^T$. where, the $a$ is used to represent the unknown value of $f''(0)$.

Since we require the condition, $f'(y_\infty) = 1$, that is, $(e_2, w(y_\infty)) = 1$ or, $(e_2, w(y_\infty)) - 1 = f(a) \approx 0$. Newton's method is implemented to adjust $a$ so that $f(a) \to 0$. Now, application of Newton's method requires the derivative of $w$.

Let $W = \frac{d}{dx}(w)$. Then, the following non-linear system can be set up

$$
\frac{d}{dy} W = \begin{pmatrix} W_2 \\ W_3 \\ -\frac{1}{2} w_1 W_3 - \frac{1}{2} w_3 W_1 \end{pmatrix} \quad \text{with} \quad W(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .
$$
We can thus solve the six dimensional system

\[
\frac{d}{dy} Z = \begin{pmatrix}
    w_2 \\
    w_3 \\
    -\frac{1}{2} w_1 w_3 \\
    W_2 \\
    W_3 \\
    -\frac{1}{2} w_1 W_3 - \frac{1}{2} w_2 W_1
\end{pmatrix},
\]

where \( Z = (w_1, w_2, w_3, W_1, W_2, W_3)^T \). The initial vector for \( Z \) is taken as \( Z(0) = (0, 0, a, 0, 0, 1)^T \).

Now, from Howarth [65], \( f''(0) = a \approx 0.33206 \). Note that for use of the Blasius equation without the coefficient of \( \frac{1}{2} \) in front of the term, \( f f'' \), then \( a \approx 0.46960 = \sqrt{2} \times 0.33206 \).

Thus, we shall use this as our starting value and use Newton’s convergence method in the following form

\[
a_{n+1} = a_n - \left( \frac{w_2(y_{\infty}, a_n) - 1}{W_2(y_{\infty}, a_n)} \right)
\]

For the Orr-Sommerfeld problem, we need the Blasius velocity profile and its second derivative, that is, \( U = f'(y) \) and \( U'' = f''(y) = -\frac{1}{2} f(y) f''(y) \). For all values of \( y \) we need the \( w_1, w_2 \) and \( w_3 \) values. To find these values, we first find \( a \) by using Newton’s iteration and then re-run the program with the correct value of \( a \) to obtain the required values, \( w_1, w_2 \) and \( w_3 \). This method seems very easy to implement, though, a problem arises from the use of the implicit Runge Kutta procedure with the compound matrix method for solving the Orr-Sommerfeld equation. The implicit RK method requires the value of \( U \) to be evaluated at \( y + (\frac{1}{2} + \frac{\sqrt{3}}{6}) \times h \) and not at \( y \). However, there are two possible solutions for overcoming this problem. The first possible solution makes use of interpolation. Basically, from solving the Blasius equation we obtain values for \( U(y) \) and \( U(y + h) \). Now, the use of the Gauss-Legendre method requires values of the velocity profile at the points \( y + a_1 h \) and \( y + a_2 h \), where, \( a_1 = \frac{1}{2} - \frac{\sqrt{3}}{6} \) and \( a_2 = \frac{1}{2} + \frac{\sqrt{3}}{6} \). So, we could use a two point interpolation (weighted average) or Legendre interpolation such that

\[
a_2 U(y) + (1 - a_1) U(y + h) = U(y + a_1 h).
\]

The second possible solution was to solve the Blasius equation using Chebyshev polynomials. This method was considered because of the fast convergence properties of Chebyshev polynomials on the interval \(-1 \leq z \leq 1\). The method involves expanding the Blasius equation in Chebyshev polynomials and solving these numerically to obtain numerical values.
for the Chebyshev coefficients, which in turn can be used to solve the Blasius boundary layer problem. An outline of the method follows.

9.1.1 Chebyshev polynomials

Since we wish to solve the Blasius equation

\[ f'''' + \frac{1}{2} f f'' = 0, \quad \text{on} \quad 0 \leq y \leq L \]

with boundary conditions

\[ f(0) = f'(0) = 0, \quad f'(L) = 1, \]

we first need to make a variable transformation to the Chebyshev domain, \(-1 \leq z \leq 1\).

Let \( z = \frac{y}{L} - 1 \), then \(-1 \leq z \leq 1\) and \( \frac{dy}{dz} = \frac{2}{L} \frac{dz}{dy} \). The Blasius equation then transforms to

\[ f'''' + \frac{L}{4} f f'' = 0, \quad -1 \leq z \leq 1 \]

and the boundary conditions become

\[ f(-1) = f'(-1) = 0, \quad f'(1) = 1. \]

We can now expand this smooth function in a series of Chebyshev polynomials

\[ f(z) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n T_n(z), \]

where the Chebyshev polynomials are defined to be

\[ T_0(z) = 1, \quad T_1(z) = z, \quad T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad \text{for} \quad n \geq 1, \]

and

\[ a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(z)T_n(z)}{\sqrt{1 - z^2}} dz. \]

Approximations to the \( a_n \)'s are obtained by solving numerically. The \( f(z) \)'s are then calculated and substituted into the Blasius equation

\[ f'''' + \frac{L}{4} f f'' = 0. \]

This leads to a system of algebraic equations for \( a_0 \cdots a_N \). Now, we have a nonlinear algebraic system for \( a = a_0, \cdots, a_N \), thus we solve this using Newton's method. Once the
system of polynomials is solved, we have numerical values for \(a_0, \ldots, a_N\). Thus, given any \(y_0 \in [0, L]\), we let \(z_0 = \frac{2}{L} y_0 - 1\) and \(f\) at that value of \(z_0\) is given by summing

\[
f(z_0) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n T_n(z_0).
\]

Since this method is also extremely easy to implement once the Chebyshev coefficients have been calculated, it is the method chosen and implemented in this report for solving the Blasius equation.

### 9.2 Deriving the initial vector for \(U(y_\infty)\)

In this section, we shall apply the theory of subspaces that are determined by finite sets of vectors, since these subspaces arise naturally as solution spaces of homogeneous linear systems of equations and are, thus, ideal for use with our homogeneous linear system of boundary conditions.

Basically, what is required is to find a basis for the two-dimensional space \((\xi_1, U) = 0\) and \((\xi_2, U) = 0\).

Before we derive the initial vector from the full asymptotic boundary conditions, the question of how accurate the boundary conditions have to be will be briefly addressed, using an example. Below is an example of a naive boundary condition at \(Z_0\):

\[
\phi' + a \phi = 0, \\
\phi' + b \phi = 0.
\]

These conditions are not accurate, but have been used before and found to work pretty well (cf. Drazin and Reid [38]).

From the first boundary condition, the general solution would be

\[
\phi(y) = A e^{\alpha y} + B e^{-\alpha y}
\]

and so,

\[
\phi'(y) = \alpha A e^{\alpha y} - \alpha B e^{-\alpha y}.
\]

Thus, the first boundary condition results in setting \(A\) to zero to get rid of the growth term. Similarly for the second boundary condition.
However, the boundary conditions are coupled and so, the full correct asymptotic boundary conditions should be used.

The asymptotic boundary conditions at \( y = y_\infty \) were found previously to be

\[
\begin{align*}
\phi'''' + \beta \phi'' - \alpha^2 \phi' - \alpha^2 \beta \phi &= 0, \\
\phi'''' + \alpha \phi'' - \beta^2 \phi' - \beta^2 \alpha \phi &= 0,
\end{align*}
\]

where \( \beta^2 = \alpha^2 + i\alpha \Re(U_\infty - c) \) and \( U_\infty = U(y_\infty) = 1 \). Assuming a spanning set can be found, let \( a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \) where

\[
a_1 = \phi, \quad a_2 = \phi', \quad a_3 = \phi'', \quad a_4 = \phi'''.
\]

then our system of boundary conditions can be re-written as

\[
\begin{align*}
a_4 + \beta a_3 - \alpha^2 a_2 - \alpha^2 \beta a_1 &= 0, \\
a_4 + \alpha a_3 - \beta^2 a_2 - \beta^2 \alpha a_1 &= 0.
\end{align*}
\]

Writing this system of equations in vector-matrix notation we have

\[
\begin{pmatrix}
-\alpha^2 \beta & -\alpha^2 & \beta & 1 \\
-\beta^2 \alpha & -\beta^2 & \alpha & 1 
\end{pmatrix}
\begin{pmatrix}
a_1 \\ a_2 \\ a_3 \\ a_4 
\end{pmatrix} = 0.
\]

Noticing that the solution of this system is a subspace in \( \mathbb{C}^4 \), we can find a spanning set for this space. By writing down the augmented matrix of this system and performing row operations so that the new system of equations formed will have the same solution set as the original system, we obtain

\[
\begin{pmatrix}
-\alpha \beta (\alpha - \beta) & (\alpha + \beta) (\beta - \alpha) & \beta - \alpha & 0 & 0 \\
-\beta^2 \alpha & -\beta^2 & \alpha & 1 & 0 
\end{pmatrix}.
\]

Now, it can clearly be seen that this system of equations has two linearly independent equations but four unknowns, thus we have two degrees of freedom, and so, can choose two arbitrary constants for two variables of our choice. From this system we can write both \( a_3 \) and \( a_4 \) in terms of \( a_1 \) and \( a_2 \). Thus we choose the following arbitrary constants:

\[
a_1 = c_1, \quad \text{and} \quad a_2 = c_2.
\]
Thus we have

\[ \mathbf{a} = (c_1, c_2, -(\alpha + \beta)c_2 - \alpha\beta c_1, (\alpha(\alpha + \beta) + \beta^2)c_2 + \alpha\beta(\alpha + \beta)c_1), \]

and by factoring out the \( c_i \)'s, we can write the general solution as

\[ \mathbf{a} = c_1 \begin{pmatrix} 1 \\ 0 \\ -\alpha \beta \\ \alpha \beta (\alpha + \beta) \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -(\alpha + \beta) \\ (\alpha^2 + \alpha \beta + \beta^2) \end{pmatrix}. \]

Thus, for any choice of real numbers \( c_1 \) and \( c_2 \), we get a solution \( \mathbf{a} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \) of the original boundary condition system, and furthermore, every solution of the system of boundary conditions is of this form for some choice of real numbers \( c_1 \) and \( c_2 \). \( \{\mathbf{x}_1, \mathbf{x}_2\} \) is called the **Canonical spanning set** and comprises of the two orthogonal starting vectors

\[
\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -\alpha \beta \\ \alpha \beta (\alpha + \beta) \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -(\alpha + \beta) \\ (\alpha^2 + \alpha \beta + \beta^2) \end{pmatrix}.
\]

The compound matrix method only requires one starting vector, formed from some complex constant multiplied by the wedge product of these two orthogonal vectors. Thus, let

\[ \mathbf{w} = \mathbf{x}_1 \wedge \mathbf{x}_2 \] (which has the property \( u \wedge v = -v \wedge u \)), then

\[
\mathbf{w} = \begin{align*}
(1 \times \mathbf{e}_1 + 0 \times \mathbf{e}_2 - \beta \mathbf{e}_3 + \alpha \beta (\alpha + \beta) \mathbf{e}_4) \wedge (0 \times \mathbf{e}_1 + 1 \times \mathbf{e}_2 - (\alpha + \beta) \mathbf{e}_3 + (\alpha^2 + \alpha \beta + \beta^2) \mathbf{e}_4) \\
= [1] \mathbf{e}_1 \wedge \mathbf{e}_2 + [- (\alpha + \beta)] \mathbf{e}_1 \wedge \mathbf{e}_3 + [(\alpha^2 + \alpha \beta + \beta^2)] \mathbf{e}_1 \wedge \mathbf{e}_4 + [- \alpha \beta] \mathbf{e}_3 \wedge \mathbf{e}_2 \\
+ [\alpha \beta (\alpha + \beta)] \mathbf{e}_3 \wedge \mathbf{e}_2 + [- \beta \alpha (\alpha^2 + \alpha \beta + \beta^2)] \mathbf{e}_3 \wedge \mathbf{e}_4 + [\alpha \beta (\alpha + \beta) \times -(\alpha + \beta)] \mathbf{e}_4 \wedge \mathbf{e}_3 \\
= [1] \mathbf{e}_1 \wedge \mathbf{e}_2 + [- (\alpha + \beta)] \mathbf{e}_1 \wedge \mathbf{e}_3 + [(\alpha^2 + \alpha \beta + \beta^2)] \mathbf{e}_1 \wedge \mathbf{e}_4 + [\alpha \beta] \mathbf{e}_3 \wedge \mathbf{e}_2 \\
- [\alpha \beta (\alpha + \beta)] \mathbf{e}_3 \wedge \mathbf{e}_2 + [- \beta \alpha (\alpha^2 + \alpha \beta + \beta^2)] \mathbf{e}_3 \wedge \mathbf{e}_4 + [(\alpha + \beta) \alpha \beta (\alpha + \beta)] \mathbf{e}_4 \wedge \mathbf{e}_3
\end{align*}
\]

\[ \Rightarrow \mathbf{w} = [1] \mathbf{e}_1 \wedge \mathbf{e}_2 + [- (\alpha + \beta)] \mathbf{e}_1 \wedge \mathbf{e}_3 + [(\alpha^2 + \alpha \beta + \beta^2)] \mathbf{e}_1 \wedge \mathbf{e}_4 + [\alpha \beta] \mathbf{e}_3 \wedge \mathbf{e}_2 + [- \beta \alpha (\alpha^2 + \alpha \beta + \beta^2)] \mathbf{e}_3 \wedge \mathbf{e}_4 + [(\alpha + \beta) \alpha \beta (\alpha + \beta)] \mathbf{e}_4 \wedge \mathbf{e}_3 + \mathbf{e}_4.
\]

Therefore, an appropriate starting vector in \( \bigwedge^2 (\mathbb{C}^4) \) is

\[
\mathbf{w}(0) = \begin{pmatrix} 1 \\ -(\alpha + \beta) \\ \alpha^2 + \alpha \beta + \beta^2 \\ \alpha \beta \\ -\alpha \beta (\alpha + \beta) \\ \alpha^2 \beta^2 \end{pmatrix}.
\]
This agrees with the starting vector used by Davey [34], derived using an alternative method. There, the constant tail initial conditions for $\phi_1$ and $\phi_2$ at $Z = Z_\infty$, namely

$$\phi_1 = [1, -\alpha, \alpha^2, -\alpha^3]^T \quad \text{and} \quad \phi_2 = [1, -\beta, \beta^2, -\beta^3]^T,$$

are substituted into the $2 \times 2$ minors of the solution matrix, $\phi = [\phi_1, \phi_2]$ (equations (9.4) to (9.9))

\[
\begin{align*}
y_1 &= \phi_1\phi'_2 - \phi'_1\phi_2 \\
y_2 &= \phi_1\phi''_2 - \phi''_1\phi_2 \\
y_3 &= \phi_1\phi'''_2 - \phi'''_1\phi_2 \\
y_4 &= \phi'_1\phi''_2 - \phi'''_1\phi_2 \\
y_5 &= \phi'_1\phi'''_2 - \phi'''_1\phi'_2 \\
y_6 &= \phi''_1\phi''_2 - \phi'''_1\phi_2.
\end{align*}
\]

Since Newton’s method shall be used to converge to roots of the system, the derivative of the initial vector must also be calculated. Thus the appropriate starting vector becomes

\[
w(0) = \left(1, -(\alpha + \beta), \alpha^2 + \alpha\beta + \beta^2, \alpha\beta, -\alpha\beta(\alpha - \beta), \alpha^2\beta^2, 0, \right.
\]

\[
\left.-\frac{R_c}{\alpha}, \frac{R_c}{\beta}, R_c, \frac{R_c}{\alpha}, \frac{R_c}{\beta}(-\alpha^2 - 2\alpha\beta), \alpha^2R_c \right)^T.
\]

### 9.3 Neutral curve for the Blasius boundary layer

In order to check our numerical methods were accurate, an investigation was made of the stability of the Blasius flow over a rigid flat plate and the results obtained using the new methods were compared with theoretical data from the literature.

From the paper of Ng & Reid [93], the nose of the neutral curve occurred at a Reynolds number of approximately 302.19 and wavenumber, $\alpha = 0.1728452040$, producing a value for the wave-speed, $c$ with the imaginary part, $1.3 \times 10^{-9}$. In comparison, using the same Reynolds number and wavenumber, $\alpha$, we obtained a value for $c$ with imaginary part $2.5 \times 10^{-7}$ whereas much larger value of $c_i$ were obtained from calculations using values around this point. Taking into consideration the use of an approximate value for the Reynolds number, the close correspondence of my results to those in the literature lead me to conclude that my theory and coding produces accurate results.
The critical point, *nose* of the neutral curve occurs at the approximate wavenumber, \( \alpha \approx 0.179 \) and Reynolds number, \( R_e \approx 302.0 \). This agrees with results in the literature (Jordinson [66], Ng & Reid [93]) when the scaling, \( \gamma = 1.72078766 \), is used to account for the different vertical length scales used (see chapter 13 for details). As with the Poiseuille flow field problem, the flow perturbations decay outside the curve and as they pass across the loci of the curve \( c_i \approx 0 \) the flow becomes unstable. The stability of the Blasius flow over a flat rigid plate can now be compared to the stability of the Blasius flow over a compliant surface. Part III will lead the discussion for the model of the compliant surface analysis.
PART III

Modelling compliant surfaces
Overview

The study of the boundary-layer flow past a flexible surface has two primary motivations. Firstly, it is a fundamental model for the fluid flow past a dolphin and other aquatic species (cf. Kramer [74]). Secondly, coated surfaces of man-made water-borne vehicles with a compliant surface has been proposed as a mechanism for delaying transition and drag reduction (cf. Carpenter [21]). The aim of this section of work is to introduce a model for a compliant surface designed by Carpenter & Garrad [23] to represent Kramer’s [70] best physical compliant coatings. The theoretical model designed by Carpenter & Garrad is two-dimensional allowing for a two-dimensional analysis of the interaction with a two-dimensional flow field. Thus, the model can be used directly for the Blasius flow problem. For study of the more realistic three-dimensional problems, the model is extended to allow for the three-dimensionality, firstly by the inclusion of wall displacement derivatives with respect to both $x$ and $y$ and secondly by introducing the possibility of an anisotropic surface, thus generalizing the type of surface permitting greater flexibility in the possible analyses pursued. The first section will give the historical background for the introduction of the type of compliant surface model studied in this report. Following this, the boundary conditions at the surface for the two and three-dimensional flow problems will be derived and parameter values for the mechanics of the wall model will be discussed.

10.1 Historic background for the plate-spring model

Since Kramer [70, 71] first described his pioneering experiments on compliant coatings, both theoretical and experimental investigations yielded no evidence for the drag-reducing capabilities of Kramer’s coatings, thus the credibility of his coatings was rather low. However, Carpenter & Garrad [23] supposed that the case against Kramer’s coatings was not
as strong as popularly thought. Kramer believed that the transition delaying properties of his coatings were responsible for the drag reductions achieved in his tests. Carpenter & Garrad showed that these subsequent independent tests carried out on the Kramer coatings were in fact completely unsuitable for investigating transition (a brief account of which is given below) and go on to describe a complete theoretical study of the hydrodynamic stability of flows over Kramer-type surfaces using linear hydrodynamic stability theory in order to see whether there is any theoretical basis for Kramer's view.

10.1.1 Kramer's experimental investigation

Figure 10.1: Cross section of Kramer's coating. All dimensions in mm. (Drawings are based on those given by Kramer [71].)

Figure 10.1 shows the first type of compliant surface designed by Kramer [70, 71]. An inner rigid wall was covered by a flexible inner skin connected to a 2mm outer flexible diaphragm by a closely spaced array of stubs. All these components were made of the same soft natural rubber. The cavity between the outer diaphragm and inner skin was filled with a highly viscous damping fluid, supposed by Kramer to damp out boundary layer waves - presumably he was referring to TS instabilities. Kramer also considered that if the stiffness and inherent damping of the heavy outer diaphragm were properly chosen, then it could act as distributed damping and thereby reduce the damaging effects of the local turbulent disturbances which have a much higher frequency than TS instabilities. Thus, his coatings were regarded as consisting of two wide-frequency band dampers [23].

According to Babenko, Gintetski & Kozlov [3] and Babenko, Kozlov & Pershin [4], Kramer's coatings would not function in the same way as dolphin's skin. They stated that
the papillae in dolphin’s skin are more readily deformable than the representative stubs in Kramers coating. Babenko et al [3] also claimed that dolphin skin is subject to a certain amount of active control, whereas Kramer’s coatings are purely passive. Nevertheless, Kramer’s results showed considerable drag reductions with his coatings.

A sketch of the coated model used for Kramer’s tests is shown in figure 10.2. This model had a streamlined body of revolution consisting of a 470 mm long tip, a tabulated contour followed by a cylindrical aft section 673 mm in length and 63.5 mm in diameter. The first 152 mm of the tip, a 13 mm transition section between tip and cylinder and the last 38 mm of the cylinder were not covered with the compliant coating. The model was sling-mounted to an after-body and towed in the sea at speeds up to 18 m/s [73, 74].

![Figure 10.2: Kramer's model. All dimensions in mm. Shaded regions were coated.](image)

At least four attempts have been made to carry out tests in order to provide independent verification of Kramer’s results. These were reported by Puryear [105], Nisewanger [95], Ritter & Messum [112] and Ritter & Porteous [113]. No significant drag reductions were observed in any of the four sets of tests. Carpenter & Garrad [23] explain this, stating that the above tests were not without fault. Below is a brief account of the explanations given by Carpenter & Garrad.

Puryear [105] obtained a two to six per cent drag increase with his compliant coatings and attributed these disappointing results to problems encountered in making a smooth join between the rigid nose and coating.

The models tested by Nisewanger [95] had a blunt nose made of rubber, giving an unfavourable pressure gradient over much of the compliant coating, thus, unsuccessful results are not surprising.

Ritter & Messum [112] carried out their tests on a small flat plate model covered with compliant coatings. This is not really comparable to Kramer’s original model so it was, perhaps, not altogether surprising that no significant drag reduction was observed.

The previous work was continued by Ritter & Porteous [113] using a cylindrical model
with an elliptical nose. The nose was fitted to the leading edge of the coated cylinder in such a way that the boundary layer could be removed by suction through a slot between the nose and cylinder, thus ensuring laminar flow at the start of the coated cylinder. We assume their coating was very soft explaining the failure to yield any significant drag reduction since Kramer’s softest coating appeared to be much more unstable [23]. Ritter & Porteous also experienced trouble with the leading edge of the coating on their models.

Thus, from this extremely brief review of the early attempts to confirm Kramer’s original results it can be concluded that the Kramer coatings have not been subjected to a satisfactory independent test. Therefore, the tests described above should not be taken as conclusive evidence that Kramer’s coatings are not capable of delaying transition under favourable conditions. There is, of course, still the possibility that Kramer’s observed drag reductions result from a favourable interaction between the compliant surface and a fully turbulent boundary layer. An additional explanation made by Carpenter & Garrad [23] is that the drag reductions could have come about owing to favourable changes to pressure drag occurring due to surface discontinuities at the coated-uncoated interfaces and/or to a favourable modification of the flow near the rear of the body, thus leading to lower base drag.

10.1.2 Review of stability analyses for boundary layers over compliant surfaces

The effects of a flexible boundary on hydrodynamic stability were first studied by Benjamin [9]. He remarked that the motion of the flexible surface could greatly affect the thin friction layer at the wall. Now since viscosity in the friction layer destabilizes the TS instabilities, then surface flexibility could possibly have a favourable effect on boundary-layer stability. He showed how the boundary conditions at the wall could be formulated for a flexible surface with the main motion normal to the wall. The response of the flexible surface to the pressure generated by the fluid motion was characterised by introducing a response coefficient, namely, a complex compliance, $Z$, which corresponds to the deflection of the surface. By extending the linear stability theory of Tollmien [131], Schlichting [115] and Lin [84], Benjamin showed how a flexible non-dissipative wall tends to stabilize TS instabilities. When $Z_r > 0$ (note that $Z_i = 0$ for non-dissipative walls), the neutral curves are shifted to a lower wavenumber and higher Reynolds number as shown in figure
10.3. However, in the case of dissipative flexible walls, he showed that TS instabilities could be destabilized by internal damping. A similar conclusion was also reached by Betchov [9].

Figure 10.3: The effect of surface compliance on the neutral curve for a non-dissipative flexible wall according to Benjamin's theory. $\alpha$ is wavenumber, $\delta^*$ is boundary layer displacement thickness and $Re$ is the Reynolds number.

The response of the flexible surface was characterised by Landahl [78] as an admittance defined as

\[ Y = \frac{\text{normal wall velocity}}{\text{wall pressure}} \]

Note that this formulation does not have any theoretical advantages over that of Benjamin's, however, it is convenient for numerical work.

Landahl approximately simulated the principal properties of a Kramer-type coating obtaining curves of neutral stability numerically. He confirmed Benjamin's conclusion regarding the destabilizing effect of internal damping on TS instabilities and clarified the effects of damping on the various types of instability. For Kramer coatings, Landahl concluded that, since the theoretically predicted critical Reynolds numbers were at best only modestly improved by wall compliance, it was unlikely that the drag reductions observed by Kramer were a result of delaying transition.

In 1967, Gyorgyfalvy [58] used Landahl's theory to extensively study, parametrically, the stability and transition of boundary layers over spring-backed membranes with internal damping. He used the $\phi^0$ method of Smith & Gamberoni [121] to calculate the transitional Reynolds number. Gyorgyfalvy found that the favourable effect on transition with a
flexible surface was due to a reduction in amplification rates rather than an increase in the critical Reynolds number, and large reductions were only available for a comparatively small range of Reynolds number. His results appeared to confirm Landahl's views on the theoretical unsuitability of Kramer's coatings for transition delay.

Landahl & Kaplan [79] extended the formulation of the wall boundary conditions to cases where the surface velocity could have a significant stream-wise motion by introducing an additional surface admittance for the stream-wise motion. The OS equation was then integrated numerically obtaining accurate solutions for various problems. In addition to spring-backed membranes, compliant surfaces formed by a non-dissipative elastic medium and by a viscoelastic medium were studied. The effects of the pressure gradient on boundary layer stability over flexible surfaces were investigated and the effects of a flexible surface on a secondary instability was briefly studied. Results indicated that a flexible surface would only slightly reduce the growth rate of the secondary instability, confirming a similar conclusion reached by Benjamin [9] using a simpler flow model.

In order to achieve a significant favourable effect on transition, the use of a light, highly flexible wall was considered necessary. However, calculations by Carpenter and Garrad [23] place some doubt on Landahl and Kaplan's results for spring-backed membrane surfaces although their overall conclusions for these surfaces appear to be sound.

Conventional linear-stability theory has been used for the majority of the work reviewed above. Bushnell & Hefner [19] questioned the validity of this approach for flows over Kramer-type flexible surfaces. They argued that the modulation produced by such a wall can be sufficient to alter the effective-mean-velocity profile, and recommended the use of stability analysis for periodically time-varying mean flows. However, Carpenter & Garrad [23] disagreed with this, stating that the argument would only be valid if the wall motion were independent of the instability under consideration. This would clearly be so in the case of active walls. However, for passive walls, this situation would only arise if the effects of a flow induced surface instability (FISI) on a TS instability were considered. Two main types of behaviour would seem to be possible: either the two modes of instability would be well separated and distinct or some sort of first order modal interaction would occur. They go on to suggest that in the first case, provided the amplitude of the separate instabilities are small enough for linearization to be valid, the instabilities could be treated independently and then brought together to give their combined effect. Any
interaction between the two modes of instability could certainly be greatly stabilizing or destabilizing. However, such an interaction would be a non-linear effect. Nevertheless, provided linearization remains valid, there would appear to be no obvious reason why conventional linear theory cannot be used to investigate such interactions.

In summary, the theoretical evidence seems to indicate that it is indeed possible to postpone transition using a compliant surface, but Kramer coatings are not suitable for this purpose. It is firmly established that internal damping destabilizes TS instabilities, thus Kramer's explanation for the action of the damping fluid would appear to be incorrect.

In the past, Kramer coatings were regarded as spring-backed tensioned membranes, until Carpenter and Garrad [23] thought to model them better as spring-backed plates with finite bending stiffness and, with this model, obtained a substantial delay in transition in certain circumstances.

It must finally be noted that historically, TS instabilities have been the focus of research, even though in many cases there are two or more modes of instability present.
For our investigations, the theory introduced in the preceding chapters will be applied to the surface-based model proposed by Carpenter & Garrard [23] for the stability of two-dimensional boundary-layer flow past a Kramer-type compliant surface. We use a surface-based model since these are simpler to implement, with the equations for the wall motion replacing the rigid-wall boundary conditions. This ensures that the continuity of the velocity across the interface between the wall and fluid is automatically satisfied.

The Kramer-type compliant surface is a simple plate-spring surface-based model. It is assumed to be constructed of an elastic plate (or tensioned membrane) supported above a rigid surface by a vertically aligned array of springs. The plate can also be backed by a viscous fluid substrate having, in general, a density and viscosity different from the mainstream fluid. The motion of the substrate fluid is assumed to be unaffected by the presence of the springs and is determined by solving the linearized Navier-Stokes equations. The visco-elastic properties of the plate and springs are taken into account approximately by using a complex elastic modulus which leads to complex flexural rigidity and spring stiffness. This should be a reasonable approximation provided that the wavelength of the surface instabilities considerably exceeds the distance between neighbouring springs. Various parameters characterising the surface properties are estimated for the actual Kramer coatings (cf. [23]). This type of model is an idealistic representation for the wall and is not an accurate representation for the types of wall used in experimental studies. Nevertheless, it is a start enabling us to estimate what effect wall compliance may have on TS instabilities. It has the advantage of presenting a problem which is computationally efficient to solve.
11.1 Boundary Conditions for the two-dimensional Kramer-type compliant Surface

We assume the wall is isotropic and is free to move under the influence of the fluid flow and becomes displaced by the fluid motion. For an isotropic wall of this type, the displacement is restricted to motion in the normal direction only such that for travelling wave disturbances, the wall displacement can take the form

\[ W(x,t) = \delta W_0 e^{i \alpha x + \lambda t}, \]

where \( \alpha \) is the stream-wise wavenumber, \( \delta \) is the boundary-layer thickness, \( \lambda = -i \omega c \) is the stability exponent with \( c \) the complex wave speed, and \( W_0 \) is the dimensionless plate displacement amplitude.

If such a surface undergoes two-dimensional disturbances, then by neglecting the horizontal displacement of the surface, the motion of the compliant wall given by the vertical position of the surface denoted by \( W(x,t) \) is governed by the following equation [23]:

\[ \rho_m \frac{\partial^2 W}{\partial t^2} + d \frac{\partial W}{\partial t} + B \frac{\partial^4 W}{\partial x^4} - T \frac{\partial^2 W}{\partial x^2} + k_E W = \delta p_s - \delta p_e. \]  

(11.1)

Here, \( k_E = k - g(\rho_e - \rho_s) \) is an equivalent spring stiffness, \( k \) is the spring stiffness, \( \rho_m \) is the density of the plate material, \( \rho_e \) is the fluid density, \( \rho_s \) is the substrate density, \( b \) is the plate thickness, \( d \) is the damping coefficient, \( B \) is the flexural rigidity of the plate, \( T \) is the longitudinal tension, \( \delta p_s \) is the perturbation in pressure acting on the plate from the fluid above, and \( \delta p_e \) is the perturbation in the pressure acting on the plate from the substrate below.
So, \( \frac{\partial \delta p_t}{\partial t} \) is a damping term, \( \delta p_s - \delta p_e \) is a forcing term, and the term \( (k - g(\rho_e - \rho_s))W \) is due to the difference in densities of water and the surface, with \( g = 9.8 \text{ m/s} \).

Note that equation (11.1) involves both beam and wave equation terms. \( W_{tt} - W_{xxt} = 0 \) is the wave equation, \( W_{tt} + W_{xxxx} = 0 \) is the beam equation and \( W_{tt} + dW_t - W_{xx} = 0 \) is the damped wave equation.) The equation was derived by Carpenter & Garrad [23] using classical thin plate theory by considering the different forces per unit plate area acting on the wall. The first term in the equation represents the plate mass per unit area multiplied by acceleration and so represents the driving forces due to the boundary layer disturbances. The remaining terms represent the restorative structural forces due to the compliant wall.

By substituting the form for the surface displacement into (11.1) and noting that
\[
\frac{\partial W}{\partial t} = -i\alpha c W
\]
and
\[
\frac{\partial W}{\partial x} = i\alpha W,
\]
we obtain
\[
[-\rho_m b\alpha^2 c^2 W_o - i\alpha c d W_o + B\alpha^4 W_o + T\alpha^2 W_o + k_E W_e] e^{i\alpha(x-ct)} = \delta p_e - \delta p_s.
\]
Now, we substitute the dimensionless expressions for the dynamic pressure perturbations given by Carpenter & Garrad [23], namely
\[
\delta p_e = \rho_e U_o^2 \hat{p}_e \tilde{W} \quad \text{and} \quad \delta p_s = \rho_s U_o^2 \hat{p}_s \tilde{W},
\]
where \( U_o \) is the free-stream velocity, and
\[
\tilde{W} = \frac{W}{\delta} \quad \text{with} \quad W = W_o e^{i\alpha(x-ct)}.
\]
Let us also non-dimensionalize the wavenumber and complex wave-speed such that
\[
\alpha = \frac{\alpha}{\delta} \quad \text{and} \quad c = \tilde{c} U_o,
\]
where the \( \tilde{\text{bar}} \) notation represents a non-dimensional parameter, then we obtain
\[
\left[ -\rho_m b\alpha^2 c^2 U_o^2 \frac{\tilde{W}_o}{\rho_o e^{i\alpha(x-ct)}} - i\frac{\alpha}{\delta} \tilde{c} U_o^2 \frac{\tilde{W}_o}{e^{i\alpha(x-ct)}} + B\alpha^4 \frac{\tilde{W}_o}{\rho_o e^{i\alpha(x-ct)}} + T\alpha^2 \frac{\tilde{W}_o}{\rho_o e^{i\alpha(x-ct)}} + k_E \frac{\tilde{W}_o}{e^{i\alpha(x-ct)}} \right] e^{i\alpha(x-ct)} = \rho_o U_o^2 \hat{p}_e \tilde{W} - \rho_s U_o^2 \hat{p}_s \tilde{W}.
\]
Now, dividing through by $\rho_eU_\infty^2\bar{W}$, assuming $|\bar{W}| \neq 0$ and simplifying, gives
\[
-\rho_m b \bar{a}^2 \bar{c}^2 \frac{\rho_e}{\rho_eU_\infty^2} \frac{i\bar{a}\bar{c}d}{\rho_eU_\infty} + \frac{B\bar{a}^4}{\delta^3 U_\infty^2 \rho_e} + \frac{T\bar{a}^2}{\rho_eU_\infty^2 \rho_e} + \frac{k_B \delta}{\rho_eU_\infty^2} = \bar{p}_s - \bar{p}_e. \tag{11.4}
\]
Equation (11.4) can be written in a totally non-dimensional form by defining and substituting in the non-dimensional coefficients describing the mechanical properties of the coating:
\[
C_m = \frac{b \rho_m}{\rho_e \delta}, \quad C_D = \frac{d}{\rho_e U_\infty}, \quad C_B = \frac{B}{\delta^3 U_\infty^2 \rho_e}, \quad C_T = \frac{T}{\delta^2 U_\infty^2 \rho_e}, \quad C_{KE} = \frac{k_B \delta}{\rho_e U_\infty^2}. \tag{11.5}
\]
Thus, the governing equation for the compliant wall motion can be written as
\[
-\bar{a}^2 \bar{c}^2 C_m - i\bar{a}\bar{c}C_D + \bar{a}^4 C_B + \bar{a}^2 C_T + C_{KE} = \bar{p}_s - \bar{p}_e. \tag{11.6}
\]
Following Carpenter and Garrad [23], the two boundary conditions at the compliant surface are taken to be of the following form.

