FUNCTIONAL ANALYSIS AND IDENTIFICATION
OF SEPARABLE NONLINEAR CONTROL
SYSTEMS USING PSEUDORANDOM INPUTS

by

E.L. MOORE, BSc.

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ABSTRACT

The analysis and identification of separable nonlinear single valued systems is carried out from a functional standpoint, by modifying the Volterra series to separate bias and steady state gain from dynamic effects. This analysis is applied to the development of generalised expressions for output bias, variance and correlation functions of nonlinear systems with Gaussian or pseudo-random inputs. An identification procedure is then developed and applied to the testing of both simulated systems, and an electrohydraulic servomotor. An error analysis is carried out showing the limitations of the method, and procedures derived designed at eliminating the effects of random and cyclic noise.
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CHAPTER 1

INTRODUCTION
In the last thirty years the field of control has grown from being an empirical art to a mathematical discipline in its own right. It has become so important that all engineers, regardless of specialization, require to understand at least its basic principles. This has been largely due to the pressures of modern industrial economics, with automation playing an ever increasing part in our lives.

It is impossible to point to any particular path of research which has led to present day knowledge. Possibly the greatest contributor to control theory has been the computer engineer. The availability of fast, cheap processing power has resulted in the conception of more complex systems. Their design and optimisation in terms of cost and performance has led to the necessity for adaptive controllers of ever increasing sophistication, in turn leading to advances in optimisation, modelling and identification techniques.

Within the broader field of control, systems modelling and identification are subjects of very large scope; the first is used in design and behaviour prediction, and the latter for the construction of models of actual plant. The two subjects are often considered separately, but very strong ties exist between them. Until recently all identification methods were based on algorithms to fit the parameters of a model of predetermined structure, such that its behaviour matched in some way that of the system being identified. The impulse response function is regarded by the author as being non-parametric. This is, however, a point which is open to discussion. Non-parametric methods, such as the impulse response determination from correlation analysis of linear systems, also rely on a model to transform the results into data from which a physical system can be constructed. Recently, however, with the advent of hypothesis testing, and the modelling of uncertainty [1], identification methods have been designed which rely far less on fixed model structure; the algorithms fit model structure as well as model parameters [2]. Perhaps ironically, these techniques have been derived using advanced modelling and analysis theory.

Modern control theory stems from the adoption of state space system representation [3]; the application of statistics and probability concepts to such models has culminated in modern filtering, estimation
and prediction theory. The basic theoretical ideas are not new, Kalman's first paper on the subject [4] having been published in 1960, but the refinements have been many. The interest which these methods has aroused is shown by the number of publications on the topic (reference [1] contains over two hundred).

Choice of identification method is dictated by a wide range of considerations; sometimes one technique is given preference simply due to the availability of expertise and/or equipment. Identification methods differ according to the class of models, the class of input signals, and the criteria used. Implementation and data processing requirements can provide other subdivisions. Such structure differences are considered in some detail by Åström and Eykhoff in their survey paper on system identification [61].

Rated among the most popular and easily applied identification methods are those based on one-shot techniques. Correlation and deconvolution schemes fall within this category. Such methods have been mostly applied to linear systems. Nonlinear identification using such methods has also been studied, but with less success apart from simple cases.

Iterative techniques are more recent and becoming very popular. These have the advantage of being applicable to on-line identification. Within this category fall model-reference, hill climbing and stochastic approximation methods. Kalman filtering, Bayesian estimation and Maximum Likelihood can also be regarded as iterative identification methods. Challenging applications, requiring combinations of the above methods, occur when system as well as state parameters are to be identified simultaneously. Probabilistic control theory, as well as application of sophisticated dynamic optimisation techniques [1][62] are often needed to tackle such problems.

If the concept of identification is examined in some detail, obstacles are met of a philosophical as well as a mathematical nature. According to Zadeh [66], 'identification is the determination, on the basis of input and output, of a system within a specified class of systems, to which the system under test is equivalent'. As formulated by Åström, equivalence is often defined in terms of a criterion or a
loss function which is a functional of the process output $y$ and the model output $y_m$, i.e.

$$V = V(y, y_m)$$

where $V$ is the loss function. The identification then becomes an optimisation problem, solution being achieved when the global minimum of $V$ has been found. This raises the familiar, but not easily solved, class of problems involving global minima, uniqueness and convergence. The choice of the loss function is often critical in the solution of the identification problem. The importance of a suitable choice is demonstrated by the way in which a linear system is identified in the frequency or time domains. Small differences in the estimated impulse response can correspond to large differences to the system transfer function. The loss function, therefore, should be chosen to bear relationship to the desired system performance.

Model reference identification schemes all rely on a loss function in conjunction with an adaptation of one of the popular optimisation algorithms such as steepest descent, conjugate gradients, Newton or least squares. All such algorithms can be modified to take into account the stochastic degradation of derivatives and other measurements, whereby a weight is associated with each new measurement which, typically exponentially, decreases with each new observation. Stochastic approximation methods fall within this category.

Conceptually, Kalman linear system identification is more difficult, though it can be shown to be related to recursive least squares identification. The Kalman filtering theory is applied to identification by introducing the parameters of the identification problem as state variables. Application of Bayesian statistics allows the extension of the Kalman filter to nonlinear system identification, though approximate analysis is needed for practical calculations. Reference [1] gives a very good background to the theory of state space, recursive estimation, prediction and filtering theory in the time domain.

Less in the limelight have been the methods of identification, based on correlation techniques and specialized test sequences. These methods were originally restricted to linear, single input-single output systems, but subsequently such limitations have been lifted. The
extensions to the more complex cases have been possible due to the continued research into the generation and the properties of specialized test signals, as well as into more refined and flexible nonparametric representations of nonlinear systems. Of significant impact to correlation methods of identification have also been the availability of hardware allowing faster computation, and the discoveries of more efficient computation algorithms. One line of research is that into the relationships between dyadic time and linear time domains; the results are only now beginning to be fully exploited and already crosscorrelation is almost as efficient as FFT algorithms [5].

Among the specialized system test signals, possibly the most popular and widely used one is the pseudo-random binary sequence, or PRBS. It is easily generated by hardware [6] or, using Galois Field difference equations [7], by software. PRBS has only two possible values at any instant (arbitrarily +1 or 0), and has the property of being a good approximation to band limited white noise. Literature describing its uses [8][49] is extensive and available in many standard textbooks. PRBS is very attractive as a test signal if correlations are needed, because its binary nature removes the need for multiplications; the only operations required are additions and possibly a single division. This applies for both software and hardware correlations. Other useful features of PRBS include the ease with which uncorrelated sequences, still possessing the properties of band limited white noise, can be generated from it. Briggs and Godfrey [9] showed that if a PRBS of bit length N is multiplied by S different Rademacher functions [10] of length $2^k$ bits, forming S new sequences, the crosscorrelation function between any two of the new sequences, with integration carried out over $2^k N$ lags, is zero for all lags. Further, the sequences have the band limited white noise property, their autocorrelation functions being similar to that of PRBS, but with period $2^k N$ rather than N. A disadvantage of the new sequences is the increased integration times required, but nevertheless they are very useful for testing multivariable systems.

Bell [11] gives an indication of the methods by which uncorrelated sequences can be applied in the case of multivariable systems which are also nonlinear. The theory is also covered in detail by Atherton [12].
Such methods as outlined by Bell employ an extension to the statistical describing function [12] approach to the analysis of nonlinear elements. The quasilinearizations are, however, coarse if such theory is directly applied to systems containing linear as well as nonlinear elements. It disregards the fact that, although the first order crosscorrelation function between two signals can be zero for all lags, higher order crosscorrelation functions do not necessarily have the same property [13]. The theoretical results obtained by Bell are similar in fact to the ones obtained in this thesis, the difference being that Bell considered nonlinear elements distinct from linear dynamic ones, and then applied his theoretical results to actual composite systems, containing both linear as well as nonlinear elements. The experimental results he obtains are accurate because he tests only Hammerstein type systems. These involve systems where the nonlinearities are concentrated in a single block preceding all the linear dynamics of the system. It will be shown in this thesis that such arrangements are very particular; they exhibit behaviour which in some respects can be considered linear when the dynamic components of the system inputs are discrete sequences.

A different line of study has been concerned with the optimal use of PRBS in minimizing errors due to measurement drift and nonreversible system dynamics. The two problems are very different, but both have been tackled by employing the characteristics of PRBS itself. The most common method of correcting for drift has been the use of reference phase PRBS. Notable among the papers produced are those by Davies [14], Ream [15] and Barker [16]. It was found that if the PRBS used has a certain starting point in the sequence or phase, then low frequency polynomial drift is eliminated by the correlation process. The optimal starting phases were determined, as well as those for which minimum errors are obtained in the presence of higher order drift. D J Moore [17] used the reference phase method in conjunction with a modified correlation algorithm to improve further the drift immunity with small added computation costs. Other methods relying solely on special algorithms, such as those of Nikiforuk [18], are mostly centred around convolution summations, using alternating sign binomial coefficients, operated on the measured system outputs. These methods, however, require large computational effort.
Another problem which was tackled by optimising the PRBS, is that of nonreversible system dynamics. It was found by D J Moore [19] that the presence of such effects produces echoes of the system impulse response at various time delays within the otherwise normal cross-correlation function. He discovered that the echoes appear at different points along the time axis, according to the position of the feedback loops around the shift register used to generate the PRBS. The best location of the echoes is obviously along the time axis at lags greater than the system's settling time; Moore found that the echoes can be positioned at optimum lag values by the use of particular feedback paths. These in fact correspond to the setting up of an optimum second order Galois Field difference equation. D J Moore [17] tabulated the optimum feedback paths for PRBS lengths of up to 1023 bits. Godfrey and Moore [50] further discuss aspects of the dynamics of direction dependent nonlinear systems, their principal contribution being the analytical explanation of the discontinuities occurring in the experimental crosscorrelation functions obtained from gas turbine tests. Their analysis, however, is confined to first order systems.

Other test signals with pseudo-noise properties have been studied for a long time [49]. The most popular among these is the three-level maximum length sequence, known also as the ternary pseudo-random sequence, or TPRS. This is a member of the broader class of p-level maximal length sequences, p being odd [20], but applications have been mostly limited to TPRS. In linear systems testing TPRS is rarely used, its only advantage in an ideal test situation being that it is unbiased. However, this advantage is not great as the effects of such bias on the estimated system impulse response are easily eliminated. On the other hand, TPRS does offer particular attractions when noise is present on the system output. A method based on TPRS is developed in this thesis for the elimination of the effects of cyclic noise, and one due to Brown [21] deals with those due to drift; he uses the inverse repeat property of TPRS in an algorithm for the elimination of polynomial drift effects up to the rth degree, by crosscorrelating the system output over (r + 2) half periods, appropriate weights being associated with each half period.

The inverse repeat property of TPRS can be stated mathematically; as
\[ x\left(t + \frac{kT}{2}\right) = -x(t) ; \quad k \text{ integer} \]  

(1.1.1)  

T being the period of \( x(t) \). TPRS is used extensively in nonlinear systems identification. It is as a consequence of the inverse repeat property that all its even order autocorrelation functions are zero. Mathematically,

\[ \frac{1}{T} \int_{0}^{T} x(t)x(t + \tau_1)x(t + \tau_2)...x(t + \tau_{2k}) = 0 \quad k \text{ integer} \]  

(1.1.2)  

This relationship also holds for any other inverse repeat signal.

The problem of identifying nonlinear systems using correlation techniques is complex and even in ideal noise free situations existing methods are still approximate. In fact it is unlikely that exact techniques will ever become available, unless tailored to a particular situation. This is because nonlinearities themselves cannot be described by a single set of mathematical expressions. Linear systems can be described by systems of linear differential equations. Nonlinear ones, on the other hand, require different types of equations, and the types change from case to case. The situation is further complicated when the nonlinearity is an integral part of the system, the nonlinear and linear dynamics being inseparable. Making the problem even more difficult is the fact that mathematical description of the behaviour of nonlinear systems usually involves the determination of high order statistical properties of their inputs. This has proved practically impossible even for digital sequences [44][45]. The exception has been the case of white noise inputs, but even for this case difficulties still occur [46][47][48]. Some simplifications can be made when the methods proposed in this thesis are adopted, and extensions have been made to cover more the general cases of non-white Gaussian inputs.

Nevertheless, providing the system is single-valued, the functional representation of systems [22] does offer a generalized, nonparametric method of writing an expression for the system output. Practically all correlation identification methods for nonlinear systems are based on such representations. Simpson and Power [23] reviewed these methods in
1972. It shows that, up to the publishing date, practically all algorithms designed to identify nonlinear systems fit the parameters of one of the models shown in Fig.1.1.1.

Volterra's original functional representation [63] is non-parametric and very general, but also difficult to use directly in the design of an identification algorithm. The restriction to separable systems, where the linear and nonlinear elements are uncoupled, and then to a particular combination of these elements with all the nonlinear effects lumped into a single element, converts the identification problem into a semiparametric one. The linear elements are still represented by impulse response functions, but the nonlinear one is restricted to being a single valued, memoryless transformation. The fact that some algorithms are only suited to Hammerstein and others to Wiener Models, as defined in Fig.1.1.1 imposes further parametric nature on the identification. In fact the Wiener Model is much more general than the Hammerstein one; with certain input conditions as shown in this thesis, Hammerstein Models exhibit some linear system characteristics.

The fact that a system is fitted to a model does not necessarily mean that the system has the same structure as that of the model. It means that, with the input conditions imposed by the identification, the system performance is equivalent to that of the model. If the system is nonlinear different system input conditions result in different parameters being fitted to the model. For this reason, test signals should be either the normal system inputs, or be superimposed onto such inputs with the minimum disturbance to the system. Large test signals, when these are different from the normal operating ones, should not be used as the 'identified' model will behave differently from the real system under normal running conditions. Thus random test signals are usually to be preferred to deterministic ones. It will be shown in this thesis that the randomness should extend to the amplitude, and not just the time and frequency domains.

Not all the nonlinear system identification algorithms so far discovered rely on a specific model. In fact in 1964 Lee and Schetzen published a report [24] describing a method of identification, quite independent from assumed model structure. The method is simple, but the computation costs are very high. In 1972 French [25] further
Fig. 1.1.1 Definition of separable nonlinear system types.
extended Schetzen's theoretical ideas. The unsuitability of Schetzen's approach lies in the fact that \( n \)-dimensional crosscorrelation functions have to be evaluated for the identification of the \( n^{th} \) term of the functional series. French applied the same ideas, but in the frequency domain, employing multidimensional Fourier transforms. Due to the efficiency of the FFT algorithms, as compared to time crosscorrelation, the method is much more viable. The result of the identification is the derivation of the Wiener kernels. The Wiener functional series \([38]\) is a functional expansion for the output of nonlinear systems with the property that the kernels are orthogonal when the system inputs are white noise. In the identification of the \( n^{th} \) Wiener kernel, for an \( N \) bit input, French showed that the saving in time denoted by \( \Delta_1 \), using the frequency domain rather than time domain methods, is given by

\[
\Delta_1 = \frac{1}{2} Nn!
\]

For a 1024 bit sequence, the third order kernel is identified 3072 times faster in the frequency domain. Even so, the amount of computation required is enormous, requiring approximately \( k_o \) multiplications, with FFT procedures, where \( k_o \) is given by

\[
k_o = \frac{2N^n}{n!}
\]

In the search for even more efficient methods of identifying the terms of the Wiener functional series, French and Butz next used Walsh functions \([10]\) to expand the kernels \([26]\). The Walsh functions form an orthogonal set with properties not unlike the sine and cosine functions, but in dyadic time rather than linear time domain. As an example of the difference, the linear time shift \((t + \tau)\) is in fact \((t \oplus \tau)\) in dyadic time, the symbol \(\oplus\) representing modulo-2 addition. The results obtained by French involve kernels in the dyadic domain. At that time the relationships between the two domains were not very well understood, and even now much research needs to be carried out. However, the advantages of working in the dyadic domain are enormous due to the efficiency of the Walsh-Fourier transforms \([10]\), which involve only additions or subtractions. The availability of the 'Fast Walsh Transform', a parallel to the FFT, increases the efficiencies even more.
FWT's are particularly useful where more than one dimension is required, and this fact is extensively used in image processing [10]. References [27] to [34] give some idea of the work which is being undertaken to find relationships between dyadic and linear time domains.

This research has also given some very useful by-products, such as the discovery by Pitassi [5] of an algorithm to perform convolution or crosscorrelation in the linear time domain needing less multiplications than those required by an equivalent FFT; the FFT procedures become more efficient when time series longer than 1024 samples need to be processed. However, apart from these very useful by-products, the research into the relationships between linear and dyadic time domains has not produced, as yet, a complete answer and the very efficient techniques, such as those employed by French and Butz to obtain the functional expansion in dyadic time domain create results which are difficult to interpret.

Other generalized methods of obtaining the functional expansion of nonlinear systems without making restrictions on the model being fitted have been tried [35][36], but the computation costs are very high indeed. Furthermore the methods which do rely on a model have produced some good results, as demonstrated by the publications given as references in this thesis.

Whether a simplified or the generalized functional description is used, the output from any single valued nonlinear system can be written in the form [37][38]

\[ y(t) = A y_1(t) + A^2 y_2(t) + \ldots + A^i y_i(t) + \ldots A^n y_n(t) \]  \hspace{5em} (1.1.3)

where \( A \) is an amplitude measure of the input and \( y_1(t) \) to \( y_n(t) \) are time functions. It can also be shown that

\[ y_1(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) \, d\tau \] \hspace{5em} (1.1.4)

where \( h(\tau) \) is an impulse response function, and \( x(t) \) is a normalised system input of unit amplitude. The actual system input is \( A x(t) \).
Expression (1.1.3) is a power series, and (1.1.4) a convolution integral. In the special case of separable linear and nonlinear effects, and when no feedback appears around the nonlinear elements, \( h(t) \) is the impulse response function of the linear part of the system [39]. Proofs of these statements are also given in this thesis. The representation of the system output given by equation (1.1.3) is very general.

Most nonlinear identification methods based on random signal excitation rely on TPRS inputs to eliminate selectively either all the even or all the odd terms from this equation. As pointed out by Gardiner [40], carrying out two tests at two different amplitudes, say \( A \) and \( B \), employing in both cases three level sequences, enables the selective elimination of one other term from (1.1.3), usually chosen to be \( y_3(t) \). This is because an implicit assumption is made that \( |y_i(t)| \) decreases with 'i'. Krempl [41] and Tunis [42] have extended and refined Gardiner's approach, testing with 'r' different amplitudes, eliminating 'r' terms. The objectives of such procedures are twofold. Either to eliminate all terms for 'i' greater than or equal to two, obtaining in that way an estimate of the linear dynamics of the system, or by taking the procedure just one step further, obtain estimates of all the first \( y_r(t) \) functions. It is easily shown [43] that such functions are related to the coefficients of a power law which describes the nonlinear, dc system input-output function. Krempl and Tunis carry out exactly this procedure. These methods were subsequently applied, with some success, to the identification of the nonlinear gain characteristics of various large scale industrial plants [64][65]. With certain conditions good results can be achieved but, according to the systems being tested, the method can be considered to have strong disadvantages. Consider the diagram in Fig. 1.1.2:

Fig. 1.1.2 Superimposition of test signal \( x_T(t) \) on normal operating signal \( x_0(t) \).
This represents a system under working conditions, \( x_0(t) \) being the normal operating signal, usually a slowly changing function of time. By 'slowly changing' is meant with comparison with \( x_I(t) \), the test signal used to identify the system at its operating point. The actual system input is thus \( x_1(t) \). To apply Gardiner's theory would require changing the amplitude of \( x_1(t) \); since \( x_0(t) \) is usually biased, changing the amplitude of the system input \( x_1(t) \) involves also changing the system operating point. Situations can be quickly imagined when such a test is not allowed. Further, with a biased system input, the elimination of all even or all odd terms from the series (1.1.1) is no longer possible because the system input is then no longer a simple unbiased TPRS, and the identification becomes significantly less accurate. Another case when the method seems lacking is illustrated in Fig. 1.1.3. In this case the system is fed with unbiased TPRS signals with different amplitudes, the object being to identify the nonlinear characteristic shown. To obtain estimates of the gain at the extremes of the operating range, very large amplitude test signals are needed; in such cases these will still be affected by the deadband shown about null. Large amplitudes, and a nonlinear effect which is felt on every test, are not good characteristics. Further, a bias cannot be imposed on the test signal without the significant cost of losing the beneficial effects of the three level sequence.

The work described in this thesis is concerned with the development of a functional series which can be applied to separable nonlinear system identification without the disadvantages mentioned above. It will be shown how the linear and nonlinear characteristics may be separately determined independently of the bias level of the system input, using TPRS signals of small amplitude. System excitation, data collection and correlation procedures have all been carried out automatically using a mini-computer system developed by the author. Accordingly, the next chapter describes, in broad terms, the software used and the system organization. The three chapters following it are concerned with theoretical development of the functional series, its properties and application for specific system inputs. Then follow three chapters on experimental procedures and results with, finally, the concluding remarks.
Fig. 1.1.3 Injection of unbiased TPRS signals into (L-NL-L) system.
CHAPTER 2

SYSTEM ORGANISATION
2.1 Introduction

This chapter contains a description of the mini-computer system, and the supporting software which was used to configure and carry out systems identification experiments. The role of the computer was two-fold. Firstly, to automate the system identification process involving random signal generation, data acquisition of the system response and correlation procedures for the separation of linear and nonlinear characteristics within the system. Secondly, to provide exact linear and nonlinear system models against which the identification process could be tested under controlled conditions. The chapter is divided into two main parts: system organization of the hardware and supporting software for use in conjunction with the identification algorithms described in Chapter 6.

