Some Low Reynolds Number Flows Involving Solid Boundaries And Porous Media

being a thesis submitted for the degree of Master of Philosophy in the University of Surrey by Frank A. N. STENGEL M. ès Sc., Agrégé de Mathématiques.

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Abstract

After a short introduction defining the notion of fluid and deriving the equation for the slow creeping motion of an incompressible Newtonian viscous fluid, the Stokes equation, the thesis moves on to describe flows around hollow surfaces. The surfaces studied are cylinders and surfaces of revolution and quantities such as drag and flux through the surface are calculated, whenever possible.

The next chapter studies the slow creeping flow of a fluid in a porous medium and studies the different ways to model that flow. After finding the equations of the motion it goes on to investigate the boundary conditions arising by having a fluid passing from a "free" medium to a porous medium.

Chapter four describes the slow axisymmetric rotation of a solid above a porous bed of infinite thickness. Using Green's functions it derives an integral equation giving the surface stress on the solid, enabling therefore to find the drag on the solid, and to prove, in this case Brenner's (Brenner [2]) asymptotic formula for the drag. It then goes on to make a complete study of the case of a disk by computing the velocity field and looking at the behaviour of the fluid far from the disk. Various graphs giving velocity profiles and functions associated with the drag are given.

Chapter five describes the flow created by an axisymmetric stokeslet above a porous bed. It compares two models for the flow in the porous part, the Darcy and Brinkman models. At the same occasion it derives a new representation for the velocity field, in the case of the Binkman model.

The last chapter sketches a description of the asymmetric flow generated by a stokeslet or a rotelet and of the problems encountered while trying to compute this flow. It criticizes in particular the new representation found in the previous chapter.
To my parents

Heureux qui, comme Ulysse, a fait un long voyage
Ou comme c'estuy là qui conquit la toison ...
Joachim du Bellay (1522-1560) Regrets XXXI

I would also like to thank

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<th>Symbol</th>
<th>Description</th>
<th>Dimension</th>
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<tbody>
<tr>
<td>$L$</td>
<td>Unit of length</td>
<td></td>
</tr>
<tr>
<td>$M$</td>
<td>Unit of mass</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>Unit of time</td>
<td></td>
</tr>
<tr>
<td>$(\rho, \varphi, z)$</td>
<td>Cylindrical polar coordinate system</td>
<td></td>
</tr>
<tr>
<td>$\delta_0$</td>
<td>Dirac $\delta$ distribution</td>
<td></td>
</tr>
<tr>
<td>$\delta_{ab}$</td>
<td>Kronecker $\delta$ symbol</td>
<td></td>
</tr>
<tr>
<td>$F, F_{\infty}$</td>
<td>Drag</td>
<td>$MLT^{-2}$</td>
</tr>
<tr>
<td>$h$</td>
<td>Distance between object and interface</td>
<td>$L$</td>
</tr>
<tr>
<td>$J_v$</td>
<td>Bessel function of order $v$</td>
<td></td>
</tr>
<tr>
<td>$k, K$</td>
<td>Resistance factor, operator</td>
<td>$L^2$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$\sqrt{k}$, Brinkman constant</td>
<td>$L^{-1}$</td>
</tr>
<tr>
<td>$L, L_{\infty}$</td>
<td>Torque</td>
<td>$ML^2T^{-2}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Kinematic viscosity</td>
<td>$L^{-1}T^{-1}$</td>
</tr>
<tr>
<td>$v$</td>
<td>Strength of stokeslet</td>
<td>$ML^2T^{-1}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Strength of rotelet</td>
<td>$ML^3T^{-1}$</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure field</td>
<td>$ML^{-1}T^{-2}$</td>
</tr>
<tr>
<td>$p, \tilde{p}, p_j$</td>
<td>Pressure associated with a stokeslet</td>
<td></td>
</tr>
<tr>
<td>$\tilde{R}$</td>
<td>$\sqrt{\rho^2 + z^2}$</td>
<td>$L$</td>
</tr>
<tr>
<td>$R$</td>
<td>$\sqrt{\rho^2 + (z - h)^2}$</td>
<td>$L$</td>
</tr>
<tr>
<td>$r$</td>
<td>$\sqrt{\rho^2 + h^2}$</td>
<td>$L$</td>
</tr>
<tr>
<td>$Re$</td>
<td>Reynolds number</td>
<td></td>
</tr>
<tr>
<td>$T(u, p)$</td>
<td>Stress tensor</td>
<td>$ML^{-1}T^{-2}$</td>
</tr>
<tr>
<td>$t, t_i, t_{ij}$</td>
<td>Stokeslet</td>
<td></td>
</tr>
<tr>
<td>$(u, v, w)$</td>
<td>Velocity field in cylindrical coordinates</td>
<td>$LT^{-1}$</td>
</tr>
<tr>
<td>$u$</td>
<td>Velocity field</td>
<td>$LT^{-1}$</td>
</tr>
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Chapter One

Introduction

1.1: Fluids and solids, what are they?

The purpose of this thesis is to study the flow of fluids in various conditions. The first word to define is the word fluid. After a little thought one discovers that defining the concept of a fluid is not an easy affair. Indeed, a "good" definition should be broad enough to consider and include all the fluids we encounter in everyday life such as water, air, washing-up liquid et al. and remain precise enough so as not to include physical objects that, intuitively at least, cannot be considered as being fluids. The easiest definition for a fluid that comes to mind is the following:

«A fluid is a continuous medium which is not solid.»

The question that arises immediately is "What is solid?". Before answering this rather fundamental question, let us define what a continuous medium is. A continuous medium, for the time being is defined as being a substance that to the eye and instruments of an external observer appears to be smooth and continuous, and does not show any fine internal structure. This definition of a continuous medium is going to be refined later on but is essentially accurate.

Let us expand on that rather short and imprecise definition, by first defining what a solid is. A solid is a continuous medium that undergoes small elastic deformations when subjected to stress and has a preferred shape to which it returns, or tries to return to, when the stress is removed. One should add that a main property of a solid is to be able to hold a rigid shape when subjected to no stress. Indeed the first part of the definition of a solid tends to exclude solids that have no elastic phase, that is that, as soon as they start to show deformations, these are permanent or more precisely, plastic deformations as, for example plasticine. Actually most objects considered as being solids show an elastic phase. However this elastic phase can be very short (i.e. you need a very small amount of stress to have a plastic deformation). As we have seen it, two keywords, characteristic of solids, are:

- A solid is generally elastic
- A solid has a preferred shape, and holds a shape.

So a fluid can now be defined in the following way:
«A fluid is a continuous medium that undergoes permanent (i.e. non reversible) deformations when subjected to stress (i.e. it flows) and that has no preferred shape.»

This definition of a fluid is still very vague. Here are some examples of fluids fitting this definition:

- **Liquids and gases.** These are the obvious choices, and also the simpler cases. Their general physical properties are well known, and well described. In this category one finds monomolecular liquids and gases (water and hydrogen for instance), but also mixtures of various monomolecular liquids and gases (e.g. air or medicinal alcohol which is 95% ethanol 5% water).

- **Solutions.** These are the mixing of a solvent (usually water) and molecules of a substance that generally does not exist in a totally pure form or that is not a fluid when pure. Examples are all the acido-basic solutions of chemistry.

- **Suspensions.** These are the mixing of a solvent and small particles or molecules which do not interact with each other. The basic example is paint or ink: pigment powder suspended in volatile solvent. More complex examples are sand suspended in water for water jets used in cleaning, detergent molecules suspended in water.

- **Colloids.** These are the very viscous suspension of macromolecules in a solvent, where the (macro-)molecules still interact with each other. A good example is colloidal starch, which is a starch-water mixture having been heated and then cooled, giving a rather sticky and unsavory substance.

These fluids are obvious ones, that is they occur relatively often in the real world. But the definition of fluid is not limited to the list given and a pathological example is blood. It is a liquid tissue with a suspension of blood particles, macromolecules, and gases whose properties vary according to the various concentration of its components and external factors.

Looking at the above mentioned examples an apparent contradiction between the physical reality of a fluid and its theoretical modelling as a continuous medium seems to appear. The real fluid has a fine, microscopic, structure, whereas a continuous medium has no fine structure. To be able to reconcile the theoretical definition (no fine structure) and practice (a fine structure), one has to make a hypothesis. This hypothesis is called the continuum hypothesis. It states the following:

There is a characteristic length \( l \) above which the fluid can be considered as a true continuous medium. That is, as long as one is only interested in phenomena whose sizes are larger than \( l \) or in physical quantities which are measured over distances larger than \( l \), one can model the physical
fluid as a continuous medium, and consider these phenomena or physical quantities (such as velocity or density) to be smooth with respect to space and time.

To put it in other words, we suppose, that as long as we are not interested in the microscopic aspect of things, we can model a physical fluid as a continuous medium.

This characteristic length \( l \) has obviously to be much larger than the size of the constituents of the fluid. For instance, water molecules have sizes of the order of 10Å, so one can take \( l = 1 \mu m = 10000 \text{Å} \), to be able not to notice any more individual molecules and their microscopic properties (e.g. Brownian movement...).

An example is density. If one fixes \( x \) a point in space and \( t \) an instant in time, and if one call \( B(x,d) \) the sphere of diameter \( d \) around \( x \), then the estimated density \( \rho(x,t)_d \) is given by

\[
\rho(x,t)_d = \frac{\text{mass of the fluid inside } B(x,d) \text{ at the instant } t}{\text{volume of } B(x,d)},
\]

is almost constant for values of \( d \) of the order of \( l \), and becomes erratic and oscillates wildly for values of \( d \) tending towards 0 (the emergence of the fine structure). When one increases \( d \), \( \rho(x,t)_d \) is bound to vary, but these variations are no longer due to the neighbourhood of \( x \), but more to the variations in density of the fluid in space.

From now on we will assume that by applying the continuum hypothesis all fluids are true continuous media. A few basic physical quantities need to be defined.

1.2 : The physical quantities

The physical quantities we will have to define beforehand are : velocity, density, viscosity, and Reynolds number. Let us take a region containing a fluid, and let us take a point \( x \) inside the fluid at the instant \( t \). We suppose that \( x \) is a fixed point, i.e. it does not move with the fluid.

- The velocity \( u(x,t) \) (dimension \( LT^{-1} \)) is basically the velocity of a small volume (of diameter the order of the characteristic distance \( l \)) of fluid surrounding \( x \) at \( t \), or the average of the velocities of the constituents of the fluid inside a small volume surrounding \( x \) at \( t \).

- The density \( \rho(x,t) \) (dimension \( ML^{-3} \)) is defined as the ratio between the mass of fluid contained in a small volume and its volume. Density is a function of time and space, that is a fluid can have variable density, in which case it is said to be compressible. A fluid with density that does not depend on time and space is called incompressible. An example of compressible fluid is air. A whole class of incompressible fluids is liquids in general (e.g. water et al.).
• The static pressure $p(x,t)$ (dimension $ML^{-1}T^{-2}$) can only be defined when the fluid is at rest, that is only when for all $x$ in the fluid $u(x,t) = 0$ (i.e. when it is not moving). In that case the pressure measures the internal surface stresses. If one considers an oriented elementary surface centred around $x$ at the instant $t$, $dS = n \, dS$, where $n$ is a normal to $dS$, then the force $F$ the fluid would exert on $dS$ if the fluid on the side towards which $n$ points was replaced by void is $F = p(x,t) \, n$. If one considers the fact that the fluid has to be at rest for one to define the static pressure, and if one calls $X(x,t)$ the volume forces acting on the fluid in $x$ at $t$, then the equating of the forces will lead to the equation, $X(x,t) = - \nabla p(x,t)$. That is one condition for the fluid to be at rest is for the volume forces to derive from a potential (viz. $- p(x,t)$).

• The viscosity $\mu$ (dimension $ML^{-1}T^{-1}$) is the quantity measuring the forces resisting deformation. It is defined as being the ratio

$$
\mu = \frac{\text{Shear stress}}{\text{Rate of deformation}},
$$

where shear stress is the force resisting deformation. The viscosity plays a very important role in the nature of the fluid. It may vary with many parameters. It usually varies with temperature and static pressure, but some other variables may affect the viscosity. For instance with thixotropic fluids, the viscosity varies with the amount of stress applied. A good example is printer's ink. When at rest it is an extremely viscous fluid, so much that it actually resembles jelly. When subjected to stress (shaken, stirred etc.) its viscosity drops considerably, and the more stress that is applied the easier it flows, which is the desired effect. Indeed one wants ink to be easily applicable on a surface (easy flow when stress is applied) and to stay on that surface until printing (i.e. very high viscosity when at rest, therefore very slow movement, if any).

A very important class of fluids is Newtonian fluids. For these fluids, shear stress is linearly dependent on the rate of deformation (strain). That is for a Newtonian fluid viscosity is independent of the magnitude of the deformation (or shear stress).

• The Reynolds number $Re$ is a dimensionless number giving an idea of the relative importance of viscous forces and inertial forces when an object, or a collection of objects, are moving within the fluid. It was named after Osborne Reynolds (1842-1912). Viscous forces are the forces generated in resistance to deformation, whereas inertial forces are generated by the elastic properties of the fluid. A basic example of an inertial force is the fluid bouncing off a surface, or an opposing flow. Let us suppose we have an object, or a collection of objects, moving with an average velocity $v$, of characteristic size $a$ in a fluid. The Reynolds number associated with this movement is
\[ Re = \frac{\rho v a}{\mu}, \]

where \( \rho \) and \( \mu \) are the characteristic density and viscosity of the fluid. \( Re \) is directly linked to the complexity of the flow of the fluid. The greater \( Re \), the more turbulent the flow will be. As long as \( Re \) is small enough (of the order 1) the flow will be relatively simple (i.e. non turbulent, or laminar). These flows are called slow viscous flows (the viscous forces are predominant). To give an idea of Reynolds numbers, here are a few samples

Speck of dust settling in the air : \( a = 0.1 \text{ mm}, v = 1 \text{ mm/s}, \rho = 1.202 \text{ kg/m}^3, \mu = 1.80 \times 10^{-5} \text{ Pa s}, Re = 3.3 \times 10^{-2} \)

Cannon ball falling in water : \( a = 10 \text{ cm}, v = 10 \text{ m/s}, \rho = 1000 \text{ kg/m}^3, \mu = 0.997 \times 10^{-3} \text{ Pa s}, Re = 10^5 \)

Jet flying at cruise speed at sea level : \( a = 100 \text{ m}, v = 800 \text{ km/h}, \rho = 1.202 \text{ kg/m}^3, \mu = 1.80 \times 10^{-5} \text{ Pa s}, Re = 1.9 \times 10^{10} \)

Reynolds numbers, not only give an idea of the complexity (i.e. the amount or existence of turbulence) of a flow, but also give a way to compare flows. The flows to be compared have to be similar. For instance two different sized torpedoes at different speeds in water. If two Reynolds numbers computed with corresponding values are equal, or very near, then the flow will be similar. An example of similar configurations with similar Reynolds numbers is:

Cannon ball falling in water : \( a_1 = 10 \text{ cm}, v_1 = 10 \text{ m/s}, \rho_1 = 1000 \text{ kg/m}^3, \mu_1 = 0.997 \times 10^{-3} \text{ Pa s}, Re_1 = 10^5 \)

A baseball having just been pitched : \( a_2 = 3 \text{ cm}, v_2 = 50 \text{ m/s}, \rho_2 = 1.204 \text{ kg/m}^3, \mu_2 = 1.80 \times 10^{-5} \text{ Pa s}, Re_2 = 10^5 \)

Very often, to simplify calculation, another quantity called kinematic viscosity, \( \nu \), is introduced. It is the ratio between the viscosity \( \mu \) and the density \( \rho \). An alternate expression for the Reynolds number is:

\[ Re = \frac{\nu a}{\nu}. \]

The tables [App 1-1,2,3] found in appendix 1 give a listing of densities, dynamic and kinematic viscosities for selected fluids.

In the main body of our study we will study fluids that have the following properties
• They are \textbf{Newtonian} fluids, that is the viscosity is independent of the deformation tensor
• They are \textbf{incompressible} fluids, that is their density is constant
• Their flow is sufficiently \textit{slow} so that the Reynolds number associated with it is very small
  \hspace{1cm} (of order one)

Let us now study the derivation of the main equation used in this thesis: the Stokes equation.

\textbf{1.3 : Basic equations}

Let us use Einstein's convention for sums involving cartesian tensors, that is, if an index
occurs twice in a product then one sums with respect to that index. For instance : \(a_{ij} x_j\) stands
for \(\sum_j a_{ij} x_j\). This simplification is only possible if we keep to cartesian coordinate systems.

Let us introduce a cartesian coordinate system of coordinates \((x_1,x_2,x_3)\) on the space in
which the fluid is moving. Let us call \(u_i\), \(i=1,2,3\) the components of the velocity \(u\) of the fluid
particle at \(x = (x_1,x_2,x_3)\) and at the instant \(t\). Let us call \(V(t)\) a volume of fluid containing the
particles that were in \(V(t_0)\) at \(t = t_0\), and \(S(t) = \partial V(t)\) its boundary. Newton's law of motion
states that

\[
\int_{V(t)} \left( \rho u_i \right)_t \, dx - \int_{V(t_0)} \left( \rho u_i \right)_{t_0} \, dx = \int_{t_0}^{t} d\tau \int_{V(t)} X_i \, dx + \int_{t_0}^{t} d\tau \int_{S(t)} F_i \, dS(x), \tag{1-1}
\]

where, \(X = (X_1,X_2,X_3)\) is the field of external volume forces acting at the instant \(t\) inside
\(V(t)\), and \(F = (F_1,F_2,F_3)\) is the field of surface forces acting on \(S(t)\) at the instant \(t\). The equation
\[1-1\), called the impulse equation, is simply saying that the difference in impulse for the volume
of fluid between \(t_0\) and \(t\) is exactly equal to the time integral of the forces acting in \(V(t)\) and on
\(S(t)\) between \(t_0\) and \(t\). Let us now turn to the equation governing the change in fluid mass at the
time \(t\) in the volume \(V\) of surface \(S\). The flux of matter passing through the oriented elementary
surface \(dS = n \, dS\) is exactly \(\rho \, u_i n_i\), where \(n = (n_1,n_2,n_3)\) is the outward oriented normal to \(S\).
The total variation of matter inside the volume \(V\), at the time \(t\) is :

\[
- \frac{\partial}{\partial t} \int_V \rho \, dx = \int_S \rho \, u_i n_i \, dS(x) \quad (\omega \text{ is the volume of } V). \tag{1-2}
\]

This equation, when rewritten using Green's theorem, leads to the usual continuity equation
\[-\frac{\partial \rho}{\partial t} = \text{div}(\rho \mathbf{u}) = \frac{\partial}{\partial x_1} (\rho u_1).\] As we supposed that the fluid is incompressible,

\[\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho = 0 \text{ so that } \text{div} \mathbf{u} = 0 \quad [1-3]\]

Let us turn back to the impulse equation [1-1]. Oseen (Oseen [15] pp 4-8) studies the surface forces, and comes to the conclusion that they are linear functions of the normal drawn on the surface at the point at which they are computed. Oseen, in the same chapter goes further, and links the surface forces with the tensor of pure deformations \(D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\). The careful analysis done by Oseen (Oseen [15], pp 4-8) show that the surface forces on \(S(t)\) are of the form:

\[F_i = \varphi_1 n_i + \varphi_2 n_j D_{ij} + \varphi_3 n_j D_{ik} D_{kj} + \ldots \quad [1-4]\]

The first term, \(\varphi_1 n_i\), is a familiar term. It is very similar to the static pressure force. In the case of a fluid at rest, we would have \(F_i = -p n_i\), where \(p\) is the static pressure. The obvious conclusion is to call the function \(p(x,t) = -\varphi_1(x,t)\) the dynamic pressure. The same way, by analogy with the definition of the viscosity, one can write \(\varphi_2 = 2\mu\), i.e. equate the viscous forces with the second term of the expansion for the surface forces.

The first approximation that is made is to neglect in equation [1-4] the terms involving powers of the deformation tensor larger or equal to 2. This approximation is made on the assumption that the deformations are small. This assumption seems to be verified in practice for slow flows. It is obvious that if one increases the velocity, the deformation tensor is going to increase linearly with the velocity. This implies that there is a threshold beyond which one cannot ignore any more the higher order terms in [1-4]. The main result is that for incompressible slow Newtonian flows, the surface forces are of the form

\[F(x,t)_i = -p(x,t)n_i + \mu n_j \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)(x,t) \quad [1-5]\]

The formula [In - 5] can also be justified by comparing the expansion [1-3] with the definition of a Newtonian fluid. Equating term by term the shear stress with the definition of the viscosity, equation [1-5] appears naturally. Now that we have determined, or rather imposed, the surface forces, let us look back at the impulse equation [1-1]. This equation, for the time being, considers that the volume \(V(t)\) always contains the same amount of fluid as there was at the instant \(t_0\) in \(V(t_0)\). Let us modify [1-1] for the case of a fixed volume \(V\), independent of \(t\). In this case the right-hand side of the impulse equation, the action of the forces on the volume \(V\), is
unchanged. For the left-hand side, the variation in impulse, we have to take into account the quantity of fluid that escaped from the volume \( V \) between \( t_0 \) and \( t \). During the instant \( dt \), an amount (mass) \( \rho u_j n_j dS dt \) of fluid escapes through \( dS \). The amount of impulse that vanishes through \( dS \) during \( dt \), is then \( u_i \rho u_j n_j dS dt \). The modified version of the impulse equation is now:

\[
\int_{V(t)} (\rho u_i)_t dx - \int_{V(t_0)} (\rho u_i)_{t_0} dx + \int \int_{S(t)} u_i \rho u_j n_j dS dS(x)
\]

\[
= \int_{t_0}^{t} dt \int_{S} X_i dx + \int_{t_0}^{t} dt \int_{S} (-\rho n_i + \mu n_j \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)) dS(x), \tag{1-6}
\]

Rewriting [1-6] with \( t_0 = t - dt \) and dividing by \( dt \), and using Green's theorem, one then has

\[
\frac{\partial}{\partial t} (\rho u_i) + \rho u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{\partial \rho}{\partial x_i} + \mu \Delta u_i, \tag{1-7}
\]

where \( \Delta f = \frac{\partial^2 f}{\partial x_i^2} \). This equation ([1-7]) describes the flow of an incompressible Newtonian fluid, that is a flow for which one can approximate surface forces as in [1-5]. This equation contains five different terms.

- **Viscous term** : \( \mu \Delta u_i \)
- **Pressure term** : \( -\frac{\partial \rho}{\partial x_i} \)
- **Inertial term** : \( \rho u_j \frac{\partial u_i}{\partial x_j} \)
- **Time dependence** : \( \frac{\partial}{\partial t} (\rho u_i) \)
- **External forces** : \( X_i \)

Various simplifications will be necessary to get rid of most of the terms (apart from the viscous and pressure terms).

The first simplification that can be done is to consider the flow to be steady. For a flow to be steady, it has to show no time dependence. Therefore, for a steady flow one has \( \frac{\partial}{\partial t} (u_i) = 0 \).

The second simplification that can be done is to suppose that the system (fluid flowing in a certain medium) is isolated, that is is not subjected to any external action. In this case the force
field $X$, which is the field of external volume forces, has to be equal to zero. Therefore, for an isolated system, there are no external forces term.

The third and last simplification that can be done, is to suppose the inertial forces to be negligible compared with the viscous forces, that is to suppose that the Reynolds number associated with the problem is small. In this case one can the ignore the inertial term $\rho \frac{\partial u_i}{\partial x_j}$. The flows for which this simplification can be made are called slow viscous flows.

In this thesis we will mainly work on incompressible, isolated, Newtonian fluids where the flow that is steady, slow and viscous. In these cases the equation for the motion of the fluid is

$$\mu \Delta u_i = \frac{\partial p}{\partial x_i}, \text{ or in vector form } \mu \Delta \mathbf{u} = \text{grad } p$$

This equation is generally known as the Stokes equation. It is named after Sir G. G. Stokes (1819-1903).

The equation describing the fact that the fluid is continuous and incompressible is

$$\frac{\partial (u_i)}{\partial x_i} = 0, \text{ or in vector form } \text{div}(\mathbf{u}) = 0$$

We now know which equation models the flow of a slow moving incompressible Newtonian fluid.

This equation is also used in the case where one approximates the flow generated by a slowly translating or rotating body by a steady flow. That is one prescribes the velocity of the on the surface of the solid and supposes that the solid remains fixed in space. This approximation of the flow is called a 'quasi-steady' flow. It is the approximation that will be used throughout the thesis.
Chapter Two

Flows past cylinders and other objects

Summary

After a short exposition of the problems studied, and a historical study of the derivation of the boundary integral equations by Oseen and Odqvist (Oseen [15], Odqvist [14]), this chapter is going to dwell on the flow past solid objects. It will be assumed that at the surface of these objects there is no slip. A study of the symmetries of the boundary integral equation will then be conducted. The rest of the chapter will show the application of the boundary integral equation to compute the flow past cylinders and surfaces of revolution. A last example is the asymmetrical flow past a spherical cap.

The aim of this chapter is to study the low Reynolds number flow of a viscous fluid past certain classes of solid surfaces. It is an expansion of the work carried out by I. Warrilow in his PhD. thesis (Warrilow [30]). In his work Warrilow studied a limited type of surfaces: Circular cylinders, square plates etc. The surfaces studied will be of two kinds (classes): hollow cylinders and hollow surfaces of revolution. The configuration in which we will work is the following:

The problem is set in a cartesian coordinate system with coordinates \((x, y, z)\) or \((x_1, x_2, x_3)\). An object, symmetrical around the three coordinate planes, is placed in an unbounded fluid filled region. This object can be any solid, but, in the cases that will be studied, it will actually be an open surface. The fluid is a viscous fluid, of dynamic viscosity \(\mu\), and is subjected to a uniform motion, that is that at a point sufficiently far away from the solid, the velocity of the fluid \(\mathbf{u}\) at that point will be of the form

\[
\mathbf{u} = \mathbf{u}_\infty = \alpha \mathbf{e}_z,
\]

where \(\alpha\) is a constant independent of the position, and \(\mathbf{e}_z\) is the base vector corresponding to the \(z\)-axis. (we have chosen convenient axes for this problem)
We will assume that the flow is sufficiently slow, that is that the Reynolds number associated with the problem is small, so that we can neglect the inertia forces in comparison with the viscous forces. We will furthermore assume that the flow is steady, and that, at the surface of the object there is no slip, and that there is conservation of the mass. The equation of the motion of the fluid will then be the steady Navier-Stokes equation, or, simply, the Stokes equation. That is, if we call $u$ the velocity field, and $p$ the dynamic pressure field, the equation of the motion of the fluid is

$$\mu \Delta u = \nabla p.$$  

The same way, the conservation of mass gives:

$$\text{div} \, u = 0.$$  

As has been said we suppose the Reynolds number associated with the problem to be small. This number is

$$Re = \frac{d \, u_\infty \, a}{\mu},$$

where $d$ is the density of the liquid (mass per volume), $u_\infty = |u_\infty| = \alpha$ is the velocity at infinity, and $a$ the diameter of the solid.

We would like to be able to compute two kind of physical quantities:

- The velocity and pressure fields $u, p$.
- The drag $D$ exerted by the fluid on the solid.

Before actually working on the problem of the uniform flow past an object, let us look at the tool that will permit that study:
2.1 : The boundary integral transformation

2.1.1 : history

A transformation of Stokes's equation into a boundary value problem involving integral equations was first made, at about the same time, by Oseen (Oseen [15]) and Odqvist (Odqvist [14]). The aim of this transformation was to convert the usual Stokes, conservation of mass equation pair

\[
\begin{align*}
\mu \Delta \mathbf{u} &= \nabla \cdot \mathbf{p} \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

where \( \mathbf{u} \) is the velocity field and \( \mathbf{p} \) the pressure field of a viscous fluid, into an equation, not only giving solutions to [2-1], but also solutions satisfying certain boundary conditions. These boundary conditions, were a combination of prescribing either the velocity \( \mathbf{u} \) on a surface, or the surface stress on the same surface, or a combination of the two. These transformations were first defined for the inner problem, that is the region where the fluid in motion is bounded.

Let us look at Oseen's transformation.

2.1.1.1 : Oseen's Transformation for the inner problem

Oseen starts with the following : One considers a volume \( V \) having a smooth boundary (or surface) \( \partial V = S \).

Let \( \mathbf{u}, p \) and \( \mathbf{u}', p' \) be two solutions of [2-1] regular everywhere inside \( V \) (and on \( S \)). The following identity can be then established (The Einstein convention for summation will be used):

Extended Green's formula (Oseen version)
If \( \text{div } \mathbf{u} = \text{div } \mathbf{u}' = 0 \) then we have:

\[
\int_{S} \left[ u'_{j} \left( \mu \frac{\partial u_{i}}{\partial n} - p \, n_{j} \right) - u_{j} \left( \mu \frac{\partial u'_{i}}{\partial n} - p' \, n_{j} \right) \right] dS
= \int_{V} \left[ u'_{j} \left( \Delta u_{j} - \frac{\partial p}{\partial x_{j}} \right) - u_{j} \left( \Delta u'_{j} - \frac{\partial p'}{\partial x_{j}} \right) \right] dV \tag{2-2}
\]

or

\[
\int_{S} \left[ \left( \mu \frac{\partial \mathbf{u}}{\partial n} - p \, \mathbf{n} \right) \mathbf{u}' - \left( \mu \frac{\partial \mathbf{u}'}{\partial n} - p' \, \mathbf{n} \right) \mathbf{u} \right] dS
= \int_{V} \left[ \left( \mu \Delta \mathbf{u} - \text{grad } p \right) \mathbf{u}' - \left( \mu \Delta \mathbf{u}' - \text{grad } p' \right) \mathbf{u} \right] dV \tag{2-3}
\]

where \( \mathbf{n} = (n_{1}, n_{2}, n_{3}) \) is the outward directed normal on \( S = \partial V \).

As already said the formulae [2-2,3] only hold for a pair of functions regular everywhere inside \( V \). Let us introduce now the following solutions: \( t_{j} = (t_{1j}, t_{2j}, t_{3j}), p_{j}, j = 1, 2, 3 \) defined as follows:

\[
t_{ji}(x, x^{(0)}) = \frac{\delta_{ij}}{|x - x^{(0)}|} + \frac{(x_{i} - x^{(0)}_{i})(x_{j} - x^{(0)}_{j})}{|x - x^{(0)}|^{3}}, \tag{2-4}
\]

\[
p_{j}(x, x^{(0)}) = 2 \mu \frac{x_{j} - x^{(0)}_{j}}{|x - x^{(0)}|^{3}} \tag{2-5}
\]

As it can be seen \( t_{ij}, p_{j} \) have singularities in \( x = x^{(0)} \), Let us modify the original configuration to exclude a sphere of radius \( \varepsilon \) around \( x^{(0)} \)

![Fig 2 - 2](image)

[Fig 2 - 2] The new configuration

In this configuration, equation [2-2] is still true. But, a close analysis of the inner sphere, as \( \varepsilon \) tends towards zero yields:
\[ u_i(x^{(0)}) = \frac{1}{8\pi \mu} \int_S \left[ t_{ij}(x, x^{(0)}) \left( \mu \frac{du_i}{dn} - p n_j \right)(x) - u_j(x) \left( \mu \frac{dt_{ij}}{dn} - p_i n_j \right)(x, x^{(0)}) \right] dS(x) \] [2-6]

or

\[ u(x^{(0)}) = \frac{1}{8\pi \mu} \int_S \left[ \left( \mu \frac{du}{dn} - p n \right)(x) \cdot t(x, x^{(0)}) - \left( \mu \frac{dt}{dn} - p n \right)(x, x^{(0)}) \cdot u(x) \right] dS(x) , \] [2-7]

where

\[ t(x, x^{(0)}) = ((t_{ij}))(x, x^{(0)}), \quad p(x, x^{(0)}) = ((p_{ij}))(x, x^{(0)}) . \] [2-8]

In the same way, we have:

\[ p = \frac{1}{4\pi} \int_S \left[ \tilde{p}_{ij}(x, x^{(0)}) \left( \mu \frac{du_i}{dn} - p n_j \right)(x) - \mu u_j(x) \frac{d\tilde{p}_{ij}}{dn}(x, x^{(0)}) \right] dS(x) \] [2-9]

\[ = \frac{1}{4\pi} \int_S \left[ \tilde{p}(x, x^{(0)}) \cdot \left( \mu \frac{du}{dn} - p n \right)(x) - \mu \frac{d\tilde{p}}{dn}(x, x^{(0)}) \cdot u(x) \right] dS(x) , \] [2-10]

where

\[ \tilde{p} = \text{grad} \left( \frac{1}{|x-x^{(0)}|} \right) . \] [2-11]

The interesting quantity is:

\[ \mu \frac{du}{dn} - p n = (\mu \frac{du_i}{dn} - p n_j) . \] [2-12]

This vector is closely related to the surface stress vector, which has the following components:

\[ (T(u, p))_{kj} n_k \equiv \left( \begin{array}{c} p \delta_{jk} + \mu \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \end{array} \right) n_k = \mu \frac{du_j}{dn} p n_j + \mu \frac{du_k}{\partial x_j} n_k \] [2-13]

One can also define \( T(u, p) \) as being:

\[ T(i_i u_i, p) = i_i \left( \begin{array}{c} -p \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \end{array} \right) i_j \]
As a matter of fact the component \( \frac{\partial u_k}{\partial x_j} n_k \) (the difference between \((T(u,p))_{kj} n_k\) and \(\mu \frac{du_i}{dn} - p n_j\)) is such that:

\[
\int_S \frac{\partial u_k}{\partial x_j} n_k \, dS = \int_V \frac{\partial^2 u_k}{\partial x_j \partial x_k} dV = \int_V \frac{\partial}{\partial x_j} [\text{div } u] \, dV = 0. \tag{2-14}
\]

Therefore, globally both [2-12] and [2-13], when integrated on the surface \(S\) give the total stress exerted by the liquid on the surface \(S\). The difference \(\frac{\partial u_k}{\partial x_j} n_k\) can be seen as the surface stress caused by forces internal to the surface \(S\).