1. The kinematic boundary condition is
   \[U'(0)\phi(0) + c\phi'(0) = 0, \tag{11.7}\]
   where $U'(0)$ is the derivative of the Blasius velocity at the wall (in our calculations we used the value 0.3320573371, calculated using the Chebyshev polynomial method discussed in chapter 9).

2. The dynamic condition at the wall is
   \[i \bar{a}(\phi'''(0) - \alpha^2 \phi'(0)) + \alpha R(\alpha^2 \bar{a}^2 C_m + i\alpha \bar{a} C_D - \alpha^4 C_B - \alpha^2 C_T - C_{KE})\phi(0) = 0. \tag{11.8}\]

These complicated boundary conditions show that the problem is coupled, e.g. the fluid flow induces the compliant wall to "buckle" and then the effect of the deformed wall changes the way the fluid flows over it.

Note that the reference length scale is taken to be $\delta$, the boundary layer thickness, and the reference velocity scale is the constant free stream velocity, $U_\infty$. The Reynolds number is then defined to be
\[R_e = \frac{U_\infty \delta}{\nu}.\]
11.1.1 Derivation of Boundary Condition 11.7

Figure 11.2 shows a schematic diagram of a compliant surface. As fluid moves over it, the surface may deform. \( \xi \) and \( \eta \) are the new deformed coordinates

\[
\begin{align*}
\xi(t) &= u(x_0 + \xi, y_0 + \eta, t), \\
\eta(t) &= v(x_0 + \xi, y_0 + \eta, t).
\end{align*}
\]

These are the velocities of the moving wall in the \( x \) and \( y \) direction, respectively.

Now, \( y_0 \) is at the wall, therefore, \( y_0 = 0 \). Suppose \( \xi \) and \( \eta \) are small, we expand \( u \) and \( v \) in Taylor series

\[
\begin{align*}
\xi(t) &= u(x_0 + \xi, 0, t) + \frac{\partial u}{\partial x}(x_0, 0, t)\xi + \frac{\partial u}{\partial y}(x_0, 0, t)\eta + \cdots \\
\eta(t) &= v(x_0 + \xi, 0, t) + \frac{\partial v}{\partial x}(x_0, 0, t)\xi + \frac{\partial v}{\partial y}(x_0, 0, t)\eta + \cdots
\end{align*}
\]

We assume \( \xi \approx 0 \) and \( \xi_0 \approx 0 \) and \( |\eta| << 1 \). That is, no elasticity in the \( x \)-direction, so that it doesn’t deform in the \( x \)-direction, and the deformation in the \( y \)-direction is very small. Then from (11.9) and (11.10), we get

\[
\begin{align*}
0 &= u(x, 0, t) + \frac{\partial u}{\partial y}(x, 0, t)\eta + \cdots \\
\eta(t) &= v(x, 0, t) + \frac{\partial v}{\partial y}(x, 0, t)\eta + \cdots
\end{align*}
\]

We now neglect the higher order terms and linearize about the Blasius basic state \((U(y), 0, 0)\).

\[
\begin{align*}
\dot{u} &= U + \dot{u} \\
\dot{v} &= \dot{v} \\
\dot{p} &= P + \dot{p}.
\end{align*}
\]
Part III: Two-dimensional model for the compliant wall

Note that \( \hat{u} \), \( \hat{v} \) and \( \eta \) are unknown. Therefore, \( \frac{\partial \hat{u}}{\partial y} \eta \) and \( \frac{\partial \hat{v}}{\partial y} \eta \) are both higher order terms, so we neglect these terms. However, \( U \) is known, and so \( U\eta \) isn’t a higher order term. Thus, we obtain

\[
0 = \hat{u}(x,0,t) + U_y(0)\eta \quad (11.11)
\]

\[
\eta_t = \hat{v}(x,0,t). \quad (11.12)
\]

Now, introducing a Fourier transformation to the velocities gives

\[
\hat{u} = \bar{u}e^{ia(x-ct)}, \quad (11.13)
\]

\[
\hat{v} = \bar{v}e^{ia(x-ct)}, \quad (11.14)
\]

\[
\eta = \bar{\eta}e^{ia(x-ct)}. \quad (11.15)
\]

The continuity equation thus reduces to

\[
 i\alpha \bar{u} + \bar{v}_y = 0. \quad (11.16)
\]

Now, substituting, (11.13), (11.14) and (11.15) into equation (11.11) we get

\[
0 = \bar{u}e^{ia(x-ct)} + U_y(0)\bar{\eta}e^{ia(x-ct)}
\]

\[
= \bar{u} + U_y(0)\bar{\eta}.
\]

Using equation (11.16) to give \( \bar{u} \) and substituting this in, we obtain

\[
\frac{i}{\alpha} \bar{v}_y = -U_y(0)\bar{\eta}. \quad (11.17)
\]

Substituting (11.14) and (11.15) into (11.12) gives

\[
-i\alpha c\bar{\eta}e^{ia(x-ct)} = \bar{\eta}e^{ia(x-ct)} \quad \Rightarrow \quad -i\alpha c\bar{\eta} = \bar{\eta}, \quad (11.18)
\]

so, from (11.17) and (11.18) we obtain

\[
\frac{i}{\alpha} \bar{v}_y = -U_y(0)\frac{\bar{v}}{-i\alpha c} \quad \Rightarrow \quad \frac{i}{\alpha} \bar{v}_y = -U_y(0)\frac{\bar{v}}{i\alpha c}.
\]

Thus, if we let the Fourier amplitude of the vertical perturbation velocity \( \bar{v} \) be denoted by \( \phi \) and, rearranging, we obtain the boundary condition (cf. equation (20.4))

\[
c\phi_y + U_y(0)\phi = 0. \quad (11.19)
\]

Note that (11.19) is a linearized boundary condition, since we have expanded by Taylor Series and neglected higher order terms. This was done since we are assuming very small displacements.
11.1.2 Derivation of boundary condition 11.8

From equation (3.28), replacing \( \bar{v} \) with \( \phi \), we have

\[
\frac{i\alpha \bar{p}}{\alpha} = \frac{1}{\alpha} (U\alpha - \omega)\phi' - U_y \phi + \frac{i}{Re\alpha} (\phi''' - \alpha^2 \phi').
\]

Now, at the wall, the boundary conditions state that \( U(0) = 0 \) and \( -U_y(0)\phi(0) = c\phi'(0) \), using equation (11.19), thus, noting that \( \frac{\omega}{\alpha} = c \) at \( y = 0 \), we can show that \( p_e \) is related to \( \phi(0) \) by

\[
\frac{i\alpha p_e}{R_e\alpha} = \frac{i}{\alpha^2 R_e} (\phi'''(0) - \alpha^2 \phi'(0)),
\]

where \( p_e = \bar{p}|_{y=0} \).

Following Carpenter & Garrad [23], the dynamic boundary condition at the surface is expressed in terms of an equality between the surface and boundary admittances, such as to allow the determination of stabilizing or destabilizing effects on the TS instability when the mechanical properties of the surface are changed. Thus

\[
Y_0 = Y_1,
\]

where

\[
Y_0 = - \frac{\bar{a}(0)}{p_c(0)} = - \frac{\phi(0)}{\frac{1}{\alpha^2 R_e} (\phi'''(0) - \alpha^2 \phi'(0))} = - \frac{\phi(0)}{\frac{\alpha^2 R_e \phi(0)}{\phi'''(0) - \alpha^2 \phi'(0)}},
\]

and

\[
Y_1 = - \frac{\partial \bar{W}}{\partial \bar{p}} \bigg|_{(\rho U_{\infty}^2)}.
\]

\( Y_0 \) and \( Y_1 \) are respectively the admittances of the boundary layer fluid and flexible surface. When suitable expressions are derived for \( Y_0 \) and \( Y_1 \) then, \( Y_0 = Y_1 \) acts as the boundary condition at the flexible wall replacing \( \phi(0) = 0 \), used for the rigid wall.

From equation (11.2) and noting that \( \bar{W} = \frac{W}{\bar{f}} \), we obtain

\[
Y_1 = \frac{i\alpha \bar{c} \bar{W}}{\rho U_{\infty}^2}.
\]

If we let

\[
\bar{W}(\bar{x}, \bar{t}) = \bar{W}_0 e^{i(\bar{x} - ct)}
\]

\[
\delta p_e = \bar{\delta p}_e e^{i(\bar{x} - ct)}
\]
and assuming no fluid substrate, \( \delta p_e = 0 \), then equation (11.1) becomes

\[
\delta(-\rho_m b\alpha^2 c^2 W_0 - i\alpha c d W_0 + B\alpha^4 W_0 + T\alpha^2 W_0 + k_E W_0)e^{i\alpha(x-\delta)} = -\delta p_e e^{i\alpha(x-\delta)}.\]  

(11.20)

So,

\[
Y_1 = \frac{i\alpha c W_0 e^{i\alpha(x-\delta)}}{\delta p_e e^{i\alpha(x-\delta)}} = \frac{i\alpha c \rho_e U^2_\infty W_0}{\delta p_e},
\]

substituting in for \( \frac{W_0}{\delta p_e} \) from equation (11.20) we obtain

\[
Y_1 = \frac{i\alpha c \rho_e U^2_\infty}{\delta(\rho_m b\alpha^2 c^2 + i\alpha c d - B\alpha^4 - T\alpha^2 - k_E)}.
\]

The boundary condition is thus,

\[
Y_1 = Y_0
\]

\[
\Rightarrow \frac{i\alpha c \rho_e U^2_\infty}{\delta(\rho_m b\alpha^2 c^2 + i\alpha c d - B\alpha^4 - T\alpha^2 - k_E)} = \frac{c^2 R_e \phi(0)}{\phi''(0) - \alpha^2 \phi'(0)}.
\]

Non-dimensionalizing the rest of the dimensional parameters using \( \alpha = \frac{\xi}{\delta} \) and \( c = U_\infty \) yields

\[
\frac{i\alpha c \rho_e U^2_\infty}{\delta(\rho_m b\alpha^2 c^2 + i\alpha c d - B\alpha^4 - T\alpha^2 - k_E)} = \frac{c^2 R_e \phi(0)}{\phi''(0) - \alpha^2 \phi'(0)}
\]

\[
\Rightarrow \frac{\delta p_m b\alpha^2 c^2 U^2_\infty}{i\alpha c \rho_e U^2_\infty} + \frac{i\alpha c U_\infty d}{i\alpha c \rho_e U^2_\infty} - \frac{B\alpha^4}{\delta^2} - \frac{T\alpha^2}{\delta^2} - \frac{k_E \delta}{\delta^2} = \frac{\phi''(0) - \alpha^2 \phi'(0)}{c^2 R_e \phi(0)}.
\]

Using equations (11.5) this can be reduced to

\[
\frac{C_m \alpha \xi}{i} + C_D - \frac{C_B \alpha^3}{i \xi} = \frac{C_T \alpha}{i \xi} = -\frac{\phi''(0) - \alpha^2 \phi'(0)}{c^2 R_e \phi(0)}.
\]

Rearranging this yields the required boundary condition

\[
\alpha R_e \phi(0)(C_m \alpha^2 c^2 + iC_D \alpha c - C_B \alpha^4 - C_T \alpha^2 - C_K E) = -i\alpha(\phi''(0) - \alpha^2 \phi'(0)),
\]

where the bar notation has been dropped from the dimensionless variables to simplify notation.
Three-dimensional model for the compliant surface

The surface-based plate-spring model of Carpenter & Garrad [23] described in the preceding chapter will be modified for use in our three-dimensional problem. This model shall then be used since the equations of motion of the wall can be used as a direct replacement of the rigid wall. Thus, only the boundary conditions of the system are required to be adjusted. The difference in the modified model occurs in the terms where derivatives with respect to \( x \) only, in the two-dimensional case, will now be taken both with respect to \( x \) and \( y \).

12.1 Boundary conditions for the three-dimensional Kramer-type compliant surface - Ekman layer problem

As with the two-dimensional flexible surface, we assume that the plate is isotropic and free to move under the influence of the fluid flow becoming displaced by the fluid motion. The boundary conditions at the surface, \( z = 0 \), are derived in the following sections.

12.1.1 Kinematic boundary conditions

These arise from equating the velocity at the wall with the velocity of the fluid. Now as fluid moves over a compliant surface, the surface may deform. Let \( \xi \), \( \eta \) and \( \zeta \) be the new deformed coordinates in the \( x \), \( y \) and \( z \) directions respectively, then the velocities of the moving wall in the \( x \), \( y \) and \( z \) directions are given by

\[
\begin{align*}
\xi^* &= u^*(x_0 + \xi, y_0 + \eta, z_0 + \zeta, t), \\
\eta^* &= v^*(x_0 + \xi, y_0 + \eta, z_0 + \zeta, t), \\
\zeta^* &= w^*(x_0 + \xi, y_0 + \eta, z_0 + \zeta, t).
\end{align*}
\]
Now, $z_0$ is at the wall and therefore, $z_0 = 0$. Assuming $\xi^*$, $\eta^*$ and $\zeta^*$ are small, we expand $u^*$, $v^*$ and $w^*$ in a Taylor series to give

\begin{align}
\xi^* &= u^*(x^*, y^*, 0, t^*) + \frac{\partial u^*}{\partial x^*}(x^*, y^*, 0, t^*)\xi + \frac{\partial u^*}{\partial y^*}(x^*, y^*, 0, t^*)\eta + \frac{\partial u^*}{\partial z^*}(x^*, y^*, 0, t^*)\zeta + \ldots \quad (12.1) \\
\eta^* &= v^*(x^*, y^*, 0, t^*) + \frac{\partial v^*}{\partial x^*}(x^*, y^*, 0, t^*)\xi + \frac{\partial v^*}{\partial y^*}(x^*, y^*, 0, t^*)\eta + \frac{\partial v^*}{\partial z^*}(x^*, y^*, 0, t^*)\zeta + \ldots \quad (12.2) \\
\zeta^* &= w^*(x^*, y^*, 0, t^*) + \frac{\partial w^*}{\partial x^*}(x^*, y^*, 0, t^*)\xi + \frac{\partial w^*}{\partial y^*}(x^*, y^*, 0, t^*)\eta + \frac{\partial w^*}{\partial z^*}(x^*, y^*, 0, t^*)\zeta + \ldots \quad (12.3)
\end{align}

It will be assumed that $|\xi| \approx 0$, $|\eta| \approx 0$, $|\zeta_\xi| \approx 0$ and $|\eta_\xi| \approx 0$, that is, no elasticity so that the wall doesn’t deform in the $x$ or $y$ direction. Let us also assume that the deformation in the $x-$direction is very small. Then from (12.1), (12.2) and (12.3) we obtain

\begin{align}
0 &= u^*(x^*, y^*, 0, t^*) + u^*_x(x^*, y^*, 0, t^*)\xi^* + \ldots \\
0 &= v^*(x^*, y^*, 0, t^*) + v^*_x(x^*, y^*, 0, t^*)\xi^* + \ldots \\
\zeta^* &= w^*(x^*, y^*, 0, t^*) + w^*_x(x^*, y^*, 0, t^*)\xi^* + \ldots
\end{align}

We can now neglect the higher order terms and linearize about the Ekman layer basic state, $(U(x), V(x), 0)$, to give

\begin{align}
u^* &= U^* + \bar{u}^* \\
v^* &= V^* + \bar{v}^* \\
w^* &= \bar{w}^* \\
p^* &= P^* + \bar{p}^*.
\end{align}

Note that $\bar{u}^*$, $\bar{v}^*$, $\bar{w}^*$ and $\zeta^*$ are unknown. Therefore, $\frac{\partial \bar{u}^*}{\partial x^*}\zeta^*$, $\frac{\partial \bar{v}^*}{\partial x^*}\zeta^*$ and $\frac{\partial \bar{w}^*}{\partial x^*}\zeta^*$ are all higher order terms and, since we are conducting a linear stability analysis, these terms can be neglected. However, $U^*$ and $V^*$ are both known, so $U^*\zeta^*$ and $V^*\zeta^*$ are not higher order terms and so cannot be neglected. Hence we obtain

\begin{align}
0 &= \bar{u}^*(x^*, y^*, 0, t^*) + U^*_x(0)\zeta^* \quad (12.4) \\
0 &= \bar{v}^*(x^*, y^*, 0, t^*) + V^*_x(0)\zeta^* \quad (12.5) \\
\zeta^* &= \bar{w}^*(x^*, y^*, 0, t^*) \quad (12.6)
\end{align}

Let us non-dimensionalize by taking the following dimensionless variables:

\begin{align}
u &= \frac{\bar{u}^*}{v_g}, \quad v = \frac{\bar{v}^*}{v_g}, \quad w = \frac{\bar{w}^*}{v_g}, \quad U = \frac{U^*}{v_g}, \quad V = \frac{V^*}{v_g}
\end{align}
Part III: Three-dimensional model for the compliant surface

\[ x = \frac{x^*}{L}, \quad y = \frac{y^*}{L}, \quad z = \frac{z^*}{L}, \quad t = \Omega t^*, \quad \zeta = \frac{\zeta^*}{L}, \]

(see chapter 17 for definitions of \( L, v_g \) and \( \Omega \)). This form is consistent with the non-dimensionalization for the perturbation equations for the Ekman boundary layer stability analysis in part V of this report. Substituting these into equations (12.4) to (12.6) we obtain

\[ \begin{align*}
0 &= u v_g + U_z(0) \zeta v_g \\
0 &= v v_g + V_z(0) \zeta v_g \\
\frac{1}{R_o} \zeta_t &= w(x, y, 0, t),
\end{align*} \]

where \( R_o = \frac{v_g}{\Omega L} \) is the Rossby number and is the ratio of convective to Coriolis accelerations.

We now assume \( |\zeta| \neq 0, U_z(0) \neq 0 \) and \( V_z(0) \neq 0 \), otherwise the equations would reduce to the two-dimensional case. By combining the equations (12.7) and (12.8), we obtain the first kinematic boundary condition, namely

\[ u V_z(0) = v U_z(0). \]  \hspace{1cm} (12.10)

This boundary condition states that the \((u, v)\) velocity vector has to be parallel to the basic velocity gradient.

To find the second kinematic boundary condition, let us introduce the following Fourier transforms.

\[ \begin{align*}
u &= \hat{u} e^{i(\alpha x + \beta y - \omega t)} \\
v &= \hat{v} e^{i(\alpha x + \beta y - \omega t)} \\
w &= \hat{w} e^{i(\alpha x + \beta y - \omega t)} \\
\zeta &= \hat{\zeta} e^{i(\alpha x + \beta y - \omega t)},
\end{align*} \]

where \( \alpha \) and \( \beta \) are the corresponding wave numbers in the \( x \) and \( y \) directions respectively, and \( \omega \) is the wave frequency supplying the stability characteristics. Substituting equations (12.11) to (12.14) into (12.7), (12.8) and (12.9) we obtain

\[ \begin{align*}
0 &= \hat{u} + U_z(0) \hat{\zeta} \\
0 &= \hat{v} + V_z(0) \hat{\zeta} \\
\hat{\zeta} &= \frac{\hat{w} i R_o}{\omega}.
\end{align*} \]

(12.15)  \hspace{1cm} (12.16)  \hspace{1cm} (12.17)
Substitution of equations (12.11) to (12.14) into the continuity equation, gives

$$i\alpha\dot{u} + i\beta\dot{v} + \dot{w}_z = 0,$$

(12.18)

and so, by substitution of $\dot{u}$ and $\dot{v}$ from equations (12.15) and (12.16) into equation (12.18) we obtain

$$i\alpha(-U_z(0)\dot{\zeta}) + i\beta(-V_z(0)\dot{\zeta}) + \dot{w}_z = 0.$$  

(12.19)

Then, by using equation (12.17) to eliminate $\dot{\zeta}$ from (12.19), we have

$$\frac{\alpha R_0}{\omega}U_z(0)\dot{w} + \frac{\beta R_0}{\omega}V_z(0)\dot{w} + \dot{w}_z = 0.$$  

Finally, by letting $c = \frac{w}{\gamma}$, where, $\gamma^2 = \alpha^2 + \beta^2$. Multiplying through by $\frac{w}{\gamma}$ we obtain our second kinematic boundary condition

$$cw_z + R_0\left(\frac{\alpha U_z(0)}{\gamma} + \frac{\beta V_z(0)}{\gamma}\right) = 0.$$  

(12.20)

Let us introduce new coordinates so that we are essentially rotating the velocity field in a horizontal plane.

$$\tilde{V} = \frac{\alpha}{\gamma}U + \frac{\beta}{\gamma}V, \quad \Rightarrow \quad \tilde{V}_z = \frac{\alpha}{\gamma}U_z + \frac{\beta}{\gamma}V_z,$$

(12.21)

$$\tilde{U} = \frac{\beta}{\gamma}U - \frac{\alpha}{\gamma}V, \quad \Rightarrow \quad \tilde{U}_z = \frac{\beta}{\gamma}U_z - \frac{\alpha}{\gamma}V_z,$$

(12.22)

$$\tilde{u} = \frac{\beta}{\gamma}u - \frac{\alpha}{\gamma}v,$$

(12.23)

$$\tilde{v} = \frac{\alpha}{\gamma}u + \frac{\beta}{\gamma}v.$$  

(12.24)

From rearranging the first of these equations, (12.21) we have

$$\frac{\gamma}{\beta}\tilde{V}_z - \frac{\alpha}{\beta}U_z = V_z,$$

(12.25)

substituting this into (12.22) gives

$$U_z = \frac{\beta}{\gamma}\tilde{U}_z + \frac{\alpha}{\gamma}\tilde{V}_z.$$  

(12.26)

Substituting (12.26) back into (12.21) we get

$$\frac{\beta}{\gamma}\tilde{V}_z - \frac{\alpha}{\gamma}\tilde{U}_z = V_z.$$  

(12.27)

Now, from rearranging equation (12.23) we have

$$\tilde{u} = \frac{\gamma}{\beta}\tilde{u} + \frac{\alpha}{\beta}\tilde{v},$$
substitution of this into (12.24) gives
\[ \theta = \frac{\beta}{\gamma} \bar{v} - \frac{\alpha}{\gamma} \bar{u}, \quad (12.28) \]
and substituting (12.28) back into (12.23) yields
\[ \frac{\beta}{\gamma} \bar{u} + \frac{\alpha}{\gamma} \bar{v} = \bar{u}. \quad (12.29) \]
Now, by substituting (12.26), (12.27), (12.28) and (12.29) back into the expression for the first kinematic boundary condition, (12.10), gives
\[ \bar{V}_z \bar{u} = \bar{U}_z \bar{v}. \quad (12.30) \]
Introducing a new variable \( \psi = i \gamma \bar{u} \), the transformed continuity equation can be expressed as
\[ i \gamma \bar{v} + \bar{w}_x = 0. \]
Finally, using this and letting \( \bar{w} = \phi \), (12.30) becomes our first kinematic boundary condition:
\[ \bar{V}_z(0)\psi + \bar{U}_z(0)\phi_x = 0, \quad \text{at} \quad z = 0. \quad (12.31) \]
Now from (12.20) and using (12.21), we find that our second kinematic boundary condition can be expressed in the following form
\[ c \phi_x + R_x \bar{V}_z(0)\phi = 0. \quad (12.32) \]

12.1.2 Dynamic boundary condition

We shall start with the wall equation in three-dimensions, such that we obtain derivatives with respect to both \( x \) and \( y \) in the restorative structural forces of the wall motion equation. Hence,
\[ \rho_m \frac{\partial^2 W}{\partial t^2} + d \frac{\partial W}{\partial t} + B \Delta^2 W - T \Delta W + [k - g(\rho_x - \rho_y)]W = \delta p_x - \delta p_y, \quad (12.33) \]
where \( \Delta W = W_{xx} + W_{yy} \) and \( \Delta^2 W = W_{xxxx} + 2W_{xxyy} + W_{yyyy} \). For a three-dimensional isotropic wall of this type, the displacement is restricted to motion in the normal direction only, such that the three-dimensional wall displacement can take the following form
\[ W(x, t) = W_0 e^{(ax + \beta y - \omega t)} + \text{c.c.}, \]
where $\alpha$ is the stream-wise wave number in the $x$-direction, $\beta$ is the stream-wise wave number in the $y$-direction, and $W_o$ is the plate displacement amplitude.

By substituting the form for the surface displacement into (12.33) we obtain
\[
(-\rho_m \omega^2 b W_o - i \omega d W_o + (\alpha^4 + 2\alpha^2 \beta^2 + \beta^4) B W_o + (\alpha^2 + \beta^2) T W_o + k E W_o) e^{i(\alpha x + \beta y - \omega t)} = \delta p_s - \delta p_e. 
\] (12.34)

Now, by adjusting the dimensionless expression for the dynamic pressure perturbations given by Carpenter and Garrad [23], used in the preceding chapter for the two-dimensional problem, to account for three-dimensionality we have
\[
\delta p_e = \rho e v_g \Omega L \rho e \tilde{W},
\]
where the velocity terms $v_g \Omega L$ have replaced the $U_\infty^2$ term used in the two-dimensional dynamic pressure perturbation expression. $v_g$ is the velocity characteristic reference in the normal direction and $\Omega L$ is the corresponding velocity reference in the stream-wise direction and the product of these expressions is also used to non-dimensionalize the pressure term in the Navier-Stokes equations in section 17.4. Also, $\tilde{W} = \frac{W}{L}$ and $W = W_o e^{i(\alpha x + \beta y - \omega t)}$.

Let us non-dimensionalize the rest of the parameters by:
\[
\begin{align*}
\alpha &= \frac{\tilde{\alpha}}{L}, \quad \beta = \frac{\beta}{L}, \quad c = \varepsilon \Omega L, \\
t &= \frac{t}{\Omega}, \quad \omega &= \alpha \Omega \Rightarrow \omega = \gamma \varepsilon \Omega,
\end{align*}
\]
where the $\bar{}$ notation represents a non-dimensional parameter. Note that $\varepsilon = \frac{L}{L}$ is a dimensionless length and so $\tilde{\alpha} = \alpha L$ is a dimensionless inverse length, and so $\alpha \varepsilon$ is dimensionless. Substituting the non-dimensional parameters into equation (12.34) we obtain
\[
\begin{align*}
\left(-\rho_m \gamma^2 \varepsilon^2 \Omega^2 b - i \gamma \varepsilon c d + \frac{\alpha^4 + 2\alpha^2 \beta^2 + \beta^4}{L^4} B + \frac{\alpha^2 + \beta^2}{L^2} T + k E \right) \frac{\tilde{W} e^{i(\alpha x + \beta y - \omega t)}}{e^{i(\alpha \varepsilon + \beta \varepsilon - \omega t)}} \\
= \rho_e v_g \Omega L \rho e \tilde{W} - \rho_e v_g \Omega L \rho e \tilde{W}. \\
\end{align*}
\]

Dividing through by $\tilde{W} v_g \Omega L \rho e$, assuming $|\tilde{W}| \neq 0$ gives
\[
\frac{-\rho_m \gamma^2 \varepsilon^2 \Omega^2 b}{\rho_e v_g \Omega} - \frac{i \gamma \varepsilon c d}{\rho_e v_g} + \frac{B \gamma^4}{\rho_e v_g \Omega L^4} + \frac{T \gamma^2}{\rho_e v_g \Omega^2 L^2} + \frac{k e}{\rho_e v_g \Omega} = \tilde{p}_s - \tilde{p}_e.
\]

Now, the Rossby number is defined as, $R_o = \frac{\gamma}{\Omega L}$. Using this we obtain
\[
\frac{-\rho_m \gamma^2 \varepsilon^2 b}{\rho_e R_o L} - \frac{i \gamma \varepsilon c d}{\rho_e R_o \Omega} + \frac{B \gamma^4}{\rho_e R_o \Omega^2 L^4} + \frac{T \gamma^2}{\rho_e R_o \Omega^2 L^2} + \frac{k e}{\rho_e R_o \Omega^2 L} = \tilde{p}_s - \tilde{p}_e.
\]
Let us define the dimensionless coefficients describing the ratios between the mechanical properties of the coating and properties of the flow to be as follows.

\[ C_m = \frac{\rho_m b}{\rho_e L R_o}, \quad C_D = \frac{d}{\rho_e L \Omega R_o}, \quad C_B = \frac{B}{\rho_e R_o \Omega^2 L^5} \]

\[ C_T = \frac{T}{\rho_e R_o \Omega^2 L^3}, \quad C_{KE} = \frac{k_0}{\rho_e R_o \Omega^2 L}. \]

Substituting these in, we obtain

\[ -\gamma^2 \varepsilon^2 C_m - i \gamma \varepsilon C_D + \gamma^4 C_B + \gamma^2 C_T + C_{KE} = \bar{p}_s - \bar{p}_e. \]

Now we take the governing equations (17.41) and (17.42) for the linear analysis from section 17.4, namely

\[ i \alpha \bar{p} = -\gamma^2 E_k \bar{u} + E_k \bar{u}_{zz} + 2 \bar{u} - R_o \bar{w} \bar{U}_z - i \gamma (R_o \bar{V} - c) \bar{u} \]

\[ i \beta \bar{p} = -\gamma^2 E_k \bar{v} + E_k \bar{v}_{zz} - 2 \bar{u} - R_o \bar{w} \bar{V}_z - i \gamma (R_o \bar{V} - c) \bar{v}, \]

where \( E_k = \frac{\eta}{\Omega L^3} \) is the Ekman number and is the ratio of viscous to Coriolis accelerations. Multiplying (12.38) by \( i \alpha \) and (12.39) by \( i \beta \), adding, and using equations (12.21) to (12.24) we obtain

\[ -\gamma^2 \bar{p} = \gamma^2 (R_o \bar{V} - c) \bar{v} - i R_o \bar{w} \gamma \bar{V}_z - 2i \gamma \bar{u} - i \gamma E_k \bar{v} + i \gamma E_k \bar{v}_{zz}. \]

Now using \( \bar{u} = \frac{i}{\gamma} \bar{w}_z \) and \( \psi = i \gamma \bar{u} \) we obtain

\[ \bar{p} = \frac{-i}{\gamma} (R_o \bar{V} - c) \phi_x + \frac{i R_o \bar{V}_z}{\gamma} \phi + \frac{2}{\gamma^2} \psi + \frac{E_k}{\gamma^2} (\phi_{zzz} - \gamma^2 \phi_z). \]

At the wall, the boundary conditions state that \( \bar{V}(0) = 0 \). From the kinematic boundary conditions we have

\[ \bar{V}_z(0) \psi + \bar{U}_z(0) \phi = 0, \quad \text{at} \quad z = 0, \]

\[ c \phi_x + R_o \bar{V}_z(0) \phi = 0. \]

From (12.41) at \( z = 0 \) we obtain

\[ p_e = \frac{-i}{\gamma} (R_o \times 0 - c) \phi_x + \frac{i R_o \bar{V}_z(0) \phi(0)}{\gamma} + \frac{2}{\gamma^2} \psi(0) + \frac{E_k}{\gamma^2} (\phi_{zzz}(0) - \gamma^2 \phi_z(0)), \]

where \( p_e = \bar{p}|_{z=0} \). Using (12.43) we get

\[ p_e = \frac{2}{\gamma^2} \psi(0) + \frac{E_k}{\gamma^2} (\phi_{zzz}(0) - \gamma^2 \phi_z(0)). \]
Now, from Carpenter & Garrad [23], the dynamic boundary condition at the surface is expressed in terms of an equality between the surface and boundary admittances so that stabilizing or destabilizing effects on the instabilities can be determined when the mechanical properties of the surface are changed. That is, \( Y_0 = Y_1 \), where \( Y_0 \) and \( Y_1 \) are the admittances of the boundary layer fluid and flexible surface, respectively, and are defined to be

\[
Y_0 = -\frac{v_e(0)}{p_e(0)} = -\frac{\phi(0)}{\frac{1}{v} \psi(0) + \frac{E_k}{v^2} (\phi_{zzz}(0) - \gamma^2 \phi_z(0))}
\]  

(12.45)

and

\[
Y_1 = \frac{\rho_e \gamma W}{\rho_e \Omega_{\nu g}}.
\]  

(12.46)

When suitable expressions are derived for \( Y_0 \) and \( Y_1 \), then \( Y_0 = Y_1 \) acts as the boundary condition at the flexible wall replacing \( \phi(0) = \phi'(0) = \psi(0) = 0 \) used for the rigid wall.