2.2 System organization

The most important tool for an engineer engaged in signal processing work is a computer. Not only can the computer carry out data acquisition from an experiment, it can also control the experiment itself as well as undertake subsidiary calculations. Minicomputers are ideally suited to this type of work as they are relatively inexpensive, retaining nevertheless reasonable processing power. Such a machine has been used in connection with the research into identification methods described in this thesis. The acquisition of such peripherals as disc units and line printers further helped with the task of software development.

The major part of this chapter is devoted to the methods which were used to maximise the processing power of the system and provide flexibility in structuring experiments.

2.2.1 Description of computer system

The minicomputer used was an Interdata Model 7/16, with 32 KB of memory. This computer features an IBM System 360/370 like instruction set, direct addressing of 32 KB (expandable to 64 KB) of core,
Fig. 2.2.1 Hardware system organization.
16 general registers, and queued interrupts. The arithmetic instruction set includes Floating Point Multiply/Divide, but these operations are carried out by software routines.

The peripherals which were available included a twin disc drive with 10 MB capacity, line printer, CRT and teletype terminals, twin cassettes with 5 MB capacity and an analogue I/O module. This provides eight multiplexed inputs and one output, with 12-bits precision. Also provided are two 10-bit D/A channels and a pen-control module, which were used for graphics work. The maximum throughput rates are of the order of 10 kHz.

The software included an operating system (denoted DOS) to handle device interrupts, an Assembler and an editor program. In Fig. 2.2.1 is shown a block diagram representation of the hardware system organization. The most important points relate to the transmission paths of data to and from the discs, the analogue I/O channels and core. As the diagram shows, data transmission between core and discs is through the selector channel. This has the characteristic of being a direct memory access device. Data transfers occur without the need for processor intervention, requiring only an initialization phase. The multiplexor bus is, in contrast, a request/response communication link, and therefore data transfers are more time consuming than via the selector channel. The analogue I/O module is connected to the multiplexor bus, and this places stringent requirements on the software needed to drive it, in order to achieve high throughput rates and still allow time for some inter-sample processing. However, it also means that spooling techniques to transfer sampled data to the disc while more sampling is being performed can be carried out without difficulty.

2.2.2 Prime objectives

Prior to the successful use of the minicomputer in an experimental environment several problems of an organizational nature had to be solved. These arose as a consequence of the type of work which was planned. The requirements were for a system which would allow the user to configure any desired experiment quickly, to carry out that experiment with maximum control, to process the resulting data as quickly and as accurately as possible, to display the results in graphical form on an X-Y plotter or
oscilloscope and finally, to save the results in any desired form on one of the available storage devices.

2.2.3 General system concepts

The most efficient programs are those written in Assembler. They are faster in execution time, most economical in core utilisation, give the greatest versatility, and allow the computer's processing power to be fully exploited. Accordingly, a policy decision was made to make exclusive use of Assembler in preference to high level languages such as FORTRAN. To meet at the same time the requirement of efficient experiment configuration and data handling, it was decided to employ a software interface between the user and the program library; this interface program was called 'Resident Support Program' (RSP). RSP is described in detail in the next section; briefly its functions can be classed under the headings of memory management, experiment configuration and data handling.

2.3 RSP

Prior to a summary of the facilities made available by RSP and the software involved, a brief description is necessary to explain how an operator communicates with the interface program.

With reference to the memory map in Fig. 2.3.1, it is seen that the operating system (DOS) and RSP, reside in low core. Together they occupy approximately 16 KB of memory, leaving only one half of the total available core for other programs and data. This proved sufficient even for complex matrix routines, due to the efficient core utilisation of Assembler code and small data arrays. The use of integer rather than floating point numbers whenever possible was also helpful in achieving this objective.

When the system is initialised, the user is automatically placed in communication with RSP. Any commands issued to the computer are thus made through the facilities provided by the interface program. The flow diagram for the RSP command decoding subroutine is shown in Fig. 2.3.2. The basic structure of this routine is simple, but the
details are complicated by the many special cases for which it must cater. Basically the computer is placed in the WAIT state ready to receive input from the system console whenever the processor is not engaged in program execution. This is the normal mode of operation, but with the difference that the console input is routed through the RSP command processor, rather than DOS.

![Memory map showing location of RSP and DOS.](image)

**Fig. 2.3.1** Memory map showing location of RSP and DOS.

The command structure of RSP itself allows direct access to DOS so that both command sets are available. RSP allows batch processing of commands as well as the setting up of any physical device as a system console; the response to console inputs can be made to log on any device. This is especially useful in the setting up of long sets of experiments, requiring different programs and intermediate data storage and/or retrieval.

The command decoding subroutine flow diagram is simplified as it does not show the location of all the error traps, used to prevent
external corruption of software or data. Such traps involve memory searches to detect overwrite conditions, format and parameter size checks. Some of the routines involved in providing these fail-safe conditions are described in the following sections.

2.3.1 Memory management

By memory management is meant the allocation of core, its partitioning and the procedures by which programs are loaded from their bulk storage device into core.

During the running of a typical experiment or a complex program execution, a common requirement is the handling of data banks. For speed such data should be resident in core, but its location can affect the efficient loading or running of other programs. An optimal location for data banks, or fields as they are sometimes called, is the top of core. RSP is designed to handle such fields.

A facility is incorporated to allow the definition of new fields, and the allocation to them of symbolic names. Upon RSP initialisation the top of core is free. At any time after initialisation the user can reserve an area of core for a new field and allocate to it a symbolic name; this is done with a single command. RSP upon request reserves subsequent fields starting from the variable location 'UT'; see Fig.2.3.3. When no fields are defined, 'UT' corresponds to the top of core. The memory map shown in Fig. 2.3.3 shows the effect of defining a new field called N1, of 3 KB, when the core situation is as in (a) and the effect of erasing field A3 when the core situation is as in (b).

A table is kept by RSP of all fields and their characteristics; this is called FTAB. Upon receiving the appropriate command a list is logged to the system console giving a memory map which includes the allocated symbolic names. The user can reference data banks by their allocated symbolic name rather than their addresses, as RSP carries out, when needed, a search for the locations in FTAB. FTAB is also used to prevent the same symbol being used for two different fields, or the reservation of an area of core of such a size that it would overwrite RSP, or other programs loaded above it.
Fig. 2.3.2 Flow diagram for command decoding subroutine.
2.3.2 Experiment configuration

Unless an experimental procedure has been well established, it is likely that several types of analysis have to be carried out on any set of experimental data. It is also possible that such data is insufficient and more tests are required.

The processing of the data, as well as its acquisition involves a number of programs and each one will, in general, require externally set parameters. RSP provides a method of achieving this objective together with easy access to the programs. To make this possible all programs have to be in the form of subroutines, and must be stored on the discs. The subroutines must be written in a standard form; this was chosen to
conform with the Interdata Assembler specifications which also makes them compatible with FORTRAN IV.

To execute such an Assembler program, the user types in a command having the format

\[ C \circ PROG, p_1, p_2, \ldots, p_n \]

where 'PROG' is the actual program name, and \((p_1, \ldots, p_n)\) are the parameters. RSP handles the disc search for 'PROG', its loading and the transfer of the parameters input from the console into the subroutine parameter table. The parameters may be either numbers or field names. In the first case RSP converts the numbers to binary format, prior to their passage as arguments to 'PROG', and in the second case it searches 'FTAB' for the address of the referenced field and passes that to 'PROG'. Once the parameters are decoded, RSP assembles a simulated call to 'PROG' which is then executed, causing the loaded subroutine to be run. Upon its termination, control is returned to the calling program, namely RSP.

An average program, occupying perhaps 1 KB of core is loaded and execution begun in one tenth of the time taken to type in the calling command. This is a very rough measure of speed, serving only as an indication of order of magnitude.

2.3.3 Data handling

Under this heading fall the display facilities as well as inter-peripheral and memory data transfer routines including in RSP.

The display options are extensive. They include graphis routines for analogue plotters, with fully decoded alphanumerics as well as those for oscilloscope display. Scaling is automatic by default, to yield maximum utilisation of the D/A converters. This facility can be bypassed allowing the user to fix the scales at any desired set of values; this is useful when plotting more than one function on the same graph. There are several other utility routines available to help in the selection of appropriate scales and the annotation of graphs. These include axis drawing and lettering, 'zero' and 'calibrate' commands to set up the analogue plotter amplifiers, manual pen-up and pen-down commands, and so on. All graphs can be drawn in either vector or 'dot'
modes. Plotting on to an oscilloscope can be either in the continuous refresh mode, or the interrupt mode. The latter allows the execution of other programs concurrently to the display operation. An immediate visualisation of the progress of an experiment can thus be obtained although flickering occurs when more than 50 data values are plotted. Flicker free display for up to 500 points can be obtained using the dedicated continuous refresh mode.

The data transfer options, for inter-peripheral communication are also extensive, including ASCII-binary routines for floating point as well as integers. Data transfers between memory and disc are particularly efficient, making full use of the facilities available. A limited set of arithmetic instructions is also included in RSP. It is possible to multiply or divide the numbers in any field by an integer power of 2, and to add or subtract constants. This applies only to fields containing integers as the arithmetic routines provided by RSP are intended only to help scaling in connection with plotting. A full set of arithmetic subroutines is kept on disc, available by the 'C' command already described.

2.3.4 Limitations and concluding remarks on RSP

The present version of RSP has two limitations. The first and major one is that the plotting routines are written for integer arrays only. It was thought that with efficient integer to floating point routines this would not matter, especially if appropriate scaling were provided prior to conversion. However this has proved laborious at times, and has led to inefficient coding due to the presence of round-off errors. The second major disadvantage lies with the size of RSP. Such routines as the alphanumerics package are rarely used and wasteful in core. An improvement would be the utilisation of overlay techniques, based on the disc system.

2.4 Scaling of integer arrays

A method of reducing core usage which has been applied extensively, is to keep data in integer form as much as possible. The use of integer data formats has a further important advantage in that the software to
handle such data is more efficient; in the computer used, the floating point instruction set is based on software routines whose efficiency cannot be compared to that of the microcoded instructions handling fixed point software. In such machines, without floating point hardware, the speed advantage is of the order of one hundred.

The Interdata 7/16 computer is a 16 bit machine, which utilizes 2's complement arithmetic. This means that the range of integer numbers which can be used is 32767 to -32768. One method of increasing this range is to use double precision integer arithmetic, but that is also time consuming and does not help in the case of numbers smaller than one. The problem was overcome by the adoption of block exponent scaling, which is now briefly described. Let a number 'A' be written as

\[ A = N \times 2^K \]

where \( N \) and \( K \) are integers in the range 32767 to -32768. Then \( A \) does not have a practical restriction on its range, or its smallest size. In some computers this is the basis of floating point data storage. The block exponent technique is different in as much as whole arrays have the same value of \( K \) at any time, and \( K \) is chosen so that the numbers within the array can be kept within specified limits, maintaining integer format. Whenever a number becomes too large to remain within the specified limits, the whole array is divided by 2, and \( K \) for that array increased by one; conversely, whenever a number within the array becomes less than one, the opposite procedure is employed. This technique works particularly well if all the numbers are within five orders of magnitude of each other.

In the case of the organization of the computer for this experimental work, the number \( K \) is kept by RSP, and a different value is associated with each field. One of the characteristics stored in FTAB is the scale of the field and a facility is included to allow running programs to update \( K \).

2.5 Supporting software

Four of the programs most used for identification experiments are described below in some detail. These are the analogue I/O routines,
simulators, correlation and general systems test programs. In the following sections capital letters appearing as program parameters denote field symbolic names. The term \( XX(i) \) refers to the \( i \)th element of field \( XX \).

### 2.5.1 Analogue I/O routine

The RSP calling sequence is

\[
C \langle ADC, IN, OU, n, \Delta t, a, p \rangle
\]

This program is used to simultaneously excite a system with a signal contained digitally in OU, and sample its response storing it in field IN. The program is ideally suited to the situation when a system needs to be excited with a repetitive signal, and the response averaged over several runs. Referring to the calling sequence, \( 'n' \) is the number of points in the fields IN and OU, \( '\Delta t' \) the sampling interval in units of 100 \( \mu s \), \( 'a' \) the number of averaging runs required. If the system initial transients are not desired a few starting runs are required to allow these to die away. The number of these 'preruns' is set by \( 'p' \). If the transients are required \( 'p' \) is set to zero. In a typical correlation experiment employing PRBS, \( p \) can be set to one as the length of the PRBS period must be longer than the system's settling time. The flow diagram in Fig. 2.5.1 helps with the explanation of some of the details of program ADC.

To enable the averaging operation it is necessary to use a dummy array of double precision integer numbers. The A/D converters have 12-bit resolution with 2's complement outputs, which means that the full scale inputs have a magnitude of 2047. Since the largest integer which can be handled is 32767, if more than 16 averages are required there is danger of overflow. For this reason the sampled data is added in to a double precision integer array, and divided after all sampling is over. The resulting sequence of numbers is then stored in array 'IN'. The sample rate is fixed by a hardware, programmable clock, which sends interrupts at the specified frequency. The interrupts are used to trigger the machine into a routine which outputs and then, 7 \( \mu s \) later, inputs a data point. The 7 \( \mu s \) delay is due to the execution time of the software which outputs the preceding data point. This routine also increments all relevant indexes, and adds in the sampled data to the
Fig. 2.5.1 Flow diagram for analogue I/o program 'ADC'.
dummy arrays providing the specified number of preruns has been executed. When these operations are finished the routine puts the machine in the WAIT state, ready for the next clock interrupt. The maximum sample rate of 10 KHz is set by the execution time of the software. Other analogue I/O programs which require less arithmetic operations between samples can achieve faster throughput rates.

When the sampling and averaging operation is over, the clock interrupts are disabled, the double precision data array is divided by the number of averaging runs performed, and the data stored in 'IN'.

The subroutine ADC is then terminated, and control returned to the calling program. This is usually RSP.

2.5.2 Simulators. Floating point programs

Two simulators were extensively used; one for nonlinear and one for linear systems. They are described separately below.

2.5.2.1 NSIM; nonlinear gain element simulator

The RSP calling sequence is

\[
\text{C @ NSIM, AA, BB, NN, n}
\]

AA is the name of an array containing a set of 'n' values which are inputted to the nonlinear gain element resulting in the array named BB. The nonlinear element is an amplitude dependent gain function with the input-output behaviour of the element determined by the values given in the array named NN. Either a piecewise linear or a polynomial type system are simulated, according to whether NN(1) = 1 or 0. If NN(1) = 1 then the output \( y \) is a function of the input \( x \) given by

\[
y = \sum_{i=2}^{12} \frac{\text{NN}(i)}{i-1} x^{i-1}
\]

Thus, up to an 11th order polynomial nonlinearity can be simulated.

If NN(1) = 0 then the simulated input-output behaviour is as shown in Fig. 2.5.2. The parameters of the piecewise linear relationship are set by the values in NN according to:
Fig. 2.5.2 *Nonlinear element simulated by program 'NSIM'.*
NN(2) = G1
NN(3) = R1
NN(4) = G2
NN(5) = R2
NN(6) = h

See Fig. 2.5.2 for the meaning of these variables.

2.5.2.2 DSIM; linear system simulator

The RSP calling sequence is

C \@ DSIM, NU, DE, LG, XT, YT, n, o, p

This program simulates the linear system with transfer function \( T(S) \) whose z-transform is \( T(z) \) given by

\[
T(z) = \frac{y(z)}{x(z)} = \left( \frac{a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{20} z^{-20}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{20} z^{-20}} \right) z^{-k}
\]

The coefficients \( a_i \) of the numerator are given in NU, \( b_i \) of the denominator in DE and the pure lag in LG. In XT are 'n' data values representing the excitation signal, resulting in 'n' data values for the response in YT.

The program has a memory of the past 20 sample values of input and output. It should therefore be initialized the first time that it is called so that all past inputs and outputs are made equal to a preset value, given by p. Any preset value can be chosen except -10000.0 which signifies that the past should be remembered.

A special feature of DSIM is that up to four systems can be simulated continuously. This implies that four separate sets of past events are remembered, the choice between them being made by the value of 'O'; this can be either 1, 2, 3 or 4, depending on which system is being simulated. These facilities allow the type of problem outlined in the flow diagram given in Fig. 2.5.3 to be handled very easily.
The situation described by the flow diagram is the typical one in which two linear systems are cascaded onto each other and are driven by a repetitive signal $x(t)$. On the first few passes, depending on time constants, the initial transients are included in their response, settling to the steady state response. Note that the input and output waveforms can be put in the same array once the input is no longer desired. This is done for system (2). The intermediate processing of
y(t) in between system (1) and system (2) could have been replaced by an operation modifying y(t), as for example that carried out by a nonlinear element.

2.5.3 Correlation programs. Integer routine

The RSP calling sequence is

\[ C @ COR, PR, YT, AC, n \]

and the operation which is carried out is

\[ AC(i) = \left( \frac{1}{n} \sum_{j=i}^{n} PR(j) YT(j + i) \right) 2^R \]

The value of R is chosen by the computer so that the resulting sequence in AC is composed of the largest possible numbers. This is done to maintain precision. The program is written for integers; the corresponding floating point programs are rarely used because, although more accurate, the execution times are much longer. The integer version will complete a 256 point correlation in 2 seconds, and a 128 point correlation in 1/8 of a second.

2.5.4 General purpose test program (GPTP)

This is a program which has been designed to test out the algorithms developed in later chapters. It contains a range of processing routines including the simulation, correlation and sampling programs described in the last sections. The flow diagram given in Fig. 2.5.4 shows the method in which these programs are linked together to form a powerful research and systems test tool. The options available enable the setting up of a wide range of experiments; the options are chosen by setting parameters in the calling sequence of GPTP. In the flow chart the options are set by the conditional boxes with 'OP' as arguments. By referring to the flowchart it can be seen that the three stage system representing the general nonlinear model (L-NL-L) can be tested or simulated in any combination required.
Fig. 2.5.4 Simplified flow diagram of general purpose test program simulation and system excitation subroutine.
The RSP calling sequence is

\[ C @ GPTP, a, m1, m2, \ell, KK, n, RR, \Delta t, s \]

where

- \( a \) = amplitude of maximal length digital system driving sequence;
- \( m1, m2 \) = initial and final bias values;
- \( \ell \) = level number of maximal length sequence;
- \( KK \) = array of values which sets up parameters of simulated system;
- \( RR \) = array to hold results of test;
- \( n \) = number of averages to be taken per sample;
- \( \Delta t \) = sample rate
- \( s \) = interval between bias values.

The program enables a set of correlation tests to be carried out over a range of operating points. These are set to scan from \( m1 \) to \( m2 \) with step 's'. At each operating point the correlation test yields a value which is stored in \( RR \). This is the small signal gain of the system. The correlation procedures used to obtain this result are described in Chapter 6. In some cases two tests are carried out at each operating point, with different amplitudes. In such a case the user must give two amplitudes \( a1 \) and \( a2 \) rather than simply \( a \). The array \( KK \) holds data which effectively sets up the software switches shown in Fig. 2.5.4, as well as the parameters of the simulated systems. It is also possible to set up excitation of an external system rather than that of a simulated one.

2.6 Conclusions

In this chapter a brief description has been given of the principal components of the minicomputer system which was used in the research into nonlinear system analysis and identification carried out by the author.

The following chapters are concerned with the description of the theory developed and the experimental results obtained using such theory as well as the system just described.
CHAPTER 3

MODIFIED FUNCTIONAL ANALYSIS OF SEPARABLE NONLINEAR SYSTEMS
3.1 Introduction to the Volterra analysis of nonlinear systems

For a linear time invariant (LTI) system, given the impulse response \( g(t) \), the response \( y(t) \) to any input \( x(t) \) may be computed by means of the convolution integral [51]

\[
y(t) = \int_{-\infty}^{\infty} g(\tau) x(t-\tau) \, d\tau \quad (3.1.1)
\]

For physically realizable systems \( h(\tau) = 0 \) when \( \tau < 0 \) and hence

\[
y(t) = \int_{0}^{\infty} g(\tau) x(t-\tau) \, d\tau \quad (3.1.2)
\]

If \( x(t) = 0 \) when \( t < 0 \), \( t = 0 \) being an arbitrary time instance, then

\[
y(t) = \int_{0}^{t} g(\tau) x(t-\tau) \, d\tau \quad (3.1.3)
\]

For the case of a system subjected to a repetitive signal of period \( T \), and if the transient response is not required, the mathematical convention is to allow the input to have started in the infinite past, and equation (3.1.2) is used. The Laplace Transform (single sided) may be taken of both sides of (3.1.2) yielding

\[
Y(s) = G(s) X(s) \quad (3.1.4)
\]

where \( G(s) \) is a Transfer Function \( (s = j\omega) \), and \( Y(s), X(s) \) are the Laplace Transforms of the response and input to the linear system.

Equations (3.1.1) to (3.1.3), for linear systems, may be regarded as a specific case of more general representations of nonlinear systems notable among which is the Volterra functional series and its counterpart in multidimensional Laplace Transforms. The response of a single valued, time invariant, nonlinear system may be represented by a series of the type

\[
y(t) = \sum_{i=1}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} g_1(\tau_1, \tau_2, \ldots, \tau_i) x(t-\tau_1)x(t-\tau_2)\ldots x(t-\tau_i) \, d\tau_1 \, d\tau_2 \cdots d\tau_i \quad (3.1.5)
\]
where $g_i(\tau_1,\tau_2,\ldots,\tau_i)$ is the $i^{th}$ order Volterra kernel, and the symbol $\int \ldots \int$ represents $i!$ identical integral signs. A comparison between equations (3.1.2) and (3.1.5) shows that any LTI system is a member of the class of systems describable by (3.1.5). This is seen if the first few terms of the infinite series (3.1.5) are expanded.

$$y(t) = \int_0^\infty g_1(\tau)x(t-\tau)d\tau + \int_0^\infty \int_0^\infty g_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1d\tau_2 + \ldots$$

$$+ \int_0^\infty \ldots \int_0^\infty g_i(\tau_1,\tau_2,\ldots,\tau_i)x(t-\tau_1)x(t-\tau_2)\ldots x(t-\tau_i)d\tau_1d\tau_2\ldots d\tau_i \quad (3.1.6)$$

(i)

The first term in this series is the convolution integral as used in linear systems analysis. An extensive investigation by George [52] shows how the higher order terms arise, and introduces the concept of multidimensional Laplace Transforms when applied to systems. Two fundamental properties of the Volterra series which have been discussed by George and others [51][39] are mentioned here for completeness.