Let us now look at Odqvist's formulae, that is at Odqvist's version of [2-6,7] and [2-9,10].

2.1.1.2 : Odqvist's transformation for the inner problem

Odqvist uses basically the same method, with a nuance though. He uses a different version of the extended Green's formula:

Extended Green's formula (Odqvist version, the same notations as in 2.1.1,1 are used): if \(\text{div } u = \text{div } u' = 0\) then:

\[
\left[ u'_j (T(u,p))_{kj} n_k - u_j (T(u',p'))_{kj} n_k \right] dS
= \left[ u'_j \left( \mu \delta_{jk} + \mu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \right) n_k - u_j \left( \mu \delta_{jk} + \mu \left( \frac{\partial u'_i}{\partial x_k} + \frac{\partial u'_k}{\partial x_j} \right) \right) n_k \right] dS
= \int_V \left[ \mu \Delta u_j - \frac{\partial p}{\partial x_j} \right] - u_j \left[ \mu \Delta u'_j - \frac{\partial p'}{\partial x_j} \right] dV = 0 \tag{2-15}
\]

or

\[
\left[ n \cdot T(u,p) \cdot u' - n \cdot T(u',p') \cdot u \right] dS
= \int_V \left[ (\mu \Delta u - \text{grad } p) \cdot u' - (\mu \Delta u' - \text{grad } p') \cdot u \right] dV = 0. \tag{2-16}
\]

Using the same tensors \(t,p\) Odqvist writes:

\[ u_i(x^{(0)}) = \]
The components of $T(t,p)(x,x(0))$ are:

$$
(T(t,p)(x,x(0)))_{jk} = -6\mu \frac{(x_i - x_i(0)) (x_j - x_j(0)) (x_k - x_k(0))}{|x-x(0)|^5}.
$$

Thus, if one defines, for $x$ on $S$:

$$
K_{ij}(x,x(0)) = \frac{1}{8\pi \mu} (T(t,p)(x,x(0)))_{jk} n_k = \frac{3}{4\pi} \frac{(x_i - x_i(0)) (x_j - x_j(0)) (x_k - x_k(0))}{|x-x(0)|^5} n_k
$$

$$
\psi_j(x) = \frac{1}{8\pi \mu} (-p \delta_{jk} + \mu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right)) n_k(x),
$$

then one has:

$$
u_j(x(0)) = \int_{S'} t_{ji}(x,x(0)) \psi_j(x) \, dS(x) + \int_{S'} u_j(x) K_{ji}(x,x(0)) \, dS(x).
$$

The functions $\psi_j$ and $u_j$ are functions associated in the boundary conditions. Indeed $\psi_j$ is proportional to the surface stress exerted by the fluid on the surface and, obviously, $u_j$ is the velocity of the fluid on the surface $S$. If one had defined $u_i$ using [2-21], then $u_i$ would be defined in the whole space, except, perhaps, on the surface $S$.

Odqvist then goes on to study the properties of equation [2-21] as $x(0)$ moves across the interface. His main result is:

$$
V_i = \int_{S'} t_{ji}(x,x(0)) \psi_j(x) \, dS(x) \text{ is continuous across } S,
$$

$$
W_i = \int_{S'} u_j(x) K_{ji}(x,x(0)) \, dS(x) \text{ is discontinuous across } S. \text{ One actually has, for } x(0) \text{ in } S:
$$
\begin{align}
W_i(x^{(0)})_{\text{in}} &= -\frac{1}{2} u_i(x^{(0)}) + \int_S \left( u_j(x) - u_j(x^{(0)}) \right) K_{ji}(x,x^{(0)}) \, dS(x), \\
W_i(x^{(0)})_{\text{out}} &= -\frac{1}{2} u_i(x^{(0)}) + \int_S \left( u_j(x) - u_j(x^{(0)}) \right) K_{ji}(x,x^{(0)}) \, dS(x),
\end{align}

where \( W_i(x^{(0)})_{\text{in}} \) (resp. \( W_i(x^{(0)})_{\text{out}} \)) is the limit of \( W_i(x^{(0)}) \) as \( x \to x^{(0)} \) inside \( V \) (resp. outside \( V \)). This result has the drawback that the integral:

\[
\int_S \left( u_j(x) - u_j(x^{(0)}) \right) K_{ji}(x,x^{(0)}) \, dS(x)
\]

is very difficult to compute (even numerically). Let us now move on to look at the boundary conditions in order to get a boundary integral equation.

\textbf{2.1.2. The integral equation}

From formulae [2-6,7] or [2-17,18] one easily sees that there are still unknowns in the problem: that is, in order to know \( u \) inside \( V \), one has to know \( u \) on \( S \) or \( n \cdot T(u,p) \) on \( S \) or a combination of the two. Another remark that can be made is that one might know what \( u \) would have been, if \( V \) had been large enough. An example of this is a Stokeslet inside a sphere, one knows the velocity, and pressure field for a stokeslet in the unbounded case, and the aim of the problem is actually to compute these fields in the case of the sphere. In these cases the expression for \( u \) is of the form:

\[
\begin{split}
u(x^{(0)}) &= \tilde{u}(x^{(0)}) + \frac{1}{8\pi\mu} \int_S \left[ n \cdot T(u,p)(x) \cdot t(x,x^{(0)}) - n \cdot T(t,p)(x,x^{(0)}) \cdot u(x) \right] \, dS(x),
\end{split}
\]

where \( \tilde{u}(x^{(0)}) \) is the velocity in the case of an infinite \( V \). In this case one can see that [2-25] is actually expressing \( u \) as the sum of a "forcing" velocity (\( \tilde{u} \)) and a "perturbation" velocity.

In most of the studied cases one supposes that \( u = 0 \) on \( S \), i.e. no slip on \( S \). Equation [2-25] then reduces to:

\[
0 = \tilde{u}(x^{(0)}) + \frac{1}{8\pi\mu} \int_S n \cdot T(u,p)(x) \cdot t(x,x^{(0)}) \, dS(x) \quad \text{for all } x^{(0)} \text{ on } S
\]

or

\[
0 = \tilde{u}(x^{(0)}) + \frac{1}{8\pi\mu} \int_S \theta(x) \cdot t(x,x^{(0)}) \, dS(x) \quad \text{for all } x^{(0)} \text{ on } S.
\]
In this case the unknown is $\theta(x) = n \cdot T(u,p)(x)$, the stress exerted by the fluid on the surface. Actually all the solutions of [2-26] will have the form

$$\theta(x) = n \cdot T(u,p)(x) + f(x), \text{ where } \text{div}(f) = 0 .$$  \[2-27\]

Therefore, as $n \cdot T(u,p)(x) - (\mu \frac{du}{dn} - p n)(x)$ is of zero divergence (see [2-14]),

$$\theta(x) = \left(\mu \frac{du}{dn} - p n\right)(x)$$  \[2-28\]

is also a solution of [2-26] (here the vector $f$ defined in [2-27] is the vector of components $-\frac{\partial u_k}{\partial x_j} n_k$). As we are only interested in the drag (i.e. the sum of $\phi$ on $S$ times a constant), and on the velocity and pressure fields, physical quantities for which $\phi$ needs only to be determined modulo a function of zero divergence, one can define $\phi$ either way. That is what will be done throughout the rest of this chapter. The definition chosen for $\phi$, in the rest of this chapter, will be:

$$\theta(x) = \left(\mu \frac{du}{dn} - p n\right)(x) \text{ for all } x \text{ on } S$$  \[2-29\]

The integral equation [2-26] is a relatively tame equation: the kernel $t(x,x^{(0)})$ is integrable everywhere, and its singularity in $x = x^{(0)}$ can easily be removed.

2.1.3 : The outer problem

Up to now we only looked at inner problems, that is problems inside a volume $V$. Let us now look at the problems in which the fluids occupies a region extending to infinity outside a volume $V$. That is one knows what the velocity $u$ or the surface stress $n \cdot T(u,p)(x)$ are on the boundary of $V : \partial V = S$. The problem comes from the behaviour of $u$ at infinity. One has to add a condition at infinity. That is one works supposing there is an outer shell surrounding $V$ and expanding towards infinity, on which conditions are prescribed. The usual type of condition is:

$$u \to u_\infty \text{ where } u_\infty \text{ is a constant, i.e. uniform flow past } S,$$  \[2-30\]

$$u \sim u_\infty \text{ at "infinity", for instance: stokeslet (point source) placed outside } S.$$  \[2-31\]
By a detailed study (done by Oseen [15]), solutions of [2-26] satisfy [2-31] (which is stronger than [2-30]) without change. One can therefore say that the solution to the outer problem are solutions of [2-26], that is solutions of:

\[ \theta = \tilde{u}(x^{(0)}) + \frac{1}{8\pi \mu} \int_S \phi(x) \cdot t(x,x^{(0)}) \, dS(x) \text{ for all } x^{(0)} \text{ on } S. \]

the solution being \( \theta(x) = \left( \mu \frac{du}{dn} - p \, n \right)(x) \).

2.2 : The equation [2-26]

Let us consider equation [2-26]. To simplify the computing of solutions to [2-26] it would be interesting to study the way these solutions are affected by certain symmetry operations preserving the boundary and the boundary conditions. This will prove of great use when looking at surfaces, or solids, which are symmetrical about the three cartesian coordinate planes.

2.2.1 : Properties - symmetries

Let us consider a solution \( u,p \) of Stokes's equation [2-1], satisfying the following boundary conditions:

\[
\begin{align*}
    u_{|S} &= 0 \\
    u(x) &\sim u^\infty(x) \text{ as } |x| \to \infty ,
\end{align*}
\]

\( (u^\infty,p^\infty) \) being a solution of Stokes equation in an unbounded medium without the solid \( V \). We have the following geometrical property

\[ \text{Lemma [Th.2-1]} : \text{If } \sigma \ast u^\infty = u^\infty \ast \sigma \text{ and } \sigma(S) = S, \text{ where } \sigma \text{ is an affine isometry, then } \sigma \ast u = u \ast \sigma. \]

That is if a transformation "keeps the symmetries of the problem" then the velocity field is not affected by this transformation.

Let us consider the mapping :

\[ Q(x;\eta) : \mathbb{R}^3 \times \mathbb{R}^3 \to L(\mathbb{R}^3,\mathbb{R}^3), \]
\[
Q(x;\eta)(u) = \left( \sum_{j=1}^{3} t_{jk}(x;\eta) u_j \right)_{k=1..3}
\]

where
\[
t_{jk}(x;\eta) = \frac{\delta_{jk}}{|x-\eta|} + \frac{(x_j - \eta_j) (x_k - \eta_k)}{|x-\eta|^3}
\]

Lemma [Th.2-2] : Let \( \sigma \) be the orthogonal symmetry around the plane perpendicular to the \( x_1 \) axis (\( l = 1..3 \)). We have : \( \forall \, x, \eta \in \mathbb{R}^3 \, \, Q(\sigma(x),\eta) = \sigma \cdot (Q(x,\sigma(\eta))) \cdot \sigma \).

Proof :

We already know that if \( x = (x_j) \) then \( \sigma(x) = ((-1)^{\delta_{ij}} x_j) \). Thus :

\[
t_{jk}(\sigma(x);\eta) = \frac{\delta_{jk}}{|\sigma(x)-\eta|} + \frac{((-1)^{\delta_{ij}} x_j - \eta_j) ((-1)^{\delta_{kl}} x_k - \eta_k)}{|\sigma(x)-\eta|^3}
\]

\[
= \frac{\delta_{jk}}{|x-\sigma(\eta)|} + \frac{((-1)^{\delta_{ij}} x_j - \eta_j) (x_k - (-1)^{\delta_{kl}} \eta_k)}{|x-\sigma(\eta)|^3}
\]

\[
= (-1)^{\delta_{ij}}((-1)^{\delta_{kl}} t_{jk}(x;\sigma(\eta)))
\]

which is what we required.

Let us consider the vector field \( \theta \), solution of the following equation :

\[
0 = u^\infty(x) + \int Q(x;\eta)(\theta(\eta)) \, d\Sigma(\eta) \quad \text{for all } x \text{ on } \Sigma,
\]

where \( \Sigma \) is a surface, not necessarily closed or connected. \( \theta \) is closely associated with the surface stress, or the difference in surface stress across \( \Sigma \).

Lemma [Th.2-3] : If \( \rho \cdot u^\infty = u^\infty \cdot \sigma \) and \( \sigma(\Sigma) = \Sigma \) (\( \sigma \) still being the same orthogonal symmetry as in [Th. 2-2]) and \( \rho = \epsilon \sigma \, (\epsilon^2 = 1) \), then \( \theta \cdot \sigma = \rho \cdot \theta \).

Proof :

20
We have:

\[ 0 = u^\infty(x) + \int Q(x;\eta)(\theta(\eta)) \, d\Sigma(\eta) \quad [2-39] \]

\[ = u^\infty \cdot \sigma(x) + \int Q(\sigma(x);\eta)(\theta(\eta)) \, d\Sigma(\eta) \quad [2-40] \]

\[ = \rho \cdot u^\infty(x) + \int Q(x;\sigma(\eta)) \cdot \sigma(\theta(\eta)) \, d\Sigma(\eta) \quad [2-41] \]

\[ = \rho \cdot u^\infty(x) + \rho \cdot \int Q(x;\sigma(\eta))(\rho(\theta(\eta))) \, d\Sigma(\eta) . \quad [2-42] \]

Thus:

\[ 0 = u^\infty(x) + \int Q(x;\sigma(\eta))(\rho(\theta(\eta))) \, d\Sigma(\eta) \quad [2-44] \]

\[ = u^\infty(x) + \int Q(x;\eta)(\rho(\theta(\eta))) \, d\Sigma(\eta) , \quad [2-45] \]

i.e.:

\[ 0 = \int Q(x;\eta)[\rho(\theta(\sigma(\eta))) - \theta(\eta)] \, d\Sigma(\eta) . \quad [2-46] \]

The last line remains true for every \( \Sigma \) such as \( \sigma(\Sigma) = \Sigma \), so by taking the limit when the area of \( \Sigma \) tends towards 0 (\( \Sigma \) still does not need to be connected), we have: \( \rho(\theta(\sigma(\eta))) = \theta(\eta) \), that is \( \theta \cdot \sigma = \rho \cdot \theta \). Q.E.D.

Our intention is to study the flow past objects \( S \) which are either hollow surfaces of revolution or hollow cylinders having both the following main properties:

- \( S \) is open, i.e. there is no volume \( V \) such that \( S = \partial V \) \quad [2-47]
- \( S \) is symmetrical around the three coordinate planes \quad [2-48]

Equation [2-26] was derived with closed surfaces in mind. Let us see how this equation can be modified to accommodate [2-47], with the symmetries [2-48].
2.2.2 : The surfaces studied

As it happens we are interested in the flow past hollow cylinders or hollow surfaces of revolution. These surfaces are not closed, that is they are not the boundary of a volume. Let us first look at the transformation of equation [2-26] for the case of a two-sided surface.

2.2.2.1 : Two-sided surfaces

Let us consider a surface $S$ which is not the boundary of a volume, that is such that there is no $V$ such that : $S = \partial V$, and also let us suppose that $S$ has two sides. We do not consider one sided surfaces like Moebius rings or Klein bottles. In these cases an analysis like the one conducted later will not work.

The derivation of equation [2-26], in the case of an open surface, is not valid any more. One of the reasons is that one cannot any more speak of an outer normal to the surface (what is inside or outside for a non closed surface ?). To alleviate this problem one could approximate $S$ by $S_e$, the set (surface) of points at a distance exactly $e$ of $S$. This surface is closed. It is the boundary of $V_e$, the set of points at a distance $e$ at most from $S$.

![Diagram of Se, S, Ve](image)

[Fig 2 - 3] The approximating surface and volume, $S_e, V_e$

In the case where $S$ is replaced by $S_e$, equation [2-26] holds :

$$0 = \vec{u}(x^{(0)}) + \frac{1}{8\pi\mu} \int_{S_e} \theta(x) \cdot t(x, x^{(0)}) \, dS(x) \text{ for all } x^{(0)} \text{ on } S_e, \quad [2-49]$$

$$\theta(x) = \left( \mu \frac{d\vec{u}}{dn} - p \, \vec{n} \right)(x)$$

We investigate how the equation is modified as $e \to 0$. Two things can be said :

- (i) $\theta$ is strongly dependent on the existence of a normal, and near the edge of the surface $S$ (i.e. the boundary of the two dimensional manifold) the normal becomes undefined.
- (ii) When $e \to 0$, the fact the $S$ has two sides becomes apparent. The surface actually behaves as if it were a set of two surfaces $S^+$ and $S^-$, with two corresponding opposed normals.
\( n^+ \) and \( n^- = -n^+ \). The two surfaces \( S^+ \) and \( S^- \) are identical in every point of view apart from orientation (thus the two normals).

After the limiting process \( \varepsilon \to 0 \) equation [2-49] becomes

\[
0 = \tilde{u}(x^{(0)}) + \frac{1}{8\pi \mu} \int_{S^+} \theta^+(x) \cdot t(x,x^{(0)}) \, dS^+(x) + \frac{1}{8\pi \mu} \int_{S^-} \theta^-(x) \cdot t(x,x^{(0)}) \, dS^-(x) \tag{2-50}
\]

i.e.

\[
0 = \tilde{u}(x^{(0)}) + \frac{1}{8\pi \mu} \int (\theta^+(x) + \theta^-(x)) \cdot t(x,x^{(0)}) \, dS(x), \tag{2-51}
\]

where \( \theta^+(x) = \left( \mu \frac{du}{dn^+} - p n^+ \right)^+(x) \), \( \theta^-(x) = \left( \mu \frac{du}{dn^-} - p n^- \right)^+(x) \) for \( x \) on \( S \),

the bracketed quantities being defined by:

\[
\left( \mu \frac{du}{dn^+} - p n^+ \right)^+(x) = \mu \lim_{t \to 0^+} \left[ \frac{1}{t} \left( u(x + tn^+) - u(x) \right) \right], \quad \lim_{t \to 0^+} [p(x + tn^+)]n^+ \tag{2-52}
\]

\[
\left( \mu \frac{du}{dn^-} - p n^- \right)^+(x) = \mu \lim_{t \to 0^+} \left[ \frac{1}{t} \left( u(x + tn^-) - u(x) \right) \right], \quad \lim_{t \to 0^+} [p(x + tn^-)]n^- \tag{2-53}
\]

That is, for \( \theta^+(x) \) (resp. \( \theta^-(x) \)), one only considers values of \( u,p \) on the \( \{+\} \) (resp. \( \{-\} \)) side of \( S \). If one looks more closely at [2-51], and if one defines \( \theta(x) \) as being

\[
\theta(x) = \theta^+(x) + \theta^-(x) = \left( \mu \frac{du}{dn^+} - p n^+ \right) + \left( \mu \frac{du}{dn^-} - p n^- \right) \text{ for } x \text{ on } S, \tag{2-54}
\]

then \( \varphi \) is the solution of

\[
0 = \tilde{u}(x^{(0)}) + \frac{1}{8\pi \mu} \int \theta(x) \cdot t(x,x^{(0)}) \, dS(x), \tag{2-55}
\]

\[
\theta(x) = \theta^+(x) + \theta^-(x) = \left( \mu \frac{du}{dn^+} - p n^+ \right) + \left( \mu \frac{du}{dn^-} - p n^- \right) \text{ for } x \text{ on } S,
\]

which is very similar (apart from the exact definition of the solution) to equation [2-26]. As a matter of fact equation [2-26] still holds for non closed surfaces, but the interpretation of its solutions has to be modified. The quantity \( \theta \), solution of [2-55] is no longer the surface stress, but the sum of the surface stresses exerted by the fluid on \( (S^+,n^+) \) (Surface \( S^+ \) with the normal \( n^+ \)), on the side indicated by \( n^+ \) and on \( (S^-,n^-) \), on the side indicated by \( n^- \). If one now defines the normal on \( S \) to be \( n = n^+ \) one has
\[ \theta(x) = \left( \mu \frac{du}{dn^+} - p n^+ \right) + \left( \mu \frac{du}{dn^-} - p n^- \right) = \left( \mu \frac{du}{dn^-} - p n \right)^+ - \left( \mu \frac{du}{dn} - p n \right) \]  

[2-56]

where the suffices indicate on which side of the surface one takes the limit. It is to be noted that \( \theta \) is independent of the normal chosen (see its definition [2-48]). The integral of \( \theta \) on \( S \), seen as a 'one sided surface', still gives the drag on \( S \). Now that [2-26], or at least the solutions to [2-26] have been modified for open surfaces by giving [2-55], let us look how [2-26], or [2-55] can be simplified in practice by using the symmetries of the configuration, namely we suppose that \( S \) is symmetrical around the three cartesian coordinate planes.

2.2.2.2 : To begin with let us suppose that our surface \( S \) is divided in eight parts \( S_i \) \( i=1..8 \), corresponding to the intersection of the surface with the eight octants. As \( S \) is symmetrical around the three cartesian coordinate planes, we have:

\[ S = \bigcup_{i=1}^{8} S_i, \text{ with } S_i \cap S_j = \emptyset \text{ if } i \neq j, \]  

[2-57]

which is a convoluted way of describing the decomposition, and

\[ S_i = \rho_i(S_1) \text{ with } \begin{cases} \rho_1 = \text{id} & \rho_2 = \sigma_1 \\ \rho_3 = \sigma_1 \cdot \sigma_2 & \rho_4 = \sigma_2 \\ \rho_5 = \sigma_3 & \rho_6 = \sigma_1 \cdot \sigma_3 \\ \rho_7 = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 & \rho_8 = \sigma_2 \cdot \sigma_3 \end{cases} \]  

[2-58]

where the \( \sigma_k \) are the orthogonal symmetries around the plane perpendicular to the \( x_k \) axis (the \( x_k \) coordinate plane). The \( \rho_k \) are the isometries mapping \( S_1 \) into \( S_k \) (the surface \( S \) is symmetrical around the three cartesian coordinate planes), they are products of the orthogonal symmetries around the three cartesian coordinate planes.

2.2.2.3 : Let us rewrite the equation [2-26] (or [2-55]), keeping the above stated symmetries in mind. We have, for \( x \in S_1 \):

\[ \theta = u^\infty(x) + \int_{S_1} \left[ \sum_{k=1}^{8} Q(x;\rho_k(\eta))(\theta(\rho_k(\eta))) \right] dS(\eta). \]  

[2-59]

Now, using [Th 2-1,3], we have, by writing \( \tau_k = \rho_k \) for \( k = 1..4 \), and \( \tau_k = -\rho_k = -\sigma_3 \cdot \rho_k \cdot 4 \) for \( k = 5..8 \), that [2-59] transforms into:

24
0 = u^\infty(x) + \int_{S_1} \left[ \sum_{k=1}^{8} Q(x; \rho_k(\eta)) \cdot \tau_k(\theta(\eta)) \right] dS(\eta). \quad [2-60]

= u^\infty(x) + \int_{S_1} \left[ \sum_{k=1}^{4} Q(x; \rho_k(\eta)) \cdot \rho_k - Q(x; (\sigma_3 \cdot \rho_k(\eta)) \cdot \sigma_3 \cdot \rho_k \right] (\theta(\eta)) dS(\eta) [2-61]

2.2.3: The computational aspect

The approximation that will be made in order to solve [2-60,61] dates from the first attempts to solve integral equations, or problems that today would lead to an integral equation, and it was first proposed by Newton. The approximation of the problem is effected by dividing $S_1$ into $N$ distinct subareas $(\Omega_k)_{k=1..N}$, and by supposing that the unknown field $\theta$ is constant on each of the $(\Omega_k)_{k=1..N}$. That is we set $\theta$ in $\Omega_k$ to be equal to $\theta(\eta_k)$ where $\eta_k$ is a point in the inside of $\Omega_k$. One can therefore write:

$$\theta(\eta) = \sum_{k=1}^{N} \theta(\eta_k) \mathbf{1}_{\Omega_k}(\eta).$$ \quad [2-62]

This approximation is equivalent to the one done when computing an integral by the method of rectangles. The problem posed by [2-61] is now approximated by the following set of linear equations for the $(\theta(\eta_k))_{k=1..N}$:

for all $\alpha$ in $[1..N]$:

$$0 = u^\infty(\eta_{\alpha}) + \int_{S_1} \left[ \sum_{k=1}^{4} Q(\eta_{\alpha}; \rho_k(\eta)) \cdot \rho_k - Q(\eta_{\alpha}; (\sigma_3 \cdot \rho_k(\eta)) \cdot \sigma_3 \cdot \rho_k \right] (\theta(\eta)) dS(\eta) \quad [2-63]

= u^\infty(\eta_{\alpha}) + \sum_{k=1}^{4} \int_{S_1} \left[ Q(\eta_{\alpha}; \rho_k(\eta)) \cdot \rho_k - Q(\eta_{\alpha}; (\sigma_3 \cdot \rho_k(\eta)) \cdot \sigma_3 \cdot \rho_k \right] (\theta(\eta)) dS(\eta) \quad [2-64]

After some manipulation we have that for all $\alpha$ in $[1..N]$,
\[ \theta = u^\infty(\theta_0) + \sum_{\beta = 1}^{N} \text{IT}(\alpha, \beta)(\theta(\eta_\beta)), \quad [2-65] \]

where

\[
\text{IT}(\alpha, \beta) = \int_{\Omega_\beta} \sum_{k=1}^{4} \left[ Q(\eta_\alpha; \rho_k(\eta)) \cdot \rho_k - Q(\eta_\alpha; (\sigma_3 \cdot \rho_k)(\eta)) \cdot \sigma_3 \cdot \rho_k \right] d\eta. \quad [2-66]
\]

Thus we have a system of 3N linear equations in 3N unknowns, where the unknowns are a set of N vectors of \( R^3 \): \( (\theta(\eta_\beta))_{\beta=1..N} \).

The operator \( (\text{IT}(\alpha, \beta))_{\alpha, \beta=1..N} \) is a rather unsavory object. Indeed to compute it for \( \alpha = \beta \), one has to get around a singularity in \( \eta_\alpha = \eta_\beta \). That is where the choice of the type of surface studied becomes very important.

### 2.3 : Numerical calculations

#### 2.3.1 : Description of the model - the case of a cylinder

Let us describe our surface \( S \) as being the image under \( \Gamma \) of \([0..1] \times [-/../], \) where \( \Gamma \) is the following function: \( \Gamma(s; z) = (\gamma_1(s); \gamma_2(s); z) \) with \( \gamma(s) = (\gamma_1(s); \gamma_2(s)) \) being piecewise of class \( C^1 \). In the case of a hollow circular cylinder we have \( \gamma(s) = (\cos(2\pi s); \sin(2\pi s)) \).

![Diagram of a cylinder](image)
Let us divide up the cylinder in the following way:

\[ \Omega_\alpha = \Gamma \left( \left[ a_\alpha : b_\alpha \right] ; \left[ c_\alpha : d_\alpha \right] \right), \]
\[ \eta_\alpha = \Gamma \left( \frac{a_\alpha + b_\alpha}{2} ; \frac{c_\alpha + d_\alpha}{2} \right). \]

[2-67]  [2-68]

that is the cylinder is divided into the images by \( \Gamma \) of rectangles in the \((s,z)\) plane. Let us look at \( IT(\alpha,\beta)_{jk} \), where

\[ IT(\alpha,\beta)_{jk} = \int_{a_\alpha}^{b_\alpha} \int_{c_\alpha}^{d_\alpha} t_{jk}(x,\eta) \, dS(\eta). \]

with \( \eta = \Gamma(s ; z) \) and \( dS(\eta) = \left| \frac{d\eta}{ds(s)} \right| \, ds \, dz \) [2-69]

In fact the terms \( \int t_{jk}(x;\Gamma(s ; z)) \, dz \) can be evaluated in a simple way. It is found that

\[ \int t_{11}(x,\eta) \, dz = \frac{(\eta_1 - x_1)^2(z - x_3)}{\left( (\eta_1 - x_1)^2 + (\eta_1 - x_1)^2 \right) \, |x-\eta|} + f(x-\eta), \]
\[ \int t_{12}(x,\eta) \, dz = \frac{(\eta_1 - x_1)(\eta_2 - x_2)(z - x_3)}{\left( (\eta_1 - x_1)^2 + (\eta_1 - x_1)^2 \right) \, |x-\eta|}, \]
\[ \int t_{13}(x,\eta) \, dz = -\frac{(\eta_1 - x_1)}{|x-\eta|}, \]
\[ \int t_{22}(x,\eta) \, dz = \frac{(\eta_2 - x_2)^2(z - x_3)}{\left( (\eta_1 - x_1)^2 + (\eta_1 - x_1)^2 \right) \, |x-\eta|} + f(x-\eta), \]
\[ \int t_{23}(x,\eta) \, dz = -\frac{(\eta_2 - x_2)}{|x-\eta|}, \]
\[ \int t_{33}(x,\eta) \, dz = -\frac{(z - x_3)}{|x-\eta|} + 2 \, f(x-\eta), \]

where : \((\eta_1,\eta_2) = \gamma(s), \eta = \Gamma(s,z)\), and :

\[ f(a,b,c) = \frac{\ln \left( \sqrt{a^2 + b^2 + c^2 + c} \right) - \ln \left( \sqrt{a^2 + b^2 + c^2 - c} \right)}{2}. \]

[2-70]  [2-71]  [2-72]  [2-73]  [2-74]  [2-75]

[2-76]
We are still left with the most difficult part of the affair, namely the computing of the $s$-integral. That is, we know $\int t_{jk}dz$, but, as soon as $\gamma$ is somewhat 'intricate',

$$\int \left( \int t_{jk}dz \right) \frac{dy}{ds(s)} ds$$ \[2-77\]

has no simple form. We are therefore left with numerical integration. That means, among other things, that we have to pay attention to the fact that $|x-\eta|$ can be equal to zero. More precisely: if $\eta_\alpha \in \Omega_\beta$ then $t_{jk}$ has a singularity around $\eta_\alpha$, which is of the kind: $\frac{1}{|x-\eta_\alpha|}$ and therefore can be integrated on the surface.

2.3.2: Description of the model: the case of a surface of revolution

2.3.2.1: Transforming equation [2-26]

In this case we will suppose that for each rotation $\sigma_\varphi$ of angle $\varphi$ around the $x_3 = z$ axis we have: $\sigma_\varphi \circ u_\infty = u_\infty \circ \sigma_\varphi$, and $\sigma_\varphi(S) = S$. Thus we may assume that for all $\varphi$: $\sigma_\varphi \circ u = u \circ \sigma_\varphi$, which means that in cylindrical coordinates we have:

$$u(\rho,\varphi,z) := u_\rho(\rho,z) e_\rho(\varphi) + u_\varphi(\rho,z) e_\varphi(\varphi) + u_z(\rho,z) e_z$$ \[2-78\]

where, in cartesian coordinates:

$$\begin{align*}
e_\rho(\varphi) &= (\cos(\varphi); \sin(\varphi); 0) \\
e_\varphi(\varphi) &= (-\sin(\varphi); \cos(\varphi); 0) \\
e_z &= (0; 0; 1)
\end{align*}$$ \[2-79\]

[Fig 2 - 5] The set-up for a surface of revolution
We may therefore look for an unknown $\theta$ having the same rotational symmetry properties. Let us rewrite the equation determining $\theta$; namely

\[ \theta = u^\infty(x) + \int_S T(x;\eta)(\theta(\eta)) \, dS(\eta) \quad \text{for} \quad x \in S, \text{as} \]

\[ \theta = u^\infty(p;0;z) + \int_S T((p;0;z);(p';\varphi';z')) \left( \theta_p(p';z') e_p(\varphi) + \theta_\varphi(p';z') e_\varphi(\varphi) + \theta_z(p';z') e_z \right) \, dS(p';\varphi';z') \]  

But as we are working with a surface of revolution we may take $p$ to be a function of $z$, i.e. $\Gamma(s,z) = (\rho(z)\cos(s); \rho(z)\sin(s); z)$. Thus

\[ dS(p;\varphi;z) = \rho(z) \sqrt{1 + \left( \frac{dp}{dz} \right)^2} \, dz \, d\varphi = \psi(z) \, dz \, d\varphi. \]

We have, after some calculation that

\[ \theta = u^\infty(p;0;z) + \int_{\varphi'}^{2\pi} T((p;0;z);(p';\varphi';z')) \psi(z') \, dz' , \]

i.e. $\theta = u^\infty(p;0;z) + \int_{\varphi'}^{2\pi} Q(p;z;p';z') \psi(z') \, dz' , \]

where $Q(p;z;p';z') = \int_{\varphi}^{2\pi} T((p;0;z);(p';\varphi';z')) \, d\varphi'$ and $R(\varphi)$ can be represented in the base $(e_1; e_2; e_3)$ by :

\[ \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

In fact $Q$ can be computed using complete elliptic integrals of the first, second and third kind, thus avoiding a numerical integration with respect to $\varphi$.

2.3.2.2 : An example : the spherical cap

A good example to test the method is provided by the problem of a flow past a spherical cap. That is : let us define $S$ to be, in cylindrical coordinates $(\rho,\varphi,z)$ :
\[
\cos(\alpha) \leq z \leq 1 ; \quad \rho = \sqrt{1-z^2}
\]  

[2-86]

In this case literature, papers by Dorrepaal (Dorrepaal [6],[7] and Dorrepaal, O'Neill and Ranger [8]) provides exact expressions for various quantities of physical interest, i.e. the drag on \( S \), the existence and the position of a wake within the cap, both for an axisymmetric and an asymmetric flow. In the case of the flow \( u^\infty = 2 e_z \), we have for every \( \alpha \) in \([0..\pi]\), a wake, and the drag \( D \) on \( S \) is given by the formula

\[
D = |u^\infty| \mu \left( 6\alpha + 8\sin(\alpha) + \sin(2\alpha) \right) = 2\mu \left( 6\alpha + 8\sin(\alpha) + \sin(2\alpha) \right).
\]  

[2-87]

The limiting surface of the wake is a zero-stream surface. Dorrepaal O'Neill and Ranger (Dorrepaal et al. [8]) give a table of the coordinate of its intersection with the \( z \) axis as a function of \( \alpha \).