Now, it can easily be shown that

\[
\frac{\partial W}{\partial t} = i \gamma \Omega W.
\]

If we let \( \delta p_e = \delta p_e e^{i(\alpha x + \beta y - \omega t)} \) and assume no fluid substrate, so that \( \delta p_s = 0 \), then equation (12.33) becomes

\[
-\rho_m b \omega^2 W e^{i(\alpha x + \beta y - \omega t)} - d i \omega W e^{i(\alpha x + \beta y - \omega t)} + B(\alpha^4 + 2\alpha^2 \beta^2 + \beta^4) W e^{i(\alpha x + \beta y - \omega t)} + T(\alpha^2 + \beta^2) W e^{i(\alpha x + \beta y - \omega t)} + k_E W e^{i(\alpha x + \beta y - \omega t)} = -\delta p_e.
\]

Non-dimensionalizing, we obtain

\[
-\rho_m b \omega^2 \Omega^2 \Omega L - i \omega \Omega \Omega W + B(\frac{\alpha^4}{L^3}) \Omega \Omega + T(\frac{\alpha^2}{L^2}) \Omega \Omega + k_E \Omega \Omega = -\delta p_e e^{i(\alpha x + \beta y - \omega t)}.
\]

Let us now drop the \( \bar{\gamma} \) notation since we have finished non-dimensionalizing. Thus

\[
Y_1 = \frac{i \gamma c W}{\delta p_e e^{i(\alpha x + \beta y - \omega t)}}.
\]

Substituting in for \( \delta p_e e^{i(\alpha x + \beta y - \omega t)} \) we obtain

\[
Y_1 = \frac{i \gamma c p_e \Omega \Omega}{\rho_m b \gamma^2 e^2 \Omega^2 + i \gamma c \Omega - B \frac{\alpha^4}{L^3} - T \frac{\alpha^2}{L^2} - k_E},
\]  

(12.47)

and so, calculating \( Y_0 = Y_1 \) using (12.45) and (12.47) and rearranging, yields the dynamic boundary condition

\[
-\frac{2}{E_k} \psi(0) + \phi_{zzz}(0) - \gamma^2 \phi_z(0) = -\frac{\gamma \phi(0)}{E_k} (\gamma^2 c^2 C_m + i \gamma c C_D - \gamma^4 C_B - \gamma^2 C_T - C_{KE}),
\]  

(12.48)

where \( C_m, C_D, C_B, C_T \) and \( C_{KE} \) are defined as stated in equations (12.35) and (12.36).
12.2 Attachment-line boundary layer over a compliant surface

Unlike the previous two representations for the plate-spring model of the compliant surface given in this part, this final representation will produce an amended model to incorporate an anisotropic wall.

12.2.1 Kinematic boundary conditions

Using the same notation as that used for the Ekman layer problem for the deformed wall coordinates in the previous section, equating the velocity at the wall with that in the fluid, expanding in Taylor series and assuming no elasticity we obtain

\begin{align}
0 &= u^*(x^*, y^*, 0, t^*) + u^*_z(x^*, y^*, 0, t^*)\zeta^* + \ldots \\
0 &= v^*(x^*, y^*, 0, t^*) + v^*_z(x^*, y^*, 0, t^*)\zeta^* + \ldots \\
\zeta^*_t &= w^*(x^*, y^*, 0, t^*) + w^*_z(x^*, y^*, 0, t^*)\zeta^* + \ldots ,
\end{align}

where * denotes dimensional parameters. Let us non-dimensionalize by taking

\begin{align}
&u = \frac{u^*}{V_0}, \quad v = \frac{v^*}{V_0}, \quad w = \frac{w^*}{V_0} \\
x = \frac{x^*}{L}, \quad y = \frac{y^*}{L}, \quad z = \frac{z^*}{L} \\
t = \frac{t^* V_0}{L}, \quad \zeta = \frac{\zeta^*}{L}.
\end{align}

Substituting these into equations (12.49) to (12.51) we obtain

\begin{align}
0 &= u + u_2\zeta + \ldots \\
0 &= v + v_2\zeta + \ldots \\
\zeta_t &= w + w_2\zeta + \ldots ,
\end{align}

Linearizing about the attachment line basic state, \((\frac{\tilde{u}(z)}{R_e}, \tilde{v}(z), \frac{\tilde{w}(z)}{R_e})\) discussed in chapter 22, by introducing the following perturbations to the basic flow

\begin{align}
u &= \frac{\tilde{u}(z)x}{R_e} + xU(x, y, 0, t) \\
v &= \tilde{v}(z) + V(x, y, 0, t) \\
w &= \frac{\tilde{w}(z)}{R_e} + W(x, y, 0, t) ,
\end{align}
we obtain

\begin{align}
0 &= xU(x, y, 0, t) + \frac{\partial_x \zeta}{R_e} \tag{12.55} \\
0 &= V(x, y, 0, t) + \bar{v}_z \zeta \tag{12.56} \\
\zeta_t &= W(x, y, 0, t) + \frac{\partial_z \zeta}{R_e}. \tag{12.57}
\end{align}

Now, our first kinematic boundary condition is obtained by combining equations (12.55) and (12.56) to give

\[ \bar{v}_z U = \frac{\bar{u}_z V}{R_e}. \tag{12.58} \]

Introducing the following Fourier transforms such that the disturbance has wavelength \( \frac{2\pi}{a} \) and propagates along the attachment line with speed \( c \),

\[ U = \tilde{U} e^{ia(y-ct)}, \quad V = \tilde{V} e^{ia(y-ct)}, \quad W = \tilde{W} e^{ia(y-ct)}, \quad \zeta = \tilde{\zeta} e^{ia(y-ct)}, \tag{12.59} \]

and substituting these into equation (12.58), we obtain

\[ \bar{v}_z \tilde{U} = \frac{\bar{u}_z \tilde{V}}{R_e}. \]

Now, from the continuity equation we have

\[ \tilde{U} + i\alpha \tilde{V} + \tilde{W}_z = 0. \]

Thus, using this to substitute for \( \tilde{V} \) from the boundary condition we obtain

\[ \bar{v}_z \tilde{U} = \bar{u}_z (\tilde{W}_z + \tilde{U}) \frac{i}{\alpha}. \]

Finally, changing notation so as to be consistent with our disturbance equations, given in section 22.3, \( \phi = \tilde{W} \) and \( \psi = \tilde{U} \), we obtain the final form of the first kinematic boundary condition

\[ (i\alpha \bar{R}_e \bar{v}_z(0) + \bar{u}_z(0)) \psi + \bar{u}_z(0) \phi_z = 0, \quad \text{at} \quad z = 0. \tag{12.60} \]

To find the second kinematic boundary condition, using equations (12.55) to (12.57), let us introduce the Fourier transforms given in equation (12.59) to obtain

\begin{align}
0 &= \tilde{U} + \frac{\bar{u}_z \zeta}{R_e} \tag{12.61} \\
0 &= \tilde{V} + \bar{v}_z \zeta \tag{12.62} \\
-\alpha \zeta &= \frac{\tilde{W} + \bar{w}_z \zeta}{R_e}. \tag{12.63}
\end{align}
Now, from the continuity equation we have

$$\dot{U} + i\alpha \dot{V} + \dot{W}_z = 0.$$  

Substituting in (12.61) and (12.62) gives

$$-\frac{\dot{u}_z}{R_c} \dot{\zeta} - i\alpha \dot{u}_z \dot{\zeta} + \dot{W}_z = 0,$$

and substituting for $\dot{\zeta}$ by using (12.63) yields

$$(\dot{u}_z + i\alpha R_c \dot{u}_z) \dot{W} + (\ddot{w}_z + i\alpha R_c \dot{c}) \ddot{W}_z = 0.$$  

Now, from the boundary conditions, we know that $\ddot{w}_z(0) = 0$. Using this, and finally changing notation to be consistent with the previous notation, $\ddot{W} = \phi$, we obtain

$$(\dot{u}_z(0) + i\alpha R_c \dot{u}_z(0)) \phi + i\alpha R_c \phi_z = 0.$$  

(12.64)

**12.2.2 Dynamic boundary condition**

Our assumption that the elastic properties of the plate material are the same in all directions, known as isotropy, used in the preceding sections, will now be modified to treat many practical applications.

If a homogeneous material has three mutually perpendicular planes of symmetry with respect to its elastic properties, it is called orthotropic, or in other words, orthogonally anisotropic. This type of plate is required in the analysis for realistic applications to numerous problems including marine and aerospace engineering. The modification to the behaviour of the plate occurs in the flexural rigidity term. A brief outline of the theory is described and further details can be found in Szilard [125].

In the derivation of Kirchhoff's small deflection plate theory, the number of independent elastic constants was two, namely, Young's modulus of elasticity, $E$ and Poisson's ratio, $\nu$. If we assume that the principal directions of orthotropy coincide with the $x$ and $y$ coordinates axes, it becomes evident that the four elastic constants, $(E_x, E_y, \nu_x, \nu_y)$, are required for the description of the orthotropic stress-strain relationships:

$$\epsilon_x = \frac{\sigma_x}{E_x} - \nu_y \frac{\sigma_y}{E_y}, \quad \epsilon_y = \frac{\sigma_y}{E_y} - \nu_x \frac{\sigma_x}{E_x}, \quad \gamma = \frac{\tau}{G_{xy}}.$$

(12.65)

where $\sigma$ is the stress and $\epsilon$ is the strain. The shear modulus, $G_{xy}$ of the orthotropic material can be expressed in terms of $E_x$ and $E_y$ as follows

$$G_{xy} \approx \frac{\sqrt{E_x E_y}}{2(1 + \sqrt{\nu_x \nu_y})} \approx \frac{E}{2(1 + \sqrt{\nu_x \nu_y})}.$$
From Szilard [125], solving equations (12.65) for \( \sigma_x, \sigma_y \) and \( \tau \) it is found that

\[
\sigma_x = \frac{E_x}{1 - \nu_x \nu_y} (\epsilon_x + \nu_x \epsilon_y), \\
\sigma_y = \frac{E_y}{1 - \nu_x \nu_y} (\epsilon_y + \nu_y \epsilon_x), \\
\tau = G_{xy} \gamma.
\]

The strain terms can be expressed in terms of lateral deflections:

\[
m_x = -D_x \left( \frac{\partial^2 W}{\partial x^2} + \nu_y \frac{\partial^2 W}{\partial y^2} \right), \\
m_y = -D_y \left( \frac{\partial^2 W}{\partial y^2} + \nu_x \frac{\partial^2 W}{\partial x^2} \right), \\
m_{xy} = -2D_t \frac{\partial^2 W}{\partial x \partial y}, \tag{12.66}
\]

where \( D_x \) and \( D_y \) are the flexural rigidities of the orthotropic plate, while \( 2D_t = (1 - \nu_{xy}) D_{xy} \) represents its torsional rigidity. Now for an orthotropic plate of uniform thickness, the torsional rigidity can be written as

\[
D_t = G_{xy} \frac{b^3}{12},
\]

where \( b \) is the plate thickness. Substitution of (12.66) into the equilibrium equation of a plate element (eqn 1.2.9 of [125]) yields the following governing differential equation for orthotropic plates:

\[
D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = P_z(x, y). \tag{12.67}
\]

Here, \( H = \frac{1}{2}(\nu_y D_x + \nu_x D_y + 4D_t) \) is called the effective torsional rigidity of the orthotropic plate.

Starting with our wall equation (12.33) for an isotropic plate, the only term that requires modification to represent an orthotropic plate is the flexural rigidity term

\[
B \Delta^2 W = BW_{xxxx} + 2BW_{xxyy} + BW_{yyyy},
\]

which now becomes

\[
D_x W_{xxxx} + 2HW_{xxyy} + D_y W_{yyyy} = D_x (W_{xxxx} + \frac{2H}{D_x} W_{xxyy} + \frac{D_y}{D_x} W_{yyyy}).
\]
Now let us assume that
\[ \frac{H}{D_x} = (1 + \chi_1) \quad \text{and} \quad \frac{D_y}{D_x} = (1 + \chi_2), \]
where \( \chi_1, \chi_2 \) can be either positive or negative. Then if we let \( D_x = \tilde{B} \), we obtain
\[
\tilde{B}(W_{xxxx} + 2(1 + \chi_1)W_{xxyy} + (1 + \chi_2)W_{yyyy}) = \tilde{B}(W_{xxxx} + 2W_{xxyy} + 2\chi_1 W_{xxyy} + W_{yyyy} + \chi_2 W_{yyyy})
\]
\[ = \tilde{B}\Delta^2 W + 2\tilde{B}\chi_1 W_{xxyy} + \tilde{B}\chi_2 W_{yyyy}. \]

Thus substituting this into equation (12.33) to replace the flexural rigidity term \( B\Delta^2 W \) gives the wall equation in three-dimensions for an orthotropic surface:
\[
\rho_m b \frac{\partial^2 W}{\partial t^2} + d \frac{\partial W}{\partial t} + \tilde{B}\Delta^2 W + 2\tilde{B}\chi_1 W_{xxyy} + \tilde{B}\chi_2 W_{yyyy} - T\Delta W + k_B W = -\delta p_e. \quad (12.68)
\]

Let the surface displacement \( W(y, t) \) take the form
\[ W = W_0 e^{i\alpha(y-ct)} + c.c., \]
where \( \alpha \) is the wavenumber, and \( W_0 \) is the plate displacement amplitude. Substituting this into (12.68) and assuming \( \delta p_e = 0 \) (no fluid substrate), we obtain
\[
\left( \rho_m b c^2 W_0 - \text{d} \alpha W_0 + \tilde{B}\alpha^4 W_0 + \tilde{B}\chi_2\alpha^4 W_0 + T\alpha^2 W_0 + k_B W_0 \right) e^{i\alpha(y-ct)} = -\delta p_e.
\]
\[ \quad (12.69) \]

Let the corresponding dimensionless expression for the dynamic pressure perturbation be given by
\[ \delta p_e = \rho_e V_o^2 \tilde{p}_e \tilde{W}, \quad \text{where} \quad \tilde{W} = \frac{W}{L}, \quad \text{and} \quad W = W_0 e^{i\alpha(y-ct)}, \]
and let us also non-dimensionalize the rest of the dimensional parameters by taking
\[ \alpha = \frac{\tilde{\alpha}}{L}, \quad c = \tilde{c} V_o, \quad t = \frac{tL}{V_o}. \]

Substituting these into equation (12.69) and dividing through by \( \tilde{W} \rho_e V_o^2 \) we obtain
\[
-\rho_m b c^2 \alpha^2 \frac{\partial^2 W}{\partial t^2} - i \frac{d}{\rho_e V_o} \alpha \frac{\partial W}{\partial t} + \frac{\tilde{B}\alpha^4}{\rho_e V_o^2 L^3} + \frac{\tilde{B}\chi_2\alpha^4}{\rho_e V_o^2} + \frac{T\alpha^2}{\rho_e LV_o^2 V_o} + \frac{k_B L}{\rho_e V_o^2} = -\tilde{p}_e.
\]

Now, we can simplify further by defining the following dimensionless parameters
\[
C_m = \frac{\rho_m b}{\rho_e L}, \quad C_D = \frac{d}{\rho_e V_o}, \quad C_{\tilde{B}} = \frac{\tilde{B}}{\rho_e V_o^2 L^3} \\
C_T = \frac{T}{\rho_e V_o^2 L}, \quad C_{KB} = \frac{k_B L}{\rho_e V_o^2}. \]
By substituting these in we obtain the dimensionless equation

\[-\alpha^2 e^2 C_m - i\alpha \delta C_D + (1 + \chi \alpha) \alpha^4 C_B + \alpha^6 C_T + C_{KE} = -\dot{\rho}_e.\] (12.70)

Now, only one of our disturbance equations for the attachment line boundary layer stability analysis carried out in chapter VI involves the pressure term, namely equation (22.23), which can be rearranged to give

\[
\tilde{p} = -\frac{i}{\alpha R_e} \tilde{V}_{zz} + \frac{i\alpha}{R_e} \tilde{V} + c\tilde{V} - \tilde{v}\tilde{V} + \frac{i}{\alpha} \tilde{v}_z \tilde{W} + \frac{i}{\alpha R_e} \tilde{w} \tilde{V}_z.
\]

We wish to eliminate \( \tilde{V} \) thus, using the continuity equation (22.25), we can substitute out \( \tilde{V} \) using

\[
\tilde{V} = \frac{i}{\alpha} (\tilde{W}_z + \tilde{U}),
\]

thus obtaining

\[
\tilde{p} = -\frac{i}{\alpha R_e} \left( \frac{i}{\alpha} (\tilde{W}_{zz} + \tilde{U}_{zz}) \right) + \frac{i\alpha}{R_e} \left( \frac{i}{\alpha} (\tilde{W}_z + \tilde{U}) \right) + \frac{c}{\alpha} (\tilde{W}_z + \tilde{U}) - \frac{\tilde{v}_z}{\alpha} (\tilde{W}_z).
\]

Now, our boundary conditions at the wall state that \( \tilde{u}(0) = \tilde{w}(0) = \tilde{v}(0) = \tilde{v}(0) = 0 \) and so, setting \( z = 0 \) in our pressure equation yields

\[
p_e = \frac{1}{\alpha R_e} \left( \tilde{W}_{zz}(0) + \tilde{U}_{zz}(0) \right) - \frac{1}{R_e} \left( \tilde{W}_z(0) + \tilde{U}(0) \right) + \frac{c}{\alpha} \left( \tilde{W}_z(0) + \tilde{U}(0) \right) - \frac{\tilde{v}_z(0)}{\alpha} \tilde{W}(0),
\]

where \( p_e = \tilde{p}|_{z=0} \). From our two kinematic boundary conditions (12.20) and (12.10) we also have,

\[
\tilde{W}_z(0) = \frac{\tilde{u}_z(0) + i\alpha R_e \tilde{v}_z(0)}{\lambda R_e} \tilde{W}(0)
\]

and

\[
\tilde{U}(0) = \frac{-\tilde{u}_z(0) \tilde{W}(0)}{(i\alpha R_e \tilde{v}_z(0) + \tilde{u}_z(0))} \Rightarrow \tilde{U}(0) = -\frac{\tilde{u}_z(0)}{\lambda R_e} \tilde{W}(0),
\]

respectively. Substituting these two expressions into our equation for the pressure at the wall, we obtain

\[
p_e = \frac{1}{\alpha^2 R_e} \left( \tilde{W}_{zz}(0) + \tilde{U}_{zz}(0) \right) - \frac{i\alpha}{\lambda R_e} \tilde{v}_z(0) \tilde{W}(0).
\]

(12.73)

Following Carpenter and Garrad [23], the dynamic boundary condition is defined by the equality of fluid to surface admittances.

\[ Y_0 = Y_1, \]
where $Y_0$ is the ratio of the normal velocity component to pressure at the wall in the fluid, and is defined by

$$Y_0 = -\frac{\bar{W}(0)}{p_e},$$

and $Y_1$ is the ratio of the vertical velocity component to the pressure at the wall in the wall, and is defined by

$$Y_1 = \frac{\partial \bar{V}}{\partial t} \frac{1}{W p_e}.$$ 

Thus our dynamic boundary condition can be calculated by

$$Y_0 = Y_1$$

$$\Rightarrow \frac{-\phi(0)}{\alpha^2 R_e (\phi''(0) + \psi''(0)) - \frac{i\alpha v_2(0)\psi(0)}{\lambda R_e}} = \frac{i\alpha c_p V_2^2}{\rho_m b_0^2 c^3 (\frac{R_e}{L})^2 L + i\alpha c D d L - (1 + \chi_2) \frac{c_B}{c_{T}} B L - \frac{c_{K}}{c_{T}} T - \frac{c_{K}}{c_{E}} L} \frac{1}{(1 + \chi_2) \frac{c_B}{c_{T}} B L - \frac{c_{K}}{c_{E}} L - \frac{c_{K}}{c_{T}} T - \frac{c_{K}}{c_{E}} L}$$

this implies

$$ic (\phi''(0) + \psi''(0)) = -\phi(0) \left[ \alpha Re \left( \alpha^2 c^2 C_m + i\alpha c C_D - \alpha^4 (1 + \chi_2) C_B - \alpha^2 C_T - C_{KE} \right) + i\alpha^2 v_2(0) \right].$$

(12.74)

We wish to substitute for the term in $\psi''$. To do this we use the second order equation, (22.35) for the disturbance, namely

$$\psi'' - \bar{\psi} \psi' - a(z) \psi - \bar{u}_z \phi = 0,$$

where $a(z) = \alpha^2 + i\alpha R_e \bar{\eta} + \lambda R_e + 2\bar{u}$. So, at $z = 0$ we have

$$\psi''(0) = a(0) \psi(0) + \bar{u}_z(0) \phi(0),$$

where $a(0) = \alpha^2 + \lambda R_e$.

Thus our dynamic boundary condition finally takes the form

$$ic (\phi''(0) + a(0) \psi(0)) = -\phi(0) \left[ \alpha Re (\alpha^2 c^2 C_m + i\alpha c C_D - \alpha^4 (1 + \chi_2) C_B - \alpha^2 C_T - C_{KE}) \right. \left. + i\alpha v_2(0) + i\alpha^2 \bar{v}_z(0) \right].$$

(12.75)
Numerical values for the wall parameters

In this section, expressions for the wall parameters used in the numerics are given. The following values are used following Carpenter & Garrad [23]: The kinematic viscosity takes the value \( \nu = 1.37 \times 10^{-6} \text{ m}^2\text{s}^{-1} \), which may be deduced from the values Kramer cites for the Reynolds number. The free stream velocity is taken to be \( U_\infty = 18 \text{ m s}^{-1} \), which was the maximum speed for Kramer’s tests, and the plate thickness \( b = 0.002 \text{ m} \). The density of the plate material, \( \rho_m = 945 \text{ kg m}^{-3} \) (i.e. that of natural rubber), the density of the main flow is \( \rho_e = 1025 \text{ kg m}^{-3} \), corresponding to the density of sea water, and \( g = 9.807 \text{ m s}^{-2} \).

The damping fluids used by Kramer were silicone fluids, and for Kramer’s best coating, the density of this substrate is about 970 \( \text{ kg m}^{-3} \). And so, for the best coating, the non-dimensional substrate to plate density ratio, \( \frac{\rho_s}{\rho_p} \approx 0.946 \).

Kramer [73] states that the softest natural rubbers available for manufacturing his coatings had an elastic modulus, \( E_s \), of around 0.4 \( \text{ N mm}^{-2} \). It was also apparent from this paper that the hardest of these rubbers used for his original coatings had an elastic modulus of about 1.0 \( \text{ N mm}^{-2} \). Carpenter & Garrad assumed that the pressure acting on the surface is supported by a large number of stubs, and from the geometric parameters for the coating, they calculated that the total cross-sectional area of the stubs was \( \frac{1}{44} \) times the total surface area of the coating. Thus, the pressure supported by the stubs is approximately 4.4 times the surface pressure. Then, given the undeformed height of the stubs is 1 mm, the parameter for the spring stiffness could be estimated as

\[
K = \frac{E}{4.4 \times 10^{-3}} = 230E \text{ N m}^{-3}.
\]

The parameter \( E \) is our main varying parameter representing wall stiffness. An increase in \( E \) represents an increase in wall flexibility, with \( E = \infty \) corresponding to a rigid wall.
Now, the flexural rigidity of the plate is given by

\[ B = \frac{Eb^3}{12(1 - \nu^2)} \]

Carpenter & Garrad [23], and for natural rubber, the density is about 945 kgm\(^{-3}\) and the Poisson ratio is close to 0.5. Thus, with a plate thickness of 2 mm, we obtain

\[ B = 8.9 \times 10^{-10} \times ENm. \]

Kramer made no mention of tension being applied to his coatings, so following Carpenter and Garrad, we will initially assume that only sufficient tension is applied to keep the coating firmly and smoothly attached to the rigid part of the model. Thus we shall initially assume \( T = 0 \). The Reynolds number used in the calculations of the neutral curves for the rigid wall case and the compliant surface is based on the length scale, \( \delta \), that is, the boundary layer thickness:

\[ R_e = \frac{U_{\infty} \delta}{\nu}. \]

This Reynolds number is directly proportional to that defined by the displacement thickness:

\[ R_e^* = \gamma R_e, \]

where the approximate value of the constant \( \gamma \) used in the calculations is 1.72078766.

The parameters defined in Carpenter & Garrad [23] depend on the displacement thickness, \( \delta_\ast \). Thus, we transformed from the displacement thickness variables to the \( \delta \)-based variables as follows. By definition,

\[ R_e^* = \frac{U_{\infty} \delta_\ast}{\nu} \Rightarrow \delta_\ast = \frac{\nu R_e^*}{U_{\infty}} = \frac{\nu R_e \gamma}{U_{\infty}} = \delta \gamma. \]

Thus the results obtained in the following section differ from those found by Carpenter and Garrad by only a simple scaling factor, and this will be taken into account when comparing numerical results.
PART IV

Two-dimensional boundary layers interacting with a compliant surface
Overview

The aim of this part of the report is to study the effect of wall compliance on the two-dimensional Tollmien-Schlichting instability. This problem has been thoroughly investigated by Carpenter and Garrad [23]. However, the methods used for the stability analysis in the present report are different and supposed more accurate. The new numerical framework based on exterior algebra, discussed in Part II is used to carry out a linear stability analysis of the Orr-Sommerfeld equation with the Blasius velocity profile interacting with the two-dimensional plate-spring model discussed in Part III. Curves of neutral stability are created for variations in the system parameters, finalised by a discussion of the results.

The two boundary conditions (11.7) and (11.8) can be written in the form

\[ a_0 \phi(0) + a_1 \phi'(0) = 0, \]  
\[ b_0 \phi(0) + b_1 \phi'(0) + b_2 \phi''(0) = 0. \]  

Or, in terms of the vector variable, the boundary conditions at \( y = 0 \) can be written in the following form

\[ \langle \eta_1(\lambda), u(0, \lambda) \rangle_R, \quad \langle \eta_2(\lambda), u(0, \lambda) \rangle_R, \]  

by taking

\[ \eta_1(\lambda) = \begin{pmatrix} a_0 \\ a_1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \eta_2(\lambda) = \begin{pmatrix} b_0 \\ b_1 \\ 0 \\ b_3 \end{pmatrix}, \]

where

\[ a_0 = U'(0), \quad a_1 = c, \]
\[ b_0 = \alpha Re(\alpha^2 c^2 C_m + i \alpha c C_D - \alpha^4 C_B - \alpha^2 C_T - C_{KE}), \quad b_1 = -i \alpha^2 c, \quad b_3 = i c. \]
14.1 Boundary conditions on $\bigwedge^2(\mathbb{C}^4)$

The above form of the boundary conditions are in standard form, thus we can apply the theory of exterior algebra and compound matrices. With the new numerical framework, we require just one boundary condition for the spring-plate model of the Kramer-type compliant surface. Suppose that the boundary conditions at $y = 0$ for the linear system

$$u_y = A(y, \lambda)u \quad u \in \mathbb{C}^4,$$  \hspace{1cm} (14.4)

are as in (14.3), i.e

$$\langle \eta_1(\lambda), u(y, \lambda) \rangle_R |_{y=0} = 0 \quad (14.5)$$

$$\langle \eta_2(\lambda), u(y, \lambda) \rangle_R |_{y=0} = 0, \quad (14.6)$$

where $\eta_1(\lambda)$ and $\eta_2(\lambda)$ are linearly independent sets that depend analytically on $\lambda$. If a complex inner product is used then the conjugates of $\eta_1(\lambda)$ and $\eta_2(\lambda)$ are used in (14.5) and (14.6). The boundary condition, at $y = 0$, associated with the induced system on $\bigwedge^2(\mathbb{C}^4)$ is obtained as follows.

The conditions (14.5) and (14.6) form a two dimensional subspace of $\mathbb{C}^4$. Let $\{\eta_1(\lambda), \eta_2(\lambda)\}$ be an analytic basis for this space. The 2-form

$$\eta_1(\lambda) \wedge \eta_2(\lambda) \in \bigwedge^2(\mathbb{C}^4),$$

or any complex multiple of it is a characterising form for the space. However, we must express this form in terms of the basis used for constructing $A^{(2)}$ in chapter 7. Let $\omega_1, \ldots, \omega_6$ be an orthonormal basis for $\bigwedge^2(\mathbb{C}^4)$. The above 2-form can then be expanded as

$$\eta_1(\lambda) \wedge \eta_2(\lambda) = \sum_{j=1}^6 w_j \omega_j,$$

then the boundary condition imposed on $U(y, \lambda)$ at $y = 0$ is

$$\langle w, U(y, \lambda) \rangle_R |_{y=0} = 0.$$

Considering $\bigwedge^2(\mathbb{C}^4)$ as a complex 6-dimensional vector space, the inner product $\langle \cdot, \cdot \rangle_R$ is the standard inner product on $\mathbb{R}^6$. This suggests the introduction of a complex analytic function $D(\lambda)$ such that

$$D(\lambda) = \langle w, U(0, \lambda) \rangle_R,$$
whose zeros correspond to eigenvalues of the original boundary value problem.

We fix the standard basis $e_1, \ldots, e_4$ for $\mathbb{C}^4$, and write $\eta_1(\lambda)$ and $\eta_2(\lambda)$ with respect to the standard basis

\[
\eta_1(\lambda) \wedge \eta_2(\lambda) = (a_0 e_1 + a_1 e_2) \wedge (b_0 e_1 + b_1 e_2 + b_3 e_4)
= (a_0 b_1 - a_1 b_0) e_1 \wedge e_2 + (a_2 b_3) e_1 \wedge e_4 + (a_1 b_3) e_2 \wedge e_4.
\]

By letting

\[
\omega_1 = e_1 \wedge e_2, \omega_2 = e_1 \wedge e_3, \ldots, \omega_5 = e_3 \wedge e_4,
\]

the boundary condition at $y = 0$ (i.e. at the Kramer type compliant surface) is defined to be

\[
D(\lambda) = (a_0 b_1 - a_1 b_0) U_1(0) + a_0 b_3 U_3(0) + a_1 b_3 U_5(0) = 0. \quad (14.7)
\]

Hence

\[
w(\lambda) = \begin{pmatrix} a_0 b_1 - a_1 b_0 \\ 0 \\ a_0 b_3 \\ 0 \\ a_1 b_3 \\ 0 \end{pmatrix}.
\]

Therefore, the proposed algorithm is to fix values for $\alpha$, $Re$, the wall parameters and $\lambda \in \Lambda$ and integrate the ODE (14.4) from $y = L_\infty$ to $y = 0$ using an implicit GL-RK method, with starting vector (9.10). A value $\lambda \in \Lambda$ is an eigenvalue if $D(\lambda) = 0$ with $D(\lambda)$ defined in (14.7). Roots of $D(\lambda)$ are then refined using Newton’s method as discussed in section 7.2.1. The results presented here have been computed using $L_\infty = 10.0$ and the fourth-order implicit GL-RK method.

### 14.2 Calculating the dimensionless parameters required for the boundary conditions

Using the numerical values for the dimensional wall parameters given in chapter 13 the dimensionless wall parameters are as follows:

\[
C_m = \frac{b \rho_m}{\rho_0 \delta} = \frac{24226.420899}{Re}.
\]

The damping parameter is defined by

\[
C_D = \frac{d}{\rho_0 U_\infty},
\]
where $d$ is the damping coefficient. To compare with Figure 11 of [23], $C_D$ is initially set to zero.

The tension parameter is

$$C_T = \frac{T}{\rho_u U_{\infty}^2 \delta},$$

where $T$ is the longitudinal tension per unit width. To compare with Figure 11 of [23] $C_T$ was also initially set to zero.

The flexural rigidity parameter is

$$C_B = \frac{B}{\rho_u U_{\infty}^2 \delta^3} = 6078227.413 \frac{E}{R_f},$$

using $B = 8.9 \times 10^{-10} E$ Nm for the flexural rigidity of the plate (cf. eqn (3.29) of [23]) and $E$, is the variable elastic modulus of the plate in Nm$^{-2}$. $C_B$ is dimensionless when $E$ is given in units of Nm$^{-2}$. The rigid wall corresponds to $E = \infty$, and in the results reported below the cases of $E$ varying from 1.0 Nmm$^{-2}$ to 0.007065 Nmm$^{-2}$ are also considered. The spring stiffness per unit width is taken as $k = 230E$ Nm$^{-3}$ (cf. eqn(3.28) of [23]). If we include a substrate then the ratio of the substrate density to the main flow density, $\frac{\rho_s}{\rho_e} = 0.946$ (cf. p.483 of [23]). Thus $\rho_s = 0.946 \times 1025 = 969.65$ kgm$^{-3}$ and $k_B = (230E - g(\rho_e - \rho_s)) = (230E - 542.81745)$ Nm$^{-3}$. However, since we have assumed there to be no substrate fluid in our analysis, it is tantamount to using the plate spring model as an approximate model for a two layer, completely solid compliant wall. The contribution of the body-force perturbation terms involving density is only non-zero if the main fluid and substrate fluid have different densities, so we can take $k_B = k$. Hence,

$$C_{KE} = \frac{k_B \delta}{\rho_u U_{\infty}^2} = 2.291813 \times 10^{-13}(230ER_e).$$

$C_{KE}$ is dimensionless when $E$ is input in units of Nm$^{-2}$. 
Of interest in applications, are curves of neutral stability which correspond to curves in the $\alpha - R_e$ plane where $\text{Im}(c) = 0$. Inside the curve corresponds to instability. In figure 15.1, the computed effect of wall flexibility on stability is shown. When $E$ is very large, the neutral curve for the Blasius boundary layer is recovered. All the other parameters are fixed, and $C_D = C_T = 0$. The effect of varying $E$ is quite pronounced, the region of instability becomes progressively smaller as $E$ is reduced. The neutral curves for the plate-spring model begin to depart from the curve corresponding to the rigid wall by being displaced downwards and to the right. A reduction in $E$ implies a corresponding reduction in spring stiffness, $k$, and flexural rigidity, $B$. Thus, it is the spring stiffness that determines the stability characteristics at high Reynolds numbers. Figure 15.2 shows a blowup of the region near the nose of the neutral curve as $E$ approaches $E_c$. The point $E_c$, which we have computed to be $E_c = 0.007065$, is the value of $E$ at which the neutral curve collapses to a point. This point is important in applications because for values of $E < E_c$ the flow is extraordinarily stable: the transition Reynolds number has been increased dramatically. This effect suggests that compliant surfaces could reduce drag by delaying transition to turbulence. The results in figure 15.1 agree qualitatively with figure 11 of Carpenter & Garrad [23]. In Figures 15.3 and 15.4 we present the effects of damping and varying tension. The results of figure 15.3 show that damping does not have a significant effect on the minimum Reynolds number for instability, but it does have a noticeable effect on the shape of the region of instability. The effect of non-zero $C_T$ shown in Figure 15.4 shows that nonzero tension has an insignificant effect on the instability characteristics of the compliant-wall boundary-layer interaction. All the results in this section were computed using the fourth-order GL-RK method presented in the previous
chapters, and a value of $L_\infty = 10$ was used.

Figure 15.1: Effect of $E$ on the neutral curves, plotted in the $\alpha - R_e$ plane: values inside a curve correspond to instability $\text{Im}(c) > 0$.