The first one arises quite naturally from analysis. It states that the order of the kernels is directly related to the order of the nonlinearity being described; the second order kernel arises from a squaring operation, the third from a cubing one and so on.

The second property is one of limitation. For a general nonlinear system, the output can be expressed by an infinite series. However there are no easily applied convergence rules or guide lines to help estimate the errors involved by its truncation. The convergence problem is definitely not trivial; when the series converges it yields the correct result. A divergent series however does not necessarily indicate system instability [51]. There is always the possibility of finding a more general functional series that will converge; George calls this the 'iteration series'. However such problems are beyond the scope of this thesis as only separable nonlinear systems are considered, when convergence is assured.[39]: For such systems the kernels of the functional series are themselves separable and may be simplified.
For the $i^{th}$ kernel

$$g_i(t_1, t_2, \ldots, t_i) = c_i g(t_1) g(t_2) \ldots g(t_i) \quad (3.1.7)$$

where $c_i$ is a constant. George, through his operational method, shows how to combine the various elements of systems so as to decompose cascade or feedback arrangements into other equivalent simplified ones. The importance of his work is that it shows that an identification procedure capable of dealing with the system shown in Fig. 3.1.1 is really quite general, provided that the system is separable.

The combination of nonlinear systems has also been investigated by Barret [67] and more recently by Kielkiewicz [39]; they give several guidelines on the determination of the convergence properties of the combined systems in particular cases. An exact or general set of convergence properties is not known to exist at the present time.

### 3.2 Some properties of Volterra kernels

Without loss of generality the kernels $g_i(t_1, t_2, \ldots, t_i)$ can be considered symmetric functions of $t_1, t_2, \ldots, t_i$. If this were not so the kernels could be made symmetric by linear transformations without affecting the Volterra series prediction of the system response [53]. As an example, consider the second order kernel $g_2(t_1, t_2)$. If this were not symmetric it could be replaced by

$$g_2(t_1, t_2) = \frac{1}{2} [g_2(t_1, t_2) + g_2(t_2, t_1)] \quad (3.2.1)$$
The kernels $g_2(\tau_1, \tau_2)$ and $g_2'(\tau_1, \tau_2)$ have the same effect when placed in the Volterra series, and the latter is obviously symmetric. This implies that, in general

$$g_1(\tau_1, \tau_2, \ldots, \tau_i) = g_1(\tau_2, \tau_1, \ldots, \tau_i)$$

$$= g_1(\tau_2, \tau_1, \ldots, \tau_i) \text{ etc} \quad (3.2.2)$$

For the case of separable kernels

$$g_1(\tau_1, \tau_2, \ldots, \tau_i) = c_1 g(\tau_1)g(\tau_2), \ldots, g(\tau_i) \quad (3.2.3)$$

and symmetry is obvious; $g(\tau)$ is usually an impulse response function.

A new notation is now introduced. Define the functional

$$H^i_g(t, x_1, x_2, \ldots, x_i) = \int g(\tau_1)g(\tau_2), \ldots, g(\tau_i)x(t-\tau_1)x(t-\tau_2), \ldots, x_i(t-\tau_i)$$

$$\delta \tau_1, \delta \tau_2, \ldots, \delta \tau_i \quad (3.2.4)$$

The limits of integration must be assumed to be 0 to $\infty$ unless otherwise specified. In the special case when $x_1 = x_2 = \ldots = x_i (= x \text{ say})$ then the operator can be written $H^i_g(t, x);$

$$H^i_g(t, x) = \int g(\tau_1)g(\tau_2), \ldots, g(\tau_i)x(t-\tau_1)x(t-\tau_2), \ldots, x_i(t-\tau_i)d\tau_1d\tau_2\ldots d\tau_i \quad (3.2.5)$$

When there is no ambiguity as to which system is being described the '$g$' is dropped from this notation.

Consider now the case when $x(t)$ is formed by the sum of two signals $x_1(t)$ and $x_2(t);

$$x(t) = x_1(t) + x_2(t) \quad (3.2.6)$$

It will be shown that the functional $H^i(t, x)$ takes a particular and predictable form. The case for $i = 1$ is examined first.
\[ H(t,x) = H(t,x_1 + x_2) = \int g(\tau) [x_1(t-\tau)+x_2(t-\tau)]d\tau \]
\[ = H(t,x_1) + H(t,x_2) \quad (3.2.7) \]

For \( i = 2; \)

\[ H^2(t,x) = H^2(t,x_1 + x_2) \]
\[ = \int \left[ g(\tau_1)g(\tau_2)[x_1(t-\tau_1)+x_2(t-\tau_1)]d\tau_1 d\tau_2 \right] \]
\[ = \int \left[ g(\tau_1)g(\tau_2)x_1(t-\tau_1)x_1(t-\tau_2)d\tau_1 d\tau_2 \right] \]
\[ + \int \left[ g(\tau_1)g(\tau_2)x_1(t-\tau_1)x_2(t-\tau_2)d\tau_1 d\tau_2 \right] \]
\[ + \int \left[ g(\tau_1)g(\tau_2)x_2(t-\tau_1)x_1(t-\tau_2)d\tau_1 d\tau_2 \right] \]
\[ + \int \left[ g(\tau_1)g(\tau_2)x_2(t-\tau_1)x_2(t-\tau_2)d\tau_1 d\tau_2 \right] \]

\[ H^2(t,x) = H^2(t,x_1) + H^2(t,x_2) + H^2(t,x_1,x_2) + H^2(t,x_2,x_1) + H^2(t,x_2) \quad (3.2.8) \]

The symmetry property (3.2.2) is used and (3.2.8) becomes

\[ H^2(t,x_1 + x_2) = H^2(t,x_1) + 2H^2(t,x_1,x_2) + H^2(t,x_2) \quad (3.2.9) \]

The result given in (3.2.9) has a parallel in the algebraic identity
\[(x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2.\]
The concept can be extended, using such identities as

\[(x_1 + x_2)^2 = x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3 \quad (3.2.10)\]

and

\[(x_1 + x_2 + x_3)^3 = [(x_1 + x_2)^3 + (x_2 + x_3)^3 + (x_3 + x_1)^3] + x_1^3 + x_2^3 + x_3^3 + 3x_1x_2x_3 \quad (3.2.11)\]
Shetzen has utilized this property [53] to obtain relationships for decomposition of \( n \)th order kernels, and uses it to suggest a simple identification procedure. The main disadvantage is that for its implementation 'n' identical systems are needed.

A case of particular relevance to the investigation described in this thesis arises when \( x(t) \) is a biased signal, with dynamic component \( \tilde{x}(t) \) and mean \( 'm' \). Then

\[
x(t) = m + \tilde{x}(t)
\]  

(3.2.12)

If \( g(t) \) is the impulse response of a system with unity gain, then

\[
H(t,m) = m \int g(\tau)d\tau = m
\]  

(3.2.13)

\[
H^i(t,m) = m^i \int g(\tau_1)...g(\tau_i)d\tau_1...d\tau_i = m^i
\]  

(i)

(3.2.14)

Also for \( x(t) \) when \( i = 1 \);

\[
H(t,x) = H(t,m+x) = m + H(t,x)
\]  

(3.2.15)

when \( i = 2 \);

\[
H^2(t,x) = H^2(t,m+x)
\]  

(3.2.16)

\[
= H^2(t,m) + 2H^2(t,m,x) + H^2(t,x)
\]

\[
= m^2 + 2mH(t,x) + H^2(t,x)
\]

\[
= [m + H(t,x)]^2
\]  

(3.2.17)

Equation (3.2.17) follows from (3.2.16) by application of the relationships already derived in this section. The general rule

\[
H^i(t,x) = H^i(t,m+x) = [m + H(t,x)]^i
\]  

(3.2.18)

will be proved in a very simple manner.
From (3.2.5)

\[
H_i(t, m+x) = \int g(\tau_1)g(\tau_2)\ldots g(\tau_i) \prod_{r=1}^{r=i} [m+x(t-\tau_r)] d\tau_1 d\tau_2 \ldots d\tau_i \tag{3.2.19}
\]

This is simplified to become

\[
H_i(t, m+x) = \left[ \int g(\tau) [m+x(t-\tau)] d\tau \right]^i
\]

\[
= \left[ m \int g(\tau) d\tau + \int g(\tau) x(t-\tau) d\tau \right]^i
\]

\[
= \left[ m + H(t, x) \right]^i \tag{3.2.20}
\]

and (3.2.18) is proved. These relationships were derived for systems with separable kernels. Systems with non-separable kernels cannot be treated in this way and equivalent relationships do not exist. It can also be proved that

\[
H^0(t, x) = 1 \tag{3.2.21}
\]

\[
H_i(t, x) = H^{i-1}(t, x)H^{1}(t, x) \tag{3.2.22}
\]

Such relationships follow by direct application of the principles outlined in this section.

3.3 Volterra series representation of a nonlinear system's response when subjected to biased inputs. Modified Volterra series.

Fig. 3.3.1 Nonlinear system driven by signal x(t).
Fig. 3.3.1 represents a nonlinear open loop system. The linear element with impulse response $g_1(t)$ and gain $K_1$ is cascaded onto an amplitude dependent, single valued nonlinear element $N$. This in turn is cascaded onto the linear element with impulse response $g_2(t)$ and gain $K_2$. The nonlinearity is described by its input output relationship

$$v(t) = N[u(t)] = \sum_{i=0}^{n} \alpha_i[u(t)]^i$$  \hspace{1cm} (3.3.1)

and is therefore termed a "polynomial nonlinearity" (PNL). The input signal $x(t)$ can be deterministic or random. Let

$$x(t) = m + \chi(t)$$ \hspace{1cm} (3.3.2)

$$m = \mathbb{E}[x(t)]$$ \hspace{1cm} (3.3.3)

Equations (3.3.2) and (3.3.3) state that the signal $x(t)$ is composed of a dynamic component, written $\chi(t)$, and a mean level $m$. $'\mathbb{E}'$ is the time expectation operator defined by

$$\mathbb{E}[\cdot] = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [\cdot] dt$$ \hspace{1cm} (3.3.4)

From linear systems theory the output from the first linear element, $u(t)$, is given by

$$u(t) = K_1 m + K_1 \int_{0}^{t} g_1(\tau)x(t-\tau)d\tau$$ \hspace{1cm} (3.3.5)

$$= K_1 [m + H_{g_1}(t,x)]$$ \hspace{1cm} (3.3.6)

The output from the nonlinearity, $v(t)$, can be found by substitution of (3.3.6) into (3.3.1) giving

$$v(t) = \sum_{i=0}^{n} \alpha_i K_1^i [m + H_{g_1}(t,x)]^i$$ \hspace{1cm} (3.3.7)
If (3.3.7) is expanded and powers of $H_{g_1}(t,x)$ collected

$$v(t) = [a_0 + a_1 K_1 m + a_2 K_1^2 m^2 + \ldots + a_n K_1^n m^n]$$

$$+ [a_1 K_1 + 2a_2 K_1^2 m + \ldots + n a_n K_1^n m^{n-1}]H_{g_1}(t,x)$$

$$+ \ldots$$

$$+ [a_n K_1^n]H_{g_1}^n(t,x)$$  \(3.3.8\)

Equation (3.3.8) can be written as

$$v(t) = \beta_0(m) + \beta_1(m)H_{g_1}(t,x) + \beta_2(m)H_{g_1}^2(t,x) + \ldots + \beta_n(m)H_{g_1}^n(t,x)$$

$$= \sum_{i=0}^{n} \beta_i(m)H_{g_1}^i(t,x)$$  \(3.3.9\)

where the $\beta_i(m)$ terms are given by

$$\beta_i(m) = \sum_{r=i}^{n} \frac{x^r}{i!(x-i)!} a_r K_1^r m^{r-1}$$  \(3.3.10\)

The binominal coefficient $P_{c_1}$ can be used in the expression for $\beta_i(m)$ yielding

$$\beta_i(m) = \sum_{r=i}^{n} C_r a_r K_1^r m^{r-1}$$  \(3.3.11\)

The $\beta_i(m)$ coefficients are examined in detail after the derivation of the expression for the output $y(t)$ of the nonlinear system. From (3.3.9) the input to the linear element with impulse response $g_2(t)$ and gain $K_2$ is known. Its output corresponds to the output of the complete nonlinear system and is given by
\[ y(t) = K_2 \int g_2(\lambda) \sum_{i=0}^{n} \beta_i^1(m) H_1^i(t-\lambda,x) \, d\lambda \]

\[ = K_2 \sum_{i=0}^{m} \beta_i^1(m) \int g_2(\lambda) H_1^i(t-\lambda,x) \, d\lambda \]  

\[ (3.3.12) \]

The output from the nonlinear system is thus expressed in terms of a series containing two sets of terms, one being represented by the \( \beta_i^1(m) \) coefficients and the gain \( K_2 \), and the other by the multidimensional convolutions. Of importance is the fact that these convolutions involve only unbiased signals and unity gain elements, while all the effects due to bias and gain are contained in the time independent \( \beta_i^1(m) \) coefficients and the gain \( K_2 \).

The representation of the system output given in (3.3.12) will be termed 'modified Volterra series representation'. The significance of \( \beta_i^1(m) \) coefficients is examined in the next section, and recursive relations given to simplify their derivation and application.

### 3.3.1 The \( \beta_i^1(m) \) coefficients

A better understanding of the form taken by the \( \beta_i^1(m) \) coefficients can be obtained with the aid of an example. Accordingly, below are listed the coefficients for a system containing a nonlinearity whose input output equation is

\[ v(t) = \sum_{i=0}^{5} a_i [u(t)]^i \]  

\[ (3.3.13) \]

\[ \beta_0^1(m) = a_0 + a_1 K_1^m + a_2 K_1^2 m^2 + a_3 K_1^3 m^3 + a_4 K_1^4 m^4 + a_5 K_1^5 m^5 \]  

\[ (3.3.14) \]

\[ \beta_1^1(m) = \alpha_1 K_1^1 + 2 \alpha_2 K_1^2 m^2 + 3 \alpha_3 K_1^3 m^3 + 4 \alpha_4 K_1^4 m^4 + 5 \alpha_5 K_1^5 m^4 \]  

\[ (3.3.15) \]

\[ \beta_2^1(m) = \alpha_2 K_1^2 + 3 \alpha_3 K_1^3 m + 6 \alpha_4 K_1^4 m^2 + 10 \alpha_5 K_1^5 m^3 \]  

\[ (3.3.16) \]
\[ \beta_3(m) = c_3 K_1^3 + 4\alpha_4 K_1^4 m + 10\alpha_5 K_1^5 m^2 \quad (3.3.17) \]

\[ \beta_4(m) = c_4 K_1^4 + 5\alpha_5 K_1^5 m \quad (3.3.18) \]

\[ \beta_5(m) = \alpha_5 K_1^5 \quad (3.3.19) \]

These are functions of bias \( m \), gain \( K_1 \) and nonlinearity coefficients \( \alpha_i \).

By noting that the bias of the signal entering the nonlinearity is \( K_1 m \), the coefficient \( \beta_0(m) \) takes special significance. If \( m_u = K_1 m \) is considered to be the input to the nonlinear element then from (3.3.14) and (3.3.13) it is seen that its output is in fact \( \beta_0(m) \). Deriving \( \beta_0(m) \) is thus straightforward, given the nonlinear element input-output relationship.

\[ \beta_0(m) = \sum_{i=0}^{n} \alpha_i(K_1 m)^i = N(K_1 m) \quad (3.3.20) \]

Inspection of equations (3.3.14) to (3.3.19) and checking with (3.3.11) yields the following recursive relationship

\[ \beta_{i+1}(m) = \frac{1}{(i+1)} \frac{d}{dm} \beta_i(m) \quad (3.3.21) \]

Repeated application of (3.3.21) gives

\[ \beta_1(m) = \frac{1}{i!} \frac{d^i}{dm^i} \beta_0(m) \quad (3.3.22) \]

Equation (3.3.22) is useful to derive the coefficients, but is especially significant when it is used in conjunction with the nonlinear system output equation; \( y(t) \) becomes

\[ y(t) = K_2 \sum_{i=0}^{n} \frac{1}{i!} \frac{d^i}{dm^i} \beta_0(m) \int g_2(\lambda)H^i_{g_1}(t-\lambda,x) d\lambda \quad (3.3.23) \]

The output equation has been expressed in terms of convolution operations between unbiased signals and unity gain linear elements, multiplied by coefficients which are themselves weighted derivatives of the nonlinear input output d.c. relationship. The derivative weights decrease in
magnitude rapidly, the first five being 1, 0.5, 0.167, 0.042 and 0.008.

The expression for the nonlinear element output, \( v(t) \) also provides some interesting relationships.

\[
v(t) = \sum_{i=0}^{n} \frac{1}{i!} \frac{d^i}{dm^i} \beta_0(m) H_i(t, x) \quad (3.3.24)
\]

Expression (3.3.24) is a Taylor Series \([54]\). Accordingly it can be summed, yielding

\[
v(t) = \beta_0 [m + H_g(t, x)] \quad (3.3.25)
\]

Examination of the \( 'H' \) functional shows that

\[
H_g(t, x) = \frac{1}{K_1} u(t) \quad (3.3.26)
\]

and therefore

\[
v(t) = \beta_0 [m + \frac{1}{K_1} u(t)] \quad (3.3.27)
\]

This relationship follows more directly from (3.3.20). However the usefulness of the derivation as given here is that the connection is established between the modified Volterra series and the Taylor series. Of some interest is the fact that the error involved in truncating the series at the \( p \) th term is of order \( Q_p \left[ \frac{1}{K_1} u(t) \right] \) where

\[
Q_p \left[ \frac{1}{K_1} u(t) \right] = \frac{1}{(p+1)!} \left[ \frac{1}{K_1} u(t) \right]^{p+1} \frac{d^p}{dm^p} \beta_0 \left[ m + \frac{\theta}{K_1} u(t) \right] \quad ; \quad 0 < \theta < 1 \quad (3.3.28)
\]

\[
= \frac{1}{p} H_g(t, x) \beta_p \left[ m + \frac{\theta}{K_1} u(t) \right] \quad ; \quad 0 < \theta < 1 \quad (3.3.29)
\]

This follows from pure mathematics \([54]\) and application of the recursive relationships which exist between the \( \beta_1(m) \) coefficients.

The expression for the output \( y(t) \) (3.3.23) is equivalent to that for the output of the system shown in Fig. 3.3.2, drawn for the case of a system with a third order nonlinearity. This is the equivalent parallel representation of the general (L-PNL-L) system when this is described by the modified Volterra
series. The gains \( \beta_x(m) \) are functions of bias \( m \). The element 'E' is the expectation operator.

Further insight into the role of the \( \beta_x(m) \) coefficients is obtained by expressing them in matrix form.

\[
\begin{pmatrix}
\beta_0(m) \\
\beta_1(m) \\
\beta_2(m) \\
\beta_3(m) \\
\beta_4(m) \\
\beta_5(m)
\end{pmatrix} = 
\begin{pmatrix}
1 & m & m^2 & m^3 & m^4 & m^5 \\
1 & 2m & 3m^2 & 4m^3 & 5m^4 & 6m^5 \\
1 & 3m & 6m^2 & 10m^3 & 20m^4 & 30m^5 \\
1 & 4m & 10m^2 & 20m^3 & 35m^4 & 56m^5 \\
1 & 5m & 10m^2 & 21m^3 & 35m^4 & 56m^5 \\
1 & 6m & 20m^2 & 36m^3 & 63m^4 & 90m^5
\end{pmatrix}
\begin{pmatrix}
k^0_0 \\
k^0_1 \\
k^0_2 \\
k^0_3 \\
k^0_4 \\
k^0_5
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5
\end{pmatrix}
\tag{3.3.30}
\]

or

\[
\bar{\beta} = \bar{M} \cdot \bar{K} \cdot \bar{a}
\tag{3.3.31}
\]

Denoting the element of the \( i^{th} \) row and \( j^{th} \) column of the \( \bar{M} \) matrix as \( Q_{ij} \), then the elements \( Q_{1j}, Q_{2j}, \ldots Q_{jj} \) can be obtained by expansion of
\[(m+1)^{j-1}\]. When the input bias is zero, then the \( \overline{M} \) matrix becomes the unit matrix \( \overline{I} \) and

\[
\beta_i(m) = a_i \overline{K}_i \tag{3.3.32}
\]

The \( \overline{M} \) matrix shows that coupling exists between the \( \alpha_i \) coefficients due to the bias \( m \); with zero bias there is no coupling. Defining a coupling matrix \( \overline{M}_c \) as

\[
\overline{M}_c = \begin{pmatrix}
0 & m^2 & m^3 & m^4 & m^5 \\
0 & 2m & 3m^2 & 4m^3 & 5m^4 \\
0 & 3m & 6m^2 & 10m^3 & 20m^4 \\
0 & 4m & 10m^2 & 20m^3 & 30m^4 \\
0 & 5m & 10m^2 & 20m^3 & 30m^4 \\
\end{pmatrix}
\tag{3.3.33}
\]

then

\[
\overline{\beta} = \overline{I} \overline{K} \overline{\alpha} + \overline{M}_c \overline{K} \overline{\alpha} \tag{3.3.34}
\]

The first term on the RHS of (3.3.34) is the one obtained when the inputs are unbiased, and the second is the coupling term. The absence of the \( i \)-th nonlinear element coefficient \( \alpha_i \) does not mean that \( \beta_i(m) \) is also zero. Such a condition occurs only with unbiased inputs. This is an important property of the modified Volterra series.

