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A SCATTERING THEORY MODEL OF THE DIELECTRIC AERIAL

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by

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ABSTRACT

Following a critique of previous theories of the dielectric aerial, a theoretical model is proposed based on the concepts of scattering theory. It is shown that the model is essentially an extension of the theory of polarization of dielectric materials.

Application of the scattering model yields an inhomogeneous Fredholm integral equation, the solution of which is the aerial radiation pattern. Explicit equations are deduced for both TM_{01} and HE_{11} mode excitation of solid dielectric rods and numerical solutions have been obtained. The solutions compare favourably with those of previous methods. Extensions of the theory to other dielectric structures, such as tubes and inhomogeneous rods, are also discussed.
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References
This thesis comprises three main parts. These are (i) a critique of previous work in the field. (ii) The development of the scattering theory model. (iii) An assessment of the results obtained from the new model.

Chapter I is a critique of all the major work in the field up to 1967. While discussions of this earlier work exist the present discussion aims to bring out certain features which have not been previously covered. A major contribution to the subject since 1967 is to be found in the publications of J.R. James Chapter 2 has therefore been devoted to a study and assessment of both his controversial views on the Kirchhoff-Huyghens theory, and on his extension of the two-aperture theory of the dielectric rod.

Chapter 3 presents what is believed to be an entirely new approach to the theory of the dielectric rod aerial which is to regard it as a scattering problem. From the model adopted it is shown that the radiation pattern can be found from the solution of an inhomogeneous fredholm integral equation. Explicit forms of the equation are deduced for both TM_{01} and HE_{11} mode excitation. It is also shown how the equations may be modified in order to include the dielectric tube aerial.

In chapter 4 the numerical solution of the integral equations is discussed and an equivalent set of linear simultaneous equations is derived both for the TM_{01} and HE_{11} excitations. The results of the computations are compared with experimental determinations.
The final chapter critically examines the results obtained in order to assess to what extent the model is successful and concludes with a discussion of the possible extensions to the theory.
1.1 Introduction

Although the dielectric rod aerial in its various forms has been successfully employed since the 1940's, the design process is essentially empirical and a completely satisfactory theory still remains to be formulated. The different treatments of the problem which have appeared to date are each in some way open to criticism. James (1,2) has given critical reviews of work in this field but his own work is itself open to criticism. There have been two approaches to dielectric rod theory which may be described as the "continuous radiation along the length of the rod" approach and the "radiation from terminal discontinuities" approach. Not unnaturally a controversy has arisen as to which has the correct interpretation of the radiation mechanism of the dielectric rod. James (29,30) has taken a prominent part in this debate which Zucker (31) has described as "a quite unnecessary controversy".

The present work will outline and comment upon the work in this field and will present an alternative approach to the theory of the dielectric rod.

1.2 Mallach's Theory

The first theoretical model of the dielectric rod aerial appeared during World War II and is due to Mallach (3). It is the archetype of the "continuous radiation along the length of the rod" theories. The Mallach model consists of an elementary plane wave
travelling at some angle with respect to the rod axis and propagating along the axis due to reflections at the dielectric air interfaces, (fig. 1.1). It is assumed that only part of the energy is reflected while the rest is radiated. The radiation pattern is calculated by applying the Kirchoff-Huyghens principle and assuming that the rod surface can be replaced by a linear array of elementary sources with a phase shift \( \phi \) between successive sources along the length of the rod. This phase shift is calculated by considering the geometrical path difference between sources as shown in (fig. 1.1) and is readily seen to be

\[
\Delta \phi = \frac{2\pi z}{\lambda} - \frac{2\pi z}{\lambda_0} \cos \frac{2\pi z}{\lambda_0} (K - \cos \theta) \quad (1.1)
\]

where \( \lambda_0 \) is the wavelength in free space and \( \lambda \) is the wavelength in the dielectric. The radiation pattern is then calculated using standard array theory on the assumption that the amplitude of the elementary sources are identical. If \( \Delta E \) is the field due to an elementary source at the point of observation then that due to \( n \) sources with successive phase shifts \( \Delta \phi \) will be given by

\[
\begin{align*}
E_p & = \Delta E \left[ 1 + e^{i\Delta \phi} + e^{2i\Delta \phi} + \ldots + e^{i(n-1)\Delta \phi} \right] \\
& = \Delta E \left[ \frac{1 - e^{i\Delta \phi}}{1 - e^{i\Delta \phi}} \right] = \Delta E \left[ \frac{\sin(n\Delta \phi/2)}{\sin(\Delta \phi/2)} \right]
\end{align*}
\]

If \( L \) is the length of the rod and putting \( E = E/n \) then as \( n \) we can write this expression in its final form

\[
\frac{\sin \left( \frac{\pi L}{\lambda_0} \right)}{\sin \left( \frac{\pi L}{\lambda_0} \right) (K - \cos \theta)} \quad (1.2)
\]
The chief objection to this formulation is that if the energy is being lost at successive reflections then the elementary sources cannot be of constant strength but must diminish with distance along the rod. This in turn implies that the radiation pattern would eventually become independent of the length of the rod. Yet in spite of this the expression (1.2) is capable of explaining in broad terms the observed features of experimental radiation patterns. Furthermore this expression appears repeatedly in subsequent treatments of the problem.

1.3 Watson and his Collaborators

The contributions of these workers \(^{(4,5,6)}\) represent the first attempt to treat the dielectric rod as an electromagnetic field problem. Their approach was to replace the dielectric rod by an equivalent source distribution of surface electric and magnetic currents flowing over the external surfaces of the rod. The equivalent current sources are found from the actual surface electric and magnetic fields by employing the formulae

\[
\begin{align*}
\mathbf{J} &= \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) ; \\
\mathbf{M} &= \mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2)
\end{align*}
\] (1.3)

where the suffices refer to the total field values adjacent to the bounding surface.

The radiation pattern is found by applying the vector potential definitions

\[
\begin{align*}
\mathbf{A} &= \frac{1}{4\pi} \int_S \left[ \mathbf{e}^{-ikr} \right] \mathbf{r} ds \\
\mathbf{A}^m &= \frac{1}{4\pi} \int_S \left[ \mathbf{e}^{-ikr} \right] \mathbf{r} ds
\end{align*}
\] (1.4)
FIG. 1.1. RAY PATHS AND PHASE RELATIONS IN MALLACH'S MODEL.

FIG. 1.2. SUMMARY OF WATSON AND HORTON'S RESULTS ON RECTANGULAR RODS.
The great disadvantage of this approach is that the surface fields must be known. In order to find the surface fields the boundary value problem for the system must be set up and solved. However, if this were done, and it is a formidable problem, it would automatically yield the radiation pattern thus obviating the necessity of the equivalent source approach. Since the rigorous solution is not easy to obtain then equations (1.3), (1.4) and (1.5) may be employed to give an approximate radiation pattern by guessing the form of the fields at the rod surface.

1.3.1 The Rectangular Rod

Watson and Horton's original paper \(^{(4)}\) reported investigations which they carried out on partially tapered rectangular rods fed by a rectangular metal waveguide excited in the TM\(_{01}\) mode. This paper is still of interest since it represents the only available treatment to date of the rectangular dielectric rod. This is probably due to the fact that a complete solution of the rectangular dielectric waveguide has only recently been achieved \(^{(32)}\). Thus their treatment is necessarily approximate and they assume that the field in the rod is virtually identical to the feed waveguide TM\(_{01}\) mode. The radiated field is found by applying equations (1.3) to (1.5) to each side of the rod for each type of equivalent current. The manner in which they take into account each contribution to the total radiated field is inconsistent. This is highlighted
by their consideration of the magnetic current contributions from the narrow faces and the broad faces. They argue that in the case of the narrow faces, since the electric field is zero on these faces in the feed waveguide, the field at the rod surface "cannot be of appreciable magnitude" and therefore their contribution to the radiated field is negligible. By contrast there is no electric field component tangential to the broad faces and hence no magnetic current, yet they argue that since there is appreciable radiation, there must be such a component. It is suggested that this component can be produced by the presence of the tapered section where the transition from plane wavefronts to cylindrical wavefronts can create the necessary components in the rectangular section. This artificial device, apart from the question of its validity, would only seem to be effective in the vicinity of the transition effect. Yet this component is assumed to exist uniformly along the length of the rod. In hindsight this assumption seems quite surprising since although the complete solution for the rectangular dielectric rod was not known, nonetheless it had been realised that this structure could only propagate hybrid modes and thus provide the sought-for tangential electric field component.

The equivalent magnetic current distribution they used is given by

\[ M = i M_0 \cos \left( \frac{Ty}{b} \right) \sin \left( \frac{2\pi}{\lambda_g} z \right) \]  

(1.5)

This standing wave distribution arises not from total reflection at the free end of the rod, which would be highly unlikely even without
the tapers which were actually used, but by imposing a condition that the tangential field must be zero at the feed waveguide aperture.

It is evident that Watson and Horton were somewhat unhappy about this part of their work since they published soon afterwards (5) a modified treatment in which they state "- since there is appreciable radiation occurring it might be more appropriate to approximate these currents by an expression representative of travelling waves". However since the results of their computations for the travelling wave case differed insignificantly from the standing wave case, they did not repudiate their previous work.

The experiments themselves were carried out on a large number of tapered rod combinations of various lengths. The form of the tapers were such that the average thickness of all rods was approximately the same. The information given in the paper is not sufficient to make detailed comparisons between particular rods and observed patterns. In summary (fig. 1.2) agreement between theory and experiment is achieved for rods of lengths between 3\( \lambda \) and 6\( \lambda \) long, but agreement outside these limits is poor, especially for longer rods.

1.3.2. \( \text{TM}_{01} \) Excitation of the Circular Rod

Since the \( \text{TM}_{01} \) excitation of the circular rod is a well-defined and soluble boundary value problem it was a logical step to use this field as an approximation to the true field. Horton, Karal and McKinney (6) carried out measurements of the surface field distributions on rods whose lengths varied from 2\( \lambda \) to 10\( \lambda \).
An amplitude variation dependent on the length of rod used was noted as well as a small superimposed standing wave. In order to take this variation into account the integration of the surface field along the length of the rod is modified by the introduction of a so-called "shading factor" $W(z)$. This is defined as

$$W(z) = d + (1-d) \sin \left( \frac{\pi z}{L} \right)$$

where $0 \leq d \leq 1$

and the integral used is

$$\int_{0}^{L} W(z) e^{-i \kappa z} \, dz$$

The value of $d$ used was determined empirically as that value which gave the best agreement with the experimental radiation patterns. The remainder of the paper is used to demonstrate that the best agreement with experiment is achieved if the surface of integration is taken to be a cylinder of a radius somewhat less than the physical radius. This radius is typically about 0.65 of the true radius, but depends upon the length of the rod used. These authors attribute the "excellent agreement" between theory and experiment to the knowledge of the surface fields which was not the case with the rectangular rod. In fact the agreement is only achieved by the introduction of a concept with no physical justification - the effective radius and an empirically adjusted value of the shading function. James (1,2) has criticised this procedure on the grounds that any theory which has adjustable parameters, which must be determined by experiment for each particular case, cannot be said to be a theory at all.
Apart from this overall criticism there are further detailed points to be made concerning the methodology of their analysis. They state that they chose the particular form of the shading function (1.6) because Mueller and Tyrell \(^{(11)}\) had found it useful in explaining their results on dielectric aerials, and also because in the special case of \(d = 0\) it reproduced the standing wave distribution which Watson and Horton \(^{(4)}\) had adopted in their original paper (and which has subsequently been commented upon in 1.3.1 above).

A careful perusal of Mueller and Tyrell's paper shows that nowhere do they use \(W(z)\) to explain their experimental results. It only arises in their preliminary discussion of the properties of the travelling wave aerial where they discuss the effect of such a distribution on the radiation pattern. In the same sense they also discuss the pattern due to an exponential field amplitude distribution with distance along the aerials. Horton, Karal and McKinney's claim that the standing wave distribution is reproduced when \(d = 0\) is seen to be incorrect. In this case \(W(z)\) is simply a half sinusoid regardless of the length of the rod and can never reproduce the true standing-wave pattern. By the same token the empirically determined values of \(d\) which give the best agreement with the experimental patterns predict surface distributions which bear no relationship at all to the observed surface fields. It is surprising that the authors never apparently made this comparison.

Their analysis gives rise to the following expression for the far-field pattern

\[
E_p(\theta) = I_1 I_2
\]  
(1.8)
where

\[ I_1 = J_0(k\sin\theta) \sin\theta - C J_1(k\sin\theta) \]

\[ I_2 = \frac{1}{2} \int_0^L W(z) e^{iQz} \, dz = \frac{Id}{2} \left[ \frac{\pi D\cos\phi}{2(\pi)^2 - \phi^2} \right] \]

\[ \phi = \frac{kL}{2} (K - \cos\theta); \ D = \frac{1-d}{d} \]

When the empirical adjustments have been made agreement between the expression (1.8) and experiment is not perfect. The analytical expression predicts null values between successive lobes whereas only minima are observed in practice.

1.4 Chatterjee and Co-Workers

These workers describe how the surface current procedure can be extended to the circular cylindrical rod propagating the HE\textsubscript{11} mode \((12,13,14,16)\). In the first paper \((14)\) the field components of the HE\textsubscript{11} mode are stated and the equivalent surface electric and magnetic current densities are deduced. The authors conclude that only the \(\phi\)-components of these currents contribute to the radiated field. The integration required to yield the vector potentials \(\mathbf{A}\) and \(\bar{A}^m\) is carried out over the curved surface of the cylinder but neglecting the end surface. They obtain the following expressions for the radiation pattern

**E-plane**

\[ \frac{E_p}{\omega_0} = \omega_0 \kappa^H_1 \left[ \frac{\sin\left(\frac{\pi L}{\lambda_0} (K - \cos\theta)\right)}{\frac{\pi L}{\lambda_0} (K - \cos\theta)} \right] 2\pi J_1 \left( \frac{k a}{2 - \sin\theta} \right) \phi \]

\[ + \kappa \kappa^E_1 \left[ \frac{\cos\left(\frac{\pi L}{\lambda_0} (K - \cos\theta)\right)}{\frac{\pi L}{\lambda_0} (K - \cos\theta)} \right] 2\sqrt{2}\pi \frac{\pi}{k a} J_1 \left( \frac{k a}{2 - \sin\theta} \right) \theta \]
These expressions are certainly incorrect as witnessed by the presence of the terms involving \( J_1(\frac{k_0}{2}\sin\theta) \) and both \( \theta \) and \( \phi \) field components in each principle plane. The errors appear to arise from the incorrect resolution of the vector potential components and the incorrect integration of the vector potential with respect to the \( \phi \)-coordinate. In the final paper (16) of the series the analysis includes the effect of radiation from the rod end. It is interesting to note that the expression which they offer for the contribution due to the cylindrical surface is completely different from (1.9) yet this is not commented upon. Their later expression for the E-plane pattern is

\[
E = \left\{ k J_1(k_0) \right\} \left[ \frac{G J_1'(k_0\sin\theta)}{k_2} + \frac{J_1(k_0\sin\theta)}{k_2} \right] + \frac{k_2}{k_0} \left\{ J_1(k_0\rho) + \frac{k_2}{k_0} J_1'(k_0\rho) \right\} \left( \frac{G\sin(\frac{\pi L}{d_0}(K - \cos\theta))}{\frac{\pi L}{d_0}(K - \cos\theta)} \right)
\]

One paper of the series (14) is devoted to an experimental study of the rod propagating the HE_{11} mode and aimed at the verification of the predictions of the earlier paper, namely equations (1.9). The
FIG. 1.3. CHATTERJEE'S RESULTS. \( L = 2 \lambda \), \( d = 0.5 \lambda \).

FIG. 1.4. FRADIN'S RESULTS.

\[ \begin{align*}
L &= 3 \lambda \\
d &= 0.46 \lambda \\
L &= 4 \lambda \\
d &= 0.46 \lambda
\end{align*} \]
interesting feature of this part of their work is that their verification consists of comparing the measured and predicted positions of the sidelobe maxima and minima for rods of various lengths between 2\(\lambda\) and 10\(\lambda\). Not surprisingly they find good agreement owing to the dominant effect of the linear array factor

\[
\sin \left( \frac{2\pi l}{\lambda}\frac{(K - \cos \theta)}{K - \cos \theta} \right)
\]

which necessarily plays a significant part of any formulation. However they do not comment on the fact that their expression (1.9) predicts pattern zeros whereas only minima are observed. They are even able to overlook the fact that for the only radiation pattern reproduced (for the 2\(\lambda_o\) rod) neither the main lobe beam width nor the side-lobe levels agree with the predicted pattern. These results are summarized in table 1 below.

<table>
<thead>
<tr>
<th>TABLE 1</th>
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<tbody>
<tr>
<td>EQN (1.8)</td>
</tr>
<tr>
<td>BEAM WIDTH</td>
</tr>
<tr>
<td>SIDE LOBE</td>
</tr>
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</tbody>
</table>

In view of these comments it would seem reasonable to conclude that the Chatterjee contribution is not a very valuable one.