2.3.3 : Numerical results

2.3.3.1 : The hollow circular cylinder

Let us take for \( S \) a hollow circular cylinder parallel to the \( z=x_3 \) axis and of radius 1 and length \( l \). More precisely : \( S = \{ x^2 + y^2 = 1, |z| \leq l \} \). To describe it, we use the following parametrisation : \((s;z) \rightarrow \Gamma(s;z) = \left( \cos\left(\frac{\pi s}{2}\right), \sin\left(\frac{\pi s}{2}\right), z \right) \). We clearly have \( S = S((0;1); [0;l]) \).

The most important problem is finding a way to subdivide \([0;l]\) in order to minimize the error made. The following way was used for the subdivisions :

\[
[0;1] = \bigcup_{i=0}^{N_{z1}-1} \left[ \frac{i}{N_{z1}}, \frac{i+1}{N_{z1}} \right] \cup \left[ l \left( \frac{N_{z1}-1}{N_{z1}} + \frac{i}{N_{z1} N_{z2}} \right), l \left( \frac{i+1}{N_{z1} N_{z2}} \right) \right]
\]  

[2-88]

That is, the interval \([0;l]\) was divided in \( N_{z1} \) sub-parts and the last interval (sub-part) was again divided in \( N_{z2} \) sub-parts. This type of subdivision will be used for all the kinds of hollow cylinders studied.
To measure the accuracy of the method Price's (Price [16]) and Warrilow's (Warrilow [30]) numerical results on the flux through a short round cylinder, with $u^\infty = 2e_3$, will be used. That flux is the integral:

$$\Phi = \int_F n(x) \cdot u(x) \, dF(x) \quad [2-90]$$

where $F$ is an open surface whose edge lies on $S$, that is on the surface of the cylinder. For practical means I have chosen $F = \{ x^2 + y^2 \leq 1; z = 0 \}$. Transforming the integral to polar coordinates we finally have that

$$\Phi = 2\pi \int_0^1 u_3(\rho, 0, 0) \, \rho \, d\rho . \quad [2-91]$$

An interesting value to compute is the ratio between $\Phi$ and $\Phi^\infty$, the flux through the unit circle in the absence of the obstacle : $\Phi^\infty$. We have:

$$\Phi^\infty = 2\pi \int_0^1 2\rho \, d\rho = 2\pi, \text{ so that } \frac{\Phi}{\Phi^\infty} = \frac{1}{\int_0^1 u_3(\rho, 0, 0) \, \rho \, d\rho} \quad [2-92]$$

The last integral is computed using Simpson's rule of integration, with 10 points on the interval $[0; 0.9]$ and 10 points in the interval $[0.9; 1]$. The results are given with 4 significant digits to be able to evaluate the precision of the different subdivisions. They are as follows
<table>
<thead>
<tr>
<th>$l$ subs.</th>
<th>(4;4;1)</th>
<th>(4;8;1)</th>
<th>(4;4;5)</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.1137</td>
<td>0.1167</td>
<td>0.1105</td>
<td>0.1097</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1813</td>
<td>0.1790</td>
<td>0.1776</td>
<td>0.1769</td>
</tr>
<tr>
<td>0.25</td>
<td>0.2576</td>
<td>0.2551</td>
<td>0.2537</td>
<td>0.2530</td>
</tr>
<tr>
<td>0.125</td>
<td>0.3311</td>
<td>0.3286</td>
<td>0.3275</td>
<td>0.3268</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.3948</td>
<td>0.3936</td>
<td>0.3917</td>
<td>0.3925</td>
</tr>
<tr>
<td>0.03125</td>
<td>0.4463</td>
<td>0.4490</td>
<td>0.4478</td>
<td>0.4487</td>
</tr>
</tbody>
</table>

$[\text{Table 2 - 1}]: \frac{\Phi}{\Phi_{\infty}}$ as a function of $l$

2.3.3.2 : Hollow square cylinders

In the case of a square cylinder the following function $\Gamma$ has been chosen:

$$\Gamma(s,z) = \begin{cases} \langle l_x; 2 s l_y; z \rangle & \text{if } s \leq \frac{1}{2} \\ \langle 2 (1 - s) l_x; l_y; z \rangle & \text{if } s \geq \frac{1}{2} \end{cases}$$  \hspace{1cm} [2-93]

The subdivision used is (4;4;2) which seems to be a good compromise between computing speed and accuracy. It gives approximately three significant digits. In all the cases $l_y = 1$, and $l_z$ and $l_x$ are varying. One has the following table for $\frac{D}{\mu u_{\infty}}$:

<table>
<thead>
<tr>
<th>$l_z \backslash l_x$</th>
<th>1</th>
<th>0.5</th>
<th>0.25</th>
<th>0.125</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24.1</td>
<td>19.4</td>
<td>15.8</td>
<td>14.8</td>
</tr>
<tr>
<td>0.5</td>
<td>20.3</td>
<td>16.1</td>
<td>13.6</td>
<td>11.9</td>
</tr>
<tr>
<td>0.25</td>
<td>17.7</td>
<td>13.9</td>
<td>11.6</td>
<td>10.1</td>
</tr>
<tr>
<td>0.125</td>
<td>15.7</td>
<td>12.2</td>
<td>10.2</td>
<td>8.84</td>
</tr>
</tbody>
</table>

$[\text{Table 2 - 2}]: \frac{D}{\mu u_{\infty}}$ as a function of $l_z, l_x$

2.3.3.3 : Hollow elliptic cylinders

In the case of elliptic cylinders the following function $\Gamma$ has been chosen:
\[
\Gamma(s,z) = (l_x \cos\left(\frac{\pi}{2}s\right); l_y \sin\left(\frac{\pi}{2}s\right), z)
\]

The same subdivision as for the rectangular cylinders has been made. One has the following table for \(\frac{D}{\mu \infty}\):

<table>
<thead>
<tr>
<th>(l_z \backslash l_x)</th>
<th>1</th>
<th>0.5</th>
<th>0.25</th>
<th>0.125</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.5</td>
<td>17.6</td>
<td>15.3</td>
<td>14.0</td>
</tr>
<tr>
<td>0.5</td>
<td>17.9</td>
<td>14.4</td>
<td>12.4</td>
<td>11.1</td>
</tr>
<tr>
<td>0.25</td>
<td>15.3</td>
<td>12.2</td>
<td>10.5</td>
<td>9.40</td>
</tr>
<tr>
<td>0.125</td>
<td>13.5</td>
<td>10.7</td>
<td>9.17</td>
<td>8.17</td>
</tr>
</tbody>
</table>

[Table 2 - 3]: \(\frac{D}{\mu \infty}\) as a function of \(l_z, l_x\)

The comparison of the values obtained for the rectangular cylinders [Table 2 - 2] and the elliptic cylinders [Table 2 - 3] yields two results. The first is that, in general these results are not far apart. In the worst case the relative difference is 16%. The second is the way the difference is affected by the parameters. The evolution of the difference can be summed up in the following statement. The thinner (resp. the shorter) the cylinder, that is the smaller \(l_x/l_y\) (resp the smaller \(l_z/l_y\)), the smaller (resp. greater) the error committed. Both cases can be attributed to one thing, namely the presence of sharp corners and their predominant effect when the cylinder is either thin or short. In the case of the thin elliptical cylinder, the longest extremities of the ellipsis act as sharp corners.

One main result is that thin elliptical cylinders and thin rectangular cylinders are not much different when it comes to drag (in the case \(l_x = 0.125, l_y = 1, l_y = 1\) the error is of about 6%)

2.3.3.4 : Spherical cap

In this case we consider the surface as described in 1.3.2.2. The subdivision made in the \([\cos(\alpha),1]\) interval is of the same kind as the one made in [2-89], except for one detail: the second subdivision is made near \(z = \cos(\alpha)\) which is the border of the cap (i.e. the first sub-part is the one that is subdivided again). Two things can be computed: The drag, and the intersection, \(q_0\), of the limiting surface of the wake (the zero stream surface) with the z axis. [2-87] gives the value of the first one, where a table in Dorrepaal (Dorrepaal, O'Neill, Ranger [8]) gives the second. The results are given with 5 significant digits in the case of the drag, and two in the case of \(q_0\), to be able to compare with Dorrepaal's results. We have:
<table>
<thead>
<tr>
<th>$\alpha \ \text{D}/\mu$</th>
<th>(5;1)</th>
<th>(6;5)</th>
<th>(10;10)</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/3$</td>
<td>13.900</td>
<td>14.047</td>
<td>14.069</td>
<td>14.077</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>17.292</td>
<td>17.400</td>
<td>17.418</td>
<td>17.425</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>18.569</td>
<td>18.612</td>
<td>18.624</td>
<td>18.629</td>
</tr>
<tr>
<td>$\pi$</td>
<td>18.850</td>
<td>18.850</td>
<td>18.850</td>
<td>18.850</td>
</tr>
</tbody>
</table>

[Table 2 - 4]: $\frac{D}{\mu u_\infty}$ as a function of $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha \ \text{q}_0$</th>
<th>(5;1)</th>
<th>(6;5)</th>
<th>(10;10)</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/3$</td>
<td>0.20</td>
<td>0.19</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>-0.33</td>
<td>-0.35</td>
<td>-0.35</td>
<td>-0.35</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>-0.70</td>
<td>-0.72</td>
<td>-0.74</td>
<td>-0.74</td>
</tr>
<tr>
<td>$\pi$</td>
<td>~</td>
<td>~</td>
<td>~</td>
<td>~</td>
</tr>
</tbody>
</table>

[Table 2 - 5]: $\text{q}_0$ as a function of $\alpha$.

The values for $\alpha = \pi$ are not relevant. Indeed, this case correspond to the complete solid sphere. But, one important thing, is that we have:

$$q_0 \to -1 \text{ as } \alpha \to \pi$$

[2-95]

2.3.4: A further problem: The asymmetric flow past a spherical cap

2.3.4.1: To explore other problems, the case of the asymmetric flow past a spherical cap seems a good example. It has been extensively studied by Dorrepaal (Dorrepaal [5]). The configuration is the following: let us consider the spherical cap described in 1.3.2.2, this time with a different flow at infinity:

$$u_\infty = e_x.$$ 

[2-96]
In the case of the antisymmetric flow $u^\infty = e_x$, a wake exists only for values of $\alpha \geq 60.90^\circ$, the rim of the cap supporting the separation surface only if $\alpha \geq 80.85^\circ$, and the norm of the drag $D$ on $S$, $D$, is given by the formula (Dorrepaal [5]):

$$D = \mu \left( 6(\alpha + \sin(\alpha)) - \frac{8}{3} \frac{\sin^2(\alpha) \cos^4\left(\frac{\alpha}{2}\right)}{\alpha + \sin(\alpha)} \right)$$

[2-97]

The complexity of the problem can be reduced by considering the symmetry around the $xz$ plane. The general computing scheme is very similar to the one described in section 2.3, with a difference, only one symmetry is to be considered. The interval considered for $s ([0,1])$ was cut in $N_5$ equal parts and the one for $z ([\cos(\alpha),1])$ was subdivided according to the way described by [2-89], with one difference, it is the first sub-part (i.e. the one nearer to $z = \cos(\alpha)$) that is redivided, and not the last. The surface is modeled as a surface of revolution, around the $z$ axis, the equation being:

$$\begin{align*}
\begin{cases}
x = \cos(\pi s) \sqrt{1 - z^2}, \\
y = \sin(\pi s) \sqrt{1 - z^2}, \\
z = z
\end{cases}, \text{ with } \cos(\alpha) \leq z \leq 1, 0 \leq s \leq 2,
\end{align*}$$

[2-98]

i.e. $\Gamma(s,z) = (\cos(\pi s) \sqrt{1 - z^2}, \sin(\pi s) \sqrt{1 - z^2}, z)$.

The only problems came when computing $\Gamma'(\alpha, \alpha)$. To overcome the problem caused by the singularity in $\eta_\alpha$, which is of the form

$$|x - \eta_\alpha|^{-1},$$

[2-99]
the following polar transformation had to be used, if \( \eta_\alpha = \Gamma(s_0, z_0) \) then \( s = s_0 + \xi \sin(\tau) ; z = z_0 + \xi \cos(\tau) \). In that case the functions \( \xi \cdot t_{jk} \cdot \Gamma(s, z) \) would have no singularity around \( \eta_\alpha \). In fact

\[
IT(\alpha, \alpha) = \int \int \left( \xi \cdot t_{jk} \cdot \Gamma(s_0 + \xi \sin(\tau) ; z_0 + \xi \cos(\tau)) \psi(\xi, \tau) \right) \, d\xi \, d\tau,
\]

where \( \psi(\xi, \tau) \xi \, d\xi \, d\tau = dS(s_0 + \xi \sin(\tau) ; z_0 + \xi \cos(\tau)) \), and the bounds of integration being chosen such as \( \Gamma(s_0 + \xi \sin(\tau) ; z_0 + \xi \cos(\tau)) \) covers \( \Omega_\alpha \) when \( \xi \) and \( \tau \) are varying.

Some interesting results came out: in the case of a small angle \( \alpha \) (i.e. when no wake is theoretically present) the matrix \( A = (IT(a, b))_{a, b} \) is severely ill conditioned, thus the results can be questioned. Indeed when computing \( IT(a, b) \) we make an approximation \( ITA(a, b) \) which leads to a triple error, the one committed while computing \( A \), the one occurring when we solve \( Ax = y \), and the one done while approximating \( \theta \).

2.3.4.2: The values computed are the norm of the scaled drag, \( D_{\mu} \), and the point \( q_v \) on the \( z \) axis where \( u_x \) vanishes (this being the centre of the wake). It can be seen that for \( \alpha = 3\pi/3 < 60.90^\circ \) we still seem to have a wake. The apparent contradiction is solved by the ill conditioning of the problem. Indeed calculations, using a finer approximation of \( IT(a, b) \) show a disappearance of that "artificial" wake, the cost of that additional precision being extreme time and memory consumption.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \mu = \mu_{\infty} )</th>
<th>( \mu = \mu_{\infty} )</th>
<th>( \mu = \mu_{\infty} )</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/3 )</td>
<td>10.700</td>
<td>10.857</td>
<td>10.866</td>
<td>10.891</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>14.937</td>
<td>15.162</td>
<td>15.166</td>
<td>15.165</td>
</tr>
<tr>
<td>( 2\pi/3 )</td>
<td>17.541</td>
<td>17.704</td>
<td>17.708</td>
<td>17.720</td>
</tr>
<tr>
<td>( \pi )</td>
<td>18.850</td>
<td>18.851</td>
<td>18.856</td>
<td>18.850</td>
</tr>
</tbody>
</table>

[Table 2-6]: \( D_{\mu_{\infty}} \) as a function of \( \alpha \)
<table>
<thead>
<tr>
<th>$\alpha \backslash q_v$</th>
<th>$(5;5;1)$</th>
<th>$(5;6;5)$</th>
<th>$(8;8;5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/3$</td>
<td>0.94</td>
<td>0.93</td>
<td>0.94</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0.44</td>
<td>0.39</td>
<td>0.38</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>-0.07</td>
<td>-0.12</td>
<td>-0.13</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\sim$</td>
<td>$\sim$</td>
<td>$\sim$</td>
</tr>
</tbody>
</table>

[Table 2-7] : $q_v$ as a function of $\alpha$. 

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Chapter Three

The problem of the boundary conditions for a free/porous media interface

Summary:

In this chapter two models will be described for the flow in a porous medium, the Brinkman and Darcy models. A particular stress will be put on a generalization, of Brinkman's model, described by Saffman (Saffman [18]): The generalized Brinkman equation. After a review of the models, attention will be focused on the behaviour of the flow at the interface and on the coupling, at the interface, of a free flow solution with a porous flow solution. For the Darcy model an experimentally deduced condition will be considered.

Before starting to model the flow in a porous medium, let us define what a porous or a free medium is.

• A free medium, as opposed to a porous or a solid medium, is a medium (i.e. region of space) filled with a viscous fluid generally containing no obstacles, that is solids on which one prescribes the velocity of the fluid. If there are any obstacles, these are in a finite number and their size is also finite, so that one can actually consider the obstacles to be a late addition to the configuration. In the case of the presence of obstacles one could consider them to be a perturbation of the flow, that is the velocity can be computed as being the sum of the velocity we would have had if there had been no obstacles plus the velocity perturbation caused by the obstacles.

• A porous medium, is a region of space containing an infinity of obstacles taking up a substantial amount of volume. Furthermore an important feature of a porous medium are pores, that is, in a porous medium a fluid can flow in channels also called pores. These pores, though, do not have a simple shape, they usually twist, intersect, separate and are of no uniform width. The problem is that one can either consider the pore walls to be solids and try to solve the equation of the motion, the 'microscopic' point of view, which is impossible analytically as soon as the pores are of a complicated shape (i.e. more complicated than circular hollow cylinders), or one can try to model an averaged version of the velocity taking into account the porous nature of the medium, but being much smoother than the velocity in the 'microscopic' point of view.
This dichotomy is very similar to the one made when modelling the flow of a fluid in a free medium. The two alternatives are, modelling the flow by computing the velocity of its constituents, or by computing averaged quantities, representative of the fluid at large scales. The first aim of this chapter is to show a way to go from the equation for the free medium in the pores, and a no slip condition on the boundary of the pores, to an equation for the averaged velocity and pressure fields.

Let us suppose that the unbounded space contains a free medium part, and a porous medium part. In this space a fluid flows, wetting completely all surfaces (that is all the surfaces that are boundary to solids touch the fluid, and the fluid occupies as much space as possible). The porous medium is supposed to permit a flow, that is to have actual pores. We are supposing that the flow is very slow and that it is viscous. If necessary, we are supposing the Reynolds number (if it can be defined) associated with the flow to be small. We are using the Stokes equation for the flow in the free medium and the flow inside the pores (that is for the non averaged flow).

3.1 : The different models of a flow in a porous medium

Let us divide the space the following way : We have three regions :

• Region(I) : The free medium where the equation of the motion is the usual Stokes equation :

\[
\begin{align*}
\mu \Delta u^I &= \nabla p^I \\
\text{div } u^I &= 0
\end{align*}
\]  

[3-1]

where \( u^I(x) \), \( p^I(x) \) are, respectively, the velocity and pressure fields.

• Region (II) : The porous medium

• The interface : A smooth surface \( S \), which has a normal drawn onto it everywhere. \( S = \partial(\text{Region(I)}) = \partial(\text{Region(II)}) \)
There are several ways to model the motion of the fluid in the porous medium. In that part, Region(II), we are interested, not in the velocity one would actually have by considering the pores to be separate and well defined entities (the microscopic velocity), but, more into an averaged quantity taking into account the general properties of the porous medium. That is we do not consider the velocity at the point \( x \), but more an average of this velocity around \( x \), thus creating a relatively smooth field. Let us call \( \mathbf{u}^\text{II}, p^\text{II} \) the averaged velocity and pressure fields in Region(II). Let \( \mathbf{u}^\text{II}, p^\text{II} \) be the 'microscopic' velocity and pressure fields in Region(II), that is the velocity and pressure fields inside the pores if one had considered the flow inside the pores to be a slow creeping flow modeled by the Stokes equation (the velocity field outside the pores being the zero field), \( \gamma, \varepsilon > 0 \) small real numbers, \( B(x,\gamma) \) the sphere \( |x-y| \leq \gamma \), and \( \chi_\gamma \) a function such that:

\[
\chi_\gamma(x-y) = \text{const. for } y \in B(x,\gamma), \chi_\gamma(x-y) = 0 \text{ for } |x-y| > \gamma + \varepsilon
\]

\( \chi_\gamma(x-y) \) is \( C^\infty \), \( \int \chi_\gamma(x-y) \, dy = 1. \)

Then one could define \( u^\text{II}, p^\text{II} \) the following way:

\[
u^\text{II}(x) = \int \chi_\gamma(x-y) \, \mathbf{u}^\text{II}(y) \, dy = \chi_\gamma \ast \mathbf{u}^\text{II}(x)
\]

\[
p^\text{II}(x) = \int \chi_\gamma(x-y) \, p^\text{II}(y) \, dy = \chi_\gamma \ast p^\text{II}(x)
\]

where \( \gamma \) is still large enough for \( B(x,\gamma) \) to contain several "grains" of the porous structure, that is large enough to contain a representative part of the porous medium around \( x \).
In a paper (Saffman [18]) Saffman describes a way to find an averaged equation for $u^{II}$, $p^{II}$. Let us examine this equation.

3.1.1: The generalized Brinkman equation

Saffman (Saffman [18]) derives the following equation for the motion in a porous medium:

$$\mu \Delta u - \nabla p = \mu \mathbf{K}(u), \text{ where } \mathbf{K}(u) = \int K_{ij}(x,y) u_j(y) \, dy,$$

[3-4]

the integral being taken in the generalized sense (actually $\mathbf{K}$ can be considered as a distribution...). The term $\mu \mathbf{K}(u)$ is the extra force density produced by the motion in the medium. It is obvious that by taking:

$$K_{ij}(x,y) = K_{ij}(x) \delta(x-y),$$

[3-5]

one reproduces a non-isotropic version of the usual Brinkman equation (Saffman [18], and see [3-10] later):

$$\mu \Delta u - \mu K_{ij}(x) u_j(x) = \nabla p,$$

[3-6]

the coefficients $K_{ij}(x)$ representing the "fine" structure of the medium around $x$. To indicate, for instance a medium whose pores permit easy flow (i.e. producing no friction forces) in the direction $t$ at the point $x$ we choose $K_{ij}(x)$ such that:

$$K_{ij}(x)t_j = 0$$

[3-7]
Furthermore if we were to assume that the medium is spatially homogeneous then we would have the coefficients $K_{ij}(x)$ forming a symmetric, positive matrix. The usual case of the Brinkman equation actually assumes that the porous medium is isotropic, i.e. there is no "preferred" direction, as in [3-7], which minimizes the force, that is the $K_{ij}(x)$ are the coefficients of a dilatation, that is:

$$K_{ij}(x) = k \delta_{ij} \quad [3-8]$$

A quick dimensional analysis tells us that the dimension of $k$ is

$L^{-2}$ (the inverse of a surface).

One can interpret $\lambda^{-1} = \sqrt{k^{-1}}$ as being a constant proportional to the average radius of the pores, or more precisely a constant proportional to the hydraulic radius of the pores (i.e. the radius of the pore if it were a cylinder). The constant $\lambda$ which will be called the Brinkman constant for the porous medium measures the permeability (or the porosity, i.e. the amount and size of the pores) of the medium. The greater $\lambda$ the less permeable the medium. The extreme case $\lambda = \infty$ corresponding to the solid medium (i.e. no flow inside it) with a no-slip condition at its boundary. The other extreme $\lambda = 0$ corresponds to the case where the porous medium does not contain any solid parts any more, i.e. when the porous medium is not any more porous but free. Let us now look a two special cases : The classical Brinkman equation, and the Darcy equation.

3.1.2 : Special cases

3.1.2.1 : The Brinkman equation

The Brinkman equation is defined using [3-4], with a special condition for $K$, namely

$$K_{ij}(x,y) = k \delta_{ij} \delta(x-y), \quad [3-9]$$

that is the medium is spatially homogeneous, isotropic and uniform. The equation we get, is the one called Brinkman's equation (Saffman [18]), namely

$$\mu A u^\Pi - \mu k u^\Pi = \text{grad} \, p^\Pi. \quad [3-10]$$

As one can immediately see, it has the advantage that as $k \to 0$, it becomes the classical Stokes equation. Let us call $\lambda = \sqrt{k}$ the Brinkman constant. Rewriting [3-10] for a zero pressure
yields \( \Delta u^{II} - \lambda^2 u^{II} = 0 \). This equation is very near to the Helmholtz equation \( (\Delta + k^2) f = 0 \). This property will be extensively used in chapters III and IV.

Another interesting case is the one where \( \mathbf{K}(\mathbf{u}) \) is very large compared with \( \Delta \mathbf{u} \), that is when the behaviour of \( \mathbf{u} \) is dominated by the behaviour of \( \mathbf{K}(\mathbf{u}) \). This is the object of the next paragraph.

3.1.2.2 : Darcy's equation

If, we assume:

\[
|\mathbf{K}(\mathbf{u})| >> |\Delta \mathbf{u}| , \quad [3-11]
\]

then, by neglecting the \( \Delta \mathbf{u} \) term, Brinkman's equation reduces to Darcy's equation:

\[
- \mu \mathbf{K}_{ij}(x) \mathbf{u}_j = \frac{\partial p}{\partial x_i} , \quad [3-12]
\]

which implies that \( \mathbf{K}(\mathbf{u}) \) derives from a potential proportional to \( p \), i.e. that the problem reduces to finding one harmonic function (\( p \) or a multiple of \( p \)), whereas, in the Brinkman (or Stokes) case one has to produce one harmonic function and two solutions to the equation \( (\Delta - k)f = (\Delta - \lambda^2)f = 0 \) (See chap. IV). There is a difference in the amount of information (i.e. number of functions) required by Darcy's and Brinkman's equation. This will be the source of various problems.

Again the Darcy equation that occurs the most often is the one for an isotropic, spatially homogeneous and uniform medium. That is the one for which: \( \mathbf{K}_{ij}(x,y) = k \delta_{ij} \delta(x-y) \), we then have:

\[
\mathbf{u}^{II} = - \frac{1}{\mu k} \nabla p^{II} \quad [3-13]
\]

3.1.3 : summary

To summarize we have defined three models for the flow in Region(II):

The Generalized Brinkman model:

\[
\mu \Delta \mathbf{u}^{II} - \mu \mathbf{K}(\mathbf{u}^{II}) = \nabla p^{II}
\]

The Brinkman model:

\[
\mu \Delta \mathbf{u}^{II} - \mu \mathbf{k} \mathbf{u}^{II} = \nabla p^{II}
\]
The Darcy model:

\[ \mathbf{u}^{\text{II}} = -\frac{1}{\mu_k} \nabla p^{\text{II}} \]

Let us now examine the way to link the solutions in Region(I) and the solutions in Region(II).

3.2: The boundary conditions

3.2.1: Specifying the problem

A question arises: What are the conditions which link the solutions in Region(I) and the solutions in Region(II). What we would like is a series of relationships linking, at the interface, the values of the various quantities we know of, that is velocity, pressure, surface tension, etc. For the models we are going to look at, the following quantities, defined on the interface S are being used (the superscript indicates in which region the quantity is being measured, \( i = \text{I..II} \)):

- The velocity: \( \mathbf{u}^i \)
- The component of the velocity normal to the interface ("normal velocity"): \( u_n^i \), on S
- The component of the velocity tangent to the interface ("tangential velocity"): \( u_t^i \), on S
- The normal derivative of the tangential velocity on the interface: \( \frac{\partial}{\partial n} \left[ u_t^i \right] \), on S
- The pressure: \( p^i \)
- The surface stress: \( T(u^i,p^i) \cdot \mathbf{n} \), on S, where \( T(u^i,p^i) \) is the stress tensor and \( \mathbf{n} \) a normal drawn on S (chosen once and for all if S is open, or outward directed normal if S is closed).

It is obvious that in certain cases there will be redundant conditions. Take, for instance, the fact that continuity of the pressure and continuity of the differential of the velocity field imply the continuity of the stress tensor across S.

Throughout the literature various models have been laid out. Before actually looking at models answering directly the boundary conditions problem, it would be interesting to look at a development by Saffman (Saffman [18]).

3.2.2: The generalized Brinkman approach

Let us modify slightly the set-up defined in 1.
We now consider that the boundary is actually a "thick object". That is, not only do we consider S, but also all the region $B = \{ x / d(x,S) \leq \varepsilon \}$, where $\varepsilon$ is small, as being the boundary. Let us look, again, at the generalized Brinkman equation, more specifically at the operator $K(u)$. Two remarks can be made:

- One can see that if $K = 0$ then Brinkman's equation is actually the Stokes equation!

- When, while still staying in Region(II), one moves nearer to the interface, the operator $K$ is bound to change quite dramatically. Indeed, if it was supposed that inside (that is well inside) Region(II) we had $K_{ij}(x,y) = k \delta_{ij} \delta(x-y)$ (i.e. the medium is isotropic, uniform, etc..), it cannot be supposed any more, that near the interface the medium is isotropic or uniform. The only supposition that can be made is the following:

$$K(u)(x) \text{ is continuous in the whole space}$$
$$K(u)(x) = 0 \text{ in Region(I)}$$
$$K(u)(x) = k u(x) \text{ in Region(II)}$$

Saffman (Saffman [18]), takes advantage of these remarks to propose the following model: One approximates the problem laid out in 1, by considering the following problem:

$$\begin{cases}
\mu \Delta u - \nabla p = \mu K(u) & \text{in the whole space} \\
\text{div } u = 0 & \\
K(u)(x) \text{ is continuous in the whole space} \\
K(u)(x) = 0 \text{ in Region(I') = Region(I) \setminus B} \\
K(u)(x) = k u(x) \text{ in Region(II') = Region(II) \setminus B}
\end{cases} \quad [3-14]$$

The advantage of this model is that all the physical quantities considered, are continuous across the boundary $S$. The problem with that model is that one does not know how $K$ is constructed inside $B$, the boundary layer. To alleviate the problem, one could make the following
hypothesis: Assume the physical quantities to be continuous, and have $K$ become discontinuous at the boundary $S$. This leads to:

3.2.3: The accepted Brinkman condition

The usual boundary condition for Brinkman's equation, in the configuration described in 1 (see Fig 3-1) is the following:

- The velocity is continuous across $S$
- The pressure is continuous across $S$
- The surface stress is continuous across $S$

Rewritten more rigorously, the conditions at $S$ in the Brinkman case are

\[
\begin{align*}
    u^I &= u^{II} & \text{[3-15]} \\
    p^I &= p^{II} & \text{[3-16]} \\
    T(u^I,p^I) \cdot n &= T(u^{II},p^{II}) \cdot n & \text{[3-17]}
\end{align*}
\]

For Darcy's model things are a little more complicated. As we have already seen in 1.2.2, there is a loss of information when crossing the boundary that is the solutions to the Stokes equation require three harmonic functions, whereas the solutions to Darcy's equation only require one harmonic function, and therefore some special boundary condition has to be devised. This is the aim of the next section.

3.2.4: The experimental Darcy condition

Historically speaking there are two successive models of a boundary condition at the interface $S$. Both introduce unusual constraints. The first one was introduced by Joseph and Tao (Joseph, Tao [12]).

3.2.4.1: The Joseph and Tao model

In a series of papers, Joseph with Tao, and later Shir (Joseph, Tao [12] and Shir, Joseph [28]) described boundary conditions for the free/porous media interface. Basically these conditions are the following:

- Continuity of the pressure.
- Continuity of the normal velocity on $S$. 

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- A no-slip condition for the tangential velocity on S,

that is

\[
\begin{align*}
    p^I &= p^{II} \\
    u_n^I &= u_n^{II} \quad \text{on } S. \\
    u_t^I &= u_t^{II} = 0
\end{align*}
\]  

[3-18]

The appearance of the flow is:

![Diagram](image)

There is an intuitive problem with this model. Indeed it is very difficult to understand why porous media have to have their pores aligned in the normal direction at and near the interface, this is more a condition for a membrane (one could consider that this model is actually considering the interface to be a membrane).

In a subsequent paper Beavers and Joseph (Beavers and Joseph [1]) tried, through experiment, to find an empirical boundary condition for $u_t$, the tangential velocity.

3.2.4.1: The experimental condition

By experimenting with laminar flows parallel to the interface (which was supposed to be plane) Beavers and Joseph (Beavers and Joseph [1]) found that there was a 'slip-flow', therefore invalidating condition [18-iii]. In these experiments it was found that there seemed to be a boundary layer, and that if one neglected that boundary layer, there would be a discontinuity of the tangential velocity at the boundary.
The discontinuity condition found was

\[ \frac{\partial u_1^I}{\partial n} = \alpha (u_1^I - u_1^{II}) \]  

[3-19]

In a paper published later (Saffman [18]) Saffman actually proceeded to prove this condition. His proof relied on one fact, the velocity \( u \) had to be of the form: \( u = u_x(z) e_x \) where \( e_x \) is parallel to the interface, and \( z \) corresponds the normal to \( S \). He proved, that, if \( k \), as defined by [3-10], was large enough, one has:

\[ \frac{1}{\alpha'} \sqrt{\frac{\mu}{k}} \frac{\partial u_1^I}{\partial n} + O\left(\frac{1}{k}\right) = u_1^I. \]  

[3-20]

\( \alpha' \) is a constant that can be interpreted as being

\[ \alpha' = \alpha \sqrt{\frac{\mu}{k}}. \]  

[3-21]

This prompted the use of the following boundary condition in the case of Darcy's model for Region(II):
where $\sigma = 0$ or -1, is a constant that combines [3-19] ($\sigma = -1$) and [3-20] ($\sigma = 0$). It is interesting to look at the behaviour of Darcy's and Brinkman's models when $k$ (for the Brinkman model) or $k, \alpha, \sigma$ (For the Darcy model) vary.

3.2.5 : Empirical discussion of the two models

Two cases can be considered : $k \to 0$, $k \to \infty$.

In the first case we are considering the limit case of the free medium. It is obvious that Darcy's equation, [3-13], becomes singular. That is we have to have a pressure tending towards zero to be able to have a finite velocity in Region(II). This seems rather awkward, since only very few cases of Stokes flows show a zero pressure (axisymmetric rotation is an example). In the case of Brinkman's equation, it seems as if, the equation was "converging" towards Stokes's equation. That is the solutions could be converging towards the solutions for a free medium.