Another property of the \( \beta_i(m) \) coefficients which is useful in the analysis of feedback and multivariable systems is summarised by (3.3.35). If \( \Delta \) is in units of bias 'm' then it can be shown that

\[
\begin{pmatrix}
\beta_0(m+\Delta) \\
\beta_1(m+\Delta) \\
\beta_2(m+\Delta) \\
\beta_3(m+\Delta) \\
\beta_4(m+\Delta) \\
\beta_5(m+\Delta)
\end{pmatrix} = \begin{pmatrix}
1 & \Delta & \Delta^2 & \Delta^3 & \Delta^4 & \Delta^5 \\
1 & 2\Delta & 3\Delta^2 & 4\Delta^3 & 5\Delta^4 \\
1 & 3\Delta & 6\Delta^2 & 10\Delta^3 \\
1 & 4\Delta & 10\Delta^2 \\
1 & 5\Delta \\
1
\end{pmatrix} \begin{pmatrix}
\beta_0(m) \\
\beta_1(m) \\
\beta_2(m) \\
\beta_3(m) \\
\beta_4(m) \\
\beta_5(m)
\end{pmatrix} \tag{3.3.35}
\]
The matrix containing the Δ values is identical to the M matrix, with Δ substituted for m. These relationships can be applied to prove the following;

\[ \beta_2(m) = \frac{1}{2!} \frac{\beta_1(m+\Delta) - \beta_1(m)}{\Delta} \quad ; \Delta^2 \text{ small} \] (3.3.36)

\[ \beta_3(m) = \frac{1}{3!} \frac{\beta_1(m+\Delta) - 2\beta_1(m) + \beta_1(m-\Delta)}{\Delta^2} \quad ; \Delta^3 \text{ small} \] (3.3.37)

\[ \beta_4(m) = \frac{1}{4!} \frac{\beta_1(m+2\Delta) - 3\beta_1(m+\Delta) + 3\beta_1(m) - \beta_1(m-\Delta)}{\Delta^3} \quad ; \Delta^4 \text{ small} \] (3.3.38)

Equations (3.3.36) to (3.3.38) are the finite difference equivalents of the derivative relationships connecting the \( \beta_1(m) \) coefficients given in (3.3.21). In each case, if a division is effected with \( \Delta P \), then \( \Delta^{P+1} \) is assumed negligible. These relationships can be used to study the behaviour of systems with discontinuous nonlinearities, as is pointed out in Chapter 7.

Before developing some relationships for the crosscorrelation functions, another set of coefficients is defined which are used in later Chapters.

3.3.2 The \( B_i(m) \) coefficients

These are defined by

\[ B_i(m) = \beta_i(m) \bigg|_{k=1} \] (3.3.39)

It can be proved that

\[ B_i(mK) = \frac{1}{K^i} \beta_i(m) \] (3.3.40)

If a signal \( x(t) = m + x(t) \) is input to a nonlinear element with input-output relationship given by

\[ v(t) = \sum_{i=0}^{n} a_i [u(t)]^i \] (3.3.41)
then the nonlinear element output can be shown to be given by

\[ v(t) = \sum_{i=0}^{n} R_i(m) [x(t)]^i \]  

(3.3.42)

3.4 The crosscorrelation function. Repetitive inputs

![Crosscorrelation arrangement](image)

In Fig. 3.4.1 is shown the correlation arrangement. In this section is developed a general formula for \( \phi_{xy}(\tau) \). The system output bias is denoted \( m_y \). For the general case

\[ \phi_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t)x(t+\tau)dt \]  

(3.4.1)

If \( x(t) \) is a repetitive signal, and \( y(t) \) the steady state response, then (3.4.1) can be simplified becoming

\[ \phi_{xy}(\tau) = \frac{1}{T} \int_{0}^{T} y(t)x(t+\tau)dt \]  

(3.4.2)
where 'T' is now the period of \( x(t) \). If \( x(t) \) is given by

\[
x(t) = m + x(t)
\]  

(3.4.3)

then substitution of \( x(t) \) into (3.4.2) yields

\[
\phi_{xy}(\tau) = m y + \phi_{xy}(\tau)
\]  

(3.4.4)

The expression for \( \phi_{xy}(\tau) \) can be found from application of (3.3.12)

\[
\phi_{xy}(\tau) = K_2 \sum_{i=0}^{n} \beta_i(m) \int g_2(\lambda) \cdot \frac{1}{T} \int_{0}^{T} H_1^{\tau}(t-\lambda, x) x(t+\tau) dt d\lambda
\]  

(3.4.5)

It follows from (3.3.6) that

\[
H_1(t, x) = \frac{1}{K_1} u(t)
\]  

(3.4.6)

where \( u(t) \) is the dynamic component of the nonlinear element input signal. Having established (3.4.6) the expression for \( \phi_{xy}(\tau) \) becomes

\[
\phi_{xy}(\tau) = K_2 \sum_{i=0}^{n} \beta_i(m) \int g_2(\lambda) \cdot \frac{1}{T} \int_{0}^{T} [u(t-\lambda)]^i x(t+\tau) dt d\lambda
\]  

(3.4.7)

The term \( \phi_{xy}(\tau) \) is zero since \( x(t) \) is an unbiased signal. The series in (3.4.7) can be written with the term for \( i=1 \) outside the summation sign.

\[
\phi_{xy}(\tau) = K_2 \beta_1(m) \cdot \frac{1}{K_1} \int g_2(\lambda) \phi_{xy}(\tau-\lambda) d\lambda
\]  

(3.4.8)
In the special case of white or pseudo-white system inputs of variance
\( \sigma^2_x \), \( \phi_{xx}(\tau) \) is approximately a delta function at zero lag. In such a case (3.4.9) becomes

\[
\phi_{xy}(\tau) = K_2 \beta_1(m) \sigma_x^2 \int g_2(\lambda) g_1(\tau-\lambda) d\lambda \\
+ K_2 \sum_{i=2}^{n} \beta_i(m) \frac{1}{K^i} \int g_2(\lambda) \phi_{xi}(\tau-\lambda) d\lambda
\]  

(3.4.10)

Let

\[
g(\tau) = \int g_2(\lambda) g_1(\tau-\lambda) d\lambda
\]  

(3.4.11)

then \( g(\tau) \) is the impulse response of two elements with impulse response functions \( g_1(\tau) \) and \( g_2(\tau) \) in cascade. Thus

\[
\phi_{xy}(\tau) = K_2 \beta_1(m) \sigma_x^2 g(\tau) + K_2 \sum_{i=2}^{n} \beta_i(m) \frac{1}{K^i} \int g_2(\lambda) \phi_{xi}(\tau-\lambda) d\lambda
\]  

(3.4.12)

In the special case of white or pseudo-white noise inputs the expression for the crosscorrelation function is simplified, the first term of the series then corresponding to the impulse response of the linear part of the system. This result is common to conventional Volterra analysis.

By considering the significance of the \( \beta_0(m) \) coefficient and the fact that \( \beta_1(m) \) is its first derivative with respect to 'm', it can be shown (see Chapter 6) that \( \beta_1(m) \) is the small signal gain of the system shown in Fig. 3.4.2 with input bias 'm'. It follows that \( K_2 \beta_1(m) \) is the small signal gain of the general (L-PNL-L) system. This is also the magnitude of the constant amplifying \( g(\tau) \) in expression (3.4.12). It is important to note that the words 'small signal' apply only to the dynamic components of the signals; a more exact terminology is thus 'small variance'.
The first term in the expression for the cross-correlation function \( \phi_{xy}(\tau) \) is thus the impulse response of the nonlinear system when its input bias is 'm' and its input variance is mathematically 'small'.

If an estimate of \( g(\tau) \) is required, white or pseudo-white noise can be injected into the system and cross-correlation carried out in the classical way; however, the nonlinearity distorts the estimate of \( g(\tau) \) because the variance of the test signals cannot in practice be made very small.

Of interest is the fact that the cross-correlation equations can be written in terms of the bivariate probability density function \( p(x,u,\tau) \).

Equation (3.4.13) will have particular relevance in the next Chapter dealing with Gaussian inputs to the nonlinear systems. A short-hand notation will be used. Let

\[
D_1(\tau) = \frac{1}{K_1} \int g_2(\lambda) \phi_{xu} i(\tau-\lambda) \, d\lambda \tag{3.4.14}
\]

which is equivalent to

\[
D_1(\tau) = \frac{1}{K_1} \int g_2(\lambda) \left[ \int_{-\infty}^{\infty} x \, u^i p(x,u,\tau-\lambda) \, dx \, du \right] \, d\lambda \tag{3.4.15}
\]
then the expression for the crosscorrelation $\phi_{xy}(\tau)$ can be written

$$\phi_{xy}(x) = K_2 \sum_{i=1}^{n} \beta_i(m) D_i(t)$$

(3.4.16)

and that for $\phi_{xy}(\tau)$

$$\phi_{xy}(\tau) = m m + K_2 \sum_{i=1}^{n} \beta_i(m) D_i(t)$$

(3.4.17)

### 3.4.1 The $D_i(\tau)$ functions

The physical interpretation of the $D_i(\tau)$ functions is of some interest. The output of the system shown in Fig. 3.4.3 is equivalent to $D_i(\tau)$.

![Physical interpretation of the $D_i(\tau)$ functions](image)

**Fig. 3.4.3 Physical interpretation of the $D_i(\tau)$ functions.**

$D_i(\tau)$ is thus a filtered version of the crosscorrelation function between the dynamic component of the system input, and the $i$th power of the filtered output, mathematically $\phi_{xui}(\tau)$. The filtering of the system input is carried out by the element preceding the nonlinearity, and the filtering of the crosscorrelation function $\phi_{xui}(\tau)$ by the linear element following the nonlinearity. It follows that $x(t)$ and $D_i(\tau)$ have the same period and if $x(t)$ is an inverse repeat signal then so is $D_i(\tau)$. Further, since $x(t)$ is unbiased, then so is $\phi_{xui}(\tau)$ and $D_i(\tau)$. Note that $D_i(\tau)$ is independent of gains $K_1$ or $K_2$. 

3.5 Feedback around a Wiener type system

![Diagram](attachment:image.jpg)

Fig. 3.5.1 Simple feedback around Wiener type system.

The diagram in Fig. 3.5.1 shows a simple feedback arrangement around a Wiener type model. The input to the Wiener model is \( e(t) \), the error signal and not the system input \( x(t) \).

If the system output and input bias values are \( m_y \) and \( m_x \) respectively, then the bias of the signal entering the Wiener model is \( m_e \) where

\[
m_e = m_x - m_y
\]

(3.5.1)

and the dynamic component of the error signal is \( e(t) \) where

\[
e(t) = x(t) - y(t)
\]

(3.5.2)

The system output is then

\[
y(t) = \sum_{i=0}^{n} \beta_i(m_e)H^i_{g}(t,e)
\]

(3.5.3)

\[
= \sum_{i=0}^{n} \beta_i(m_e)H^i_{g}(t,x-y)
\]

(3.5.4)

The theory on the 'H' functional developed in section (3.2) shows that

\[
H^i_{g}(t,x-y) = \sum_{r=0}^{n} \frac{i!}{r!} H^i_{g}[t,x^{i-r},(-y)^r]
\]

(3.5.5)
where \( ^i C_r \) is the binomial coefficient and

\[
H^i_g(t, x^{i-r}, (-y)^r) = (-1)^r \int g(\tau_1), \ldots, g(\tau_{i-r}) x(t-\tau_1), \ldots, x(t-\tau_{i-r}) d\tau_1 \ldots d\tau_{i-r}
\]

\[
\int g(\theta_1), \ldots, g(\theta_{r}) y(t-\theta_1), \ldots, y(t-\theta_{r}) d\theta_1 \ldots d\theta_{r}
\]

This can be written as

\[
H^i_g[t, x^{i-r}, (-y)^r] = (-1)^r [w(t)]^{i-r} [z(t)]^r
\]

the physical interpretation of \( w(t) \) and \( z(t) \) being given in Fig. 3.5.2.

---

**Fig. 3.5.2** Physical interpretation of \( H^i_g(t, x^{i-r}, (-y)^r) \).
The signals $w(t)$ and $z(t)$ are the unbiased filtered versions of the system input and output respectively. In both cases the filtering operation is carried out by a linear element with an impulse response identical to that of the linear element in the Wiener model.

The system output $y(t)$ can be written in terms of $z(t)$ and $w(t)$. From (3.5.4) and (3.5.6)

\begin{equation}
    y(t) = \sum_{i=0}^{n} \beta_i(m_e) \sum_{r=0}^{i} \gamma_r w^{i-r}(t) z^r(t) \tag{3.5.8}
\end{equation}

\begin{align*}
    y(t) &= \beta_0(m_e) + \\
    &= \beta_0(m_e) + \\
    &+ \beta_1(m_e) [w(t) - z(t)] + \\
    \beta_2(m_e) [w^2(t) - 2w(t)z(t) + z^2(t)] + \\
    \beta_3(m_e) [w^3(t) - 3w(t)z(t) + 3w(t)z^2(t) - z^3(t)] + \ldots \tag{3.5.9}
\end{align*}

The expression for $y(t)$ can be further simplified

\begin{align*}
    y(t) &= \beta_0(m_e) + \\
    &= \beta_0(m_e) + \\
    &+ \beta_1(m_e) [w(t) - z(t)] + \\
    \beta_2(m_e) [w(t) - z(t)]^2 + \\
    \beta_3(m_e) [w(t) - z(t)]^3 + \ldots \\
    &= \sum_{i=0}^{n} \beta_i(m_e) [w(t) - z(t)]^i \tag{3.5.10}
\end{align*}

It would have been possible to obtain this result directly from inspection of the feedback system interpreting $e(t)$ as the input to the Wiener system and applying convolution directly. The derivation given is however more general, and serves as an example of the use of the $H_g(t,x)$ functional.
The expected result follows directly from the interpretation of $w(t)$ and $z(t)$

$$y(t) = \sum_{i=0}^{n} \beta_i (m_e) \left[ \int g(\tau)x(t-\tau)d\tau - \int g(\tau)y(t-\tau)d\tau \right]^i$$  

(3.5.11)

This equation is shown graphically in Fig. 3.5.3. The signal $f(t)$ is defined as

$$f(t) = w(t) - z(t)$$  

(3.5.12)

Since $w(t)$ and $z(t)$ are unity-gain filtered versions of the feedback system input and output signals, then

$$E[f(t)] = m_f$$  

(3.5.13)

$$= E[w(t)] - E[z(t)]$$  

(3.5.14)

$$= E[x(t)] - E[y(t)]$$  

(3.5.15)

$$m_f = m_x - m_y = m_e$$  

(3.5.16)

The outputs of system 1 and 2 are both $y(t)$. System 2 is in fact the parallel representation of a Wiener model. The physical interpretation of the feedback system output equation (3.5.1) is useful both to demonstrate the versatility of the modified Volterra analysis, and because the composite system shown in Fig. 3.5.3 can be used as the basis of a model reference identification algorithm. If system 1 contains unknown parameters, system 2 can be used as a reference model and its output compared with the actual system output, to produce a tracking algorithm. The impulse response $g(t)$ as well as the nonlinearity coefficients $\alpha_i$ can be identified at the same time. The $\beta_i (m)$ coefficients have recursive relationships and their form is fixed for specific $\alpha_i$ coefficients and gain $K_1$. 
Fig. 3.5.3 Simple feedback around Wiener type system and its parallel representation, for the case of a third order nonlinear element.
3.5.1 Crosscorrelation and feedback. Linear system

The linear case is considered first.

\[ y(t) = K \int g(\theta) x(t-\theta) d\theta \]
\[ = K \int g(\theta) x(t-\theta) d\theta - K \int g(\theta) y(t-\theta) d\theta \]

and the crosscorrelation function \( \phi_{xy}(\tau) \) is

\[ \phi_{xy}(\tau) = K \int g(\theta) \phi_{xx}(\tau-\theta) d\theta - K \int g(\theta) \phi_{xy}(\tau-\theta) d\theta \]

The functions \( \phi_{xx}(\tau) \) and \( \phi_{xx}(\tau) \) are equivalent. If \( x(t) \) is a pseudo-noise signal, with the usual perquisites on its period, then

\[ \phi_{xy}(\tau) = \sigma_x^2 [Kg(\tau) - \int g(\theta) K_L g_L(\tau-\theta) d\theta] \]

where \( g_L(t) \) is the impulse response and \( K_L \) the gain of the closed loop system, and \( \sigma_x^2 \) is the variance of \( x(t) \).

\[ \phi_{xy}(\tau) = \sigma_x^2 [Kg(\tau) - KK_L \int g(\theta) g_L(\tau-\theta) d\tau] \]

\[ = \sigma_x^2 [K_L g_L(\tau)] \]
This leads to the equation

\[ K_L g_L(\tau) = Kg(\tau) - KK_L g_o(\tau) \]  \hspace{1cm} (3.5.23)

where \( g_o(\tau) \) is the impulse response of the elements with impulse responses \( g(\tau) \) and \( g_L(\tau) \) in cascade. Since \( g_L(\tau), g(\tau) \) and \( g_o(\tau) \) are all unity gain impulse response functions both sides of (3.5.23) can be integrated with respect to \( \tau \) yielding

\[ K_L = K - KK_L \]  \hspace{1cm} (3.5.24)

\( K_L \) is the value obtained by integrating \( \phi_{xy}(\tau) \) over the period of the input signal \( x(t) \), and is equal to the closed loop gain; an expression for \( K_L \) is obtained by solving (3.5.24);

\[ K_L = \frac{K}{1+K} \]  \hspace{1cm} (3.5.25)

If the Laplace Transform is taken of both sides of (3.5.23) then

\[ K_L G_L(s) = KG(s) - KK_L G_L(s)G(s) \]  \hspace{1cm} (3.5.26)

where \( G(s) \) denotes the Laplace Transform of \( g(t) \). Solving for \( G_L(s) \) yields

\[ K_L G_L(s) = \frac{KG(s)}{1+KG(s)} \]  \hspace{1cm} (3.5.27)

The Transfer Function of the closed loop system, normalised to unity gain is \( G_L(s) \), or

\[ G_L(s) = \frac{(1+K)G(s)}{1+KG(s)} \]  \hspace{1cm} (3.5.28)

Expression (3.5.28) is that of a unity gain Transfer Function. The Inverse Laplace Transform of \( G_L(s) \) is \( g_L(t) \), the impulse response of the linear closed loop system, normalised to unity gain.

The expression for the crosscorrelation function becomes

\[ \phi_{xy}(\tau) = \sigma_x^2 \left[ \frac{1}{1+K} \right] g_L(\tau) \]  \hspace{1cm} (3.5.29)
The open loop gain $K$ affects the dynamics of the closed loop system.

3.5.2 Crosscorrelation expressions for nonlinear feedback systems.

Nonlinear element in forward path.

![Diagram of nonlinear feedback system]

Consider Fig. 3.5.5; an expression for the output $y(t)$ has already been derived (3.5.5). Let the input $x(t)$ be a pseudo-noise signal of variance $\sigma_x^2$. From (3.5.11)

$$y(t) = \sum_{i=0}^{n} \beta_i(m_e) \left[ w(t) - z(t) \right]_i$$  \hspace{1cm} (3.5.30)

$$= \sum_{i=0}^{n} \beta_i(m_e) \left[ f(t) \right]_i$$  \hspace{1cm} (3.5.31)

The crosscorrelation function $\phi_{xy}(\tau)$ follows directly.

$$\phi_{xy}(\tau) = \beta_i(m_e) \left[ \phi_{wx}(\tau) - \phi_{xz}(\tau) \right] + \sum_{i=2}^{n} \beta_i(m_e) \phi_{xi}(\tau)$$  \hspace{1cm} (3.5.32)

The signals $w(t)$ and $z(t)$ are the result of unity gain filtering operations carried out respectively on the system input and output signals, $x(t)$ and $y(t)$. The integral $\phi_{wx}(\tau)$ must be unity and that of $\phi_{xy}(\tau)$ must be equal to the gain of the closed loop system when the inputs have a mathematically
small variance. Denoting this integral by $K_C(m)$, and assuming small variances, integration of (3.5.32) yields

$$K_C(m) = \beta_1(m_e)[1-K_C(m)]$$  \hspace{1cm} (3.5.33)

The integrals are carried out over the range zero to $T$, where $T$ equals the system's settling time.

With the input signal variance small, mathematically, the higher order crosscorrelation terms are negligible, and the system can be considered to behave linearly.

The expression for $K_C(m)$, the equivalent small signal gain, is thus

$$K_C(m) = \frac{\beta_1(m_e)}{1+\beta_1(m_e)}$$  \hspace{1cm} (3.5.34)

$\beta_1(m_e)$ is the open loop 'small signal' system gain and $K_C(m)$ is the closed loop equivalent. The form of the expression for $K_C(m)$ is familiar from linear systems analysis. For the case of mathematically small input variance, the expression for the crosscorrelation function can be reduced as shown below.

$$\phi_{xy}(\tau) = \beta_1(m_e)[\phi_{xw}(\tau) - \phi_{xz}(\tau)]$$  \hspace{1cm} (3.5.35)

$$= \beta_1(m_e)[g(\tau) - K_C(m) \int g_C(\lambda)g(\tau-\lambda) d\lambda]g^2_X$$  \hspace{1cm} (3.5.36)

where $g_C(\lambda)$ is the impulse response function of the linear part of the closed loop system. It follows therefore that

$$K_C(m)g_C(\tau) = \beta_1(m_e)[g(\tau) - K_C(m) \int g_C(\lambda)g(\tau-\lambda) d\lambda]$$  \hspace{1cm} (3.5.37)

To find an expression for $g_C(\tau)$, as before Laplace Transforms are used, yielding

$$K_C(m)G_C(s) = \beta_1(m_e)G(s) - \beta_1(m_e)K_C(m)G_C(s)G(s)$$  \hspace{1cm} (3.5.38)
from which

\[
K_C(m)G_C(s) = \frac{\beta_1(m_e)G(s)}{1 + \beta_1(m_e)G(s)}
\]  

(3.5.39)

\[
G_C(s) = \frac{[1 + \beta_1(m_e)]G(s)}{1 + \beta_1(m_e)G(s)}
\]  

(3.5.40)

The Inverse Laplace Transform of the Transfer Function \(G_C(s)\) is the small signal system impulse response \(g_C(t)\) normalized to unity gain. The expression for the crosscorrelation function becomes

\[
\phi_{xy}(\tau) = \sigma^2 \left[ \frac{\beta_1(m_e)}{1 + \beta_1(m_e)} \right] g_C(\tau) + \sum_{i=2}^{n} \beta_1(m_e) \phi_{xi}(\tau)
\]  

(3.5.41)

The expressions for the linear, (3.5.27) to (3.5.29), and nonlinear, (3.5.39) to (3.5.41), cases should be compared. Strong similarities are evident; the closed loop gain expressions have almost predictable form. Of importance is the fact that the impulse response function \(g_C(t)\), appearing as the first term in the series for the crosscorrelation function, is itself a function of the nonlinear open loop gain. Thus an identification procedure designed at estimating the system gain function and the overall impulse response of the linear elements, will yield two types of data, depending on the existence of feedback around the nonlinearity.