Figure 15.2: Blow up of the nose of the neutral curve in Figure 15.1 near the critical value of $E$. 
Part IV: Computed neutral curves

Neutral curves for the Blasius boundary layer over a compliant surface $E=0.7$ with various damping

Figure 15.3: Effect of compliant surface damping $C_D$ on the neutral curve

Neutral curves for the Blasius boundary layer over a compliant surface $E=0.7$ with varying tension

Figure 15.4: Effect of tension $C_T$ variation on the neutral curve
PART V

Three-dimensional problem: Rotating flows and the Ekman layer
Overview

A central aim of the project is to study the instability of fluid flow past a compliant surface as a model for the flow past a dolphin. To implement a more realistic setting, three-dimensionality is incorporated. One of the primary ways that three-dimensionality can occur is when rotation is present, leading to Coriolis effects. Therefore, in this section, the stability of the Ekman layer will be studied which is a prototype for three dimensional rotating flows and has the added advantage that an explicit solution for the basic state is available.

16.1 Ekman Spiral

Surface winds and surface ocean currents are intimately related, but how winds drive currents is not so obvious. Earth's rotation plays an important role.

F. Nansen observed that the drift of surface ice was angled at 20-40 degrees to the right of the wind direction, in the northern hemisphere. Basically, the process begins when winds blow across water and drag on the surface. This surface drag sets into motion a thin layer of water, a few centimetres thick, which in turn drags on the thin layer beneath, setting it in motion. This process continues downwards by the same mechanism, where the stress between each layer and the next lower layer provides a transfer of momentum to successively deeper layers resulting in another deflection at each layer. Such transfer of momentum from one layer to the next is inefficient, however, and therefore energy is lost in the process. As a result, current speed decreases with increasing depth.

In an infinite ocean on a non-rotating Earth, water would always move in the same direction as the wind that set it in motion. However, because Earth rotates, surface waters are deflected to the right of the wind (Northern Hemisphere). Nansen attributed this effect
to the Coriolis forces, stating that these forces, introduced by the rotation of the earth, are not negligible compared to the slow drift velocities.

Based on this suggestion, in 1905 the Swedish physicist V.W. Ekman [39] first explained the phenomenon by analysing the problem of a wind-driven rotating flow, resulting from a balanced pressure gradient, and Coriolis and frictional forces. Ekman showed that the flow has a boundary layer structure and to explain the shift, he assumed a simple uniform ocean with no boundaries. In such an ocean, the motion of each deeper layer is deflected to the right of the one above. The mean velocity can be represented by a vector that changes length exponentially with depth and changes angle linearly with depth forming a spiral when viewed from above. This is called the Ekman Spiral, Figure 16.1. Note that a spiral for the southern hemisphere has the opposite sense of deflection, but current speeds still decrease with increasing depth. Ekman spiral flows can be created in the laboratory and have been observed in both atmospheric and oceanic flows.

Figure 16.1: Ekman spiral illustrates geostrophic oceanic flows induced by wind stress. Northern hemisphere flow is shown here. (Diagram reproduced following J. Vanyo [132].)
Boundary layer velocity profiles that approximate the Ekman layer occur in the atmospheric boundary layer to a height of perhaps 1000 m, and in wind driven surface layers of the ocean to a depth of the order of 50 m. However, oceanic and atmospheric examples always involve turbulence due to the rough boundary surfaces. Unsteadiness of the mean flow and thermal effects may also be important. The numerical tests carried out in this part of the report, are concerned only with the steady laminar mean flow of a uniform fluid, having applications to geophysical flows.

The Ekman layer is an exact solution of the Navier-Stokes equations, and is thus, an attractive feature for theoretical analyses. It also has the added advantage that it is strictly parallel since it has a constant boundary layer thickness and also has a constant geostrophic velocity, and so, theory in the literature assumes a single Reynolds number definition of the flow. Therefore, the need to account for downstream growth of the boundary layer or to make a local parallel flow approximation to reduce the governing partial differential equations to a more amenable ordinary differential set is unnecessary. Hence, the Ekman layer is a good model flow for theoretical studies.

The stability of flow of incompressible viscous fluid in the Ekman layer on an impermeable surface has been thoroughly researched. Experimental work showed, and theoretical research confirmed, the existence of two forms of instability in the Ekman layer with respect to small disturbances, these are referred to as type-1 and type-2 instabilities. The type-1 instability is an inviscid class B type and type-2, a viscous class A type. A selection of the major experimental and theoretical works now follow.

In 1955 Gregory et al. [54] used the China clay technique for the flow induced by a rotating circular disk in still air. They confirmed the velocity profiles deduced theoretically, and determined a critical Reynolds number for the onset of instability. Furthermore, Stuart's mathematical analysis provided physical insight into the nature of the instability and successfully explained certain features of the evolved flow. The instability observed had the form of stationary roll vortices. The theoretical results of Stuart presented in their paper are based on a transformation which allows the three-dimensional problem to be reduced to a two-dimensional one. Stuart worked with a local Cartesian coordinate system obtained from resolving the two coordinates in the plane of the disk through an angle $\epsilon$ and two-dimensional mean flow profiles were then formed from a combination of radial and tangential velocity components. Gregory & Walker [55] extended the study of
instability to a rotating disk with suction.

A while later, Faller [40] and Tatro & Mollo-Christensen [126] carried out experiments, in which a rotating fluid contained a sink on the central axis of rotation and distributed sources around the outer edge. Faller was concerned with the stability of the steady laminar boundary-layer flow of a homogeneous fluid which occurs in a rotating system when the relative flow is slow compared to the basic speed of rotation. Ekman flow was produced in a large cylindrical rotating tank by withdrawing water from the centre and introducing it at the rim. This created a steady-state symmetrical vortex in which the flow from the rim to the centre took place entirely in the shallow viscous boundary-layer at the bottom. Faller detected an unstable boundary layer above a critical Reynolds number of 125 ± 5, which was later attributed to the type-1 instability.

Further experimental and theoretical studies of the motion produced by sources and sinks in a fluid rotating about a vertical axis in a tank were performed by Barcilon [6] and Hide [64]. Hide showed that, if there is a closed curve in the plane perpendicular to the axis of rotation through which there is a net flux of mass, then the net transport of fluid occurs within both the side-wall layers and the horizontal-wall layers, having an Ekman-layer structure at the outer edge. These experiments showed the presence of instability waves in the Ekman layers, where the Rossby number (ratio of convective to Coriolis terms) was small. They have been successfully analysed by neglecting the non-linear inertial terms from the mean flow equations and assuming a constant geostrophic velocity (Lilly [83], Faller & Kaylor [42], Spooner & Criminale [124], Marlatt & Birinjen [89]). Then in 1995 Marlatt & Birinjen went on to describe a numerical simulation of secondary instability of the type-2 mode.

The theoretical studies of the stability of the steady Ekman layer were carried out a few years later. It has been shown by Lilly (in 1966) [83] and Melander (in 1983) [90] that the most unstable mode for the Ekman layer is a travelling branch-2 (type-2) mode with a critical Reynolds number of around 55 and angle of orientation, $\epsilon$, approximately 7.5 degrees (Lilly) and $R_{ec} = 54.2$ with $\epsilon \approx 7.2$ degrees (Melander). This disturbance mode is associated with a Coriolis-viscous force balance and is a class A disturbance. The onset of the second mode, branch-1 (type-1) stationary wave instability is initiated at Reynolds number of about 115 (Lilly [83]) or 116 (Lingwood [86]). This mode is a class B, inviscid cross-flow instability and is associated with a velocity profile inflexion point.
Lilly [83] used a perturbation analysis and numerically solved both the complete set of equations involving Coriolis effects, and also the Orr-Sommerfeld equations \( (Re_c = 85) \). Lilly was the first to suggest that the substantially lower critical Reynolds number for the complete set, along with changes in the shape of the compared computed neutral curves for both the OS equation and the extended set, and indicated the existence of the separate instability mechanism dependent on the Coriolis effects and viscosity, namely, the type-2 instability. This hypothesis was verified by means of a simplified analytic approach to the problem, where using various assumptions, Lilly demonstrated that energy could be extracted from the mean flow component parallel to the disturbance bands. Disturbance growth from this energy was found to be reliant on the Coriolis terms, which are of comparable magnitude to the viscous terms. As a result of this finding, he designated this as a parallel instability and suggested that it was essentially of the viscous type since its presence vanishes at high Reynolds numbers. Characteristic features associated with this mode were higher phase speeds and orientation at negative angles.

At about the same time, Faller & Kaylor [42], obtained numerical solutions to the time dependent non-linear equations of motion starting with a perturbation on the finite difference equivalent of a laminar Ekman solution. Their results confirmed the presence of the two distinct modes of instability.

The critical Reynolds number for the type-1, convective instability of stationary waves in the Ekman layer agrees quite closely with the experimentally observed Reynolds numbers of Faller [40]. The instabilities in the Ekman layer observed by Faller & Kaylor [42] had a wave angle of about 14 degrees, which is again consistent with the stationary branch-1 inviscidly-unstable waves. Tatro & Mollo-Christensen [126] observed branch-1 modes in the Ekman layer with almost a constant wave angle of 14.6 degrees, and by using the hot wire anemometry technique, with a relatively fast response and greater sensitivity than flow visualization techniques, were able to detect the more rapidly travelling, small wave number disturbances. Wave traces from the hot wire measurements showed a relatively uniform periodic signal at \( Re = 126 \), but as the Reynolds number increased to \( Re = 130 \), the character of the signal was shown to change, where modulation of the wave form gave evidence for the existence of two modes of instability at this Reynolds number. Within this study, the type-1 mode was always detected before any type-2 mode, and unlike the type-2 disturbance, which remained within the boundary layer, the type-1 instability was found
to persist beyond the boundary layer edge at higher Reynolds numbers. Thus, although branch-2 travelling waves can have significantly lower critical Reynolds numbers than stationary waves, it is the latter that are more commonly observed in experiments. Analysis of the relative growth rates of the two modes reveals a more rapid growth associated with the inviscid type-1 mode, which would suggest that even though it is the viscous type-2 mode which first becomes unstable, the growth of the inviscid mode would be sufficiently greater than the former to dominate at higher Reynolds numbers. This could provide some explanation for the difficulty in detecting the type-2 mode at higher Reynolds numbers in experiments. However, it could also be due to experimental measurement techniques that filter out travelling waves (Lilly [83], section 5). Faller [40], did in fact, detect the parallel instability (p572) and noted qualitative features essentially corresponding to Lilly’s numerical results, although, it was observed erratically. Nevertheless, it was considered unlikely to have resulted from inadequacies of the experimental apparatus and control. It became apparent that Faller’s observational technique was inadvertently designed to filter out evidence of the parallel instability region. This same comment applies more forcefully to the results of Gregory et al. [54] on the rotating disk, since the china clay technique is only sensitive to stationary bands.

The laminar Ekman layer has similarities with the Von Karman boundary layer (steady axisymmetric incompressible flow due to an infinite disk rotating in a still fluid; Von Karman 1921). Whilst studying the stability of the Von Karman boundary layer, Lingwood [85] discovered an absolute instability produced by a coalescence of the inviscidly-unstable mode and a third mode that is spatially damped and inwardly propagating. This absolute instability was then confirmed experimentally by Lingwood [87], whom suggested that this mechanism is responsible for the onset of non-linear behaviour and laminar-turbulent transition. This discovery along with the general similarities between the Von Karman and Ekman boundary layer, promoted Lingwood [86] to investigate the types of instability modes for the laminar Ekman layer. By following the work of Briggs [15] and Bers [8] in the field of plasma physics, Lingwood observed that absolute instability can be identified by singularities in the dispersion relationship that occurs when modes associated with wave propagation in opposite directions coalesce. Such points have become known as pinch points. The existence of this absolute instability has been confirmed by the numerical simulations of Davies and Carpenter [36].
16.2 Outline of Part V

No work known to date has been done experimentally or numerically on the analysis of instability of the Ekman boundary layers interaction with a compliant coating. The work reviewed in the preceding sections indicates that the eigenvalue spectrum for the Ekman layer stability problem is considerably complex, with up to three families of eigensolutions present for the case of a rigid wall. The introduction of a compliant boundary greatly adds to this with an exuberant number of possible eigenmodes. As well as the modified versions of the rigid wall instabilities, there is the possibility of the onset of TWF and divergence, and then of course any modal interactions between these eigensolutions. It would, therefore, be an enormous task to investigate all of these forms of instability and so a decision has to be made on which area to focus our attention.

Throughout this part, interest lies in the analysis of the viscous, type-2 mode of instability generated by an Ekman boundary layer. The effects of the type of compliant wall modelled by Carpenter & Garrad [23] on this parallel disturbance will be investigated. The structure of this part is as follows. A brief description of the linear Ekman solution of the Navier-Stokes equation for the mean flow is given below followed by a discussion on the derivation of the coupled disturbance equations. The classical non-dimensionalization technique of Lilly [83] is rejected and instead a new generalized method relative to a rotating coordinate system is used, allowing the dimensionless Rossby number to be incorporated in the generalized non-dimensionalization parameters. The numerical framework described in chapter 7 is then extended in order to create an induced system on $\Lambda'(c^6)$. Asymptotic boundary conditions are then derived and the initial vector is generated for the numerical stability analysis of the Ekman layer problem. Curves of neutral stability are produced for the rigid wall case. Finally, the effects of the modified plate-spring model in three-dimensions on the stability characteristics of the Ekman flow are discussed. This model for the compliant surface is used since it is relatively simple to implement computationally.

16.3 Ekman layer solution

In 1905, Ekman [39] considered the flow in terms of a constant eddy viscosity. However, the analysis is directly applicable to laminar flows by replacing the eddy viscosity with a
constant dynamic viscosity. Furthermore, the analysis can be applied to a layer above a rigid boundary (atmospheric flow above earth’s surface) as well as below a free surface.

Given a large body of fluid at rest relative to a uniformly rotating boundary set in motion by a uniform pressure gradient (modified to incorporate the effects of gravity and centrifugal forces) which is then balanced by the Coriolis force. If the pressure gradient lies in the \((x^*, y^*)\)-plane (with the plane rotating about the \(x^*\)-axis at \(\Omega^*\) and the asterisks denote dimensional quantities) with components \((P_{x^*}, 0)\), then the equations giving the mean velocity components \((U^*, V^*)\) in the Ekman layer near the boundary, \(z^* = 0\), are

\[
-2\Omega^* U^* = \frac{P_{x^*}}{\rho^*} + \nu^* \frac{\partial^2 V^*}{\partial z^*^2},
\]

\[
2\Omega^* V^* = \nu^* \frac{\partial^2 U^*}{\partial z^*^2},
\]

where \(\rho^*\) and \(\nu^*\) are fluid density and kinematic viscosity, respectively [86]. The boundary conditions are no-slip at \(z^* = 0\) and the geostrophic velocity for large \(z^*\) is \(U^*(z^* \to \infty) \equiv U^*_\infty = -P_{x^*}/(2\rho^*\Omega^*)\). The following section will derive the non-dimensional analytic solution for the Ekman layer.

### 16.3.1 Ekman solution

The Ekman solution shall be derived using complex analysis. Let us impose the mean flow as

\[
U^* = U^*_\infty + u
\]

\[
V^* = v.
\]

Substituting these expressions into equations (16.1) and (16.2) we obtain

\[
-2\Omega^* u = \nu^* V^*''
\]

\[
2\Omega^* v = \nu^* U^*''.
\]

where prime denotes differentiation with respect to \(z\). Now let us define a complex variable, \(W = v + iu = V^* + iU^* - iU^*_\infty\), then

\[
\nu^* W'' = \nu^* V^*'' + i\nu^* U^*'' = 2i\Omega^*(v + iu)
\]

\[
\Rightarrow W'' = \frac{2i\Omega^*}{\nu^*} W.
\]
The general solution of equation (16.7) is of the form

\[ W = A \cos \sqrt{\frac{2i\Omega^*}{\nu^*}} z + B \sin \sqrt{\frac{2i\Omega^*}{\nu^*}} z, \quad (16.8) \]

where the constants \( A \) and \( B \) are to be found from the boundary conditions of the Ekman system. At the wall, \( z = 0 \), we have the no-slip condition, thus, \( U^*(0) = V^*(0) = 0 \). Using this boundary condition yields

\[ W(0) = A = V^*(0) + iU^*(0) - iU^*_\infty \]

\[ \Rightarrow A = -iU^*_\infty. \]

Re-writing the expression, \( \sqrt{\frac{2i\Omega^*}{\nu^*}} \) as \( (1 + i)\sqrt{\frac{\Omega^*}{\nu^*}} \), let us write the general solution (16.8) in terms of exponentials such that

\[ W(z^*) = e^{-\frac{z^*}{L}} \left[ e^{i\frac{z^*}{L}} \left( A + i U^* \right) + e^{-i\frac{z^*}{L}} \left( A - i U^* \right) \right], \quad (16.9) \]

where \( L = \sqrt{\frac{\Omega^*}{\nu^*}} \) is a length scale. The second term in equation (16.9) produces an unbounded solution as \( z^* \rightarrow \infty \). A requirement of our system is that the solution be bounded, thus we must eliminate this term by ensuring that \( \frac{A}{2} = B = \frac{B}{2i} \), that is

\[ Ai = B \Rightarrow B = U^*_\infty, \]

from \( A = -iU^*_\infty \). Thus, the solution of \( W(z^*) \) can be simplified to obtain

\[ W(z^*) = -iU^*_\infty e^{i\frac{z^*}{L}} e^{-\frac{z^*}{L}}, \quad (16.10) \]

that is,

\[ W(z^*) = v + iu = U^*_\infty e^{-\frac{z^*}{L}} \left( \sin \frac{z^*}{L} + i \cos \frac{z^*}{L} \right). \]

From this, we can extract out the real and imaginary parts to form \( v \) and \( u \) respectively. Hence,

\[ v = U^*_\infty e^{-\frac{z^*}{L}} \sin \frac{z^*}{L}, \quad (16.11) \]

\[ u = -U^*_\infty e^{-\frac{z^*}{L}} \cos \frac{z^*}{L}. \quad (16.12) \]

Finally, by substituting equations (16.11) and (16.12) into equations (16.3) and (16.4), and non-dimensionalizing, we obtain the non-dimensional analytic solution of the Ekman
where $z = \frac{r}{L}$ and $L = (\frac{\rho^*}{\mu^*})^{\frac{1}{2}}$. $U$ and $V$ are the parallel and perpendicular components of the flow respectively. Figure 16.2 shows the mean velocity components plotted against the axial coordinate and figure 16.3 shows the mean velocity components plotted as an Ekman spiral.

Figure 16.2: Mean velocity profiles for the Ekman layer flow

Figure 16.3: Mean velocity profiles plotted as an Ekman profile

The velocity near $z = 0$ is linear in $z$ and inclined at 45 degrees in a clockwise direction from the direction of the body force due to the applied pressure gradient. The Reynolds number for the flow takes the form $Re = \frac{U^* L}{\nu^*}$.
16.4 Three dimensional instability

There are some subtle differences that arise in the linear stability of three-dimensional disturbances in comparison with the two-dimensional case. The general form of the three-dimensional disturbance in Cartesian coordinates can be written as

$$F(x, y, z, t) = \tilde{F}(z)\exp\{i(\alpha x + \beta y - \omega t)\} + c.c,$$

where $\alpha$ and $\beta$ are wave-numbers in the $x$ and $y$ directions respectively, and $\omega$ is the disturbance frequency.

For a temporal approach to the problem, both $\alpha$ and $\beta$ are taken to be real and the frequency becomes the complex eigenvalue, $c = i\omega$ is to be determined. In general, the eigenvalue problem takes the form

$$\mathcal{F}(c, \alpha, \beta) = 0,$$

where $\mathcal{F}$ is a complex-valued function. The calculation of temporal eigenvalues is the most fundamental approach to the problem.
Consider a boundary layer flow over a flat plate extending infinitely in the stream-wise, \( x \), and span-wise, \( y \), directions with the boundary conditions on \([0, \infty)\), which is subjected to rotation about the vertical, \( z \) axis at a constant dimensional rotation rate, \( \Omega^* \). The motion of the fluid in this system is governed by the Navier-Stokes equations together with the continuity equation which enforces the condition of fluid incompressibility. Now, the majority of flow problems are attacked in a fixed, or inertial, coordinate system. However, for the geophysical boundary-layer on a rotating earth, we may wish to use non-inertial coordinates moving with the accelerating system.

17.1 Non-inertial coordinate systems

Newton's second law of motion, \( \mathbf{F} = m \mathbf{a} \), is only valid if \( \mathbf{a} \) is the absolute acceleration of the particle relative to inertial coordinates, thus this law must be modified for the rotating coordinate system.

Suppose that \((X, Y, Z)\) are in the inertial frame and that our chosen coordinates \((x, y, z)\) are translating and rotating relative to that frame. Let \( \mathbf{R}^* \) and \( \Omega^* \) be the displacement and angular velocity vectors of the \((x, y, z)\) system relative to \((X, Y, Z)\). Then by vector calculus, Greenwood [53] states that we can relate the absolute acceleration, \( \mathbf{a}^* \), of a particle to its displacement \( \mathbf{x}^* \) and velocity, \( \mathbf{u}^* \), relative to the moving system in the following way,

\[
\mathbf{a}^* = \frac{d^2 \mathbf{R}^*}{d \mathbf{t}^*} + \frac{\partial \mathbf{R}^*}{\partial \mathbf{t}^*} \times \mathbf{x}^* + \mathbf{u}^* \times (\Omega^* \times \mathbf{x}^*) + \frac{d \mathbf{u}^*}{d \mathbf{t}^*} + 2 \Omega^* \times \mathbf{u}^*,
\]

where \( \frac{d \mathbf{R}^*}{d \mathbf{t}^*} \approx 0 \), and \( \frac{\partial \mathbf{R}^*}{\partial \mathbf{t}^*} \times \mathbf{x}^* \approx 0 \) for the earth. Thus, the extra terms arising as a result of the rotation are the Coriolis acceleration, \( \Omega^* \times \mathbf{u}^* \) and the centripetal acceleration,
\( \mathbf{\Omega}^* \times (\mathbf{\Omega}^* \times \mathbf{x}^*) \), with \( \mathbf{x}^* \) being the dimensional position vector and \( \mathbf{\Omega}^* = [0, 0, \mathbf{\Omega}]^T \), and the asterisk represents dimensionality. Thus, if \( \mathbf{u} \) is a non-inertial velocity vector, then the entire right hand side of (17.1) must replace the derivative \( \frac{\partial \mathbf{u}}{\partial t} \) in the Navier-Stokes equations. Thus, the Navier-Stokes equations are written in the rotating frame so that the velocity vector, \( \mathbf{u}^* \) and the pressure, \( p^* \) in that frame satisfy

\[
\frac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{u}^* + 2\mathbf{\Omega}^* \times \mathbf{u}^* + \mathbf{\Omega}^* \times (\mathbf{\Omega}^* \times \mathbf{x}^*) = -\frac{1}{\rho} \nabla p^* + \nu \nabla^2 \mathbf{u}^*,
\]

(17.2)

\[
\nabla \cdot \mathbf{u}^* = 0,
\]

(17.3)

where \( \rho \) is the fluid density and \( \nu \) is the kinematic viscosity. Now, we can simplify this equation by introducing a modified pressure, \( p_m \), which incorporates the centripetal term by using the following vector relation:

\[
\nabla \times (\mathbf{\Omega}^* \times \mathbf{x}^*) = -\nu \nabla \times (\mathbf{\Omega}^* \times \mathbf{x}^*) = 0. 
\]

The pressure can then be redefined as

\[
p_m = p - \frac{1}{2} \rho (\mathbf{\Omega}^* \times \mathbf{x}^*)^2.
\]

Hereafter, the subscript on \( p_m \) will be dropped, and any further expressions involving \( p \) will refer to the modified pressure. Using this expression we obtain

\[
\frac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{u}^* + 2\mathbf{\Omega}^* \times \mathbf{u}^* = -\nabla p^* + \nu \nabla^2 \mathbf{u}^*,
\]

(17.4)

\[
\nabla \cdot \mathbf{u}^* = 0.
\]

(17.5)

The boundary conditions arise from the requirement of zero velocity at the wall. That is, if \( \mathbf{u} = [u, v, w]^T \) then

\[
u = v = w = 0 \text{ at } z = 0.
\]

(17.6)

Also, the flow at infinity is represented by the Ekman spiral which satisfies

\[
\lim_{z \to \infty} U(z) = U_\infty \text{ and } \lim_{z \to \infty} V(z) = 0,
\]

(17.7)

where \( U_\infty \) is the speed in the free-stream.

The full, nonlinear problem described by equations (17.4), (17.5), (17.6) & (17.7) has an exact solution which defines the mean state. This mean flow field is obtained from
Ekman's exact solution to the Navier-Stokes equations and the mean velocity components are written

\[ u(x^*, y^*, z^*, t^*) = U(z^*) = 1 - e^{-z^*} \cos z^*, \]  
\[ v(x^*, y^*, z^*, t^*) = V(z^*) = e^{-z^*} \sin z^*, \]  
\[ w(x^*, y^*, z^*, t^*) = 0. \]

Below, I will describe two different formulations for non-dimensionalizing the Navier-Stokes equations in a rotating frame. The differences occur in the choice for the length, velocity, pressure and time scales.

17.2 Governing stability equations - Non-dimensionalizing the Ekman problem


The second section is based on a variation on an approach proposed by James P. Vanyo [132] on page 147 of 'Rotating Fluids in Engineering and Science', and it will be argued that this is the more general and thus, more appropriate form of non-dimensionalization.

17.3 Classical Ekman non-dimensionalization (Lilly, 1966)

Starting with the Navier-Stokes equations, in the form of (17.4), we assume the Ekman steady state mean flow and introduce a perturbation to the Ekman boundary layer solution,

\[ u^* = U^* + \hat{u}^* \quad \text{and} \quad p^* = P^* + \hat{p}^*. \]

The perturbed flow field is substituted into the Navier-Stokes equations and linearization about the mean flow field with respect to the perturbations is performed to yield the following set of linear equations for \( \hat{u}^* \) and \( \hat{p}^* \):

\[ \frac{\partial \hat{u}^*}{\partial t^*} + U^* \cdot \nabla \hat{u}^* + \hat{u}^* \cdot \nabla U^* + \nabla \hat{p}^* + 2\Omega^* \hat{k} \times \hat{u}^* = \nu \nabla^2 \hat{u}^* \]

\[ \nabla \cdot \hat{u}^* = 0, \]
where $U^*$ is the Ekman velocity field given in equations (17.8) to (17.10) above. Boundary conditions appropriate to the rigid wall problem are that the perturbation velocities vanish at the wall ($z = 0$) and at $z = \infty$.

We now make all terms dimensionless by choosing scales

$$D = \left( \frac{\nu}{\Omega} \right)^{\frac{1}{2}} \text{ (Length scale)}, \quad \nu = \text{ (Velocity scale)}, \quad Re = \frac{v_{bg}D}{\nu},$$

and time scale $= \frac{\text{length scale}}{\text{velocity scale}}$ (cf. Lilly [83]). In this formulation, the same length scale is used for the horizontal and vertical directions.

A critical issue here is the definition of $v_g$. In Lilly’s paper, $v_g$ is defined to be the geostrophic velocity, associated with the inviscid flow in the farfield, which satisfies

$$-2\Omega u_I = \frac{1}{\rho} \frac{\partial p_I}{\partial x} \quad \text{and} \quad 2\Omega u_I = \frac{1}{\rho} \frac{\partial p_I}{\partial y},$$

and so

$$u_I = -\frac{1}{2\rho\Omega} \frac{\partial p_I}{\partial y}, \quad v_I = \frac{1}{2\rho\Omega} \frac{\partial p_I}{\partial x},$$

where the subscripts, $I$ indicate inviscid flow. Suppose $\nabla p_I$ is a constant with magnitude $P_I$, then Lilly [83] defines $v_g$ by

$$v_g = \frac{1}{2\rho\Omega} P_I.$$

Thus, it appears that Lilly assumes $v_g$ to be a given constant.

Proceeding with the above characteristic scales leads to the following dimensionless variables,

$$u = \frac{\hat{u}^*}{v_g}, \quad v = \frac{\hat{v}^*}{v_g}, \quad w = \frac{\hat{w}^*}{v_g}, \quad U = \frac{U^*}{v_g}, \quad V = \frac{V^*}{v_g},$$

and

$$x = \frac{x^*}{D}, \quad y = \frac{y^*}{D}, \quad z = \frac{z^*}{D}, \quad t = \frac{t^* v_g}{D}, \quad p = \frac{p^*}{\rho v_g^2}.$$

Note also that $U(z) = (U(z), V(z), 0)$ and $u^*(x, y, z, t) = (u^*, v^*, w^*)$. So, for the first component of the Navier-Stokes equations we obtain

$$u_t + \frac{v_g^2}{D} u_x + U v_g \frac{u_g v_g}{D} + V v_g \frac{v_g y}{D} + w v_g \frac{v_g z}{D} + p_x - 2 \frac{\nu}{D^2} v_g v_g - \frac{v_g^2}{D^2} \nabla^2 u.$$

Multiplying by $D$ and dividing by $v_g^2$ gives
Part V: Rotating boundary layer

\[ u_t + U u_x + V u_y + w U_z + p_x - \frac{2 \nu}{\nu} \frac{D u}{D y} = -\frac{\nu}{D y} \nabla^2 u. \]

Now, \( Re = \frac{\nu D}{\nu} \), so, substituting this in we obtain

\[
\begin{align*}
\boxed{u_t + U u_x + V u_y + w U_z + p_x - \frac{2 \nu}{Re} \frac{D u}{D y} = \frac{1}{Re} \nabla^2 u.}
\end{align*}
\]

Similarly, we obtain

\[
\begin{align*}
\boxed{v_t + U v_x + V v_y + w V_z + p_y + \frac{2 \nu}{Re} \frac{D v}{D x} = \frac{1}{Re} \nabla^2 v}
\end{align*}
\]

and

\[
\begin{align*}
\boxed{w_t + U w_x + V w_y + p_z = \frac{1}{Re} \nabla^2 w.}
\end{align*}
\]

with the continuity equation

\[ u_x + v_y + w_z = 0, \]

where \( \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \). Non-dimensionalizing the Ekman mean velocity field we obtain

\[ U(z) = 1 - e^{-z} \cos z \quad \text{and} \quad V(z) = e^{-z} \sin z, \]

where \( z \) is the dimensionless vertical coordinate.

Equations (17.11), (17.12) and (17.13) are linear equations, and are equivalent to those equations obtained after non-dimensionalizing and linearizing the Navier-Stokes equations in part II. As in the Orr-Sommerfeld equation derivation, we look for solutions taking the form of normal modes so that

\[
\begin{align*}
\begin{pmatrix}
 u(x, y, z, t) \\
 v(x, y, z, t) \\
 w(x, y, z, t) \\
 p(x, y, z, t)
\end{pmatrix}
 = \text{Real}
\begin{pmatrix}
 \hat{u}(z) \\
 \hat{v}(z) \\
 \hat{w}(z) \\
 \hat{p}(z)
\end{pmatrix}
 e^{i(\alpha x + \beta y - \omega t)} + \text{c.c.},
\end{align*}
\]

that is, the velocity and pressure depend only on the \( z \)-direction. Note that by taking the form in (17.14) for the general solution, we cover all possible bounded solutions by varying real \( \alpha \) and \( \beta \) in the \( x \) and \( y \) directions.
Substitution of this disturbance form into equations (17.11)-(17.13) gives the following non-dimensional governing stability equations.

\[
\begin{align*}
    i(\alpha U + \beta V - \omega) \hat{u} + i\alpha \hat{\beta} - \frac{2\theta}{Re} &= \frac{1}{Re} \hat{u}_{zz} - \frac{1}{Re} (\alpha^2 + \beta^2)\hat{u}, \\
    i(\alpha U + \beta V - \omega) \hat{v} + i\beta \hat{\beta} + \frac{2\hat{\theta}}{Re} &= \frac{1}{Re} \hat{v}_{zz} - \frac{1}{Re} (\alpha^2 + \beta^2)\hat{v}, \\
    i(\alpha U + \beta V - \omega) \hat{w} + \hat{p}_z &= \frac{1}{Re} \hat{w}_{zz} - \frac{1}{Re} (\alpha^2 + \beta^2)\hat{w},
\end{align*}
\]

with the continuity equation as

\[
i\alpha \hat{u} + i\beta \hat{v} + \hat{w}_z = 0.
\]

Now, we want to eliminate the pressure, \( \hat{p} \), from equations (17.15), (17.16) and (17.17), since we don't have straightforward boundary conditions for this variable. This can be achieved through the introduction of a new variable, \( \gamma = (\alpha^2 + \beta^2)^{\frac{1}{2}} \), that is, the modulus of the wavenumber. Let us introduce new coordinates

\[
\begin{align*}
    \tilde{U} = \frac{\beta}{\gamma} U - \frac{\alpha}{\gamma} V, \\
    \tilde{V} = \frac{\alpha}{\gamma} U + \frac{\beta}{\gamma} V, \\
    \tilde{\psi} = \frac{\beta}{\gamma} \hat{u} - \frac{\alpha}{\gamma} \hat{v}, \\
    \tilde{\psi} = \frac{\alpha}{\gamma} \hat{u} + \frac{\beta}{\gamma} \hat{v},
\end{align*}
\]

where the ratios of \( \alpha \) and \( \beta \) to \( \gamma \) are dimensionless. By introducing these coordinates, we are essentially rotating the velocity fields in the horizontal plane. This can be seen by substituting \( \beta = \gamma \cos \varepsilon \) and \( \alpha = -\gamma \sin \varepsilon \) where \( \alpha \) and \( \beta \) are the wave numbers in the \( x \) and \( y \)-directions respectively, and \( \varepsilon \) is the rotation angle. Let us also introduce the wave speed \( c = \frac{\psi}{\gamma} \). Substituting these new variables and coordinates into equations (17.15), (17.16) and (17.17) we obtain the following set of equations:

\[
\begin{align*}
    i\gamma(\tilde{V} - c) \hat{u} + \hat{w}_z + i\alpha \hat{\beta} - \frac{2\hat{\theta}}{Re} &= \frac{1}{Re} \hat{u}_{zz} - \frac{1}{Re} \gamma^2 \hat{u}, \\
    i\gamma(\tilde{V} - c) \hat{v} + \hat{w}_z + i\beta \hat{\beta} + \frac{2\hat{\theta}}{Re} &= \frac{1}{Re} \hat{v}_{zz} - \frac{1}{Re} \gamma^2 \hat{v}, \\
    i\gamma(\tilde{V} - c) \hat{w} + \hat{p}_z &= \frac{1}{Re} \hat{w}_{zz} - \frac{1}{Re} \gamma^2 \hat{w},
\end{align*}
\]

with \( i\gamma \hat{v} + \hat{w}_z = 0 \) from the continuity equation.