In the second case we are considering the limit case of the solid medium. In both cases, we have to have a zero velocity in Region(II) in order to have a finite pressure. This does not seem contrary to intuition.
Chapter Four

The rotating solid above a porous bed of infinite thickness

Summary

In this chapter a study of the rotating particle in an "axisymmetric" setting will be made. The study will take advantage of the knowledge of Green's function for the equations considered. Then using a method similar to the one used by Shail and Gooden (Shail, Gooden [23]) an integral equation for the stress on the surface of the solid will be derived. The torque exerted by the particle on the fluid will then be computed, and approximated. A special case: the disk, will be considered and various quantities will be computed. An asymptotic study will then be conducted, and it will be shown that this problem can be approximated by a rotelet of suitable strength.

The aim of this chapter is the study of the low Reynolds number flow generated by the rotation of an axisymmetric solid around its axis of symmetry, above a porous bed of material of infinite thickness (the radius being the maximum of the distance between the axis of rotation and the boundary of the solid). We suppose the angular speed \( \Omega \) to be small, so that the Reynolds number

\[
Re = \frac{d \, \Omega \, a^2}{\mu},
\]

where \( d \) is the density of the fluid, \( a \) a typical body dimension and \( \mu \) its viscosity, is small. In this case the flow is viscous, and the Stokes equation applies in the free medium part. For the porous bed of material (the porous medium) we will assume that Brinkman's equation is valid.

As we are looking at a problem having rotational symmetries around the axis of revolution of the solid, let us use a cylindrical polar coordinate system of coordinates \((\rho, \varphi, z)\), where the \( z \)-axis is the axis of symmetry of the solid.

4.1: Description of the problem
4.1.1 : The configuration

We consider the following configuration : in a fluid filled space we have two regions, and a rotating solid described by the surface of revolution S and of radius a placed around z = h (that is the point \( \rho = 0, z = h \) is inside the solid), and rotating at the angular velocity \( \Omega \). Let us call \( u^i, p^i \) the velocity and pressure fields associated with the flow in region (i). We are assuming that the flow is a slow viscous flow and that the viscosity of the fluid is \( \mu \). The two regions (I) and (II) are defined as follows (See [Fig. 4-1]) :

- Medium (I) in Region(I) : Defined as the half-space \( z \geq 0 \), it is a free medium, the equation of motion for the fluid is the Stokes equation: \( \mu \Delta u^I = \nabla p^I \), where \( \mu \) is the viscosity of the fluid.

- Medium (II) in Region(II) : Defined as the half-space \( z \leq 0 \), it is a porous medium ; the equation of motion for the fluid is Brinkman's equation : \( \mu \Delta u^{II} - \mu k u^{II} = \nabla p^{II} \), where \( k \) is the Brinkman constant of the porous medium. Let us define \( \lambda = \sqrt{k} \) and whenever possible replace \( k \) by \( \lambda^2 \)

4.1.2 : The coordinate system

The problem is finding the flow created by a slowly rotating small particle (or a rotelet) by using a method appropriate to each case. We suppose that the particle is rotating around the z-axis and is located in \( z = h \). Furthermore, we suppose that the particle is rotationally symmetric. Therefore, we can safely assume that the flow will have a symmetry of rotation. Thus we will use a cylindrical system of coordinates \((\rho, \phi, z)\).
4.1.3: The equations

On the boundary \( z = 0 \) we have the following conditions (see chap II)

- Continuity of the velocity; i.e. \( u^I = u^{II} \) on \( z = 0 \).
- Continuity of the surface stress; i.e. \( T^e = T^{II}e_z \) on \( z = 0 \).

As there is a rotational symmetry, and because the flow is created by a rotating particle, we can write \( u = (0, v, 0) \), and knowing that the flow lines are circles centred on the \( z \)-axis, we have \( p^I = p^{II} = 0 \). \( v^I \), and \( v^{II} \), will satisfy the following equations:

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial v^I}{\partial \rho} \right) - \frac{v^I}{\rho^2} + \frac{\partial^2 v^I}{\partial z^2} = 0 \quad [4-1]
\]

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial v^{II}}{\partial \rho} \right) - \left( \frac{1}{\rho^2} + \lambda^2 \right) v^{II} + \frac{\partial^2 v^{II}}{\partial z^2} = 0 \quad [4-2]
\]

The boundary conditions rewrite as

\[
v^I = v^{II} \text{ on } z = 0 \quad [4-3]
\]

\[
\frac{\partial v^I}{\partial z} = \frac{\partial v^{II}}{\partial z} \text{ on } z = 0 \quad [4-4]
\]

\[
v^I = \Omega \rho \text{ on } S \text{ (in the case of a rotating solid).} \quad [4-5]
\]

To solve equations [4-1,2], it is convenient to use Green’s functions adapted to the problem. It is the aim of the next paragraph to rewrite the solutions of [4-1,2] as suitable integrals involving Green’s functions.

4.2: The Green’s functions method (for a rotating particle)

4.2.1: The Green’s functions

To start with let us find a Green’s function for [4-1] defined in the half-space \( z \geq 0 \) being such that, at the interface \( z = 0 \), \( \frac{\partial G}{\partial z} = 0 \). Such a Green’s function, \( G(r, r') \), which is a solution of \( \Delta_r G = -4\pi \delta(r-r') \), where \( r \) is the position vector corresponding to the point of coordinates \( (\rho, \phi, z) \), and \( \Delta \) is the operator \( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \), can be written the following way
where \( |r - r'| \) is the distance between the points that are at the extremities of \( r \) and \( r' \).

The same way, let us construct a Green's function for [4-2], \( H(r,r') \), defined in the half space \( 0 < z < 0 \), such as 0 on the interface \( z = 0 \). \( H \) is a solution of \( \Delta r H - \lambda^2 H = -4\pi \delta(r-r') \), with \( \lambda = \sqrt{k} \). We have:

\[
H(r,r') = \frac{e^{-\lambda |r-r'|}}{|r-r'|} + H'(r,r'), \quad H' \text{ being regular everywhere in } z,z' < 0.
\]  

[4-7]

Let us define \( w^i = v^i \cos(\phi) \) ( \( i = I, II \)). \( w^I \) is a solution of \( \Delta w^I = 0 \) (see [4-1]), and \( w^{II} \) is a solution of \( (\Delta - \lambda^2)w^{II} = 0 \) (see [4-2]). Using Green's theorem in the half space \( z < 0 \), we can write:

\[
\int \left[ \frac{\partial H}{\partial z} + H \Delta w^{II} \right] d\mathbf{r}' = \int \left[ \left( \frac{\partial H}{\partial z} + \lambda^2 H \right) w^{II} - H \Delta w^{II} - \lambda^2 w^{II} \right] d\mathbf{r}'
\]

\[
= 4\pi w^{II}, \quad \text{because } \Delta r H - \lambda^2 H = -4\pi \delta(r-r')
\]

\[
= \int \left[ \frac{\partial H}{\partial z} w^{II} - H \frac{\partial w^{II}}{\partial z} \right] dS' \text{ (Green's theorem)}.
\]

Therefore, as \( \frac{\partial H}{\partial z} = 0 \) on \( z = 0 \),

\[
w^{II} = \frac{1}{4\pi} \int_{z=0} H(r,r') \frac{\partial w^{II}}{\partial z} dS' = \frac{1}{4\pi} \int_{\rho'=0}^{\infty} \int_0^{2\pi} \left( \int_{\rho''=0}^{\infty} H(r,r') \cos(\phi - \phi') d\phi' \right) \rho' d\rho'
\]

[4-8]

If we call \( H^{(1)}(\rho,z;\rho',z') \) the coefficient of \( \cos(\phi - \phi') \) in the Fourier expansion of \( H \), we have

\[
v^{II}(\rho,z) = \frac{1}{4} \int_0^{\infty} H^{(1)}(\rho,z;\rho',0) f(\rho') \rho' d\rho', \quad \text{with } f(p) = \frac{\partial v^{II}}{\partial z}(\rho,0)
\]

[4-9,10]

The same way as for [4-8], we have:

\[
w^I = \frac{1}{4\pi} \int_{z=0} G(r,r') \frac{\partial w^I}{\partial z} dS' + \frac{\Omega}{4\pi} \int S \rho' \cos \phi' \frac{\partial G}{\partial n}(r,r') dS'
\]

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\[-\frac{1}{4\pi} \int_S G(r,r') \frac{\partial w^I}{\partial n} dS', \text{ where } n \text{ is the outwards directed normal to } S. \]  

[4-11]

Let us now transform [4-11]. To do this let us define

\[\sigma(p,z) = -\frac{1}{4\pi} \rho \frac{\partial}{\partial n}(\frac{\partial v^I}{\partial z}) = \frac{1}{4\pi} \frac{\partial}{\partial n}(\Omega \rho - v^I) \text{ for } (p,z) \text{ in } C \]

[4-12]

where C is the curve which is the intersection of the boundary of the solid S with the plane \[\varphi = 0\]. That is, S can be defined the following way

\[S = \{ (p; \varphi; z) / (p; z) \text{ in } C; 0 \leq \varphi \leq 2\pi \}. \]

Defining \[G^{(1)}\] the same way we defined \[H^{(1)}\], and using [4-12],[4-11] rewrites as

\[v^I(p,z) = -\frac{1}{4\pi} \int_0^\infty G^{(1)}(p,z;p',0) \frac{\partial v^I}{\partial z}(p',0) \rho' \, dp' + \pi \int_C G^{(1)}(p,z;p',z') \sigma(p',z') \rho' \, dl'.\]

[4-13]

Using equations [4-8,13], the boundary conditions [4-3..5] rewrite as

\[f(p) = \frac{\partial v^I}{\partial z}(p,0).\]
\[-\frac{1}{4\pi} \int_0^\infty G^{(1)}(p,0;p',0) f(p') \rho' \, dp' + \]
\[\int_C G^{(1)}(p,0;p',z') \sigma(p',z') \rho' \, dl' = \frac{1}{4\pi} \int_0^\infty H^{(1)}(p,0,p',0) f(p') \rho' \, dp' \text{ for } \rho \geq 0. \]

[4-14]

\[\int_C G^{(1)}(p,z;p',z') \sigma(p',z') \rho' \, dl' - \frac{1}{4\pi} \int_0^\infty G^{(1)}(p,z;p',0) f(p') \rho' \, dp' = \Omega \rho \]

[4-15]

for \((p,z) \text{ in } C.\)

Equations [4-14,15] give a system of integral equations. To solve it we will have to use a representation for both \[G^{(1)}\] and \[H^{(1)}\]. This representation will make use of the Hankel transform since Bessel functions are functions associated with the separation of variables of the \[\Delta\] and \[\Delta - \lambda^2\] operators.

4.2.2: The representations

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Let us use the following representations for $G^{(1)}$ and $H^{(1)}$, we have

$$G^{(1)}(\rho,z;\rho',z') = 2 \int_0^\infty J_1(s\rho) J_1(s\rho') \left( e^{-s|z - z'|} + e^{-s(z + z')} \right) \, ds \quad [4-16]$$

$$H^{(1)}(\rho,z;\rho',z') = 2 \int_0^\infty A_\lambda(s) J_1(s\rho) J_1(s\rho') \left( e^{-\sqrt{s^2 + \lambda^2} |z - z'|} + e^{\sqrt{s^2 + \lambda^2} (z + z')} \right) \, ds \quad [4-17]$$

where $A_\lambda(s) = \frac{s}{\sqrt{s^2 + \lambda^2}} \quad [4-18]$.

The result given in [4-17] results from the following formula showing the complete Fourier decomposition of $\frac{e^{-\lambda |r-r'|}}{|r-r'|}$ (Morse, Feshbach [13] p 888)

$$\frac{e^{-\lambda |r-r'|}}{|r-r'|} = \sum_{n \in \mathbb{N}} (2 - \delta_{0n}) \cos(n(\varphi - \varphi')) \int_0^\infty A_\lambda(s) J_n(s\rho) J_n(s\rho') e^{-\sqrt{s^2 + \lambda^2} |z - z'|} \, ds.$$  

Let us use [4-16,17] to transform equations [4-14,15]. Equation [4-14] gives:

$$\pi \int_C \int_0^\infty J_1(s\rho) J_1(s\rho') e^{-sz'} \sigma(\rho',z') \rho' \, dl' \, ds$$

$$= \frac{1}{4} \int_0^\infty (1 + A_\lambda(s)) J_1(s\rho) \int_0^\infty J_1(s\rho') f(\rho') \rho' \, d\rho' \, ds. \quad [4-19]$$

By defining $F(s)$ as being the Hankel transform of $f$ of order 1, that is,

$$F(s) = \int_0^\infty J_1(s\rho') f(\rho') \rho' \, d\rho', \quad [4-20]$$

[4-14] rewrites as

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\[ 4\pi \int_0^\infty J_1(sp) \int J_1(sp') e^{-sz'} \sigma(p',z') \rho' \, dl' = \int_0^\infty \left( 1 + A_\lambda(s) \right) J_1(sp) F(s) \, ds , \] \[ [4-21]\]

which is standing for all positive \( \rho \), therefore we have:

\[ F(s) = \frac{4\pi}{1 + A_\lambda(s)} \int J_1(sp') e^{-sz'} \sigma(p',z') \rho' \, dl' \]

\[ [4-22]\]

Indeed if for all \( \rho > 0 \), \( \int J_1(sp) f(s) \, ds = 0 \) then for all \( s \geq 0 \), \( f(s) = 0 \). Let us work on equation [4-15]. By expanding the second term in the left hand side and substituting \( F \) where necessary, we have

\[ \pi \int G^{(1)}(p,z;p',z') \sigma(p',z') \rho' \, dl' - \int J_1(sp) e^{-sz} F(s) \, ds = \Omega \rho . \]

\[ [4-23]\]

But

\[ \int_0^\infty J_1(sp) e^{-sz} F(s) \, ds = \int_0^\infty 4\pi \int_0^\infty \frac{1}{1 + A_\lambda(s)} J_1(sp) J_1(sp') e^{-s(z + z')} \sigma(p',z') \rho' \, dl' \, ds \]

\[ [4-24]\]

thus, if we define

\[ G^{(1)}_0(p,z;p',z') \equiv 2 \int_0^\infty J_1(sp) J_1(sp') e^{-s|z - z'|} \, ds \]

\[ [4-25]\]

\[ K(p,z;p',z') \equiv 2 \int_0^\infty J_1(sp) J_1(sp') \frac{A_\lambda(s) - 1}{A_\lambda(s) + 1} e^{-s(z + z')} \, ds \]

\[ [4-26]\]

we have the following equation for \( \sigma \):

\[ \Omega \rho = \pi \int G^{(1)}_0 + K(p,z;p',z') \sigma(p',z') \rho' \, dl' \text{ for all } (p,z) \in C \]

\[ [4-27]\]
The interesting thing about the knowledge of $\sigma$ is that the torque exerted by the liquid on the particle is a simple integral of $\sigma$. Indeed $\sigma$ is linearly dependent on the surface stress on the surface of the solid $S$. The next section will study the torque exerted by the fluid on the solid, and make an asymptotic expansion of the torque, when the solid if far from the interface.

4.3: Torque exerted on the rotating particle by the fluid

Let us call $L$, the torque exerted by the fluid on the particle $S$. We have $L = -Le_z$ with $L = 8\pi^2\mu \int_C \rho^2 \sigma(\rho,z) \, dl$. Let us study $L$ for $\frac{h}{a} \gg 1$. Two cases can occur: $\lambda h = O(1)$ or $\lambda h \gg 1$. In both cases the analysis is very similar.

4.3.1: The case $\lambda h \gg 1$

Let us first study the case $\lambda h \gg 1$. Let us expand $K$ as a power series for large $h$, we have

$$K = \frac{1}{h^3} A \rho \rho' + \frac{1}{h^4} B \rho \rho' \left( (\rho + 2h) + \frac{2}{\lambda} \right) + O\left( \left( \frac{a}{h} \right)^5 \right) \tag{4-29}$$

with, in our case: $A = -\frac{1}{8}$, $B = \frac{3}{16}$ \tag{4-30}

Let us also expand $\sigma$ as a power series in $h$, we have

$$\sigma = \sigma_0 + \frac{1}{h^3} \sigma_1 + \frac{1}{h^4} \sigma_2 + ... \tag{4-31}$$

The terms in $h^{-1}, h^{-2}$ are equal to zero, a justification can be seen in the power series for $K$. Equation [4-23] becomes now a system of equations with the unknowns $\sigma_0, \sigma_1, \sigma_2, ...$. We have

$$\pi \int_C G_0^{(1)}(\rho,z;\rho',z') \sigma_0(\rho',z') \, \rho' \, dl' = \Omega \rho \tag{4-32}$$

$$\int_C G_0^{(1)}(\rho,z;\rho',z') \sigma_1(\rho',z') \, \rho' \, dl' = -A \rho \int_C \rho' \sigma_0(\rho',z') \, dl' \tag{4-33}$$

$$\int_C G_0^{(1)}(\rho,z;\rho',z') \sigma_2(\rho',z') \, \rho' \, dl' = -B \rho \int_C \rho' \left( \rho' + 2h \right) \sigma_0(\rho',z') \, dl' \tag{4-34}$$
Let us define $L_\infty = 8\pi^2 \mu \int_\mathcal{C} \rho^2 \sigma_0(\rho, z) \, dl$ [4-35], the torque in the absence of porous medium (or for $h = \infty$), and let us rearrange the terms in [4-32,33]. [4-33] is now equivalent to

$$\sigma_1(\rho, z) = -\frac{\pi}{\Omega} A \int_\mathcal{C} \rho^2 \sigma_0(\rho', z') \, dl' \sigma_0(\rho, z) = -\frac{\pi}{\Omega} A \frac{L_\infty}{8\pi^2 \mu} \sigma_0(\rho, z).$$  \[4-36\]

This will permit us now to find an asymptotic expansion in $\frac{1}{h}$ of $L$, and this as a simple function of $A$, $h$, $\Omega$, $L_\infty$.

### 4.3.2: The general case

Let us return to $L$, using [4-31,36], we have

$$L = 8\pi^2 \mu \int_\mathcal{C} \rho^2 \sigma_0(\rho, z) \, dl +$$

$$\frac{1}{h^3} \int_\mathcal{C} \rho^2 \sigma_1(\rho, z) \, dl + \frac{1}{h^4} \int_\mathcal{C} \rho^2 \sigma_2(\rho, z) \, dl + O\left(\left(\frac{\alpha}{h}\right)^5\right)$$

$$= L_\infty - \frac{A}{8\pi \mu h^3 \Omega} L_\infty^2 + \frac{8\pi^2 \mu}{h^4} \int_\mathcal{C} \rho^2 \sigma_2(\rho, z) \, dl + O\left(\left(\frac{\alpha}{h}\right)^5\right).$$ \[4-37\] \[4-38\]

But, if we compute, using [4-34], the following integral

$$\int_\mathcal{C} \rho \sigma_0(\rho, z) \int_\mathcal{C} G_0^{(1)}(\rho, z; \rho', z') \sigma_2(\rho', z') \rho' \, dl' \, dl =$$

$$\frac{\Omega}{\pi} \int_\mathcal{C} \rho^2 \sigma_2(\rho, z) \, dl =$$

$$-B \int_\mathcal{C} \int_\mathcal{C} \rho^2 \rho'^2 \sigma_0(\rho, z) \sigma_0(\rho', z') \left((z + z' - 2h) + \frac{2}{\lambda}\right) \, dl' \, dl =$$

$$-2B \int_\mathcal{C} \int_\mathcal{C} \rho^2 \rho'^2 \sigma_0(\rho, z) \sigma_0(\rho', z') \left((z - h) + \frac{1}{\lambda}\right) \, dl' \, dl =$$

$$-2B \int_\mathcal{C} \rho^2 \sigma_0(\rho, z) \, dl \int_\mathcal{C} \rho^2 \left(z - h + \frac{1}{\lambda}\right) \sigma_0(\rho, z) \, dl =$$
\[-\frac{2B}{8\pi^2\mu}L_\infty \int_C \rho^2 \left(z - h + \frac{1}{\lambda}\right) \sigma_0(\rho,z) \, dl.\]

That is:
\[
\int_C \rho^2 \sigma_2(\rho,z) \, dl = -\frac{2B}{8\pi^2\mu}L_\infty \int_C \rho^2 \left(z - h + \frac{1}{\lambda}\right) \sigma_0(\rho,z) \, dl, \quad [4-39]
\]

thus [4-37] is now equivalent to
\[
L = L_\infty - \frac{A}{8\pi\mu h^3\Omega} \frac{L_\infty^2}{h^4} - \frac{2B}{8\pi\mu}L_\infty \int_C \rho^2 \left(z - h + \frac{1}{\lambda}\right) \sigma_0(\rho,z) \, dl + O\left(\frac{1}{h^5}\right). \quad [4-40]
\]

Looking at the integral \( \int_C \rho^2 (z - u) \sigma(\rho,z) \, dl \), considered as a function of \( u \), it is obvious that there is a value of \( u \), inside the interval \([z_{min}, z_{max}]\), where \( z_{min} \) (resp. \( z_{max} \)) is the minimal (resp maximal) value \( z \) can take on \( C \), for which this integral is equal to zero. In the case where \( C \) is symmetrical around a line perpendicular to the \( z \) axis, then \( u \) will be the \( z \) coordinate of the intersection point (i.e. will define the centre of the surface of the solid \( S \)). Therefore if we choose \( h \) to be that value (this is the actual definition of \( h \), see 1.1 for a description of \( h \)), we have
\[
\int_C \rho^2 (z - h) \sigma(\rho,z) \, dl = 0, \therefore \int_C \rho^2 (z - h) \sigma_0(\rho,z) \, dl = O\left(\frac{1}{h^3}\right). \quad [4-41]
\]

The reason for this awkward condition will become apparent later (see 4.5...). Thus, the final formula for the drag, which is [4-40] rewritten, is
\[
\frac{L}{L_\infty} = 1 - \frac{1}{8\pi\mu h^3\Omega}L_\infty \left(\frac{2B}{\lambda h}\right) + O\left(\frac{1}{h^3}\right). \quad [4-42]
\]

That formula being very near to Brenner's formula (which is supposedly only true in the case of solid boundaries). As a matter of fact we will see in the next section that we have the same formulae.

4.3.3: The case \( \lambda h = O(1) \)

Let us now look at the case \( \lambda h = O(1) \), the same way as we derived [4-29], we have for \( K \)
\[ K = \frac{1}{h^3} A_1 \rho p' + \frac{1}{h^4} B \rho p'(z + z' - 2h) + O\left(\frac{a^5}{h}\right) \] \tag{4-43}

with, this time:

\[
\begin{align*}
A_1 &= \frac{1}{2} \int_0^\infty t^2 e^{-2t} \frac{A_\lambda(t)}{A_\lambda(h) + 1} \, ds = \frac{1}{2} \int_0^\infty t^2 e^{-2t} \frac{t - \sqrt{t^2 + \lambda^2 h^2}}{t + \sqrt{t^2 + \lambda^2 h^2}} \, ds \\
B &= \frac{3}{16}.
\end{align*}
\tag{4-44}
\]

Actually, when \( \lambda h \to \infty, A_1 \to A = -\frac{1}{8}. \) Using the same method as the one described in 3.2, we get

\[
\frac{L}{I_{\infty}} = 1 - \frac{1}{8\pi h^3} \Omega I_{\infty} A_1 + O\left(\frac{a^5}{h}\right) \tag{4-45}
\]

We have now computed the torque and a power series in \( h \) of the torque for large \( h \) for axisymmetric solids having a non-zero volume. Let us now look at the case of a disk, which is a zero volume solid.

\section*{4.4: The case of the disk}

\subsection*{4.4.1: The particularity of the problem}

Let us now look at a disk centred in \( z = h \) and of radius \( a \). Since a disk is a solid without interior, the derivation of equation [4-27] is not any more valid. It relied heavily on Green's formula, which necessitates a solid with a non-zero volume. Nevertheless, using the same method, as in chapter 2-§2.2.2, we can rewrite [4-27] as follows

\[
\Omega \rho = \int_0^a \left( G_0^{(1)} + K \right)(\rho; h; \rho'; h') \sigma^* (\rho') \rho' \, d\rho', \tag{4-46}
\]

where \( G_0^{(1)} \) and \( K \) are defined as in [4-25,26], and
\[ \sigma^*(p) \equiv -\frac{1}{4\pi} \rho \frac{\partial^2 [v^\dagger]}{\partial z^2} \rho - \frac{1}{4\pi} \rho \frac{\partial [v^\dagger]}{\partial z} \rho = \sigma^* - \sigma = -\frac{1}{4\pi} \rho \frac{\partial^2 [v^\dagger]}{\partial e^2} \rho - \frac{1}{4\pi} \rho \frac{\partial [v^\dagger]}{\partial (-e^2)} \rho. \] \hspace{1cm} [4-47]

The bracketed quantities are defined by,
\[ \frac{\partial^\epsilon}{\partial u^\epsilon}[f] = \lim_{h \to 0^\epsilon} \left[ \frac{f(x+hu) - f(x)}{h} \right], \] where \( \epsilon \) is the symbol + or - \hspace{1cm} [4-48]

and,
\[ \frac{\partial^\epsilon}{\partial (-u)^\epsilon}[f] = -\frac{\partial^\epsilon}{\partial u^\epsilon}[f]. \hspace{1cm} [4-49] \]

Let us define the following functions
\[ B_\lambda(s) = \frac{A_\lambda(s) - 1}{A_\lambda(s) + 1} = \frac{s - \sqrt{s^2 + \lambda^2}}{s + \sqrt{s^2 + \lambda^2}}, \quad C_\lambda(s) = \frac{A_\lambda(s)}{A_\lambda(s) + 1}. \hspace{1cm} [4-50] \]

The analysis made in §3 still holds for \( \sigma^* \). The solution \( \sigma^* \) to the equation [4-46] is a function that intuitively shows a discontinuity of the form \( \frac{1}{\sqrt{a - \rho}} \) near the edge of the disk. To "tame" \( \sigma^* \), Williams (Williams [34]) rewrites [4-46] by changing the unknown in the equation. Let us look at this transformation.

4.4.2: Williams's transformation

Let us transform equation [4-46] using Williams's (Williams [34]) method. Let us set :
\[ \zeta(p) \equiv \frac{2\pi}{\Omega} \rho \sigma^*(p). \hspace{1cm} [4-51] \]

Let us now change the unknown \( \zeta \) into the unknown \( W \):
\begin{align*}
W(p) &= p \int_{\rho}^{a} \frac{\zeta(t)}{t} \frac{1}{\sqrt{t^2 - \rho^2}} \, dt \quad \text{(the actual unknown change)} \\
\zeta(p) &= -\frac{2p}{\pi} \frac{d}{dp} \left[ \int_{\rho}^{a} \frac{W(x)}{\sqrt{x^2 - \rho^2}} \, dx \right] \quad \text{(the inversion formula)}. 
\end{align*}

The dimensions are: \([W] = \ell, \quad [\zeta] = \ell\). [4-46] is now equivalent to

\[ W(p) = 2p - \int_{0}^{a} L(t, p) W(t) \, dt, \quad [4-53] \]

where

\[ L(t, p) = \frac{2}{\pi} \int_{0}^{\infty} \sin(st) \sin(sp) B_\lambda(s) e^{-2sh} \, ds. \quad [4-54] \]

A quick dimensional analysis yields the following results: if \(W(p)\) is a solution for the set of values \((a, h, \lambda)\), then \(\frac{1}{a} W\left(\frac{D}{a}\right)\) is a solution for the set of values \((1, \frac{h}{a}, \lambda a)\). This reduces the effective number of parameters to two. The non-dimensional physical quantities are: \(\frac{1}{a} W, \frac{h}{a}, \lambda a, \frac{1}{\Omega a} u^i, \frac{1}{\mu \Omega a^3} L\)

The formula for the drag \(L\) can now be rewritten as a function of \(W\). We have

\[ L = 8 \pi^2 \mu \int_{0}^{a} \rho^2 \sigma^*(\rho) \, d\rho = 4 \Omega \pi \mu \int_{0}^{a} \rho \zeta(\rho) \, d\rho = 16 \Omega \mu \int_{0}^{a} x W(x) \, dx. \quad [4-55] \]

The drag in the case of an infinite free medium, \(L_\infty\), can easily be computed. Indeed, in this case we have \(\lambda = 0\), thus \(A_\lambda = 1, B_\lambda = 0\), therefore \(W(p) = 2p\), so, eventually we get:

\[ L_\infty = \frac{32}{3} \mu \Omega a^3. \quad [4-56] \]

Therefore the formula [4-42] giving the ratio between \(L\) and \(L_\infty\), becomes, in the case of a disk,
\[ L_{\infty} = 1 - \frac{4}{3\pi h^3} a^3 \tilde{A}_1 + o\left(\frac{a^5}{h}\right) \quad [4-57] \]

Where \( \tilde{A}_1 \) is either \( A_1 \) [4-44] (case \( \lambda h = O(1) \)) or \( A + \frac{B}{h} \) [4-30] (case \( \lambda h \gg 1 \)). Let us now compute the interfacial velocity, that is the velocity at the interface \( z = 0 \).

4.4.3: The interfacial velocity

Let us rewrite [4-13] to get the velocity for \( z = 0 \) (i.e. interfacial velocity). We have

\[ v^I(p,0) = -\frac{1}{4} \int_0^\infty G^{(1)}(\rho,0;\rho',0) f(\rho') \rho' \, d\rho' + \pi \int_0^a G^{(1)}(\rho,0;\rho',h) \sigma^*(\rho') \rho' \, d\rho'. \quad [4-58] \]

Let us expand \( G^{(1)} \), using the definition of \( F(s) \) [4-20,22], we get

\[ v^I(p,0) \]

\[ = -\int_0^\infty \int_0^\infty J_1(s\rho) J_1(s\rho') \, ds \, f(\rho') \rho' \, d\rho' + \pi \int_0^a G^{(1)}(\rho,0;\rho',h) \sigma^*(\rho') \rho' \, d\rho'. \quad [4-59] \]

\[ = -\int_0^\infty J_1(s\rho) F(s) \, ds + \pi \int_0^a G^{(1)}(\rho,0;\rho',h) \sigma^*(\rho') \rho' \, d\rho'. \quad [4-60] \]

Using equation [4-22] to replace \( F(s) \), we have now

\[ v^I(p,0) = -4\pi \int_0^\infty \int_0^a \frac{1}{1 + A^2(s)} J_1(s\rho) J_1(s\rho') e^{-\lambda h} \, ds \, \sigma^*(\rho') \rho' \, d\rho' \]

\[ + 4\pi \int_0^\infty \int_0^a J_1(s\rho) J_1(s\rho') e^{-\lambda h} \, ds \, \sigma^*(\rho') \rho' \, d\rho'. \quad [4-61] \]

or, more concisely,
\[ v^i(p,0) = 4\pi \int_0^a \left[ \int_0^\infty C_\lambda(s) \ J_1(s \rho) \ J_1(s \rho') \ e^{-hs} \ ds \right] \sigma^c(\rho') \ \rho' \ d\rho', \tag{4-62} \]

where \( C_\lambda \) is defined as in [4-50], i.e. \( C_\lambda(s) = \frac{A_\lambda(s)}{A_\lambda(s) + 1} \)

The method used to compute the interfacial velocity can be used again, although with some slight modifications in the case of arbitrary axisymmetric solids. The following steps are specific to the case of a disk.

Let us use the method described by Shail (Shail [20]) to transform the equation [4-62] into a formula involving the function \( W \) defined by [4-52]. This involves a lengthy calculation which follows. We have

\[ v^i(p,0) = 2\Omega \int_0^a \left[ \int_0^\infty C_\lambda(s) \ J_1(s \rho) \ J_1(s \rho') \ e^{-hs} \ ds \right] \zeta(\rho') \ d\rho' \]  
\[ = -\frac{4\Omega}{\pi} \int_0^a \left[ \frac{d}{d\rho'} \rho' \right] C_\lambda(s) \ J_1(s \rho) \ J_1(s \rho') \ e^{-hs} \ ds \ \frac{d}{d\rho'} \left[ \int_0^a \frac{W(t)}{t^2 - \rho'^2} \ dt \right] d\rho' \]  
\[ = \frac{4\Omega}{\pi} \int_0^a \left[ \int_0^\infty C_\lambda(s) \ J_1(s \rho) \ J_1(s \rho') \ e^{-hs} \ ds \right] \int_0^a \frac{W(t)}{t^2 - \rho'^2} \ dt \ d\rho' \]  
\[ = \frac{4\Omega}{\pi} \int_0^a \left[ \int_0^\infty C_\lambda(s) \ J_1(s \rho) \ J_0(s \rho') \ e^{-hs} \ ds \right] \left( \int_0^a \frac{W(t)}{t^2 - \rho'^2} \ dt \right) d\rho' \]  
\[ = \frac{4\Omega}{\pi} \int_0^\infty \left( \int_0^a \frac{W(t)}{t^2 - \rho'^2} \rho' \ J_0(s \rho') \ dt \ d\rho' \right) s C_\lambda(s) \ J_1(s \rho) \ e^{-hs} \ ds \tag{4-67} \]
\[ \frac{4\Omega}{\pi} \int_0^\infty \left( \int_0^t \frac{p'}{\sqrt{t^2 - p'^2}} \, dp' \, dt \right) s \, C_\lambda(s) \, J_1(sp) \, e^{-hs} \, ds. \]  \[4-68\]

Let us use the following formula to simplify the innermost integral in \([4-68]\). This formula is:

\[ \int_0^x \frac{p \, J_0(p\rho)}{\sqrt{x^2 - \rho^2}} \, dp = \frac{\sin(tx)}{t}. \]  \[4-69\]

\([4-68]\) can now be rewritten to give the final formula for the interfacial velocity,

\[ v_1(p) = v_I(p,0) = \frac{4\Omega}{\pi} \int_0^a W(t) \left[ \int_0^\infty C_\lambda(s) \, J_1(sp) \, \sin(st) \, e^{-hs} \, ds \right] \, dt \]  \[4-70\]

The method we just used to compute the interfacial velocity ([4-63..70]) can, with slight modifications, be used to compute the velocities \(v_I, v_{II}\) everywhere. Equation \([4-71]\) was omitted.