1.5 Fradin's Approach

Fradin (8) has introduced what he calls the 'second equivalence
principle' by which the radiation pattern of a dielectric rod may be calculated from an equivalent current system distributed throughout the volume of the rod. If $E$ is the field at any point within the dielectric then the equivalent volume current density $J_{eq}$ is given by

$$J_{eq} = i\omega (\varepsilon - \varepsilon_0) \vec{E}$$  \hspace{1cm} (1.11)

where the field quantities vary with time as $e^{i\omega t}$. The radiation pattern is found by applying the vector potential relationships (1.4) and (1.5) and Fradin deduces the following expression for the $E$-plane pattern

$$\vec{E}_E = 30k\omega (\varepsilon - \varepsilon_0) \cos \theta \int_0^{2\pi} \int_0^L \int_0^\infty \frac{e^{-ikr}}{r} \rho d\rho d\phi dz$$  \hspace{1cm} (1.12)

The attractive feature of this approach is that, unlike the equivalent surface current formulation, it is closely related to its physics of the radiation process since $J_{eq}$ is simply the polarization current density of the material. In practice this approach offers no advantage over the surface current approach since the field distribution within the rod has to be specified in the same arbitrary manner.

Fadin considers a rod excited by the circular waveguide TE$_{11}$ mode. It is interesting to note, however, that he does not use the HE$_{11}$ field within the rod to calculate the radiation pattern. Instead he proceeds to deduce a completely empirical field distribution. He starts with a field distribution of the form

$$\vec{E}_y = A(z) f(\rho, \phi) e^{-iyz}$$  \hspace{1cm} (1.13)
where $\gamma$ is propagation constant for the $HE_{11}$ mode while $f(\rho, \phi)$ is taken as the feed waveguide $TE_{11}$ mode distribution. He adopts an axial variation $A(z) = A_o e^{-\beta z}$ and argues that since radiation implies energy loss then $\beta > 0$. The definitive expression for the far-field pattern is found by substituting the explicit expressions for (1.13) into (1.12) and using the usual transformation

$$r = R - z \cos\theta - \rho \sin\theta \cos(\phi - \phi')$$  \hspace{1cm} (1.14)

for the far field approximation. Fradin actually uses the incorrect $r = R - z \cos\theta - \rho \sin\theta$ and carries this through the corresponding integrals. However the results of the integrations are correctly given and yield the final expression

$$E_E = \cos\theta \frac{J_{1/2}(ka \sin\theta)}{(ka/2 \sin\theta)} \sinh \frac{[\frac{B_L}{2} + i \frac{kL}{2} (K-\cos\theta)]}{[\frac{B_L}{2} + i \frac{kL}{2} (K-\cos\theta)]} \sinh[\frac{B_L}{2} + \frac{i}{2} kL (K-1)]$$  \hspace{1cm} (1.15)

In order to compute the pattern the exponent $\beta$ must be determined. Fradin deduces a value for $\beta$ by noting the maximum to minimum ratio for the sidelobes of experimental patterns and deduces a value such that $\beta L/2 = 0.5$. Unfortunately even with this empirical adjustment the comparison of the computed patterns with the experimental patterns is seen (Fig. 1.4) to be not very impressive. However Fradin's contribution is notable for his attempt to account for the observed pattern minima.

1.6 Brown and Spector

A completely different approach to the problem is offered by
Brown and Spector (10) who consider the dielectric rod as a member of the class of surface wave aerials. Following earlier work by Zucker (17) on the radiation mechanisms of surface wave aerials they argue that the travelling wave supported by the rod is a surface wave mode which propagates without attenuation and hence cannot lose energy continuously by radiation as it propagates. They consider that a small amount of energy is radiated at the feed aperture discontinuity. The majority of the incident energy is propagated by the surface wave mode without loss to the terminal end of the rod where, apart from a small reflection, it is radiated. The observed radiation pattern results from the superposition of the feed end and terminal end radiation with due account taken of the path difference of these two sources to the point of observation.

In order to calculate the pattern the ends of the rod are used to define equivalent aperture planes over which field distributions are assumed. Standard aperture field integration formulae (22) are applied to give the radiation from each discontinuity and the total pattern can be found provided that the relative proportions of the total energy radiated from each discontinuity is known.

Brown and Spector of necessity proceed empirically. They assume that the field distribution over the terminal and aperture is the surface wave mode, the $E$ components of which are given for the radial distances less than ($\rho < a$) and greater than ($\rho > a$) the radius of the rod

$$E_\rho = \begin{cases} \frac{J_1(k_{\rho_1} \rho)}{A} + B \frac{J_1(k_{\rho_1} \rho)}{k_{\rho_1} \rho} & \sin \phi \ e^{-\gamma z} \\ A J_1(k_{\rho_1} \rho) + B \frac{J_1(k_{\rho_1} \rho)}{k_{\rho_1} \rho} & \cos \phi \ e^{-\gamma z} \end{cases}$$

$$E_\phi$$
FIG. 1.5: BROWN AND SPECTORS RESULTS FOR CYLINDRICAL RODS OF DIAMETER 0.46 λ₀.
In order to calculate the radiation from the feed end discontinuity the equivalent aperture field distribution was assumed to be that of the undisturbed feed waveguide mode - an assumption which these authors acknowledge as being probably unrealistic. The fraction of the total power radiated from each aperture was deduced by assuming a set of values for this quantity and selecting that which gave a computed pattern which gave the best agreement with experimental values. This was found to be of the order of 6% of the total from the feed end aperture. No analytical expressions for the radiation pattern is given but some computed patterns are reproduced (fig. 1.5). It can be seen that detailed agreement between the computed and experimental patterns is not very good but the general features of the experimental patterns are reproduced. In particular the predicted patterns exhibit minima.

Brown and Spector's work has been attacked in the final paper of Chatterjee et al's series previously referred to. They argue that the radiation pattern is the sum of contributions radiated from the length of the rod and from its end surface and describe a series of experiments which they claim supports their assertion. They obtain the radiation pattern eq. (1.10) from the HE_{11} mode which is taken as the source field - derived by solving Maxwell's equations with proper boundary conditions but IGNORING RADIATION FROM THE ROD. This apparent incompatibility of assumptions is
never resolved.

Their experimental investigations include a series of measurements of the radiated field intensity at a distant point on the rod axis and of the standing wave patterns along the rod surface with the free end wholly or partially covered by metal foil discs or annuli. Variations in the radiated power were observed but it is obvious that such experiments are inherently incapable of deciding between the two postulated mechanisms since the foil covering is itself a discontinuity. Their argument is valueless and confidence in their work is further reduced by the completely erroneous expression which they obtain for the ratio of its received powers when the rod end is covered and uncovered. From standard transmission-line theory the power reflected from the termination is given by

\[ P_R = \left( \frac{S - 1}{S + 1} \right)^2 \]

and the fraction of power transmitted is

\[ P_T = 1 - P_R = \frac{4}{2 + \frac{1}{S} + S} \]

where \( S \) is the standing wave ratio.

Now the incident energy is carried partly within the dielectric \( (P_i) \) and partly by the surface wave field \( (P_o) \) so that

\[ P_i + P_o = 1 \]

These authors make a fundamental error by assuming that a metal disc completely covering the end of the rod will reflect the power
carried within the rod completely while not affecting the power carried by the surface wave. They use this assumption to state that

$$\frac{P_i}{P_o} = \frac{P_R}{P_T}$$

and proceed to compare the values of $\frac{P_R}{P_T}$ deduced from their standing wave measurements with the theoretical values of $\frac{P_i}{P_o}$ which can be deduced from the field configurations within and exterior to the rod. Not surprisingly there are large discrepancies between the values of these two ratios. These authors conclude that the assumption that the rod behaves as a non-radiating surface-wave transmission line which was used to calculate $\frac{P_i}{P_o}$ is not justified.

They further derive expressions for the power received when the rod end is partially covered. The total field at a point on the axis is given by the sum of the contributions from the rod and $E_1$ and the cylindrical surface $E_2$, i.e.

$$E = E_1 + E_2$$

where

$$E_1 = \int_0^{\infty} k_\rho \rho J_0(k_\rho \rho) d(k_\rho \rho)$$

and

$$E_2 = \frac{\sin \left( \frac{\pi L}{\lambda_0} (K-\cos \theta) \right)}{\left( \frac{\pi L}{\lambda_0} (K-\cos \theta) \right)}$$

If the rod end is partially covered to a radius $\rho < 0$ they state erroneously that the contribution from the end will be given by

$$\int_0^{\rho} k_\rho \rho J_0(k_\rho \rho) d(k_\rho \rho) = \frac{1}{\omega E_1}$$
while incorrect for a disc this would be appropriate for an annulus. The fraction of power received is then given by

\[
\frac{[(1 - \alpha) E_1 + E_2]^2}{[E_1 + E_2]^2} \quad \text{for discs}
\]

and

\[
\frac{[\alpha E_1 + E_2]^2}{[E_1 + E_2]^2} \quad \text{for annuli}
\]

Values of \( \alpha \) as a function of \( \rho \) are not reported and despite the presentation of several graphs and tables it is difficult to correlate the various results presented. However in one case a theoretical curve is given which indicates that when the end is completely covered the received power is 68% of the power when the end is completely uncovered, whereas the experimental value was about 25%. The authors are quite unable to account for the discrepancy and we must conclude that their attack on Brown and Spector, as presented, has no substance.

1.7 Concluding Remarks

The papers just discussed reflect the state of knowledge up to 1967. They have all approached the problem in a semi-empirical manner and close investigation has revealed inconsistencies in assumptions and errors in the physics and the mathematics of some of the work. The broad agreement between the computed and experimental patterns is due to the presence of the linear array factor (1.2) in all formulations. However good agreement is only achieved by arbitrary modifications.
After a full discussion of the recent work of J.R. James in the next chapter I shall suggest an alternative approach to the problem based on scattering theory. This will be seen to entail the minimum of assumptions and will, in effect, attempt to solve the boundary value problem.

Note Added in Proof

From a further consideration of Brown and Spector's results shown in Fig. (1.5) the lack of agreement shown for the case of the $2\lambda_0$ rod is unnecessarily severe since it is apparent that they have erroneously compared their calculations for this case with the experimental values appropriate to a rod of length $3\lambda_0$. 
CHAPTER 2

A Critique of J.R. James' Work

2.1 Introduction

The dielectric aerial debate was re-opened after a lapse of some years by the work of J.R. James (1,2,17,18,19,29). He has developed a useful critique of some of the earlier work discussed in the previous chapter and has expounded his own views in a series of papers and technical reports and culminating in his doctoral thesis (1). He accepts the end-discontinuity theory and gives considerable attention to what has been the least satisfactory part of the theory, namely the radiation from the feed end discontinuity. He also extends his arguments to a consideration of tapered dielectric rods.

2.2 The Schelkunoff Equivalence Principle and the Vector Kirchoff Formula

2.2.1 A Justification of the End Discontinuity Approach

In discussing Brown and Spector's paper James (1) notes that these authors deduce their radiation pattern by using the fictitious current sources (eqn. 1.3)

\[ \mathbf{J} = \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) ; \quad \mathbf{M} = \mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) \]  

(1.3)

and claims that this is consistent with the end discontinuity postulate. He says "the physical discontinuity which is postulated at the rod end is expressed mathematically by (eqns. 1.3)". But the
discontinuity expressed by (eqns. 1.3) is an artefact of the fictitious source representation and exists for any chosen bounding surface. It therefore cannot be used as an argument to justify its equation with the "physical discontinuity at the rod end".

2.2.2. James' Analysis of the Schelkunoff Equivalence Principle

James commences his investigation by showing that the Schelkunoff equivalence principle and the vector-Kirchhoff equations are equivalent.

Starting with eqn. (1.5)

\[
\mathbf{E} = -i\omega \mu_0 \mathbf{A} + \left( \frac{1}{i\omega \epsilon_0} \right) \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \mathbf{A}^m \\
(1.5)
\]

he substitutes the vector potential definitions (1.4) but takes into account volume distributions of current. This yields

\[
4\pi \mathbf{E} = -i\omega \mu_0 \int_{V} J_d \, dv + \frac{1}{i\omega \epsilon_0} \nabla \left( \nabla \cdot \int_{V} J_d \, dv \right) - \nabla \times \left[ \int_{V} \nabla \psi \, dv \right] \\
- i\omega \mu_0 \int_{S} J_s \psi \, ds + \frac{1}{i\omega \epsilon_0} \nabla \left( \nabla \cdot \int_{S} J_s \psi \, ds \right) - \nabla \times \left[ \int_{S} M_s \psi \, ds \right] \\
(2.1)
\]

where \( \psi = \frac{e^{-ikr}}{r} \).

By applying some well known vector identities he transforms eqn. (2.1) into the corresponding vector Kirchhoff equation in its most general form,
An important feature of James' argument is that the vector potentials in eqn. (1.5) and hence the expressions derived from them are solutions of the inhomogeneous wave equations

\[ \nabla^2 \begin{bmatrix} A_m^i \\ A_m^o \end{bmatrix} = -\omega^2 \mu \epsilon \begin{bmatrix} A_m^i \\ A_m^o \end{bmatrix} - \begin{bmatrix} J_m \\ M_m \end{bmatrix} \] (2.3)

and he argues that the significance of this has not been previously realised since the dielectric waveguide mode used as the source field by previous authors is derived from the solution of the homogeneous wave equation

\[ \nabla^2 \begin{bmatrix} A_m^i \\ A_m^o \end{bmatrix} = -\omega^2 \mu \epsilon \begin{bmatrix} A_m^i \\ A_m^o \end{bmatrix} \] (2.4)

The core of his argument is contained in the following somewhat lengthy quotation from his thesis. He writes

"... the significance lies in the fact that the regions occupied by sources are regions of field discontinuity and this is expressed for surface sources in eqn. (1.3). Now \( E \) and \( H \) in the surface integral of eqn. (2.2) satisfy \( (J^S = n \times H ; M^S = n \times E) \) and are discontinuous on \( S \). We may ask what the consequences will be if we integrate with FIELDS THAT ARE CONTINUOUS ACROSS \( S \) and the
outcome will be that radiation may be predicted from non-radiating surfaces. For instance it is well established that waves gliding along a continuous dielectric rod do not radiate, yet there is no condition which prevents the application of eqn. (2.2) to any arbitrary surface within the guided waves and THIS WILL RESULT IN THE PREDICTION OF FAR FIELD RADIATION. In this case it appears that the guided waves are continuous everywhere and are deduced from the homogeneous vector potential wave equations as opposed to eqn. (2.2) which, as a consequence of eqn. (1.3), asserts that over the surface of integration the fields are discontinuous .... although eqn. (2.3) is a sufficient restriction in the above example we find in general that it presents an apparently impossible situation because physically existing fields are always continuous everywhere. That this is so is well substantiated by field measurements and the fact that the mathematical boundary conditions acknowledge the continuous nature of realistic fields. We are confronted by the existence of a condition eqn. (2.3) which implies that the integration of physically realistic fields in the vector kirchhoff formulation is NOT STRICTLY VALID IN THE MATHEMATICAL SENSE".

James has argued somewhat differently elsewhere (2). In a volume containing current sources the vector potential satisfies the inhomogeneous equation (2.3) "The homogeneous equation (2.4) is only applicable when the spaces occupied by the sources are excluded from the range of integration in which case THE FIELD WILL BE ZERO I.E. THERE WILL BE NO RADIATION FIELD". In other words he is suggesting that by enclosing a source with a hypothetical surface the source is "switched off"! He continues "From a physical standpoint this amounts to the obvious fact that surfaces or volumes must not be subjected to radiation formulae unless it is certain that they cause radiation". This last statement has been criticised elsewhere ( ).
2.2.3 Remarks on Equivalent Sources and the Kirchhoff-Huyghens Principle

This is a convenient point to re-assert certain features of the equivalence principles. The principles state that if the field distribution is known over a closed surface then the field at ANY exterior point can be calculated provided that the surface field satisfies certain continuity conditions. The principles are thus more general than being simply radiation formulae. As James says, the real field is necessarily continuous across the hypothetical surface whereas the equivalent current distribution over this surface which produces the identical field necessarily gives rise to a field distribution which is discontinuous across the chosen surface. Both the real field and the field due to the equivalent field distribution, being identical, are solutions of the homogeneous wave equation exterior to, BUT EXCLUSIVE OF, the boundary surface. The solution of the inhomogeneous equation (2.3) however is appropriate BOTH to the exterior AND the boundary surface itself (23). The fact that the surface wave is a solution of the homogeneous wave equation cannot of itself exclude such a field as a candidate for an equivalent source distribution.

The actual surface chosen to evaluate the distant field is purely a matter of convenience and identical results should be obtained whatever the form of surface employed. Iiguchi (37) however has claimed that different surfaces of integration do not give identical results and that the end-aperture approach is superior. He attributes this to a better knowledge of the field over this surface despite the fact that his predicted pattern, having no side-lobe structure, does not reproduce experimental results, whereas integration along the
length of the rod does produce an approximation to the observed pattern. The expected equivalence between the end plane aperture and the SEMI-INFINITE rod has been verified by other workers \(10,31,35\). The basic reason why radiation is predicted when the surface wave approximation is used is because a system of finite length is being considered. By definition an infinite rod supporting a surface wave does not radiate and if the surface integrations are carried out correctly then the equivalence principles will correctly predict zero radiation.