In the open loop case, the identified impulse response function will remain constant with input operating point \(m\). However when feedback exists around the nonlinearity, the identified impulse response function will change with \(m\). This is shown by (3.5.40) and will now be demonstrated by a simple example. Consider the system shown in Fig. 3.5.6.
When the nonlinear element input has a very small variance the system behaves linearly, with a gain given by \( K_c(m) \)

\[
K_c(m) = \frac{\beta_1(m_e)}{1 + \beta_1(m_e)} \quad (3.5.42)
\]

where \( \beta_1(m_e) \) is the small signal open loop system gain.

The system Transfer Function is accordingly

\[
K_c(m)G_c(s) = \frac{\beta_1(m_e)}{s + [1 + \beta_1(m_e)]} \quad (3.5.43)
\]

The Transfer Function shows that the cutoff frequency and not only the gain, are functions of operating point.

When the nonlinearity does not exist \( \beta_1(m_e) = 1 \) and the Transfer Function becomes

\[
K_cG_c(s) = \frac{1}{s+2} \quad (3.5.44)
\]

showing the familiar effect of putting a position feedback around a first order lag system; the gain is halved and the cutoff frequency doubled. In the nonlinear case however, gain and cutoffs are variable. In Fig. 3.5.7
are shown two functions giving a graphic view of the variation of these parameters with small signal open loop gain. It must be emphasized that by 'small signal' here is meant always 'small variance'. It is the dynamic component of the signal which affects the accuracy of the results and not its bias. One other point of interest is that in open loop systems one has good control, even if not the freedom of choice, of the variance of the signal entering the nonlinearity. If feedback exists however, this control is partially lost as the system output contributes to the nonlinear element input variance.

Fig. 3.5.7 Variation of closed loop gain $K_c$ and closed loop cut off frequency $f_c'$ with open loop gain $\beta_1(m_e)$. All variances are assumed small.
3.5.3 Crosscorrelation expressions for nonlinear feedback systems. Nonlinear element in feedback path.

From the diagram, and linear theory it follows that

\[ u(t) = K \int g(\tau) e(t-\tau) d\tau \]  \hspace{1cm} (3.5.45)

The error signal \( e(t) \) can be expressed as

\[ e(t) = m_e + x(t) - y(t) \]  \hspace{1cm} (3.5.46)

where \( m_e \) is its bias. Substitution into (3.5.45) yields

\[ u(t) = K \int g(\tau) \bar{x}(t-\tau) d\tau - K \int g(\tau) y(t-\tau) d\tau + K m_e \]  \hspace{1cm} (3.5.47)

Crosscorrelating the output \( u(t) \) with \( x(t) \) yields

\[ \phi_{xu}(\theta) = K \int g(\tau) \phi_{xx}(\theta-\tau) d\tau - K \int g(\tau) \phi_{xy}(\theta-\tau) d\tau \]  \hspace{1cm} (3.5.48)

\[ = \sigma_x^2 K g(\theta) - K \int g(\tau) \phi_{xy}(\theta-\tau) d\tau \]  \hspace{1cm} (3.5.49)

Equation (3.5.49) is obtained using the assumption of pseudo-noise system inputs. If the small signal closed loop gain is denoted by \( K_N(m) \), where \( m \) is the system input bias, then if all variances are small, integration of (3.5.49) yields
Equation (3.5.50) uses the expression derived in the last section, for the integral of $\phi_{xy}(\tau)$, obtained by making the assumption of small nonlinear element input variance. The small signal equivalent gain $K_N(m)$ can thus be expressed as

$$K_N(m) = K - K \frac{\beta_1(m_e)}{1 + \beta_1(m_e)}$$ (3.5.50)

Note that the open loop small signal gain is $\beta_1(m_e)$ while the forward path gain is $K$. Expression (3.5.51) thus conforms to the form of the closed loop linear system gain $K_L$ given by

$$K_L = \frac{\text{forward path gain}}{1 + \text{open loop gain}}$$ (3.5.52)

The expression for the equivalent small signal Transfer Function of the Wiener system with nonlinear feedback can be obtained by taking Laplace Transforms of both sides of (3.5.49). Then, denoting the unity gain normalised equivalent closed loop Transfer Function by $G_N(s)$

$$K_N(m)G_N(s) = KG(s) - KG(s) \frac{\beta_1(m_e)G(s)}{1 + \beta_1(m_e)G(s)}$$ (3.5.53)

Use has again been made of an expression derived from the last section for the Laplace Transform of $\phi_{xy}(\tau)$. This is given by (3.5.39). It follows that

$$K_N(m)G_N(s) = \frac{KG(s)}{1 + \beta_1(m_e)G(s)}$$ (3.5.54)

$$G_N(s) = \frac{[1 + \beta_1(m_e)]G(s)}{1 + \beta_1(m_e)G(s)}$$ (3.5.55)

$G_N(s)$ is the unity gain normalised small signal Transfer Function for the nonlinear system with nonlinear feedback. The expression for $G_N(s)$ is identical to that for the case of linear feedback, the nonlinear element
appearing in that case in the forward path. It can be concluded that the gain of the closed loop system has a functional dependence on the nonlinear element which differs according to whether this occurs in the feedback or forward paths of the control loop; on the other hand the other system parameters are independent of nonlinear element position.

The variation of closed loop gain with $\beta_1(m_e)$ for the cases of nonlinear element in the feedback and forward paths is shown in Fig. 3.5.9.

![Variation of closed loop small signal gain as a function of nonlinear element position.](image_url)

It is important to note that $\beta_1(m_e)$ is a function of $K$ as well as the nonlinear element coefficients.

Regarding the Transfer Function, it has been shown that $G_N(s) = G_C(s)$ [$= H(s)$ say]. Let the open loop small signal Transfer Function be

$$G(s) = \frac{Q(s)}{R(s)} = \frac{\sum_{i=0}^{m_1} q_i s^i}{\sum_{i=0}^{m_2} r_i s^i}$$

(3.5.56)
Then it can be shown from the formulae for $G_N(s)$ or $G_C(s)$ that

$$H(s) = \frac{Q(s) + \beta_1(m_e)Q(s)}{R(s) + \beta_1(m_e)Q(s)} \quad (3.5.57)$$

The Inverse Laplace Transform of (3.5.57) yields the impulse response of the equivalent, small signal, closed loop system.

This concludes the analysis feedback circuits. It has been included here to show the flexibility of the modified Volterra analysis rather than as a detailed account of nonlinear system feedback functional theory.

### 3.6 Summary

In this Chapter is described a modification, developed by the author, to the Volterra functional series for the special case of separable kernels. The conventional Volterra functional expansion for the output of separable nonlinear system involves a series of multidimensional convolutions between non-unity gain impulse response functions and the system inputs. One of the major disadvantages of this representation is that the bias of the input signals appears implicitly in the convolution operations making their theoretical treatment much more difficult. The modified Volterra functional expansion, on the other hand, involves only convolution operations between unity gain impulse response functions and the dynamic component of the input signal, these being weighted by coefficients which contain all the effects due to bias and gain. The modified Volterra series representation is then used in the derivation of output and crosscorrelation expressions for open as well as closed loop nonlinear systems. The analysis is very much simplified compared with that for the derivation of the corresponding expressions, using conventional functional series expansions.
CHAPTER 4

SEPARABLE NONLINEAR SYSTEMS WITH GAUSSIAN INPUTS
4.1 Introduction

The last chapter was devoted to the development of a body of theory for the analysis of nonlinear open and closed loop systems. The theory is generalized in that no restrictions are imposed on the system inputs. The formulae which are obtained are thus generally applicable, but as a consequence complicated. In particular, it is not possible to obtain explicit expressions for the system output bias, variance or correlation functions. In this chapter the system inputs are restricted to a particular class of signals, namely normally distributed random variables. This enables the otherwise complicated expressions for the system output parameters to be simplified; in some cases such simplifications are significant.

The treatment of nonlinear systems with normally distributed random inputs is a subject which is of general importance since in many practical situations the Gaussian inputs assumption is valid. Present theory, which has been restricted to systems with unbiased Gaussian inputs and symmetric nonlinearities, is extended using the modified Volterra functional analysis. A link is then established between the Volterra functional description of nonlinear system response and the Random Biased Describing Function.
4.2 Gaussian signal excitation of a linear system

\[ x(t) \rightarrow K \rightarrow g(t) \rightarrow u(t) \]

Fig. 4.2.1 Linear system, gain \( K \).

In Fig. 4.2.1 is shown a linear system with gain \( K \), unity-gain normalised impulse response \( g(t) \), excited with a biased signal \( x(t) \). Let

\[ x(t) = m_x + \tilde{x}(t) \tag{4.2.1} \]

where \( m_x \) is the bias of \( x(t) \) and \( \tilde{x}(t) \) its dynamic component. Let the input and output variances be \( \sigma_x^2 \) and \( \sigma_u^2 \) respectively. If the ratio of the two variances when \( K = 1 \) is denoted by \( \Omega_g \), then, for the general case

\[ \frac{\sigma_u^2}{\sigma_x^2} = K^2 \Omega_g^2 \tag{4.2.2} \]

\( \Omega_g^2 \) is termed the normalised power transfer ratio for the system with impulse \( g(t) \).

From spectral analysis theory

\[ \Omega_g^2 = \frac{\int_{-\infty}^{\infty} |G(j\omega)| S_{xx}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega} \tag{4.2.3} \]

where \( G(j\omega) \) is the Transfer Function of the linear system with impulse response \( g(t) \), and \( S_{xx}(\omega) \) denotes the power spectral density function of \( x(t) \).
The ratio of output to input variances can also be obtained in terms of autocorrelation functions. It can be shown that

\[ \phi_{uu}(\tau) + m_u^2 = K^2 \int_{-\infty}^{\infty} \phi_{gg}(\theta) \phi_{xx}(\tau-\theta) d\theta + (m_x K)^2 \int_{-\infty}^{\infty} \phi_{gg}(\theta) d\theta \]

\[ = K^2 \int_{-\infty}^{\infty} \phi_{gg}(\theta) \phi_{xx}(\tau-\theta) d\theta + (m_x K)^2 \]

(4.2.4)

\( K \) is the gain of the linear system and \( \phi_{gg}(\theta) \) is given by

\[ \phi_{gg}(\theta) = \frac{1}{T} \int_{0}^{T} g(t)g(t+\theta) dt \quad 0 \leq \theta \leq T \]

(4.2.5)

where \( T \) is the settling time of the system with impulse response \( g(t) \). The function \( \phi_{gg}(\theta) \) can be non-zero only inside the range \( 0 \leq \theta \leq T \).

By setting \( \tau = 0 \) in expression (4.2.4), the output mean square and variance expressions follow, making use of the symmetry of autocorrelation functions

\[ g_u^2 + m_u^2 = K^2 \int_{-\infty}^{\infty} \phi_{gg}(\theta) \phi_{xx}(\theta) d\theta + (m_x K)^2 \]

(4.2.6)

The RHS of (4.2.6) is the output mean square value or output power. The output bias is obviously \( m_u \) where

\[ m_u = m_x K \]

(4.2.7)

The expression for the linear system output variance becomes

\[ g_u^2 = K^2 \int_{-\infty}^{\infty} \phi_{gg}(\theta) \phi_{xx}(\theta) d\theta \]

(4.2.8)
The normalized power transfer ratio can thus be written as

\[ \Omega^2_g = \frac{\int_{-\infty}^{\infty} \phi_{gg}(\theta) \phi_{xx}(\theta) d\theta}{\phi_{xx}(0)} \]  

(4.2.9)

In the special case of white noise inputs, \( \phi_{xx}(\theta) = \sigma_w^2 \delta(\theta) \), where \( \sigma_w^2 \) is the white noise variance, and then (4.2.9) reduces to

\[ \frac{\sigma_w^2}{g^2} = \frac{\int_{-\infty}^{\infty} \phi_{gg}(\theta) \delta(\theta) d\theta}{\sigma_w^2 g^2} = \Phi_{gg}(0) \]  

(4.2.10)

\[ T \lim_{T \to \infty} \int_{-T}^{T} g^2(t) dt \]

\[ = \frac{\sigma_w^2}{g} \]  

(4.2.11)

and \( \frac{\sigma_w^2}{g} \) is in fact the 'power' of the impulse response function \( g(t) \).

If there are two linear systems in cascade then it is possible to obtain expression for the power transfer ratios of each one. Consider Fig. 4.2.2 which represents two such systems; \( K_1, K_2 \) and \( g_1(t), g_2(t) \) are respectively their gains and impulse response functions. Making reference to the diagram it follows that

\[ \frac{\sigma_y^2}{\sigma_x^2} = \frac{\sigma_y^2}{\sigma_x^2} \frac{\sigma_y^2}{\sigma_u^2} \]  

(4.2.12)

and using the definition of power transfer ratio

\[ \frac{\sigma_y^2}{\sigma_x^2} = (K_1 K_2)^2 \Omega^2_{g_{12}} \]  

(4.2.13)

\[ \frac{\sigma_u^2}{\sigma_x^2} = K_1^2 \Omega^2_{g_1} \]  

(4.2.14)
Substitution of (4.2.13) to (4.2.15) into (4.2.12) yields

\[ \Omega_{12}^2 = \frac{\Omega_1^2}{g_{12}} \frac{\Omega_2^2}{g_2} \]  

(4.2.16)

This relationship states that the power transfer ratio of the second linear element is dependent on that one of the first, as well as the variance and spectrum of the system input \( x(t) \). This is confirmed by writing \( \Omega_2^2 \) in terms of other parameters.

\[ \Omega_{12}^2 = \frac{\Omega_1^2}{g_{12}} \frac{\Omega_2^2}{g_2} = \frac{\int |G_1(j\omega)G_2(j\omega)|^2 S_{xx}(\omega) d\omega}{\int |G_1(j\omega)|^2 S_{xx}(\omega) d\omega} \]  

(4.2.17)

The power transfer ratio is a function of input signal power spectral density; by necessity the spectrum of the input to the second linear element will depend on the Transfer Function of the linear element proceeding it, as well as on the spectrum of the overall system input.
In simple cases it is possible to obtain analytical expressions for the power transfer ratios. Consider the system shown in Fig. 4.2.3 representing a second order oscillatory system with an input of variance $\alpha$ and power spectral density $S_{xx}(\omega)$, where

$$S_{xx}(\omega) = \frac{\alpha \sqrt{2}}{\alpha^2 + \omega^2}$$  \hspace{1cm} (4.2.18)

![Diagram of second order oscillatory system](image)

Fig. 4.2.3 Second order oscillatory system excited by signal $x(t)$, with spectral density $S_{xx}(\omega)$.

From spectral theory

$$S_{yy}(\omega) = |G(j\omega)|^2 S_{xx}(\omega) \, d\omega$$  \hspace{1cm} (4.2.19)

and

$$\sigma^2_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) \, d\omega$$  \hspace{1cm} (4.2.20)

By performing the integration, and noting that $\sigma^2_x = \alpha$, it can be shown that

$$\frac{\sigma^2_y}{\sigma^2_x} = \frac{2\zeta + \frac{\alpha}{\omega_n}}{2\zeta [1 + 2\zeta \left(\frac{\alpha}{\omega_n}\right) + \left(\frac{\alpha}{\omega_n}\right)^2]}$$  \hspace{1cm} (4.2.21)
One interesting point which immediately arises is that $\sigma_y^2$ is greater than $\sigma_x^2$ when

$$\frac{\alpha}{\omega_n} < \frac{1}{2\zeta} [1 - 4\zeta^2] \quad (4.2.22)$$

Hence the linear system output variance can be greater than that of its input. Plotting $\frac{1}{2\zeta} [1 - 4\zeta^2]$ against damping ratio $\zeta$ produces the function shown in Fig. 4.2.4. Since both $\alpha$ and $\omega_n$ must be positive, the condition for a possible increase in variance is that the damping ratio be smaller than 0.5.

This analysis has yielded an interesting formula and a conclusion, but it is restricted to the particular case analysed. The integrations required in cases when the order of the systems are greater than three are numerically simple due to the existence of tables to aid the calculations, but it is virtually impossible to obtain in such cases analytical relationships such as that given in (4.2.21). The use of computers however significantly reduces the work load. A program was written to analyse the dependancy of $\sigma_y^2$ and $\sigma_x^2$ on the parameters of the system shown in Fig. 4.2.5. This consists of a second order oscillatory system $[g_1(t)]$, followed by a low pass filter $[g_2(t)]$ with cut off frequency 'b'. The gain of both systems is unity. As before, the spectral density of the input $x(t)$ is $S_{xx}(\omega)$ where

$$S_{xx}(\omega) = \frac{\alpha v^2}{\alpha^2 + \omega^2} \quad (4.2.23)$$

and the variance of $x(t)$ is $\alpha$. 

![Fig. 4.2.4 Plot of function $\frac{1}{2\zeta}[1 - 4\zeta^2]$.](image-url)
This is a situation where the development of an analytical relationship between $\Omega_g^2$ and the other system parameters is very difficult. On the other hand, an expression for $\Omega_g^2$ can be derived; this is given by equation (4.2.21).

Plots of $\Omega_g^2$ against $(\alpha/\omega_n)$ are shown in Fig. 4.2.6. These curves confirm the deductions about variance amplification mentioned earlier. In Fig. 4.2.7 is shown the variation of $\sigma_u^2$ as a function of $[\omega/\omega_n]$. It can be seen that as the variance of the input signal is increased, that of the second order system output also increases, rapidly at first, and then asymptotically to a value which can be shown to be given by $[\omega_n/4\zeta]$. The physical explanation for these curves lies in the fact that the variance of the system input is proportional to its bandwidth. As this bandwidth increases above that of the second order system, the efficiency of variance transmission across it is reduced. This effect is expressed by the $\Omega_g^2$ curves. At the same time the variance transmitted across the second order system reaches a maximum, and any increase in input signal variance has no effect on that of the output. Both $\Omega_g^2$ and $\sigma_u^2$ are independent of the cut off frequency 'b'. On the other hand $\sigma_y^2$ and thus the overall system output variance $\sigma_y^2$, depends on both $(\alpha/\omega_n)$ as well as 'b'. This can be predicted from analysis of equation (4.2.17) and was shown numerically by the computer solution for $\sigma_y^2$. In Fig. 4.2.8 are shown curves indicating the dependencies of $\Omega_g^2$. 

**Fig. 4.2.5** Cascade connection of second order oscillatory system with low pass filter, excited by signal $x(t)$ of power spectral density $S_{xx}(\omega)$. 

$$\frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$ 

$$\frac{b}{s + b}$$
Fig. 4.2.6 Power transfer ratio \( \frac{\Omega^2}{g_1} \) as a function of \( \frac{a}{\omega_n} \) and damping \( \zeta \).

Fig. 4.2.7 Output variance \( \sigma_u^2 \) as a function of \( \frac{a}{\omega_n} \) and damping \( \zeta \).
on the parameters \( \frac{a}{\omega_n} \) and 'b'. From such curves and those for \( \frac{g_1^2}{b_{1}} \) given earlier, it is possible to construct ones for \( \frac{g_1^2}{b_{1}} \) and hence the output variance \( \sigma_y^2 \) can be determined. For the example being considered, it can be seen that as the cut off frequency 'b' is increased, the effect of the low pass filter is reduced. At values of cut off frequency which are of the order of \( \frac{a}{\omega_n} \), the normalised power transfer ratio is approximately inversely proportional to \( \frac{a}{\omega_n} \). It must again be emphasised that these results are particular to the system considered.

![Graph showing Power transfer ratio \( \Omega_2^2 \) as a function of \( \frac{a}{\omega_n} \) and normalised cut off frequency \( b/\omega_n \).]

The efficiency of variance transmission of the second of two linear systems in cascade, depends on the spectrum of its input; this in turn depends on the spectrum of the overall system input as well as the Transfer Functions of both systems. This is of importance in the analysis of variance and power transmission in nonlinear systems.

It must be emphasised that the general formulae for \( \frac{g_2^2}{b_{2}} \) and \( \frac{g_1^2}{b_{1}} \) are independent of the statistical properties of the signals involved. The example of the two linear systems in cascade was used to shown how \( g_1^2 \) and \( \Omega_2^2 \) can be derived for a particular case. Power transfer ratios will be used in the analysis of nonlinear systems with Gaussian inputs.
It has been shown [55] that if the input to a linear system with gain \( K \) is Gaussian, then the output will also be Gaussian; if the input p.d.f. is given by \( p_x(x) \), where

\[
p_x(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[ \frac{(x-m_x)^2}{2\sigma_x^2} \right]
\]

(4.2.24)

then the output p.d.f. is given by \( p(u) \) where

\[
p_u(u) = \frac{1}{K\Omega_g \sigma_x \sqrt{2\pi}} \exp \left[ \frac{(u-Km)_x^2}{2K^2\Omega^2_g \sigma_x^2} \right]
\]

(4.2.25)

These are shown in Fig. 4.2.9 for the case when \( \Omega_g^2 \) and \( K \) are both greater than unity.