4.4.4 : The velocity everywhere

4.4.4.1 : Using the same method as the one we used to compute \(v_I\), we can actually compute \(v^I(p,z)\) and \(v^II(p,z)\). Three cases have to be distinguished: \(z \geq h, h \geq z \geq 0,\) and \(0 \geq z.\)

- (i) In the first case, the case where \(z \geq h \geq 0\) we have

\[ v^I(p,z) = \Omega \int_0^a \left[ \int_0^\infty J_1(sp) \, J_1(sp') \left( e^{-s(z - h)} + e^{-s(z + h)\frac{A_\lambda(s) - 1}{A_\lambda(s) + 1}} \right) \, ds \right] \, \zeta(p') \, dp' \]  \[4-72\]
\[ = 2\Omega \int_{0}^{a} \left[ \int_{0}^{\infty} J_{1}(sp) J_{1}(sp') e^{-sz} \left( \text{ch}(sh) - \frac{e^{-sh}}{1 + \lambda^2 A(s)} \right) ds \right] \zeta(\rho') d\rho' \quad [4-73] \]
\[ = \frac{4\Omega}{\pi} \int_{0}^{a} W(t) \left[ \int_{0}^{\infty} J_{1}(sp) \sin(st) e^{-sz} \left( \text{ch}(sh) - \frac{e^{-sh}}{1 + \lambda^2 A(s)} \right) ds \right] dt \quad [4-74] \]

• (ii) In the case where \( h \geq z \geq 0 \):

\[ v^{I}(\rho, z) \]
\[ = \Omega \int_{0}^{a} \left[ \int_{0}^{\infty} J_{1}(sp) J_{1}(sp') \left( e^{-s(h - z)} + e^{-s(z + h)} \frac{A(s)}{A(s) + 1} \right) ds \right] \zeta(\rho') d\rho' \quad [4-75] \]
\[ = 2\Omega \int_{0}^{a} \left[ \int_{0}^{\infty} J_{1}(sp) J_{1}(sp') e^{-sh} \left( \text{ch}(sz) - \frac{e^{-sz}}{1 + \lambda^2 A(s)} \right) ds \right] \zeta(\rho') d\rho' \quad [4-76] \]
\[ = \frac{4\Omega}{\pi} \int_{0}^{a} W(t) \left[ \int_{0}^{\infty} J_{1}(sp) \sin(st) e^{-sh} \left( \text{ch}(sz) - \frac{e^{-sz}}{1 + \lambda^2 A(s)} \right) ds \right] dt \quad [4-77] \]

• (iii) In the case where \( 0 \geq z \):

\[ v^{II}(\rho, z) \]
\[ = 2\Omega \int_{0}^{a} \left[ \int_{0}^{\infty} C_{\lambda}(s) J_{1}(sp) J_{1}(sp') e^{-sh} e^{sz} ds \right] \zeta(\rho') d\rho', \text{ where } e = \sqrt{s^2 + \lambda^2} \quad [4-78] \]
\[ = \frac{4\Omega}{\pi} \int_{0}^{a} W(t) \left[ \int_{0}^{\infty} C_{\lambda}(s) J_{1}(sp) \sin(st) e^{-sh} e^{sz} ds \right] dt \quad [4-79] \]
4.4.4.2: In a similar fashion, we can compute the velocity fields for any kind of axisymmetric solid. We have

\[ v^I(\rho, z) = 2\pi \int_{C} \left[ \int_{0}^{\infty} J_1(s\rho) J_1(s\rho') \left( e^{-s|z - z'|} + e^{-s(z + z')} \frac{A\lambda(s) - 1}{A\lambda(s) + 1} \right) ds \right] \sigma(\rho', z') \rho' \, dl' \]  

\[ v^II(\rho, z) = 4\pi \int_{C} \left[ \int_{0}^{\infty} C_\lambda(s) J_1(s\rho) J_1(s\rho') e^{-s^2} e^{sz} ds \right] \sigma(\rho', z') \rho' \, dl' \]  

where \( \varepsilon = \sqrt{s^2 + \lambda^2} \)

4.4.4.3: One can now check the consistency of results [4-70,74,77,79..81] by simply computing \( v^I(\rho, 0) \) and \( v^II(\rho, 0) \) (they are equal to \( v_1(\rho) \)). In the case of the disc, computing the value of \( v^I(\rho, h) \), creates slight computational problems. As a matter of fact the kernel

\[ e^{-s\rho} \left( \frac{e^{-sz}}{1 + A\lambda(s)} \right), \]  

becomes equal to

\[ \frac{1}{2} \left( 1 + e^{-2sh} \frac{A\lambda(s) - 1}{A\lambda(s) + 1} \right) = \frac{1}{2} \left( 1 + e^{-2sh} B_\lambda \right). \]  

But, we have

\[ \int_{0}^{\infty} J_1(s\rho) \sin(st) ds = \frac{t}{\rho\sqrt{\rho^2 - t^2}} 1_{\{\rho > t\}}(t), \]  

which derives from (See Watson [31], Gradstheyn [9] p763 f : 6.752.1)

\[ \int_{0}^{\infty} e^{-as}J_1(s\rho) \sin(st) ds = \frac{2t}{r_1r_2 (r_1 + r_2)} \frac{\left( \rho \left( r_1 + r_2 \right) - t \left( r_1 - r_2 \right) \right)}{\sqrt{(r_1 + r_2)^2 - 4t^2}}, \]  

where
\[ r_1^2 = (p + t)^2 + a^2 \quad \text{[4-86]} \]
\[ r_2^2 = (p - t)^2 + a^2. \quad \text{[4-87]} \]

\( v^I \) can now be rewritten as

\[
v^I(p,h) = \frac{2\Omega}{\pi} \int_0^a W(t) \left[ \frac{t}{\rho^2 - t^2} \mathbf{1}_{\{p>t\}}(t) + \int_0^\infty J_1(sp) \sin(st) e^{-2sh} B_\lambda \, ds \right] \, dt, \quad \text{[4-88]} \]

that is, if \( p < a \)

\[
v^I(p,h) = \frac{2\Omega}{\pi} \left[ \int_0^p W(t) \frac{t}{\rho^2 - t^2} \, dt + \int_0^a \left[ \int_0^\infty J_1(sp) \sin(st) e^{-2sh} B_\lambda \, ds \right] \, dt \right] \quad \text{[4-89]} \]

(= \( \Omega \rho \)),

and if \( p \geq a \)

\[
v^I(p,h) = \frac{2\Omega}{\pi} \int_0^a W(t) \left[ \frac{t}{\rho^2 - t^2} + \int_0^\infty J_1(sp) \sin(st) e^{-2sh} B_\lambda \, ds \right] \, dt. \quad \text{[4-91]} \]

A good way to convince oneself of the validity of the formulae given in [4-89,91] is to compute the values in the case of the free-medium (i.e. \( \lambda = 0 \)). In this case \( W(t) = 2t \) and \( B_\lambda = 0 \). The formulae [4-89-91] transform to

\[
v^I(p,h) = \frac{2\Omega}{\pi} \int_0^p 2t \frac{t}{\rho^2 - t^2} \, ds = \frac{2\Omega}{\pi} \rho \arcsin(1) = \Omega \rho, \text{ for } p \leq a, \quad \text{[4-92]} \]

68
\[ v^I(\rho, h) = \frac{2\Omega}{\pi} \int_0^a 2t \frac{t}{\rho \sqrt{\rho^2 - t^2}} \, ds \]  

[4-93]

\[ = \frac{2\Omega}{\pi} \left( \rho \arcsin \left( \frac{a}{\rho} \right) - \frac{a}{\rho} \sqrt{\rho^2 - a^2} \right) = \frac{2\Omega}{\pi} \left( \rho \arcsin \left( \frac{a}{\rho} \right) - a \sqrt{1 - \left( \frac{a}{\rho} \right)^2} \right) \text{ for } \rho \geq a. \]  

[4-94]

4.4.4.4: Another interesting feature is $\frac{\partial v}{\partial \rho}$ near $\rho = a$, where, in this section $v = v^I(\rho, h)$. It is 'obvious' that we will have a discontinuity. Let us prove it; we have

\[ \frac{\partial v}{\partial \rho} = \Omega, \text{ for } \rho \leq a, \]

\[ \frac{\partial v}{\partial \rho} = \Omega \left( \arcsin \left( \frac{a}{\rho} \right) - \frac{a}{\rho} \frac{a^2 + \rho^2}{\sqrt{\rho^2 - a^2}} \right) \text{ for } \rho \geq a. \]  

[4-95]

One sees immediately that, not only do we have a discontinuity of $\frac{\partial v}{\partial \rho}$ near $\rho = a$, but actually we have a singularity for $\frac{\partial v}{\partial \rho}$ as $\rho \to a^+$. More precisely

\[ \lim_{\rho \to a^+} \left[ \frac{\partial v}{\partial \rho} \right] = -\infty. \]  

[4-96]

As a matter of fact, one can expand this study to the case $\lambda > 0$. To start with it is clear that, if $W$ exists, then it is of class $C^\infty$. Let us have another look at the right hand sides of formulae [4-89-91]. These are in two terms. The part

\[ \int_0^a W(t) \left[ \int_0^\infty J_1(s\rho) \sin(st) e^{-2sh} B_\lambda \, ds \right] \, dt \]  

[4-97]

is of class $C^\infty$. Therefore, only the first term has to be examined. This integral can be rewritten the following way:

\[ \int_0^a W(t) \frac{t}{\rho \sqrt{\rho^2 - t^2}} \left\{ \rho > t \right\} (t) \, dt = \frac{1}{a} \int_0^a W\left( \frac{p}{a} s \right) \frac{s}{\sqrt{a^2 - s^2}} \left\{ a > s \right\} (s) \, ds. \]  

[4-98]
We have, for \( p < a \)

\[
\frac{d}{dp} \left[ \int_0^a W(t) \frac{t}{p \sqrt{p^2 - t^2}} \mathbf{1}_{\{p > t\}}(t) \, dt \right]
\]

\[
= \frac{1}{a^2} \int_0^a W(\frac{p}{a} s) \frac{s}{\sqrt{a^2 - s^2}} \mathbf{1}_{\{a > s\}}(s) \, ds
\]

\[
= \frac{1}{a^2} \int_0^a W'(\frac{p}{a} s) \frac{s^2}{\sqrt{a^2 - s^2}} \, ds
\]

for \( p > a \)

\[
\frac{d}{dp} \left[ \int_0^a W(t) \frac{t}{p \sqrt{p^2 - t^2}} \mathbf{1}_{\{p > t\}}(t) \, dt \right]
\]

\[
= \frac{1}{a^2} \int_0^a W(\frac{p}{a} s) \frac{s}{\sqrt{a^2 - s^2}} \, ds
\]

\[
= \frac{1}{a^2} \int_0^a W'(\frac{p}{a} s) \frac{s^2}{\sqrt{a^2 - s^2}} \, ds - \frac{a^2}{p^2 \sqrt{p^2 - a^2}} W(a).
\]

One can again see very well the discontinuity, an singularity of \( \frac{\partial v}{\partial p} \) near \( a \). This singularity is of the form \( \frac{1}{\sqrt{p^2 - a^2}} \).
We have now computed the velocity field for any solid. Let us study its behaviour at infinity. The scheme we will use for this study is the same as the one used for the velocity. We will first look at the case of the disk to be able to understand the general method, then we will generalize the results found for the disk to an arbitrary axisymmetric solid.

### 4.5. Asymptotic evaluations

Let us study the asymptotical properties of \( v \) as \( R \to \infty \) (\( R^2 = \rho^2 + z^2 \)). Two cases have to be considered: \( \lambda > 0 \) and \( \lambda = 0 \). In the easiest \( \lambda = 0 \) by using the actual formulae for the velocity \([4-74,79]\), we have

\[
v \sim \frac{1}{8\pi \mu} \frac{\rho}{R^5}.
\]  

[4-104]

In the case \( \lambda > 0 \), we have to distinguish the two cases \( z > 0 \) and \( z < 0 \). But before starting with the actual calculations, let us study the behaviour when \( z \to +\infty \) of a class of integrals.

### 4.5.1. Equivalents, for certain integrals

Let us study, for very large \( z \geq 0 \), the following class of integrals

\[
v_f(\rho,z) = \int_0^\infty J_1(s \rho) e^{-sz} \left( 1 + e^{-2sh} B_\lambda(s) \right)f(s) \, ds,
\]  

[4-105]

where \( f \) is a function such that

\[
g(x) = \begin{cases} f'(0) & \text{if } x = 0 \\ \frac{1}{x} f(x) & \text{if } x > 0 \end{cases} \text{ is continuous and bounded by } M_f \text{ in } \mathbb{R}^+.
\]  

[4-106]

Let us look at \( v_f \) when \( z \) is very large, and such that: \( z \geq \eta \rho, \eta > 0 \). We have the following general properties for the functions \( B_\lambda \) and \( C_\lambda \) defined in \([4-50]\)

\[1 + e^{-2sh} B_\lambda(s) = 2C_\lambda(s) + \left( 1 - e^{-2sh} \right) B_\lambda(s), \quad B_\lambda(\frac{s}{2}) = B_{\lambda z}(s), \quad C_\lambda(\frac{s}{2}) = C_{\lambda z}(s).\]  

[4-107]

The integral \([4-105]\) can now, using the variable change \( s' = sz \) be rewritten as follows
Let us study the various components of this integral, and let us suppose that $\frac{\rho}{z}$ has a limit, $\tan(\omega_\infty)$ when $z \to +\infty$, that is that if we had written

$$\rho = \tilde{\rho} \sin \omega, \quad z = \tilde{\rho} \cos \omega, \quad \tilde{\rho}^2 = \rho^2 + z^2,$$

then we would have $\omega \to \omega_\infty$ as $z \to +\infty$ (or $\tilde{\rho} \to +\infty$). We have, as $z \to +\infty$

$$\left| \frac{z}{\rho} J_1(s \frac{\rho}{z}) \right| \leq \frac{s}{2},$$

$$\frac{z}{\rho} J_1(s \frac{\rho}{z}) \to \Xi(s, \omega_\infty) = \begin{cases} \cotan(\omega_\infty) J_1(s \tan(\omega_\infty)) & \text{if } \omega_\infty > 0 \\ \frac{s}{2} & \text{if } \omega_\infty = 0 \end{cases},$$

$$\left| z \left( 1 - e^{-2s \frac{h}{z}} \right) B_\lambda z(s) \right| \leq 4sh \text{ and } z \left( 1 - e^{-2s \frac{h}{z}} \right) B_\lambda z(s) \to 2sh,$$

$$|z C_\lambda z(s)| \leq \frac{s}{\lambda} \text{ and } z C_\lambda z(s) \to \frac{s}{\lambda},$$

$$\left| z f\left(\frac{s}{z}\right) \right| \leq M_f s \text{ and } z f\left(\frac{s}{z}\right) \to f(0) s,$$

therefore, using the Lebesgue dominated convergence theorem, we have, as $z \to +\infty$,

$$\frac{z^4}{\rho} v_f(\rho, z) \to 2 f(0) \int_0^\infty \Xi(s, \omega_\infty) e^{-s} \left( \frac{1}{\lambda} - h \right) s^2 \, ds.$$  \[4-115\]

Let us apply this formula to equations [4-74] (we are still looking at the case $z \geq 0$) to find an asymptotic equivalent of $v^I$ as $z \to +\infty$.

4.5.2 : Application to the rotating disk

The velocity field produced by the rotating disk is (see equations [4-72..74])
\[ v^I = \frac{2\Omega}{\pi} \int_0^a W(t) \sin(st) \, dt \left( e^{-iz+\h\lambda s} + B_\lambda(s) e^{-z+h\lambda s} \right) J_1(s) \, ds \]

\[ = \frac{2\Omega}{\pi} v_f, \text{ where } f = \int_0^a W(t) \sin(st) \, dt. \]  

We can therefore apply [4-115] to [4-116]. Thus, as \( z \to +\infty \), we have, in the case of the disk,

\[ v^I \sim \frac{\mathbf{L}}{4\pi\mu} \int_0^\infty \Xi(s,\omega_\infty) e^{-s} \left( \frac{1}{\lambda} - h \right) s^2 \, ds \]  

[4-117]

Let us now generalize [4-117] to the case of an arbitrary axisymmetric solid \( S \).

4.5.3 : the general case

The general case is a little more complicated. Indeed, we do not have any more a one dimensional object. Let us suppose that \( z > z' \) for all \( z' \) such that \((p',z') \in C, \) and \( z \geq \eta p \).

[Fig 4-2] : The region \( z \geq z', \ z \geq \eta p \)

Rewriting [4-80], we have

\[ v^I(p,z) = 2\pi \int_0^\infty J_1(s\rho) e^{-sz} \left[ \int_C \left( e^{sz'} + B_\lambda(s) e^{-sz'} \right) J_1(s\rho') \sigma(p',z') \rho' \, dl' \right] ds. \]  

[4-118]

The same way as for [4-113,115] we can prove that, as \( z \to +\infty \), we have
\[ \frac{z^4}{\rho} v^I(\rho,z) \rightarrow 2\pi \int_0^\infty \Xi(s,\omega_\infty) e^{-s} s^2 \left[ \int_\mathcal{C} \left( \frac{1}{\lambda} - z' \right) \sigma(\rho',z') \rho'^2 \, dl' \right] \, ds, \quad [4-119] \]

but, in §4.3, we had assumed that
\[ \int_\mathcal{C} (z-h) \sigma(\rho,z) \rho^2 \, dl = 0, \quad [\text{eqn. 4-41}] \]

therefore, [4-117] is equivalent to
\[
\frac{z^4}{\rho} v^I(\rho,z) \rightarrow 2\pi \int_0^\infty \Xi(s,\omega_\infty) e^{-s} s^2 \left[ \int_\mathcal{C} \left( \frac{1}{\lambda} - h \right) \sigma(\rho',z') \rho'^2 \, dl' \right] \, ds \quad [4-120] \\
\quad \rightarrow 2\pi \int_0^\infty \Xi(s,\omega_\infty) e^{-s} s^2 \left( \frac{1}{\lambda} - h \right) \frac{L}{8\pi^2 \mu} \, ds. \quad [4-121] 
\]

Thus, if we called
\[ F^I(\omega_\infty) \equiv \int_0^\infty \Xi(s,\omega_\infty) e^{-s} \left( \frac{1}{\lambda} - h \right) s^2 \, ds. \quad [4-122] \]

We would have, whatever the solid,
\[ v^I(\rho,z) \sim \frac{\rho}{z^4} \frac{L}{4\pi \mu} F^I(\omega_\infty) \text{ as } z \rightarrow \infty \text{ and } \rho = O(z) \quad [4-123] \]

Using a similar method, for the case \( z < 0 \), we have, for \( z \leq -\eta \rho \) and if we assume that \( \frac{\rho}{z} \) has a limit as \( z \rightarrow -\infty \)
\[ v^H(\rho,z) \sim \frac{\rho}{\lambda z^4} \frac{L}{4\pi \mu} F^H(\omega_\infty) \text{ as } z \rightarrow -\infty \text{ and } \rho = O(-z). \quad [4-124] \]

Where
We just studied the behaviour of the velocity field created by a rotating solid above a porous bed of material. It would be interesting to see whether the approximation of a rotating solid by a point torque, the rotelet, made in the case of solid no-slip surfaces can still be used. This is the aim of the next section.

4.6 : The rotelet problem

4.6.1 : The new problem

Let us suppose that, instead of having a rotating solid, we have a rotelet in \( z = h \). Let us call \( \vec{v}^I \) (resp. \( \vec{v}^{II} \)) the velocity field created by a rotelet of strength \( v \) in Region(I) (resp. Region(II)). The equations describing the motion of the fluid are somewhat different. Indeed, we write \( \vec{v}^I \) as being the sum of the velocity field created by the rotelet \( \vec{v}_0^I(\rho, z) \), and a perturbation field \( \vec{v}^I_\ast \) expressed directly as a Hankel transform. These equations are

\[
\vec{v}^I(\rho, z) = \vec{v}_0^I(\rho, z) + \vec{v}^I_\ast(\rho, z) \quad \text{for} \quad z \geq 0,
\]

where \( \vec{v}_0^I(\rho, z) = \frac{v \rho}{R^3} \), with \( R = \sqrt{\rho^2 + (z-h)^2} \), represents a rotelet of strength \( v \) in the unbounded free space. The boundary conditions [4-3] and [4-4] are still valid. Let us rewrite them as follows

\[
[4-3] \text{gives} : \left. \frac{2}{\rho} \vec{v}_0^I(\rho, 0) \right|_{\rho = 0} + \int_0^\infty M^I(s) J_1(s \rho) \, ds = \int_0^\infty M^{II}(s) J_1(s \rho) \, ds, \quad [4-128]
\]

\[
[4-4] \text{gives} : \frac{2}{\rho} \frac{\partial}{\partial z} \vec{v}_0^I(\rho, 0) - \int_0^\infty s M^I(s) J_1(s \rho) \, ds = \int_0^\infty \sqrt{s^2 + \lambda^2} M^{II}(s) J_1(s \rho) \, ds, \quad [4-129]
\]

4.6.2 : The solutions
Let us use a Hankel transform to extract $M^I$ and $M^{II}$ from $[4-128,129]$; to do this we compute the Hankel transforms of $\tilde{V}_0^I(p,0)$ and $\frac{\partial}{\partial z}(\tilde{V}_0^I)(p,0)$.

We have, by setting $r = R(p,0) = \sqrt{\rho^2 + h^2}$,

\[ \int_0^\infty \rho \tilde{v}_0^I(p,0) J_1(sp) \, d\rho = \int_0^\infty \frac{\rho p^2}{r^3} J_1(sp) \, d\rho = \nu e^{-hs}, \quad [4-130] \]

\[ \int_0^\infty \rho \frac{\partial}{\partial z}(\tilde{V}_0^I)(p,0) J_1(sp) \, d\rho = -3 \int_0^\infty \frac{\rho p^2 (0 - h)}{r^5} J_1(sp) \, d\rho = \nu s e^{-hs}, \quad [4-131] \]

$M^I$ and $M^{II}$ will satisfy the following system:

\[
\begin{cases}
- M^I + M^{II} = \nu e^{-hs} \\
\sqrt{s^2 + \lambda^2} M^I + M^{II} = \nu s e^{-hs}.
\end{cases} \quad [4-132]
\]

The solutions to $[4-132]$ are

\[ M^I = \nu s \frac{s - \sqrt{s^2 + \lambda^2}}{s + \sqrt{s^2 + \lambda^2}} e^{-hs}, \quad [4-133] \]

\[ M^{II} = 2\nu \frac{s^2}{s + \sqrt{s^2 + \lambda^2}} e^{-hs}. \quad [4-134] \]

Let us, again, look at the behaviour of $\tilde{V}^I$ and $\tilde{V}^{II}$ as $z$ tends to infinity.

4.6.3: Asymptotic results

Let us do the same analysis as was done in section 5, for the velocity field produced by the rotelet. The velocity field produced by a rotelet is

\[ \tilde{V}^I = \nu \frac{p}{R^3} + \nu \int_0^\infty s \, B_\lambda(s) \, J_1(s) \, e^{-(z+h)s} \, ds \quad [4-135] \]

\[ = \nu \int_0^\infty s \left( e^{-|z+h|s} + B_\lambda(s) \, e^{-(z+h)s} \right) J_1(s) \, ds \quad [4-136] \]
\( = \nu \nu_T \), where \( f = s \)

It is obvious that by taking \( \nu = \frac{L}{8\pi \mu} \) we have:

\[
\tilde{v}^I \sim v^I(\rho,z) \sim \frac{\rho}{z^4} \frac{L}{4\pi \mu} I^I \text{ as } z \to \infty \text{ and } \rho = O(z). \tag{4-137}
\]

The same way, one sees that

\[
\tilde{v}^{II}(\rho,z) \sim v^{II}(\rho,z) \sim \frac{\rho}{\lambda z^4} \frac{L}{4\pi \mu} I^{II} \text{ as } z \to -\infty \text{ and } \rho = O(-z). \tag{4-138}
\]

The conclusion one can draw, for the time being, is that, as long as one is not interested in knowing the exact distribution of the flow near the solid then one can make the approximation of using a rotelet of the strength \( \frac{L}{8\pi \mu} \). In this case there will be a matching of the asymptotic velocity in the case of a rotelet (formulae [4-137,138]) with the asymptotic formulae [4-124,125]. As we will see, using the same value for the strength of the rotelet will give a matching in the case of the drag.

4.6.4 : Drag (in the case of a rotelet)

We assume that the rotelet is used to model a rotating particle of characteristic dimension \( a \) and turning with an angular velocity \( \Omega \). Let us prove that the asymptotic formula for the torque, as computed by Brenner (Brenner [2]), holds. I.e. that we have

\[
\frac{L}{I_{\infty}} = \frac{1}{1 - \frac{1}{8\pi \mu \Omega a^3} e^3 + O(e^5)} \text{ with } e = \frac{a}{h}, \tag{4-139}
\]

and let us compute \( K = \frac{8\pi \mu h^3 \omega_1}{I_{\infty}} \). To do this we have to compute

\[
\omega_1 = -\frac{1}{2} \text{curl}(\tilde{v}^{I*}(0,h)) e_z = -\frac{1}{2\rho} \frac{\partial}{\partial \rho} (\rho \tilde{v}^{I*})(0,h).
\]

We have:

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\[ \omega_1 = -\frac{v}{2} \left[ \int_0^\infty J_0(s \rho) \frac{s^2 - \sqrt{s^2 + \lambda^2}}{s + \sqrt{s^2 + \lambda^2}} e^{-(h+z)s} \, ds \right] \]

\[ \rho = 0 \quad z = h \]

\[ = -\frac{v}{2} \int_0^\infty s^2 - \sqrt{s^2 + \lambda^2} \frac{e^{-2hs}}{s + \sqrt{s^2 + \lambda^2}} \, ds \]

\[ = -\frac{v}{2h^3} \int_0^\infty t^2 e^{-2t} \frac{A_\lambda(t) - 1}{A_\lambda(t) + 1} \, dt \]

As for section 3, we have separate two cases, either \( \lambda h \gg 1 \) or \( \lambda h = O(1) \). Let us use the value for \( v \) found in 6.3, [4-137] \( v = \frac{L}{8\pi \mu} \), we have for \( \lambda h \gg 1 \)

\[ K = -(A + \frac{2B}{\lambda h})(A, B \text{ defined by [4-29,30]}), \text{ which leads to} \]

\[ \frac{L}{L_\infty} = \frac{1}{1 + \frac{L_\infty}{8\pi \mu \Omega a^3}(A + \frac{2B}{\lambda h}) e^3 + O(e^5)} \quad \text{with } e = \frac{a}{h} \]

\[ \frac{L}{L_\infty} = 1 - \frac{1}{8\pi \mu \Omega h^3} L_\infty (A + \frac{2B}{\lambda h}) + O\left(\frac{a}{h}\right)^3 \]

for \( \lambda h = O(1) \), \( K = -A_1 \) (\( A_1 \text{ defined by [4-44]} \)),

\[ \frac{L}{L_\infty} = \frac{1}{1 + \frac{L_\infty}{8\pi \mu \Omega a^3} A_1 e^3 + O(e^5)} \quad \text{with } e = \frac{a}{h} \]

\[ \frac{L}{L_\infty} = 1 - \frac{1}{8\pi \mu \Omega h^3} L_\infty A_1 + O\left(\frac{a}{h}\right)^3 \]

These results are exactly the same as those found in §3. This means that we have verified, a posteriori, in the case of a rotating solid Brenner's formula. One can sum up the results of this section as follows. An axisymmetric particle of radius a rotating at an angular velocity \( \Omega \), at a height \( h \) above a porous medium of Brinkman constant \( \lambda \), in a fluid of viscosity \( \mu \), can be modeled, for the study of the movement of the fluid far from the solid by a rotelet of strength \( \frac{L}{8\pi \mu} \) where \( L \) is the drag the fluid exerts on the particle. The rotelet model only provides the
general behaviour (i.e. far from the solid) of the fluid and does not give any precision of the movement of the fluid near the solid.

4.7 : Numerical calculations

Three things have been extensively computed, for the rotating disk and the rotelet:

- Contour plots of the velocity, with various values of \( \lambda \), and \( \frac{h}{a} = 1 \).
- Interfacial velocity plots, for various values of \( \lambda \) and \( h \).
- Various functions associated with the drag, for various \( \lambda \) and \( h \).

4.7.1 : Contour plots of the velocity

These contour plots [Plots 4-1..4] have been made with the following values of \( h \) and \( \lambda \):

- \( \frac{h}{a} = 1 \)
- \( \lambda a = 0, 1, 10, 100 \)

The following contours have been chosen:

\[
\frac{V}{\Omega a} = 0.8, 0.6, 0.4, 0.2, 0.1, 0.05, 0.01, 0.005
\]

These plots have been made to show the sensitivity of the velocity to the factor \( \lambda \) (The Brinkman constant). Two things can be deduced: The greater \( \lambda \), the more the fluid gets 'bounced' off the porous medium, and, the greater \( \lambda \) gets the slower the fluid moves into the porous medium. These properties stem from the fact that the larger \( \lambda \) gets, the flatter the contours get for the half space \( z < h \).

Another interesting feature is the shape of the intersection of the contour lines with the segment representing the intersection of the disk with the plane \( \phi = 0 \). It seems that we have:

\[
\lim_{z \to h^+} \frac{\partial V}{\partial z} = - \lim_{z \to h^-} \frac{\partial V}{\partial z}, \text{ or }
\]

\[
\sigma^+ = - \frac{1}{4\pi} \rho \frac{\partial^2 V}{\partial z^2} \rho = - \sigma^- = \frac{1}{4\pi} \rho \frac{\partial^2 V}{\partial z^2} \rho
\] (see [4-48])
Additionally, to give an idea of the scales, a three-dimensional plot of the scaled velocity, in
the case $\frac{h}{a} = \lambda a = 1$, has been made. [Plot 4-5]

4.7.2 : Interfacial velocity plots

two kinds of plots have been made:

- Scaled interfacial velocity as a function of the scaled radius. [Plots 4-6..13]
- Scaled interfacial velocity as a function of $\frac{h}{a}$. [Plots 4-14,15]

To start with a study of the interfacial velocity as a function of the radius has been made. On a
same plot $\frac{h}{a}$ has been fixed, and curves have been plotted for selected values of $\lambda a$. One
noteworthy object is the hump (maximum) that can be observed. It is the image of the
geometrical properties of the configuration. That is the maximum in velocity is the the image of
the edge of the disk. It is interesting to look how that maximum varies with $\lambda$ and $h$. [Plot 4-17]
shows this variation. Two features are to be noted. The first is the bottle-neck near the disk, the
point for the maximum hits a minimum for $\frac{h}{a}$ around 0.4. The second, less visible on the plot is
the tangent of the curve at $h = 0$. This tangent is infinite. The shape of these curves is quite
interesting, since one can see that there is a value of $h$ for which the radius for the maximum is
minimal.

The second set of plots [Plots 4-14,15] is more there to be used a a very rudimentary abacus
to find $\lambda$. Indeed it gives values of the scaled interfacial velocity for $\frac{h}{a} = 1$, for various $\lambda$, as a
function of $h$. As a matter of fact it seems (I am alas not an experimentalist) simple to find out the
interfacial velocity, and one usually knows the value of $h$.