2.2.4 The Invalidity of the Kirchhoff-Huyghens Principle

In view of his previous arguments it is not surprising that James should question the validity of the Kirchhoff-Huyghens principle. It could always be argued that the lack of agreement between the calculated and observed radiation patterns is due to the fact that the assumed field over the surface of integration differs from the actual field. Again it has always been assumed that if the field is measured over the chosen surface of integration (i.e. known accurately) then the radiation pattern can be accurately predicted. James concludes that even if the field were known accurately over the appropriate boundary surface, this still would not enable the field to be predicted accurately. Since the Kirchhoff-Huyghens principle plays a fundamental role in the theory of aerials, the arguments leading to this startling conclusion should be carefully examined.

James chooses a spherical surface to enclose a system of sources. The solution of the appropriate homogeneous wave equation can be represented by a superposition of spherical harmonics whose coefficients are determined by the field distribution over the reference
sphere. The far-field pattern is found from the asymptotic values of the modes when the radial co-ordinate is made very large. He also applies the vector-kirchhoff formula using the field over the reference sphere as the equivalent source, and by comparing the results of both procedures finds that they do not agree.

The essential steps of the analysis are given below. In order to simplify the work he chooses a field represented by a single symmetrical mode but notes that real patterns with azimuthal symmetry can always be represented by a summation of such symmetrical modes. The field of a single mode is given by

\[ E_\phi = \sum_{n=1}^{\infty} \frac{i}{Z_0} \left( \frac{A_n + iB_n}{P_n} \right) \cos \theta \frac{H_n(r)}{kr} e^{-iwt} \quad H_\phi = 0 \]

\[ H_\phi = \sum_{n=1}^{\infty} -\frac{1}{Z_0} -\frac{1}{Z_0} (A_n + iB_n) P_n (\cos \theta) H_n(r) e^{iwt} \quad E_\phi = 0 \]  

The far field is calculated from an asymptotic form of the vector kirchhoff equation due to Silver (22)

\[ E_p = \frac{-ik}{4\pi R} e^{-ikR} R_L x \iint_S \left\{ (n x E) - Z_0 R_L x (n x H) \right\} x e^{ik\phi} \cdot R_1 dS \]

After expressing the field (2.4) in cartesian components eqn. (2.6) can be written

\[ E_p = \frac{-ik}{4\pi R} e^{-ikR} R_L x \iint_S \left\{ \cos \phi (E_{x} + Z_0 H_{x}) - \sin \phi (E_{y} + Z_0 H_{y}) \right\} \times e^{ik\phi} \cdot R_1 dS \]
In the far field the approximation \( R = Z \cos \theta + \sin \theta \cos (\phi - \phi') \) can be made to the phase factor of eqn. (2.7). After evaluation of the integral and expressing in spherical coordinates the following result is obtained

\[
E_\theta = \left\{ \frac{\pi}{2ka} J_{n+\frac{1}{2}}(ka) \right\} e^{-iKR} \frac{P'_n(\cos \theta')}{R} e^{-i\omega t} \quad (2.8)
\]

Whereas the asymptotic value of the field of the nth mode from eqn. (2.5) is given by

\[
E_\theta = P'_n(\cos \theta') e^{-i\omega t} e^{-iKR} \quad (2.9)
\]

In order to investigate the discrepancy between these two results (2.8) and (2.9) it will be helpful to solve the problem by an alternative and rigorous procedure, namely the boundary value problem for the field of a spherical current shell. Following Weeks we require to solve the equations

\[
\begin{align*}
\nabla \times H &= i\omega E + J_n(\cos \theta) \delta(r-a) \\
\nabla \times E &= -i\omega H
\end{align*}
\quad (2.10)
\]

where \( \delta(r-a) \) is the Dirac \( \delta \)-function.

Considering the nth T.M. mode only the current density can be written as \( J_n(\cos \theta) = b_n P'_n(\cos \theta) \). The fields within and outside the shell must satisfy the following conditions at the shell boundary \( r = a \),
where the suffices I and \( \pi \) represent the interior and exterior of the shell respectively and \( K_n \) is the surface current density. This may be more conveniently expressed as

\[
K_n = \frac{b_n I_0}{2\pi a} P_n'(\cos \theta)
\]

where \( I_0 \) is the total current at the equator (\( \theta = \pi/2 \)). If the medium within and exterior to the shell is assumed to be free space, the general solutions for the fields in spherical co-ordinates of the form

\[
A_n J_n'(kr) + B_n N_n'(kr) \quad \text{for } r < a
\]

\[
C_n H_n^{(1)}(kr) + D_n H_n^{(2)}(kr) \quad \text{for } r > a
\]

are substituted into eqn. (2.12) to give

\[
\begin{align*}
A_n J_n(k a) + B_n N_n(k a) &= O_n H_n^{(1)}(k a) + D_n H_n^{(2)}(k a) \\
C_n H_n^{(1)}(k a) + D_n H_n^{(2)}(k a) - A_n J_n(k a) - B_n N_n(k a) &= -\frac{b_n I_0}{2\pi}
\end{align*}
\]
Since the fields at the origin cannot be infinite then $B_n = 0$ and since we assume there are no incoming waves from infinity then $C_n = 0$. This enables the solution of (2.13) to be written

$$D_n = -\frac{b_n J_0(ka)}{2\pi} \left[ \frac{J_1'(ka)}{H_1'(ka) J_1(ka) - H_1''(ka) J_1(ka)} \right] .$$

The denominator can be shown to have the value $\sqrt{-1}$ so that

$$D_1 = -\frac{b_1 I_0}{2\pi} J_1(ka)$$

and hence the field

$$E_\theta = \frac{k}{\omega_0} b_n J_n(ka) H_n'(kr) P_n^{l} (\cos \theta)$$

which for large values of $R$ can be written in the form

$$E_\theta \propto J_n(ka) P_n^{l} (\cos \theta) \frac{e^{-ikR}}{R} .$$

Thus the solution of the boundary value problem (2.14) and the result of the Kirchhoff-Huyghens integration (2.8) are identical and predict zero field for certain values of the radius of the equivalent spherical surface while the asymptotic value of a single mode field (2.9) shows no such effect. James accepts this latter solution since he feels that the prediction of a zero field for certain values of sphere’s radius cannot be a valid result. There is no direct experimental evidence for or against the result (2.8) and (2.14) since real sources cannot be adequately represented by a single mode.

In view of James' emphasis on the significance of the inhomogeneity at the origin, the solution of (2.13) can be written in the form

$$D_n = -\frac{b_n J_0(ka)}{2\pi} \left[ \frac{J_1'(ka)}{H_1'(ka) J_1(ka) - H_1''(ka) J_1(ka)} \right] .$$

The denominator can be shown to have the value $\sqrt{-1}$ so that

$$D_1 = -\frac{b_1 I_0}{2\pi} J_1(ka)$$

and hence the field

$$E_\theta = \frac{k}{\omega_0} b_n J_n(ka) H_n'(kr) P_n^{l} (\cos \theta)$$

which for large values of $R$ can be written in the form

$$E_\theta \propto J_n(ka) P_n^{l} (\cos \theta) \frac{e^{-ikR}}{R} .$$

Thus the solution of the boundary value problem (2.14) and the result of the Kirchhoff-Huyghens integration (2.8) are identical and predict zero field for certain values of the radius of the equivalent spherical surface while the asymptotic value of a single mode field (2.9) shows no such effect. James accepts this latter solution since he feels that the prediction of a zero field for certain values of sphere’s radius cannot be a valid result. There is no direct experimental evidence for or against the result (2.8) and (2.14) since real sources cannot be adequately represented by a single mode.
geneous wave equation it is interesting to note that the solution he rejects is that of this equation while he accepts the homogeneous wave equation solution.

More recently James has adopted a somewhat less radical view of the validity of the Kirchhoff-Huyghens integration and has suggested that in applying the principle surfaces of integration which contain surface wave fields should be avoided.

2.3 James' Solution of the Dielectric Rod Problem

It has been stated earlier that James' solution is essentially an extension of Brown and Spector's method of equivalent radiating apertures. He attempts to improve their results by using more accurate aperture field expressions and to resolve the uncertainty about the proportion of power radiated directly at the feed aperture. The feed is assumed to be a semi-infinite circular metal waveguide propagating the appropriate mode. For $HE_{11}$ mode excitation of the dielectric rod this is the waveguide $TE_{11}$ mode. The radiation due to the feed aperture is calculated on the assumption that the presence of the dielectric rod has no effect on the radiated pattern.

The radiation from the free end of the rod is calculated by applying the aperture radiation formula, eqn. (2.6), separately to the field distribution over the rod end, and the surface wave distribution exterior to the rod. The treatment so far is identical to Brown and Spector who then employed approximate expressions for the end aperture fields. James uses the full field expressions and extends the generality of the treatment by including the reflected
wave components due to the feed and terminal end discontinuities.

His resulting expression for the H-plane pattern of a rod excited in the HE_{11} mode is given by

\[
\frac{E_p}{\rho} = -\{(1+K \cos \theta) + |\Gamma_1| e^{i\Psi_1(1-K \cos \theta)}\} D \frac{J_1(X_3) J_1(k \sin \theta)}{\sin \theta} \\
\left[ \frac{BKka(K+n^2 \cos \theta)}{l - \frac{k \sin \theta}{X_1}} \right] \left\{ J - \frac{k \sin \theta}{X_1} \right\} + \frac{X_2^2 BKka(K+\cos \theta)}{l + \frac{k \sin \theta}{X_2}} \left\{ J - \frac{k \sin \theta}{X_2} \right\} \\
+ \left[ 1 + \frac{X_2^2}{X_1^2} \right] \left[ 1 + \frac{K \cos \theta}{\sin \theta} \right] e^{ikL(K-\cos \theta)} \\
+ |\Gamma_1| e^{i\Psi_1} \left[ \frac{BKka(K-h^2 \cos \theta)}{l - \frac{k \sin \theta}{X_1}} \right] \left\{ J - \frac{k \sin \theta}{X_1} \right\} + \frac{X_2^2 BKka(K-\cos \theta)}{l + \frac{k \sin \theta}{X_2}} \left\{ J - \frac{k \sin \theta}{X_2} \right\} \\
+ \left[ 1 + \frac{X_2^2}{X_1^2} \right] \left[ 1 + \frac{K \cos \theta}{\sin \theta} \right] e^{ikL(K+\cos \theta)} \left\{ J_1(X_1) J_1(k \sin \theta) \right\}
\]

(2.13)

where

\[
J = \frac{J'(k \sin \theta)}{J_1(k \sin \theta)} ; \quad Q = \frac{J'(X_1)}{J_1(X_1)} ; \quad H = \frac{H_1^{(1)'}(X_2)}{iH_1^{(1)}(X_2)}
\]

\[X_1 = k_{p1} \alpha , \quad X_2 = k_{p2} \alpha , \quad X_3 = k_{p3} \alpha ,\]
\[ |r_i e^{i \psi_i} | \] is the reflection coefficient of the feed aperture discontinuity.

\[ |r_t e^{i \psi_t} | \] is the reflection coefficient of the terminal end discontinuity.

D is an arbitrary constant representing the field amplitude in the feed waveguide.

James' method of evaluating the power radiated directly from the feed, and hence the constant D, is achieved by considering the maximum to minimum ratio of the sidelobes and the envelope of the radiation pattern. This procedure is very similar to that employed by Fradin and discussed in Chapter 1.

James uses the published measurements of several authors as well as measurements of his own to establish an empirical relationship for the percentage of power radiated directly as a function of the diameter of the rod. It can be seen from figure 2.1 that the accuracy of the method is not high and an estimate for a given diameter of g (40±10)% is typical.

James states that the % power radiation loss was computed from the 'best' values of D using the relationship

\[
\% \text{ Power loss} = \left( \frac{D_g^2}{D^2 + D_E^2} \right) \times 100\
\]

where \( D_E \) is the value of D for which the power radiated by the feed waveguide and the total surface wave power are equal. He also finds that, within the limits of confidence of his D-values, the power loss function is well represented by using the "chopped surface wave" distribution. James states that the
FIG. Z.1. JAMES EMPIRICAL RADIATION LOSS FUNCTION WITH EXPERIMENTAL ESTIMATES.
\[ \% \text{Power loss} = \left( \frac{P_1}{P_1 + P_0} \right) \times 100\% \]

where \( P_1 \) and \( P_0 \) are the powers carried within and exterior to the rod by the surface wave. The ratio \( P_1/P_0 \) is well known from the solution of the boundary value problem.

Since the quantities \( P_1 \) and \( P_0 \) depend only upon the rod parameters the total expression for the radiated field eqn. (2.14) depends only on the physical dimensions of the rod, its relative permittivity and the operating frequency. James obtained this expression by using the result that the surface wave receiving efficiency is equal to the surface wave excitation efficiency and assuming that the power received in the feed guide is simply equal to the power carried within the rod. He is in effect regarding the feed waveguide as a surface launcher of zero height. Now while it is true that the chopped surface wave calculation predicts that the launching efficiency tends to zero as the height of the launching aperture tends to zero, Wenger (42) has shown by means of a rigorous Wiener-Hopf solution that the former method is unreliable for such small apertures. Furthermore the argument cannot be reversed by considering the rod excited by the feed waveguide. This serves to emphasise the point made when Chatterjee's paper 5 was being discussed in Chapter 1, namely that the guided wave power carried within and external to the rod cannot be treated as independent entities.

2.3.1 Discussion of James' Results

The ultimate criticism of any proposed theory rests in its agreement, or otherwise, with experiment. In James' estimation his procedure gives as good agreement with experiment as aperture theory
does for conventional aperture problems. In order to test his claims, figs. 2.2 to 2.10 inclusive reproduce the comparisons of his computations with experimentally observed patterns. Figs. 2.3 to 2.6 show Horton and McKinney's results for rods of a fixed length 6\(\lambda\) but progressively increasing diameters. It can be seen that eqn. (2.14) gives a reasonable prediction for the 0.5\(\lambda\) and 1.0\(\lambda\) diameter rods with the percentage power loss stated but is quite unable to predict the observed patterns for the two largest rod diameters. James has variously attributed this non-agreement to the interaction of the effective end apertures or to the presence of higher order modes which could, if excited, propagate freely for the larger diameters. In either case eqn. (2.14) would not be able to account for these effects without further modification.

Of more interest are the comparisons with Keily's experiments for rods of diameter 0.46\(\lambda\) but varying lengths (figs. 2.7 to 2.10). Here it can be said that although the predicted and experimental patterns are similar in each case the actual agreement is not very close, especially for the longer rods. It is noteworthy that for the 2\(\lambda\), 3\(\lambda\) and 4\(\lambda\) rods a radiation loss of 38% has been assumed while in the case of the 6\(\lambda\) rod a value of 49% has been adopted when one would have expected the same value to be used throughout. The comparison of figures 2.3 and 2.10 is significant since both refer to rods of the same length and material and of closely similar diameter (0.5\(\lambda\) and 0.46\(\lambda\)) yet they represent the best and poorest agreements respectively with James' predictions. Note also the difference between the assumed values for the radiation loss, which are 23% and 49% respectively. It would appear from the examples quoted that James has not used the radiation loss formula based on the chopped
surface wave distribution but has used estimates determined from each set of data considered. This is borne out by fig. 2.2 which compares the calculation with MacLean and Williams (39) measurements for a rod 2.75λ long and 0.38λ in diameter. The assumed radiation loss is 40% whereas the empirical formula would suggest a value close to 80% which cannot be encompassed even by the rather wide confidence limits admitted by James.

A final but important feature of the calculations is noted. In figures 2.2, 2.7, 2.8 and 2.9 the patterns are computed using values for the rod lengths which are significantly shorter than the stated physical lengths. James attributes this difference to aperture coupling effects and has attempted to estimate its importance by employing a reciprocity theorem calculation, but his results indicate a negligible effect for the rods in question. His method is open to question since he assumes that he is dealing with the interaction between two radiating apertures. Kane (46) has shown that the interaction will only proceed via multiply reflected surface waves and the magnitude of this effect is very small.

It is evident that the foregoing discussion casts considerable doubt on James' claim to have "developed a practical radiation formula ...... which depends solely on the physical dimensions and material of the rod, together with the operating frequency". In particular his use of the best estimate for the radiated power rather than his empirical formula, and the adoption of values of rod length to give the most favourable agreement with experiment, lays him open to the identical criticism which he has himself made against the work of Horton and Mackinney and which has been discussed in Chapter 1.
FIG. 2.2. COMPARISON OF COMPUTED AND MEASURED PATTERNS (H PLANE) FOR $d = 0.38\lambda_0$ AND $l = 2.75\lambda_0$. 

COMPUTED RADIATION LOSS AT MOUTH OF GUIDE TAKEN AS 40%. 

$1 = 2\lambda_0$. 

EXPERIMENTAL MACLEAN AND WILLIAMS. 

$l = 2.75\lambda_0$. 

Db 

180° 120° 60° 0° 60° 120° 180°
FIG 2.3 COMPARISON OF COMPUTED AND MEASURED PATTERNS (H PLANE) FOR $d = 0.5 \lambda_0$ AND $l = 6 \lambda_0$.

FIG 2.4 COMPARISON OF COMPUTED AND MEASURED PATTERNS (H PLANE) FOR $d = 1.0 \lambda_0$ AND $l = 6 \lambda_0$. 