---

\( \Omega_g^2 \)

\( K \)

\( \Omega_g \)

\( K \)
4.3 Passage of a Gaussian signal through a nonlinear system

The modified Volterra analysis of the system shown in Fig. 4.3.1 has been carried out in Chapter 3. The results will now be particularized for the special case of Gaussian input signals.

Let \( m \) and \( \tilde{x}(t) \) denote the bias and dynamic component of \( x(t) \). Then from (3.3.12)

\[
y(t) = K_2 \beta_0(m) + K_2 \sum_{i=1}^{n} \beta_i(m) \int g_2(\lambda) h_i(t-\lambda, x) d\lambda
\]

and

\[
\phi_{xy}(\tau) = K_2 \sum_{i=1}^{n} \beta_i(m) \frac{1}{k_1} \int g_2(\lambda) \phi_{xu_i}(\tau-\lambda) d\lambda
\]

4.3.1 The Crosscorrelation Function

If \( x(t) \) is a Gaussian process then (4.3.2) can be simplified. The crosscorrelation functions \( \phi_{xu_i}(\tau) \) are evaluated first, by introducing the bi-variate probability function \( p(x, u, t) \).

It is known that [55]

\[
\phi_{xu_i}(\tau) = \int \int x^i u^i p(x, u, \tau) dx \, du
\]
The p.d.f. takes a simple form in this case, since both signals are normally distributed and unbiased. Thus

\[ p(x,u,\tau) = \frac{1}{F} \exp \left( \frac{-x^2}{2\sigma_x^2} - 2 \left( \frac{x}{\sigma_x} \right) \left( \frac{u}{\sigma_u} \right) \rho_{xu}(\tau) + \left( \frac{u}{\sigma_u} \right)^2 \right) \]  

where

\[ F = 2\pi \sigma_x \sigma_u \sqrt{1-\rho_{xu}(\tau)} \]

and \( \rho_{xu}(\tau) \) is the normalized crosscorrelation coefficient defined by

\[ \rho_{xu}(\tau) = \frac{\phi_{xu}(\tau)}{\sigma_x \sigma_u} \]

If (4.3.4) is substituted in (4.3.3) and the double integration is carried out it can be shown that [47]

\[ \phi_{xui}(\tau) = C_i \sigma_x \sigma_u^i \rho_{xu}(\tau) \]  

\[ = 0 \]  

where \( C_i \) is given by

\[ C_i = \frac{1}{\pi} \frac{2^{(i+1)/2}}{\Gamma\left(\frac{i+2}{2}\right)} \]

and \( \Gamma(p) \) is the gamma function given by

\[ \Gamma(p) = (p-1)! = \int_0^\infty x^{p-1} e^{-x} \, dx \]

Relationship (4.3.7) is not restricted to integer values of \( i \).

A switch function is now defined

\[ S(i) = \frac{1}{2} [1+(-1)^i] = 1 \]  

\[ = 0 \]  

where \( i \) is even integer

\( \) 

where \( i \) is odd integer
Expression (4.3.7) can be rewritten in terms of the crosscorrelation function \( \phi_{xy}(\tau) \) rather than the correlation coefficient, and if use is made of (4.2.2) to write \( \sigma_u \) in terms of \( \sigma_x \) then

\[
\phi_{xy}(\tau) = S(i+1) C_i \frac{K_i}{g_x^1} \sigma_x^{-1} \sigma_x^{-1} \phi_{xy}(\tau) \tag{4.3.11}
\]

This can be substituted into the series for the crosscorrelation function (4.3.2) to yield

\[
\phi_{xy}(\tau) = K_2 \sum_{i=1}^{n} S(i+1) \beta_i(m) C_i \frac{K_i}{g_x^1} \sigma_x^{-1} \int g_2(\lambda) \phi_{xy}(\tau-\lambda) d\lambda \tag{4.3.12}
\]

or more concisely

\[
\phi_{xy}(\tau) = \left[ K_2 \sum_{i=1}^{n} S(i+1) \beta_i(m) C_i \frac{K_i}{g_x^1} \sigma_x^{-1} \phi_{xy}(\tau-\lambda) \right] f(\tau) \tag{4.3.13}
\]

The function \( f(\tau) \) is now considered in detail. \( x(t) \) is a Gaussian signal, and can therefore be considered as being the result of a white noise process passed through a linear filter. The impulse response of such a filter is denoted here as \( h(t) \); its gain can be arbitrarily set to unity. This implies that the system being analyzed, shown in Fig. 4.3.1, is equivalent to that in Fig. 4.3.2, where \( w(t) \) is a white process.

**Fig. 4.3.2** Nonlinear system driven by signal \( x(t) \); \( x(t) \) derived from white noise process \( w(t) \).
The white noise process input to $h(t)$ has a bias equal to that of $x(t)$, denoted $m$. From linear systems theory and (4.3.13)

$$ f(t) = \frac{1}{K_1} \int g_2(\lambda) \phi_{xu}(\tau-\lambda) d\lambda $$

$$ = \int g_2(\lambda) \int g_1(\theta) \phi_{xx}(\tau-\lambda-\theta) d\theta \ d\lambda \quad (4.3.14) $$

$$ (4.3.15) $$

Also from linear systems theory

$$ \phi_{xx}(\tau-\lambda-\theta) = \int \phi_{hh}(\xi) \phi_{WW}(\xi-\tau+\lambda+\theta) d\xi \quad (4.3.16) $$

Substituting this expression into (4.3.15) yields

$$ f(t) = \int g_2(\lambda) \int g_1(\theta) \int \phi_{hh}(\xi) \phi_{WW}(\xi-\tau+\lambda+\theta) d\xi \ d\theta \ d\lambda \quad (4.3.17) $$

$w(t)$ is a white noise process, and therefore its autocorrelation function is nonzero only at zero lag. Thus, by changing first the integration order, and using the fact that

$$ \phi_{xx}(t) = \phi_{xx}(-t) \quad (4.3.18) $$

The following transformations can be operated on $f(t)$.

$$ f(t) = \int g_2(\lambda) \int \phi_{hh}(\xi) \int g_1(\theta) \phi_{WW}(\xi-\tau+\lambda+\theta) d\theta \ d\xi \ d\lambda $$

$$ = \sigma_w^2 \int g_2(\lambda) \int \phi_{hh}(\xi) g_1(\tau-\xi-\lambda) d\xi \ d\lambda $$

$$ = \sigma_w^2 \int \phi_{hh}(\xi) \int g_2(\lambda) g_1(\tau-\xi-\lambda) d\lambda \ d\xi $$

$$ = \sigma_w^2 \int \phi_{hh}(\xi) g_{12}(\tau-\xi) d\xi $$

and $f(t)$ is then

$$ f(t) = g_{h12}(t) \sigma_w^2 \quad (4.3.19) $$

where $\sigma_w^2$ is the variance of the white noise process and

$$ g_{12}(t) = g_2(t) * g_1(t) \quad (4.3.20) $$
The symbol \( * \) denotes convolution, implying that \( f(t) \) is the impulse response of the system shown in Fig. 4.3.3

\[
\text{Fig. 4.3.3 Cascade connection of linear components of nonlinear system.}
\]

All three systems have unity gain and thus

\[
\int f(t) \, dt = \sigma_w^2 \quad (4.3.22)
\]

The expression for the crosscorrelation function given in (4.3.13) can now be written as

\[
\phi_{xy}(\tau) = K(m) \, f(\tau) \quad (4.3.23)
\]

where

\[
K(m) = K_2 \sum_{i=1}^{n} S(i+1) \ C_i \ \beta(m) \ \Upsilon_i^{-1} \ \sigma_x \quad (4.3.24)
\]

Thus the crosscorrelation function across a nonlinear system with a Gaussian input is proportional to the impulse response of the pre-conditioning filter \( h(t) \), cascaded on to the linear part of the system. The gain which the system exhibits to the Gaussian input is given by \( K(m) \sigma_w^2 \).

The system \( h(t) \) needed to transform the theoretic white input \( w(t) \) into the actual one \( x(t) \), can be determined by noting that
where $S_{xx}(\omega)$ and $S_{ww}(\omega)$ denote power spectral densities. $S_{ww}(\omega)$ is thus the power spectral density of the white noise process; this is a constant over its bandwidth, with the requirement that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ww}(\omega) d\omega = \sigma_w^2 = \frac{\sigma^2}{\sigma_n^2}$$

(4.3.26)

Of some relevance to systems testing is the value of the integral of the crosscorrelation function over the range 0 to $T$, where $T$ is the settling time of the linear part of the system. In particular when $\sigma_w^2 = 1$

$$\int_0^T \phi_{xy}(t) dt = K(m) \int_0^T f(t) dt = K(m)$$

(4.3.27) (4.3.28)

The first few terms of $K(m)$ are expanded;

$$K(m) = K_2 \beta_1(m) + 3 K_2 \sigma_x^2 \Omega_1^2 \beta_3(m) + 15 K_2 \sigma_x^4 \Omega_1^4 \beta_5(m) + \ldots$$

(4.3.29)

The first term of the expression corresponds to the small signal system gain at the operating point $m$; this is denoted by $T(m)$. The higher order terms can be regarded as distortions. When $\sigma_u^2 (= \sigma_x^2 \Omega_1^2)$ approaches zero, the variance of the signal entering the nonlinear element approaches zero, and $K(m)$ then equals $T(m)$.

$$K(m) = T(m) + K_2 \sum_{i=3}^{n} \frac{S(i+1)}{C_i(\sigma_x \Omega_1)} \beta_i(m)$$

(4.3.30)

The magnitude of the distortion terms increase with increasing input signal variance and with $\Omega_1^2$, and therefore, in a testing environment where it is desired to find $T(m)$, the variance $\sigma_x^2$ should be kept as small as possible. $\Omega_1^2$ can be reduced by filtering the input signal prior to its injection into the system. The gain term $K_1$ is however unalterable, as are the
coefficients $\beta_i(m)$. Note that $K(m)$ is independent of $\frac{\Omega^2}{\sigma^2}$.

4.3.2 The propagation of bias in nonlinear systems with Gaussian inputs

![Fig. 4.3.4 Nonlinear system driven by signal $x(t)$.

The output of the nonlinear system shown in Fig. 4.3.4 is given by (4.3.1), and repeated here for completeness

$$y(t) = K_2 \sum_{i=0}^{n} \beta_i(m) \int g_2(\lambda) \frac{H_i}{g_1} (t-\lambda, x) d\lambda$$

(4.3.31)

where $m$ is the system input bias. The output bias $y$ is given by $E[y(t)]$, where $E$ is the expectation operator. $H(t,x)$ is equivalent to $K_1^{-1} u(t)$, thus

$$E[y(t)] = K_2 \sum_{i=0}^{n} \beta_i(m) \frac{1}{k_i^1} \int g_2(\lambda) E[u(t-\lambda)]^i d\lambda$$

(4.3.32)

Using the fact that $u(t)$ is an unbiased Gaussian variable, it can be shown that

$$E[u(t-\lambda)]^i = S(i) C_{i-1} \sigma_u^i$$

(4.4.33)

where $C_i$ is the coefficient defined in (4.3.8) and $\sigma_u^2$ is the variance of $u(t)$. This variance can be expressed in terms of the system input variance and the characteristics of the linear system preceding the nonlinear element, yielding
The first few terms of the output bias expression (4.3.35) may be expanded as follows

\[ m_y = K_2 \sum_{i=0}^{n} S(i) \beta_i(m) \frac{1}{K_1^i} \int g_2(\lambda) [C_{i-1} K_1^i \Omega_i^i \sigma_x^i] d\lambda \quad (4.3.34) \]

\[ = K_2 \sum_{i=0}^{n} S(i) \beta_i(m) C_{i-1} \Omega_i^i \sigma_x^i \quad (4.3.35) \]

The output bias is, as expected, a function of \( m \). However it is also a function of the higher order input statistics, and it is by virtue of the fact that the input is Gaussian that these are completely determinate by its bias and variance. The output bias increases with the input variance, and with \( \Omega_x^2 \). The first terms in the series (4.3.36) is the output bias with \( \sigma_x^2 = 0, g_1^2 \) which is the condition for a constant input. The terms containing \( \beta_2(m) \) and higher are dynamic effects.

The effect of the linear element following the nonlinearity is to amplify its input bias by its gain \( K_2 \), but that of the linear element preceeding the nonlinearity is more complex. The gain \( K_1 \) appears in the \( \beta_i(m) \) coefficients and the dynamics of the element are included in the value of \( \Omega_x^2 \); this term appears raised to the same powers as \( \sigma_x^2 \), the input variance. Dynamic and gain effects appearing prior to the nonlinear element, nonlinearly affect the output bias which, in contrast, is unaffected by the linear dynamics of elements after the nonlinearity, and is proportional to their gain.

It is possible to find equivalent output bias relationships when either or both the linear elements are nonexistent by replacing \( K_1, K_2 \) and \( g_1^2 \) by the value 1, as appropriate. The output bias is independent of \( \Omega_x^2 \).
4.3.3 The propagation of power in nonlinear systems with Gaussian inputs

To determine the power output from a linear system it is necessary to know its transfer characteristics as well as the spectrum of its input. For this reason it is not possible to find a relationship for the output power of a (L-NL-L) system without first obtaining the nonlinear element output power spectral density. This can be determined from the autocorrelation function. Consider Fig. 4.3.5.

Using the theory developed in Chapter 3

\[ v(t) = \sum_{i=0}^{n} \beta_i^{(m)} \frac{1}{k_1^i} u_i(t) \]  \hspace{1cm} (4.3.37)

\[ \phi_{vv}(\tau) = \sum_{i=0}^{n} \sum_{j=0}^{n} \beta_i^{(m)} \beta_j^{(m)} \frac{1}{k_1^{i+j}} \mathbb{E}[u_i^*(t) u_j^*(t+\tau)] \]  \hspace{1cm} (4.3.38)

It has been shown by Smith [55] how the mixed autocorrelation function can be evaluated, producing a power series in terms of the autocorrelation function \( \phi_{uu}(\tau) \) of the signal entering the nonlinear element. The output autocorrelation function \( \phi_{vv}(\tau) \) can be written as

\[ \phi_{vv}(\tau) = \sum_{i=0}^{n} \sum_{j=0}^{n} \beta_i^{(m)} \beta_j^{(m)} \frac{1}{k_1^{i+j}} \mathbb{E}[u_i^*(t) u_j^*(t+\tau)] \]

\[ \left[ S(i) S(j) F_e + S(i+1) S(j+1) F_o \right] \]  \hspace{1cm} (4.3.39)
where the functions $F_e$ and $F_o$ contain respectively only even and odd powers of $\phi_{uu}(\tau)$. In particular they are defined by the series

$$F_e = C_i C_j \left[ 1 + \frac{1}{\sigma_u^2} \frac{i(j-1)}{2!} \phi_{uu}^2(\tau) + \frac{1}{\sigma_u^4} \frac{i(i-2) j(j-2)}{4!} \phi_{uu}^4(\tau) + \ldots \right]$$  \hspace{1cm} (4.3.40)

$$F_o = C_i C_j \left[ \frac{1}{\sigma_u^2} \phi_{uu}(\tau) + \frac{1}{\sigma_u^6} \frac{(i-1)(j-1)}{3!} \phi_{uu}^3(\tau) + \right.$$

$$\left. + \frac{1}{\sigma_u^{10}} \frac{(i-1)(i-3)(j-1)(j-3)}{5!} \phi_{uu}^5(\tau) + \ldots \right]$$  \hspace{1cm} (4.3.41)

where 'C' is the constant as defined in (4.3.8). The series are terminated by the first zero term. The expression for the nonlinear element output autocorrelation function can be slightly simplified by noting that

$$\sigma_u^2 = k^2 \Omega_x^2 \sigma_x^2$$  \hspace{1cm} (4.3.42)

and then $\phi_{vv}(\tau)$ becomes

$$\phi_{vv}(\tau) = \sum_{i=0}^{N} \sum_{j=0}^{N} \beta_i(m) \beta_j(m) \Omega_x^{i+j} \sigma_x^{i+j} \left[ S(i)S(j)F_e + S(i+1)S(j+1)F_o \right]$$  \hspace{1cm} (4.3.43)

This is still cumbersome in appearance. Unfortunately further simplifications do not produce results which are more easily interpreted for the general case. The usefulness of the equation lies in the fact that numerical values of $\phi_{vv}(\tau)$ can be obtained easily from a knowledge of $\Omega_x^2$ and $\sigma_x^2$ and the form of the nonlinear element instantaneous gain or d.c. function. Note that $\sigma_x^2$ is the variance of the system input signal and not its power.

The power spectral density at the output from the nonlinear element can be calculated by applying the Wiener-Khinchine relationship to $\phi_{vv}(\tau)$. Hence

$$S_{vv}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{vv}(\tau) e^{-j\omega \tau} d\tau$$  \hspace{1cm} (4.3.44)
From \( S_{vv}(\omega) \) can be calculated \( S_{yy}(\omega) \) since

\[
S_{yy}(\omega) = K_2 |G_2(j\omega)|^2 S_{vv}(\omega) \tag{4.3.45}
\]

where \( G_2(j\omega) \) is the unity gain normalized Transfer Function of the linear element following the nonlinear one. The power output from the complete nonlinear system follows by applying Parseval's theorem; if the output power is denoted \( \psi_y^2 \) then

\[
\psi_y^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega \tag{4.3.46}
\]

The power at the output of the nonlinear element can also be calculated by finding \( \phi_{vv}(\omega) \), and the variance by subtracting from this value \( m_v^2 \); \( m_v \) is the bias output from the nonlinear element. Hence existing theory for variance and power transmission in nonlinear systems with unbiased inputs and symmetric nonlinearities can now be applied to the more general cases. The \( \beta_i(m) \) coefficient products, automatically cater for bias values as they appear.

As an example of the use of the power expression, below are given relationships for a system containing a third order nonlinear element.

\[
\phi_{vv}(\tau) = [\beta_0^2(m) + 2\beta_0(m)\beta_2(m)\sigma_u^2 + \beta_2^2(m)\sigma_u^4] + [\beta_1^2(m) + 6\beta_1(m)\beta_3(m)\sigma_u^2 + 9\beta_3^2(m)\sigma_u^4] \phi_{uu}(\tau) + 2\beta_2^2(m) \phi_{uu}^2(\tau) + 6\beta_3^2(m) \phi_{uu}^3(\tau) \tag{4.3.47}
\]

From this relationship and that for the nonlinear element output bias \( m_v \), can be calculated the variance \( \sigma_v^2 \):

\[
m_v = \beta_0(m) + \beta_2(m) \sigma_u^2 \tag{4.3.48}
\]
\[ \sigma_v^2 = [\beta_1^2(m) + 2\beta_0(m)\beta_2(m)]\sigma_u^2 + [2\beta_2^2(m) + \beta_3(m)]\sigma_u^4 + [15\beta_3^3(m)]\sigma_u^6 \] (4.3.49)

Whether the nonlinear element is symmetric or not, the even order \( \beta_i(m) \) coefficients will appear in the expression for the variance, unless the system input bias is zero. In such a case, and for a symmetric nonlinear element, the variance becomes

\[ \sigma_v^2 = \sigma_u^2 + 6\alpha_1\alpha_3\sigma_u^4 + 15\alpha_3^3\sigma_u^6 \] (4.3.50)

The relationships have been given in terms of \( \sigma_u^2 \), the variance of the signal entering the nonlinear element. By using

\[ \sigma_u^2 = K_2^2 \Omega_2^2 \sigma_x^2 \] (4.3.51)

they could be rewritten in terms of the system input variance and the characteristics of the linear element preceding the nonlinear one. For any set of Gaussian system input conditions, the determination of nonlinear element output is thus fully determinate from the above equations. Slight complications arise when it is necessary to find the variance output from the complete nonlinear system shown in Fig. 4.3.5, due to the necessity of calculating \( \Omega_2^2 \). This entails finding the spectral density function of the signal entering the second linear element; this spectral density can be calculated from the autocorrelation function \( \phi_{vv}(\tau) \) by application of the Wiener-Khinchine Theorem, as stated earlier, or alternatively from frequency convolution theorems. It has been shown in this section that the autocorrelation function \( \phi_{vv}(\tau) \) can be expressed as

\[ \phi_{vv}(\tau) = \sum_{i=0}^{n} \gamma_i(m) \phi_{uu}(\tau) \] (4.3.52)

where \( \gamma_i(m) \) are product functions of the \( \beta_i(m) \) coefficients. Define now

\[ U(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{uu}(\tau) e^{-j\omega\tau} d\tau \] (4.3.53)
Then $U_n(\omega)$ is the power spectral density function of $u(t)$. Let

$$U_n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^{i}_{uu}(\tau) e^{-j\omega\tau} d\tau \tag{4.3.54}$$

Then it can be shown that

$$U_1(\omega) = \int_{-\infty}^{\infty} U_{1-1}(\omega_1) U(\omega-\omega_1) d\omega_1 \tag{4.3.55}$$

The power spectral density of the signal output from the nonlinear element is $S_{vv}(\omega)$ which, from (4.3.52), can be written as

$$S_{vv}(\omega) = \sum_{i=0}^{n} \gamma_i(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^{i}_{uu}(\tau) e^{-j\omega\tau} d\tau \tag{4.3.56}$$

and if (4.3.55) is substituted into this expression then

$$S_{vv}(\omega) = \sum_{i=0}^{n} \gamma_i(\omega) \int_{-\infty}^{\infty} U_{1-1}(\omega_1) U(\omega-\omega_1) d\omega_1$$

$$= \gamma_1(\omega) U(\omega) + \gamma_2(\omega) \int_{-\infty}^{\infty} U(\omega_1) U(\omega-\omega_1) d\omega_1$$

$$+ \cdots + \gamma_n(\omega) \int_{-\infty}^{\infty} U_{n-1}(\omega_1) U(\omega-\omega_1) d\omega_1 \tag{4.3.57}$$

In the case of the system containing the third order nonlinear element

$$S_{vv}(\omega) = \gamma_1(\omega) U(\omega) + \gamma_2(\omega) \int_{-\infty}^{\infty} U(\omega_1) U(\omega-\omega_1) d\omega_1$$

$$+ \gamma_3(\omega) \int_{-\infty}^{\infty} U(\omega_1) U(\omega-\omega_1) d\omega_1 \tag{4.3.58}$$
where

\[ U_2(\omega) = \int_{-\infty}^{\infty} U(\omega_1) U(\omega - \omega_1) d\omega_1 \]  

(4.3.59)

Using (4.3.45) the power spectral density of the nonlinear system output \( S_{yy}(\omega) \) can be calculated, and from this, its variance \( \sigma^2_y \). Note that \( U_1(\omega) = U(\omega) \).