4.7.3 : Drag, and associated functions

The functions plotted, in the case, when applicable, of the disk, are:

- The scaled drag $\frac{L}{\Omega ma^2}$ for the disk, as a function of $h$, for different $\lambda$ [Plot 4-16]

- The drag factor $A_1 = \frac{1}{2} \int_0^\infty s^2 e^{-2s} B_\lambda \left( \frac{s}{h} \right) ds$ as a function of $h$ and $\lambda$ [Plots 4-18,19]

- The relative difference between $A_1$ and $\frac{1}{8} \left( 1 - \frac{3}{\lambda h} \right)$ [Plots 4-20,.22]
The relative error between $\delta_{\text{actual}} = h^3 \left( \frac{L}{L_{\infty}} - 1 \right)$ and $\delta_{\text{approx}} = \frac{4}{3\pi} a^3 \Lambda_1$ [Plot 4-23]

The interesting point in these plots is in [Plots 4-20..23]: One can see the influence of $\lambda a$ and $\frac{h}{a}$ on the matching of the asymptotic formula with the actual values. As a matter of fact one has: the larger $\lambda h$ the better the approximation. Furthermore it can also be seen that $\lambda$ and $h$ are two independent parameters. This has to be contrasted with the case of the rotelet where there is only one parameter, $\lambda h$. Therefore, for the exact behaviour of the velocity field and torque, one has to consider $\lambda$ and $h$ to be separate parameters, but for the general behaviour (i.e. very far from the solid) only $\lambda h$ matters.
Contour Plot of Velocity

For $\lambda a = 100, \frac{b}{a} = 1$
Scaled interfacial velocity for:
- $h/a = 1.0$
- $\lambda a = \{ 0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \ 1.2 \ (4) \ 1.6 \ (5) \ 2.0 \ (6) \}$

Scaled interfacial velocity for:
- $h/a = 2.0$
- $\lambda a = \{ 0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \ 1.2 \ (4) \ 1.6 \ (5) \ 2.0 \ (6) \}$
Scaled interfacial velocity for:
- $\frac{b}{a} = 4.0$
- $\lambda a = \{0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \ 1.2 \ (4) \ 1.6 \ (5) \ 2.0 \ (6)\}$

Scaled interfacial velocity for:
- $\frac{b}{a} = 8.0$
- $\lambda a = \{0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \ 1.2 \ (4) \ 1.6 \ (5) \ 2.0 \ (6)\}$

![Plot 4 - 8](image)

![Plot 4 - 9](image)
Scaled interfacial velocity for:

- $\frac{h}{a} = 1.0$
- $\lambda_a = \{0.0, 0.4, 0.8\}$

Scaled interfacial velocity for:

- $\frac{h}{a} = 0.5$
- $\lambda_a = \{1.2, 0.4, 1.6\}$
Scaled interfacial velocity for:
- $h/a = 0.25$
- $\lambda a = \{ 0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \}
- $\lambda a = \{ 1.2 \ (4) \ 1.6 \ (5) \ 2.0 \ (6) \}$

Scaled interfacial velocity for:
- $h/a = 0.125$
- $\lambda a = \{ 0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \}
- $\lambda a = \{ 1.2 \ (4) \ 1.6 \ (5) \ 2.0 \ (6) \}$
Scaled entrance velocity with \( \frac{p_0}{A} \): 1.0 for:

- \( \lambda_2 = \{0.0, 0.2, 0.4, 0.6\}\) for \( \lambda_2 = 0.0 \)

- \( \lambda_2 = \{0.0, 0.6\}\) for \( \lambda_2 = 0.6 \)

[Plot 4.14] [Plot 4.15]
Scaled drag for:
- \( \lambda_a = \{0.1 \ (1) \ 0.5 \ (2) \ 0.9 \ (3) \}

Scaled radius for maximum interfacial velocity for:
- \( \lambda_a = \{0.0 \ (1) \ 0.4 \ (2) \ 0.8 \ (3) \}

[Plot 4 - 16]

[Plot 4 - 17]
Contour plot of the resistance factor - A.
Surface plot of the resistance factor - A
Relative error between $A_1$ and $\frac{1}{g}$ for:

- $A_1 = \int_0^\infty s^2 e^{-2s} B_{2,1}(\frac{s}{h})\,ds$
- $\lambda_a = \{ 0.1 (1) 1.1 (2) 2.1 (3) 3.0 (4) 4.0 (5) 5.0 (6) \}$

Relative error between $A_1$ and $\frac{1}{g}$ for:

- $A_1 = \int_0^\infty s^2 e^{-2s} B_{2,1}(\frac{s}{h})\,ds$
- $\lambda_a = \{ 0.1 (1) 1.1 (2) 2.1 (3) 3.0 (4) 4.0 (5) 5.0 (6) \}$

[Plot 4 - 20] [Plot 4 - 21]
Relative error between $\delta_{\text{actual}}$ and $\delta_{\text{approx}}$

for:

- $\delta_{\text{actual}} = h^3 \left( \frac{L}{32} \Omega \mu a^3 - 1 \right)$
- $\delta_{\text{approx}} = \frac{4}{3\pi} \pi^3 A_1$
- $\lambda_a = \{ 0.1 (1), 1.1 (2), 2.1 (3), 3.0 (4), 4.0 (5), 5.0 (6) \}$

Relative error between $A_1$ and $\frac{1}{8} \left( 1 - \frac{3}{\lambda h} \right)$

for:

- $A_1 = \int_0^\infty s^2 e^{-2s} B_{\lambda a}(s) ds$

(Plot 4 - 23)
Chapter Five

Axisymmetric stokeslet above a porous medium

Summary

In this chapter a study of the stokeslet above a porous medium will be made, both in the Darcy and the Brinkman case. In order to do the second a new representation for the velocity and pressure fields will be derived. In both cases a study will be conducted in the limiting cases of the porous medium being either free or solid.

5.1 : Introduction

This chapter is devoted to the study of a slow steady axisymmetric creeping flow in an incompressible Newtonian fluid created by a point force above a porous bed of material of infinite thickness. Since the flow is created by a point force it is very difficult to compute a Reynolds number. Indeed point forces, that is elementary solutions of the Stokes equation, are singular at a point in space, their origin. However as we use a stokeslet of strength \( V \) to model a particle of radius \( a \) moving with a velocity \( u \) along its axis of symmetry, then the Reynolds number associated with the problem is

\[
Re = \frac{4 d V}{3 \mu}
\]

where \( d \) is the density of the fluid and \( \mu \) its viscosity. the constant \( \frac{4}{3} \) placed in front is there to provide a matching, in the case of a sphere moving in a free medium between the number computed above and the classical definition of the Reynolds number (\( du/a \mu \)). Since the Reynolds number gives only a qualitative measurement of the complexity of the flow (i.e. presence or not of vortices etc.) the constant \( \frac{4}{3} \) can be discarded. As we are working with slow creeping viscous flows we have to take \( Re \) very small (that is of the order of 1).

This chapter, as already said, is divided in two parts, in which the study will use respectively the Darcy and Brinkman models to model the flow in the porous bed. The method used in each case is roughly the same. It is as follows:
• Defining the boundary conditions.
• Representing the velocity and pressure fields.
• Transforming the representations using Hankel transforms.
• Solving the linear system obtained.
• Compute and evaluate the limit cases.
• Assuming Brenner's formula, find the drag.

In order to do this for the Brinkman model one has, first to devise a new representation for the velocity and pressure fields consistent with the Brinkman model. But first let us work on the Darcy model.

5.2 : The Darcy case

5.2.1 : The initial configuration - the boundary conditions

We have the following configuration : in a fluid filled space we have two regions, and a slowly falling solid modeled by a stokeslet of strength \( v \) placed in \( z = h \). Let us call \( u^I, p^I \) (resp \( u^{II}, p^{II} \)) the velocity and pressure fields in region (I) (resp. region (II)). The two regions (I) and (II) are defined as follows :

• Region I : Defined as the half-space \( z \geq 0 \) is a "free" medium. The equation of motion for the fluid is Stokes's equation: \( \mu \Delta u^I = \nabla p^I \), where \( \mu \) is the viscosity of the fluid.
• Region II : Defined as the half-space \( z \leq 0 \) is a porous medium. The equation of motion for the fluid is Darcy's law : \( -\kappa \kappa u^{II} = \nabla p^{II} \), where \( k \) is the Darcy constant of the porous medium.

In the two media we still have, of course, conservation of mass, i.e. : \( \text{div}(u^i) = 0, i = I, II \)
The problem is to find the flow generated by that particle, and the resistance opposed to it by the fluid. As we are supposing the problem to be axisymmetric, we will work in a cylindrical coordinate system : \((\rho, \phi, z)\). The velocities are written as : \(u^i = (u^i_1, 0, w^i)\), \(i = I, II\). The second, \(\phi\) component of the velocity is zero since we are working with an axisymmetric flow.

Let us use the boundary conditions described in Chap. 3 §3.2.4.1 eqn. [3-21]. On the boundary \(z = 0\) we have the following conditions :

- Continuity of the pressure : \(p^I = p^{II}\) on \(z = 0\) \[5-1\]
- Continuity of the normal velocity : \(w^I = w^{II}\) on \(z = 0\) \[5-2\]
- Plus the following boundary condition : 
  \[
  \frac{\partial u^I_1}{\partial z} = \alpha \left( u^I_1 - \sigma u^{II}_1 \right) \]
  \[5-3\]
where \(\alpha, \sigma\) are two constants (\(\sigma\) can take the values 0 or 1, and \(\alpha\) any value between 0 and \(\infty\)).

Before going further let us rewrite the velocity and pressure fields as functions of axially harmonic functions.

5.2.2 : The representation

Let us write the following representation (Shail [19], Shail, Packham [25,26] ...) for \(u^I, p^I\) :

\[
  u^I = u_{\infty} + z \text{ grad } \psi^I - \psi^I e_z + z \text{ grad } \left( \frac{\partial \chi^I}{\partial z} \right) - \frac{\partial \chi^I}{\partial z} e_z + \text{ grad } \chi^I
\]
\[5-4\]
\[
  p^I = p_{\infty} + 2\mu \left( \frac{\partial \psi^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} \right)
\]
\[5-5\]
where \( u_\infty, p_\infty \) are the velocity and pressure fields created by a stokeslet of strength \( v \) in an unbounded fluid, in a free medium, and \( \psi^I, \chi^I \) are two axially harmonic functions, i.e. such that:

\[
\Delta \psi^I = \Delta \chi^I = 0, \quad \Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2}
\]

For \( u^\Pi, p^\Pi \), let us apply the mass conservation law to Darcy's law. We obtain \( \Delta p^\Pi = 0 \). Therefore \( u^\Pi \) derives from a potential proportional to \( p^\Pi \). Thus we can write:

\[
p^\Pi = 2\mu \psi^\Pi \text{ with } \Delta \psi^\Pi = 0 \tag{5-7}
\]
\[
u^\Pi = -\frac{2}{k} \text{ grad } \psi^\Pi = -\frac{2}{\lambda^2} \text{ grad } \psi^\Pi = -\beta \text{ grad } \psi^\Pi, \beta = \frac{2}{\lambda^2}. \tag{5-8}
\]

Let us rewrite [5-10.3] knowing that:

\[
u = v \left( \frac{\rho (z-h)}{R^3} e_\rho + \left( \frac{1}{R^3} + \frac{(z-h)^2}{R^3} \right) e_z \right) \text{ and } p_\infty = 2 \mu v \frac{z-h}{R^3}; R = \sqrt{\rho^2 + (z-h)^2} \tag{5-9}
\]

We have, on \( z = 0 \):

\[
[5-1] \Rightarrow -v \frac{h}{R^3} + \frac{\partial \psi^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} = \psi^\Pi, \text{ where } r = \sqrt{\rho^2 + h^2} \tag{5-10}
\]
\[
[5-2] \Rightarrow v \left( \frac{1}{r^3} + \frac{h^2}{r^3} \right) = \psi^I - \beta \psi^\Pi \tag{5-11}
\]
\[
[5-3] \Rightarrow v \frac{\partial}{\partial \rho} \left( \frac{1}{r^3} + \frac{h^2}{r^3} \right) + \frac{\partial}{\partial \rho} \left( \psi^I + 2 \frac{\partial \chi^I}{\partial z} \right) = \alpha \frac{\partial}{\partial \rho} \left( v \frac{h}{r} + \chi^I + \sigma \beta \psi^\Pi \right) \tag{5-12}
\]

As \( \psi^I, \chi^I, \psi^\Pi \) vanish at infinity, we have:

\[
[5-12] \Leftrightarrow v \left( \frac{1 + \alpha h}{r} + \frac{h^2}{r^3} \right) + \psi^I + 2 \frac{\partial \chi^I}{\partial z} = \alpha \left( \chi^I + \sigma \beta \psi^\Pi \right) \tag{5-13}
\]

Equations [5-10,11,13] are the equations coupling the flow in region (I) and the flow in region (II). Let us transform these equations by applying Hankel transforms of order 0. The Hankel transforms of order 0 actually decompose axially harmonic functions by separating the coordinates \( \rho \) and \( z \).
5.2.3 : Solving the equation

Let us rewrite \( \psi^I, \chi^I, \psi^{II} \) in the form of Hankel transforms of order 0:

\[
\psi^I = \int_0^\infty A^I(s) J_0(sp) e^{-sz} \, ds \quad [5-14]
\]
\[
\chi^I = \int_0^\infty B^I(s) J_0(sp) e^{-sz} \, ds \quad [5-15]
\]
\[
\psi^{II} = \int_0^\infty A^{II}(s) J_0(sp) e^{sz} \, ds \quad [5-16]
\]

and let us use the following formulae:

\[
\int_0^\infty \frac{P}{r} J_0(sp) \, dp = \frac{1}{s} e^{-sh} \quad [5-17]
\]
\[
\int_0^\infty \frac{P}{r^3} J_0(sp) \, dp = \frac{1}{\pi} e^{-sh} \quad [5-18]
\]

Equations [5-10,11,13], using [5-14..18] rewrite now as:

\[ [5-10] \Rightarrow -v e^{-sh} - A^I(s) + s B^I(s) = \frac{1}{s} A^{II}(s) \quad [5-19] \]
\[ [5-11] \Rightarrow v \left( \frac{1}{s} + h \right) e^{-sh} = \frac{1}{s} A^I(s) - \beta A^{II}(s) \quad [5-20] \]
\[ [5-13] \Rightarrow v \left( -\frac{1 + \alpha h}{s} + h \right) e^{-sh} = -\frac{1}{s} A^I(s) + \left( 2 + \frac{\alpha}{s} \right) B^I(s) + \frac{\lambda \sigma \beta}{s} A^{II}(s) \quad [5-21] \]

Equations [5-19..21] are equivalent to the system:

\[
\begin{pmatrix}
-s & s^2 & -1 \\
1 & 0 & -\beta s \\
-1 & 2s + \alpha & \alpha \beta \sigma
\end{pmatrix}
\begin{pmatrix}
A^I \\
B^I \\
A^{II}
\end{pmatrix}
= v
\begin{pmatrix}
\frac{s}{h(s-\alpha)-1} \quad 1 + sh
\end{pmatrix} e^{-sh} \quad [5-22]
\]

with solutions
\[ A^I(s) = v \frac{hB_s^4 + \beta(h\alpha(\sigma - 1) - 3)s^3}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} e^{-sh} \]
\[ + \frac{(\beta(h\alpha(\sigma - 1) - 2h)s^2 + (2h\alpha + 2)s + \alpha)}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} e^{-sh} \]
\[ = v A^I(s) e^{-sh} \] \[ [5-23] \]

\[ B^I(s) = v \frac{hB_s^3 + \beta(h\alpha(\sigma - 1) - 2)s^2 + 2(\beta \alpha + h)s - h\alpha}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} e^{-sh} = v B^I(s) e^{-sh} \] \[ [5-24] \]

\[ A^II(s) = -v \frac{2(h\alpha + 2)s^2 + 2\alpha s}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} e^{-sh} = v A^{II}(s) e^{-sh} \] \[ [5-25] \]

Let us now rewrite the velocities, as functions of \( A^I, B^I, A^{II} \). If we write \( u^I = u_{\infty} + u^I^* \), we have:

\[ u^I^* = -v \int_0^\infty s (zA^I(s) + (1 - zs)B^I(s)) J_1(sp) e^{-s(h + z)} ds = \]
\[ -v \int_0^\infty s \left( \frac{\beta(h - z)s^3 + \beta(\alpha(h - z) - h - z - 2)s^2}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} + \frac{2(hz\alpha + h + z + \beta \alpha \sigma)s + \alpha(z - h)}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} \right) J_1(sp) e^{-s(h + z)} ds \] \[ [5-26] \]

\[ w^I^* = v \int_0^\infty (1 + sz)A^I(s) + zsB^I(s)) J_0(sp) e^{-s(h + z)} ds = \]
\[ -v \int_0^\infty \left( \frac{\beta(h - z)s^4 + \beta(\alpha(h - z) - h - z - 3)s^3 + (2hz\alpha + 2h + 2z + \beta \alpha \sigma - \beta \alpha)s^2}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} + \frac{\alpha(h + z) + 2)s + \alpha}{\beta s^3 + \beta \alpha(\sigma + 1)s^2 + 2s + \alpha} \right) J_0(sp) e^{-s(h + z)} ds \] \[ [5-27] \]

The same way, for \( u^{II} \), we have:
\[ u_\| = \beta v \int_0^\infty s \, A_{II}^\gamma(s) J_1(s \rho) \, e^{s(z - h)} \, ds \]

\[ = -2\beta v \int_0^\infty s^2 \frac{(h \alpha + 2)s + \alpha}{\beta s^3 + \beta \alpha (\sigma + 1)s^2 + 2s + \alpha} J_1(s \rho) \, e^{s(z - h)} \, ds \quad [5-28] \]

\[ w_\| = -\beta v \int_0^\infty s \, A_{II}^\gamma(s) J_0(s \rho) \, e^{s(z - h)} \, ds \]

\[ = 2\beta v \int_0^\infty s^2 \frac{(h \alpha + 2)s + \alpha}{\beta s^3 + \beta \alpha (\sigma + 1)s^2 + 2s + \alpha} J_0(s \rho) \, e^{s(z - h)} \, ds \quad [5-29] \]

Now that we have expressions for \( u^I \) and \( u^\| \), let us deal with the limit cases, that is when \( \beta \) and \( \alpha \) take extreme values (i.e. 0 or \( \infty \)).

### 5.2.4: Limiting cases

Let us look at the solutions [5-26..29] in the limit cases, that is for \( \beta = 0, \alpha = \infty \) (solid wall) and \( \beta = \infty \) (an possible approximation of the free medium). To begin with let us tackle the case of the solid wall. If we substitute 0 for \( \beta \) in [5-26..29] we have:

\[ u^I = -v \int_0^\infty s \frac{2(h \alpha z + h + z)s + \alpha(z - h)}{2s + \alpha} J_1(s \rho) \, e^{-s(h + z)} \, ds \quad [5-30] \]

\[ w^I = -v \int_0^\infty s \frac{2(h z \alpha + h + z)s^2 + (\alpha(z + h) + 2)s + \alpha}{2s + \alpha} J_0(s \rho) \, e^{-s(h + z)} \, ds \quad [5-31] \]

\[ u^\| = w^\| = 0 \quad [5-31,32] \]

The solid wall configuration forces us to take \( \alpha = \infty \). Indeed \( \frac{\partial v^I}{\partial z} \) can take any value and, in the case of the solid wall, both \( u^I \) and \( u^\| \) are equal to zero for \( z = 0 \). We have, by defining \( R_2 = \sqrt{\rho^2 + (z + h)^2} \), and by taking the limit of [5-30,31] as \( \alpha \to \infty \),

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\[ u^{*} = -v \int_{0}^{\infty} (2hzs + z - h)s J_{1}(sp) e^{-s(h + z)} ds = -v \left( \frac{\rho(z - h)}{R_{2}^{3}} + \frac{6hz(h + z)\rho}{R_{2}^{5}} \right) \] [5-33]

\[ w^{*} = -v \int_{0}^{\infty} \left( 2hzs^2 + (z + h)s + 1 \right) J_{0}(sp) e^{-s(h + z)} ds \]
\[ = -v \left( \frac{1}{R_{2}^{2}} + \frac{(z + h)^2}{R_{2}^{5}} - \frac{2hz}{R_{2}^{5}}(p^2 - 2(z + h)^2) \right) \] [5-34]

Thus \( u_{\infty} + u^{*} = 0 \) on \( z = 0 \) as wanted. Let us have a look at \( p^{II} = 2\mu \psi^{II} \):

\[ p^{II} = -4\nu \int_{0}^{\infty} (zs + 1)s e^{-s(h - z)} J_{0}(sp) ds = -4\nu \left( -\frac{z}{R^{3}} + \frac{3h(z - h)^2}{R^{5}} \right) \] [5-35]

Since the boundary \( z = 0 \) is a solid boundary, an integral of \( p^{I} = p^{II} \) over the interface \( z = 0 \) gives the force the fluid exerts on the surface. Its value \( F_{S} \) is

\[ F_{S} = \int_{z = 0} p^{I} dS = -8\pi\mu \nu \] (this works because \( u^{I} = u^{II} = 0 \) on \( z = 0 \)).

which is not surprising, since, if one calls \( F \) the force the fluid exerts on the moving solid, we must have \( F + F_{S} = 0 \), and \( F \) is modeled as being \( F = 8\pi\mu \nu \) (Newton's third law of dynamics).

Let us now have a look at the approximative "free-medium" conditions: Let us substitute \( \beta = \infty \) in [5-26..29]

\[ u^{*} = -v \int_{0}^{\infty} \frac{(h - z)s^2 + \left( \alpha(\sigma(h - z) - h - z) - 2 \right)s + 2\alpha \sigma}{s + \alpha(\sigma + 1)} J_{1}(sp) e^{-s(h + z)} ds \] [5-36]

\[ w^{*} = -v \int_{0}^{\infty} \frac{(h - z)s^2 + \left( \alpha(\sigma(h - z) - h - z) - 3 \right)s + \alpha(\sigma - 1)}{s + \alpha(\sigma + 1)} J_{0}(sp) e^{-s(h + z)} ds \] [5-37]
\[ u^I = -2v \int_0^\infty s^2 \left( \frac{h \alpha + 2}{s + \alpha(\sigma + 1)} \right) J_1(sp) e^{-s(h + z)} \, ds \]  
[5-38]

\[ w^I = 2v \int_0^\infty s^2 \left( \frac{h \alpha + 2}{s + \alpha(\sigma + 1)} \right) J_0(sp) e^{-s(h + z)} \, ds \]  
[5-39]

One sees immediately that there is no way, as in the case of a actual free medium, of ensuring that \( v^I = w^I = 0 \). This conclusion is fairly obvious since the equation for the flow in the porous medium is not the Stokes equation. Let us now examine the resistance the fluid opposes to the movement of the particle.

5.2.5: The drag

We have seen in chapter 4 that Brenner's asymptotic formula for the drag (Brenner [2]) still works in the case of an axially symmetric solid rotating above a porous medium. Although we did not prove it let us assume that Brenner's formula is valid for the case of a solid translating above a porous medium. The aim of this section is, therefore more to compute the coefficients of the Brenner formula rather than actually computing the drag. The asymptotic formula, for large \( h \), for the drag on the particle, according to Brenner is:

\[
\frac{F_{\infty}}{F} = 1 - \kappa \frac{F_{\infty}}{8\pi\mu U} + O\left(\left(\frac{a}{h}\right)^3\right) \]  
[5-40]

where \( F \) is the drag in the \( z \) direction, \( F_{\infty} \) the drag if there was no wall, \( U \) the velocity of the particle, and \( a \) the characteristic dimension of the particle. If we define the strength of the stokeslet as being \( v = \frac{F}{8\pi\mu} \), we have:

\[
\kappa = \frac{8\pi\mu h}{F} u^I(0,h) e_x = \frac{8\pi\mu h}{F} w^I(0,h) \\
= -h \int_0^\infty \left( \frac{2(\alpha h + 3)s^3 + (2h^2\alpha + 4h + \beta\alpha\sigma - \beta\alpha)s^2}{\beta s^3 + \beta\alpha(\sigma + 1)s^2 + 2s + \alpha} + \frac{2(\alpha h + 1)s + \alpha}{\beta s^3 + \beta\alpha(\sigma + 1)s^2 + 2s + \alpha} \right) e^{-2sh} \, ds \]  
[5-41]
\[
\begin{aligned}
&=-\int_0^\infty \left(-\beta(2\alpha h+3)t^3 + h(2h^2\alpha + 4h + \beta \alpha \sigma - \beta \alpha)t^2 + \beta t^3 + \beta \alpha h(\sigma + 1)t^2 + 2h^2 + \alpha h^3 \right) e^{-2t} dt
\end{aligned}
\] [5-42]

Two cases are now at hand (we still suppose \( h \gg 1 \)), either \( \alpha h = O(1) \), and in that case, the safest way to compute \( \kappa \) is to use [5-42] directly, or \( \alpha h \gg 1 \), and in that case we actually have:

\[
\kappa = -\left( \frac{3}{2} - \frac{3}{2\alpha h} + O\left(\left(\frac{1}{\alpha h}\right)^3\right)\right) \] [5-43]

in the case \( \beta = 0, \alpha = \infty \) (solid wall), we have (cf. [5-42]):

\[
\kappa = -\int_0^\infty (2t^2 + 2t + 1) e^{-2t} dt = -\frac{3}{2} \] [5-44]

which is what we expected (cf Brenner [2]).

Let us turn now to the Brinkman case. Before even attempting to solve the problem of the particle falling above a porous medium, with the Brinkman model modelling the flow in the porous medium, we have to find a representation of the solutions to Brinkman's equation. This is the aim of the next section.

5.3 : Representing solutions of the Brinkman equation

Let us look at the solutions of the following partial differential equations:

\[
\begin{aligned}
\Delta u - \lambda^2 u &= \text{grad} \ p \\
\text{div} \ u &= 0
\end{aligned}
\] [5-45]

i.e. a non dimensional form of the Brinkman equation, where \( \lambda \) is the Brinkman constant, and the equation of conservation of mass. The solutions \( u, p \) are a three dimensional field, and a scalar field that vanish at infinity. Let us first define some symbols. Let us call:
Lemma [Th5-1]: Let $u$ be a function of $C^\infty$, then the function $g$ defined as

$$g(t) = e^{\lambda t} \int_{\infty}^{t} e^{-2\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda x} u(x) \, dx \, d\tau,$$

has the following properties:

(a): $g$ is a solution of: $y'' - \lambda^2 y = u$
(b): $g \in C^\infty$

Proof:

Let us prove the the lemma sequentially. It is obvious, to start with that $g$ is properly defined, and in $C^\infty$. Let us differentiate, using the standard rules:

$$g' = \lambda g - e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx$$

$$g'' = \lambda^2 g - \lambda e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx + \lambda e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx - u(t) = \lambda^2 g - u(t) \quad \text{Q.E.D.}$$

For part (b) we will have to make the distinction between $+\infty$ and $-\infty$. Let us actually prove the following:

$$e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \to 0 \text{ as } x \to \pm\infty$$

Let $M$ be such that for all $x$, $|u(x)| \leq M$. Let us now choose $\epsilon > 0$; since $u \to 0$ as $x \to +\infty$, there exists $x_0$ such that: $x > x_0 \Rightarrow |u(x)| \leq \epsilon$, thus for $t \geq x_0$, we have:

$$\left| e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \right| \leq e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} |u(x)| \, dx \leq e^{\lambda t} \left[ \int_{-\infty}^{x_0} e^{\lambda x} |u(x)| \, dx + \int_{x_0}^{t} e^{\lambda x} |u(x)| \, dx \right]$$

$$\leq e^{\lambda t} \left( e^{\lambda x_0} M + (e^{\lambda t} - e^{\lambda x_0}) \epsilon \right) = e^{\lambda t} \left( N + e^{\lambda t} \epsilon \right) = N e^{\lambda t} + \epsilon, \quad N = e^{\lambda x_0} (M - \epsilon)$$
It is now clear that there exists \( t_0 \geq x_0 \) such that for all \( t \geq t_0 \), we have:

\[
\left| \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \right| \leq 2e \text{ which proves that }: e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \to 0 \text{ as } x \to +\infty
\]

The same way, for all \( \varepsilon > 0 \), there exists an \( x_0 \), such that for all \( x \leq x_0 \), \( |u(x)| \leq \varepsilon \). Therefore, for \( t \leq x_0 \), we have:

\[
\left| \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \right| \leq e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda x} |u(x)| \, dx \leq e^{-\lambda t} \int_{-\infty}^{t} \varepsilon \, dx \leq \varepsilon. \text{ This simply means :}
\]

\[
e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \to 0 \text{ as } x \to -\infty
\]

We now have proved that, for all \( u \) in \( C_0^\infty \), \( h_u(t) = e^{\lambda t} \int_{-\infty}^{t} e^{\lambda x} u(x) \, dx \in C_0^\infty \). Let us look at \( h_u(-t) \):

\[
h_u(-t) = e^{\lambda t} \int_{-\infty}^{-t} e^{\lambda x} u(x) \, dx = e^{\lambda t} \int_{t}^{\infty} e^{-\lambda y} u(-y) \, dy = e^{\lambda t} \int_{t}^{\infty} e^{-\lambda y} \bar{u}(y) \, dy = \bar{h}_u(t)
\]

which proves that : \( \bar{h}_u(t) = \bar{h}_u(-t) \in C_0^\infty \), for all \( u \) in \( C_0^\infty \). That is : \( \bar{h}_u = \bar{h}_u \) vanishes at infinity.

As \( g = \bar{h}_u \) with \( v = h_u \), we have proved that \( g \) vanishes at infinity, that is we have proved (b). Actually if one looks further, then it is easy to prove that \( g \) is the only solution of \( y'' - \lambda^2 y = u \) in \( C_0^\infty \).

Let us now proceed to actually construct a representation for a solution of Brinkman's equation. This is the aim of the following theorem which will coin a representation for a limited case of the Brinkman equation, the case where \( p \) is a constant.

**Theorem [Th5-2] :** Let \( u \) be a solution of:

\[
\left\{ \begin{array}{l}
\Delta u - \lambda^2 u = 0 \\
\text{div } u = 0
\end{array} \right. \text{ which is in } C_0^\infty (\mathbb{R}^3)^3.
\]

Then there exist two function \( \chi \) in \( C_0^\infty (\mathbb{R}^3) \), \( \theta \) in \( C^\infty (\mathbb{R}^3) \), such that:

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\[ u = \text{grad} \left( \frac{\partial \chi}{\partial z} \right) - \lambda^2 \chi \mathbf{e}_z + \text{curl} (\theta \mathbf{e}_z) \], and \[ \Delta \chi - \lambda^2 \chi = \Delta \theta - \lambda^2 \theta = 0 \]

**Proof:**

Let us write \( u \) the following way: \( u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \). Using Lemma [Th5-1], let \( \chi(z) \) be a solution in \( C^\infty_0(z) \) of:

\[ \frac{\partial^2 \chi}{\partial z^2} - \lambda^2 \chi = u_z \]

Saying that \( \Delta u - \lambda^2 u = 0 \), implies that \( \Delta u_z - \lambda^2 u_z = 0 \), which in turn implies that:

\[ \left[ \frac{\partial^2}{\partial z^2} - \lambda^2 \right] (\Delta \chi - \lambda^2 \chi) = 0 \]

But if \( \chi \in C^\infty_0(\mathbb{R}^3) \), then \( \Delta \chi - \lambda^2 \chi \in C^\infty_0(\mathbb{R}^3) \), the same holds also when one isolates \( z \). But the only solution, in \( C^\infty_0(z) \) of \( y'' - \lambda^2 y = 0 \) is 0. Therefore we must have \( \Delta \chi - \lambda^2 \chi = 0 \). Let us now define \( \alpha \) and \( \beta \) by writing the following:

\[ u_x = \frac{\partial^2 \chi}{\partial x \partial z} + \alpha, \quad u_y = \frac{\partial^2 \chi}{\partial y \partial z} + \beta. \]

It is clear that \( \alpha \) and \( \beta \) are both functions of \( C^\infty_0(\mathbb{R}^3) \). The conservation of the mass, \( \text{div} \ u = 0 \) implies that:

\[ \frac{\partial^3 \chi}{\partial x^2 \partial z} + \frac{\partial \alpha}{\partial x} + \frac{\partial^3 \chi}{\partial x^2 \partial z} + \frac{\partial \beta}{\partial y} + \frac{\partial^3 \chi}{\partial y \partial z} - \lambda^2 \frac{\partial \chi}{\partial z} = 0 \iff \frac{\partial}{\partial z} \left( \Delta \chi - \lambda^2 \chi \right) + \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} = 0 \]

\[ \iff \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} = 0 \iff \frac{\partial \alpha}{\partial x} = -\frac{\partial \beta}{\partial y}. \]

This implies that there exists a function \( \theta \) in \( C^\infty(\mathbb{R}^3) \) and a function of only \( x \) (resp \( y \)), \( k_x \) (resp \( k_y \)) such that:
\[ \alpha = \frac{\partial}{\partial y} + k_y(y), \quad \beta = -\frac{\partial}{\partial x} + k_x(x). \]
But as \( \beta \) is in \( C^\infty(R^2) \), we must have \( \beta \to 0 \) as \( (x,y) \to \infty \). This implies \( k_x = 0 \). Indeed if we have \( y \to \infty \) we have \( \beta \to 0 \) and \( \frac{\partial\theta}{\partial x} \to 0 \), but \( k_x \) is independent of \( y \), therefore we must have \( k_x = 0 \) for all \( x \). A similar argument is used for \( \alpha \) and \( k_y \) (i.e. \( k_y = 0 \)). That permits us to rewrite \[
\begin{pmatrix}
\alpha \\
\beta \\
0
\end{pmatrix} = \text{curl}(\theta e_z).
\]

We also have, from \( \Delta u_x - \lambda^2 u_x = \Delta u_y - \lambda^2 u_y = 0 \) that \( \Delta \alpha - \lambda^2 \alpha = \Delta \beta - \lambda^2 \beta = 0 \), which implies that \( \Delta \theta - \lambda^2 \theta = 0 \). Q.E.D.

Let us now turn to a representation of the complete version of the non dimensional Brinkman equation

**Theorem [Th5-3] (Corollary to Theorem [Th5-2])**: Let \((u,p)\) be a solution of Brinkman's equation:

\[
\begin{cases}
\Delta u - \lambda^2 u = \text{grad} \ p \\
\text{div} \ u = 0
\end{cases}
\]

Then there exist three functions \( \varphi \) in \( C^\infty(R^3) \), \( \chi \) in \( C^\infty(R^3) \), \( \theta \) in \( C^\infty(R^3) \), such that:

\[
\begin{cases}
u = \text{grad} \ \varphi + \text{grad} \left( \frac{\partial\chi}{\partial z} \right) - \lambda^2 \chi \ e_z + \text{curl} \left( \theta \ e_z \right), \\
p = - \lambda^2 \ \varphi
\end{cases}
\]

**Proof**:

Let us compute \( \text{div}(\Delta u - \lambda^2 u) = \Delta(\text{div} \ u) - \lambda^2 \ \text{div}(u) = 0 \). This proves that \( \text{div}(\text{grad} \ p) = \Delta p = 0 \).

Let us now take \( \varphi \) such that \( p = - \lambda^2 \ \varphi \), we have \( \varphi \in C^\infty(R^3) \), and \( \Delta \varphi = 0 \). Let us rewrite \( u \) as

\[ u = \bar{u} + \text{grad} \ \varphi. \]

We will have:

\[ \Delta u - \lambda^2 u = \text{grad} \ p \iff \Delta \bar{u} - \lambda^2 \bar{u} + \Delta(\text{grad} \ \varphi) - \lambda^2 \text{grad} \ \varphi = - \lambda^2 \ \text{grad} \ \varphi \iff \Delta \bar{u} - \lambda^2 \bar{u} = \theta \]
Therefore, by applying theorem [Th5-2], we have $\chi, \theta$ such that:

$$\bar{u} = \text{grad} \left( \frac{\partial \chi}{\partial z} \right) - \lambda^2 \chi e_z + \text{curl} (\theta e_z).$$

That is:

$$u = \text{grad} \varphi + \text{grad} \left( \frac{\partial \chi}{\partial z} \right) - \lambda^2 \chi e_z + \text{curl} (\theta e_z),$$

and $p = -\lambda^2 \varphi$. Q.E.D.