- COMPUTED WITH 23% RADIATION LOSS AT MOUTH.
- EXPERIMENTAL

- COMPUTED WITH 4% RADIATION LOSS AT MOUTH.
- EXPERIMENTAL HORTON AND McKINNEY
FIG. 2.5. COMPARISON OF COMPUTED AND MEASURED PATTERNS (H PLANE) FOR $d = 1.5 \lambda_0$ AND $l = 6 \lambda_0$.

FIG. 2.6. COMPARISON OF COMPUTED AND MEASURED PATTERNS (H PLANE) FOR $d = 2.4 \lambda_0$ AND $l = 6 \lambda_0$. 

- COMPUTED WITH 2% RADIATION LOSS AT MOUTH.
- EXPERIMENTAL.

- COMPUTED WITH 1% RADIATION LOSS AT MOUTH.
- EXPERIMENTAL.
FIG. 2.7
COMPUTED AND MEASURED PATTERNS FOR \( d = 0.46 \lambda_0 \) AND \( l = 2 \lambda_0 \).

- - - - - - - EXPERIMENTAL
- - - - - - - COMPUTED.

(\( l = 1.7 \lambda_0 \), RADIATION LOSS AT JUNC. = 38%.)

FIG. 2.8
COMPUTED AND MEASURED PATTERNS FOR \( d = 0.46 \lambda_0 \) AND \( l = 3 \lambda_0 \).

- - - - - - - EXPERIMENTAL
- - - - - - - COMPUTED.

(\( l = 2.3 \lambda_0 \), RADIATION LOSS AT JUNC. = 38%.)

FIG. 2.9
COMP AND MEAS. PATTERNS FOR \( d = 0.46 \lambda_0 \) AND \( l = 4 \lambda_0 \).

- - - - - - - EXP
- - - - - - - COMP

(\( l = 3.5 \lambda_0 \), RADIATION LOSS AT JUNC. = 38%.)

FIG. 2.10
COMP AND MEAS. PATTERNS FOR \( d = 0.46 \lambda_0 \) AND \( l = 6 \lambda_0 \).

- - - - - - - EXP
- - - - - - - COMP

(\( l = 6 \lambda_0 \), RADIATION LOSS AT JUNC. = 49%.)
It is appropriate at this point to ask why the model has not realized its expectations. The root of the difficulties lie in the fact that three major quantities must be known. These are the exact radiation patterns from each equivalent aperture, the correct power division between the feed end radiation, and surface wave power. The rod end aperture is not thought of as a problem since the field distribution regarded as well known being simply the total HE_{11} mode within and exterior to the rod. Yet this is only an approximation to the true field. The E_p component is discontinuous over the chosen aperture plane and thus violates one of the necessary conditions for the valid application of the Vector kirchhoff formula. Neuman (24,43) has carried out a series of near field measurements in the vicinity of the free end of a dielectric rod and compared them with field distributions calculated for the HE_{11} mode. The measurements and calculations are very similar (fig. 2.11). However the results are not identical and it is the difference between the results which are of more significance than their similarity. The point here is that an approximate field and an invalid procedure must introduce a measure of uncertainty in the accuracy of the final result. An accurate solution for the rod end field distribution, and the corresponding radiation pattern, has now become available due to Yaghjian and Kornhauser (27) who used a complete model solution approach. Their field distributions are naturally continuous over the aperture and their computed radiation patterns are similar to those computed by Brown and Spector.

The question of the feed end aperture is more serious. James is of the opinion that the modifying effect of the rod on the feed
aperture pattern is negligible, yet the work of Brick \(^{(40)}\) and Chung \(^{(41)}\) on the radiation of dipole sources within dielectric slabs shows that the radiation pattern is considerably modified. Duncan \(^{(26)}\) has obtained the following expression for the radiated field of ring current source within and concentric with a cylindrical dielectric rod

\[
F(\theta) = \frac{J_1(k_a)}{k \rho J_0(k_b) H_1^{(1)}(k_b \cos \theta) - \varepsilon_r \cos \theta J_1(k_b) H_1^{(1)}(k_b \cos \theta)}
\]

where \(a\) and \(b\) are the radii of the source and rod respectively.

This expression is obviously radically different to that for a ring source in free space. Blakey \(^{(25)}\) has considered the dielectric slab excited by a parallel-plate guide and has deduced a feed pattern correction factor by means of a physical optics model as well as an expression for the feed end radiated power. His results which broadly agree with the other authors quoted indicate that the field strength in the direction of the aerial axis is zero as opposed to the maximum to be expected for the undisturbed field. The criticism of the two model approach is really one of the assumptions made rather than the model itself. Although it can claim to represent a physical interpretation of the radiation mechanism, it is on a par with the earlier kirchhoff integration approaches in that a rigorous solution can only be obtained if the fields are known accurately over the apertures, and these fields can only be obtained by the solution of two somewhat complicated boundary value problems.

However a semi-rigorous solution is in principle available. Angulo and Chang \(^{(20,21)}\) have solved more or less rigorously the separate boundary value problems of the rod end and feed end, as
well as the excitation efficiency of the \( \text{TM}_{01} \) surface wave mode, but explicit expressions for the radiation patterns are not given. Presumably this work could be adapted to the \( \text{HE}_{11} \) mode and combined with the recent work of Yaghjian and Kornhauser, but would certainly present a formidable computational problem.

2.4 The Work of J. Bach Andersen

During the final stages of writing this thesis the author was able to obtain a copy of the recently published book by J. Bach Andersen (57). This book includes the definitive exposition of his work on the dielectric rod, whose existence has only been known up until now as a conference paper abstract. It represents an extension of his earlier work on the theory of planar surface wave aerials (58).

Andersen adopts the view that the presence of a guiding structure modifies the radiation characteristics of the primary source. He argues this point of view by considering the effect on the radiation pattern of a slot in an infinite plane conductor when the conductor is covered by an infinite sheet of dielectric. He calculates the resulting radiation pattern by using the geometric optics approach identical to that of the present author (25) and obtains comparable results.

His approach to the surface wave aerial and in particular the dielectric rod is to set up and solve a well-defined boundary value problem. In the latter case he considers a magnetic current ring source concentric with an infinite circular cylindrical rod and solves the problem by an integral transform method. In order to take into account the finite length of the real rod he employs the compensation
theorem due to Monteith \(^{(59)}\). This theorem is essentially an extension of the well known reciprocity relationships. As Andersen points out it is essentially a perturbation technique and as such produces an approximate solution to the problem.

Andersen gives an account of experimental work carried out to verify the theory. The rods used in the experiment were fed from a co-axial aperture in a large ground plane such that the rods formed an extension to the centre conductor and the aperture between the inner and outer conductors acted as the magnetic ring source. The experimental patterns in general are in close agreement with the theory up to about 30° from the rod axis but beyond 50° there is an increasing disagreement. This is certainly due in part to the fact that in the experimental arrangement the feed aperture is large and therefore departs considerably from the source of infinitessimal thickness assumed by the theory. This is unfortunate from the point of view of estimating the true validity of the approximations involved.

As it stands the theory cannot be used to calculate the patterns of waveguide-fed rods. However by re-working the problem with the source within the dielectric, as has been done by Duncan \(^{(26)}\), and using the superposition principle for the fields due to elementary ring sources spanning the feed aperture another method is thus available, in principle.

The significant contribution which Andersen has made lies in the insight which has been gained about the relative roles of the radiated waves and surface waves and their interactions with the end discontinuity to produce the observed far-field patterns. As such his work represents the best discussion of the physics of the dielectric rod regarded as a transmission line problem.
3.1 Introduction

The current treatments of the dielectric aerial, in the absence of a rigorous solution of the boundary value problem, are essentially empirical. In this chapter another approach will be adopted which can be summed up by the question: 'How will the field of a given source be modified by the presence of a dielectric body?' The particular problem we shall investigate is the extent to which scattering theory may be applied to the dielectric rod aerial. The real aerial system of dielectric cylinder plus feed system will be idealized in the following way. The rod will be assumed to be isolated in space but subjected to an incident field identical to that produced by the feed system. The model thus neglects any possible interaction between the rod and the feed system by multiple scattering.

3.2 Formulation of the Scattering Problem

Scattering problems are very commonly solved by expressing the incident and scattered fields as series expansions of the solutions to the wave equation in the appropriate geometry and matching the fields at the boundaries of the scattering object. The problem of the scattering of a plane wave at an arbitrary angle of incidence to an infinite dielectric cylinder has been treated, for example, by Wait \(^{47}\). The expressions which Wait deduces for this simple case are far from simple and suggest that the effort required to obtain a solution for the finite dielectric rod by this approach would be formidable.
The scattering process can be formulated in a way which appeals directly to the physics of the process. A dielectric body is one which exhibits an induced polarization \( \mathbf{P} \) when subjected to an applied electric field. This polarization represents the combined effects of the individual polarizations of the atoms and molecules comprising the dielectric. The dielectric can be regarded as a collection of oscillating dipoles which are the source of the scattered field. The scattered field is thus the summation of the individual induced dipole fields. However the induced polarization of any particular atom is due to the combined effect of the incident field and that of the remaining atoms. The scattering process can thus be treated as an extension of the well-known theory of dielectric polarization. Following Born and Wolf \(^{48}\) we may express the total field \( \mathbf{E}_p \) at any point as

\[
\mathbf{E}_p = \mathbf{E}^{(i)} + \sum_k \mathbf{E}_k
\]  

(3.1)

where \( \mathbf{E}^{(i)} \) is the incident field and \( \mathbf{E}_k \) is the field due to the \( k \)th induced dipole moment. The Hertz vector \( \mathbf{\pi} \) of a dipole of moment \( \mathbf{p} \) is given by

\[
\mathbf{\pi} = \frac{\mathbf{p}(t - \mathbf{r})}{r^3}
\]  

(3.2)

and the corresponding field is found from the relationship

\[
\mathbf{E} = \nabla \times \nabla \times \mathbf{\pi}
\]  

(3.3)

Using (3.2) and (3.3) in (3.1) we obtain

\[
\mathbf{E}_p = \mathbf{E}^{(i)} + \sum_k \left[ \nabla \times \nabla \times \frac{\mathbf{p}_k(t - \mathbf{r}_{kp})}{r_{kp}^3} \right]
\]  

(3.4)
Since we are considering a macroscopic system it is convenient to assume that the number density \( N \) and the induced dipole moment per atom \( \rho \) are continuous functions of the position vector \( \mathbf{r} \) and to allow the above summation to become an integral over the volume occupied by the material, whence

\[
E_p (\mathbf{r}', t) = E^{(i)} + \int V \times \nabla \times \left[ N \alpha \frac{(\mathbf{r}', t - \mathbf{r})}{\mathbf{r}} \right] \text{d}V \tag{3.5}
\]

where \( \rho = \alpha \mathbf{E} \); \( \alpha \) is the atomic polarizability and \( r = |\mathbf{r} - \mathbf{r}'| \).

Equation (3.5) can be brought into a slightly more useful form by assuming the usual time variation \( e^{i\omega t} \) for the field quantities. Since we are considering fields external to the dielectric the vector operators may be brought outside the integral. Furthermore the relationship between the atomic polarizability and the relative permittivity is invoked, namely \((\varepsilon_r - 1) = N\alpha\). The resulting expression is

\[
E_p = E^{(i)} + \frac{i\omega}{4\pi} V \nabla \times \left[ (\varepsilon_r - 1) \int \frac{E_p e^{-ik(\mathbf{r} - \mathbf{r}')}}{|\mathbf{r} - \mathbf{r}'|} \text{d}V \right] \tag{3.6}
\]

This is the fundamental Integro-differential equation of the scattering process from a finite volume of an inhomogeneous dielectric and is an expression of the physics of the dielectric rod aerial.

### 3.3 Application of the theory to the Circular Cylindrical Rod

Since we are concerned with radiation patterns rather than the entire field, the vector operations can be carried out in eqn. (3.6) and the usual far-field approximations applied. In terms of the cylindrical co-ordinate system employed here (fig. 3.1) this gives finally
\[ E_p(\theta, \phi) = E^1(\theta, \phi) + \frac{k^2(e^{-1})}{4\pi} \int E_p(p', \phi', z') x \]
\[ \times e^{ik[p'\sin\theta(\cos(\phi-\phi')+z'\cos\theta)]} d\rho d\phi dz \]

where the common factor \( e^{-ikR} \) has been eliminated from both sides of the equation. This is an integral representation of the total field rather than a true integral equation since the unknown fields \( E_p \) are expressed as functions of different variables. However it will be shown that the field under the integral sign may be transformed to yield a true integral equation for the scattering process.

3.3.1 \( \text{TM}_{01} \) Excitation

It is well known that a cylindrical dielectric rod is able to support and propagate without attenuation the \( \text{TM}_{01} \) mode above a cut-off frequency which is determined by its radius and relative permittivity. The formulation just given is perfectly general however and its application should be independent of whether a propagating mode exists in the rod or not.

The unknown field can be conveniently derived from a z-directed vector potential \( A_z = \psi \). The scalar function \( \psi \) is a solution of the homogeneous Helmholtz equation in cylindrical co-ordinates
\[ (\nabla^2 + k^2)\psi = 0. \]

The form of solution which has been found most convenient in this application is the representation by an "angular spectrum" of cylindrical waves introduced by Hill (33). The use of this representation will enable eqn. (3.7) to be transformed to an integral equation. The solution for \( \psi \) is expressed as
\[ \psi = \int_{-1}^{+1} G(C) J_0(kp\sqrt{1-C^2}) e^{-ikCz} \frac{dC}{\sqrt{1-C^2}} \quad (3.8) \]
where $C = \cos \alpha$ and $G(C)$ is the angular spectrum. The limits of integration include only the radiated waves. The components of the electric field within the rod are found from the potential function (3.8) from the following relationships (49)

$$
E_{\phi} = \frac{1}{i \omega C} \frac{\partial^2 \psi}{\partial \rho \partial \phi} = \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] \frac{k^2 \sqrt{1-C^2}}{i \omega C} G(C) J_o'(k \rho \sqrt{1-C^2}) e^{-i k C z} \frac{dC}{\sqrt{1-C^2}} \\
E_{\phi} = 0 \\
E_z = \frac{1}{i \omega C} \left[ \frac{\partial^2}{\partial z^2} + k^2 \right] \psi = \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] \frac{k^2 \sqrt{1-C^2}}{i \omega C} G(C) J_o'(k \rho \sqrt{1-C^2}) e^{-i k C z} \frac{dC}{\sqrt{1-C^2}} \\
$$

(3.9)

In order to determine the scattered fields we require to resolve the field within the rod into its rectangular co-ordinates. These are

$$
E_x = E_{\rho \cos \phi} = \frac{k^2}{i \omega C} \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] G(C) J_o'(k \rho \sqrt{1-C^2}) \cos \phi e^{-i k C z} \frac{dC}{\sqrt{1-C^2}} \\
E_y = E_{\rho \sin \phi} = \frac{k^2}{i \omega C} \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] G(C) J_o'(k \rho \sqrt{1-C^2}) \sin \phi e^{-i k C z} \frac{dC}{\sqrt{1-C^2}} \\
E_z = \frac{-k^2}{i \omega C} \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] \sqrt{1-C^2} G(C) J_o(k \rho \sqrt{1-C^2}) e^{-i k C z} \frac{dC}{\sqrt{1-C^2}} \\
$$

(3.10)

The corresponding components of the scattered field are found from the integral in equation (3.7) and are

$$
E_x(s) = \frac{k^2}{4 \pi} \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] \int_0^{2 \pi} \int_0^{2 \pi} \int_0^L E_x e^{i k [\rho \sin \phi \cos (\phi - \phi') + z \cos \theta]} x \rho' d\rho' d\phi' dz' \\
E_y(s) = \frac{k^2}{4 \pi} \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] \int_0^{2 \pi} \int_0^{2 \pi} \int_0^L E_y e^{i k [\rho \sin \phi \cos (\phi - \phi') + z \cos \theta]} x \rho' d\rho' d\phi' dz' \\
E_z(s) = \frac{k^2}{4 \pi} \left[ \begin{array}{c} +1 \\ -1 \end{array} \right] \int_0^{2 \pi} \int_0^{2 \pi} \int_0^L E_z e^{i k [\rho \sin \phi \cos (\phi - \phi') + z \cos \theta]} x \rho' d\rho' d\phi' dz' \\
$$

(3.11)

Since $E_{\phi} = 0$ the total scattered field expressed in spherical polar coordinates will be found from
\[ E(s)(\theta) = (E^x(s)\cos \phi + E^y(s)\sin \phi) \cos \theta - E^z(s)\sin \theta \quad (3.12) \]

Thus using eqns. (3.10) in (3.11) we obtain

\[
E_x(s) = \frac{k^4(e_r-1)}{4\pi\omega e_0^2} \int_{-1}^{+1} \int_{0}^{2\pi} \int_{0}^{\pi} \text{CG}(CG) J'_0(kp\sqrt{1-C^2}) e^{i[k\rho \sin \theta \cos(\phi - \phi')] + (\cos \theta - C)z} \cos \phi \rho' d\phi' dz' \frac{dC}{\sqrt{1-C^2}} \quad (3.13)
\]

with similar expressions for the other components. The integrations with respect to the co-ordinates \( \phi, z \) and \( \rho \) may now be carried out.