4.4 Gaussian inputs to a nonlinear feedback system

![Nonlinear feedback system with Gaussian inputs](image)

Fig. 4.4.1 Nonlinear feedback system with Gaussian inputs.

When feedback exists around a nonlinear system, the calculation of its output bias and variance involves solution of integral equations. Exact solution is rarely possible even if the signal entering the nonlinear element is normally distributed. The statistical properties of the error signal \( e(t) \) are of paramount importance. The p.d.f. of the error signal \( p_e(e) \) is a function of the joint properties of the input and output signal \( x(t) \) and \( y(t) \). In particular

\[ p_e(e) = \int_{-\infty}^{\infty} p_{xy}(z+e, z) dz \]  

(4.4.1)

In the special case of \( x(t) \) and \( y(t) \) being statistically independent then

\[ p_e(e) = \int_{-\infty}^{\infty} p_x(z+e) p_y(z) dz \]  

(4.4.2)
$p_{xy}(x,y)$ is the bi-variate p.d.f. of the signals $x(t)$ and $y(t)$. The bias error signal is $m_e$ where

$$m_e = m_x - m_y \quad (4.4.3)$$

and the variance of $e(t)$ is $\sigma_e^2$ where

$$\sigma_e^2 = \sigma_x^2 + \sigma_y^2 - 2E[x\ y] \quad (4.4.4)$$

As before the bar underneath a symbol indicates dynamic component. Note that $\sigma_e^2$ can be expressed as

$$\sigma_e^2 = \sigma_x^2 + \sigma_y^2 - 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\ y\ p(x,y)\ dx\ dy \quad (4.4.5)$$

If $x(t)$ and $y(t)$ are independent the above equation reduces to

$$\sigma_e^2 = \sigma_x^2 + \sigma_y^2 \quad (4.4.6)$$

If the system contains nonlinear elements it is unlikely that $e(t)$ is exactly Gaussian. The presence of other linear elements in the circuit however can produce a distribution at the input to the nonlinear element, which closely approximates the normal one. This is because most physically occurring nonlinear elements generate harmonics, reducing low frequency and increasing high frequency power. The linear elements however tend to cut out the high frequency components and smooth intermodulation effects. This depends on the location and bandwidth of the elements concerned. In general experiments have shown [16] that the Gaussian inputs assumption is reasonable in many cases, especially when the nonlinear element is buffered between linear ones. In the case of $x(t)$ being Gaussian there is an additional normalization process carried out at the summing junction. This is indicated by the convolution operations between the p.d.f.'s of $x(t)$ and $y(t)$, as given by (4.4.1) and (4.4.2). The problems associated with obtaining an exact solution for values of output bias and variance with the feedback system are great, involving integral equations. This is so because bias transmission in the feedback system involves knowledge of nonlinear element input variance, and this in turn depends on the spectral density function of the error signal $S_{ee}(\omega)$. This is given by
\[ S_{ee}(\omega) = S_{xx}(\omega) + S_{yy}(\omega) - 2S_{xy}(\omega) \]

\( S_{yy}(\omega) \) can be written in terms of a series involving frequency convolutions as shown in the last section. The resulting equation can only be solved by means of approximate methods. Alternatively the nonlinear feedback system can be replaced by an equivalent open loop one, involving the cascade of nonlinear subsystems. Such work is covered by George [22]. Once the feedback system has been replaced by an open loop equivalent, the theory already outlined can be applied to it. Such work is however outside the scope of this thesis.

4.5 The statistical describing function for biased inputs (RBDF)

The letters RBDF stand for 'Random Biased Describing Function'. This is the terminology used by Atherton [12] when referring to a describing function first introduced by Booton [59]. The RBDF models single-valued nonlinearities by two linear gains, one for the bias and one for the dynamic component of its input signal, which is assumed to be random (and also Gaussian for most cases). It will be shown that the RBDF is closely connected to the Volterra modified functional description of nonlinear systems.

Fig. 4.5.1 The Statistical Describing Function (RBDF).
Consider Fig. 4.5.1. The nonlinear element input and output signals are \( x(t) \) and \( y(t) \), related by

\[
y(t) = g[x(t)] \tag{4.5.1}
\]

and \( g(x) \) is a lagless, single-valued transformation. The RBDF approximates the output \( y(t) \) by \( r(t) \) where

\[
r(t) = f_m m + f_K[x(t)] \tag{4.5.2}
\]

The constants \( f_m \) and \( f_K \) can be found by minimising the cost function \( E[e^2(t)] \), given by

\[
E[e^2(t)] = \int_{-\infty}^{\infty} [y(t)-r(t)]^2 p_x(x)dx \tag{4.5.3}
\]

\( p_x(x) \) is the probability density function of the input signal. Substitution of (4.5.2) into (4.5.3) and carrying out the minimization yields

\[
f_m = \frac{1}{m} \int_{-\infty}^{\infty} g(x) p(x)dx \tag{4.5.4}
\]

\[
f_K = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} (x-m) g(x) p(x)dx \tag{4.5.5}
\]

A very simple derivation given by Atherton [12], as well as others, shows that the linear model provided by the RBDF is an optimum linear model of the nonlinearity when the input process is separable. The implications of separability are investigated by Nuttal [60]. A process is said to be separable if it satisfies

\[
\int_{-\infty}^{\infty} x_1 p(x_1, x_2, \tau)dx_1 = x_2 \ p(x_2) \ \phi_{xx}(\tau)/\phi_{xx}(0) \tag{4.5.6}
\]
If the equations for $f_m$ and $f_K$ are written in terms of the correlation functions it can be shown that [12]

$$f_m = \frac{\phi_{my}(0)}{\phi_{mm}(0)} = \frac{m}{m^2} = \frac{m}{m}$$

$$f_K = \frac{\phi_{xy}(0)}{\phi_{xx}(0)} = \frac{1}{\sigma_x^2} \frac{\phi_{xy}(0)}{\sigma_y^2}$$

(4.5.7)

(4.5.8)

It has been shown that Gaussian and random phase sinusoidal processes are separable. Since nonlinearities often appear within systems of complex nature, preceded by linear elements the nonlinear element inputs are often considered Gaussian. A great deal of work has been carried out to justify this approximation and to examine its limitations [55][46]. The analytical simplicity which results by making the Gaussian inputs assumption often outweighs the resulting errors, and further it has been shown that these are small, especially in the common situation when the nonlinearity appears within a feedback loop.

In the following sections the Gaussian inputs assumption is made and some results derived linking the RBDF to the modified Volterra series and to correlation analysis.

4.5.1 The (RBDF) for a polynomial nonlinearity

Let the nonlinear element input-output relationship be

$$v(t) = \sum_{i=0}^{n} a_i [u(t)]^i$$

(4.5.9)

where $u(t) = m_u + \tilde{u}(t)$

(4.5.10)

Then the output $v(t)$ can be written

$$v(t) = \sum_{i=0}^{n} B_i (m_u) u(t)]^i$$

(4.5.11)
where the coefficients $B_i(m)$ are defined in section (3.3.2). If (4.5.11) is used to find $f_m$ and $f_K$, then assuming $u(t)$ is a Gaussian variable with variance $\sigma^2_u$

$$f_m = \frac{1}{m_u} \sum_{i=0}^{n} S(i) C_i B_i(m) \sigma_u^i$$  \hspace{1cm} (4.5.12)$$

$$f_K = \sum_{i=1}^{n} S(i+1) C_i B_i(m) \sigma_u^{i-1}$$  \hspace{1cm} (4.5.13)$$

Equations (4.5.12) and (4.5.13) are stated without the intermediate steps as these involve only mathematical manipulation. In the next section these results are extended to a general (L-PNL-L) system.

4.5.2 The RBDF applied to a (L-PNL-L) system

![Diagram of system A](image1)

Fig. 4.5.2 Nonlinear system driven by signal $x(t)$.

The nonlinear system shown in Fig. 4.5.2 can be considered as being made of two subsystems, denoted A and B, cascaded on to each other. Subsystem A is that within the dotted line.

The value of $f_m$ and $f_K$ for 'A' can be found by modifying the formulae for $f_m$ and $f_K$ for the nonlinearity alone, in such a way that the parameters in terms of $u(t)$ are written in terms of $x(t)$, and multiplying the result by $K_1$, this being the gain of the linear element preceeding the nonlinear one. Note that, from (3.3.10)
and therefore
\[ u = \Omega_k \sigma_x \]  (4.5.15)

If (4.5.14) and (4.5.15) are substituted into (4.5.12) and (4.5.13), then

\[
\begin{align*}
\sum_{i=0}^{n} S(i) C_i \beta_i(m) \Omega_i \sigma_i \rho_i
\end{align*}
\]  (4.5.16)

\[
\begin{align*}
\sum_{i=1}^{n} S(i+1) C_i \beta_i(m) \Omega_i \sigma_i \rho_i
\end{align*}
\]  (4.5.17)

Having determined the (RBDF) for the nonlinear sub-system A, the determination of the (RBDF) for the sub-system A cascaded on to B is very simple as B is linear; it involves direct multiplication by \( K_2 \). Thus for the complete system

\[
\begin{align*}
\sum_{i=0}^{n} S(i) C_i \beta_i(m) \Omega_i \sigma_i \rho_i
\end{align*}
\]  (4.5.18)

\[
\begin{align*}
\sum_{i=1}^{n} S(i+1) C_i \beta_i(m) \Omega_i \sigma_i \rho_i
\end{align*}
\]  (4.5.19)

If the expressions for the output bias and the equivalent gain of the nonlinear system, with Gaussian inputs are compared with (4.5.18) and (4.5.19) then the following relationships are obtained

\[
\begin{align*}
\frac{1}{m_x} \text{(Output bias)}
\end{align*}
\]  (4.5.20)

\[
\begin{align*}
\frac{1}{K_2} \text{(Equivalent Gain)}
\end{align*}
\]  (4.5.21)
If (4.5.20) and (4.5.21) are substituted into (4.5.2), the output from the nonlinear system as predicted by the (RBDF) is

\[ r(t) = \text{(System output bias)} + \text{(Equivalent Gain)}x(t) \]  

Equation (4.5.22) provides a link between the RBDF and the modified Volterra series.

The RBDF is ideally suited to dealing with systems containing discrete nonlinearities. The link which has been shown to exist between the modified Volterra series and RBDF analysis can be applied in an intuitive argument to support the use of an approximate modified Volterra analysis when the nonlinearities are discrete; the nonlinearities can be approximated by polynomials. Alternatively methods based on the polynomial assumptions of nonlinear forms, using Volterra analysis, can be used directly on systems with discrete nonlinearities, and some degree of error accepted. Such approaches yield very good results, some of which are given in Chapter 7.

Estimates of the errors inherent in the RBDF prediction of the output variance of nonlinear open loop systems are now examined.

4.6 An example of the accuracy of the RBDF method in estimating output variance

![Fig. 4.6.1 RBDF equivalent to (L-PNL-L) system.](image-url)
The describing function approach of analyzing the response of the (L-PNL-L) system, is to replace it by an equivalent one as shown in Fig. 4.6.1, where the gains $f_m$ and $f_K$ correspond to those given by equations (4.5.18) and (4.5.19). The system output is thus predicted as

$$y_p(t) = K_2 \sum_{i=0}^{n} S(i) C_{i-1} \beta_i(m) \frac{s_i}{g_1} \sigma_x^i$$

$$+ (K_2 \sum_{i=0}^{n} S(i+1) C_i \beta_i(m) \frac{s_i-1}{g_1} \sigma_{x-1}^i) \int g(\tau) \chi(t-\tau) d\tau \quad (4.6.1)$$

where

$$g(\tau) = \int g_2(\lambda) g_1(\tau-\lambda) d\lambda \quad (4.6.2)$$

Expression (4.6.1) is exactly equivalent to (4.5.22), but with all the relevant constants inserted.

From (4.6.1) it follows immediately that the predicted output variance is, for a Gaussian, non-white input

$$\sigma_p^2 = K_2^2 \sum_{i=0}^{n} \sum_{j=0}^{n} S(i+1) S(j+1) C_i C_j \beta_i(m) \beta_j(m) \frac{s_i+j-2}{g_1} \sigma_x^{i+j} \quad (4.6.3)$$

The describing function approach thus obviates the need for the derivation of the nonlinear element output power spectral density. The relationships obtained are however approximate. Errors are also present in the estimate of the nonlinear element output variance $\sigma_v^2$. As an example of the errors involved, expressions for $\sigma_v^2$ are derived with the describing function and the exact methods, for a system with a third order nonlinearity.

Expression for $\sigma_v^2$ obtained from describing function theory:

$$\sigma_v^2 = \beta_1^2(m) \sigma_u^2 + 6 \beta_1(m) \beta_3(m) \sigma_u^4 + 9 \beta_3^2(m) \sigma_u^6 \quad (4.6.4)$$
Expression for $\sigma_v^2$ obtained from exact theory:

$$\sigma_v^2 = \left[ \beta_1^2(m) + 2\beta_0(m)\beta_2(m) \right] \sigma_u^2 + \left[ 2\beta_2^2(m) + 6\beta_1(m)\beta_3(m) \right] \sigma_u^4 + 15\beta_3^3(m)\sigma_u^6$$  \hspace{1cm} (4.6.5)

In both cases $\sigma_u^2$ represents the variance at the input to the nonlinear element. The magnitude of the error using the describing function approach is $|\Delta \sigma^2|$, given by

$$|\Delta \sigma^2| = 2\beta_0(m)\beta_2(m)\sigma_u^2 + 2\beta_2^2(m)\sigma_u^4 + 6\beta_3^3(m)\sigma_u^6$$  \hspace{1cm} (4.6.6)

Since the nonlinearity is of third order, $\beta_3(m)$ is in fact $\sigma_3$, a constant value. Most of the error is due to the even order $\beta_i(m)$ coefficients. Depending on the nonlinear element input-output relationship, such errors can obviously be large.

The cost of obtaining exact solutions is increased computation costs. The statistical describing function is used successfully in many applications but, especially if the systems are complex, the build up in error may be considerable. In such circumstances the exact analysis, perhaps aided by a computer, may be necessary. The work carried out in this chapter is a step towards more exact system behaviour prediction.

4.7 **Summary**

This chapter is concerned with the application of the modified Volterra functional analysis to nonlinear systems with Gaussian inputs. In particular, the concept of power transfer ratios has been introduced and analyzed for linear systems, and then applied to the derivation of analytical relationships for the calculation of equivalent gain, bias and variance transmission, in nonlinear open loop systems. A relationship has then been found between the biased statistical describing function and the modified Volterra series expansion for nonlinear system response. An example is given of the errors involved in estimation of output variance by using the RBDF method.
CHAPTER 5

SEPARABLE NONLINEAR SYSTEMS WITH PSEUDO-RANDOM INPUTS
5.1 Introduction. Definition of signal waveforms

As shown in Chapters 3 and 4, when a system's input signal is white noise, the analysis of its behaviour is simplified and, for linear cases, the crosscorrelation is a map of the system's impulse response. White noise is difficult to generate and requires long signal periods before its statistical properties converge to the desired forms and therefore approximations to it are commonly used. Pseudo-random sequences of various types have been applied now for several years. In this chapter the use of 2-level (PRBS), 3-level (TPRS) and 5-level (FPRS) sequences are considered, with particular emphasis on the effect of input bias on the crosscorrelation function.

Part of a 3-level sequence, superimposed on a bias \( m \), is shown in Fig. 5.1.1. The figure serves to define the term 'amplitude' as applied to discrete level-numbered signals, the amplitude 'A' being defined as the magnitude of the smallest step in the signal. The peak-to-peak amplitude of the sequence, denoted by \( A_p \), is given by

\[
A_p = (\ell - 1)A \quad (5.1.1)
\]

where \( \ell \) is the number of levels.

Thus

\[
\begin{align*}
A_2 &= A ; \quad \text{PRBS} \\
A_3 &= 2A ; \quad \text{TPRS} \\
A_5 &= 4A ; \quad \text{FPRS}
\end{align*}
\]
If $x_\ell(t)$ denotes an $\ell$-level pseudo-random signal of amplitude $A$, superimposed on a bias $m$, then

$$x_\ell(t) = m + x_\ell(t)$$

$$= m + A z_\ell(t) \quad \ell \neq 2$$ (5.1.3)

and $z_\ell(t)$ is an $\ell$-level pseudo-random signal of unit amplitude. Using this definition

$$H^i(t, x_\ell) = H^i(t, A z_\ell)$$

$$= A^i H^i(t, z_\ell)$$ (5.1.4)

When $\ell$ is odd $z_\ell(t)$ is an unbiased signal.

In the special case of PRBS, ($\ell = 2$) then

$$x_2(t) = m + A z_2(t)$$

$$= m + A m_2 + A z_2(t)$$ (5.1.5)

where $m_2$ is the bias inherent in the nature of the PRBS signal. It can be shown that

$$m_2 = \frac{1}{N_2 \Delta t}$$ (5.1.6)

where $N_2$ is the bit number of the PRBS signal, and $(\frac{1}{\Delta t})$ the PRBS clock frequency.
5.2 Some correlation properties of maximal length sequences, (3 and 5 levels), applied to systems with polynomial nonlinearities

Fig. 5.2.1 General nonlinear system.

If expression (5.1.3) is substituted into those for the output \( y(t) \) and the crosscorrelation function \( \phi_{x_y}(\tau) \) then they become

\[
y(t) = K_2 \sum_{i=0}^{n} \beta_i(m) A^i \int g_2(\lambda) H_{g_1}(t-\lambda, z_x) d\lambda \tag{5.2.1}
\]

\[
\phi_{x_y}(\tau) = K_2 \sum_{i=1}^{n} \beta_i(m) \frac{1}{K_1^i} A^{i+1} \int g_2(\lambda) \phi_{z_y} u_i(\tau-\lambda) d\lambda \tag{5.2.2}
\]

In (5.2.1) and (5.2.2) '\( \lambda \)' is odd. The special case of PRBS inputs is considered separately in the next section. If the modified definition of \( D_1(\tau) \) is used as given below

\[
D_1(\tau) = \frac{1}{K_1^i} \int g_2(\lambda) \phi_{z_y} u_i(\tau-\lambda) d\lambda \tag{5.2.3}
\]

then

\[
\phi_{x_y}(\tau) = K_2 \sum_{i=1}^{n} \beta_i(m) A^{i+1} D_1(\tau) \tag{5.2.4}
\]
Since only inverse repeat sequences are considered, having the property

\[ D_{21}(\tau) = 0 \]  \hspace{1cm} (5.2.5)

the crosscorrelation function becomes

\[ \phi_{xy}(\tau) = K_z \sum_{i=1}^{n} S(i+1) \beta_1(m) A^{i+1} D_{1}(\tau) \]  \hspace{1cm} (5.2.6)

where \( S(i) \) is the switch function defined in Chapter 3, having the property

\[ S(i) = 1 \quad \text{; } i \text{ even} \]

\[ = 0 \quad \text{; } i \text{ odd} \]  \hspace{1cm} (5.2.7)

Fig. 5.2.2 shows the autocorrelation function for 3 and 5-level sequences, clocked at frequency \( \lambda \). The 5-level sequence autocorrelation has the same form as that of the 3-level one except for the spike height which is different in the two cases.
With reference to Fig. 5.2.2,

\[ F_0 |_{3 \text{ level}} = \frac{2}{3} \left( \frac{N + 1}{N} \right) A^2 \]  
\[ F_0 |_{5 \text{ level}} = 2 \left( \frac{N + 1}{N} \right) A^2 \]  

(5.2.8) \hspace{2cm} (5.2.9)

N is the bit length of the sequences. Extensive literature on the subject of maximal length sequences covers their properties in detail; only the basic information is given here which is relevant to the development of correlation relationships.

5.3 Injection of 3 and 5-level sequences into Hammerstein type systems

In the special case of a system of Hammerstein type, as shown in Fig. 5.3.1, the expressions for the output \( y(t) \), and crosscorrelation function \( \phi_{xy}(\tau) \), take particularly simple forms. The nonlinearity input-output relationship is assumed to be of polynomial type. Hence

\[ v(t) = \sum_{i=0}^{n} a_i [x(t)]^i \]  

(5.3.1)
If \( x(t) \) is given by (5.1.3) then

\[
v(t) = \sum_{i=0}^{n} B_i(m) A^i [z_3(t)]^i
\]

(5.3.2)

where \( B_i(m) \) are the coefficients defined by (3.3.3). The cases of 3 and 5-level inputs are considered separately.