Multiply (17.18) by \( \beta \) and (17.19) by \( \alpha \) and subtract to obtain

\[
i\gamma^2(\tilde{V} - c) \tilde{u} + \gamma \hat{w}_z - \frac{2\hat{\theta}}{Re} = \frac{1}{Re} \gamma \hat{v}_{zz} - \frac{1}{Re} \gamma^3 \hat{u}.
\]

Now, let us multiply (17.18) by \( \alpha \) and (17.19) by \( \beta \) and add, to obtain

\[
i\gamma^2(\tilde{V} - c) \tilde{v} + \gamma \hat{w}_z + \gamma^2 \hat{\theta} + \frac{2\hat{\theta}}{Re} = \frac{1}{Re} \gamma \hat{v}_{zz} - \frac{1}{Re} \gamma^3 \hat{v}.
\]
Part V: Rotating boundary layer

Working with (17.21), we eliminate $\tilde{v}$ by using the continuity equation, $i\gamma \tilde{v} + \tilde{w}_z = 0$, obtaining

$$-\frac{\gamma^3}{Re} \tilde{u} + \gamma \tilde{u}_{zz} - i\gamma^2 (\tilde{V} - c) \tilde{u} - \tilde{w}\gamma \tilde{U}_z + \frac{2i}{Re} \tilde{w}_z = 0.$$  \hspace{1cm} (17.23)

We can now combine (17.20) and (17.22) to eliminate $\tilde{p}$ and also use the continuity equation, $\tilde{u} = \frac{i}{\gamma} \tilde{w}_z$. From (17.22) we have

$$\gamma^2 \tilde{p} = \frac{\gamma^3}{Re} \tilde{w}_z + \frac{i}{Re} \tilde{w}_{zzz} - \frac{2\gamma}{Re} \tilde{u} - \tilde{w}\gamma \tilde{V}_z + \gamma (\tilde{V} - c) \tilde{w}_z .$$

Differentiating with respect to $z$ gives

$$\dot{\tilde{p}}_z = \frac{1}{Re} \tilde{w}_{zz} + \frac{1}{\gamma^2 Re} \tilde{w}_{zzzz} + \frac{2i}{\gamma^2 Re} \tilde{u}_z + \frac{i}{\gamma} \tilde{w}_z \tilde{V}_{zz} - \frac{i}{\gamma} (\tilde{V} - c) \tilde{w}_{zz} .$$ \hspace{1cm} (17.24)

Substituting (17.24) into (17.17) and using $\phi = \tilde{w}$ and $\psi = i\gamma \tilde{u}$, in the formed equation and also using equation (17.23), we arrive at the following generalization of the Orr-Sommerfeld equation in the form of two coupled equations:

$$\phi'''' - b(z) \phi'' - a(z) \phi + 2\psi' = 0$$  \hspace{1cm} (17.25)

$$\psi'' + (\gamma^2 - b(z)) \psi - i\gamma Re \dot{U}' \phi - 2\phi' = 0$$  \hspace{1cm} (17.26)

where

$$a(z) = -\gamma^4 - i\gamma^3 Re (\tilde{V} - c) - i\gamma Re \tilde{V}_{zz}$$

$$b(z) = 2\gamma^2 + i\gamma Re (\tilde{V} - c) .$$

These are the equations studied by Lilly [83]. Now, Lilly introduces the angular velocity, $\Omega$, within his geostrophic velocity term, used to non-dimensionalize the system of equations. However, he appears to set this equal to a constant thus eliminating the introduction of another dimensionless variable other than the Reynolds number. In the following non-dimensionalization method, we follow Vanyo [132] and make a change so as to allow for variations in the rotation rate.
17.4 Non-dimensionalizing the N-S equations with emphasis on the Rossby and Ekman numbers

Taking the Navier-Stokes equation relative to a rotating frame as follows:

\[
\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{U}^* \cdot \nabla \mathbf{u}^* + \mathbf{u}^* \cdot \nabla \mathbf{U}^* + 2 \Omega \mathbf{k} \times \mathbf{u}^* = -\frac{1}{\rho} \nabla \rho^* + \nu \nabla^2 \mathbf{u}^*.
\]

All quantities are defined in, and expressed as components in the rotating frame, with angular velocity, \( \Omega \), and \( \mathbf{k} \) is the unit vector in the \( z \) direction. The velocity vector, \( \mathbf{u}^* \) has components \((u^*, v^*, w^*)\) and the mean velocity has the form \( \mathbf{U}^* = (U^*, V^*, 0) \).

For the non-dimensionalization I shall an arbitrary length scale, \( L \). Thus, both the horizontal and vertical length scales shall be non-dimensionalized by this value, which although arbitrary, examples could include: \( L = \frac{U_{\infty}}{\Omega} \), or \( L = (\frac{v_{\infty}}{\Omega})^{1/2} \). Time is to be made dimensionless by \( \frac{1}{\Omega} \), which assumes that the equation will be used when rotational phenomena dominate the solution. Pressure is to be made dimensionless by \( \rho \Omega U_{\infty} L \). Thus the following dimensionless variables are introduced:

\[
\begin{align*}
    u &= \frac{u^*}{v_g}, & v &= \frac{v^*}{v_g}, & w &= \frac{w^*}{v_g}, \\
    x &= \frac{x^*}{L}, & y &= \frac{y^*}{L}, & z &= \frac{z^*}{L}, \\
    t &= t^* \Omega, & p &= \frac{p^*}{\rho \Omega v_g L}, \\
    U &= \frac{U^*}{v_g}, & V &= \frac{V^*}{v_g}.
\end{align*}
\]

With these definitions, an explanation is required. The reason for non-dimensionalizing pressure using \( L \) in the form of the horizontal length scale, \( L = \frac{U_{\infty}}{\Omega} \). This is because the pressure gradient for the mean flow depends only on the horizontal direction. By introducing this non-dimensionalization for pressure our equations can be simplified. Writing out the components of the governing dimensional equations,

\[
\begin{align*}
    \frac{\partial u^*}{\partial t^*} + U^* u^* + V^* u^*_y + w^* U^*_z - 2 \Omega v^* &= -\frac{1}{\rho} p^*_x + \nu \nabla^2 u^* & (17.27) \\
    \frac{\partial v^*}{\partial t^*} + U^* v^* + V^* v^*_y + w^* V^*_z + 2 \Omega u^* &= -\frac{1}{\rho} p^*_y + \nu \nabla^2 v^* & (17.28) \\
    \frac{\partial w^*}{\partial t^*} + U^* w^* + V^* w^*_y + w^* w^*_z &= -\frac{1}{\rho} p^*_z + \nu \nabla^2 w^* & (17.29)
\end{align*}
\]

where \( \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \).
Part V: Rotating boundary layer

From (17.27) and substituting for the dimensional parameters we obtain
\[
\begin{align*}
    u_t + \frac{v_0}{\Omega L} u_x + \frac{v_0}{\Omega L} V u_y + \frac{v_0}{\Omega L} w U_z - 2v + p_x = \frac{\nu}{\Omega L^2} u_{xx} + \frac{\nu}{\Omega L^2} u_{yy} + \frac{\nu}{\Omega L^2} u_{zz},
\end{align*}
\] (17.30)

Now, if we let the Rossby number be defined as \( R_o = \frac{\omega L}{v} \) and the Ekman number defined by \( E_k = \frac{\nu}{\Omega L^2} \) then we obtain
\[
\begin{align*}
    u_t + R_o U u_x + R_o V u_y + R_o w U_z - 2v + p_x = E_k (u_{xx} + u_{yy} + u_{zz}).
\end{align*}
\] (17.31)

Similarly, by substitution of the non-dimensional equations into (17.28) and (17.29) we obtain the following two equations:
\[
\begin{align*}
    v_t + R_o U v_x + R_o V v_y + R_o w V_z + 2u + p_y = E_k (v_{xx} + v_{yy} + v_{zz})
\end{align*}
\] (17.32)
\[
\begin{align*}
    w_t + R_o U w_x + R_o V w_y + p_z = E_k (w_{xx} + w_{yy} + w_{zz}),
\end{align*}
\] (17.33)

with the equation of continuity defined to be
\[
\begin{align*}
    u_x + v_y + w_z = 0.
\end{align*}
\]

The above set of equations can be rearranged to give
\[
\begin{align*}
    u_t + R_o (U u_x + V u_y + w U_z) - 2v + p_x = E_k \nabla^2 u
\end{align*}
\] (17.34)
\[
\begin{align*}
    v_t + R_o (U v_x + V v_y + w V_z) + 2u + p_y = E_k \nabla^2 v
\end{align*}
\] (17.35)
\[
\begin{align*}
    w_t + R_o (U w_x + V w_y) + p_z = E_k \nabla^2 w,
\end{align*}
\] (17.36)

where \( R_o = \frac{\omega L}{v} \) is the Rossby number and is the ratio of convective to Coriolis accelerations. \( E_k = \frac{\nu}{\Omega L^2} \) is the Ekman number and is the ratio of viscous to Coriolis accelerations.

Then the Reynolds number is the ratio \( \frac{R_o}{E_k} \) as follows:
\[
\begin{align*}
    R_e = \frac{R_o}{E_k} = \frac{UL^2}{ \Omega L} \frac{\nu}{\nu} = \frac{UL}{\nu}.
\end{align*}
\]

Equations (17.34), (17.35) and (17.36) are linear equations. Let us look for solutions taking the normal-mode form
\[
\begin{align*}
    \begin{pmatrix}
    u(x, y, z, t) \\
    v(x, y, z, t) \\
    w(x, y, z, t) \\
    p(x, y, z, t)
    \end{pmatrix}
    = \text{Real} \begin{pmatrix}
    \hat{u}(z) \\
    \hat{v}(z) \\
    \hat{w}(z) \\
    \hat{p}(z)
    \end{pmatrix} e^{i(\alpha x + \beta y - \omega t)},
\end{align*}
\] (17.37)
thus covering all possible bounded solutions by varying real $\alpha$ and $\beta$ in the $x$ and $y$ directions.

Substituting equation (17.37) into (17.34) we obtain

\[
'i(R_0 U \alpha + R_0 V \beta - \omega)\dot{u} + R_0 \dot{w} U_z + i\alpha \dot{\phi} - 2\theta = E_k \ddot{u}_{zz} - E_k(\alpha^2 + \beta^2)\dot{u}.
\]  

(17.38)

Similarly, by substitution of (17.37) into (17.35) and (17.36) we obtain

\[
'i(R_0 U \alpha + R_0 V \beta - \omega)\dot{\theta} + R_0 \dot{w} V_z + i\beta \dot{\phi} + 2\dot{\theta} = E_k \dot{\theta}_{zz} - E_k(\alpha^2 + \beta^2)\dot{\theta}.
\]  

(17.39)

\[
'i(R_0 U \alpha + R_0 V \beta - \omega)\dot{\phi} + \dot{p}_z = E_k \dot{\phi}_{zz} - E_k(\alpha^2 + \beta^2)\dot{\phi}.
\]  

(17.40)

respectively, with the continuity equation as

\[
i\alpha \dot{u} + i\beta \dot{\theta} + \dot{w} = 0.
\]

Let us now introduce new coordinates

\[
\tilde{U} = \frac{\beta}{\gamma} U - \frac{\alpha}{\gamma} V, \quad \tilde{V} = \frac{\alpha}{\gamma} U + \frac{\beta}{\gamma} V, \quad \tilde{u} = \frac{\beta}{\gamma} \dot{u} - \frac{\alpha}{\gamma} \dot{\theta}, \quad \tilde{v} = \frac{\alpha}{\gamma} \dot{u} + \frac{\beta}{\gamma} \dot{\theta}.
\]

where since the ratios of $\alpha$ and $\beta$ to $\gamma$ are dimensionless, we can use dimensionless values for $\alpha$, $\beta$ and $\gamma$ separately. Thus, we have the following set of equations

\[
i\gamma(R_0 \tilde{V} - c)\tilde{u} + R_0 \dot{w} U_z + \alpha \dot{\phi} - 2\tilde{\theta} = E_k \ddot{\tilde{u}}_{zz} - E_k \gamma^2 \tilde{u}
\]  

(17.41)

\[
i\gamma(R_0 \tilde{V} - c)\tilde{\theta} + R_0 \dot{w} V_z + \beta \dot{\phi} + 2\tilde{\theta} = E_k \dot{\tilde{\theta}}_{zz} - E_k \gamma^2 \tilde{\theta}
\]  

(17.42)

\[
i\gamma(R_0 \tilde{V} - c)\tilde{\phi} + \dot{p}_z = E_k \dot{\tilde{\phi}}_{zz} - E_k \gamma^2 \dot{\tilde{\phi}}
\]  

(17.43)

with $i\gamma \tilde{v} + \dot{w} = 0$ from the continuity equation.

After some similar manipulation to eliminate the pressure terms, as carried out for the classical case, these equations reduce to the Orr-Sommerfeld type form

\[
\phi''' - b(z)\phi'' - a(z)\phi + \frac{2}{E_k} \psi' = 0,
\]  

(17.44)

\[
\text{Coupling term}
\]

where

\[
a(z) = -\gamma^4 - \frac{i\gamma^3}{E_k} (R_0 \tilde{V} - c) - \frac{i\gamma R_0}{E_k} \tilde{V}_{zz}
\]

\[
b(z) = 2\gamma^2 + \frac{i\gamma}{E_k} (R_0 \tilde{V} - c)
\]

and

\[
\psi'' + (\gamma^2 - b(z))\psi - \frac{2}{E_k} \tilde{U}_z \phi' = 0.
\]  

(17.45)

\[
\text{Coupling term}
\]
17.5 Comparison of the two forms of the equations

From above we have

\[ \psi'' + \left( \gamma^2 - b(z) \right) \psi - \frac{i\gamma R_e \bar{V}_z \psi}{E_k} - \frac{2\psi'}{E_k} = 0 \]  
\[ \text{(17.46)} \]

Coupling term

\[ \phi''' - b(z)\phi'' - a(z)\phi + \frac{2\psi'}{E_k} = 0 \]  
\[ \text{(17.47)} \]

where

\[ a(z) = -\gamma^4 - \frac{i\gamma^3}{E_k} (R_e \bar{V} - c) - \frac{i\gamma R_e \bar{V}^2}{E_k} \]

Coupling term

\[ b(z) = 2\gamma^2 + \frac{i\gamma}{E_k} (R_e \bar{V} - c) \]

From Lilly (1966)

\[ \psi'' + \left( \gamma^2 - b(z) \right) \psi - \frac{i\gamma R_e \bar{V}_z \psi}{E_k} - 2\psi' = 0 \]  
\[ \text{(17.48)} \]

Coupling term

\[ \phi''' - b(z)\phi'' - a(z)\phi + 2\psi' = 0 \]  
\[ \text{(17.49)} \]

where

\[ a(z) = -\gamma^4 - \frac{i\gamma^3}{E_k} R_e (\bar{V} - c) - \frac{i\gamma R_e \bar{V}^2}{E_k} \]

\[ b(z) = 2\gamma^2 + \frac{i\gamma R_e (\bar{V} - c)}{E_k} \]

Comparing the two sets of equations, we notice that equations (17.46) and (17.48) become equal, and equations (17.47) and (17.49) become equal if the two forms of \( a(z) \) and \( b(z) \) agree. If the Ekman number is set to \( E_k = 1.0 \) and the Rossby number is equal to the Reynolds number, \( R_e = R_e \), then the only difference between Lilly’s equations and the equations derived in the previous section occurs in the scaling of wave-speed, \( c \), i.e. \( c_{old} = c_{new}/R_e \) for the coupled equations derived here.

There is still the question of whether the geostrophic velocity can be considered constant or \( \Omega \) dependent.

The boundary conditions at the wall corresponding to the coupled system are

\[ \phi(0) = \phi'(0) = \psi(0) = 0 \]  
\[ \text{(17.50)} \]
The sixth order system of equations retains the effects of the Coriolis acceleration and defines an eigenvalue problem of the form

$$\mathcal{F}(\alpha, \beta, c, R_0, E_k) = 0.$$ 

If the Coriolis effects are neglected, then the equations (17.48) and (17.49) decouple and the stability problem is reduced to a fourth order equation, that is an extension of the usual Orr-Sommerfeld equation

$$\frac{i}{\Re} \left( \phi''' + 2\gamma \phi'' + \gamma^4 \phi \right) + \left( \alpha U + \beta V - \frac{\omega}{R_0} \right) \left( \phi'' - \gamma^2 \phi \right) - \left( \alpha U'' + \beta V'' \right) = 0$$

and a second order equation, termed Squire’s equation

$$\frac{i}{\Re} \left( \psi'' - \gamma^2 \psi \right) + (\alpha U + \beta V) \psi - \frac{\omega}{R_0} \psi = 0.$$
A Numerical linear system and the Compound matrix method

The sixth order system of equations defined in the previous section is solved numerically using an extension to the numerical framework developed in part II of this report. To be consistent with numerical analyses in part IV, the eigenvalue problem will be considered in the temporal manner. The numerical scheme begins by writing our coupled perturbation equations for the Ekman boundary layer stability analysis (17.44) and (17.45), namely

\[
\phi''' - b(z)\phi'' + a(z)\phi + \frac{2}{E_k}\psi' = 0 ,
\]

\[
\psi'' - (b(z) - \gamma^2)\psi - \frac{iR_0}{E_k}\gamma\tilde{U}_z\phi - \frac{2}{E_k}\psi' = 0 ,
\]

where

\[
b(z) = \frac{i\gamma}{E_k}(R_0\tilde{V} - c) + 2\gamma^2
\]

\[
a(z) = \frac{iR_0}{E_k}\gamma\tilde{V}_z + \frac{i\gamma^3}{E_k}(R_0\tilde{V} - c) + \gamma^4
\]

as a sixth order system comprising a set of six first order ordinary differential equations in the following transformed variables.

\[
u_1 = \phi, \quad u_2 = \phi', \quad u_3 = \phi''
\]

\[
u_4 = \phi''', \quad u_5 = \psi, \quad u_6 = \psi'
\]

then, the perturbation equations are defined to be

\[
u_1' = u_2, \quad u_4' = \left( b(z)u_3 - a(z)u_1 - \frac{2}{E_k}u_6 \right)
\]

\[
u_2' = u_3, \quad u_5' = u_6
\]

\[
u_3' = u_4, \quad u_6' = \left( b(z) - \gamma^2 \right) u_5 + \frac{iR_0}{E_k}\gamma\tilde{U}_z u_1 + \frac{2}{E_k}u_2 .
\]
Part V: A numerical linear system and the compound matrix method

Thus, the linear system can be written as

\[ u_z = A(z, \lambda)u, \quad u \in \mathbb{C}^6, \]

where the eigenvalue \( \lambda = -i\gamma c \) and

\[
A(z, \lambda) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-a(z) & 0 & b(z) & 0 & 0 & -\frac{2}{\delta_k} \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{i_R e z}{\delta_k} \tilde{U}_z & \frac{2}{\delta_k} & 0 & 0 & (b(z) - \gamma^2) & 0
\end{bmatrix},
\]

with the coordinate \( z \) representing the vertical direction in the physical problem.

The boundary conditions associated with a rigid surface at \( z = 0 \) are

\[ \phi(0) = \phi'(0) = \psi(0) = 0. \]

Now for the numerical analysis, we wish to reformulate the system incorporating the parameters, \( R_0 \) and \( R_e \) only. That is, we wish to write the Ekman number, \( E_k \) in terms of the Rossby and Reynolds numbers. Now, \( R_0 = \frac{v_p}{\Omega L} \), \( E_k = \frac{\nu}{\Omega L^2} \) and \( R_e = \frac{\nu L}{v} \). So, we can write the Ekman number as

\[ E_k = \frac{R_0}{R_e}. \]

Using the above substitution, the coupled equations and, subsequently, the linear system thus become

\[
\begin{align*}
\phi''' - b(z)\phi'' - a(z)\phi + 2\left(\frac{R_e}{R_0}\right)\psi' &= 0 \\
\psi'' + (\gamma^2 - b(z))\psi - iR_e\gamma\tilde{U}_z\phi - 2\left(\frac{R_e}{R_0}\right)\phi' &= 0,
\end{align*}
\]

where

\[
\begin{align*}
a(z) &= -iR_e\gamma\tilde{V}_{zz} - i\gamma^3R_e\left(\tilde{V} - \frac{c}{R_0}\right) - \gamma^4 \\
b(z) &= i\gamma R_e\left(\tilde{V} - \frac{c}{R_0}\right) + 2\gamma^2,
\end{align*}
\]

with \( \tilde{V} \) defined as

\[
\tilde{V} = \frac{\alpha}{\gamma}U + \frac{\beta}{\gamma}V \\
= \frac{\alpha}{\gamma}(1 - e^{-2z}\cos z) + \frac{\beta}{\gamma}e^{-2z}\sin z.
\]
Substituting out $\alpha$, $\beta$ and $\gamma$ using the following definitions

$$
\alpha = -\gamma \sin \varepsilon, \quad \beta = \gamma \cos \varepsilon,
$$

we obtain

$$
\dot{V} = -\sin \varepsilon (1 - e^{-z} \cos z) + \cos \varepsilon e^{-z} \sin z.
$$

From this, note that

$$
\dot{V}_{zz} = -2e^{-z} \cos(z + \varepsilon),
$$

and $\bar{U}$ is defined as

$$
\bar{U} = \frac{\beta}{\gamma} U - \frac{\alpha}{\gamma} V
$$

Again, using the definitions for $\alpha$ and $\beta$ above, we obtain

$$
\bar{U} = \cos \varepsilon (1 - e^{-z} \cos z) + \sin \varepsilon e^{-z} \sin z,
$$

and from this, note that

$$
\bar{U}_z = e^{-z} (\sin(z + \varepsilon) + \cos(z + \varepsilon)).
$$

Our linear system now takes the form

$$
u_z = A(z, \lambda)u \quad \text{with} \quad u \in \mathbb{C}^6,
$$

where

$$
A(z, \lambda) =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-a(z) & 0 & b(z) & 0 & 0 & -2\frac{R_s}{R_c} \\
0 & 0 & 0 & 0 & 0 & 1 \\
iRe\gamma\bar{U}_z & 2\frac{R_s}{R_c} & 0 & 0 & (b(z) - \gamma^2) & 0
\end{bmatrix}.
$$

For use of the numerical framework, which restricts $u$ onto $\Lambda^3(\mathbb{C}^6)$, we need to convert this $6 \times 6$ matrix, $A(z, \lambda)$, to a $20 \times 20$ matrix. This is calculated effectively with the aid of the program written in Maple in appendix A. The numerics for which were developed in part II for the restriction of $u$ onto $\Lambda^3(\mathbb{C}^4)$ and extended in the next section, where an algorithm is presented for the restriction onto $\Lambda^3(\mathbb{C}^5)$ and for the creation of $A^{(9)}(z, \lambda)$. 
18.1 Compound matrix for restriction to $\wedge^3(\mathbb{C}^6)$

It is straightforward to now transform the sixth order system, \((18.4)\), into the standard form for the numerical framework. The procedure for this new framework on a fourth order system given in Part II shall be modified to create a corresponding procedure for this sixth order problem.

From the preceding section, the system of ODEs can be expressed as a linear system of the form

\[
uz = A(z, \lambda)u \quad u \in \mathbb{C}^6 \quad \text{and} \quad 0 \leq z \leq L, \tag{18.5}
\]

with the three boundary conditions

\[
(e_1, u(0, \lambda)) = (e_2, u(0, \lambda)) = (e_5, u(0, \lambda) = 0,
\]

where $e_j$ is the standard unit vector in $\mathbb{C}^6$, and $L$ is some acceptably large fixed value approximating the upper boundary at infinity. To form the compound matrix algorithm, the natural space to integrate this system is $\wedge^3(\mathbb{C}^6)$ which has dimension 20. We proceed by introducing any orthonormal basis for $\mathbb{C}^6$, for example, $e_1, \ldots, e_5$ and by implementing the standard lexically ordered basis for $\wedge^3(\mathbb{C}^6)$, namely, $w_1 = e_1 \wedge e_2 \wedge e_3$, $w_2 = e_1 \wedge e_2 \wedge e_4$, \ldots, $w_{20} = e_4 \wedge e_5 \wedge e_6$. We can then construct the following induced ODE.

\[
W_z = A^{(3)}(z, \lambda)W \quad W \in \wedge^3(\mathbb{C}^6),
\]

where $A^{(3)}(z, \lambda)$ is a $20 \times 20$ matrix.

The induced boundary condition at $z = 0$ is that the component of $W$ in the direction $e_1 \wedge e_2 \wedge e_5$, denoted by $D(\lambda)$, should be zero.

The inner product on $\mathbb{C}^6$, denoted $\langle \cdot, \cdot \rangle$ induces an inner product on each vector space $\wedge^3(\mathbb{C}^6)$ as follows. Let

\[
x = x_1 \wedge x_2 \wedge x_3 \quad \text{and} \quad y = y_1 \wedge y_2 \wedge y_3
\]

be any decomposable 3-forms. Then the inner product of $x$ and $y$ is defined by

\[
[x, y]_3 \overset{\text{def}}{=} \det \begin{bmatrix}
\langle x_1, y_1 \rangle & \langle x_1, y_2 \rangle & \langle x_1, y_3 \rangle \\
\langle x_2, y_1 \rangle & \langle x_2, y_2 \rangle & \langle x_2, y_3 \rangle \\
\langle x_3, y_1 \rangle & \langle x_3, y_2 \rangle & \langle x_3, y_3 \rangle
\end{bmatrix}, \quad [x, y]_3 \in \wedge^3(\mathbb{C}^6), \tag{18.6}
\]
Part V: A numerical linear system and the compound matrix method

where $[x, y]_3$ represents an inner product. This definition extends by (bi)-linearity to any 3-form (i.e. not necessarily decomposable). The induced matrix $A^{(3)} : \bigwedge^3(C^6) \to \bigwedge^3(C^6)$ can be identified with a $20 \times 20$ matrix with entries defined by

$$
\{A^{(3)}\}_{i,j} = \{w_i, Aw_j\}_3, \quad i, j = 1, \ldots, 20,
$$

(18.7)

where, for any decomposable $x = x_1 \wedge x_2 \wedge x_3 \in \bigwedge^3(C^6)$,

$$
Ax \overset{\text{def}}{=} Ax_1 \wedge x_2 \wedge x_3 + x_1 \wedge Ax_2 \wedge x_3 + x_1 \wedge x_2 \wedge Ax_3.
$$

An advantage of this definition of the induced matrix is that it is easily automated.

For example, suppose $A$ is an arbitrary $6 \times 6$ matrix with complex entries $a_{i,j}$, $i, j = 1, \ldots, 6$. Let $e_1, \ldots, e_6$ be the standard basis for $C^6$ and use the basis made from triple combinations of these, $w_1, \ldots, w_{20}$ for $\bigwedge^3(C^6)$. It is easy to check that this basis is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$. Therefore

$$
\{A^{(3)}\}_{1,1} \overset{\text{def}}{=} \{w_1, Aw_1\}
= \begin{bmatrix}
\langle e_1, Ae_1 \rangle & \langle e_1, e_2 \rangle & \langle e_1, e_3 \rangle \\
\langle e_2, Ae_1 \rangle & \langle e_2, e_2 \rangle & \langle e_2, e_3 \rangle \\
\langle e_3, Ae_1 \rangle & \langle e_3, e_2 \rangle & \langle e_3, e_3 \rangle
\end{bmatrix}
+ \begin{bmatrix}
\langle e_1, e_1 \rangle & \langle e_1, Ae_2 \rangle & \langle e_1, e_3 \rangle \\
\langle e_2, e_1 \rangle & \langle e_2, Ae_2 \rangle & \langle e_2, e_3 \rangle \\
\langle e_3, e_1 \rangle & \langle e_3, Ae_2 \rangle & \langle e_3, e_3 \rangle
\end{bmatrix}
+ \begin{bmatrix}
\langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \langle e_1, Ae_3 \rangle \\
\langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle & \langle e_2, Ae_3 \rangle \\
\langle e_3, e_1 \rangle & \langle e_3, e_2 \rangle & \langle e_3, Ae_3 \rangle
\end{bmatrix}
= \langle e_1, Ae_1 \rangle + \langle e_2, Ae_2 \rangle + \langle e_3, Ae_3 \rangle = a_{11} + a_{22} + a_{33}.
$$

By repeating this process for elements $i = 1 \ldots 20$, $j = 1 \ldots 20$, we can produce the following induced $20 \times 20$ matrix:
### Part V: A numerical linear system and the compound matrix method

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a_{11} + a_{22} + a_{33}</strong></td>
<td><strong>a_{34}</strong></td>
<td><strong>a_{35}</strong></td>
<td><strong>a_{36}</strong></td>
<td><strong>a_{44}</strong></td>
<td><strong>a_{45}</strong></td>
<td><strong>a_{46}</strong></td>
</tr>
<tr>
<td><strong>a_{43}</strong></td>
<td><strong>a_{11} + a_{22} + a_{44}</strong></td>
<td><strong>a_{45}</strong></td>
<td><strong>a_{46}</strong></td>
<td><strong>a_{56}</strong></td>
<td><strong>a_{55}</strong></td>
<td><strong>a_{54}</strong></td>
</tr>
<tr>
<td><strong>a_{53}</strong></td>
<td><strong>a_{54}</strong></td>
<td><strong>a_{11} + a_{22} + a_{55}</strong></td>
<td><strong>a_{56}</strong></td>
<td><strong>a_{66}</strong></td>
<td><strong>a_{65}</strong></td>
<td><strong>a_{64}</strong></td>
</tr>
<tr>
<td><strong>a_{63}</strong></td>
<td><strong>a_{64}</strong></td>
<td><strong>a_{65}</strong></td>
<td><strong>a_{11} + a_{22} + a_{66}</strong></td>
<td><strong>a_{66}</strong></td>
<td><strong>a_{65}</strong></td>
<td><strong>a_{64}</strong></td>
</tr>
<tr>
<td><strong>−a_{42}</strong></td>
<td><strong>a_{33}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{11} + a_{33} + a_{44}</strong></td>
<td><strong>a_{45}</strong></td>
<td><strong>a_{46}</strong></td>
</tr>
<tr>
<td><strong>−a_{52}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{32}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{54}</strong></td>
<td><strong>a_{11} + a_{33} + a_{55}</strong></td>
<td><strong>a_{56}</strong></td>
</tr>
<tr>
<td><strong>−a_{62}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{32}</strong></td>
<td><strong>a_{64}</strong></td>
<td><strong>a_{56}</strong></td>
<td><strong>a_{11} + a_{33} + a_{66}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>−a_{52}</strong></td>
<td><strong>a_{42}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{53}</strong></td>
<td><strong>a_{43}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>−a_{62}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{42}</strong></td>
<td><strong>−a_{53}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{43}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{62}</strong></td>
<td><strong>a_{52}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{63}</strong></td>
<td><strong>a_{53}</strong></td>
</tr>
<tr>
<td><strong>a_{41}</strong></td>
<td><strong>−a_{31}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{21}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>a_{51}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{31}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{21}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>a_{61}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{31}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{21}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>a_{51}</strong></td>
<td><strong>−a_{41}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>a_{61}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{41}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{61}</strong></td>
<td><strong>−a_{41}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{61}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>14</strong></td>
<td><strong>a_{15}</strong></td>
</tr>
<tr>
<td><strong>−a_{25}</strong></td>
<td><strong>−a_{26}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{13}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>a_{24}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{25}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{13}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>a_{24}</strong></td>
<td><strong>a_{25}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{13}</strong></td>
</tr>
<tr>
<td><strong>−a_{35}</strong></td>
<td><strong>−a_{36}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{12}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>a_{34}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{36}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{12}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>a_{34}</strong></td>
<td><strong>a_{35}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{12}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>a_{11} + a_{44} + a_{55}</strong></td>
<td><strong>a_{56}</strong></td>
<td><strong>−a_{46}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{12}</strong></td>
</tr>
<tr>
<td><strong>a_{65}</strong></td>
<td><strong>a_{11} + a_{44} + a_{66}</strong></td>
<td><strong>a_{45}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>−a_{64}</strong></td>
<td><strong>a_{54}</strong></td>
<td><strong>a_{11} + a_{55} + a_{66}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{22} + a_{55} + a_{44}</strong></td>
<td><strong>a_{45}</strong></td>
<td><strong>a_{46}</strong></td>
<td><strong>−a_{35}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{54}</strong></td>
<td><strong>a_{22} + a_{55} + a_{55}</strong></td>
<td><strong>a_{56}</strong></td>
<td><strong>a_{54}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{54}</strong></td>
<td><strong>a_{65}</strong></td>
<td><strong>a_{22} + a_{53} + a_{66}</strong></td>
<td><strong>a_{56}</strong></td>
</tr>
<tr>
<td><strong>a_{21}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{33}</strong></td>
<td><strong>a_{43}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{22} + a_{44} + a_{55}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>a_{21}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{63}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{43}</strong></td>
<td><strong>a_{65}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{21}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{63}</strong></td>
<td><strong>a_{53}</strong></td>
<td><strong>−a_{64}</strong></td>
</tr>
<tr>
<td><strong>a_{31}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{52}</strong></td>
<td><strong>−a_{42}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{32}</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>a_{31}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{62}</strong></td>
<td><strong>0</strong></td>
<td><strong>−a_{42}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{31}</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{62}</strong></td>
<td><strong>−a_{52}</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td><strong>a_{61}</strong></td>
<td><strong>−a_{51}</strong></td>
<td><strong>a_{41}</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>0</strong></td>
<td><strong>a_{62}</strong></td>
</tr>
</tbody>
</table>
As in the analysis for the two-dimensional Blasius boundary layer problem, Newton's method is required to converge the eigenvalue, \( \lambda \), such that the boundary conditions at the wall are satisfied. Thus the following expanded system is integrated to obtain the values of \( W \) and \( \frac{\partial}{\partial \lambda} W \),

\[
\frac{d}{dz} \begin{pmatrix} W \\ \frac{\partial}{\partial \lambda} W \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{(3)}(z, \lambda) & 0 \\ \frac{\partial}{\partial \lambda} \mathbf{A}^{(3)}(z, \lambda) & \mathbf{A}^{(3)}(z, \lambda) \end{pmatrix} \begin{pmatrix} W \\ \frac{\partial}{\partial \lambda} W \end{pmatrix},
\]

with the boundary condition at the wall defined as

\[ D(\lambda) = 0, \]

where the derivative of \( \mathbf{A}^{(3)}(z, \lambda) \) with respect to \( \lambda \) is obtained easily by differentiating the original 6 \times 6 matrix, \( \mathbf{A}(z, \lambda) \) with respect to \( \lambda \) and using the compound matrix algorithm described above to produce the induced 20 \times 20 matrix \( \frac{\partial}{\partial \lambda} \mathbf{A}^{(3)}(z, \lambda) \). Finally, the asymptotic boundary conditions at infinity are to be defined so that an initial vector for the integration can be obtained. This method is described in the next section.