The \( \phi \)- integral (50)

\[
\int_{0}^{2\pi} \{ \cos n\phi \} \{ \sin n\phi \} e^{i k p \sin \theta \cos(\phi - \phi')} = \pm (i)^n 2\pi J_n(kp\sin \theta) \{ \cos n\phi \} \{ \sin n\phi \} \quad (3.14)
\]

The \( z \)- integration is elementary

\[
L \int_{0}^{\pi} e^{ik[\cos \theta - C]z} dz = \frac{\{ \cos kL(\Gamma - C) - 1 \} + i \sin kL(\Gamma - C)}{ik(\Gamma - C)} \quad (3.15)
\]

where for convenience we now put \( \Gamma = \cos \theta \) from now on. The total scattered field can now be expressed as

\[
E(s)_0 = -\frac{k^3(e_r-1)}{2} \int_{-1}^{+1} \text{G(C)}[\text{CFI}_1 - \sqrt{1-C^2} \Gamma I_0] \quad (3.16)
\]

\[
\frac{[\cos kL(\Gamma - C) - 1] + i \sin kL(\Gamma - C)}{(\Gamma - C)} dC
\]

where \( I_0 \) and \( I_1 \) are the results of the \( \rho \)- integration. These are given by

\[
I_0 = \int_{0}^{a} J_0(kp\sqrt{1-C^2}) J_0(kp\sqrt{1-\Gamma^2}) dp \quad (3.17)
\]
I_1 = - \int_0^a J_o(k\rho\sqrt{I-C^2}) J_1(k\rho\sqrt{I-T^2}) d\rho = \int_0^a J_1(k\rho\sqrt{I-C^2}) J_1(k\rho\sqrt{I-T^2}) d\rho
(3.18)

These integrals are of the same standard form

\[ \int_0^z J_\mu (pz) J_\nu (qz) dz \]
and may be evaluated following Maclachlan (50) to give

I_o = \frac{a}{k[\tau^2-C^2]} \left[ \sqrt{I-C^2} J_o(k\nu\sqrt{I-C^2}) J_1(k\nu\sqrt{I-T^2}) - \sqrt{I-T^2} J_1(k\nu\sqrt{I-C^2}) J_0(k\nu\sqrt{I-T^2}) \right]
(3.19)

I_1 = \frac{a}{k[\tau^2-C^2]} \left[ \sqrt{I-C^2} J_o(k\nu\sqrt{I-C^2}) J_1(k\nu\sqrt{I-T^2}) - \sqrt{I-T^2} J_1(k\nu\sqrt{I-C^2}) J_0(k\nu\sqrt{I-T^2}) \right]
(3.20)

Equations (3.16), together with expressions (3.19) and (3.20), express the scattered field in terms of the angular spectrum. It is now necessary to express the total field in the same way.

In terms of the scalar potential function \( \psi \) the far field is given by (49)

\[ E_0 = - i \omega \psi \hat{\theta} = i \omega \psi \sin \theta \hat{\hat{z}} \]
(3.21)

where \( \hat{\theta} \) and \( \hat{\hat{z}} \) are unit coordinate vectors.

We now require to evaluate

\[ \psi = - \int_{-1}^{+1} G(C) J_o(k\rho\sqrt{I-C^2}) e^{-ikCz} \frac{dC}{\sqrt{I-C^2}} \]
(3.8)
for large values of \( p \) and \( z \). It will be convenient to use the method of stationary phase to do this. The integral is expressed in terms of spherical co-ordinates by using the substitutions \( p = R \sin \theta \), \( z = R \cos \theta \), and using the integral representation for the Bessel function (3.14). The potential \( \psi \) can now be written as

\[
\psi = \frac{1}{2\pi} \int_{-1}^{1} \int_{-\pi}^{\pi} G(C)e^{-ikR \sin \theta \sqrt{1-C^2}} \cos(\phi - \phi') e^{-ik(R \cos \theta)} \frac{d\phi}{\sqrt{1-C^2}} \frac{dC}{\sqrt{1-C^2}}
\]  

(3.22)

The integral is of the form

\[
\iint e^{ikf(x,y)} g(x,y) \, dx \, dy
\]

and can be evaluated by expanding the integrand as a Taylor series about the stationary point \( (x_0, y_0) \) if it exists. The integral can then be shown to have the value

\[
\frac{ikf(x_0, y_0)}{\kappa} \frac{2\pi}{\sqrt{f''_x f''_y}} e^{-i\pi/2}
\]  

(3.23)

where \( f''_x = \frac{\partial^2}{\partial x^2} \); \( f''_y = \frac{\partial^2}{\partial y^2} \).

In the case under consideration it can be shown that the integral does indeed have stationary values and these correspond to the co-ordinates of the point of observation in the far field. The asymptotic value of (3.22) obtained in this way is

\[
\psi \to -\frac{G(\cos \theta)}{\sin \theta} \frac{e^{-ikR}}{kR} e^{-i\pi/2}
\]  

(3.24)

whence

\[
E_0 = -\frac{\iomega \omega_0}{k} \frac{e^{-ikR}}{R} e^{-i\pi/2}
\]  

(3.25)
Now the incident field can be written in the form 

\[ E^{(i)}(\theta) = F(\Gamma) \frac{e^{-ikR}}{R} \]

so that by combining this with eqns. (3.16) and (3.25) the integral representation (3.7) can now be written in the form

\[ G(\Gamma) = -F(\Gamma) + \lambda \int_{-1}^{+1} G(C) \kappa(\Gamma, C) \, dC \]  

(3.26)

Thus the scattering problem has been expressed as the well-known Inhomogeneous Fredholm integral equation.

Now the incident field in the case of TM \(_{01}\) mode propagation is the TM \(_{01}\) radiation pattern of the circular waveguide. This is given approximately (Silver (22)) by

\[ F(\Gamma) = -\frac{ika\kappa_{01}}{2\sqrt{1-\Gamma^2}} \left( \beta_{01} / k + \Gamma \right) J_1'(\kappa_{01} a) \frac{J_0(ka\sqrt{1-\Gamma^2})}{1 - \left( \frac{\kappa_{01}}{k\sqrt{1-\Gamma^2}} \right)^2} \]  

(3.27)

Hence the integral equation to be solved can be written more explicitly as

\[ G(\Gamma) + \frac{ika\kappa_{01}}{2\sqrt{1-\Gamma^2}} (\beta_{01} / k + \Gamma) J_1'(\kappa_{01} a) \frac{J_0(ka\sqrt{1-\Gamma^2})}{1 - \left( \frac{\kappa_{01}}{k\sqrt{1-\Gamma^2}} \right)^2} = \frac{(\epsilon - 1)}{2} k^2 \int_{-1}^{+1} G(C) \left[ C_{11} - \sqrt{C_{11}^2 - C_{12}^2} I_1 \right] \left[ \cos kL(\Gamma - C) - 1 \right] i \sin kL(\Gamma - C) \, dC \]  

(3.28)

where \( I_0 \) and \( I_{11} \) are given by (3.19) and (3.20) respectively.

### 3.3.1 The HE\(_{11}\) Mode Excitation

The procedure just discussed may be applied to the case of the most commonly employed excitation of the dielectric rod, namely the
HE\textsubscript{11} mode. This will eventually yield a pair of simultaneous integral equations. In terms of vector potentials of the Electric and Magnetic type the Electric field is given by

\[
E = \frac{1}{\imath \omega_0} \nabla (\mathbf{V} \cdot \mathbf{A}) - \imath \omega_0 \mathbf{A} - \nabla \times \mathbf{F}
\]

Since a hybrid mode can be regarded as the superposition of a TE and a TM mode and the HE\textsubscript{11} mode is hybrid in its \(z\) direction following Kelly (7) we may choose \(z\) directed electric and magnetic vector potentials such that

\[
A = \psi z ; \quad F = \phi z
\]

The Electric field components in cylindrical co-ordinates are then given by

\[
\begin{align*}
E_\rho &= \frac{1}{\imath \omega_0} \frac{\partial^2 \psi}{\partial \rho \partial z} - \frac{1}{\rho} \frac{\partial \phi}{\partial \phi} \\
E_\phi &= \frac{1}{\imath \omega_0} \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z} + \frac{\partial \phi}{\partial \phi} \\
E_z &= \frac{1}{\imath \omega_0} \left[ \frac{\partial^2 \psi}{\partial z^2} + \frac{k^2}{\rho} \psi \right]
\end{align*}
\]

In order to obtain a field distribution of the same form as the HE\textsubscript{11} mode, i.e.

\[
\begin{align*}
E_\rho &= \left[ \frac{J_0(k_\rho \rho)}{k_\rho} + B J_1(k_\rho \rho) \right] \sin \phi e^{-\gamma z} \\
E_\phi &= \left[ A J_1(k_\rho \rho) + B \frac{1}{k_\rho} J_0(k_\rho \rho) \right] \cos \phi e^{-\gamma z} \\
E_z &= -\frac{k_\rho}{\gamma} B J_1(k_\rho \rho) \sin \phi e^{-\gamma z}
\end{align*}
\]
Potentials of the following forms are required

\[ \psi = - \int_{-1}^{+1} G(C) J_1(k\sqrt{1-C^2}) \sin\phi e^{-ikCz} \frac{dC}{\sqrt{1-C^2}} \]  
\[ \phi = - \int_{-1}^{+1} H(C) J_1(k\sqrt{1-C^2}) \cos\phi e^{-ikCz} \frac{dC}{\sqrt{1-C^2}} \]  

(3.31)

When these are substituted in the component equations (3.8) above the following are obtained

\[ E = \int_{-1}^{+1} \left[ \frac{k^2C}{\omega_o} G(C) J_1(k\sqrt{1-C^2}) - kh(C)J_1(k\sqrt{1-C^2}) \right] \sin\phi' e^{-ikCz} dC \]  
\[ E = \int_{-1}^{+1} \left[ \frac{k^2C}{\omega_o} G(C) J_1(k\sqrt{1-C^2}) - kh(C)J_1(k\sqrt{1-C^2}) \right] \cos\phi' e^{-ikCz} dC \]  
\[ E_z = \int_{-1}^{+1} \left[ \frac{k^2C}{\omega_o} G(C) J_1(k\sqrt{1-C^2}) \sin\phi' e^{-ikCz} dC \right] \]  

(3.32)

This field is now resolved into rectangular components using the relationships \( E_x = E_x \cos\phi' - E_y \sin\phi' \) and \( E_y = E_x \sin\phi' + E_y \cos\phi' \) and are given by

\[ E_x = - \int_{-1}^{+1} \left[ \frac{k^2C}{\omega_o} G(C) + kh(C) \right] J_2(k\sqrt{1-C^2}) \sin2\phi' e^{-ikCz} dC \]  
\[ E_y = \int_{-1}^{+1} \left[ \frac{k^2C}{\omega_o} G(C) - kh(C) \right] J_0(k\sqrt{1-C^2}) + \right. \]  
\[ \left. + \frac{1}{2} \left[ \frac{k^2C}{\omega_o} + kh(C) \right] \cos2\phi' \right] e^{-ikCz} dC \]  

(3.33)
As in the TM\textsubscript{01} mode case these components may be integrated over the volume of the rod to give the corresponding components of the scattered field. These are

\[
E_x(s) = \frac{k(\varepsilon_r-1)^{+1}}{2i} \int \left[ \frac{k^2 C G(C) + kH(C)}{\omega_e} \right] \frac{[(\cos kl(C)-1) + isinkl(C)]}{(l-C)} \times \sin 2\phi \, dC
\]

\[
E_y(s) = \frac{k(\varepsilon_r-1)^{+1}}{2i} \int \left[ \frac{k^2 C G(C) - kH(C)}{\omega_e} \right] \frac{[(\cos kl(C)-1) + isinkl(C)]}{(l-C)} \times \left[ \frac{kH(C)}{\omega_e} \frac{1}{2} \frac{k^2 C G(C)}{\omega_e} + kH(C) I_2 \cos 2\phi \right] \, dC
\]

\[
E_z(s) = \frac{k(\varepsilon_r-1)^{+1}}{2i} \int \left[ k \sqrt{1-C} G(C) I_1 \sin \phi \frac{[(\cos kl(C)-1) + isinkl(C)]}{(l-C)} \right] \, dC
\]

(3.34)

The scattered far-field may now be expressed in terms of the usual spherical co-ordinates by means of the transformations

\[
E_0(s) = (E_x(s) \cos \phi + E_y(s) \sin \phi) \cos \theta - E_z(s) \sin \theta
\]

\[
E_\phi(s) = -E_x(s) \sin \phi + E_y(s) \cos \phi
\]

(3.35)

which then yield on substitution of eqns. (3.34)

\[
E_0(s) = \frac{k(\varepsilon_r-1)^{+1}}{2i} \int \left[ - \frac{kH(C)}{\omega_e} \frac{1}{2} \frac{k^2 C G(C)}{\omega_e} \frac{(I_0-I_2)}{2} \right] \cos \theta
\]

\[
- \sqrt{1-C} \frac{k^2 G(C)}{\omega_e} I_1 \sin \phi \left[ \frac{[(\cos kl(C)-1) + isinkl(C)]}{(l-C)} \right] \, dC \sin \phi
\]

\[
E_\phi(s) = \frac{k(\varepsilon_r-1)^{+1}}{2i} \int \left[ \frac{kH(C)}{\omega_e} \frac{1}{2} \frac{k^2 C G(C)}{\omega_e} \frac{(I_0+I_2)}{2} \right]
\]

\[
x \left[ \frac{[(\cos kl(C)-1) + isinkl(C)]}{(l-C)} \right] \, dC \cos \phi
\]

(3.36)
The total far field of the vector potentials $A$ and $F$ are given by

$$
E_0 = -i\omega_0 A_0 - ikF_0
$$

$$
E_\phi = -i\omega_0 A_\phi + ikF_0
$$

(3.37)

Since $A = \psi \hat{z}$ and $F = \phi \hat{z}$ the far field can be expressed as

$$
E_0 = i\omega_0 \psi \sin \theta \quad E_\phi = -ik \phi \sin \theta
$$

(3.38)

The asymptotic values of the potentials at large distances can be determined, as before, by employing the method of stationary phase. This yields

$$
\psi \sim -G(\cos \theta) \frac{e^{-ikR}}{R} e^{-i\pi/2} \sin \phi
$$

$$
\phi \sim -H(\cos \theta) \frac{e^{-ikR}}{R} e^{-i\pi/2} \cos \phi
$$

(3.39)

whence the corresponding far field components are from (3.37)

$$
E_0 = -\frac{\omega_0}{k} G(\cos \theta) \frac{e^{-ikR}}{R} e^{-i\pi/2} \sin \phi
$$

$$
E_\phi = H(\cos \theta) \frac{e^{-ikR}}{R} e^{-i\pi/2} \cos \phi
$$

(3.40)

Combining these latter expressions with those for the scattered fields deduced above (3.36) and using the appropriate incident field patterns one then obtains two simultaneous fredholm equations. In practice only the principle $E$- and $H$-planes are of interest. Thus by setting $\phi = 0$ and $\pi/2$ the resultant equations become:
\[ G(r) + \frac{k^2(\varepsilon r - 1)}{2} \int_{-1}^{+1} \left\{ \left[ \frac{\mu_0}{\varepsilon_0} \frac{(I_0 + I_2)}{2} \right] - \sqrt{1 - \varepsilon r} G(C) \frac{(I_0 - I_2)}{2} \right\} r \]

\[ - \sqrt{1 - \varepsilon r} \sqrt{1 - \varepsilon r} G(C) I_1 \left\{ \frac{\cos k L (r - C) - 1}{(r - C)} + i \sin k L (r - C) \right\} J_0(\text{kin} \theta) \]

\[ = \frac{i^2 \omega_0}{\sqrt{2}} \left[ 1 + \frac{\beta_{11}}{k} \cos \theta \right] \frac{J_1(\text{kin} \theta)}{\sin \theta} \]

\[ H(r) - \frac{k^2(\varepsilon r - 1)}{2} \int_{-1}^{+1} \left\{ \left[ \frac{\mu_0}{\varepsilon_0} \frac{(I_0 - I_2)}{2} \right] - H(C) \frac{(I_0 + I_2)}{2} \right\} \]

\[ \left\{ \frac{\cos k L (r - C) - 1}{(r - C)} + i \sin k L (r - C) \right\} dC \]

\[ = \frac{i^2 k a \omega_0}{2} \left[ \frac{\beta_{11}}{k} + \cos \theta \right] \frac{J_1'(\text{kin} \theta)}{1 - \left( \frac{\varepsilon_0}{\mu_0} \right)^2} \]

where \( I_0 \) and \( I_1 \) have been defined previously (3.17) (3.18) and

\[ I_2 = \int_0^\infty J_2(k\sqrt{1-C^2}) J_2(k\sqrt{1-C^2}) d\theta. \]

3.3.3 The Dielectric Tube Aerial

The theory outlined above is readily extended to the dielectric tube aerial. The integral equations for TM_{01} and HE_{11} excitation will be of exactly the same form as those just derived for the solid dielectric rod. The only essential difference between the rod and the tube is that the limits of integration with respect to the radial co-ordinate are different. The integral functions \( I_0, I_1 \) and \( I_2 \) defined previously are consequently modified. The appropriate functions \( I_0', I_1' \) and \( I_2' \) are defined by
where \( a \) and \( b \) are the outer and inner radii of the tube respectively. These integrals are again of the standard form and readily evaluated. \( I_0 \) is given by

\[
I_0' = \int_a^b J_0(kp\sqrt{1-C^2}) J_0(\frac{kp}{\sqrt{1-C^2}}) dp
\]

\[
I_1' = \int_a^b J_1(kp\sqrt{1-C^2}) J_1(\frac{kp}{\sqrt{1-C^2}}) dp
\]

\[
I_2' = \int_a^b J_2(kp\sqrt{1-C^2}) J_2(\frac{kp}{\sqrt{1-C^2}}) dp
\]

(3.42)

Thus in order to discuss dielectric tubes the integral equations for dielectric rods are used, but with the integral functions \( I_0', I_1', \) and \( I_2' \) replaced by the functions \( I_0, I_1, \) and \( I_2 \) where appropriate.
The Solution of the Integral Equations

4.1 Preliminary Remarks

The scattering formulation of the dielectric rod problem has shown that the radiation pattern $G(r)$ is the solution of the inhomogeneous Fredholm integral equation

$$G(r) + \lambda \int G(c) K(r,c) dc = F(r)$$  \hspace{1cm} (3.26)

The properties of this equation are well-known and there exists an extensive literature on the solution of integral equations \(^{(51)}\). The complexity of the kernel function $K(r,c)$ virtually precludes the possibility of obtaining an analytical solution for the present problem. In order to obtain a solution it is necessary to employ one of the several possible approximation methods. There are three important approximation techniques available and each one was considered in turn.