We now have a way to represent solutions to the Brinkman equation as functions of a harmonic function and two solutions of $(\Lambda - \lambda^2)f = 0$. Let us apply that knowledge to the second part of this chapter, the study of the flow when one models the motion in the porous medium using the Brinkman equation.

### 5.4: The Brinkman case

Let us use the same configuration as for the Darcy case. What will change are the various equations associated with region II, and boundary conditions. Let us define these new equations.

#### 5.4.1: The new configuration and boundary conditions

We have the following configuration: in a fluid filled space we have two regions, and a slowly falling solid modeled by a stokeslet of force $v$ placed in $z = h$. The two regions, region I and region II are defined as follows:

- **Region I**: Defined as the half-space $z \geq 0$, is a "free" medium. The equation of motion for the fluid is Stokes's equation: $\mu \Delta u^I = \text{grad} p^I$, where $\mu$ is the viscosity of the fluid.
- **Region II**: Defined as the half-space $z \leq 0$, is a porous medium. The equation of motion for the fluid is Brinkman's equation: $\mu \Delta u^{II} - \mu k u^{II} = \text{grad} p^{II}$, where $k = \lambda^2$ is the Brinkman constant of the porous medium.

In both regions we still have, of course, conservation of mass, that is: $\text{div}(u) = 0$.

The new boundary conditions write as (see chap 3 §3.2.3 eqn. [3-15,17])

- Continuity of the velocity: $u^I = u^{II}$ on $z = 0$.
- Continuity of the stress exerted on the surface $z = 0$: $T^I e_z = T^{II} e_z$ on $z = 0$

We can write $u^i = (u^i, 0, w^i)$, $i = I, II$. The above mentioned conditions will rewrite as:
\[ u^I = u^{II} \text{ on } z = 0 \] [5-46]
\[ w^I = w^{II} \text{ on } z = 0 \] [5-47]
\[ \frac{\partial u^I}{\partial z} + \frac{\partial w^I}{\partial \rho} = \frac{\partial u^{II}}{\partial z} + \frac{\partial w^{II}}{\partial \rho} \text{ on } z = 0 \] [5-48]
\[ - p^I + 2\mu \frac{\partial w^I}{\partial z} = - p^{II} + 2\mu \frac{\partial w^{II}}{\partial z} \text{ on } z = 0. \] [5-49]

Let us reduce these to simpler expressions. The continuity of \( w \) across the boundary \( z = 0 \) implies the continuity of \( \frac{\partial w}{\partial \rho} \) i.e.

\[ \frac{\partial w^I}{\partial \rho} = \frac{\partial w^{II}}{\partial \rho} \text{ on } z = 0, \] [5-50]

therefore [5-48] becomes

\[ \frac{\partial u^I}{\partial z} = \frac{\partial u^{II}}{\partial z} \text{ on } z = 0. \] [5-51]

The same way, using the mass conservation equation we have:

\[ \text{div}(u^I) = \text{div}(u^{II}) = 0 \text{ on } z = 0. \] [5-52]

As we have, for the same reasons as above, continuity of \( \frac{\partial u}{\partial \rho} \) across the interface and

\[ \text{div}(u^i) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho u^i) + \frac{\partial w^i}{\partial z} = 0 \text{ i = I,II.} \]

Therefore

\[ \frac{\partial w^I}{\partial z} = \frac{\partial w^{II}}{\partial z} \text{ on } z = 0 \text{ (continuity of } \frac{\partial w}{\partial z}). \] [5-53]

Equation [5-49] now becomes

\[ p^I = p^{II} \text{ on } z = 0. \] [5-54]

Instead of using the boundary conditions described in [5-48,49] let us use those computed in [5-51,54]. Let us now use the knowledge gained in §5.3 to represent \( u^I \) and \( u^{II} \).
5.4.2: The representation

Let us represent \( u^I \) and \( p^I \) using the classical method. The representations for \( u^I \) and \( p^I \) are as follows:

\[
\begin{align*}
    u^I &= u_\infty + z \, \text{grad} \, \psi^I - \psi^I \, e_z + z \, \text{grad} \left( \frac{\partial \chi^I}{\partial z} \right) - \frac{\partial \chi^I}{\partial z} \, e_z + \text{grad} \, \chi^I \quad [5-55] \\
p^I &= p_\infty + 2\mu \left( \frac{\partial \psi^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} \right) \quad [5-56]
\end{align*}
\]

where \( \Delta \chi^I = \Delta \psi^I = 0 \) (in this case \( \Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \)) and \( u_\infty, p_\infty \) are the velocity and pressure fields created by a stokeslet of strength \( \nu \) placed at \( z = h \) in an unbounded free medium.

For the region \( z \leq 0 \) we have to use the new representation defined in §5.3

\[
\begin{align*}
    u^I &= \text{grad} \, \psi^\Pi + \text{grad} \left( \frac{\partial \chi^\Pi}{\partial z} \right) - \lambda^2 \chi^\Pi \, e_z \\
p^I &= -\mu \lambda^2 \psi^\Pi, 
\end{align*}
\]

where \( \Delta \psi^\Pi = \Delta \chi^\Pi = 0 \), and \( \lambda = \sqrt{k} \). We have define again

\[
\begin{align*}
    u_\infty \equiv v \left( \frac{p}{R^3} \, e_\rho + \left( \frac{1}{R} + \frac{(z-h)^2}{R^2} \right) \, e_z \right) \text{ and } p_\infty \equiv 2 \mu \frac{v}{R^3} \frac{z-h}{R} - \frac{2}{R} \left( \frac{z-h}{R^2} \right). 
\end{align*}
\]

Using [5-55,57] let us rewrite [5-46]. To start with let us rewrite [5-55,57] to show the components of \( u^I, u^\Pi \). We have

\[
\begin{align*}
    u^I &= u^I_\infty + z \, \frac{\partial \psi^I}{\partial \rho} + z \, \frac{\partial^2 \chi^I}{\partial \rho \partial z} + \frac{\partial \chi^I}{\partial \rho} \text{, } w^I = w^I_\infty + z \, \frac{\partial \psi^I}{\partial z} - \psi^I + z \, \frac{\partial^2 \chi^I}{\partial z^2} \\
    u^\Pi &= \frac{\partial \psi^\Pi}{\partial \rho} + \frac{\partial^2 \chi^\Pi}{\partial \rho \partial z} \text{, } w^\Pi = \frac{\partial \psi^\Pi}{\partial z} + \frac{\partial^2 \chi^\Pi}{\partial z^2} - \lambda^2 \chi^\Pi 
\end{align*}
\]

But \( u_\infty^I = \frac{\partial}{\partial \rho} \left( \frac{h-z}{R} \right) \). So, on \( z = 0 \), we have [5-46] which is equivalent to

\[
\begin{align*}
    \frac{\partial}{\partial \rho} \left( \frac{h-z}{R} \right) + \frac{\partial}{\partial \rho} \left( z \, \psi^I + z \, \frac{\partial \chi^I}{\partial z} + \chi^I \right) = \frac{\partial}{\partial \rho} \left( \psi^\Pi + \frac{\partial \chi^\Pi}{\partial z} \right) \text{ on } z = 0, 
\end{align*}
\]

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which, as \( \psi^I \) et al. vanish at infinity, is equivalent to

\[
\nabla \frac{h}{r} + \chi^I = \psi^{\Pi} + \frac{\partial \chi^{\Pi}}{\partial z}; r = \sqrt{\rho^2 + h^2}.
\]

[5-64]

The same way, for [5-47] we have

\[
\nabla \left( \frac{1}{r} + \frac{h^2}{r^3} \right) \cdot \psi^I = \frac{\partial \psi^{\Pi}}{\partial z} + \frac{\partial^2 \chi^{\Pi}}{\partial z^2} - \frac{\lambda^2}{2} \chi^{\Pi}.
\]

[5-65]

Let us rewrite equation [5-54] using equations [5-56,58]:

\[
\frac{\partial \psi^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} = - \frac{\lambda^2}{2} \psi^{\Pi} + \nabla \frac{h}{r^3}.
\]

[5-66]

Equation [5-51] gives, along with [5-55,57]:

\[
\left( \frac{u^{\infty}}{\partial z} \right)_{z=0} + \frac{\partial \psi^I}{\partial \rho} + 2 \frac{\partial^2 \chi^I}{\partial \rho \partial z} = \frac{\partial^2 \psi^{\Pi}}{\partial \rho \partial z} + \frac{\partial^3 \chi^{\Pi}}{\partial \rho \partial z^2} \iff
\]

\[
\nabla \left( \frac{\rho}{r^3} - 3 \frac{\rho}{r^5} \frac{h^2}{\partial z} \right) + \frac{\partial \psi^I}{\partial \rho} + 2 \frac{\partial^2 \chi^I}{\partial \rho \partial z} = \frac{\partial^2 \psi^{\Pi}}{\partial \rho \partial z} + \frac{\partial^3 \chi^{\Pi}}{\partial \rho \partial z^2}.
\]

For the same reasons as for [5-64] the previous expression is equivalent to

\[
\nabla \left( - \frac{1}{r} + \frac{h^2}{r^3} \right) + \psi^I + 2 \frac{\partial \chi^I}{\partial z} = \frac{\partial \psi^{\Pi}}{\partial z} + \frac{\partial^2 \chi^{\Pi}}{\partial z^2}.
\]

[5-67]

The boundary conditions [5-46,47,51,54] have now, after lengthy calculations, been transformed into the following system of partial differential equations involving \( \chi^I \) et al.
\[
\begin{align*}
\begin{cases}
\nu \frac{h}{r} + r = \psi^{II} + \frac{\partial^2 \chi^{II}}{\partial z^2} \\
\nu\left(\frac{1}{r} + \frac{h^2}{r^3}\right) - \psi^I = \frac{\partial \psi^{II}}{\partial z} + \frac{\partial^2 \chi^{II}}{\partial z^2} - \frac{\lambda^2}{2} \psi^{II} + \nu \frac{h}{r^3}
\end{cases}
\end{align*}
\]

Now that we have a set of equations involving either solutions of Laplace's equations or eigenvalues of Laplace's operator (solutions of \((\Delta - \lambda^2)f = 0\)), we will be able to use integral (Hankel) transform representations of the functions involved in the final solution of our problem.

5.4.3: Solving equations [5-64.67]

Let us use the following Hankel transform representation for \(\psi^I, \chi^I (i = I, II)\):

\[
\begin{align*}
\psi^I &= \int_0^\infty A^I(s) e^{-sz} J_0(\rho s) \, ds \\
\chi^I &= \int_0^\infty B^I(s) e^{-sz} J_0(\rho s) \, ds \\
\psi^{II} &= \int_0^\infty A^{II}(s) e^{sz} J_0(\rho s) \, ds \\
\chi^{II} &= \int_0^\infty B^{II}(s) e^{\sqrt{s^2 + \lambda^2} z} J_0(\rho s) \, ds
\end{align*}
\]

Before going any further let us compute the Hankel transforms of \(\frac{1}{r}\) and \(\frac{1}{r^3}\), with \(r = \sqrt{s^2 + \rho^2}\). We have

\[
\begin{align*}
\int_0^\infty \rho \frac{1}{r} J_0(\rho s) \, dp &= \frac{1}{s} e^{-hs}, \\
\int_0^\infty \rho \frac{1}{r^3} J_0(\rho s) \, dp &= \frac{1}{h} e^{-hs}.
\end{align*}
\]
By applying a Hankel transform of order 0 the equations of the system [5-64..67] we have

\[ 5-64 \] \iff \nu \frac{h}{s} e^{-hs} + \frac{1}{s} B^I(s) = \frac{1}{s} A^I(s) + \sqrt{\frac{s^2 + \lambda^2}{s}} B^I(s) \quad [5-74] \\
\[ 5-65 \] \iff \nu \left( \frac{1}{s} e^{-hs} + h e^{-hs} \right) = \frac{1}{s} A^I(s) + A^I(s) + s B^I(s) \quad [5-75] \\
\[ 5-66 \] \iff -A^I(s) + sB^I(s) \frac{\lambda^2}{2s} A^I(s) = \nu e^{-hs} \quad [5-76] \\
\[ 5-67 \] \iff \nu \left( -\frac{1}{s} e^{-hs} + h e^{-hs} \right) + \frac{1}{s} A^I(s) - 2 B^I(s) = A^I(s) + \frac{s^2 + \lambda^2}{s} B^I(s). \quad [5-77]

These equations leads to the linear system in the Hankel transforms of \( \psi^I \) et al.

\[
\begin{pmatrix}
0 & 1 & -1 & -\sqrt{s^2 + \lambda^2} \\
1 & 0 & s & s^2 \\
-s & s^2 & \lambda^2 & 0 \\
1 & -2s & -s & s^2 + \lambda^2
\end{pmatrix}
\begin{pmatrix}
A^I \\
B^I \\
A^I \\
B^I
\end{pmatrix}
= \nu
\begin{pmatrix}
-h \\
1 + sh \\
s \\
1 - sh
\end{pmatrix} e^{-hs} \quad [5-78]
\]

Its solutions are

\[ A^I(s) = \nu \frac{8s^4 - 2h\lambda^2 s^3 + 2\lambda^2 s^2 + h\lambda^4 s + \lambda^4 + 2s(-4s^2 + h\lambda^2 s + \lambda^2)\sqrt{s^2 + \lambda^2}}{\lambda^2 \left(6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}\right)} e^{-hs} \quad [5-79] \\
= \nu A^{I*}(s) e^{-hs} \\
B^I(s) = \nu \frac{8s^3 - 2h\lambda^2 s^2 + 4\lambda^2 s - h\lambda^4 + 2s(-4s + h\lambda^2)\sqrt{s^2 + \lambda^2}}{\lambda^2 \left(6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}\right)} e^{-hs} \quad [5-80] \\
= \nu B^{I*}(s) e^{-hs} \\
A^{II}(s) = \nu \frac{8s^2 + 4h\lambda^2 s + 4\lambda^2 + 8s\sqrt{s^2 + \lambda^2}}{\lambda^2 \left(6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}\right)} e^{-hs} \quad [5-81] \\
= \nu A^{II*}(s) e^{-hs} \\
B^{II}(s) = 4\nu s \frac{-4s + h\lambda^2}{\lambda^2 \left(6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}\right)} e^{-hs} \quad [5-82] \\
= \nu B^{II*}(s) e^{-hs}. \]
Let us define $\mathbf{u}^I = \mathbf{u}^I - \mathbf{u}_\infty$. According to equations [5-55, 68, 69, 79, 80] we have the following for the components of $\mathbf{u}^I$

$$u^I = z \frac{\partial \psi^I}{\partial p} + z \frac{\partial^2 \chi^I}{\partial z \partial p} + \frac{\partial \chi^I}{\partial p}.$$

$$= v \int_0^\infty s \left( -z A^I(s) + (sz - 1)B^I(s) \right) J_1(sp) e^{-(h + z)s} ds$$

$$= -v \int_0^\infty s \left( \frac{8s^3 - 2\lambda^2(h + z)s^2 + 2\lambda^2(hz\lambda^2 + 2)s + \lambda^4(z - h)}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} \right) +$$

$$\frac{2s(-4s + s\lambda^2(h + z))\sqrt{s^2 + \lambda^2}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} J_1(sp) e^{-(h + z)s} ds$$

$$w^I = z \frac{\partial \psi^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2}.$$

$$= v \int_0^\infty s \left( 1 + sz \right) A^I(s) + zs^2 B^I(s) \right) J_0(sp) e^{-(h + z)s} ds$$

$$= -v \int_0^\infty s \left( \frac{8s^4 - 2\lambda^2(h + z)s^3 + 2\lambda^2(hz\lambda^2 + 1)s^2 + \lambda^4(h + z)s + \lambda^4}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} \right) +$$

$$\frac{2s(-4s^2 + \lambda^2(h + z)s + \lambda^2)\sqrt{s^2 + \lambda^2}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} J_0(sp) e^{-(h + z)s} ds.$$  

For $p^I$ let us use [5-56, 68, 69, 79, 80], we have

$$p^I = 2\mu \left( \frac{\partial \psi^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} \right).$$

$$= 2\mu v \int_0^\infty s \left( -s A^I(s) + s^2 B^I(s) \right) J_0(sp) e^{-(h + z)s} ds$$

$$= -2\mu v \int_0^\infty s \frac{-2s^2 + 2h \lambda^2 s + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}}{6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}} J_0(sp) e^{-(h + z)s} ds.$$
Let us now turn towards $u^I, p^I$. Using \([5-57,58,70,71,81,82]\) we have:

$$
u^I = \frac{\partial u^I}{\partial \nu} + \frac{\partial^2 \chi^I}{\partial \nu \partial z}$$

$$= -\nu \int_0^\infty s \left( A^{II*}(s) e^{zs} + \sqrt{s^2 + \lambda^2} B^{II*}(s) e^{\sqrt{s^2 + \lambda^2}} \right) J_1(s \rho) e^{-hs} ds$$

$$= -4\nu \int_0^\infty s^2 \left( \frac{(2s^2 + h\lambda^2 s + \lambda^2)e^{zs}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} \right) +$$

$$\left( \frac{\left( -4s + h\lambda^2 \right)e^{\sqrt{s^2 + \lambda^2} + 2s^2 e^{zs}}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} \right) J_1(s \rho) e^{-hs} ds$$

$$\omega^I = \frac{\partial \omega^I}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} - \lambda^2 \chi^I$$

$$= \nu \int_0^\infty \left( sA^{II*}(s) e^{zs} + s^2 B^{II*}(s) e^{\sqrt{s^2 + \lambda^2}} \right) J_0(s \rho) e^{-hs} ds$$

$$= 4\nu \int_0^\infty s^2 \left( \frac{(h\lambda^2 s + \lambda^2)e^{zs} + h\lambda^2 s e^{\sqrt{s^2 + \lambda^2}}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} \right) +$$

$$\frac{2s\sqrt{s^2 + \lambda^2} e^{zs}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} J_0(s \rho) e^{-hs} ds$$

$$p^I = -\mu \lambda^2 \psi^I$$

$$= -\mu \nu \lambda^2 \int_0^\infty A^{II*}(s) J_0(s \rho) e^{(z - h)s} ds$$

$$= -4\mu \nu \int_0^\infty \frac{2s^2 + h\lambda^2 s + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}}{6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2}} J_0(s \rho) e^{(z - h)s} ds$$

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Now that we have computed the components of the velocity fields in the two regions of the space, let us study the two limit cases $\lambda = 0$ or $\infty$.

5.4.4 : The limit cases

Let us examine the two known limit cases. That is, free medium, corresponding to $\lambda = 0$, and solid wall, corresponding to $\lambda = \infty$. Let us examine the first case.

We have, when $\lambda \to 0$, by using L'Hôpital's rule twice

\[ v^{I*} \to 0 \]
\[ w^{I*} \to 0 \]
\[ p^{I*} \to 0, \]

and of the velocity and pressure in region (II)

\[ v^{II} \to -\nu \int_0^\infty s (h - z) J_1(sp) e^{-(h - z)s} ds = \nu \rho \frac{(z-h)}{R^3} \]
\[ w^{II} \to \nu \int_0^\infty ((h - z)s + 1) J_0(sp) e^{-(h - z)s} ds = \nu \left( \frac{1}{R} + \frac{(z-h)^2}{R^3} \right) \]
\[ p^{II} \to -4\mu \nu \int_0^\infty s J_0(sp) e^{-(h - z)s} ds = 2 \mu \nu \frac{z - h}{R^3}. \]

Now let us take the other case $\lambda \to \infty$. In that case we have

\[ v^{I*} \to \nu \int_0^\infty s (h - z) J_1(sp) e^{-(h + z)s} ds = \nu \rho \frac{(h - z)}{R^3} \frac{6hz(z + h)}{R^2} \]
\[ w^{I*} \to -\nu \int_0^\infty (1 + s(h + z)) J_0(sp) e^{-(h + z)s} ds \]
\[ = -\nu \left( \frac{2}{R^2} \frac{\rho^2}{R^3} - \frac{2hz}{R^5} \frac{\rho^2}{R^5} - 2(z + h)^2 \right) \]
\[ p^{I*} \to -2\mu \nu \int_0^\infty (1 + 2hs) J_0(sp) e^{-(h + z)s} ds = 2\mu \nu \frac{z - h}{R^3} \frac{6h(z + h)^2}{R^2} \]
\[ v^{II} \to 0 \]
\[ w^I \rightarrow 0 \]

\[ p^I \rightarrow -4\mu v \int_0^\infty s (hs + s) J_0(s\rho) e^{-\rho^2} \text{d}s = 2\mu v \left( \frac{2\pi}{R^3} - \frac{6}{R^5} \right) \frac{h(z - h)^2}{R^5} \]

where \( R_2 = \sqrt{\rho^2 + (z + h)^2} \).

We found exactly the values we expected. Let us now move on to the next section, drag forces.

### 5.4.5: The drag

Bearing in mind the remarks made in section 5.2.5, the resistance formula, according to Brenner (Brenner [2]), and supposing it is valid in the case of a porous wall, is:

\[ \frac{F_0}{F} = 1 - \kappa \frac{F_0}{8\pi \mu U} + O\left( \frac{\kappa}{h} \right) \]

where \( F \) is the drag in the \( z \) direction, \( F_0 \) the drag if there was no wall, \( U \) the velocity of the particle, and \( \kappa \) the characteristic dimension of the particle. As we did for the Darcy case let us take \( v = \frac{F}{8\pi \mu} \), we have

\[ \kappa = \frac{8\pi \mu h}{F} u^*(0,h) \quad e^2 = \frac{8\pi \mu h}{F} w^*(0,h) \]

\[ = -h \int_0^\infty \left( \frac{8s^4 - 4h^2s^2 + 2\lambda^2(h^2\lambda^2 + 1)s^2 + 2h\lambda^4s + \lambda^4}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} \right) + \]

\[ \frac{2s(-4s^2 + 2h\lambda^2s + \lambda^2)\sqrt{s^2 + \lambda^2}}{\lambda^2 \left( 6s^2 + \lambda^2 + 2s\sqrt{s^2 + \lambda^2} \right)} e^{-2hs} \text{d}s \]

\[ = -\int_0^\infty \left( \frac{8t^4 - 4h^2\lambda^2t^3 + 2h^2\lambda^2(h^2\lambda^2 + 1)t^2 + 2h^4\lambda^4s + h^4\lambda^4}{\lambda^2 \left( 6h^2t^2 + h^4\lambda^2 + h^4\sqrt{t^2 + h^2\lambda^2} \right)} \right) + \]

\[ \frac{2t(-4t^2 + 2h^2\lambda^2t + h^2\lambda^2)\sqrt{t^2 + h^2\lambda^2}}{\lambda^2 \left( 6h^2t^2 + h^4\lambda^2 + 2h^2t\sqrt{t^2 + h^2\lambda^2} \right)} e^{-2t} \text{d}t \]

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Again we have a dichotomy based on the nature of $\lambda h$. Either $\lambda h = O(1)$, and in that case [5-109] is the best way to compute $\kappa$, or $\lambda h \gg 1$, and in that case:

$$\kappa = -\left(\frac{3}{2} - \frac{3}{2\lambda h} + O\left(\frac{1}{\lambda h^2}\right)\right)$$

[5-110]

The limit cases can again be taken, giving the expected results.

5.5 : Numerical calculations

The numerical calculations made for this chapter are exclusively plots of streamlines. These plots were generated using a simple Runge-Kutta ordinary differential equations solver. The o.d.e. involved was the autonomous system:

$$\frac{dx}{dt} = u(x)$$

To ensure the smoothness of the plot, the curves have been plotted using a curve fitting algorithm, that is the plotting program tried to smooth the segments computed during the solving of the ordinary differential. This explains the various bizarre features of certain plots, namely:

- Unexpected inflexion points as in [Plot 5-1],[Plot 5-5]
- Touching streamlines as in [Plot 5-2]
- Rough looking cycles due to the fact that the curve fitting algorithm does not give the same curve for different points (even if these are on the same streamline), as in [Plot 5-5],[Plot 5-24].

Since there are three parameters ($\beta, \alpha, \sigma$) for the Darcy model, and only one ($\lambda$) for the Brinkman model, most of the plots ([Plots 5-1..18]) were made for the Darcy model. In order to be able to compare the two models one has to remember that

$$\beta = \frac{2}{\lambda^2} \text{ or } \lambda = \sqrt{\frac{2}{\beta}}$$

The comparison between a Darcy plot and a Brinkman plot should be done for equivalent $\lambda$ and $\beta$.

A thing that is common to all plots is that they all show stagnation points, apart from the few that are done for parameters sufficiently near to the "free medium" conditions, i.e. $\beta = \infty$ or $\lambda = 0$. 

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5.5.1: The Darcy model plots

These plots [Plots 5-1..18] were done for a combination of the following values for $\beta$, $\alpha$, $\sigma$, $\beta = 0, 1, 5$, $\alpha = 0, 1, 10$, and $\sigma = 0, 1$.

A few things can be said beforehand

- When $\beta = 0$ (i.e. $\lambda = \infty$) the value of $\sigma$ is irrelevant. Indeed, $\sigma$ acts only on $u^I$ at the interface, which is equal to zero everywhere on the interface.
- The same can be said for the case $\alpha = 0$, since only products of the form $\alpha \sigma$ can be found in the expressions giving the components of the velocity fields.

When this is the case ([Plots 5-1,2], [Plots 5-3,4], ...) I have tried, whenever possible, to give different points of view.

Some rather striking plots can be found in the cases $\beta = 1$, $\alpha = 10$ and $\beta = 5$. These plots all show a discontinuity in radial velocity. The usual case is $u^I$ rather weak at the interface, and $u^I$ rather strong at the same interface. It is to be noted, that whenever applicable, the case $\sigma = 1$ has a damping effect on the discontinuity. indeed the case $\sigma = 1$, as opposed to the case $\sigma = 0$, the kink in the streamline is less marked, if non-existent.

5.5.2: The Brinkman model

The curves computed where for $\lambda = 0.1, 0.5, 1, 5$ and for comparison with the Darcy case $\lambda = 0.63 = \sqrt{\frac{2}{5}} (\beta = 5)$, $1.41 = \sqrt{2} (\beta = 1)$.

It is quite interesting to note that the curves, even for equivalent $\beta$ and $\lambda$ are quite dissimilar for $\sigma = 0$, but, that for $\sigma = 1$ they show, at least superficial resemblance.

5.5.3 Conclusion

From comparing the various curves plotted two points need to be stressed

- The factor $\sigma$ seems to play a very important role when one compares the Darcy and Brinkman model for equivalent values of $\beta$ and $\lambda$. The comparison can only be done for small values of $\beta$ and large values of $\lambda$. It seems that setting $\sigma$ to 1 gives a better visual (i.e. qualitative) agreement between the Darcy and Brinkman curves. That is, although ultimately we should have the same curves, it seems that we have a better agreement when $\sigma = 1$. This tends to
imply that the $O(k^{-1})$ term in the boundary condition formula derived by Saffman (Saffman [18] and [3-20]) has more importance than was originally thought and cannot be neglected. Basically this term seems to contain, in a non-explicit fashion the $u_{I}^{II}$ term that is present in the boundary condition formula derived by Beavers and Joseph (Beavers and Joseph [1] and [3-19]). Furthermore the formula derived by Beavers and Joseph has the advantage of having been experimentally proved in the case of a few selected flows (i.e. laminar flows parallel to the interface).

- Although more complex theoretically (i.e. long and convoluted formulae), the Brinkman model is simpler when it comes to give it actual parameters. It only needs one $\lambda$, whereas the Darcy model needs three parameters $\beta$, $\alpha$, $\sigma$ which are not easy to estimate (especially the slip-parameter $\alpha$).
Streamlines plot in the Darcy model for:
\( \beta = 0, \alpha = 0, \sigma = 0 \)
The stagnation point is at:
\[ \rho = 1.66 \]
\[ z = 1.17 \]

Streamlines plot in the Darcy model for:
\( \beta = 0, \alpha = 0, \sigma = 1 \)
No stagnation point detected
Streamlines plot in the Darcy model for:
\[ \beta = 0, \alpha = 1, \sigma = 0 \]
The stagnation point is at:
\[ \{ \rho = 3.02, \ z = 2.13 \} \]

Streamlines plot in the Darcy model for:
\[ \beta = 0, \alpha = 1, \sigma = 1 \]
The stagnation point is at:
\[ \{ \rho = 1.63, \ z = 1.48 \} \]
Streamlines plot in the Darcy model

for:

$\beta = 0, \alpha = 10, \sigma = 1$

$\beta = 1.15$

$\beta = 1.28$

The stagnation point is at:

$z = 1.28$

[Plot 5.5, 6]
Streamlines plot in the Darcy model

\( \beta = 1, \alpha = 0, \sigma = 1 \)

The stagnation point is at:

\( \rho = 3.15 \)

\( z = 1.92 \)

Streamlines plot in the Darcy model

\( \beta = 1, \alpha = 0, \sigma = 0 \)

The stagnation point is at:

\( \rho = 3.15 \)

\( z = 1.92 \)
Streamlines plot in the Darcy model for:

\( \beta = 1, \alpha = 1, \sigma = 0 \)
The stagnation point is at:

\( \rho = 2.33 \)
\( z = 1.36 \)

Streamlines plot in the Darcy model for:

\( \beta = 1, \alpha = 1, \sigma = 1 \)
The stagnation point is at:

\( \rho = 2.58 \)
\( z = 1.55 \)
Streamlines plot in the Darcy model

for:
\[ \beta = 1, \alpha = 10, \sigma = 0 \]
The stagnation point is at:
\[ \rho = 2.10 \]
\[ t = 1.15 \]

Streamlines plot in the Darcy model

for:
\[ \beta = 1, \alpha = 10, \sigma = 1 \]
The stagnation point is at:
\[ \rho = 2.45 \]
\[ t = 1.46 \]
Streamlines plot in the Darcy model
for:
\[ \beta = 5, \alpha = 0, \sigma = 0 \]
The stagnation point is at:
\[ (\rho = 3.91, z = 1.06) \]

Streamlines plot in the Darcy model
for:
\[ \beta = 5, \alpha = 0, \sigma = 1 \]
The stagnation point is at:
\[ (\rho = 3.91, z = 1.06) \]

[Plot 5 - 13,14]
Streamlines plot in the Darcy model

for:
$\beta = 5, \alpha = 1, \sigma = 0$
The stagnation point is at:
$r = 3.73$
$z = 0.60$

Streamlines plot in the Darcy model

for:
$\beta = 5, \alpha = 1, \sigma = 1$
The stagnation point is at:
$r = 4.60$
$z = 1.86$

[Plot 5 - 15,16]
Streamlines plot in the Darcy model
for:
\[ \beta = 5, \alpha = 10, \sigma = 0 \]
The stagnation point is at:
\[ (r, \theta) = (3.65, 0.29) \]

Streamlines plot in the Darcy model
for:
\[ \beta = 5, \alpha = 10, \sigma = 1 \]
The stagnation point is at:
\[ (r, \theta) = (4.86, 2.08) \]
Streamlines plot in the Brinkman model for:
\[ \lambda = 0.1 \]
The stagnation point is at:
\[ \rho = 30.2 \]
\[ z = 7.60 \]

Streamlines plot in the Brinkman model for:
\[ \lambda = 0.5 \]
The stagnation point is at:
\[ \rho = 6.37 \]
\[ z = 2.30 \]
Streamlines plot in the Brinkman model for:
\[ \lambda = 1 \]
The stagnation point is at:
\[ \rho = 3.49 \]
\[ z = 1.73 \]

Streamlines plot in the Brinkman model for:
\[ \lambda = 5 \]
The stagnation point is at:
\[ \rho = 1.39 \]
\[ z = 1.31 \]
Streamlines plot in the Brinkman model for:
\[ \lambda = 0.63 \]
The stagnation point is at:
\[ \rho = 5.16 \]
\[ z = 2.07 \]

Streamlines plot in the Brinkman model for:
\[ \lambda = 1.41 \]
The stagnation point is at:
\[ \rho = 2.66 \]
\[ z = 1.57 \]
Chapter Six

Asymmetric flow above a porous medium

Summary:

This chapter considers the asymmetric point force (torque) above a porous medium in the Brinkman case. After briefly establishing the equations, the values corresponding to a rotelet and then to a stokeslet are computed. An exploration of the limiting cases shows that the representation found in chapter 5 fails to provide the expected result. To alleviate this problem a new, dual, representation is made and an alternate solution of the problem is sketched.

Introduction

This chapter, as seen in the summary, is supposed to cover a fairly large field, and is, for its major part, the summary of a progress report. To allow extensions, it has been on purpose left in a fairly rough state. Indeed the last part clearly shows that there is still a lot to be done, especially into representing the solutions to Brinkman's equation, and finding proper Hankel transforms of these representations. It has to be noted that this chapter is actually sketching the problem of the asymmetric point force or torque above a porous medium and tries only to point at the problems that arose when trying to find the limit, in the case of a "free" medium of the velocity fields found.

The general structure of this chapter is as follows:

- Section 1: Computing the base quantities.
- Section 2: The application to the rotelet and stokeslet of the quantities found in section 1
- Section 3: Constructing a 'brinklet'
- Section 4: Building a new representation and using it.