### 18.2 Boundary conditions at infinity

The most difficult part of this problem is constructing the starting values at \( y = L_\infty \). The method of which follows.
18.2.1 Starting values at \( z = L_\infty \)

To obtain the starting values at \( z = L_\infty \) we need to find asymptotic boundary condition at \( z_\infty \). Let \( A_\infty(\lambda) = \lim_{z \to \infty} A(z, \lambda) \), then the eigenvalues are determined by the characteristic polynomial for \( A_\infty(\lambda) \), which takes the form

\[
\det(\mu I - A_\infty(\lambda)) = \mu^6 - f_1(\lambda)\mu^4 + f_2(\lambda)\mu^2 - f_3(\lambda) = 0, \tag{18.9}
\]

where \( f_1, f_2 \) and \( f_3 \) are analytic functions of \( \lambda = -i\gamma c \) with the following explicit expressions.

\[
\begin{align*}
  f_1 &= 3\gamma^2 + 2\Gamma, \tag{18.10} \\
  f_2 &= 4 + 3\gamma^4 + 4\gamma^2\Gamma + \Gamma^2, \tag{18.11} \\
  f_3 &= \gamma^6 + 2\gamma^4\Gamma + \gamma^2\Gamma^2, \tag{18.12}
\end{align*}
\]

with \( \Gamma = i\gamma R_e(\bar{V}_\infty - \frac{e}{R_e}) \) and \( \bar{V}_\infty = \lim_{z \to \infty} \bar{V} = \frac{\xi}{\gamma} = -\sin \epsilon \).

Now, equation (18.9) is a sixth-order polynomial in \( \mu \) and thus has six roots. It is known that when \( \text{Real}(\lambda) > 0 \) there are exactly three roots with positive real part and three with negative real part (Bridges, personal communication). We wish to eliminate the roots with positive real part since these produce an unbounded solution as \( z \to +\infty \). However, the explicit expressions in (18.10) to (18.12) are difficult to work with, thus, the numerical algorithm proposed below is implemented to construct the starting vector.

**Numerical algorithm to construct the starting vector at \( z_\infty \)**

In this section we present an algorithm for computing the starting vector and its derivative since computing the eigenvalue of largest positive or negative real part of \( A_\infty(\lambda) \) is difficult analytically. The proposed algorithm will find the eigenvalue of \( A_\infty(\lambda) \) of largest positive/negative real part, its eigenvector and the derivative with respect to \( \lambda \) of its eigenvector. For definiteness, we shall assume it is the eigenvalue with largest positive real part that is desired and will denote it by \( \sigma(\lambda) \) with the corresponding eigenvector denoted by \( \xi(\lambda) \).

Now, since the matrix \( A(z, \lambda) \) is asymptotically constant (independent of \( z \)), in the limit as \( z \to \infty \), that is at the upper boundary, asymptotically correct boundary conditions can be derived for the numerical integration. Asymptotic conditions for integration using the compound matrix method have been derived by Ng & Reid [94] and Davey...
[34]. These results were then expanded upon by Allen & Bridges [1], where they showed that the asymptotic conditions can be derived using the induced system, and that the asymptotic matrix associated with the induced system has a unique simple eigenvalue of largest negative real part controlling the asymptotics. From this theory, it follows that the eigenvalue of largest real part of \( A_{\infty}^{(3)}(\lambda) \) is simple and therefore analytic. The eigenvalue equation

\[
A_{\infty}^{(3)}(\lambda)\xi(\lambda) = \sigma(\lambda)\xi(\lambda),
\]

is analytic and when differentiated with respect to \( \lambda \) gives

\[
\left( A_{\infty}^{(3)}(\lambda) - \sigma(\lambda)I \right) \frac{d}{d\lambda} \xi(\lambda) = -A_{\infty}^{(3)}(\lambda)\xi(\lambda) + \sigma'(\lambda)\xi(\lambda). \tag{18.13}
\]

To obtain \( \xi'(\lambda) \), it is necessary to solve this system. However, the matrix \( A_{\infty}^{(3)}(\lambda) - \sigma(\lambda)I \) is singular since \( \sigma(\lambda) \) is an eigenvalue and so, by construction its determinant vanishes.

The idea is to reformulate this system in a way that it can be solved numerically. First, note that a necessary and sufficient condition for the singular equation

\[
\frac{d}{d\lambda} \xi(\lambda) = f
\]

to be solvable is that

\[
\langle \eta(\lambda), f \rangle_C = 0,
\]

where \( \eta(\lambda) \) is the left eigenvector associated with the eigenvalue \( \sigma \). Application to (18.13) shows that \( \sigma'(\lambda) \) should satisfy

\[
\frac{d}{d\lambda} \sigma(\lambda) = \frac{\langle \eta(\lambda), A_{\infty}^{(3)}(\lambda)\xi(\lambda) \rangle_C}{\langle \eta(\lambda), \xi(\lambda) \rangle_C}.
\]

Following Allen & Bridges [1], the proposed way to solve the system in (18.13) is to reformulate it as an augmented system on \( \mathbb{C}^{d+1} \), where \( d = 20 \) is the dimension of the augmented system on \( \Lambda^3(\mathbb{C}^3) \):

\[
\begin{bmatrix}
A_{\infty}^{(3)}(\lambda) - \sigma(\lambda)I & -\xi(\lambda) \\
-\eta(\lambda)^* & 0
\end{bmatrix}
\begin{bmatrix}
\xi'(\lambda) \\
\sigma'(\lambda)
\end{bmatrix}
=
\begin{bmatrix}
-A_{\infty}^{(3)}(\lambda)\xi(\lambda) \\
0
\end{bmatrix}. \tag{18.14}
\]

This system is equivalent to the stated problem [1], and solving it yields both \( \xi'(\lambda) \) and \( \sigma'(\lambda) \).

In summary, a numerical algorithm for generating the starting vector for our basic ODE associated with asymptotic boundary conditions is as follows:
For fixed \( \lambda \), the eigenvalue (and associated eigenvector) of \( A^{(3)}(\lambda) \), of largest positive/negative real part is obtained numerically. Since this eigenvalue is simple, and has real part farther from the origin than any other eigenvalue of positive/negative real part, this numerical construction will be robust. The augmented system (18.14) is then solved for \( \xi'(\lambda) \) and the starting vector for the induced linear system of the perturbation equations, (18.8), on \( \Lambda^3(\mathbb{C}^6) \) is then given by the following forty dimensional vector.

\[
\begin{pmatrix} \xi(\lambda), \xi'(\lambda) \end{pmatrix}^T \in \mathbb{C}^{40}.
\]

Numerical results show that the algorithm is indeed robust and the results show impressive accuracy even with the implementation of the second-order implicit midpoint method.
The complex eigenvalues of our sixth order system of equations were determined for a vast number of combinations of $\gamma$, $Re$, $R_0$ and $\varepsilon$, with greater density of calculations in regions near the critical point. For the present calculations we set $Re = R_0$ so that comparisons can be made with existing results in the literature. In order to find a neutral stability point, two of the parameters, e.g. $\gamma$ and $\varepsilon$, are fixed at some reasonable values and the zeros of $c_i$ found by varying the third parameter, $Re$. Figure 19.2 shows this representation of the neutral curves. The continuation procedure adopted here was the technique whereby a simple linear approximation to the curve is assumed. On the surface of neutral stability, i.e. the surface in the $(Re, \gamma, \varepsilon)$-space along which $c_i$ vanishes, contours of constant $Re$ are also found as functions of wave number, $\gamma$, and orientation angle, $\varepsilon$. Figure 19.1 shows this representation of the neutral curves. A different continuation procedure was used to produce these contours in $Re$. The continuation package AUTO was integrated into the code where the arc-length is introduced as a new parameter (see chapter 8), and at fixed values of this parameter, a zero of $c_i$ along a line approximately orthogonal to the contour is sought. The Ekman spiral is unstable to perturbations corresponding to parameter values inside the contours. If the parameter value for a perturbation lies outside the contours the perturbation will decay. The critical point of the eigenstate is determined by minimisation of $Re$ under the constraint that $c_i = 0$. This point is marked by a (x) for the type-2 instability in figure 19.1 and is listed in table 19.1 together with some existing theoretical results from the literature. Table 19.2 shows the accuracy of the critical point obtained using our new numerical framework by comparing the results to those of Melander [90]. The code used to find the nose of the Ekman neutral surface uses the second order (mid-point) Runge-Kutta algorithm. Therefore, this must be taken
into account when comparing results. From table 19.2 it can be seen that the accuracy of the critical point increases with the choice for the value of the upper boundary, and decreased step size for the integration. However, this appears to just compensate for the less accurate mid-point method used for the numerical integration calculation.

The shape of the neutral curves as the Reynolds number increases suggests the existence of a separate instability mechanism, which appears to grow and dominate the instabilities as Reynolds number increases. The onset of this second instability occurs at around $Re = 120$, and from the previous literature, this apparition is attributed to the type-1 inviscid instability mechanism.

Figure 19.1: Curves of Neutral stability for the Ekman-layer with various Reynolds numbers
Figure 19.2: Curves of Neutral stability for the Ekman layer in the $R_e - \gamma$ plane for rotation angles: $-30^\circ$, $-14^\circ$ and $13^\circ$

<table>
<thead>
<tr>
<th>Reference</th>
<th>Wave type</th>
<th>$R_e$</th>
<th>$\alpha_c$</th>
<th>$2\pi/\alpha$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Faller (1966)</td>
<td>2</td>
<td>55</td>
<td>-</td>
<td>24</td>
<td>-15</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>118</td>
<td>-</td>
<td>11</td>
<td>10-12</td>
</tr>
<tr>
<td>Lilly (1966)</td>
<td>2</td>
<td>55</td>
<td>0.187</td>
<td>21</td>
<td>-20</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>110</td>
<td>0.050</td>
<td>11.9</td>
<td>7.5</td>
</tr>
<tr>
<td>Iooss et al. (1978)</td>
<td>2</td>
<td>54.2</td>
<td>0.195</td>
<td>19.88</td>
<td>-23.3</td>
</tr>
<tr>
<td>Melander (1983)</td>
<td>2</td>
<td>54.15504</td>
<td>0.19489</td>
<td>19.869</td>
<td>-23.3261</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>112.75847</td>
<td>0.05182</td>
<td>11.397</td>
<td>7.2021</td>
</tr>
<tr>
<td>Allen (2001)</td>
<td>2</td>
<td>54.15504</td>
<td>0.19487</td>
<td>19.872</td>
<td>-23.3261</td>
</tr>
</tbody>
</table>

Table 19.1: Summary of theoretical results for the Ekman instability
<table>
<thead>
<tr>
<th>$L_{\infty}$</th>
<th>Step size, $h$</th>
<th>Angle of orientation, $\varepsilon$</th>
<th>$C_i$</th>
<th>$C_r$</th>
<th>Reynolds no.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mogeth v. Melander 1983</td>
<td>-23.32610874647</td>
<td>0</td>
<td>0.6163019690056</td>
<td>54.15503924999</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.02</td>
<td>-23.3261087464691</td>
<td>-9.96964D-11</td>
<td>0.615278927369</td>
<td>54.1551938068</td>
</tr>
<tr>
<td>50</td>
<td>0.02</td>
<td>-23.3261087464691</td>
<td>-9.4532D-11</td>
<td>0.616072311672</td>
<td>54.1551889885</td>
</tr>
<tr>
<td>100</td>
<td>0.02</td>
<td>-23.3261087464691</td>
<td>8.0258D-11</td>
<td>0.616323500359</td>
<td>54.1550527485</td>
</tr>
<tr>
<td>500</td>
<td>0.02</td>
<td>-23.3261087464691</td>
<td>-7.21280D-11</td>
<td>0.616323500656</td>
<td>54.1550525485</td>
</tr>
<tr>
<td>300</td>
<td>0.01</td>
<td>-23.3261087464691</td>
<td>9.61736D-11</td>
<td>0.616323524212</td>
<td>54.1550413485</td>
</tr>
</tbody>
</table>

Table 19.2: Accuracy of the critical Reynolds number
Referring back to the boundary conditions derived in part III for the three-dimensional compliant plate-spring Ekman model, the two kinematic boundary conditions (12.31) and (12.32), and the dynamic boundary condition (12.48) are

\[
\begin{align*}
&\bar{V}_z(0)\psi + \bar{U}_z(0)\phi_x = 0, \quad \text{at} \quad z = 0, \\
&c\phi_x + \bar{R}_o \bar{V}_z(0)\phi = 0,
\end{align*}
\]

\[
\left[ i c \left( \frac{2}{E_k} \psi(0) + \phi_{zzz}(0) - \gamma^2 \phi_z(0) \right) = -\frac{\gamma\phi(0)}{E_k} \left( \gamma^2 \gamma^2 C_m + i \gamma C_D - \gamma^2 C_B - \gamma^2 C_T - C_{KE} \right) \right],
\]

respectively.

For programming and study purposes, the Ekman number, $E_k$ will be replaced by the expression $\frac{R_x}{\bar{R}_o}$. By doing this and also noting that $\lambda = -i\gamma c$, the two kinematic boundary conditions along with this dynamic boundary condition can be written in the following form

\[
\begin{align*}
&a_0 \phi(0) + a_1 \phi'(0) = 0 \quad (20.4) \\
&d_0 \phi(0) + d_1 \phi'(0) = 0 \quad (20.5) \\
&b_0 \phi(0) + b_1 \phi'(0) + b_3 \phi'''(0) + b_4 \psi(0) = 0, \quad (20.6)
\end{align*}
\]

where

\[
\begin{align*}
b_0 &= \gamma \frac{R_x}{\bar{R}_o} (-\lambda^2 C_m - \lambda C_D - \gamma^4 C_B - \gamma^2 C_T - C_{KE}) \quad (20.7) \\
b_1 &= \lambda \gamma \quad (20.8) \\
b_3 &= -\frac{\lambda}{\gamma} \quad (20.9)
\end{align*}
\]
Part V: Boundary conditions for the compliant surface

\[ b_4 = -2\frac{\lambda_{Re}}{\gamma R_0} \] (20.10)
\[ a_0 = \tilde{V}'(0)R_0 = (\cos(\epsilon) - \sin(\epsilon))R_0 \] (20.11)
\[ a_1 = \frac{\lambda \epsilon}{\gamma} \] (20.12)
\[ d_0 = \tilde{V}'(0) = \cos(\epsilon) - \sin(\epsilon) \] (20.13)
\[ d_1 = \tilde{U}'(0) = \sin(\epsilon) + \cos(\epsilon) \] (20.14)

20.1 Code Validation

Since there are no existing results for the Ekman stability problem incorporating a compliant boundary, with which values obtained in this report could be compared, a number of tests have been carried out in order to establish confidence in the code.

The first test is to take the limiting value of the wall parameter which makes the boundary equivalent to a rigid wall, that is, to consider the limit \( E \to \infty \). Thus we wish to check the condition of the system of equations in this limit to ensure they will behave correctly.

20.1.1 Well-posed boundary conditions

In this subsection, the boundary conditions will be checked to ensure consistency with those for the rigid wall case and also to ensure that they do in fact form a well-conditioned set of boundary conditions.

Recall that the boundary conditions, at \( z = 0 \), for the rigid wall are

\[ \phi = \phi' = \psi = 0 \, . \]

Now, we can set up the boundary conditions for the compliant surface (equations (20.4) to (20.6)) in matrix-vector notation, namely

\[ \begin{pmatrix} a_0 & a_1 & 0 \\ 0 & d_1 & d_0 \\ b_0 & b_1 & b_4 \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -b_3 \phi'' \end{pmatrix} \] (20.15)

The first point to consider is the linear independence of the boundary conditions, i.e. to check that the determinant of the matrix in (20.15) is non-zero. Thus,

\[ \text{determinant} = a_0(d_1b_4 - b_1d_0) - a_1(-b_0d_0), \]
and from equations (20.7) to (20.14) we have,

\[ \text{determinant} = R_0 (\cos \varepsilon - \sin \varepsilon) \lambda \left( \frac{c b_0}{R_0} - b_1 (\cos \varepsilon - \sin \varepsilon) - b_4 (\cos \varepsilon + \sin \varepsilon) \right). \]

Thus, if \( \cos \varepsilon = \sin \varepsilon \) i.e. at \( \varepsilon = \pm 45^\circ \) then our boundary conditions break down and become linearly dependent, hence we must ensure the conditions

\[ \cos \varepsilon \neq \sin \varepsilon, \quad \lambda \neq 0 \quad \text{and} \quad \left( \frac{c b_0}{R_0} - b_1 (\cos \varepsilon - \sin \varepsilon) - b_4 (\cos \varepsilon + \sin \varepsilon) \right) \neq 0. \]

When these conditions are not satisfied, modified boundary conditions would have to be derived. Henceforth, we will assume that the above conditions are satisfied.

The next issue to address is the importance of the consistency of the boundary conditions with those for the rigid wall case. That is, we wish to obtain the rigid wall boundary conditions when the compliance of the surface is very small, namely at very large values of \( E \), since we have the rigid wall case when \( E - \rightarrow \infty \).

Now, \( E \) only occurs in the boundary conditions in the terms \( C_{KE} \) and \( C_B \) which are both found in the expression for \( b_0 \), namely

\[ b_0 = \gamma \frac{R_e}{R_0} (\lambda^2 C_m - \lambda C_D - \lambda^4 C_B - \lambda^2 C_T - C_{KE}), \]

and also in the expressions for \( C_{KE} \) and \( C_B \), namely

\[ C_{KE} = \frac{K_E}{\rho_e R_0 \Omega^2 L} = 2.2918 D^{-13} \times 230.0 \times E \times R_e \]

\[ C_B = \frac{6078227.413 E}{R_0^3}. \]

Thus, as \( E \rightarrow \infty \), \( C_{KE} \rightarrow \infty \) and \( C_B \rightarrow \infty \) and, hence, \( b_0 \rightarrow -\infty \). And so, our matrix in (20.15) becomes ill-conditioned since all the other entries will approximate to zero in comparison to the entry, \( b_0 \) in cell position (3,1).

To eliminate this ill-conditioning problem, let us divide the row 3 of the matrix-vector equation (20.15) by \( b_0 \). That is, divide the dynamic boundary condition through by \( b_0 \). So by defining new variables

\[ \bar{b}_1 = \frac{b_1}{b_0}, \quad \bar{b}_4 = \frac{b_4}{b_0}, \quad \bar{b}_3 = \frac{b_3}{b_0}, \quad \bar{b}_0 = 1, \quad \bar{b}_1 = \bar{b}_4, \]

then our boundary conditions can be written as

\[ \begin{pmatrix} a_0 & a_1 & 0 \\ 0 & d_1 & d_0 \\ 1 & \bar{b}_1 & \bar{b}_4 \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\bar{b}_3 \phi'' \end{pmatrix}. \]

(20.17)
Then as $E \to \infty$, $b_1 \to 0$, $b_4 \to 0$ and $b_3 \to 0$. Thus, from the last equation we obtain $\phi = 0$, which is consistent with the first boundary condition for the rigid wall case. Then, from the first equation in (20.17) we have $c\phi' = 0$ and since $c \neq 0$ it implies that $\phi' = 0$ which satisfies the second boundary condition for the rigid wall. Finally, from the second equation in (20.17) we have $d_0\psi = 0$, thus as long as $d_0 = \cos \epsilon - \sin \epsilon \neq 0$ (which is the condition required for linear independence and which we have assumed to hold true), then we have $\psi = 0$ satisfying our final boundary condition for the rigid wall. Hence, our boundary conditions for the compliant surface are now consistent with those for the rigid wall as $E \to \infty$.

The boundary conditions for the compliant surface are now defined to be

$$
\begin{align*}
a_0\phi(0) + a_1\phi'(0) &= 0 \quad (20.18) \\
d_0\psi(0) + d_1\psi'(0) &= 0 \quad (20.19) \\
\phi(0) + b_1\phi'(0) + b_3\phi''(0) + b_4\psi(0) &= 0. 
\end{align*}
$$

These boundary conditions at the wall can be written in terms of vector variables in the following form:

$$
\begin{align*}
\langle \eta_1(\lambda), u(0, \lambda) \rangle_R, \quad \langle \eta_2(\lambda), u(0, \lambda) \rangle_R, \quad \text{and} \quad \langle \eta_3(\lambda), u(0, \lambda) \rangle_R 
\end{align*}
$$

by taking

$$
\eta_1(\lambda) = \begin{pmatrix} a_0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_2(\lambda) = \begin{pmatrix} 0 \\ d_1 \\ 0 \\ 0 \\ d_0 \end{pmatrix} \quad \text{and} \quad \eta_3(\lambda) = \begin{pmatrix} 1 \\ b_1 \\ 0 \\ 0 \\ d_0 \end{pmatrix},
$$

where $a_0, a_1, d_0, d_1, b_1, b_3, b_4$ are defined as in equations (20.7) to (20.14) and (20.16).

The second check was to confirm, by increasing the value of $L_{\infty}$ that the code converged sufficiently well. Results showed that $L_{\infty} = 10$ is sufficient for convergence, and thus this value is used throughout the numerics. In the next section, the boundary conditions on $\Lambda^3(\mathbb{C}^8)$ shall be derived starting from the standard form of the boundary conditions given above.
20.2 Boundary conditions on $\Lambda^3(C^6)$

The boundary conditions are now in the standard form required to apply an extended
theory of the exterior algebra and the coordinate-free compound matrices framework de­
scribed in chapter 7 for the boundary condition on $\Lambda^2(C^4)$. Now for the new numerical
framework, we will require just one boundary condition. The theory on the construction of
the boundary conditions on $\Lambda^3(C^6)$ leads us to the conclusion that the boundary condition
at $z = 0$, associated with the induced system on $\Lambda^3(C^6)$ can be constructed by consider­
ing the standard form of the three boundary condition, (20.18), (20.19) and (20.20) which
form a subspace of $C^6$. By letting $\{\eta_1(\lambda), \eta_2(\lambda), \eta_3(\lambda)\}$ be an analytic basis for this space,
then the following 3-form is analytic,

$$\eta_1(\lambda) \wedge \eta_2(\lambda) \wedge \eta_3(\lambda) \in \Lambda^3(C^6).$$

This 3-form can then be expressed in terms of the basis we used for constructing
our matrix $A^{(3)}$ for the linear system of perturbation equations on $\Lambda^3(C^6)$ by fixing the
standard basis $e_1, \ldots, e_6$ for $C^6$ and writing $\eta_1(\lambda), \eta_2(\lambda)$ and $\eta_3(\lambda)$ with respect to the
standard basis. That is,

$$\eta_1(\lambda) \wedge \eta_2(\lambda) \wedge \eta_3(\lambda) = (a_0 e_1 + a_1 e_2) \wedge (b_1 e_2 + b_2 e_3 + b_4 e_5) \wedge (d_0 e_5 + d_1 e_2)$$

$$= (a_0 b_1 e_1 \wedge e_2 + a_0 b_2 e_1 \wedge e_4 + a_0 b_4 e_1 \wedge e_5 + b_1 e_2 \wedge e_3 + a_1 b_3 e_2 \wedge e_4 + a_1 b_3 e_2 \wedge e_4)$$

$$+ a b_4 e_2 \wedge e_5 \wedge (d_0 e_5 + d_1 e_2)$$

$$= a_0 b_2 e_1 \wedge e_2 \wedge e_3 + a_0 b_3 e_1 \wedge e_4 \wedge e_3 + a_0 b_4 e_1 \wedge e_5 \wedge e_2$$

$$+ a_0 d_1 e_1 \wedge e_2 \wedge e_2 + a_1 d_0 e_1 \wedge e_2 \wedge e_3 + a_1 d_0 e_2 \wedge e_4 \wedge e_5.$$

Rearranging into lexicographical ordering using the property for wedge products, $e_j \wedge e_i =
-e_i \wedge e_j$ and by letting $w = (w_1, \ldots, w_6)^T$ and also defining $\omega_1 = e_1 \wedge e_2 \wedge e_3, \omega_2 =
e_1 \wedge e_2 \wedge e_4, \ldots, \omega_{20} = e_4 \wedge e_5 \wedge e_6$ as the induced orthonormal basis, then the boundary
condition at $z = 0$ (i.e. at the compliant surface) is defined to be

$$\Delta(\lambda) = 0,$$

where

$$\Delta(\lambda) = (a_0 d_1 b_3) w_2 + (a_0 d_1 b_4 + a_1 d_0 - a_0 d_0 b_1) w_3 - (a_0 d_0 b_3) w_3 - (a_1 d_0 b_3) w_{14}. \quad (20.21)$$
Hence, for use of Newton's method, its derivative with respect to $\lambda$ is defined to be

$$
\Delta'(\lambda) = (a_0 d_1 \bar{b}_3)w_{22} + (a_0 d_1 \bar{b}_4)w_{22} + (a_0 d_1 \bar{b}_4 + a_1 d_0 - a_0 d_0 \bar{b}_1)w_{23} + (a_0 d_1 \bar{b}_4 + a_1 d_0 - a_0 d_0 \bar{b}_1)w_{3} - \left((a_0 d_0 \bar{b}_3)w_{23} + (a_0 d_0 \bar{b}_3)w_{6}\right) - \left((a_1 d_0 \bar{b}_3)w_{34} + (a_1 d_0 \bar{b}_3)w_{14}\right),
$$

where,

$$
(a_0 d_1 \bar{b}_3)' = a_0 d_1 Db_3 \\
(a_0 d_1 \bar{b}_4)' = a_0 d_1 Db_4 \\
(a_1 d_0)' = \frac{d\delta}{\gamma} \\
(a_0 d_0 \bar{b}_1)' = a_0 d_0 Db_1 \\
(a_1 d_0 \bar{b}_3)' = a_0 d_0 Db_3 \\
(a_1 d_0 \bar{b}_3)' = d_0 Da_1 b_3,
$$

with

$$
Db_1 = \frac{R_o(\lambda^2 C_m - \gamma^4 C_B - \gamma^2 C_T - C_{KE})}{R_e(\lambda^3 C_m + \lambda C_D + \gamma^4 C_B + \gamma^2 C_T + C_{KE})^2},
$$

$$
Db_3 = -\frac{R_o(\lambda^3 C_m - \gamma^4 C_B - \gamma^2 C_T - C_{KE})}{\gamma^2 R_e(\lambda^3 C_m + \lambda C_D + \gamma^4 C_B + \gamma^2 C_T + C_{KE})^2},
$$

$$
Db_4 = -2\frac{(\lambda^2 C_m - \gamma^4 C_B - \gamma^2 C_T - C_{KE})}{\gamma^2(\lambda^3 C_m + \lambda C_D + \gamma^4 C_B + \gamma^2 C_T + C_{KE})^2},
$$

$$
Da_1 b_3 = \frac{i\lambda R_o(\lambda C_D + 2\gamma^4 C_B + 2\gamma^2 C_T + 2C_{KE})}{\gamma^3 R_e(\lambda^3 C_m + \lambda C_D + \gamma^4 C_B + \gamma^2 C_T + C_{KE})^2}.
$$

### 20.3 Numerical values for parameters

This section gives the expressions for the wall parameters to be used in the numerics. The values for the wall properties will be assigned following Carpenter and Garrad [23] and are discussed in part III, chapter 13. Consider the dimensionless parameter associated with the plate thickness, namely

$$
C_m = \frac{\rho_m b}{\rho_e L R_o},
$$

From the 2-dimensional Blasius case, of part IV, $\bar{C}_m = \frac{\rho_m b}{\rho_e \delta}$, where $\delta$ was the vertical displacement thickness. Since our length scale, $L$, is arbitrary, this scale represents the
Part V: Boundary conditions for the compliant surface

\( \delta^* \) in the 2-dimensional case. Thus for the 3-dimensional Ekman layer problem, we could write

\[
C_m = \frac{\bar{C}_m}{R_o},
\]

(20.22)

where \( \bar{C}_m \) represents the \( C_m \) from the 2-dimensional study (see section 14.2). The next dimensionless parameter to consider is that associated with the flexural rigidity of the wall, i.e.

\[
C_B = \frac{B}{\rho_e R_o \Omega^2 L^5}.
\]

Again, from the 2-dimensional Blasius case studied in part IV, \( \bar{C}_B = \frac{B}{\rho_e U_\infty \delta} \), where \( B = 8.9 \times 10^{-10} E \) with \( E \) representing the spring stiffness. In our non-dimensionalization of the coupled equations for the rotating boundary layer, we non-dimensionalized by \( v_p \Omega L \) instead of \( U_\infty \). Thus, using this and \( R_o = \frac{3}{\Omega L} \), we can rewrite the dimensionless parameter, \( C_B \), for the 3-dimensional study as

\[
C_B = \bar{C}_B,
\]

(20.23)

where \( \bar{C}_B \) represents the dimensionless parameter \( C_B \) for the 2-dimensional case. Finally, we shall consider the non-dimensional parameter that is associated with the spring stiffness of the flexible plate, namely

\[
C_{KE} = \frac{K_E}{\rho_e R_o \Omega^2 L}.
\]

Again, from the 2-dimensional Blasius case described in part IV, \( \bar{C}_{KE} = \frac{(230E-542.81745)\delta^*}{U_\infty} \), and as before, we have used \( v_p \Omega L \) as the square of the characteristic velocity scale. Thus for the 3-dimensional rotating boundary layer problem, we have

\[
C_{KE} = \bar{C}_{KE},
\]

(20.24)

where \( \bar{C}_{KE} \) represents the \( C_{KE} \) from the 2-dimensional study.

The dimensionless parameters for damping coefficient and plate material tension, namely, \( C_D \) and \( C_T \) are to be set to zero for the initial stages of the analysis, since the previous 2-dimensional study has shown that these parameters do not have a significant effect on transition delaying properties of the compliant wall-flow coupled system.

Now, equations (20.22), (20.23) and (20.24) contain a large number of parameters. However, there are only three important parameters to be varied: the spring stiffness, \( E \); the Rossby number, \( R_o \); and the Reynolds number, \( R_e \). Thus, it is reasonable to assign any
values to the remaining parameters with the only restriction that the order of magnitude be realistic for the nature of the parameter. Thus, I shall use the data from Carpenter and Garrad for the 2-dimensional, Blasius, case to determine the order of magnitude for each of the required dimensionless parameters. (See section 14.2.)

For $C_m$, the order of magnitude of $C_m$ is \( \frac{20000}{R_e} \), thus

$$C_m = \frac{20000}{R_e} \times \frac{1}{R_e}.$$  \hspace{1cm} (20.25)

For $C_B$, the order of magnitude of $C_B$ is \( \frac{6,000,000}{R_e^2} \), thus

$$C_B = \frac{6,000,000}{R_e^2} \times \frac{1}{R_e}.$$  \hspace{1cm} (20.26)

Finally, for $C_{KE}$, the order of magnitude of $C_{KE}$ is \( 1 \times 10^{-10} R_e E \), thus

$$C_{KE} = 1.0 \times 10^{-10} R_e E,$$ \hspace{1cm} (20.27)

where $E$ represents the spring stiffness.

### 20.4 Curves of neutral stability- compliant wall

Initial values for the type-2 mode for the local iteration scheme were obtained by systematically decreasing the wall stiffness parameter and tracking the solutions from the rigid wall eigenvalues. In figure 20.1, the computed effect of wall compliance on stability of the Ekman boundary layer flow is shown for the fixed Reynolds numbers $R_e = 60$. The parameter directly representing the plate spring stiffness is varied whilst all other parameters are kept fixed. The computed results show that wall compliance has an insignificant effect on the neutral curves at low Reynolds numbers, where the type-2 instability is dominant. Figure 20.2 shows a blown-up version of figure 20.1 showing the small change in size of the neutral curve as $E$ is decreased implying an increase in wall compliance. It can be noted that although this effect is insignificantly small, the decrease in size of the instability region as the wall compliance is increased is consistent with the effect obtained from the 2-dimensional, Blasius computations.

Figure 20.3 shows the effect of wall compliance on the neutral curve for the Ekman boundary layer for fixed angle of orientation, $\varepsilon = -30.0$. Thus the Reynolds number is allowed to vary to large values where the type-1 instability becomes dominant. There
appears to be a pronounced effect as the plate stiffness is decreased. The region of instability appears to become progressively smaller as $E$ is reduced. The neutral curves for the plate-spring model begin to depart from the curve corresponding to the rigid wall by being displaced downwards and start to turn round as if to form enclosed regions of instability similar to those produced by the wall compliance in the Blasius case studied in part IV.

The computed curves shown are incomplete due to possible mode coalescence. Further points were unable to be detected through an eigenvalue search scheme, due to the value of the derivative of the boundary condition at the wall required for Newton's convergence method, approaching zero, thus leading to a conjecture that mode coalescence is occurring.

Figure 20.1: Neutral curve for the Ekman layer at $Re = 60$ with varying compliance on the surface.
Part V: Boundary conditions for the compliant surface

Figure 20.2: Blow up of figure showing the small effect of wall compliance at $R_e = 60$.

Figure 20.3: Neutral curve for the Ekman layer for $\varepsilon = -30$ showing the effects of wall compliance.
PART VI

Swept-wing boundary layers: stability of the attachment line flow past a compliant surface
Overview

Three dimensionality is an essential attribute of boundary layer flows over wings of commercial air-planes, since these components are swept in order to avoid shocks at higher cruising speed. The practical problem of designing these swept-back wings has led to the need for analysis of flow over a body whose leading edge is not normal to the oncoming stream.

Not surprisingly, the motivation for this part was acquired directly from the dolphin, where evolution has moulded this sea animal in such a way that this swept wing attribute is evident on all three of its fin types. It can be seen from figure 21.1 that the dorsal (top fin used for balance), pectoral (side fins aiding maneuvers) and caudral (propulsive lunate tail flute) all resemble highly swept back wings.