4.1.1. The method of successive approximations.

This method is straightforward in principle. The unknown function $G(c)$ under the integral in eqn. (3.26) is replaced by some known function which is thought to be an approximation to the true solution. Solving this equation for $G(r)$ then yields a solution, say $G_1(r)$ which is hopefully a better approximation to the true solution. The solution $G_1(r)$ is then in turn used as the known function in the integral which, in its turn, yields a new approximation $G_2(r)$. This procedure is then repeated until a suitable convergence has been achieved. It is of interest to know under what conditions a convergent set of solutions will occur. This is most readily seen by regarding $\int K(r,c) dc$ as a linear operator $A$ acting on the unknown $G(c)$. Eqn (3.26) can then be written
\[ \mathcal{G}(r) - \lambda \mathcal{A} \mathcal{G}(r) = F(r) \]  

(4.1)

Now take the approximate solution to be \( \mathcal{G} = \mathcal{F}(r) \)

Then

\[ \mathcal{G}_1 - \lambda A \mathcal{F}(r) = \mathcal{F}(r) \quad \text{or} \quad \mathcal{G}_1(r) = (1 + \lambda A) \mathcal{F}(r) \]

(4.2)

This solution is the well-known \( \lambda \mathcal{A} \mathcal{F} \) Approximation. Repeating the procedure gives

\[ \mathcal{G}_2(r) = \mathcal{F}(r) + \lambda A \mathcal{F}(r) + \lambda^2 A^2 \mathcal{F}(r) \]

\[ \vdots \]

\[ \mathcal{G}_n(r) = \mathcal{F}(r) + \lambda A \mathcal{F}(r) + \lambda^2 A^2 \mathcal{F}(r) + \cdots + \lambda^n A^n \mathcal{F}(r) \]

(4.3)

It is evident that the successive approximations can only converge to a solution provided that \( |\lambda A| < 1 \). This condition can be written as

\[ \iint |K(\mathbf{r}, \mathbf{c})|^2 d\mathbf{r} d\mathbf{c} < \frac{1}{|\lambda|} \]

(4.4)

The complexity of the integral again precludes analytical evaluation and so a solution was attempted for the case of \( TM_01 \) excitation without evaluating the condition (4.4). Since the solution diverged after two iterations it was concluded that (4.4) was not satisfied and this procedure was quickly abandoned.

4.1.2 Approximating series solution.

In this method it is assumed that the solution can be satisfactorily approximated by a series of linearly independent functions - for example a polynomial. Thus if

\[ \mathcal{G}(\mathbf{r}) \approx \sum_{i=1}^{n} c_i \psi_i(\mathbf{r}) \]

is substituted into the integral equation (3.26) a set of \( n \) equations is obtained from which the \( n \) unknown co-efficients \( c_i \) may be determined. The use of this method raised several problems. For example, what is the most suitable set of approximating functions? and How many terms would be required to give satisfactory accuracy? The method can only, in
Thus in the case of a rod of length $2\lambda$, the radiation pattern should be representable by 12 to 14 terms of a Fourier series and this certainly appears reasonable on inspection of the measured patterns. A solution was attempted using 12 terms initially with not very satisfactory results. The successive edition of further terms did not effect any improvement nor was the application of a least-squares optimizing procedure. The computed results in all cases exhibited the features discussed earlier, namely poor agreement with observations both at the pivotal points and at intermediate values. This approach was therefore abandoned.
4.1.3 Finite Sum Approximation.

The method which has been adopted for the present investigation involves the approximation of the integral in Eqn. (3.26) by a finite sum quadrature formula, i.e.

\[ \int_{a}^{b} y(x) \, dx \approx \sum_{k=1}^{n_1} a_k y(x_k) \quad (4.1) \]

where the coefficients \( a_k \) and the pivotal points \( x_k \) depend upon the particular quadrature formula employed. Thus for a particular value of the variable \( \Gamma \), say \( \Gamma_i \), the integral equation can be written, using (4.1) as

\[ G(\Gamma_i) + \lambda \sum_{k=1}^{n_1} a_k G(C_k) K(\Gamma_i, C_k) = F(\Gamma_i) \quad (4.2) \]

If now \( \Gamma \) is made to take successively the same \( n \) values as the variable \( C_k \) then a system of \( n \) simultaneous linear equations is obtained. The notation can be conveniently modified by putting \( C = \Gamma = X \) so that the system of equations becomes

\[ G(X_i) + \lambda \sum_{k=1}^{n_1} a_k G(X_k) K(X_i, X_k) = F(X_i) \quad i = 1, 2, 3, ..., n \quad (4.3) \]

Because the quadrature (4.1) is only an approximation to the integral the solution of the equations (4.3) will only yield an approximate solution to the integral equation (3.26). The formal exact equivalent to the integral equations can be written as

\[ G(X_i) + \lambda \sum_{k=1}^{n_1} a_k G(X_k) K(X_i, X_k) + \varepsilon_i = F(X_i) \quad i = 1, 2, 3, ..., n \quad (4.4) \]

where \( \varepsilon_i \) is the quadrature error. The various methods by which the quadrature error may be reduced will not be discussed at this point except to note that the greater the number of subdivisions of the range of integration the smaller will be the quadrature error provided the function is well-behaved in this range.
then becomes two sets of equations on separating and equating real and imaginary parts. The equations to be solved are

\[ G'(x_i) - \lambda h \sum_{k=0}^{N} a_k G'(x_k) K_1(x_i, x_k) + \lambda h \sum_{k=0}^{N} a_k G''(x_k) K_2(x_i, x_k) = 0 \]

\[ G''(x_i) - \lambda h \sum_{k=0}^{N} a_k G'(x_k) K_2(x_i, x_k) - \lambda h \sum_{k=0}^{N} a_k G''(x_k) K_1(x_i, x_k) = \frac{(0.51 + x_i)\sqrt{1-x_i^2} \int_0^{0.87\pi} \left(1-x_i^2\right) - \frac{2.405}{0.87\pi}^2}{(1-x_i^2) - \frac{2.405}{0.87\pi}^2} \]  

(4.5)

where

\[ K_1 = \frac{[X_1 X_k I_2 - (1 - X_1^2)^{3/2}(1 - X_k^2)^{3/2}I_2]}{(X_1 - X_k)} \frac{[\cos 62.5\pi L(X_i - X_k) - 1]}{[\cos 62.5\pi L(X_i - X_k) - 1] + \lambda h} \]

\[ K_2 = \frac{[X_1 X_k I_1 - (1 - X_1^2)^{3/2}(1 - X_k^2)^{3/2}I_2]}{(X_1 - X_k)} \frac{[\sin 62.5\pi L(X_i - X_k)]}{[\sin 62.5\pi L(X_i - X_k)] + \lambda h} \]

\[ K_2 = 62.5\pi L[X_1 X_k I_1 - (1 - X_1^2)^{3/2}(1 - X_k^2)^{3/2}I_2] \text{ if } X_i = X_k \]

\[ \lambda = 0.78 \times (62.5\pi)^2 \]

and \( h = \frac{2}{N}; \ X_i, k = (-1 + ih); \ i, k = 0, 1, 2, ..., N. \) The numerical values adopted are appropriate to a rod of radius 0.435\( \lambda_0 \) and relative permittivity 2.56. When using the repeated trapezium rule for integration we set the \( a_k = 1. \) When the repeated Simpson's rule is employed we have \( a_0 = a_N = \frac{1}{3}; \ a_k = \frac{2}{3} \) (k odd) or \( a_k = \frac{4}{3} \) (k even). Computations of eqns. (4.5) have been carried out for 20, 40 and 60 points using both the trapezium rule and Simpson's rule. The corresponding points of the 40 and 60 point Simpson calculations were virtually identical, being in agreement to the third decimal place.
both the trapezium rule and Simpson's rule have been used. Allen's work on the numerical integration of radiation patterns has been taken as a guide. His work suggests that a subdivision of two intervals per half cycle of the pattern would yield a maximum error of 1 db in the computed pattern using the increment rule which is essentially similar to the trapezium rule. This suggests a subdivision of about twenty intervals for a rod of length $2\lambda$, as a lower limit for an acceptable solution. Allen's work can only be used as a rough guide since the integral which he evaluated and those of the present work, although similar in form, are not identical. Furthermore the integrand involves the unknown function and for this reason the error cannot be determined. In addition we are more interested in the departures of the calculated values of the function from the true values rather than the overall departure of the finite sum approximation from the true integral. Correction procedures are available but in the first instance solutions to the approximate equations (4.3) only have been sought.

4.3 The $TM_{01}$ Mode Solution

The required pattern function $G(X_i)$ will generally be a complex quantity i.e. it will vary both in amplitude and in phase. Accordingly this function is expressed in terms of its real and imaginary parts as

$$G(X_i) = G'(X_i) + i G''(X_i)$$

Inspection of the explicit form of the Kernel function $K(X_i, X_k)$ (3.28) reveals that this, too, is complex. The set of equations (4.3)
then becomes two sets of equations on separating and equating real and imaginary parts. The equations to be solved are

\[ G'(X_i) - \lambda h \sum_{k=0}^{N} a_k G'(X_k) K_1(X_i, X_k) + \lambda h \sum_{k=0}^{N} a_k G''(X_k) K_2(X_i, X_k) = 0 \]

\[ G''(X_i) - \lambda h \sum_{k=0}^{N} a_k G'(X_k) K_2(X_i, X_k) - \lambda h \sum_{k=0}^{N} a_k G''(X_k) K_1(X_i, X_k) = \frac{0.51 + X_i \sqrt{1 - X_i^2}}{(1 - X_i^2) - \left(\frac{2.405}{0.87\pi}\right)^2} \]

where

\[ K_1 = [X_iX_k I_2 - (1 - X_i^2)^{1/2}(1 - X_k^2)^{1/2}I_2] \frac{[\cos 62.5\pi L(X_i - X_k) - 1]}{(X_i - X_k)} \]

\[ K_1 = 0 \text{ if } X_i = X_k \]

\[ K_2 = [X_iX_k I_1 - (1 - X_i^2)^{1/2}(1 - X_k^2)^{1/2}I_2] \frac{\sin 62.5\pi L(X_i - X_k)}{(X_i - X_k)} \]

\[ K_2 = 62.5\pi L[X_iX_k I_1 - (1 - X_i^2)^{1/2}(1 - X_k^2)^{1/2}I_2] \text{ if } X_i = X_k \]

\[ \lambda = 0.78 \times (62.5\pi)^2 \]

and \( h = \frac{2}{N} \); \( X_i, k = (-1 + ih) \); \( i, k = 0, 1, 2, \ldots, N \). The numerical values adopted are appropriate to a rod of radius 0.435\( \lambda_o \) and relative permittivity 2.56. When using the repeated trapezium rule for integration we set the \( a_k = 1 \). When the repeated Simpson's rule is employed we have \( a_0 = a_N = \frac{1}{3} \); \( a_k = \frac{2}{3} \) (k odd) or \( a_k = \frac{4}{3} \) (k even).

Computations of eqns. (4.5) have been carried out for 20, 40 and 60 points using both the trapezium rule and Simpson's rule. The corresponding points of the 40 and 60 point Simpson calculations were virtually identical, being in agreement to the third decimal place.
This gives some assurance that a solution has been achieved. As was expected the 20 point solutions do not agree so closely with the 40 and 60 point results but do nevertheless show the same general features.

The computed solution is shown in Fig. 4.1 where it is compared with the experimental results of Horton, Karal and McKinney. It can be seen that the positions of the maxima and minima agree reasonably well with the experimental values. The pattern envelope is remarkably close to the experimental one even for angles greater than 90° from the forward direction. However it will also be noted that the pattern minima are shallower than the experimental values. Even so, the scattering model has yielded results which are closely similar to the observed pattern. This is in contrast to James' calculations using a simple two-aperture model. In order to achieve reasonable agreement with experiment he found it necessary to adopt the value 2.3λ₀ as the effective length of the rod. James states that this is justified since the rod used by Horton et al was tapered at the end and presumably the effective length is greater than the length of the uniform cylinder. However it is interesting to note that James does not apply this agreement to his calculations for the longer rods. Horton et al's description of their experiment is so ambiguous that it is perfectly possible to gain the impression that the free ends of their rods were tapered. However on closer inspection of their paper and of a later paper by McKinney it is evident that the tapers are within the feed waveguide and the length indicated is the true length of the exposed circular cylinder.

A computation has also been carried out for a rod of length 6λ₀ using the 60 point Simpson formula. This is equivalent to using a 20
FIG. 4.1. 2λ ROD TM_{01} EXITATION.
FIG 4.2. THE EFFECT OF DIFFERENT FEED PATTERNS

- DIELECTRIC FILLED W/G.
- UNFILLED W/G.
FIG. 4.3. CALCULATION FOR $6 \lambda$ ROD (TM$_{01}$) COMPARED WITH EXPERIMENT.
point formula in the $2\lambda_0$ case and would thus not be expected to yield very accurate results. It can be seen from Fig. (4.3) that this is indeed the case. The detailed agreement between the calculated and measured patterns is certainly poor but the observed and computed pattern envelopes are nevertheless very similar.

The effect of modifying the feed pattern is shown in Fig. (4.2) where the 'obliquity factor' $(0.51 + \cos \theta)$ is changed to $(0.34 + \cos \theta)$ which represents the hypothetical case of a rod excited by an air filled waveguide. As might be expected the differences between the two patterns are most marked at large angles away from the rod axis.

4.4 The $HE_{11}$ Mode Solution

The integral equations for the $HE_{11}$ mode case (3.40) are dealt with in exactly the same way and the following four sets of linear equations are obtained.

$$
\begin{align*}
G'(x_i) - \lambda h \sum_{k=0}^{N} a_k \left[ X_k K_a H'(x_k) - X_k K_H''(x_k) - M_a G'(x_k) + M_b G''(x_k) \right] &= [1 + 0.60x_i] \frac{J_1(\kappa \sqrt{1-x_i^2})}{\sqrt{1-x_i^2}} \\
G''(x_i) - h \sum_{k=0}^{N} a_k \left[ X_k K_a H'(x_k) + X_k K_H''(x_k) - M_a G'(x_k) - M_b G''(x_k) \right] &= 0 \\
H'(x_i) + \lambda h \sum_{k=0}^{N} a_k \left[ K_c H'(x_k) - K_d H''(x_k) - X_k K_a G'(x_k) + K_b G''(x_k) \right] &= [0.60 + x_i] \frac{J_1(\kappa \sqrt{1-x_i^2})}{1 - \frac{(\kappa \sqrt{1-x_i^2})^2}{x_i^2}} \\
H''(x_i) + \lambda h \sum_{k=0}^{N} a_k \left[ K_d H'(x_k) + K_c H''(x_k) - X_k K_b G'(x_k) - X_k K_a G''(x_k) \right] &= 0
\end{align*}
$$

(4.6)
where
\[
K_a = \frac{1}{2}(I_0 - I_2) \frac{[\cos kL(X_i - X_k) - 1]}{(X_i - X_k)}
\]
\[
K_b = \frac{1}{2}(I_0 - I_2) \frac{[\sin kL(X_i - X_k)]}{(X_i - X_k)}
\]
\[
K_c = \frac{1}{2}(I_0 + I_2) \frac{[\cos kL(X_i - X_k) - 1]}{(X_i - X_k)}
\]
\[
K_d = \frac{1}{2}(I_0 + I_2) \frac{[\sin kL(X_i - X_k) - 1]}{(X_i - X_k)}
\]
\[
M_a = \frac{1}{2}X_i X_k (I_0 + I_2) - \sqrt{1 - X_i^2} \sqrt{1 - X_k^2} I_1 \frac{[\cos kL(X_i - X_k) - 1]}{(X_i - X_k)}
\]
and
\[
M_b = \frac{1}{2}X_i X_k (I_0 + I_2) - \sqrt{1 - X_i^2} \sqrt{1 - X_k^2} I_1 \frac{\sin kL(X_i - X_k)}{(X_i - X_k)}
\]

Thus the solution of the HE_{11} mode case requires twice as many equations for a given number of subdivisions compared with the TM_{01} mode case. The time available on the computer only permits a maximum subdivision of 40 which in turn requires the solution of 160 simultaneous equations.