In this chapter we are considering the asymmetric movement of a particle of characteristic dimension a modeled by a point force (Stokeslet) or a point torque (Rotelet) of strength v, in a fluid of viscosity \( \mu \). As we are interested by the slow creeping viscous flows generated by the particle we have to compute the Reynolds number Re associated with the problem. We have
for the stokeslet: \( \text{Re} = \frac{d \nu}{\mu} \), where \( d \) is the density of the fluid,

for the rotelet: \( \text{Re} = \frac{d \nu}{a \mu} \)

The expressions used to compute the Reynolds number have to be compared with those used in chapters 4 and 5.
6.1 : Computing the base quantities

6.1.1 : Exposition

6.1.1.1 : The Configuration

The configuration has not changed since chapter 5. We have two regions, region (I) and region (II), in which an incompressible Newtonian fluid of viscosity $\mu$ flows. The fluid obeys the law of conservation of mass. We are supposing that the flow is slow and viscous, that is that the Reynolds number associated with it is small. The two regions are defined as follows:

- **Region (I)**: the half-space $z \geq 0$. It is a free medium, the equation of the motion for the fluid is the Stokes equation: $\mu \Delta u^I = \nabla P^I$, where $u^I, P^I$ are the velocity and pressure fields for the fluid in region (I).
- **Region (II)**: the half-space $z \leq 0$. It is a porous medium, the equation of the motion for the fluid is Brinkman's equation: $\mu \Delta u^{II} - \mu k u^{II} = \nabla P^{II}$, where $\lambda$ is the Brinkman constant for the medium.

Let us suppose that in $z = h$ we have an asymmetric motion produced by either a rotelet or a stokeslet oriented parallel to the plane $z = 0$.

![Diagram of the original configuration](Image)

6.1.1.2 : The fields
Let us work in a cylindrical polar coordinate system of coordinates \((\rho, \varphi, z)\), and let us assume that the fields created by the point force or torque in an unbounded free medium, \(\mathbf{u}_\infty = (\mathbf{u}_\infty, \mathbf{v}_\infty, \mathbf{w}_\infty)\), \(\mathbf{p}_\infty = \mathbf{p}_\infty\), is such that:

\[
\begin{align*}
\mathbf{u}_\infty(\rho, \theta, z) &= U_\infty(\rho, z) \cos(\varphi - \varphi_0), & [6.1-1] \\
\mathbf{v}_\infty(\rho, \theta, z) &= V_\infty(\rho, z) \sin(\varphi - \varphi_0), & [6.1-2] \\
\mathbf{w}_\infty(\rho, \theta, z) &= W_\infty(\rho, z) \cos(\varphi - \varphi_0), & [6.1-3] \\
\mathbf{p}_\infty(\rho, \theta, z) &= P_\infty(\rho, z) \cos((\varphi - \varphi_0). & [6.1-4]
\end{align*}
\]

that is we only take the 'first term' in a Fourier expansion of the velocity and pressure fields. We can then suppose that the velocity and pressure fields are of the following form:

\[
\begin{align*}
\mathbf{u}_1 &= \mathbf{u}_1(\rho, z) \cos(\varphi - \varphi_0), \mathbf{v}_1(\rho, z) \sin(\varphi - \varphi_0), \mathbf{w}_1(\rho, z) \cos(\varphi - \varphi_0), & [6.1-5] \\
\mathbf{p}_1 &= \mathbf{p}_1(\rho, z) \cos(\varphi - \varphi_0), i = I, II, & [6.1-6]
\end{align*}
\]

that is that they have Fourier expansions identical in form to those of the point force (torque).

6.1.1.3 : The boundary conditions

Let us use Chapter 3 §3.2.3, eqns. [3-15,17]. We have, in the Brinkman case, on the border \(z = 0\) the following boundary conditions:

\[
\begin{align*}
\mathbf{u}_1 &= \mathbf{u}_{11}, \text{ on } z = 0 \text{ (6.1.e. continuity of the velocity fields across } z = 0) & [6.1-7] \\
\mathbf{T}_1 \cdot \mathbf{e}_z &= \mathbf{T}_{11} \cdot \mathbf{e}_z, \text{ on } z = 0 \text{ (6.1.e. continuity of stress across } z = 0) & [6.1-8]
\end{align*}
\]

where \(\mathbf{T}_1\) (resp. \(\mathbf{T}_{11}\)) is the stress tensor in region (I) (resp. region (II)). According to [6.1-5], [6.1-6] rewrites, by eliminating the azimuthal dependence

\[
\begin{align*}
\mathbf{u}_1 &= \mathbf{u}_{11}, \text{ on } z = 0, & [6.1-9] \\
\mathbf{v}_1 &= \mathbf{v}_{11}, \text{ on } z = 0, & [6.1-10] \\
\mathbf{w}_1 &= \mathbf{w}_{11}, \text{ on } z = 0. & [6.1-11]
\end{align*}
\]

The same way [6.1-7] rewrites as

\[
\begin{align*}
-\mathbf{p}_1 + 2\mu \frac{\partial \mathbf{w}_1}{\partial z} &= -\mathbf{p}_{11} + 2\mu \frac{\partial \mathbf{w}_{11}}{\partial z}, \text{ on } z = 0, & [6.1-12] \\
\frac{\partial \mathbf{u}_1}{\partial z} + \frac{\partial \mathbf{w}_1}{\partial \rho} &= \frac{\partial \mathbf{u}_{11}}{\partial z} + \frac{\partial \mathbf{w}_{11}}{\partial \rho}, \text{ on } z = 0. & [6.1-13]
\end{align*}
\]
\[ \frac{\partial v^I}{\partial z} + \frac{1}{\rho} \frac{\partial w^I}{\partial \theta} = \frac{\partial x^I}{\partial z} + \frac{1}{\rho} \frac{\partial w^I}{\partial \theta}, \quad \text{on} \; z = 0. \]  \[ 6.1-14 \]

The conservation of the mass \( \text{div} \; u^I = 0 \) rewrites as

\[ \frac{\partial u^I}{\partial \rho} + \frac{1}{\rho} \frac{\partial v^I}{\partial \rho} + \frac{\partial w^I}{\partial z} = 0, \quad \text{for} \; i = I, \ II. \]  \[ 6.1-15 \]

Using the same technique as in chapter 5 (eqns [5-48..54]) using [6.1-15], we reduce [6.1-12..14] to:

\[ p^I = p^\Pi, \; \text{on} \; z = 0, \]  \[ 6.1-16 \]

\[ \frac{\partial u^I}{\partial z} = \frac{\partial u^\Pi}{\partial z}, \; \text{on} \; z = 0, \]  \[ 6.1-17 \]

\[ \frac{\partial v^I}{\partial z} = \frac{\partial v^\Pi}{\partial z}, \; \text{on} \; z = 0. \]  \[ 6.1-18 \]

6.1.2 : The first representation : The fields

6.1.2.1 : Representing the fields

Let us represent \( u^I \) and \( p^I \) using the method devised by Shail (Shail [20]). The representations for \( u^I \) and \( p^I \) are as follows:

\[ u^I = u_\infty + z \text{ grad } \Psi^I - \Psi^I e_z + z \text{ grad} \left( \frac{\partial x^I}{\partial z} - \frac{\partial x^I}{\partial z} e_z + \text{ grad } X^I + \text{ curl}(\Phi^I e_z) \right) \]  \[ 6.1-19 \]

\[ = u_\infty + u^{I*} \]  \[ 6.1-20 \]

\[ p^I = \rho_\infty + 2 \mu \left( \frac{\partial \Psi^I}{\partial z} + \frac{\partial^2 x^I}{\partial z^2} \right) \]  \[ 6.1-21 \]

where \( \Delta \Psi^I = \Delta X^I = \Delta \Phi^I = 0 \) (in this case \( \Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \)) and \( u_\infty, p_\infty \) are

the velocity and pressure fields of the motion source. In this representation the azimuthal component is implicitly included if [6.1-19,20]. For the region \( z \leq 0 \) we have to use the new representation that has been derived in chapter 5 §5.3. We have

\[ u^\Pi = \text{ grad } \Psi^\Pi + \text{ grad} \left( \frac{\partial x^\Pi}{\partial z} - \lambda^2 X^\Pi e_z + \text{ curl}(\Phi^\Pi e_z) \right) \]  \[ 6.1-22 \]
\[ p^\Pi = -\mu \lambda^2 \Psi^\Pi \]  

where \( \Delta \Psi^\Pi = \Delta X^\Pi - \lambda^2 X^\Pi = \Delta \Phi^\Pi - \lambda^2 \Phi^\Pi = 0. \)

Let us rewrite \( \Psi^i, X^i, \Phi^i \) the same way as we did for \( u^i \) in [6.1-1-3], that is let us use only the same elements in the Fourier decomposition of \( \Psi^i \) et al. as would be required by the components of \( u_{\infty}, p_{\infty} \). By decomposing \( u^I, p^I, u^\Pi, p^\Pi \), we take only

\[
\Psi^I = \psi^I(\rho, z) \cos(\varphi - \varphi_0), \\
X^I = \chi^I(\rho, z) \cos(\varphi - \varphi_0), \\
\Phi^I = \phi^I(\rho, z) \sin(\varphi - \varphi_0).
\]

Now that we have a representation for the velocity and pressure fields in both regions of space, let us rewrite the boundary condition equations.

6.1.2.2 : The new boundary equations

The boundary conditions, that is equations [6.1-9-11,16-19] now become

\[
\left. u^I_{\infty}\right|_{z=0} + \frac{\partial \Psi^I}{\partial \rho} + \frac{\phi^I}{\rho} = \frac{\partial \psi^II}{\partial \rho} + \frac{\partial^2 \chi^I}{\partial \rho \partial z} + \frac{1}{\rho} \frac{\partial \phi^\Pi}{\partial \rho} \\
\left. v^I_{\infty}\right|_{z=0} - \left( \frac{1}{\rho} \frac{\partial \chi^I}{\partial \rho} + \frac{\partial \phi^I}{\partial \rho} \right) = - \left( \frac{1}{\rho} \frac{\partial \psi^II}{\partial \rho} + \frac{1}{\rho} \frac{\partial \chi^I}{\partial z} + \frac{\partial \phi^\Pi}{\partial \rho} \right) \\
\left. w^I_{\infty}\right|_{z=0} - \frac{\partial \psi^II}{\partial z} + \frac{\partial^2 \chi^I}{\partial z^2} - \lambda^2 \chi^II \\
\left. p^I_{\infty}\right|_{z=0} + \frac{4\mu}{\left( \frac{\partial^2 \chi^I}{\partial z^2} + \frac{\partial \psi^II}{\partial z} \right)} = -\mu \lambda^2 \psi^II \\
\left. \frac{\partial u^I_{\infty}}{\partial z} \right|_{z=0} + 2 \frac{\partial^2 \chi^I}{\partial \rho \partial z} + \frac{\partial \psi^II}{\partial \rho} + \frac{1}{\rho} \frac{\partial \phi^I}{\partial z} = \frac{\partial^2 \psi^II}{\partial \rho \partial \rho z} + \frac{1}{\rho} \frac{\partial \phi^\Pi}{\partial \rho \partial \rho z} + \frac{1}{\rho} \frac{\partial \phi^\Pi}{\partial z} \\
\left. \frac{\partial v^I_{\infty}}{\partial z} \right|_{z=0} - \left( \frac{2}{\rho} \frac{\partial \chi^I}{\partial z} + \frac{1}{\rho} \frac{\partial \psi^II}{\partial \rho} + \frac{\partial \phi^I}{\partial \rho} \right) = - \left( \frac{1}{\rho} \frac{\partial \psi^II}{\partial z} + \frac{1}{\rho} \frac{\partial \chi^I}{\partial z^2} + \frac{\partial \phi^\Pi}{\partial \rho \partial \rho z} \right)
\]


\[
\left. (u^I_{\infty} + v^I_{\infty}) \right|_{z=0} + \left( \frac{\partial \chi^I}{\partial \rho} - \frac{1}{\rho} \phi^I \right) \chi^I - \phi^I = \left( \frac{\partial \psi^II}{\partial \rho} - \frac{1}{\rho} \phi^\Pi \right) \psi^II + \frac{\partial \chi^I}{\partial z} - \phi^\Pi
\]
which, by defining $u_+$ as being the solution of 
$$f - [f] = (u_d^+ + v_j^+)$$
that vanishes as $p \to \infty$, is equivalent to

$$u_+ = -\chi^I + \phi^I + \psi^II + \frac{\partial \chi^II}{\partial z} - \phi^II$$  \[6.1-34\]

The same way

$$[6.1-27] - [6.1-28] \iff u_- = -\chi^I - \phi^I + \psi^II + \frac{\partial \chi^II}{\partial z} + \phi^II$$  \[6.1-35\]

$$[6.1-29] + [6.1-30] \iff v_+ = -2 \frac{\partial \chi^I}{\partial z} - \psi^I + \frac{\partial \phi^I}{\partial z} + \frac{\partial \psi^II}{\partial z} + \frac{\partial^2 \chi^II}{\partial z^2} - \frac{\partial \phi^II}{\partial z}$$  \[6.1-36\]

$$[6.1-29] - [6.1-30] \iff v_- = -2 \frac{\partial \chi^I}{\partial z} - \psi^I - \frac{\partial \phi^I}{\partial z} - \frac{\partial \psi^II}{\partial z} - \frac{\partial^2 \chi^II}{\partial z^2} + \frac{\partial \phi^II}{\partial z}$$  \[6.1-37\]

where $u_-, v_+$ and $v_-$ are suitable solutions of $\left(\frac{\partial}{\partial p} \pm \frac{1}{p}\right)[f] = g$, for $g = u_\infty^l - v_\infty^I$, $\frac{\partial u_\infty^l}{\partial z} + \frac{\partial v_\infty^l}{\partial z}$ respectively. The meaning of suitable solution is the following: the solution should be bounded when $p \to \infty$, and have a limit when $p \to 0$. These two conditions prescribe any constant that might appear.

It is now time, as in chapter 5 to transform the boundary PDE's ([6.1-29,30,34..37]) by applying Hankel transforms.

6.1.3 : A second representation - The field components

6.1.3.1 : The Hankel transforms

Let us use a Hankel transform representation for $\psi^I, \chi^I, \phi^I$. We have

$$\psi^I = \int_0^{\infty} A^I(s) e^{-sz} J_1(ps) \, ds, \quad \psi^II = \int_0^{\infty} A^II(s) e^{sz} J_1(ps) \, ds, \quad [6.1-38,39]$$

$$\chi^I = \int_0^{\infty} B^I(s) e^{-sz} J_1(ps) \, ds, \quad \chi^II = \int_0^{\infty} B^II(s) e^{\sqrt{s^2 + \lambda^2} z} J_1(ps) \, ds, \quad [6.1-40,41]$$
\[ \phi^I = \int_0^\infty C^I(s) e^{-sz} J_1(\rho s) \, ds, \quad \phi^{II} = \int_0^\infty C^{II}(s) e^{\sqrt{s^2 + \lambda^2} z} J_1(\rho s) \, ds. \quad \text{[6.1-42,43]} \]

The same way, let us write
\[ u_+ = \int_0^\infty U_+(s) J_1(\rho s) \, ds, \quad u_- = \int_0^\infty U_-(s) J_1(\rho s) \, ds, \quad \text{[6.1-44,45]} \]
\[ v_+ = \int_0^\infty V_+(s) J_1(\rho s) \, ds, \quad v_- = \int_0^\infty V_-(s) J_1(\rho s) \, ds, \quad \text{[6.1-46,47]} \]
\[ p_{\infty}|_{z=0} = \mu \int_0^\infty P(s) J_1(\rho s) \, ds, \quad w_{\infty}|_{z=0} = \int_0^\infty W(s) J_1(\rho s) \, ds. \quad \text{[6.1-48,49]} \]

Equations [6.1-29,30,34..37] become, after the application of a Hankel transform of order one
\[ [6.1-29] \iff W = A^I + s A^{II} + s^2 B^{II} \quad \text{[6.1-50]} \]
\[ [6.1-30] \iff P = 2s A^I - 2s^2 B^I - \lambda^2 A^{II} \quad \text{[6.1-51]} \]
\[ [6.1-34] \iff U_+ = - B^I + C^I + A^{II} + \sqrt{s^2 + \lambda^2} B^{II} - C^{II} \quad \text{[6.1-52]} \]
\[ [6.1-35] \iff U_- = - B^I - C^I + A^{II} + \sqrt{s^2 + \lambda^2} B^{II} + C^{II} \quad \text{[6.1-53]} \]
\[ [6.1-36] \iff V_+ = - A^I + 2s B^I - s C^I + s A^{II} + (s^2 + \lambda^2) B^{II} - \sqrt{s^2 + \lambda^2} C^{II} \quad \text{[6.1-54]} \]
\[ [6.1-37] \iff V_- = - A^I + 2s B^I + s C^I + s A^{II} + (s^2 + \lambda^2) B^{II} + \sqrt{s^2 + \lambda^2} C^{II} \quad \text{[6.1-55]} \]

Equations [6.1-50..55] are equivalent to the system:
\[
\begin{pmatrix}
1 & 0 & 0 & s & s^2 & 0 \\
2s & -2s^2 & 0 & -\lambda^2 & 0 & 0 \\
0 & -1 & 1 & 1 & \sqrt{s^2 + \lambda^2} & -1 \\
0 & -1 & -1 & 1 & \sqrt{s^2 + \lambda^2} & 1 \\
-1 & 2s & -s & s^2 + \lambda^2 & -\sqrt{s^2 + \lambda^2} & -1 \\
-1 & 2s & s & s^2 + \lambda^2 & \sqrt{s^2 + \lambda^2} & 1
\end{pmatrix}
\begin{pmatrix}
A^I \\
B^I \\
C^I \\
A^{II} \\
B^{II} \\
C^{II}
\end{pmatrix}
= \begin{pmatrix}
W \\
P \\
U_+ \\
U_- \\
V_+ \\
V_-
\end{pmatrix} \quad \text{[6.1-56]} \]

Let us now, in the next section, compute the actual solutions to the previous systems in the case of a rotelet and a stokeslet.
6.2 : Boundary values, in the case of a rotelet or a stokeslet

Let us compute in the case of a rotelet and of a stokeslet the quantities necessary to solve the system [6.1-56], and eventually study the velocity fields.

6.2.1 : The Rotelet

For a rotelet of force \( v \) placed in \( z = h \), and oriented parallel to the \( y \) axis, we have

\[
\begin{align*}
  u_\infty^I &= v \frac{z - h}{R^3}, \\
  v_\infty^I &= -v \frac{z - h}{R^3}, \\
  w_\infty^I &= -v \frac{\rho}{R^3}, \\
  p_\infty^I &= 0.
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
  u_+ &= 0, \\
  u_- &= -\frac{2\nu}{\rho} \left( 1 - \frac{h}{r} \right), \\
  v_+ &= 0, \\
  v_- &= -2\nu \frac{\rho}{r^3}, \\
  w_+ &= 0, \\
  w_- &= -2\nu e^{-hs}, \\
  p &= 0.
\end{align*}
\]
The numerical simulation shows no surprises. That is, as \( \lambda \to 0 \), we have, as we would expect \( u^I \to 0 \) and \( u^{II} \to u_\infty \). This is confirmed by an analysis of \( A^I, B^I, C^I \) as \( \lambda \to 0 \). The same way, as \( \lambda \to \infty \) we have \( u^{II} \to 0 \). Let us now turn to the case of the stokeslet

6.2.2: The Stokeslet

For a stokeslet of force \( v \) placed in \( z = h \) and oriented parallel to the \( x \) axis, we have \( \varphi_0 = 0 \), and :

\[
\begin{align*}
  u^I_\infty &= v \left( \frac{2}{R} - \frac{(z-h)^2}{R^3} \right) \sqrt{\rho^2 + (z-h)^2}, \\
  v^I_\infty &= -\frac{v}{R}, \\
  w^I_\infty &= -\frac{vph}{R^3}, \\
  p^I_\infty &= \frac{2\mu v p}{R}.
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
  u_+ &= -\frac{vp}{r}, \quad r = \sqrt{\rho^2 + h^2}, \\
  u_- &= \frac{v}{\rho} \left( -4h + 3r + \frac{h^2}{r} \right), \\
  v_+ &= -\frac{vph}{r^3}, \\
  v_- &= \frac{v}{\rho} \left( 4 - \frac{5h}{r} + \frac{h^3}{r^3} \right).
\end{align*}
\]

and

\[
\begin{align*}
  U_+ &= -v \left( h + \frac{1}{s} \right) e^{-hs}, \\
  U_- &= -v \left( h - \frac{3}{s} \right) e^{-hs}, \\
  V_+ &= -vhs e^{-hs}, \\
  V_- &= v \left( 4 - hs \right) e^{-hs}, \\
  W &= -vhs e^{-hs}.
\end{align*}
\]
Let us now look at the limit case $X \to 0$. We have, as expected $u^\Pi \to 0$, and for $u^{I*}$, we have the same limits.

The problems start to arise for the other limit case: $X \to 0$. We have a very serious problem; we expect that as $X \to 0$, $u^{I*} \to 0$, and $u^\Pi \to u_\infty$. What happens instead is that

$$\begin{cases}
u^{I*} \to 0 \\
v^{I*} \to 0 \text{ which is a highly uncanny behaviour!} \\
w^{I*} \to 0
\end{cases}$$

A reason for that problem is that the representation becomes "singular" as $\lambda \to 0$. Analysing the term for the pressure in the representation for the flow in region (II) given by [6.1-23], shows that, since the pressure is the product of a term that vanishes as $X \to 0$, $\lambda^2$ and $\Psi^\Pi$, the function $\Psi^\Pi$ has to be unbounded for the pressure $p^\Pi$ to be non zero on $z = 0$ when $\lambda \to 0$.

An exploration of the system [6.1-56] in order to find a condition on the right hand side of [6.1-56] (i.e. on the cause of the motion of the fluid) such that, in the end, $u^{I*} \to 0$ when $\lambda \to 0$, yields the following result:

$$s(U_+ - U_-) - (V_+ - V_-) = 0(X)$$
$$2s(U_+ + U_-) - (V_+ + V_-) - 2W = 0(X)$$
$$s(V_+ + V_-) - 2sW - 2P = O(\lambda)$$
$$s^2(U_+ + U_-) - 2sW - 2P = O(\lambda)$$

That is, if the conditions [6.2-29..32] are met, then $u^{I*} \to 0$, and $u^\Pi \to u_\infty$. The conditions [6.2-29..32] can be rewritten as

$$s U_+ - V_+ = O(\lambda)$$
$$s U_- - V_- = O(\lambda)$$
$$P = O(\lambda)$$
$$s(U_+ + U_-) - sW = O(\lambda)$$

It is easy to see that the values corresponding to the rotelet satisfy [6.2-33..36] (or [6.2-29..32]) and that the values for the stokeslet do not satisfy the conditions given by [6.2-33..36] (or [6.2-29..32]). This brings the representation found in chapter 5 into question and requires the coining of a new way to represent the velocity fields. It would be very interesting to have a representation that would lead to the same system, but with a right hand side vector complying
with the conditions [6.2-33..36]. An 'easy' way to achieve this is to write the velocity in region (II) as the sum of a forcing velocity and a perturbation velocity. The perturbation velocity would be represented using the representation found in chapter 5. The forcing velocity field is the equivalent to the stokeslet, but with the Brinkman equation. So let us first construct what could be called a brinklet, an elementary solution to the Brinkman equation, that tends towards a rotelet when $\lambda \to 0$. 
6.3 : The search for a "brinklet"

6.3.1 : The Solution according to Oseen

We are looking for a solution to the following equation

\[ \Delta u_k - \lambda^2 u_k - \nabla p_k = \alpha \delta_0, \]  

[6.3-1]

where \( \alpha \) is a constant, and \( \delta_0 \) the Dirac distribution in zero. This solution is an elementary solution of the Brinkman equation corresponding to the \( k \)th coordinate axis. We will call it brinklet in reference to the construction of the name stokeslet.

Let us use the Oseen method (Oseen [15] pp25,26), to do so, we take \( u_k, p_k, \phi \) such that :

\[ u_k = \delta_{jk} \Delta \phi - \frac{\partial^2 \phi}{\partial x_j \partial x_k}, \]  

[6.3-2]

\[ p_k = -\frac{\partial}{\partial x_k} (\Delta \phi - \lambda^2 \phi), \]  

[6.3-3]

\[ \Delta (\Delta \phi - \lambda^2 \phi) = \alpha \delta_0. \]  

[6.3-4]

We would like \( u_k, p_k \) to be such that, when \( \lambda \to 0, u_k \to f_k \), the stokeslet corresponding to the \( k \)th coordinate axis (the same for the pressure fields). That is when \( \lambda \to 0 \) we would like

\[ u_k = \delta_{jk} \frac{1}{r} + \frac{x_k x_k}{r^3}, \quad r = \sqrt{x^2 + y^2 + z^2}, \]  

[6.3-5]

\[ p_k = \frac{-x_k}{r^3}. \]  

[6.3-6]

Let us take

\[ \phi = -2 \frac{1 - e^{-\lambda r}}{\lambda^2 r}. \]  

[6.3-7]

We have:
\[ u_{kk} = -2 \frac{3x_k^2 - r^2}{\lambda^2 r^5} + 2e^{-\lambda r} \frac{-2\lambda r^3 + \lambda^2 r^2 \left( r^2 - x_k^2 \right) + 3(\lambda r + 1)(r^2 - x_k^2)}{\lambda^2 r^5} \]  
\[ u_{kj} = -2 \frac{x_k x_j}{\lambda^2 r^3} \left( 3 - e^{-\lambda r} \left( \lambda^2 r^2 + 3\lambda r + 3 \right) \right), \text{ for } j \neq k \]  
\[ p_k = 2 \frac{x_k}{r^3} \]  

6.3.2: The limiting case

For small values of \( \lambda \) we have:

\[ u_{kk} = \left( \frac{1}{r} + \frac{x_k^2}{r^3} \right) - \frac{4}{3} \lambda + \frac{1}{4r} \left( 3r^2 - x_k^2 \right) \lambda^2 + \mathcal{O}(\lambda^2) \]  
\[ u_{kj} = \frac{x_k x_j}{r^3} - \frac{x_k x_j}{4r} \lambda^2 + \mathcal{O}(\lambda^2), \text{ for } j \neq k \]

It is clear that \( u_{kj} \rightarrow t_{kj} \) when \( \lambda \rightarrow 0 \). The convergences are uniform provided that \( r \geq \varepsilon > 0 \). For the pressure term \( p_k \) there is no need to explore the limits since it is always equal to the pressure for a stokeslet.
6.4 : A dual representation

Let us sketch an alternate way to obtain the solution to the problem of the asymmetric stokeslet

6.4.1 : The new representation

As already hinted in section two, a change of representation for the flow in region (II) is needed. This change is rewriting the velocity and pressure fields in region (II) as being the sum of a forcing field caused by a 'brinklet', the equivalent of a stokeslet for the Brinkman equation, and a perturbation field modeled using the initial representation, that is the one coined in chapter 5 and used in section one. We write

\[ u^I = u^I_\infty + z \text{grad} \Psi^I e_z + z \text{grad} \left( \frac{\partial X^I}{\partial z} \right) e_z + \text{grad} X^I + \text{curl} \left( \Phi^I e_z \right), \]  
\[ u^\Pi = u^\Pi_\infty + \text{grad} \Psi^\Pi + \text{grad} \left( \frac{\partial X^\Pi}{\partial z} \right) - \lambda^2 X^\Pi e_z + \text{curl} \left( \Phi^\Pi e_z \right) \]

\[ p^I = p^I_\infty + 2\mu \left( \frac{\partial \Psi^I}{\partial z} + \frac{\partial^2 X^I}{\partial z^2} \right), \]

\[ p^{\Pi} = p^{\Pi}_\infty - \mu \lambda^2 \Psi^{\Pi}, \]

where \( u^I_\infty \) is an asymmetric stokeslet, and \( u^{\Pi}_\infty \) is an asymmetric 'brinklet' such that, when \( \lambda \to 0, u^{\Pi}_\infty \to u^I_\infty \). If we now write

\[ u_\infty = u^I_\infty u^{\Pi}_\infty \]

\[ p_\infty = p^I_\infty - p^{\Pi}_\infty \]

we can conduct the same analysis as in section one with the velocity in region (I) (resp. region II) being \( u^I_\infty + u^I_\infty \) (resp. \( u^{\Pi}_\infty \)). We now proceed on to compute the actual values of the Hankel transforms \( U_+, U_-, V_+, V_-, W, P \). By looking at the expression defining a brinklet, and therefore \( u^{\Pi}_\infty \), it is obvious that, all the above mentioned Hankel transforms are of order \( \lambda \) when \( \lambda \)
Therefore the conditions \([6.2-33..36]\) are met. That is, using this representation, we have, for an asymmetric stokeslet

\[ u^1 \to 0 \text{ when } \lambda \to 0 \]  \[6.4-9\]

Alas the problem that was encountered when using the initial representation, as in section one, has changed.

6.4.2: The new problem

A problem arises when trying to compute the Hankel transforms leading to \(U_+\) etc. More precisely, we have to compute the Hankel transforms of order 0, 1 and 2 of functions having terms of the following form

\[
\frac{e^{\lambda r}}{r}, \frac{e^{\lambda r}}{r^3}, r = \sqrt{\rho^2 + h^2}
\]

A careful search in the literature (Gradshteyn, Watson, etc.) gives [Gradshteyn, equation 6.645] for \(v = 0,1,2\)

\[
\int_0^\infty \frac{e^{\lambda r}}{r} J_v(sp) dp = \int_0^\infty \frac{e^{\lambda r}}{r^2} J_v(s\sqrt{r^2 - h^2}) dr \]  \[6.4-10\]

\[
= h \int_0^\infty \frac{e^{-\lambda x}}{\sqrt{x^2 - 1}} J_v(h s \sqrt{x^2 - 1}) dx
\]

\[
= v \left(h \frac{\sqrt{\lambda^2 + s^2} - \lambda}{2} \right) K_v(h \frac{\sqrt{\lambda^2 + s^2} + \lambda}{2})
\]  \[6.4-11\]

These integrals are always defined, provided that \(\lambda > 0\). The complexity of the transforms found leads to two problems
• The expressions found for $U_+$ etc. are very difficult to manipulate even using symbolic manipulation packages such as Reduce or Macsyma, and their transcription to executable code is extremely tedious.

• The code produced to compute the value of $U_+$ etc. has proved to be unstable, that is an increase in the number of elements taken to compute the derived Bessel functions and their reverse Hankel transforms did not show a visible convergence even when using mathematical libraries such as NAG.

This rather disappointing result leaves two open problems. The first one is the question whether there is another way to represent solutions of the Brinkman equation that does not cause problems when $\lambda \to 0$. It was rather obvious from the start that, intuitively the model coined in chapter 5 had a very serious flaw (see the discussion made in section two §6.2.2). The second one is to know if it is possible to improve on the results already found by constructing a computing algorithm that converges when one wants to compute $U_+$ et al.
## Appendix 1: Density and viscosities for selected fluids

### Table [App. 1-1]: Liquids at 21.1°C (70°F):

<table>
<thead>
<tr>
<th>Name</th>
<th>Density ($\times 10^3$ kg/m$^3$)</th>
<th>Dyn. vise. ($\times 10^{-3}$ Pa s)</th>
<th>Kin. vise. ($\times 10^{-6}$ m$^2$/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon tetrachloride</td>
<td>1.582</td>
<td>0.956</td>
<td>0.604</td>
</tr>
<tr>
<td>Commercial solvent</td>
<td>0.717</td>
<td>0.839</td>
<td>1.17</td>
</tr>
<tr>
<td>Heavy fuel oil</td>
<td>0.908</td>
<td>132.6</td>
<td>148</td>
</tr>
<tr>
<td>Medium fuel oil</td>
<td>0.854</td>
<td>3.27</td>
<td>3.83</td>
</tr>
<tr>
<td>Medium lub. oil</td>
<td>0.891</td>
<td>103</td>
<td>116</td>
</tr>
<tr>
<td>Regular petrol</td>
<td>0.724</td>
<td>0.464</td>
<td>0.641</td>
</tr>
<tr>
<td>Water</td>
<td>0.998</td>
<td>0.997</td>
<td>0.984</td>
</tr>
</tbody>
</table>

### Table [App. 1-2]: Gases at 20°C (68°F) and 1013 mbar (1.013×10$^5$ Pa):

<table>
<thead>
<tr>
<th>Name</th>
<th>Density (kg/m$^3$)</th>
<th>Visc. ($\times 10^{-5}$ Pa s)</th>
<th>Kin. visc. ($\times 10^{-5}$ m$^2$/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>1.204</td>
<td>1.80</td>
<td>1.49</td>
</tr>
<tr>
<td>Ammonia</td>
<td>0.718</td>
<td>1.10</td>
<td>1.53</td>
</tr>
<tr>
<td>Carbon dioxide</td>
<td>1.841</td>
<td>1.56</td>
<td>0.845</td>
</tr>
<tr>
<td>Methane</td>
<td>0.667</td>
<td>1.12</td>
<td>1.79</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>1.165</td>
<td>1.85</td>
<td>1.59</td>
</tr>
<tr>
<td>Oxygen</td>
<td>1.329</td>
<td>2.11</td>
<td>1.59</td>
</tr>
</tbody>
</table>

### Table [App 1-3]: Liquids at 20°C (68°F):

<table>
<thead>
<tr>
<th>Name</th>
<th>Density ($\times 10^3$ kg/m$^3$)</th>
<th>Visc. ($\times 10^{-3}$ Pa s)</th>
<th>Kin. visc. ($\times 10^{-6}$ m$^2$/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benzene</td>
<td>0.879</td>
<td>0.655</td>
<td>0.745</td>
</tr>
<tr>
<td>Castor (Ricinus) oil</td>
<td>0.960</td>
<td>989</td>
<td>1030</td>
</tr>
<tr>
<td>Ethanol</td>
<td>0.789</td>
<td>1.20</td>
<td>1.53</td>
</tr>
<tr>
<td>Glycerin</td>
<td>1.262</td>
<td>834</td>
<td>661</td>
</tr>
<tr>
<td>Linseed oil</td>
<td>0.925</td>
<td>33.2</td>
<td>35.9</td>
</tr>
<tr>
<td>Turpentine</td>
<td>0.862</td>
<td>1.49</td>
<td>1.73</td>
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</tbody>
</table>
Bibliography