On a swept wing, many instability mechanisms occur that can lead to the catastrophic breakdown of laminar to turbulent flow: contamination along the leading edge; Tollmien-Schlichting (TS) waves; stationary or travelling cross-flow vortices; Taylor-Gortler vortices or combinations of these modes are among the mechanisms that can lead to this breakdown.
In addition to TS and CF disturbances which lead to transition over the wing chord, attachment-line instabilities are possible. If transition were to occur at some location on the attachment line, the outboard portion of the whole wing would have turbulent flow. This can be clearly understood from figure 21.2 which shows the attachment line region of a swept wing.

Transition along the attachment line can be prevented by designing the attachment line Reynolds number such that it does not exceed some critical value.

There appears to be no practical computation method for analysing finite-span swept wings, and design information is usually obtained by experiment. Pfenninger & Bacon [106] used a wing sweep of 45 degrees to study the attachment-line instabilities in a wind tunnel capable of reaching speeds sufficient to obtain unstable disturbances. With hot wires, they observed regular sinusoidal oscillations with frequencies comparable with the most unstable 2-dimensional modes of theory; these modes caused transition to occur at around $Re_\theta \approx 240 \Rightarrow Re_{\text{crit}} \approx 594$.

A continued interest in the transition initiated near the attachment line of swept wings led Poll [102, 103] to perform additional experiments with the swept circular model of Cumpsty & Head [31]. Like Pfenninger & Bacon [106], Poll observed disturbances that amplified along the attachment line. He noted that no unstable modes were observed below $Re_\omega \approx 230 \Rightarrow Re_{\text{crit}} \approx 569$.

Although no practical method is available to study finite-span swept wings, it is possible to study computational analyses if the yawed wing is assumed to have a constant cross-section and infinite span (i.e. no tips). In this case the velocities are independent of the span-wise coordinate, $y$, yet the flow is truly three-dimensional, since all three

Figure 21.2: Sketch of the attachment-line flow (from Wentz, Ahmed, and Nyenhuis 1985).
components, \( u, v \) and \( w \) exist.

With an eigenvalue problem approach, Hall, Malik & Poll [60] studied the linear stability of disturbances in the attachment line boundary layer flow called swept Hiemenz flow illustrated in figure 21.3.

\[
\text{Figure 21.3: Sketch of attachment line region of swept Hiemenz flow (from Wentz, Ahmed, and Nyenhuis 1985).}
\]

This three-dimensional base flow is a similarity solution to the Navier-Stokes equations; hence its use is advantageous in stability analyses. By assuming periodic disturbance modes along the attachment line, Hall, Malik and Poll determined neutral curves with and without the presence of steady suction and demonstrated that the attachment line boundary-layer can theoretically be stabilized with small amounts of suction. The approach taken by Hall, Malik and Poll is referred to as a non-parallel theory because the study accounted for all linear terms, including the wall-normal velocity component of the base flow. The mean, or base, flow referred to as swept Hiemenz flow has the fluid coming obliquely down towards the wall, it then turns away from the attachment-line into the \( x \)-direction to form a boundary layer. In the \( y \)-direction, the flow is uniform.
We consider a surface upon which there is a three-dimensional boundary layer formed from the flow of a viscous incompressible fluid of kinematic viscosity $\nu$ adjacent to an infinite swept wing and, furthermore, consider a localised region of the flow with Cartesian coordinates $(x, y, z)$. Experimental observations suggest that disturbances periodic in some direction or other can appear. The coordinates in the surface can be selected so that one of the coordinate directions is coincident with the wave fronts. Let $y$ denote the dimensionless coordinate in the direction of periodicity, $x$ the other coordinate in the surface orthogonal to $y$, and $z$ the coordinate normal to the surface.

We shall start with the three dimensional Navier-Stokes equations

\begin{align}
    &u_*^t + u^* u_*^x + v^* u_*^y + w^* u_*^z + \frac{1}{\rho} p_*^x = \nu \Delta u^* \quad (22.1) \\
    &v_*^t + u^* v_*^x + v^* v_*^y + w^* v_*^z + \frac{1}{\rho} p_*^y = \nu \Delta v^* \quad (22.2) \\
    &w_*^t + u^* w_*^x + v^* w_*^y + w^* w_*^z + \frac{1}{\rho} p_*^z = \nu \Delta w^* \quad (22.3)
\end{align}

with boundary conditions

\begin{align}
    u^* = v^* = 0, \quad w^* = 0, \quad \text{for} \quad z^* = 0, \quad (22.4) \\
    u^* \to U_0 \frac{x^*}{L_1}, \quad v^* \to V_0, \quad \text{as} \quad z^* \to \infty.
\end{align}

The $*$ notation represents dimensional parameters. Define $L_2 = \left(\frac{U_0 L_1}{V_0}\right)^{\frac{1}{2}}$, so that $L_2$ is the thickness of the boundary layer at the wall and $L_1$ is the length in the $x$-direction. We non-dimensionalize the N-S equations by introducing the following scales:

\begin{align}
    L_2 = \text{length}, \quad V_0 = \text{velocity}, \quad T = \frac{V_0}{L_2}, \quad (22.5)
\end{align}
so that the dimensionless parameter are defined by

\[ u = \frac{u^*}{V_o}, \quad v = \frac{v^*}{V_o}, \quad w = \frac{w^*}{V_o}, \]
\[ x = \frac{x^*}{L_2}, \quad y = \frac{y^*}{L_2}, \quad z = \frac{z^*}{L_2}, \]
\[ t = \frac{t^*}{L_2}, \quad p = \frac{p^*}{pV_o^2}. \] (22.6)

Substituting these dimensionless parameters into (22.1) to (22.3) we obtain

\[ u_t + uu_x + vu_y + wu_z + p_x = -\frac{1}{Re} (u_{xx} + u_{yy} + u_{zz}) \] (22.7)
\[ v_t + uv_x + vv_y + wv_z + p_y = -\frac{1}{Re} (v_{xx} + v_{yy} + v_{zz}) \] (22.8)
\[ w_t + uw_x + vw_y + wu_z + p_z = -\frac{1}{Re} (w_{xx} + w_{yy} + w_{zz}) \] (22.9)

where the parameter \( Re \) has been defined by

\[ Re = \frac{V_o L_2}{\nu}. \]

Substituting equations (22.6) into the boundary conditions gives

\[ u = v = 0, \quad w = 0, \quad \text{for} \quad z = 0 \]
\[ u \to \frac{x}{Re}, \quad v \to 1, \quad \text{as} \quad z \to \infty. \] (22.10)

### 22.1 Basic flow

The steady components of the basic state in the \( x \) and \( y \) directions are zero at the surface and approach the values \( \frac{x}{Re} \) and 1, respectively, when \( z \to \infty \). The normal velocity component is zero at the surface and grows linearly with \( z \) when \( z \to \infty \). The important simplifying feature of this flow is that it corresponds to an exact solution of the Navier-Stokes equations satisfying the conditions given by (22.10). Consider a solution of the form

\[ u = \frac{x}{Re} \bar{u}(z), \quad v = \bar{v}(z), \quad w = \frac{1}{Re} \bar{w}(z). \] (22.11)

Hence, \( \bar{u} \to 1, \quad \bar{v} \to 1 \) as \( z \to \infty \), where \( \bar{u}, \bar{v}, \bar{w} \) satisfy three equations which shall be derived below.

Substituting (22.11) into the continuity equation gives

\[ \frac{\bar{u}}{Re} + \frac{\bar{w}_z}{Re} = 0 \implies \bar{u} + \bar{w}_z = 0. \] (22.12)

Substitute (22.11) into the momentum equation. Let us start with the dimensionless \( w \) equation since this equation will determine the form of the pressure variable. Thus
substituting (22.11) into (22.9) gives
\[ \frac{\bar{w}\bar{u}_z}{R_e^2} + p_x - \frac{\bar{u}_{zz}}{R_e^2} = 0, \]
thus the equation is satisfied if
\[ p_x = \frac{\bar{u}_{zz}}{R_e^2} = \frac{\bar{w}\bar{u}_z}{R_e^2}. \quad (22.13) \]

Integrating equation (22.13) with respect to \( z \) yields a form for the pressure field
\[ p = F(x) + G(z), \quad (22.14) \]
where \( G(z) = \frac{R_e^2}{u'^2} - \frac{1}{2} \frac{u'^2}{R_e^2} \). To find the form of \( F(x) \) we substitute (22.11) into the dimensionless \( u \) momentum equation to give
\[ \frac{xu'^2}{R_e^2} + \frac{\bar{w}\bar{u}_z x}{R_e^2} + p_x - \frac{\bar{u}_{zz} x}{R_e^2} = 0 \]
\[ \Rightarrow \frac{\bar{u}_{zz} x}{R_e^2} - \frac{\bar{w}\bar{u}_z x}{R_e^2} - \frac{u'^2 x}{R_e^2} = p_x. \]

Integrating this equation with respect to \( x \) yields
\[ p = \frac{x^2}{2R_e^2} \left( \bar{u}_{zz} - \bar{w}\bar{u}_z - u'^2 \right) + \tilde{G}(z). \quad (22.15) \]

From this, equation (22.14) becomes
\[ p = \frac{x^2}{2R_e^2} \bar{p} + G(z), \quad (22.16) \]
where, using 22.14 and 22.15, \( \bar{p} = \bar{u}_{zz} - \bar{w}\bar{u}_z - u'^2 = \text{constant} \) and \( \tilde{G}(z) = G(z) + \text{constant} \).

Since \( \bar{p} \) is a constant, the pressure in the far field must be equal to this constant. From the dimensionless \( u \) equation (22.7) and using equations (22.11) we have
\[ p_x = \frac{x}{R_e} \bar{u}_{zz} - \frac{x}{R_e^2} \bar{w}\bar{u}_z - \frac{x}{R_e^2} u'^2. \]

Now, we know that \( \bar{u}_{zz} \to 0, \bar{u}_z \to 0 \) and \( u'^2 \to 1 \) as \( z \to \infty \). Using these, the pressure in the far field becomes
\[ p_x \to -\frac{x}{R_e^2} \bar{p} \quad \text{as} \quad z \to \infty. \]

From equation (22.16), \( p_x = \frac{x}{R_e^2} \bar{p} \), and so the pressure in the far field must equal this. Thus,
\[ \frac{x}{R_e^2} \bar{p} = -\frac{x}{R_e^2}, \]
and, hence, we obtain

\[ \bar{p} = -1. \]  

(22.17)

Thus substituting (22.11) into the dimensionless \( u \) equation (22.7) and using (22.16) and (22.17) we obtain

\[ \bar{u}^2 + \bar{w} \bar{u}_z - 1 - \bar{u}_{zz} = 0, \]

thus eliminating \( \bar{u} \) by using (22.12) yields

\[ \bar{w}_{zzz} + \bar{w}_z^2 - \bar{w} \bar{w}_{zz} - 1 = 0. \]  

(22.18)

Finally, by substitution of (22.11) into the dimensionless \( v \) momentum equation and using (22.15) we have

\[ \bar{v}_{zz} - \bar{w} \bar{v}_z = 0. \]  

(22.19)

Thus, \( \bar{u}, \bar{v} \) and \( \bar{w} \) from the exact solution velocity field, (22.11), must satisfy

\[ \bar{u} + \bar{w}_z = 0, \]

\[ \bar{w}_{zzz} + \bar{w}_z^2 - \bar{w} \bar{w}_{zz} - 1 = 0, \]

\[ \bar{v}_{zz} - \bar{w} \bar{v}_z = 0, \]

\[ \bar{w}_z(0) = 0, \quad \bar{v}(0) = 0, \quad \bar{w}_z(\infty) = -1, \quad \bar{v}(0) = 0, \quad \bar{v}(\infty) = 1 \]

with the solution of the above system having \( \bar{w} \sim -[z - \delta] \) as \( z \to \infty \). The displacement, \( \delta \), is the constant 0.64790, which is the result of the numerical integration of the second equation of (22.20). The system of equations (22.20) agrees with that stated by HALL ET AL [60].

22.2 Linear stability problem

By assuming the disturbance to be sufficiently small, we can restrict our attention to the linear stability of the flow (22.11) to disturbances periodic in the \( y \)-direction with wavelengths \( \frac{2\pi L_2}{\alpha} \). Since we wish to produce curves of neutral stability, the temporal stability problem is considered, thus we take \( \alpha \) to be real and search for the zeroes of the imaginary part of the complex wave-speed \( c \) for each value of \( \alpha \) and \( Re \).
We perturb the flow (22.11) by writing

\[ u = \frac{\partial u}{\partial z} + \alpha U \]

\[ v = \bar{v} + V \]

\[ w = \frac{\partial w}{\partial z} + W \]

\[ p = \frac{\partial^2 p}{\partial z^2} + G(z) + P, \]

where \( \alpha U, V, W \) and \( P \) represent the components of the disturbance. The \( x \)-dependence of the disturbance enables us to find a solution of the linear stability equations by solving ordinary differential equations. Such a simplification does not happen in more general boundary layer centrifugal instability problems where a self-consistent linear stability analysis leads to a partial differential system, Hall [59].

The basic flow described above is susceptible to centrifugal instabilities because of the curvature of the streamlines in the \( x - z \) plane. There exists a discrete spectrum of damped eigenvalues with eigenfunctions that decay exponentially when \( z \to \infty \). Now if \( V_0 \) is not zero then the span-wise velocity component is susceptible to Tollmien-Schlichting waves and our appropriate differential eigenvalue problem is of sixth order in \( z \) and will be derived below [60].

### 22.3 Derivation of the sixth order eigenrelation

Substituting the superimposed variables (22.21) into the dimensionless continuity and Navier-Stokes equations, subtracting out those terms that satisfy \( u, v \) and \( w \) exactly, and linearizing by taking \( U, V \) and \( W \) proportional to \( \exp(ia(y - ct)) \) so that the disturbance has wavelength \( \frac{2\pi}{a} \) and propagates along the attachment line with speed \( c \), we obtain three equations which govern the disturbance components \( \tilde{U}, \tilde{V} \) and \( \tilde{W} \):

\[ -iaRec\tilde{U} + 2\bar{u}\tilde{U} + iaRe\bar{u}\tilde{U} + \bar{w}\tilde{U}_x + \tilde{W}_x = \tilde{U}_{zz} - \alpha^2 \tilde{U} \]  

(22.22)

\[ -iaRec\tilde{V} + iaRe\bar{v}\tilde{V} + \bar{w}\tilde{V}_x + \tilde{W}_x + iaRe\tilde{P} = \tilde{V}_{zz} - \alpha^2 \tilde{V} \]  

(22.23)

\[ -iaRec\tilde{W} + iaRe\bar{w}\tilde{W} + \bar{w}\tilde{W}_x + \tilde{W}_x + Re\tilde{P}_x = \tilde{W}_{zz} - \alpha^2 \tilde{W}. \]  

(22.24)

Similarly the continuity equation becomes

\[ \tilde{U} + i\alpha \tilde{V} + \tilde{W}_x = 0. \]  

(22.25)

Now, by eliminating the pressure perturbation, \( \tilde{P} \) and the \( y \) component of velocity \( V \), we can systematically reduce equations (22.22) to (22.24) to a pair of coupled ODEs to
determine $\bar{U}$ and $\bar{V}$. To eliminate the pressure perturbation from the equations we note that from (22.23) we have

$$i\alpha R_e \bar{p} = -\alpha^2 \bar{V} + \bar{V}_{zz} + i\alpha R_e c \bar{V} - i\alpha R_e \bar{v} \bar{V} - R_e \bar{v}_z \bar{W} - \bar{w} \bar{V}_z.$$ 

Multiplying by $\frac{i}{\alpha}$ gives

$$R_e \bar{p} = i\alpha \bar{V} - \frac{i}{\alpha} \bar{V}_{zz} + R_e c \bar{V}$$

$$- R_e \bar{v} \bar{V} + i\frac{R_e}{\alpha} \bar{W} \bar{v}_z + \frac{i}{\alpha} \bar{w} \bar{V}_z.$$  (22.26)

Differentiating equation (22.26) with respect to $z$ gives

$$R_e \bar{P}_z = i\alpha \bar{V}_z - \frac{i}{\alpha} \bar{V}_{zzz} + R_e c \bar{V}_z - R_e \bar{v}_z \bar{V} + \bar{v} \bar{V}_z$$

$$+ \frac{R_e}{\alpha} (\bar{W} \bar{v}_{zz} + \bar{W}_z \bar{v}_z) + \frac{i}{\alpha} (\bar{w} \bar{V}_{zz} + \bar{w}_z \bar{V}_z).$$

For the elimination of the $y$ component of velocity, $\bar{V}$, we note that from the continuity equation we have

$$\bar{V} = \frac{i}{\alpha} (\bar{V}_z + \bar{U}).$$

Substituting this and its derivatives into (22.27) we obtain

$$R_e \bar{P}_z = -(\bar{W}_{zz} + \bar{U}_z) + \frac{1}{\alpha^2} (\bar{W}_{zzzz} + \bar{U}_{zzzz}) + \frac{i}{\alpha} R_e c (\bar{W}_{zz} + \bar{U}_z)$$

$$- \frac{i}{\alpha} R_e \bar{v}_z (\bar{W}_z + \bar{U}) + \frac{1}{\alpha} R_e \bar{v} (\bar{W}_{zz} + \bar{U}_z) + \frac{i}{\alpha} R_e (\bar{W} \bar{v}_{zz} + \bar{W}_z \bar{v}_z)$$

$$- \frac{1}{\alpha^2} \bar{w} (\bar{W}_{zzz} + \bar{U}_{zz}) - \frac{1}{\alpha^2} \bar{w}_z (\bar{W}_{zz} + \bar{U}_z).$$  (22.28)

Substituting (22.28) into (22.24) gives

$$\left(-iaRc + iaRe c - \bar{v}_z + \bar{U}_z + R_e c \bar{V} - R_e \bar{v} \bar{V} + \alpha^2 \bar{w} \bar{V} + \bar{w} \bar{W}_z + \bar{w}_z \bar{W}_zight) - \left(2 - \frac{i}{\alpha} R_e c + R_e \frac{\bar{v}}{\alpha} + \frac{i}{\alpha^2} \bar{w}_z \bar{v}_z \right) \bar{W}_{zz}$$

$$- \left(1 - \frac{i}{\alpha} R_e c \bar{v}_z + \frac{i}{\alpha} \bar{v}_z \bar{v}_z + \frac{1}{\alpha^2} \bar{w}_z \bar{v}_z \bar{V}_{zzz} + \frac{1}{\alpha^2} \bar{U}_{zzzz} + \frac{1}{\alpha^2} \bar{U}_z \bar{V}_{zz} + \frac{1}{\alpha^2} \bar{U}_z \bar{V}_{zz} - \frac{i}{\alpha} R_e \bar{v}_z \bar{U} - \frac{i}{\alpha^2} \bar{w} \bar{V}_{zzz} = 0. \right.$$  (22.29)

To simplify this equation we can eliminate $\bar{U}_{zzz}$. By differentiating equation (22.22) we obtain

$$\bar{U}_{zzz} = -iaRc \bar{U}_z + 2(\bar{w} \bar{U}_z + \bar{w}_z \bar{U}) + i\alpha R_c (\bar{w} \bar{U}_z + \bar{w}_z \bar{U}) + (\bar{u}_w \bar{W}_z + \bar{u}_z \bar{W}_z) + (\bar{w} \bar{U}_z + \bar{w}_z \bar{U}) + \alpha^2 \bar{U}_z.$$

Substituting this into equation (22.29) and simplifying gives

$$\bar{W}_{zzzz} - 2\alpha^2 \bar{W}_{zz} + \alpha^4 \bar{W} + i\alpha R_e c \bar{W}_{zz} - \alpha^2 R_e c \bar{W} =$$

$$\left(iaRc \bar{W}_z - i\alpha R_e \bar{v} \bar{W} - \bar{w} \bar{W}_{zz} - \alpha^2 \bar{w} \bar{W}_z \right) + \bar{w}_z \bar{W}_z - \alpha^2 \bar{w}_z \bar{W}_z - \bar{w}_z \bar{W}_z.$$  (22.30)

Let us define the operator

$$M \equiv \left(\frac{d^2}{dz^2} - \alpha^2 \right).$$
then equation (22.30) becomes

\[
(M + i\alpha R_e) M \tilde{W} = i\alpha R_e \tilde{w} M \tilde{W} - i\alpha R_e \tilde{w} z \tilde{W} + w M \tilde{W}_z + \tilde{w}_z M \tilde{W} \\
-2(\tilde{u}_z \tilde{U} + \tilde{u} \tilde{U}_z) - (\tilde{u}_z^2 \tilde{W} + \tilde{u}_z \tilde{W}_z).
\]  

(22.31)

Thus, our pair of coupled equations to determine \( \tilde{U} \) and \( \tilde{W} \) are

\[
(M + i\alpha R_e) \tilde{U} = 2\tilde{u} \tilde{U} + \tilde{u}_z \tilde{W} + \tilde{w} \tilde{U}_z + i\alpha R_e \tilde{v} \tilde{U}, \quad (22.32)
\]

\[
(M + i\alpha R_e) M \tilde{W} = i\alpha R_e \tilde{v} M \tilde{W} - i\alpha R_e \tilde{v} z \tilde{W} + w M \tilde{W}_z + \tilde{w}_z M \tilde{W} \\
-2(\tilde{u}_z \tilde{U} + \tilde{u} \tilde{U}_z) - (\tilde{u}_z^2 \tilde{W} + \tilde{u}_z \tilde{W}_z),
\]  

(22.33)

with boundary conditions

\[
\tilde{U}(0) = \tilde{W}(0) = \tilde{W}'(0) = 0, \quad \text{at} \quad z = 0. \quad (22.34)
\]

Changing the notation to more familiar variables by letting \( \phi = \tilde{W} \), the dimensionless disturbance amplitude in the \( z \)-direction and \( \psi = \tilde{U} \), the dimensionless disturbance amplitude in the \( x \)-direction, we have

\[
\psi'' - \tilde{w} \psi' - a(z) \psi - \tilde{u}_z \phi = 0, \quad (22.35)
\]

\[
\psi''' - \tilde{w} \psi'' - b(z) \psi'' + c(z) \psi' + d(z) \phi + 2\tilde{u}_z \psi + 2\tilde{u} \psi' = 0, \quad (22.36)
\]

where

\[
a(z) = \alpha^2 + i\alpha R_e \tilde{v} + \lambda R_e + 2\tilde{u}
\]

\[
b(z) = 2\alpha^2 + \lambda R_e + i\alpha R_e \tilde{v} + \tilde{w}_z
\]

\[
c(z) = \alpha^2 \tilde{w} + \tilde{u}_z
\]

\[
d(z) = \alpha^4 + \alpha^2 \lambda R_e + i\alpha^2 R_e \tilde{v}
\]

\[
+ \alpha^2 \tilde{w}_z + i\alpha R_e \tilde{v}_z + \tilde{u}_zz,
\]

and \( \lambda = -i\alpha \). Note that by setting \( \tilde{u} = \tilde{w} = 0 \) in (22.36) we obtain the Orr-Sommerfeld equation for \( \tilde{W} \). However, Hall et al. [60] point out that there does not appear to be a rational approximation to (22.35) and (22.36) that decouples the equations to produce this simplification.

The boundary conditions corresponding to our sixth order problem are

\[
\psi(0) = \phi(0) = \phi'(0) = 0, \quad \text{at} \quad z = 0. \quad (22.37)
\]
Thus, equations (22.35) and (22.36) along with the boundary conditions (22.37) is an eigenvalue problem, where the eigenrelation can be obtained as before by first writing
\[ u = [u_1, u_2, u_3, u_4, u_5, u_6]^T \]
with
\[ u_1 = \phi \quad u_2 = \phi' \quad u_3 = \phi'' \]
\[ u_4 = \phi''' \quad u_5 = \psi \quad u_6 = \psi', \]
so that \( u \) satisfies linear equation
\[ u_x = A(z, \lambda)u, \quad u \in C^6, \quad (22.38) \]
where \( A \) is the \( 6 \times 6 \) matrix defined by
\[
A(z, \lambda) = \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
d(z) & -c(z) & b(z) & -2\bar{a}_x & -2\bar{a} \\
0 & 0 & 0 & 0 & 0 & 1 \\
\bar{a}_x & 0 & 0 & 0 & a(z) & \bar{w}
\end{pmatrix}
\]
with boundary conditions
\[ (e_1, u) = (e_2, u) = (e_5, u) = 0 \quad \text{at} \quad z = 0. \]
The numerical solution of this eigenvalue problem is non-trivial because of the rapidly varying nature of the eigenfunctions. Thus using lexicographical ordering on the wedge product of the standard basis, \((e_1, \ldots, e_6)\), we use our trusted numerical framework in the calculations. We generate the 20 dimensional system
\[ w_x = A^{(3)}(z, \lambda)w, \quad w \in \bigwedge^3(C^6) \cong C^{20}, \quad (22.39) \]
where \( A^{(3)}(z, \lambda) \) is calculated using the MAPLE program in appendix A. The three boundary conditions at the wall corresponding to the original linear system (22.38) reduce to just one boundary condition at the wall for this augmented linear system, namely,
\[ [w, e_1 \wedge e_2 \wedge e_3]_3 = w_3 = 0. \quad (22.40) \]
The first step in the calculations is to integrate the equations (22.11) using Chebyshev polynomials to obtain the best state for input into the linear stability problem. Then (22.39) can be integrated from \( \infty \) to 0 subject to (22.40) together with the condition that \((\bar{U}, \bar{V}, \bar{W})\) tend exponentially to zero when \( z \rightarrow \infty \).
22.4  Boundary conditions at infinity and the initial vector

In the discussion in section 3 of Hall et al [60], it was shown that for sufficiently large values of \( \bar{z}_\infty \)

\[ \tilde{U} \sim e^{-\frac{1}{2}z^2}, \quad \tilde{W} \sim e^{-\alpha z}, \]

where \( \bar{z}_\infty = z_\infty - 0.64790. \)

Now using our notation, \( \tilde{U} = \psi \) and \( \tilde{W} = \phi \) we can obtain the following proposed form for the boundary conditions to be applied at a large value of \( z \) [60]:

\[
\begin{align*}
\phi' &= -\alpha \phi \quad \Rightarrow \quad \phi' + \alpha \phi = 0 \\
\phi'' &= -\alpha \phi' \quad \Rightarrow \quad \phi'' + \alpha \phi' = 0 \\
\psi' &= -z_\infty \psi \quad \Rightarrow \quad \psi' + \bar{z}_\infty \psi = 0.
\end{align*}
\]

Now the boundary conditions at \( z \rightarrow \infty \) depend on \( z \), so we can’t use the eigenvalue method used to find the initial vector for the Ekman layer problem, since the theory used specified that

\[ \lim_{z \rightarrow \infty} A(z, \lambda) = A_\infty(\lambda), \]

i.e. the system at infinity is independent of \( z \), for the eigenvalues of this matrix to control the asymptotic behaviour. Thus we revert back to the method used for the derivation of the initial vector for the 2-dimensional Blasius problem, and apply the appropriate boundary conditions in (22.41).

22.4.1  Derivation of the initial vector

Applying the ideas of spanning sets to study the structure of solution sets of systems of homogeneous linear equations, we start with the boundary conditions at infinity for our system:

\[
\begin{align*}
\phi' + \alpha \phi &= 0 \\
\phi'' + \alpha \phi' &= 0 \\
\psi' + \bar{z}_\infty \psi &= 0.
\end{align*}
\]

Now let us define \( \eta = (\eta_1, \ldots, \eta_6) \) where

\[
\begin{align*}
\eta_1 &= \phi & \eta_2 &= \phi' & \eta_3 &= \phi'' \\
\eta_4 &= \phi''' & \eta_5 &= \psi & \eta_6 &= \psi' ,
\end{align*}
\]
then our system of boundary conditions can be written as

\[\begin{align*}
\alpha \eta_1 + \eta_2 &= 0, \\
\alpha \eta_2 + \eta_3 &= 0, \\
\tilde{z}_\infty \eta_5 + \eta_6 &= 0.
\end{align*}\]  \hspace{1cm} (22.43)

Writing the system (22.43) in vector-matrix form we obtain

\[\begin{pmatrix}
\alpha & 1 & 0 & 0 & 0 & 0 \\
0 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{z}_\infty & 1
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 \\
\eta_5 \\
\eta_6
\end{pmatrix} = 0. \hspace{1cm} (22.44)

Noticing that the solution set of (22.43) is a subspace of \(\mathbb{R}^6\), we can find a spanning set for this space. We first write down the augmented matrix of the system, namely

\[\begin{pmatrix}
\alpha & 1 & 0 & 0 & 0 & 0 \\
0 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{z}_\infty & 1
\end{pmatrix}
\]

and perform a sequence of row operations so that the new system of equations formed will have the same solution set as (22.43). Thus we obtain

\[\begin{pmatrix}
\alpha & 1 & 0 & 0 & 0 & 0 \\
0 & \alpha^2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{z}_\infty & 1
\end{pmatrix} \cdot
\]

Note that this matrix has rank 3, that is, we have three linearly independent equations but six unknown variables, thus we have \(6 - 3 = 3\) degrees of freedom. So, we choose arbitrary constants for three variables.

From this new system, we note that we can find \(\eta_2\) in terms of \(\eta_1\), \(\eta_3\) in terms of \(\eta_1\) and \(\eta_6\) in terms of \(\eta_5\). Thus, let us choose the following arbitrary constants

\[\eta_1 = c_1, \quad \eta_4 = c_2, \quad \eta_5 = c_3.\]

Then we have

\[\eta = (c_1, -\alpha c_1, \alpha^2 c_1, c_2, c_3, -\tilde{z}_\infty c_3)\]
and by factoring out the $c_i$s we can write the general solution as

$$\eta = c_1 \begin{pmatrix} 1, -\alpha, \alpha^2, 0, 0, 0 \end{pmatrix} + c_2 \begin{pmatrix} 0, 0, 0, 1, 0, 0 \end{pmatrix} + c_3 \begin{pmatrix} 0, 0, 0, 0, 0, 1, -\bar{\alpha} \end{pmatrix}. \quad (22.45)$$

We can conclude that for any choice of real numbers $c_1, \ldots, c_3$ we get a solution $\eta = c_1 x_1 + c_2 x_2 + c_3 x_3$ of (22.43) and that every solution of (22.43) is of this form for some choice of real numbers $c_1, c_2, c_3$. Moreover, the set $\{x_1, x_2, x_3\}$ is uniquely determined by the procedure. $\{x_1, x_2, x_3\}$ is called the *canonical spanning set* and are the three orthogonal starting vectors. Thus for our numerical framework we require just one initial vector formed from some complex constant multiplied by the wedge product of these three orthonormal vectors. That is,

$$x_1 \wedge x_2 \wedge x_3 = (e_1 - \alpha e_2 + \alpha^2 e_3) \wedge e_4 \wedge (e_5 - \bar{\alpha} e_6)$$

$$= (e_1 \wedge e_4 \wedge e_5) - \bar{\alpha} e_6 (e_1 \wedge e_4 \wedge e_5) - \alpha (e_2 \wedge e_4 \wedge e_5) + \alpha \bar{\alpha} e_6 (e_2 \wedge e_4 \wedge e_5)$$

$$+ \alpha^2 e_3 \wedge e_4 \wedge e_5 - \alpha^2 \bar{\alpha} e_6 (e_3 \wedge e_4 \wedge e_5)$$

$$= W(8) - \bar{\alpha} W(9) - \alpha W(14) + \alpha \bar{\alpha} W(15) + \alpha^2 W(17) - \alpha^2 \bar{\alpha} W(18).$$

Thus, an appropriate starting vector is

$$W = C(0, 0, 0, 0, 0, 0, 0, 1, -\bar{\alpha}, 0, 0, 0, -\alpha, \alpha \bar{\alpha}, 0, \alpha^2, -\alpha^2 \bar{\alpha}, 0, 0)^T,$$

where $C$ is some complex multiple.

### 22.5 Numerical comparison with Wilson and Gladwell

The paper by Wilson and Gladwell titled 'The Stability of a Two-Dimensional Stagnation Flow to Three-Dimensional Disturbances' [136] showed that the viscous stagnation point flow is stable to infinitesimal disturbances periodic in the direction normal to the plane of the flow in the limit of infinite Reynolds number. They show that the boundary curvature is irrelevant and that the cylinder problem considered by Kestin & Wood can be reduced for $R_e \to \infty$ to the flat plate problem of determining the eigenvalues of the disturbance equations.

To come to their conclusion they presented the numerical solution of the disturbance equations proposed by Gortler [49] together with their more stringent boundary conditions at $\infty$, and found that on setting $c_i = 0$ there is no eigenvalue $\alpha$, meaning that there is
no neutral wavenumber. They then restored $c$ into their equations and solved for $c_i$ as an eigenvalue with $\alpha$ prescribed. There results showed that $c < 0$ for all values of $\alpha$, with the implication that the flow is stable.

To compare numerical values with those obtained by Wilson and Gladwell, we set $Re = 0$ in equations (22.22) to (22.24) in all places except where $Re$ is multiplied by the stability parameter, wave-speed $c$, or the pressure term. In these places we shall set $Re = 1$

By doing this, we obtain equations of the form for stagnation point flow consistent with those studied by Wilson and Gladwell. Thus, we have

\begin{align*}
-iac\ddot{U} + 2\ddot{\bar{u}} + \ddot{\bar{u}}_z + \bar{u}_z \ddot{W} &= \ddot{U}_{zz} - \alpha^2 \ddot{U} \quad (22.46) \\
-iac\ddot{V} + \ddot{\bar{w}}_z + \ddot{\bar{w}} \ddot{P} &= \ddot{V}_{zz} - \alpha^2 \ddot{V} \quad (22.47) \\
-iac\ddot{W} + \ddot{\bar{w}} \ddot{W}_z + \ddot{\bar{w}}_z \ddot{W} + \ddot{P}_z &= \ddot{W}_{zz} - \alpha^2 \ddot{W} \quad (22.48) \\
\ddot{U} + ia\ddot{V} + \ddot{W}_z &= 0. \quad (22.49)
\end{align*}

Now, by eliminating the pressure term from the equation, we reduce the equations to a sixth order system comprising of a fourth order equation coupled with a second order equation which governs the stability of the system. Hence,

\begin{align*}
\phi''' - \bar{w}\phi'' - b(z)\phi'' + c(z)\phi' + d(z)\phi + 2\bar{u}_z \psi + 2\bar{u}\psi' &= 0 \\
\psi'' - \bar{w}\psi' - a(z)\psi - \bar{u}_z \phi &= 0,
\end{align*}

where

\begin{align*}
a(z) &= \alpha^2 + 2\bar{u} + \lambda \\
b(z) &= 2\alpha^2 + \bar{w}_z + \lambda \\
c(z) &= \alpha^2 \bar{w} + \bar{u}_z \\
d(z) &= \alpha^2 \bar{w}_z + \alpha^2 \lambda + \alpha^4 + \bar{u}_{zz},
\end{align*}

and $\phi = \ddot{W}$, $\psi = \ddot{U}$.