Fig. (4.4*) compares the computed H-plane pattern for a rod of length 2\(\lambda_0\) and radius 0.23\(\lambda_0\) and \(\varepsilon_r = 2.56\) with the experimental pattern due to Keilly. These are seen to agree quite well, although in the case of the 4\(\lambda_0\) rod the number of points used in the calculation is not really sufficient to ensure reasonable accuracy. In this case, however, the results obtained still represent the measured values reasonably well.
Fig. 4. HE mode comparison with experiment

- $L = 2\lambda_0$
- $d = 0.46\lambda_0$
- $L = 3\lambda_0$
From the manner in which the basic integral equation has been derived it is evident (Eqs. (3.5) and (3.6)) that the scattering theory can be extended to situations where the dielectric permittivity of the rod is not constant throughout its volume. In this case the most general form of the integral representation (Eqn. 3.7) becomes

\[ \mathbf{E}(\theta, \varphi) = \mathbf{E}^i(\theta, \varphi) + \frac{k^2}{4\pi} \int \left[ \mathbf{E}(\theta', \varphi', z') - \mathbf{I} \right] \mathbf{E}(\theta', \varphi', z') \cdot \frac{\mathbf{K}}{e^{ikL\cos(\theta-\varphi)}} d\theta' d\varphi' dz' \]

In practice interest has usually been confined to the study of the propagation properties of dielectric structures whose permittivity is a function of one coordinate variable only. A special case of one-dimensional permittivity variation which has been investigated is that in which the longitudinal \( z \) variation of permittivity is a periodic function of the \( z \)-coordinate. Experiments have been carried out on such inhomogeneous dielectric rods excited in the \( HE_{11} \) mode. The free space portion of the rods consisted of alternate discs of perspex and P.T.E.K. threaded on a central rod of perspex. The discs were 0.68 cm in thickness and their external diameter was 2.2 cm. The diameter of the centre rod was 0.65 cm. Rods of nominal length \( 2\lambda_0 \) and \( 6\lambda_0 \) were used in these investigations.

The frequency dispersion characteristic (Brillouin diagram) of the periodic structure was determined by measuring the radial component of the electric field at the rod surface as a function of distance along the rod. The observed standing wave pattern was set up by using either the free end discontinuity of the rod or by means of a 20 cm diameter brass plate acting as a short circuit. The dielectric guide wavelength was thus directly determined for a number of frequencies in the range of interest. (8 GHz to 11 GHz). The initial measurements were carried out using a dipole probe to determine the field directly but in order
Fig. 5.1. V.SWR. of periodic rod at feed aperture plane.

Fig. 5.2. Brillouin diagram of periodic rod.
to obtain greater accuracy a technique using a modulated scattering element was finally adopted. This technique has been described by several authors (61,42). In the present investigation the modulated scattering element was a micro-miniature germanium diode connected to an audio signal generator by 42 s.w.g. copper wire. The connecting leads of the diode acted as the scattering element proper and the combination was effectively a switched elementary dipole of length $\frac{L}{10}$. As a check on the accuracy and validity of the results obtained for the periodic rod similar measurements were also carried out for a uniform perspex rod of the same dimensions. Very good agreement with Gillespie's theoretical dispersion curves was found and the results indicated a relative permittivity value of 2.56 ± 0.02.

The observed dispersion characteristic of the periodic structure is shown in Fig. (5.2). Measurements were also made of the V.S. WR. of the feed guide and $6 \lambda_0$ rod combination over the frequency range 8 → 11 GHz. The results are shown in Fig. 5.1. It can be seen that the corresponding reflection coefficient is $\approx 1$ over the frequency range 8.6 → 8.8 GHz and can thus be regarded as further support for the stop-band interpretation of the observed dispersion curve.

Radiation patterns.

A series of radiation patterns for the $2 \lambda_0$ composite rod were made at frequencies in the range 8 → 11 GHz. Measurements were also made at these same frequencies for a uniform perspex rod of the same dimensions in order to effect a direct comparison between the two rods. Both $E$ and $H$ plane patterns were determined. The corresponding $H$-plane patterns for both types of rod are compared in Fig. (5.3).

The results of the $H$-plane measurements are summarized in Fig. (5.4) where the beam width and the first two sidelobe levels of both the uniform and the composite rod are plotted as functions of frequency.

Two features are readily apparent from an inspection of the radiation patterns or the summary diagram Fig. (5.4).
Fig. 5.3 Comparison of radiation patterns of $2\lambda$, uniform and periodic rods (H plane).

$t = 8.6$ GHz.

$t = 8.4$ GHz.
(1) The rapid variation of beamwidth and, to a lesser extent, sidelobe levels in what has been identified as the "stopband" frequency range of the periodic structure.

(2) For frequencies greater than the "stopband" the beamwidth and sidelobe levels are consistently less than those of the uniform rod. Also the sidelobes of the composite rod lie closer to the main axis than those of the uniform rod.

Interpretation of the patterns.

It is reasonable to ask to what extent a theory of the dielectric rod aerial can explain the observations obtained from the composite rod. In the previous discussion of the theories involving the application of vector-Kirchhoff formulae it has been seen that the predicted radiation patterns are dominated by the linear array factor $\sin \chi /\chi$

where

$$\chi = \frac{\pi L}{A} (K - \cos \theta).$$

Furthermore, that while formulae of this form give only approximate results for any given set of conditions, never the less the general features of dielectric rod behavior for variation in dimensions, relative permittivity or wavelength, are reasonably well represented.

In the case of the composite rod, since it certainly supports a propagating surface wave outside the "stopband" region, the Kirchhoff type formulae should also be able to predict its general behavior provided that the appropriate value of $K = \frac{\lambda}{\lambda}$ is adopted. It would seem reasonable to assume that away from the "stopband" region the composite rod's behavior could be accounted for by assuming that it approximates to a uniform rod whose effective permittivity is the mean permittivity of the composite structure which in the present case would be $\epsilon_r \approx 2.3$. In Fig. (5.5) the measured values of $K$ as a function of frequency have been plotted together with the theoretical curve for a uniform rod of the same diameter. It can be seen that mean value assumption is reasonable to a first approximation and furthermore there is no violent variation of $K$ evident in the stop-band region.
Fig 5.4  Beamwidth and sidelobe levels vs. freq. for uniform and periodic $2\lambda_0$ rods
between the uniform and the composite rods should be accountable simply from the differences in their $K$-values.

For a given ratio $L/\lambda_0$, provided that $X < \gamma$, the greatest value of $\frac{\sin X}{X}$ is attained when $X = \frac{nL}{\lambda_0} (K - 1)$. It can be readily seen that the greater the value of $K$ the smaller will be the corresponding largest value of $\frac{\sin X}{X}$ which means that the relative sidelobe levels of the uniform rod will be greater than those of the composite rod, and also that the sidelobes of the uniform rod will be closer to the main beam than that of the composite rod. Both these features are observed in the experimental patterns.

Thus far the predictions of the approximate theory have been borne out in practice but apart from the 'stop-band' behaviour there is one further prediction which is not completely realized. According to the approximate theory the width of the main lobe should be smaller for the uniform rod than for the composite rod. This can be readily seen as follows. For $X < \gamma$

$$\frac{E(\theta)}{E_{\text{MAX}}} = \frac{\sin X_1}{X_1} \left[ \frac{\sin X_1}{X_1} \right]_{\text{MAX}} = \frac{\sin X_2}{X_2} \left[ \frac{\sin X_2}{X_2} \right]_{\text{MAX}}.$$

where $X_{1,2} = [K_{1,2} - \cos \theta_{1,2}^2]$, $X_{1,2} \text{ MAX } = [K_{1,2} - 1]$, and if $K_1 > K_2$, then as has been previously shown

$$\left[ \frac{\sin X_1}{X_1} \right]_{\text{MAX}} > \left[ \frac{\sin X_2}{X_2} \right]_{\text{MAX}}.$$

From which it follows

$$\frac{\sin X_1}{X_1} > \frac{\sin X_2}{X_2}.$$ This in turn implies that

$$K_1 - \cos \theta_1 > K_2 - \cos \theta_2,$$

or

$$(K_1 - K_2) > (\cos \theta_1 - \cos \theta_2).$$

Giving finally

$$\theta_2 > \theta_1.$$
Fig. 5.5: Measured dispersion of periodic rod compared with uniform rod ($\varepsilon = 2.3$)
This prediction is seen to be confirmed in the case of the composite rod only for frequencies below the "stopband" whereas for frequencies above the stopband the width of the main lobe is less than that of the uniform rod at the same frequency. Pattern variations in the "stopband" region and the reduced beamwidth above the stopband is presumably due to the inhomogeneity of the composite rod. Since the approximate theory is incapable of explaining these observations it is of interest to see to what extent the proposed scattering theory modified for an inhomogeneous permittivity variation Eqn. (5.1) can improve on the approximate theory.

In order to try to assess the effect of the longitudinal variation of relative permittivity, the following approximate treatment was adopted. The uniform central rod was neglected and the discontinuous changes in relative permittivity were approximated by a sinusoidal variation with the z-coordinate. The z-varying permittivity must therefore be included in the z-integral part of the scattered field (eqn. 3.7) and becomes

\[ \int \left[ (\varepsilon_r(z) - 1) e^{-ik(r-C)z} \right] \, dz \tag{5.1} \]

Assuming the values \( \varepsilon_r = 2.56 \) for perspex and \( \varepsilon_r = 2.10 \) for P.T.F.E., the appropriate form of the z-varying permittivity becomes

\[ \varepsilon_r(z) = 2.33 - 0.23 \cos \left( \frac{2\pi z}{h} \right) \tag{5.2} \]

where \( h \), the thickness of two discs, is the period of the structure.

The integration of (5.1) is perfectly straightforward and gives rise to an additional term. The new z-integration to be used in the integral equation (3.41) thus becomes

\[
1.33 \left[ \frac{\cos kL(r-C) - 1}{ik(r-C)} + i\sin kL(r-C) \right] \\
- 0.115 \left[ \frac{\cos kL(r-C+2.5) - 1}{ik(r-C+2.5)} + i\sin kL(r-C+2.5) \right] \\
+ \left[ \frac{\cos kL(r-C-2.5) - 1}{ik(r-C-2.5)} + i\sin kL(r-C-2.5) \right] \tag{5.3}
\]
It is also of interest to consider the effect of the inhomogeneity in the radial direction due to the presence of the perspex centre rod. Again an approximate treatment has been adopted. The outer ring structure giving rise to the $z$-variation in permittivity is assumed to be uniform and to have a value of permittivity which is the mean of that of perspex and P.T.F.E., namely $\varepsilon_r = 2.33$.

The integration in the radial direction is taken to be the sum of the integrals over the centre rod and the outer annulus, paying due regard to the permittivity of each region. All of the integrals required have been previously deduced and stated (eqns. 3.19, 3.20, 3.41 and 3.42). As a result the equations to be solved (eqn. 3.40) have to be modified as follows. The term $(\varepsilon_r - 1)$ now takes the value of 1.33.

The radial co-ordinate integration involves functions of the form

$$\int_{0}^{a} [\varepsilon_r (\rho) - 1] J_n (kp\sqrt{1-C^2}) J_n (kp\sqrt{1-\varepsilon^2}) \rho d\rho \quad (5.4)$$

Which can be treated as

$$b$$

$$(\varepsilon_1 - 1) \int_{0}^{a} J_n (kp\sqrt{1-C^2}) J_n (kp\sqrt{1-\varepsilon^2}) \rho d\rho +$$

$$a$$

$$(\varepsilon_2 - 1) \int_{b}^{a} J_n (kp\sqrt{1-C^2}) J_n (kp\sqrt{1-\varepsilon^2}) \rho d\rho \quad (5.4)$$

and gives immediately

$$(\varepsilon_1 - 1) I_n (b) + (\varepsilon_2 - 1) I_n (a, b) = (5.4)$$

Since, from eqn. (3.42) $I_n (a, b) = I_n (a) - I_n (b)$ then (5.4) can be written

$$(\varepsilon_2 - 1) \left[ I_n (a) + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2 - 1} I_n (b) \right] = (5.5)$$

In the present case the numerical values become $(\varepsilon_2 - 1) = 1.33$; $\varepsilon_1 - \varepsilon_2 = 0.173$. Thus in order to modify eqns. 3.40 it only requires all existing terms of the form $I_n (a)$ to be altered to

$$I_n (a) + 0.173 I_n (b).$$
The solution of the modified equations yielded values virtually identical to those for a uniform rod to the accuracy with which points can be plotted on the scale adopted for Fig. 5.1. Thus the centre rod would appear to have no significant effect on the pattern and this is not unreasonable on a simple cross-sectional area argument since the centre rod diameter is only about one third of the overall diameter of the rod, so that one would expect the majority of the energy within the rod to be seated within the outer annular volume.
6.1 The Experimental Results

In a previous chapter the results of the computations for the scattering model have been commented upon briefly. They are considered here in somewhat more detail in order to assess the success or otherwise of the model.

The discussion will concentrate on the HE_{11} excitation since this is the case of most practical interest as well as being the one for which detailed measurements are available. In Figs. 6.1 and 6.2 the measured patterns for the 0.22 cm. diameter uniform and inhomogeneous rods are compared with the scattering theory model for various frequencies. As in the case of the 0.46 \lambda. diameter rod the theoretical predictions are seen to be in reasonable agreement with the measured patterns with regard to general shape, beam width and the position of sidelobe maxima and minima. There is one feature which is of some importance. As can be seen from Fig. 5.4, the beam width of the uniform rod and, apart from the stop-band region, that of the inhomogeneous rod increases with increasing frequency. The scattering model, however, is seen to predict beam widths which decrease with frequency. In the case of the inhomogeneous rod there is no indication of the observed stop-band behaviour. The scattering model is thus in agreement with earlier models and with physical intuition.

6.2 Discrepancies between theory and experiment.

In order to ascertain the sources of the differences between the observed and the calculated patterns it is appropriate, at this point, to review the assumptions made in constructing the model.

The major assumption of this work is that the feed waveguide can be represented by the field distribution which it sets up in the absence of the dielectric rod. Consequently possible interaction between the rod and the feed guide has been ignored.
Fig. 6.1 Comparison of theoretical & measured H-plane patterns (uniform rod)
Fig. 6.2 Comparison of theoretical & measured H-plane patterns (periodic rod)

- $f = 8.6\,\text{GHz}$
- $f = 8.4\,\text{GHz}$

$\text{dB Relative power}$

- Measured
- Theory
The second assumption is that the field due to the feed waveguide is adequately represented by the feed pattern functions actually used. These functions, as given by Silver and suitably modified for a dielectric-filled guide, are deduced from standard aperture theory. This involves the application of Kirchhoff-Huyghen's principles to the assumed field distribution (the propagating mode) over the waveguide aperture. It is well-known that the predicted field is only approximate although adequate for most purposes. Both measurements and exact calculations show that discrepancies exist between the true feed patterns and those actually used. This was verified in the case of TE excitation of the 22mm diameter dielectric-filled feed waveguide. Fig 6.3 compares the experimental E and H plane, patterns at 8.6 GHz, with the assumed feed pattern given by the Kirchhoff approximation. It can be seen that the H-plane pattern is significantly narrower than predicted but the E-plane pattern is reasonably well approximated. The effect of using the measured and the theoretical feed patterns to predict the radiation patterns of the dielectric rod are shown in Fig. 6.4 where a comparison with the experimental rod patterns are made. It can be seen that the agreement between prediction and measurement is worsened in the H-plane when the measured feed pattern is used. The agreement between theory and experiment was found to be generally much better for the E-plane where the use of the measured feed pattern has somewhat improved the agreement. These results suggest that the effective feed pattern in the E-plane is probably reasonably close to the measured pattern but the effective H-plane pattern differs significantly from the measured pattern. This in turn implies that the major assumption is not strictly valid and that a significant interaction does occur between the dielectric rod and the feed wave-guide. The interaction is essentially a multiple scattering process, and to represent it adequately would be a formidable task - probably no easier than attempting to solve the complete boundary problem for the system.
Fig. 6.3. Measured and approximate feed patterns.
Fig. 6.4. Effect of feed pattern change on calculated pattern at 86GHz.
Fig. 6.4 Contd. E-plane patterns.

- Measured
- Theory (using measured feed pattern.)

\( f = 8.6 \text{GHz.} \)

\( f = 8.8 \text{GHz.} \)
6.3 Rod-Feed Interaction.