To compare numerical values for wavenumber, $\alpha$, and corresponding growth rates, $\lambda$, we set up the linear system as a vector-matrix system and integrate using our numerical framework. The results are shown in table 22.1.

The results given in table 22.1 show excellent consistency with those generated by Wilson and Gladwell who believed their results to be accurate to all the figures given.
Table 22.1: Comparison of numerical values for the wave number, $\alpha$ and corresponding growth rate $\lambda$ with those obtained by Wilson and Gladwell

Wilson and Gladwell checked that their values obtained were not particularly sensitive to the value chosen for infinity. They used $z_\infty = 6$ except for small values of $\alpha$ where they found it necessary to integrate out to $z_\infty = 10$ to obtain the required accuracy. For the new calculation, $z_\infty = 10$ is used at all times. These results provide convincing evidence that the Chebyshev coefficients for both $\bar{u}$ and $\bar{v}$ are accurate.

### 22.5.1 Accuracy check for $\bar{v}$

Finally, we must check the accuracy of the Chebyshev coefficients for $\bar{v}$. To do this, we shall calculate $\bar{v}(z)$ for various values of $z$ using two different methods and shall compare the results. The first method is that of which we wish to check, namely, using Chebyshev polynomials, the outline of which follows:

Using the proposed Chebyshev coefficients, we wish to find values of $\bar{v}(z)$ for various values of $z$ in the integral range $[0, L_\infty]$. Now the range for Chebyshev polynomials is $-1 \leq x \leq 1$, thus we must first transform our coordinates in such a way that at $z = L_\infty$ we have $x = 1$ and when $z = 0$ we have $x = -1$. This can be achieved using the coordinate transformation

$$x = \frac{2z}{L_\infty} - 1.$$

The Chebyshev polynomials can then be calculated to obtain $\bar{v}(z)$ for chosen values of $z$ in the integral range.

Our second method for calculating the values of $\bar{v}(z)$ involves solving the third exact equation (22.19), that is

$$\bar{v}_{zz} - \bar{w}u_z = 0,$$  \hfill (22.50)
with boundary conditions $\bar{v}(0) = 0$ and $\bar{v}(\infty) = 1$. By rearranging equation (22.50), we can set up a vector-matrix linear system for $\bar{v}_x$, that is

$$\bar{v}_{2x} = \bar{w}\bar{v}_x.$$

Let $u_1 = \bar{v}(z)$ and $u_2 = \bar{v}'(z)$ then $u'_1 = u_2$, $u'_2 = \bar{w}u_2$. Thus, we set up the linear vector-matrix system

$$u' = Au \quad u \in \mathbb{C}^2,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & \bar{w} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} \bar{v}(z) \\ \bar{v}'(z) \end{pmatrix},$$

(22.51)

and $\bar{w}$ can be calculated by using Chebyshev polynomials since we have already checked that the Chebyshev coefficients for $\bar{w}$ are accurate.

Now, the linear system (22.51) is critically stable for numerical integration, this can be shown as follows. Using the GL-RK mid-point rule for integration

$$u^{n+1} = u^n + \frac{h}{2}A(y_{n+\frac{1}{2}})(u^n + u^{n+1})$$

$$= u^n + \frac{h}{2}A(y_{n+\frac{1}{2}})u^{n+1}$$

$$\Rightarrow \left[ I - \frac{h}{2}A_{n+\frac{1}{2}} \right] u^{n+1} = \left[ I + \frac{h}{2}A_{n+\frac{1}{2}} \right] u^n.$$

If we now substitute in the $2 \times 2$ matrix $A$ we obtain

$$\begin{bmatrix} 1 & -\frac{h}{2} \\ 0 & 1 - \frac{h}{2}\bar{w} \end{bmatrix} u^{n+1} = \begin{bmatrix} 1 & \frac{h}{2} \\ 0 & 1 + \frac{h}{2}\bar{w} \end{bmatrix} u^n$$

$$\Rightarrow u^{n+1} = \frac{1}{1 - \frac{h}{2}\bar{w}} \begin{bmatrix} 1 - \frac{h}{2}\bar{w} & \frac{h}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{h}{2} \\ 0 & 1 + \frac{h}{2}\bar{w} \end{bmatrix} u^n$$

thus

$$u^{n+1} = \begin{bmatrix} 1 & \frac{h}{1 - \frac{h}{2}\bar{w}} \\ 0 & 1 + \frac{h}{1 - \frac{h}{2}\bar{w}} \end{bmatrix} u^n.$$

Now, the eigenvalues of the discrete system, namely $\lambda_1 = 1$ and $\lambda_2 = \frac{1 + \frac{h}{2}\bar{w}}{1 - \frac{h}{2}\bar{w}}$ are simple and also note that since $\bar{w}$ is always negative, we have that $|\lambda_2| < 1$. It is easy to show that

$$u^n = B^n u^0.$$
Since we have simple eigenvalues, we can use the result from linear algebra that there can be found a diagonal matrix, $D$ satisfying
\[ D = S^{-1}BS \]
\[ \Rightarrow B = SDS^{-1} \]
and so $B^n = SD^nS^{-1}$, where the diagonal matrix is found to be
\[ D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-\frac{b}{b}} \end{pmatrix} \]
Thus, the numerical stability of the system depends only on the properties of the diagonal entries. These entries have modulus less than or equal to unity and so we conclude that the system is numerically stable.

Thus, we can numerically integrate (22.51) forwards from $z = 0$ to $z = L_{\infty}$ using shooting methods.

Now, we require an initial vector $u(0) = \begin{pmatrix} \bar{v}(0) \\ \bar{v}'(0) \end{pmatrix}$. From the boundary conditions we have $\bar{v}(0) = 0$ thus, let us assume the initial vector to be
\[ u(0) = \begin{pmatrix} 0 \\ a \end{pmatrix} \]
for arbitrary $a$. At $L_{\infty}$, the boundary conditions require $\bar{v}(\infty) = 1$. Assuming the vector at $L_{\infty}$ to be
\[ u(L_{\infty}) = \begin{pmatrix} b \\ c \end{pmatrix} \]
we can scale this vector so that $b = 1$, that is $u(L_{\infty}) = \begin{pmatrix} 1 \\ \frac{c}{b} \end{pmatrix}$. Thus, by scaling our initial vector in the same way, i.e $u(0) = \begin{pmatrix} 0 \\ \frac{b}{b} \end{pmatrix}$, the code can now be rerun to produce accurate values of $\bar{v}(z)$ for comparison with those produced by the Chebyshev method.

### 22.5.2 Chebyshev accuracy results

Table 22.2 shows the value of $\bar{v}(z)$ calculated using the Chebyshev coefficients and also by solving the linear system (22.51) along with the error in the results. The small discrep-
Table 22.2: Accuracy of Chebyshev coefficients

<table>
<thead>
<tr>
<th>Value of $x$</th>
<th>$\bar{v}(x)$ chebyshev</th>
<th>$\bar{v}(x)$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.899</td>
<td>0.28634621839592</td>
<td>0.28634623518959</td>
<td>1.679367×10^{-08}</td>
</tr>
<tr>
<td>-0.799</td>
<td>0.54933210386440</td>
<td>0.54933212446577</td>
<td>2.060137×10^{-08}</td>
</tr>
<tr>
<td>-0.699</td>
<td>0.75788471107903</td>
<td>0.75788473194036</td>
<td>2.086133×10^{-08}</td>
</tr>
<tr>
<td>-0.599</td>
<td>0.89228347980418</td>
<td>0.89228363447996</td>
<td>1.546758×10^{-07}</td>
</tr>
<tr>
<td>-0.499</td>
<td>0.96102276829090</td>
<td>0.96102299135633</td>
<td>2.231654×10^{-07}</td>
</tr>
<tr>
<td>-0.399</td>
<td>0.98866097664809</td>
<td>0.98866115228359</td>
<td>1.756355×10^{-07}</td>
</tr>
</tbody>
</table>

The accuracy between the Chebyshev values and the linear system values are most probably due to the less accurate second order GL-RK algorithm used for the shooting method. With this method we can only expect accuracy to 7 or 8 d.p. Thus we can conclude that the Chebyshev coefficients are accurate to 7 or 8 d.p.

### 22.5.3 Computed neutral curve - rigid wing

Figure 22.1 shows the curve of neutral stability, that is the locus of points in the wave number, $\alpha - Re$ space where $c_i = 0$. The nose of the neutral curve occurs at the critical Reynolds number $Re(\text{crit}) = 583.1$, wave number, $\alpha = 0.287855$ and wave speed $c_r = 0.382480$. These values correspond almost exactly with those found by Hall et al [60]. Thus, the attachment line boundary layer flow is susceptible to travelling-wave instabilities that propagate along the attachment line. Inside the thumb, the perturbations become unstable whereas outside the curve, they are stable.
Figure 22.1: Neutral curve for the attachment line boundary layer over a rigid surface
Boundary conditions for the compliant surface

The two kinematic boundary conditions along with the dynamic boundary condition for the anisotropic flexible surface at the wall, \( z = 0 \), derived in part III are

\[
(i \alpha R_e \bar{v}_z(0) + \bar{u}_z(0)) \psi + \bar{u}_z(0) \phi_z = 0,
\]

\[
(i \alpha R_e \bar{v}_z(0) + i \alpha R_e \bar{\phi}_z(0)) \psi + i \alpha R_e \bar{\phi}_z(0) = 0,
\]

\[
ic(\phi''(0) + a(0) \psi(0)) = -\phi(0) \left( \alpha R_e (\alpha^2 C_m + i \alpha C_D - \alpha^4(1 + \chi_2) C_B - \alpha^2 C_T - C_{KE}) + icu_z(0) + i\alpha^2 \bar{u}_z(0) \right).
\]

These three boundary conditions can be written in the following standard form

\[
a_0 \phi(0) + a_1 \phi'(0) = 0
\]

\[
d_0 \psi(0) + d_1 \psi'(0) = 0
\]

\[
b_0 \phi(0) + b_3 \phi''(0) + b_4 \psi(0) = 0
\]

where

\[
b_0 = \alpha R_e (-\lambda^2 C_m - \lambda C_D - \alpha^4(1 + \chi_2) C_B - \alpha^2 C_T - C_{KE}) + i\alpha^2 \bar{v}_z(0) + ic\bar{u}_z(0)
\]

\[
b_3 = -\frac{\lambda}{\alpha}
\]

\[
b_4 = -\frac{\lambda a(0)}{\alpha}
\]

\[
a_0 = \bar{u}'(0) + i \alpha R_e \bar{v}'(0)
\]

\[
a_1 = -\lambda R_e
\]

\[
d_0 = \bar{u}'(0) + i \alpha R_e \bar{v}'(0) = a_0
\]

\[
d_1 = \bar{u}'(0).
\]
23.1 Well posed boundary conditions

The boundary conditions will now be checked to ensure they are consistent with those for the rigid wall case, that is, for when $E$ (proportional to the spring stiffness) tends to the value infinity (approximated by a very large value). We shall also check to ensure that the set of boundary conditions do in fact form a well conditioned set.

To ensure consistency, we note that the variable $E$ only occurs in the expressions for $C_{kE}$ and $C_B$, found in the expression for $b_0$. Now both $C_{kE}$ and $C_B$ are directly proportional to $E$, thus as $E \to \infty$, $C_{kE} \to \infty$ and $C_B \to \infty \Rightarrow b_0 \to -\infty$. Let us set up our boundary conditions for the compliant surface in vector-matrix form giving

$$
\begin{pmatrix}
a_0 & a_1 & 0 \\
0 & d_1 & d_0 \\
b_0 & 0 & b_4
\end{pmatrix}
\begin{pmatrix}
\phi \\
\phi' \\
\psi
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
-b_3 \phi'''
\end{pmatrix}.
$$

(23.5)

Thus the matrix in (23.5) becomes ill-conditioned since all other entries will approximate to zero in comparison to the value of $b_0$ in cell (3,1). To eliminate this ill-conditioning, we divide row 3 by $b_0$ and define new variables $\tilde{b}_3 = \frac{b_3}{b_0}$ and $\tilde{b}_4 = \frac{b_4}{b_0}$ thus obtaining the vector-matrix equation

$$
\begin{pmatrix}
a_0 & a_1 & 0 \\
0 & d_1 & d_0 \\
1 & 0 & \tilde{b}_4
\end{pmatrix}
\begin{pmatrix}
\phi \\
\phi' \\
\psi
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
-b_3 \phi'''
\end{pmatrix}.
$$

(23.5)

Then as $E \to \infty$, $\tilde{b}_4 \to 0$ and $\tilde{b}_3 \to 0$. Thus, from the third equation we obtain $\phi = 0$ which is consistent with the first boundary condition for the rigid wall case. Using this, from the first equation we have $a_1 \phi' = 0$. Now $a_1 = -\lambda R_e \neq 0$ and, therefore, $\phi' = 0$ which corresponds to the second boundary condition for the rigid wall. Finally, from the second equation and using the above conditions we obtain $d_0 \psi = 0$ where $d_0 = \frac{i \alpha R_e \varphi'(0) + \bar{u}'(0)}{\varphi'(0)} \neq 0$, since both $\varphi'(0)$ and $\bar{u}'(0)$ are non-zero. This leads us to the condition $\psi = 0$ which is consistent with the third boundary condition for the rigid wall case. Hence the system of boundary conditions for the compliant surface are consistent with those for the rigid wall.

The task of checking that the system of boundary conditions is well-posed we check to ensure that the three boundary conditions are linearly independent. The determinant of
the coefficient matrix for the boundary conditions is

\[ \text{det} = a_0 d_1 \tilde{b}_4 + a_1 d_0. \]

Now \( a_0 = d_0 \) thus we have

\[ a_0 (d_1 \tilde{b}_4 + a_1). \]

Since \( a_0 \neq 0 \), we just need to ensure \( d_1 \tilde{b}_4 + a_1 \neq 0 \) for our three boundary conditions to be well-posed. Thus, we assume this to hold true. If, however, this condition is not satisfied then modified boundary conditions would need to be found. The boundary conditions for the compliant surface are thus

\[ a_0 \phi(0) + a_1 \phi'(0) = 0 \]
\[ d_0 \psi(0) + d_1 \psi'(0) = 0 \]  \hspace{1cm} (23.6)
\[ \phi(0) + \tilde{b}_3 \psi''(0) + \tilde{b}_4 \psi(0) = 0. \]

### 23.2 Boundary conditions on \( \wedge^3(\mathbb{C}^6) \)

With the boundary conditions in the required standard form, we can apply the theory of chapter 7 on exterior algebra and compound matrices.

To use the numerical framework, we require just one boundary condition formed from the wedge product of the three boundary conditions above. Thus by defining standard vectors in \( \mathbb{R}^6 \), namely \( e_1, \ldots, e_6 \). Let

\[ a = a_0 e_1 + a_1 e_2 \]
\[ b = e_1 + \tilde{b}_3 e_4 + \tilde{b}_4 e_5 \]
\[ d = d_0 e_5 + d_1 e_2, \]

and also define \( \omega \) by

\[ \omega = (\omega_1, \ldots, \omega_{20})^T, \]
where \( \omega_1 = e_1 \wedge e_2 \wedge e_3, \quad \omega_2 = e_1 \wedge e_2 \wedge e_4 \)
\[ \ldots \]
\[ \omega_{19} = e_3 \wedge e_5 \wedge e_6, \quad \omega_{20} = e_4 \wedge e_5 \wedge e_6. \]

Then,

\[ a \wedge b \wedge d = (a_0 e_1 + a_1 e_2) \wedge (e_1 + \tilde{b}_3 e_4 + \tilde{b}_4 e_5) \wedge (d_0 e_5 + d_1 e_2) \]
Part VI: Boundary conditions for the compliant surface

\[ \Delta(\lambda) = a_0 \bar{b}_3 \bar{d}_0 Z_8 - a_0 \bar{b}_3 d_1 Z_2 - (a_0 \bar{b}_4 d_1 + a_1 d_0) Z_3 + a_1 \bar{b}_3 d_0 Z_{14}, \]

where \( Z \in \wedge^3(\mathbb{C}^6) \) and \([Z, \omega_i] = Z_i\). Thus our underlying equation must satisfy

\[ \Delta(\lambda) = a_0 \bar{b}_3 d_0 Z_8 - a_0 \bar{b}_3 d_1 Z_2 - (a_0 \bar{b}_4 d_1 + a_1 d_0) Z_3 + a_1 \bar{b}_3 d_0 Z_{14} = 0 \quad (23.7) \]

at the boundary between the compliant surface and the fluid, that is, at \( z = 0 \).

For use of Newton's method for convergence we also require the derivative of this function, namely

\[ \Delta'(\lambda) = -(a_0 \bar{b}_3 d_1)Z_{22} - (a_0 \bar{b}_3 d_1)' Z_2 - (a_0 \bar{b}_4 d_1 + a_1 d_0) Z_{23} - (a_0 \bar{b}_4 d_1 + a_1 d_0)' Z_3 + (a_0 \bar{b}_3 d_0) Z_{28} + (a_0 \bar{b}_3 d_0)' Z_8 + (a_1 \bar{b}_3 d_0) Z_{34} + (a_1 \bar{b}_3 d_0)' Z_{14}. \]

### 23.3 Computed neutral curves-compliant wall

Figure 23.1 shows the computed neutral curves of stability for various values of spring-stiffness. The curves were produced using a simple linear relation. Taking the values of the nose of the rigid wall neutral curve as the starting point and decreasing the spring stiffness parameter, \( E \), to various values which are then fixed and Reynolds number varied searching for neutral points in the wave number, \( \alpha \) direction.

The effect of a compliant surface appears to have a significant effect on the shape of the curve of neutral stability. The instability region appears to shrink, break away from the rigid wall neutral curve and form closed instability regions. This effect is similar to that obtained by the addition of a flexible surface in the 2-dimensional Blasius flow problem. Figure 23.2, shows a blowup of the region near the nose of the neutral curve as \( E \) approaches \( E_c \). The point \( E_c \) which we have computed to be \( E_c = 0.01236 \), is the value of \( E \) at which the neutral curve collapses to a point. This point is important in applications because for values of \( E < E_c \), the flow is extraordinarily stable: the transition Reynolds number has been increased dramatically. This effect suggests that compliant surfaces could reduce drag over a swept wing by delaying transition to turbulence.
Figure 23.1: Curves of neutral stability for the attachment line boundary layer over a compliant surface with varying spring stiffness

Figure 23.2: Blow up of the nose of the neutral curve for the attachment-line boundary layer over a compliant surface
The aim of this research was two-fold. Firstly, to develop numerical methods for solving stiff ordinary differential equations in hydrodynamic stability and secondly, to study the interaction between two-dimensional and three-dimensional viscous fluid flow and a compliant surface under chosen flow conditions. A discussion and the relevant conclusions of which are given in the following chapters.
A major achievement has been the development of a general framework for solving stiff linear ODEs in hydrodynamic stability on unbounded domains without requiring orthogonalization. This new framework uses exterior algebra as a starting point and consists of a reformulation of the compound matrix method in a coordinate-free way by using the exterior algebra spaces, $\bigwedge^k(\mathbb{C}^n)$ where $k$ is the subspace dimension and $n$ the dimension of the original system. This framework is most advantageous for numerical solution of differential eigenvalue problems on unbounded domains where significant difficulties arise in setting up matrix discretizations (the most obvious alternative to a shooting algorithm).

The original compound matrix method was relatively straightforward to set up for a four-dimensional problem, however, in the classical literature, several issues with this method were unresolved and no clear explanations of how to generalise the method for application to higher dimensions has been available. Implicit in the formulation of the compound matrix is a choice of basis for $\mathbb{C}^n$. How can the basis be changed? In principle, the idea should work for any $k$, $1 \leq k \leq n$, but how can this be done in a straight forward and implementable way? What about boundary conditions in infinite domains? Is there any advantage to using particular numerical integrators? These questions have now been answered during the course of this research. It is clear that due to the rapid increase in the order of the compound matrix system, the method is unlikely to be practical for problems of order higher than eight. Nevertheless, this new framework can be set up to solve fourth and sixth order problems with great ease and confidence, being applicable to a large class of problems many arising from the study of hydrodynamic stability. The theory has been illustrated in this report by application to three differential eigenvalue problems on unbounded intervals: hydrodynamic stability of boundary layer flow over
Part VII: Development of numerical methods

a flat plate, the eigenvalue problem associated with the stability of the Ekman layer in atmospheric dynamics and the stability problem of the attachment-line boundary layer flow past a swept wing.

The question regarding the possible advantages of using particular numerical integrators was answered by the investigation into the theory of geometric numerical integration. It was discovered that a class of implicit Runge-Kutta methods are ideal for the required preservation of a quadratic function to machine accuracy. Commonly used fourth-order Runge-Kutta algorithms will not necessarily preserve this function accurately, especially over long-range integration.

The problem of how to form boundary conditions in infinite domains has been solved by the construction of a new formulation framework for asymptotic boundary conditions. The results of Ng & Reid [93] and Davey [34] who derived asymptotic conditions for integration using the compound matrix method have been expanded using the exterior algebra approach leading to a new and straightforward method for obtaining asymptotic boundary conditions. These asymptotic conditions can be derived using the induced system and left eigenvector theory. A numerical algorithm has been created for computing starting vectors of asymptotic systems.
Interaction between two-dimensional and three-dimensional viscous flow and the compliant surface

The two-dimensional theoretical plate-spring model for a Kramer-type compliant surface developed by Carpenter & Garrad [23] was used as a starting point for developing a model for the compliant surface to investigate numerically the hydrodynamic stability of such surfaces. The two-dimensional model for the compliant wall has been extended to three-dimensions and adjusted to allow for a more generalized framework providing greater range of applicability in the stability investigations for a compliant wall by including the possibility of an anisotropic compliant surface. Although the effects of an anisotropic compliant surface have not been investigated during the course of this research, the framework for this type of wall having been derived opens up this future area of research.

The effect of the Kramer type flexible surface on the instability in the Blasius, Ekman and attachment-line boundary layer flow configurations has been investigated. The already thoroughly investigated Blasius case [23] was implemented to test the new numerical framework and thus obtain the most accurate results to date. The two-dimensional stability analysis of the Blasius boundary layer flow focuses on a relatively simple flow configuration with the aim of establishing whether wall compliance could have any effect on the powerful TS instability mechanism. Results show that damping does not have a significant effect on the minimum Reynolds number for instability, but does have a noticeable effect on the shape of the region of instability. Non-zero tension has an insignificant effect on the instability characteristics of the compliant wall boundary layer interaction.

The preceding research moves on from the two-dimensional Blasius case by investigating the effect of the wall compliance on the instability in three-dimensional boundary layers in the form of two model problems: The effects of the flexible surface on the rotating Ekman velocity profile, and the effect of compliance on the attachment line instability for
flow over a swept wing with direct reference to the dolphin. In the former case two forms of instability can arise from the flow: an inviscid type (type-1) generated by the presence of an inflexion point in the velocity profile, and a second viscous instability (type-2) which is reliant on Coriolis effects for its existence. With the inclusion of a compliant wall, we generate another instability, namely a flow induced surface instability and of course also the possibility of and combination of mode interaction.

25.1 Stability features of rotating Ekman flow

Ekman layer flow was used to study the influence of rotation on hydrodynamic stability through consideration of Coriolis effects which gives rise to a strong instability mechanism producing instability at much more modest Reynolds numbers. The boundary layer disturbances were assumed to grow temporally and the Ekman layer has been investigated using linear stability theory producing neutral curves calculated for a range of rotation rates and Reynolds numbers using our framework for numerically integrating on $\lambda^{(2)}(C^5)$. In conjunction with this numerical method, a Newton's algorithm was used in order to determine the zeros of the stability eigenvalue. The general convective nature of much of this flow is already known where the type-2 instability is first to become unstable at the relatively low Reynolds number of $R_e = 54.15504$. A disturbance in a convectively unstable flow is swept away as it grows and so the source area is ultimately left undisturbed, so the boundary later remains basically laminar until the instability wave has travelled far enough away to have grown to amplitudes sufficient to cause non-linearities. To further this line of research, the plate-spring compliant wall model has been incorporated and the stability response assessed in comparison to the rigid wall results. Differing degrees of wall compliance were considered by fixing the wall parameters and only varying the spring stiffness parameter, $E$. Curves of neutral stability for the type-2 disturbance were found to be unaffected by the presence of the compliant boundary at low Reynolds numbers where the type-2 instability is dominant. However as the Reynolds number increases to the point where the type-1 instability dominates, the curves of neutral stability appear to depart downwards from the rigid wall neutral curve and turn as if going to form closed instability loops. That is, these results indicate that compliant walls do not have a significant stabilizing effect on the high growth rates of the Ekman type-2 instability, but do
appear to have a stabilizing effect on the inflexion type-1 instability mode. The compliant surface neutral curves are, however, incomplete since mode coalescence occurs for certain parameter regions.

25.2 Stability features of the attachment line flow past a swept wing

As with the previous two problems discussed during this research, the boundary layer disturbances for the attachment-line flow were assumed to grow temporally and the linear stability theory used to produce the neutral curves in conjunction with our numerical framework on $\Lambda^{(3)}(C^d)$. This line of research being directly related to the dolphin due to the obvious 'swept wing' formation of this creatures three fin types, was thus naturally furthered by the inclusion of the plate-spring compliant surface as a model for the dolphins flexible skin, Differing degrees of wall compliance has revealed that an increase in the flexibility of the surface could indeed reduce the instability region. Furthermore, this reduction appears to be similar to that obtained for the introduction of the compliant surface in the Blasius boundary layer flow. This indicates that the introduction of a swept wing does not effect the use of a compliant surface for its transition delaying properties.
In this section suggestions for furthering the current research along with other interesting related potential research problems shall be discussed.

26.1 Global analysis

The results given in this report relating to the effects of the interaction of a compliant wall with a three-dimensional boundary layer led to issues of mode coalescence rendering incomplete neutral curves. To add to these results and to ensure also that the most unstable modes have actually been found, a global eigenvalue search scheme would be of immense assistance by allowing all modes to be tracked and thus full curves of neutral stability to be produced. Cooper [29] suggested that a similar global analysis scheme as that presented by Yeo in 1995 for a wall model in a two-dimensional temporal analysis involving the Blasius boundary layer, could potentially be applied to the coupled fluid/compliant wall rotating disk system. Thus it seems feasible that this type of global scheme could also be modified for use with the Ekman layer and attachment line problems.

26.2 The Compliant wall

Throughout this research, the compliant wall has consisted of some modified/extended version of the simple two-dimensional plate-spring model designed by Carpenter & Garrad [23]. The modification of the compliant wall model involved an extension to three-dimensionality and also the incorporation of an anisotropic surface. However, an investigation of the possible effects due to anisotropy was not considered and hence this is a natural path for further research. This study would initially involve using the model provided in
part III of the current research and studying the possible effects of an anisotropic wall by varying the ratio, $\frac{D_x}{D_y} = (1 + \chi_2)$, of the flexural rigidities in the $x$ and $y$ directions. That is, varying the parameter, $\chi_2$ in the derived dynamic boundary conditions. Continuing this area of research still further, stability investigation could be carried out using more sophisticated/realistic compliant wall models.

### 26.3 Oscillating boundary layers

The dolphin swims with the aid of the propulsive power of its tail fluke. Considerable body movement and tail movement can be observed from a swimming dolphin indicating that thrust is developed by both means. Some body movement is required, of course, to keep the net forces and movements in balance. Figure 26.1 shows a superimposed diagrammatic form of a dolphin's movement during acceleration [80]. One possible method of reducing drag is based on the effects of the body undulations (wave-like up and down movement). With this as a motivation, an area of interest for research would be initially to study the stability of an oscillating boundary layer over a compliant surface.

![Figure 26.1: Dolphin movements during acceleration](image-url)
Bibliography


[99] Owen, P.R. and Randall, D.J. (1952) *Boundary Layer Transition on the Swept Wing,* RAE TM Aero 277.


This appendix contains a MAPLE code which converts a $6 \times 6$ matrix on $\mathbb{C}^6$ to a $20 \times 20$ matrix on $\wedge^3(\mathbb{C}^6)$, given any basis $e_1, \ldots, e_6$ of $\mathbb{C}^6$. The only inputs are the entries of the $6 \times 6$ matrix and the 6 basis vectors.

```
with(linalg); E := vector(6, [e1, e2, e3, e4, e5, e6]);
A := array(1..6, 1..6, [[a11, a12, a13, a14, a15, a16], [a21, a22, a23, a24, a25, a26], [a31, a32, a33, a34, a35, a36], [a41, a42, a43, a44, a45, a46], [a51, a52, a53, a54, a55, a56], [a61, a62, a63, a64, a65, a66]]);
e1 := vector(6, [1, 0, 0, 0, 0, 0]);
e2 := vector(6, [0, 1, 0, 0, 0, 0]);
e3 := vector(6, [0, 0, 1, 0, 0, 0]);
e4 := vector(6, [0, 0, 0, 1, 0, 0]);
e5 := vector(6, [0, 0, 0, 0, 1, 0]);
e6 := vector(6, [0, 0, 0, 0, 0, 1]);
B := array(1..20, 1..20):
for i from 1 to 20 do;
  for j from 1 to 20 do;
    if member(i, $1..4$) then
```

Compound matrix conversion code

\[ \begin{align*}
aa & := E[1]; \\
bb & := E[2]; \\
cc & := E[i + 2]; \\

e\text{elif member}(i,5..7) & \text{ then} \\
aa & := E[1]; \\
bb & := E[3]; \\
cc & := E[i - 1]; \\

e\text{elif member}(i,8..9) & \text{ then} \\
aa & := E[1]; \\
bb & := E[4]; \\
cc & := E[i - 3]; \\

e\text{elif } (i = 10) & \text{ then} \\
aa & := E[1]; \\
bb & := E[5]; \\
cc & := E[i - 4]; \\

e\text{elif member}(i,11..13) & \text{ then} \\
aa & := E[2]; \\
bb & := E[3]; \\
cc & := E[i - 7]; \\

e\text{elif member}(i,14..15) & \text{ then} \\
aa & := E[2]; \\
bb & := E[4]; \\
cc & := E[i - 9]; \\

e\text{elif } (i = 16) & \text{ then} \\
aa & := E[2]; \\
bb & := E[5]; \\
\end{align*} \]
cc := E[i - 10];

elif member(i,$17..18) then
    aa := E[3];
    bb := E[4];
    cc := E[i - 12];

elif (i = 19) then
    aa := E[3];
    bb := E[5];
    cc := E[i - 13];

elif (i = 20) then
    aa := E[4];
    bb := E[5];
    cc := E[i - 14];
fi;

if member(j,$1..4) then
    dd := E[1];
    ee := E[2];
    ff := E[j + 2];

elif member(j,$5..7) then
    dd := E[1];
    ee := E[3];
    ff := E[j - 1];

elif member(j,$8..9) then
    dd := E[1];
    ee := E[4];
Compound matrix conversion code

\[ ff := E[j - 3]; \]

\text{elif} (j = 10) \text{ then}
\[ dd := E[1]; \]
\[ ee := E[5]; \]
\[ ff := E[j - 4]; \]

\text{elif member}(j, 11..13) \text{ then}
\[ dd := E[2]; \]
\[ ee := E[3]; \]
\[ ff := E[j - 7]; \]

\text{elif member}(j, 14..15) \text{ then}
\[ dd := E[2]; \]
\[ ee := E[4]; \]
\[ ff := E[j - 9]; \]

\text{elif} (j = 16) \text{ then}
\[ dd := E[2]; \]
\[ ee := E[5]; \]
\[ ff := E[j - 10]; \]

\text{elif member}(j, 17..18) \text{ then}
\[ dd := E[3]; \]
\[ ee := E[4]; \]
\[ ff := E[j - 12]; \]

\text{elif} (j = 19) \text{ then}
\[ dd := E[3]; \]
\[ ee := E[5]; \]
\[ ff := E[j - 13]; \]
elif (j = 20) then
  \[ dd := E[4]; \]
  \[ ee := E[5]; \]
  \[ ff := E[j - 14]; \]
fi;

\[ ad := \text{innerprod}(aa, dd); \]
\[ bd := \text{innerprod}(bb, dd); \]
\[ cd := \text{innerprod}(cc, dd); \]
\[ ae := \text{innerprod}(aa, ee); \]
\[ be := \text{innerprod}(bb, ee); \]
\[ ce := \text{innerprod}(cc, ee); \]
\[ af := \text{innerprod}(aa, ff); \]
\[ bf := \text{innerprod}(bb, ff); \]
\[ cf := \text{innerprod}(cc, ff); \]

\[ x := \text{evalm}(A\& * dd); \]
\[ y := \text{evalm}(A\& * ee); \]
\[ z := \text{evalm}(A\& * ff); \]

\[ aAd := \text{innerprod}(aa, x); \]
\[ aAe := \text{innerprod}(aa, y); \]
\[ aAf := \text{innerprod}(aa, z); \]
\[ bAd := \text{innerprod}(bb, x); \]
\[ bAe := \text{innerprod}(bb, y); \]
\[ bAf := \text{innerprod}(bb, z); \]
\[ cAd := \text{innerprod}(cc, x); \]
\[ cAe := \text{innerprod}(cc, y); \]
\[ cAf := \text{innerprod}(cc, z); \]

\[ B1 := \text{array}(1..3, 1..3, [[aAd, ae, aAf], [bAd, be, bf], [cAd, ce, cf]]); \]
$B2 := \text{array}(1..3,1..3, [[ad, aAe, af], [bd, bAe, bf], [cd, cAe, cf]]);$  
$B3 := \text{array}(1..3,1..3, [[ad, ae, aAe], [bd, be, bAe], [cd, ce, cAe]]);$  

$BB1 := \text{det}(B1);$  
$BB2 := \text{det}(B2);$  
$BB3 := \text{det}(B3);$  

$B[i,j] := BB1 + BB2 + BB3;$  

od;  
od;  

print(B);
Graph of the Eigenfunctions for plane Poiseuille flow

Figure B.1: The real and imaginary parts of the eigenfunction $\phi = \phi_r + i\phi_i$ when $\alpha = 1$ and $R_e = 10^6$ reproduced from Davey [35].