In order to test the validity of the rod-feed interaction postulate, a series of measurements were carried out with a nominal \( 1 \lambda_o \) (3.4cm) long rod of 2.2cm diameter. A set of measurements was carried out to determine the order of magnitude of the incident power scattered back to the feed waveguide. A standing wave method was used and the following procedure was adopted. The dielectric-filled feed waveguide radiating into free space was carefully matched by means of a 3-screw turner until no reflection was detected. The \( 1 \lambda_o \) rod was then fixed to the feed to form a normal feed-rod system and the corresponding reflection was measured. This procedure was repeated with the rod spaced at various distances from the dielectric-filled feed by means of cylindrical expanded polystyrene spacers. A preliminary experiment with the spacers alone gave no measurable reflection. The results of the measurements carried out at a frequency of 8.6 GHz are summarized in the table below.

| spacing  | V.S.W.R. | \(|\Gamma|\) |
|----------|----------|-----------|
| 0 cm     | 2.36     | 0.40      |
| 1 cm     | 1.46     | 0.19      |
| 1.5 cm   | 1.32     | 0.14      |
| 3.4 cm   | 1.16     | 0.08      |

It can be seen that the amount of energy reflected into the feed guide decreases monotonically with increasing separation of rod and feed. For the normal rod-feed combination the reflection coefficient is quite high whereas for a separation of about \( 1 \lambda_o \) it falls to a relatively small value and hence virtually no interaction is taking place. A corresponding set of radiation pattern measurements were also made for various rod-feed spacings. The most significant of these for the present discussion are those of zero spacing and a spacing of 3.4 cm as these are the cases of maximum and negligible interaction respectively.
The measured pattern for the 3.4 c.m. case is shown in Fig 6.5 where a comparison with the scattering theory predictions using the measured feed pattern is made. It can be seen from Fig 6.7 that while the normal feed-rod combination pattern is reasonably well predicted by the theory, in the case of the spaced rod (Fig 6.5) the measured and predicted patterns are very close. The significance of this is twofold. In the first place it provides some experimental justification for the concept of rod-feed interaction and secondly it provides an experimental confirmation of the scattering theory since the assumptions of the simple model are approximately satisfied in the case of the \( \frac{1}{4} \)-spaced rod. It should also be noted that neither the simple surface integration theory nor the two-aperture model can be used to predict the results of spaced rod experiments.

6.4. Theoretical considerations.

The experiments just described provide some empirical justification for the proposal that the interaction between rod and feed should not be ignored. A full theoretical treatment would be extremely complicated and would probably not be justified but will be outlined later in the chapter. A simpler approach, following the work of Midgley (55) on the re-radiation from horns is capable of providing at least an order-of-magnitude estimate. The following assumptions will be made:

1. That the complicated back-scattered field incident on the feed aperture is represented by a uniform plane wave.

2. The incident back-scattered field excites the TE 11 mode only in the feed guide; higher order modes are neglected.

The re-radiated field from the aperture is then found by demanding continuity over the aperture plane of the assumed field components. i.e.

\[ E_r = E_S + E_m \]  

(6.1)

where \( E_r \) is the re-radiated component

\( E_S \) is the back-scattered field

\( E_m \) is the excited waveguide mode
Fig 6.5. H plane pattern for 1λ rod spaced 1λ from feed
Fig. 6.1. The effect of a space between rod and feed. Rod length and spacing = 1λ₀.
Fig. 67 H-Plane pattern computations for a 1λ uniform rod.
Or using the assumptions (1) and (2) above the rectangular components of the re-radiated aperture field is given by

\[
E_{rx} = E_s \left[ 1 - J_2 \left( k_x r \right) \sin 2\phi \right]
\]

\[
E_{ry} = -E_s \left[ J_0 \left( k_r r \right) - J_2 \left( k_r r \right) \cos 2\phi \right]
\]

(6.2)

and the corresponding far field is found by aperture integration in the usual way yielding

\[
E_{\theta} \propto -E_s \frac{\cos \theta}{k} \left[ J_1 \left( k \sin \theta \right) \frac{J_0 \left( k \sin \theta \right)}{J_1 \left( k \sin \theta \right)} \right]
\]

and

\[
E_{\phi} \propto E_s \left[ \frac{\cos \theta}{k} \right] \left[ -J_1 \left( k \sin \theta \right) \frac{J_0 \left( k \sin \theta \right)}{J_1 \left( k \sin \theta \right)} \right]
\]

(6.3)

In order to proceed further we require the relative amplitudes of the incident and re-radiated fields. From the continuity conditions it is easily seen that for an incident field of unit amplitude \( E_s = \left| \Gamma \right| \) where \( \left| \Gamma \right| \) is the relative amplitude of the waveguide made due to scatter from the rod. Thus the effective field incident on the rod is the sum of the incident field plus the re-radiated field.

The relative power re-radiated is readily found. By integrating the Poynting vector of the TE\(_{11}\) mode over the guide cross section it is found that the ratio of the mode power to the back-scattered power is 0.835. The back scattered power is thus \( \frac{0.835}{\left| \Gamma \right|^2} \) and hence the re-radiated power in terms of unit incident power is simply \( \frac{0.835}{\left| \Gamma \right|^2} - \left| \Gamma \right|^2 \). Using the measured value \( \left| \Gamma \right|^2 = 0.4 \), it is then found that the re-radiated power is 3\% of the original incident power. From Middley's work and from the form of the re-radiated field in the H-plane it can be seen that the effect is minimal in the boresight direction but increases with increasing angle away from boresight where, of course, the main field is becoming progressively weaker. A calculation of the pattern using this simple model for the re-radiated field was carried out and the result is shown in Fig. 6.7., where various calculated patterns for the \( \lambda_0 \) rod are compared. The agreement with the measurements has been somewhat improved and, considering the simplicity of the assumptions made this result should be regarded as indicative rather than conclusive.

The observed broadening of the main beam with increasing frequency can be understood now in terms of the re-radiated component. As the frequency
increases the angle between bore sight and the maximum of the re-radiated pattern decreases, thus at a given angle away from bore sight the relative level of the re-radiated pattern increases thus counteracting the expected effect of narrowing the main beam. The extent of the effect must depend on how the amount of energy back scattered by the rod varies with frequency. In this respect the experimental and theoretical results found for the inhomogeneous rod are quite significant. It has been remarked earlier that the theoretical predictions of the simple model give no indication of the dramatic increase and then decrease of the main beam width as the frequency of measurement is increased and passes through the stop-band region. In fact the model predicts the behaviour to be expected of a uniform rod of lower effective permittivity. However from the standing-wave measurements previously presented (Fig 5.1) the amount of power scattered back into the feed-guide in the "stop-band" region is seen to be very large and would necessitate a correspondingly larger component of the re-radiated field.

The notion of interaction between the rod and the feed would thus appear to be reasonably well founded. The simple approximations used are capable of explaining some of the features in the observed patterns in general terms but it is not reasonable to expect absolute agreement considering the simplicity of the assumptions made.

6.5. The basis of a complete scattering theory approach.

From the foregoing discussion it follows that a fully rigorous treatment requires the application of the theory of multiple scattering. This theory has been developed formally and extensively by Twersky. He has been able to show that the multiple scattering due to a system of objects can be predicted provided that the individual scattering properties of each object is known. For the rod and feed this means that the individual scattering properties of each component must be known. The simple model of the rod scattering developed earlier solves this part of the problem. In the case of the feed guide a rigorous solution by the Wiener-hopf technique is required to determine the field scattered
by the aperture. With both these solutions available Twersky's expressions can then be used to yield two self consistent integral equations whose solution should in principle, be a completely rigorous treatment of the rod problem. The equations can then be shown to take the following symbolic form (Appendix B)

\[
G_1(\mathbf{r}) + \lambda \int [G_1(\mathbf{c}) + G_2(\mathbf{c})] K_1(\mathbf{r}, \mathbf{c}) d\mathbf{c} = -\lambda \int G^0_2 K_1(\mathbf{r}, \mathbf{c}) d\mathbf{c},
\]

\[
G_2(\mathbf{r}) + \lambda \int [G_1(\mathbf{c}) + G_2(\mathbf{c})] K_2(\mathbf{r}, \mathbf{c}) d\mathbf{c} = -\lambda \int G^0_1 K_1(\mathbf{r}, \mathbf{c}) d\mathbf{c}.
\]

Where \(G_1\) and \(G_2\) represent the additional scattered fields due to the mutual presence of the rod and feed, \(G_1^0\) and \(G_2^0\) the known scattered fields of the isolated rod and feed respectively and \(K_1\) \(K_2\) the modified kernels due to the relative positions of rod and feed with respect to the chosen system of coordinates. The solution of the problem for the far field is found by summing the quantities \(G_1^0\) \(G_1\) \(G_2^0\) \(G_2\) together with the original exciting field \(F\).

The effort required to carry out this program would not seem to be justified in the light of the reasonably satisfactory results which can be obtained from the simple scattering model.

6.6 Concluding remarks:

The simple scattering theory model developed in chapter 3 has been shown to give good agreement with the published result for 0.46 \(\lambda_0\) diameter rods of various lengths. A series of measurements have been made on larger diameter rods ( \(\sim 0.7 \lambda_0\)) with both uniform and periodic variations of relative permittivity. In general the agreement between theory and experiment was found to be reasonable both for \(E\) and \(H\) plane patterns. In all computations only the true physical dimensions of the rods were used and no attempt was made to adopt empirical "effective values" for these dimensions in order to secure better agreement between theory and experiment. The discrepancies that do exist appear to be capable of explanation in terms of an effective feed pattern which arises from the unavoidable interaction between the feed aperture and the rod itself.
APPENDIX 'A'

Stationary Phase Evaluation of the Total Field.

Potentials of the form \( \Upsilon = -\int_{-i}^{+i} G(c) J_1(kp\sqrt{1-c^2}) \sin p \, dc \sqrt{1-c^2} \)

have been used.

Expressing the Bessel function as an integral yields

\[
\Upsilon = -\frac{1}{2\pi i} \int_{-i}^{+i} dC \int_{0}^{2\pi} G(c) e^{-ikp\sqrt{1-c^2} \cos(p-p')} e^{-ikCz} \sin p \, dp'
\]

But \( p = R \sin \Theta = R\sqrt{1-r^2} \); \( z = R \cos \Theta = R \Gamma \). Hence

\[
\Upsilon = -\frac{1}{2\pi i} \int_{-i}^{+i} dC \int_{0}^{2\pi} G(c) e^{-ikR\sqrt{1-r^2} \cos(p-p')} e^{-ik\Gamma C} \sin p \, dp'
\]

The \( p \)-integral is of the form \( \int f(x) e^{ikg(x)} \, dx \).

which has the asymptotic value \( \left( \frac{2\pi}{K \left| q'(x_s) \right|} \right)^{1/2} f(x_s) e^{ikg(x_s) + \frac{i\pi}{2}} \) \( q(x) \geq 0 \),

where \( x_s \) is the stationary value i.e. \( q'(x_s) = 0 \).

In the above notation, \( f(x) \equiv G(c) e^{-ik\Gamma C} \)

and \( K = -kR\sqrt{1-r^2} \sqrt{1-\Gamma^2} \).

Then \( q'(x) = \sin(p-p') = 0 \)

\( q''(x) = -\cos(p-p') = -1 \) \{ when \( p = p' \), which is thus a stationary point \).

Thus we now have

\[
\Upsilon = -\frac{1}{2\pi i} \int_{-i}^{+i} dC \frac{G(c)}{\sqrt{1-c^2}} e^{-ikR\sqrt{1-r^2} \cos(p-p')} e^{-ik\Gamma C} \sin p
\]

This integral is again of the same standard form where now we can write \( f(x) = G(c) \left( \frac{2\pi}{-kR\Gamma(\sqrt{1-r^2})} \right)^{1/2} e^{ik\Gamma C} \) \( q(x) = (\Gamma C + \sqrt{1-r^2} \sqrt{1-\Gamma^2}) \)

Then \( q'(x) = 0 \)

\( q''(x) = -1 \) \{ when \( C = \Gamma \), the stationary point \}. 

\[
\]
This finally yields
\[ \psi = -\frac{1}{2\pi i} \frac{G(\Gamma)}{\sqrt{1-\Gamma^2}} \left( \frac{2\pi}{-kR(1-\Gamma^2)} \right)^{\frac{1}{2}} \left( \frac{2\pi (1-\Gamma^2)}{-kR} \right)^{\frac{1}{2}} e^{-ikR} e^{-\frac{i}{2} \sin \varphi} \]

which simplifies to
\[ \psi = \frac{G(\Gamma)}{\sqrt{1-\Gamma^2}} e^{-ikR} \sin \varphi. \]

Similarly, \[ \phi = \frac{H(\Gamma)}{\sqrt{1-\Gamma^2}} e^{-ikR} \cos \varphi. \]

The total far field, using Eqns (3.38) is thus

\[ \begin{align*}
E_\theta &= \frac{\mu_0 m}{k} G(\Gamma) \frac{e^{ikR}}{R} \sin \varphi \\
E_\varphi &= -i \frac{H(\Gamma)}{R} \frac{e^{ikR}}{R} \cos \varphi
\end{align*} \]  

(3.40)

which, with eqns (3.36) for the scattered field gives the integral equations (3.41) directly.
APPENDIX B

MULTIPLE SCATTERING

The integral equation
\[ G(r) + \lambda \int G(c) K(r, c) \, dc = F(r) \]
can be re-written in terms of the scattered field \( G^s \), since \( G = G^s + F \).

Whence
\[ G^s(r) + \lambda \int G^s(c) K(r, c) \, dc = -\lambda \int F(c) K(r, c) \, dc \]
or in linear operator notation
\[ G^s + \mathcal{L}(G^s) = -\mathcal{L}(F) \]

In the case of two objects \( O_1 \) and \( O_2 \), and a primary source \( F \), the scattering from each is determined by the total field acting on each one individually, i.e., the primary field plus the field scattered by the other object. Hence two equations are required

\[ G^s_1 + \mathcal{L}_1(G^s_1) = -\mathcal{L}_1(F_{T2}) \]
\[ G^s_2 + \mathcal{L}_2(G^s_2) = -\mathcal{L}_2(F_{T2}) \]

where \( F_{T1} = F + G^s_1 \); \( F_{T2} = F + G^s_2 \) are the total fields acting on each object. Using these explicitly

\[ G_{1, s} + \mathcal{L}_1(G_{1, s}) = -\mathcal{L}_1(G_{2, s}) - \mathcal{L}_1(F) \]
\[ G_{2, s} + \mathcal{L}_2(G_{2, s}) = -\mathcal{L}_2(G_{1, s}) - \mathcal{L}_2(F) \]

We now introduce the following notation:
\[ G^s = G^{s0} + G^{s1} \]

Where \( G^{s0} \) is the scattered field in absence of other object
\( G^{s1} \) is the additional scattered field due to the presence of the other object.

These relations are now substituted into the equations above.
and yield
\[
\left[ g_{1}^{s_{0}} + \mathcal{L}_{1} (g_{1}^{s_{0}}) \right] + \left[ g_{1}^{s_{1}} + \mathcal{L}_{1} (g_{1}^{s_{1}}) \right] = -\mathcal{L}_{1} (g_{2}^{s_{0}}) - \mathcal{L}_{1} (g_{1}^{s_{1}}) - \mathcal{L}_{1} (F)
\]
\[
\left[ g_{2}^{s_{0}} + \mathcal{L}_{2} (g_{2}^{s_{0}}) \right] + \left[ g_{2}^{s_{1}} + \mathcal{L}_{2} (g_{2}^{s_{1}}) \right] = -\mathcal{L}_{2} (g_{1}^{s_{0}}) - \mathcal{L}_{2} (g_{2}^{s_{1}}) - \mathcal{L}_{2} (F)
\]

But for isolated scattering, by definition
\[
g_{1}^{s_{0}} + \mathcal{L} (g_{1}^{s_{0}}) = -\mathcal{L} (F)
\]

so that the equations become,
\[
g_{1}^{s_{1}} + \mathcal{L}_{1} (g_{1}^{s_{1}}) = -\mathcal{L}_{1} (g_{2}^{s_{0}}) - \mathcal{L}_{1} (g_{2}^{s_{1}})
\]
\[
g_{2}^{s_{1}} + \mathcal{L}_{2} (g_{2}^{s_{1}}) = -\mathcal{L}_{2} (g_{1}^{s_{0}}) - \mathcal{L}_{2} (g_{1}^{s_{1}})
\]

which may be re-written as
\[
g_{1}^{s_{1}} + \mathcal{L}_{1} (g_{1}^{s_{1}} + g_{2}^{s_{1}}) = -\mathcal{L}_{1} (g_{2}^{s_{0}})
\]
\[
g_{2}^{s_{1}} + \mathcal{L}_{2} (g_{1}^{s_{1}} + g_{2}^{s_{1}}) = -\mathcal{L}_{2} (g_{1}^{s_{0}})
\]

This is the required result. Two self-consistent integral equations denote the addition fields due to their mutual interaction in terms of their (presumed known) scattering properties when isolated.

The total field is given finally by
\[
g_{1}^{s_{0}} + g_{1}^{s_{1}} + g_{2}^{s_{0}} + g_{2}^{s_{1}} + F
\]

The operators \( \mathcal{L}_{1} \) and \( \mathcal{L}_{2} \) take account of the position of the objects relative to the chosen co-ordinate system.
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