### 5.3.1 3-level inputs

The expression for \( v(t) \), the output from the nonlinear element, can be simplified by noting that

\[
[z_3(t)]^{2i+1} = z_3(t) \quad ; \quad i \text{ integer } \geq 0
\]

(5.3.3)

\[
[z_3(t)]^{2i} = [z_3(t)]^2 \quad ; \quad i \text{ integer } \geq 1
\]

(5.3.4)

As a consequence of (5.3.3) it also follows that

\[
\phi_{z_3} (0, 0, \ldots, 0,) = A^{2i-2} \phi_{z_3 z_3} (t)
\]

(5.3.5)

By using (5.3.3) and (5.3.4), the signal \( v(t) \) can be decomposed into the sum of three components, involving bias, \( z_3(t) \) and \( [z_3(t)]^2 \). Thus

\[
v(t) = B_0(m) + z_3(t) \sum_{i=1}^{n} S(i+1) A^i B_i(m)
\]

\[
+ [z_3(t)]^2 \sum_{i=1}^{n} S(i) A^i B_i(m)
\]

(5.3.6)

The signal \( [z_3(t)]^2 \) is itself biased, being composed of a sequence of numbers with values of either +1 or 0. Its bias is easily determined; denoting it by \( M_p \)

\[
M_p = E [z_3(t)]^2
\]

(5.3.7)
It is obviously the variance of \( z_3(t) \). This will be denoted by \( \sigma_{z_3}^2 \); its value in terms of the TPRS bit length \( N \) is

\[
\sigma_{z_3}^2 = \frac{2}{3} \left( \frac{N + 1}{N} \right) = M
\]  

(5.3.8)

Denoting the dynamic component of \( [z_3(t)]^2 \) by \( z_p(t) \)

\[
[z_3(t)]^2 = \frac{M}{p} + z_p(t)
\]  

(5.3.9)

The nonlinear element output can now be expressed in terms of a bias component, and the sum of two other signals.

\[
v(t) = [B_0(m) + \sigma_{z_3}^2 \sum_{i=1}^{n} S(i) A_i B_i(m)] + [z_3(t) \sum_{i=1}^{n} S(i+1) A_i B_i(m)] + [z_p(t) \sum_{i=1}^{n} S(i) A_i B_i(m)]
\]  

(5.3.10)

The output from the Hammerstein system is, from convolution

\[
y(t) = K[B_0(m) + \sigma_{z_3}^2 \sum_{i=1}^{n} S(i) A_i B_i(m)] + \sum_{i=1}^{n} S(i+1) A_i B_i(m) \int g(\tau) z_3(t-\tau) d\tau + \sum_{i=1}^{n} S(i) A_i B_i(m) \int g(\tau) z_p(t-\tau) d\tau
\]  

(5.3.11)

The system output bias is obviously given by

\[
m_y = K[B_0(m) + \sigma_{z_3}^2 \sum_{i=1}^{n} S(i) A_i B_i(m)]
\]  

(5.3.12)
which is a function of even order $B_1(m)$ coefficients, TPRS amplitude, and the variance of the unity gain TPRS. The variance $\sigma^2_{z_3}$ acts as a multiplier only to the terms $B_2(m)$, $B_4(m)$ and higher, and not on $B_0(m)$. When the system is linear all such higher terms are zero, and the variance dependency on the output bias disappears. An interesting property of the signals $z_3(t)$ and $[z_3(t)]^2$ is that they are uncorrelated. This can be easily demonstrated.

\[
\phi_{z_3[z_3]^2}(\tau) = \frac{1}{T} \int_0^T [z_3(t)]^2 z_3(t+\tau)dt
\]

\[
= \frac{1}{T} \int_0^T z_3(t) z_3(t) z_3(t+\tau)dt
\]

\[
= \phi_{z_3}(0, \tau)
\]

\[
= 0
\]  

(5.3.13)

This is proved using the property of TPRS

\[
\phi_{z_3^{2i+1}(\tau_1, \ldots, \tau_{2i})} = 0 ; \text{ for all } \tau_i
\]  

(5.3.14)

It follows that the output from the nonlinear element is in fact made up of a bias and two uncorrelated dynamic signals. The crosscorrelation function $\phi_{x_3 y}(\tau)$ is easily derived.

\[
\phi_{x_3 y}(\tau) = A_{y m} \phi_{x_3 y}(\tau)
\]

(5.3.15)

\[
= A_{y m} [ \sum_{i=1}^n S(i+1) A^{i+1} B_1(m)] \int g(\lambda) \phi_{z_3 z_3}(\tau-\lambda) d\lambda
\]  

(5.3.16)

Providing the period of $z_3(t)$ is more than twice the system settling time

\[
\phi_{x_3 y}(\tau) = A_{y m} + c_1 g(\tau) \quad 0 \leq \tau < T
\]  

(5.3.17)
where \( 2T \) is the period \( z^3(t) \) and

\[
c_1 = \sigma^2 \sum_{i=1}^{n} S(i+1) A^{i+1} B_{1}(m)
\]  

(5.3.18)

The crosscorrelation function \( \phi_{xy}(\tau) \) is thus proportional to the impulse response of the linear element in the Hammerstein model. The constant of proportionality, \( c_1 \), is a nonlinear function of input bias and amplitude. The bias on the crosscorrelation function is \( A \), and this is also a nonlinear function of input bias. The input variance also affects this value.

The theory has shown that a Hammerstein type system excited by TPRS exhibits behaviour which could be mistaken for linear, unless the tests are carried out at more than one bias level.

### 5.3.2 5-level inputs

The output from the nonlinear element is as before

\[
v(t) = \sum_{i=0}^{n} B_{i}(m) A^{i}[z_{5}(t)]^{i}
\]  

(5.3.19)

where \( \ell \) is 5 in this case. Since \( z_{5}(t) \) is a five level sequence it can take values of -2, -1, 0, 1, 2 and

\[
z^{2i+1}_{5}(t) = \begin{cases} 
  z_{5}(t) & \text{if } z_{5}(t) = \pm 1 \\
  2^{2i+1} z_{5}(t) & \text{if } z_{5}(t) = \pm 2
\end{cases} 
\]  

\[
z^{2i}_{5}(t) = \begin{cases} 
  z^{2}(t) & \text{if } z_{5}(t) = \pm 1 \\
  2^{2i} z^{2}(t) & \text{if } z_{5}(t) = \pm 2
\end{cases}
\]  

(5.3.20)
As a consequence of (5.3.19) and (5.3.20), the output from the nonlinearity cannot be expressed as a simple sum of maximal length sequences and the expressions for the system output cannot be simplified. There are two possible nonlinear element output amplitudes per system input polarity, the nonlinear element output, mapping the nonlinear effect between these two amplitudes.

The 5-level input case has briefly been mentioned here only as an example of the more general case of 'p' level input. It shows that TPRS is in fact optimal in many respects. TPRS has the inverse repeat property which enables the elimination of even order kernels, and when the system is of Hammerstein type, the dynamics of the impulse response can be exactly determined by correlation methods. The sequences with more than 3 levels do not offer the same advantages when applied to Hammerstein type systems.

5.4 Two-level inputs. PRBS

PRBS inputs are treated separately as a special case since there are unique problems associated with them. They have been used extensively in the past due to their simplicity of generation and the shorter sequence lengths involved. With 3-level inputs if T is the system settling time, the test signal has to be at least 2T time units long due to the inverted spike occurring at lag T in the autocorrelation function. With PRBS the sequence period need only exceed T by a small amount. PRBS however has the disadvantages that it is an inherently biased signal and does not have the inverse repeat property. The PRBS autocorrelation function is shown overleaf, in Fig. 5.4.1. Computation times using TPRS rather than PRBS are not necessarily longer. The inverse repeat property of TPRS can be used to half the multiplications required to compute crosscorrelation functions. The only advantage of PRBS remains shorter testing time.
As explained in Chapter 1, for linear systems the inverse repeat property is not necessary except in cases where it is desired to eliminate external effects such as those due to drift, but for nonlinear systems the automatic elimination of all the even-ordered kernels by the cross-correlation process is a very real advantage. One method of using PRBS and still eliminate the even order kernels, is to carry out two sets of correlations, the second one with an inverted PRBS input. This method is inferior compared with the one using 3-level signals. Apart from the longer sampling times required there are also inherent inaccuracies due to the bias of the PRBS itself. This will now be explained. Let $x_2(t)$ be a PRBS signal superimposed on a bias $m$. Then

$$x_2(t) = m + Az_2(t) \quad (5.4.1)$$

If the bias due to the unity amplitude PRBS is $m_2^*$ then

$$x_2(t) = [m + Am_2^*] + Az_2(t) \quad (5.4.2)$$

The actual input bias is thus $[m + Am_2^*]$. If the nonlinear system is of the general form shown in Fig. 5.2.1 then its output is

$$y(t) = K_2 \sum_{i=0}^{n} \beta_i (m + Am_2^*) A^i \int g_2(\lambda)H_1^i (t-\lambda, z_2) d\lambda \quad (5.4.3)$$
and the crosscorrelation function between the unit amplitude PRBS and the output, $\phi_{z_2y}(\tau)$, is given by

$$\phi_{z_2y}(\tau) = m \sum_{i=1}^{n} \beta_i (m + Am_2) A^i D_i(\tau)$$

(5.4.4)

where $D_i(\tau)$ is given by (5.2.3), with $\varepsilon = 2$

If a second test is then carried out with the PRBS inverted, making the new input $x_2(t)$ given by

$$x_2(t) = m - A[m_2 + z_2(t)]$$

$$= m - Am_2 - A z_2(t)$$

(5.4.5)

then the new output is

$$y'(t) = K_2 \sum_{i=0}^{n} \beta_i (m - Am_2) (-A)^i \int g_2(\lambda) H\beta_1 (t-\lambda, z_2) d\lambda$$

(5.4.6)

The new crosscorrelation function $\phi_{z_2y'}(\tau)$ is

$$\phi_{z_2y'}(\tau) = -m \sum_{i=1}^{n} \beta_i (m - Am_2) (-A)^i D_i(\tau)$$

(5.4.7)

If the crosscorrelation function $\phi_{z_2y}(\tau)$ and $\phi_{z_2y'}(\tau)$ are added, and the result called $\phi(\tau)$ then

$$\phi(\tau) = m_2 \sum_{i=1}^{n} S(i+1) [\beta_i (m + Am_2) + \beta_i (m - Am_2)] A^i D_i(\tau)$$

$$+ K_2 \sum_{i=1}^{n} S(i) [\beta_i (m + Am_2) - \beta_i (m - Am_2)] A^i D_i(\tau)$$

(5.4.8)

If the amplitude $A$ is very small, and the PRBS of long period then $(Am_2)$ is small, especially if the nonlinearity is not too pronounced about the operating point $m$, so that
and the even order terms in (5.4.8) are approximately zero. However, especially in noisy conditions, 'A' cannot be made very small, and the sequence length has to be kept to reasonably short lengths, or testing times become impractical. In such cases the relationship in (5.4.9) is considerably in error.

These arguments show that the inverse repeat PRBS test as a method of eliminating even order kernels is not very satisfactory. The operating point \( m \) could be varied for the two tests to compensate for the effect of the PRBS bias, but it is sometimes inconvenient to do so, and in any case it complicates the experiment.

5.4.1 PRBS input to Hammerstein type system

If a PRBS is the input to a Hammerstein type system, the expressions for the output \( y(t) \) and crosscorrelation function \( \phi_{xy}(\tau) \) are very simple. As before

\[
v(t) = \sum_{i=0}^{n} B_i(m) A^i [z_2(t)]^i
\]

(5.4.10)

where \( z_2(t) \) is a PRBS signal, \( A \) is the amplitude, \( m \) the bias superimposed on the PRBS; \( z_2(t) \) can take only 2 values, namely +1 or -1 and therefore
\[ [z_2(t)]^{2i+1} = z_2(t) \quad ; \quad i \text{ integer } \geq 0 \]  
\[ [z_2(t)]^{2i} = 1 \quad ; \quad i \text{ integer } \geq 0 \]  

(5.4.11)

The output from the nonlinearity is thus

\[ v(t)_2 = \sum_{i=0}^{n} S(i) B_i(m) A^i z_2(t) + \sum_{i=0}^{n} S(i+1) B_i(m) A^i \]  

(5.4.12)

\[ = c_0 + c_1 z_2(t) \]  

(5.4.13)

where \( c_0 \) and \( c_1 \) are constants depending on the amplitude \( A \), the operating point \( m \) and the coefficients \( B_i(m) \). The signal \( z_2(t) \) is biased. As before

\[ z_2(t) = m_2 + z_2(t) \]  

(5.4.14)

The expression for the output \( y(t) \) becomes

\[ y(t) = Kc_0 + Kc_1 \int g(\tau) z_2(t-\lambda) \, d\lambda \]  

\[ = Kc_0 + Kc_1 \int g(\tau) [m_2 + z_2(t-\lambda)] \, d\lambda \]  

\[ = [Kc_0 + Kc_1 m_2] + Kc_1 \int g(\tau) z_2(t-\lambda) \, d\lambda \]  

(5.4.15)

The output bias \( m_y \) is thus

\[ m_y = K[c_0 + m_2 c_1] \]  

(5.4.16)

The bias \( m_2 \), contributed by the PRBS, has thus a different effect from the superimposed bias \( m \) in the generation of the output bias \( m_y \); \( m_2 \) is amplified by odd order \( B_i(m) \) coefficients, while \( m \) is amplified by even order ones. The overall behaviour is, however, more complicated since the \( B_i(m) \) coefficients depend themselves solely on the superimposed bias \( m \).
5.5 Summary

Expressions for response and crosscorrelation functions of general (L-PNL-L) systems subjected to two and three level pseudorandom biased sequences have been derived. The expressions have also been particularised to Hammerstein type systems. It has been demonstrated that TPRS possesses properties which make it an ideal choice as a test signal when the systems are nonlinear. It has also been shown that the well known procedures of reducing nonlinearity errors based on PRBS inversion suffer from inherent errors associated with the fact that PRBS is itself biased. Some of the phenomena of Hammerstein type systems have been explained, with particular reference to their 'linear' behaviour when they are subjected to PRBS or TPRS.

The explicit form of the expressions has enabled a better understanding of the mechanisms by which solutions are obtained. This has been possible due to the particular form taken by the modified Volterra functional description of nonlinear systems.
CHAPTER 6

TEST PROCEDURES FOR SYSTEMS CONTAINING SMOOTH NONLINEAR ELEMENTS
6.1 Introduction

This chapter is concerned with experimental work carried out using digital simulations or electric networks, and the development of some methods which help with the elimination of errors commonly occurring in systems testing.

In the first part results are given which relate to theory developed in Chapter 3. Then an identification technique based on this theory is explained and tested for systems containing polynomial type nonlinearities. In Chapter 7 results are given of tests carried out on systems with piecewise linear, or discontinuous, nonlinearities. The practical limitations imposed on the identification performance by the systems are explored, with particular emphasis on noise effects. Some general conclusions and suggestions are made to help in the choosing of an optimum experimental configuration.

Prior to the main body of this chapter is a guide explaining the conventions used in the annotation of the diagrams.

6.2 Conventions used in the annotation of the diagrams

As far as possible every set of experimental results given in the following chapter has a diagram illustrating the system test configuration, with all the information pertaining to the experiment given on the diagram. To aid this certain conventions have been adopted.

6.2.1 Input signals

The symbol for the input signal is usually \( x(t) \). Next to this will be found a row vector of dimension 5. Writing this vector in symbolic form

\[
\mathbf{r} = [ \ C, L, A, m, M ]
\]

There are two possible cases depending on whether \( C = 'P' \) or \( C = 'G' \).

(i) If \( C = 'P' \), then the signal is a pseudo-noise sequence. In this case \( L \) is the level number of the sequence (usually 3, but sometimes 2 for PRBS),
A is the amplitude of the sequence in volts, \( m \) is the signal bias in volts, and \( M \) its bit length.

(ii) If \( C = 'G' \), then the signal is a Gaussian process. In this case \( L \) is set to infinity, \( A \) is the root mean square value of the process and \( m \) its bias, both in volts. \( M \) is set to infinity.

In the case of the same diagram used to characterise several tests at different mean levels, 'm' is replaced by 'variable'. In the case of two amplitude tests, the amplitude given in the vector is the smallest. The second amplitude can be considered to be a constant times the first one; this constant is denoted by \( R \), the value of which is given separately on the diagrams.

6.2.2 Noise signals

These are considered input signals to the system and as such the conventions described in section (6.2.1) apply.

6.2.3 Output signals

Next to the symbol for the output will be found a row vector of dimension 2. Writing this vector in symbolic form

\[
p = [ S, N ]
\]

\( S \) is the sample rate in units to be specified, and \( N \) is the number of averages made.

Anti-aliasing filters have been used prior to sampling in all cases, but these are only occasionally shown in the diagrams. It must be understood however that such filters are integral parts of these experiments.

6.2.4 Linear elements

The linear elements are represented by square boxes. Within these appear the symbols \( g(t) \), \( h(t) \) or \( K \), with subscripts if more than one of each type appear on the diagram.
The elements denoted by $g(t)$ or $h(t)$, are unity gain, linear elements, the functions $g(t)$ and $h(t)$ being their impulse responses; $h(t)$ is used rather than $g(t)$ when the element is a low-pass filter of order 4. Inside the box representing $h(t)$ will also appear a quantity such as $f_c = 100$ Hz. This represents the filter cut-off frequency. The symbols $g(t)$ and $K$ have in some cases the letters $A$ or $B$ as subscripts. These are used to represent particular systems, simulated within the computer using the pulse transfer function technique. Below appear the 's' and 'z' transforms of the systems $g_A(t)$ and $g_B(t)$.

<table>
<thead>
<tr>
<th>Impulse response</th>
<th>'s'-Transform</th>
<th>'z'-Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_A(t)$</td>
<td>$\frac{0.03}{s + 0.1s + 0.03}$</td>
<td>$\frac{0.032z^{-1}}{1 - 1.61z^{-1} + 0.82z^{-2}}$</td>
</tr>
<tr>
<td>$g_B(t)$</td>
<td>$\frac{1}{(s+1.2)(s+1.6)}$</td>
<td>$\frac{0.25z^{-1}}{1 - 0.5z^{-1} + 0.06z^{-2}}$</td>
</tr>
</tbody>
</table>

The symbols denoted by $K$ are pure gains. $K_A = K_B = 1$.

Hence $g_A(t)$ represents an oscillatory system impulse response, and $g_B(t)$ a non-oscillatory one.

6.2.5 Nonlinear elements

These are represented by pointed boxes. Inside the box is the input-output function for the nonlinearity. This is either a polynomial or a small graph showing pictorially the nonlinearity shape. This latter type of representation is used in the case of discontinuous nonlinearities.

6.2.6 Scaling

Most of the graphs given in this chapter are useful only to obtain pictorial information, and no actual values of ordinates are needed. Accordingly the vertical axes, especially in the cases of sampled records and correlation functions, are given in computer units, and the 'scale' value is included with the graph as a value ($s = \ldots$). The definition of the scale value for computer units is given in Chapter 2. Actual values
in units of volts etc., may be obtained if desired using the methods outlined in Chapter 2.

6.2.7 Additional abbreviations used to annotate graphs

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>IB</td>
<td>input bias (volts)</td>
</tr>
<tr>
<td>OB</td>
<td>output bias (volts)</td>
</tr>
<tr>
<td>PRBS</td>
<td>pseudo-random binary sequence</td>
</tr>
<tr>
<td>TPRS</td>
<td>ternary pseudo-random sequence</td>
</tr>
<tr>
<td>T(m)</td>
<td>small-signal, system gain function</td>
</tr>
<tr>
<td>SLT</td>
<td>single level test</td>
</tr>
<tr>
<td>RMLT</td>
<td>repeated mean level test</td>
</tr>
<tr>
<td>Thc.</td>
<td>theoretical curve</td>
</tr>
</tbody>
</table>

6.3 PRBS System Inputs

A well accepted and understood method to obtain the system impulse response of linear systems is to excite them with a PRBS signal and perform a crosscorrelation between the system output and its input. This yields, apart from determinable constants, the impulse response. As outlined in Chapter 1, this method has been refined and adapted to take into account situations such as output drift and multivariable systems, when several PRBS signals uncorrelated with each other are required. However when nonlinearities occur within the system, unless this is of Hammerstein type, the technique breaks down. The errors involved are large. By way of an example in Fig. 6.3.2 and Fig. 6.3.3 are shown the estimates of a system's impulse response function, in the first case with the system linear, and in the second case when a second order nonlinear element occurs at its output. In Fig. 6.3.1 is shown the configuration with the nonlinearity included.
The PRBS was fed into the system with zero superimposed bias, nevertheless the crosscorrelation function for the linear case shows the presence of the bias inherent in PRBS; in the nonlinear case large distortions mask such effect. Note that the distortions caused by the nonlinear element on the estimate of the impulse response are severe. These will be compared in the next section to those obtained when the inputs are three-level sequences.
Fig. 6.3.2 Impulse response approximation obtained with PRBS input; linear system.

Fig. 6.3.3 Effect on the estimate of the impulse response when the linear system is followed by a second order nonlinear term.
6.4 Response and crosscorrelation functions for nonlinear systems subjected to three-level inputs

Three-level sequences (TPRS) are a member of the broader class of signals known as 'inverse repeat sequences'. These have the characteristic of being repetitive over their period, and also being repetitive, but with a sign change, over their half-period.

The steady-state response of a linear system subjected to such an inverse repeat signal is another inverse repeat signal and further, the crosscorrelation function also has the inverse repeat characteristic. When the input is a TPRS then the crosscorrelation function is made up of two impulse response functions, the second being 'inverted'. This is subject to the usual restriction on the half-period of the TPRS that it must be longer than the effective decay time of the system's impulse response. Demonstrating these points are Fig. 6.4.2 and Fig. 6.4.3, showing the response and crosscorrelation functions obtained for the experimental set up shown in Fig. 6.4.1, representing a linear system.

![Diagram showing a linear system excited with TPRS](image)

As mentioned, the response and crosscorrelation functions have the inverse repeat property. The crosscorrelation function is proportional to the system impulse response up to the half-period lag value, and then to the inverted impulse response. Since the system input is unbiased, the response is also unbiased.
Fig. 6.4.2 Linear system response to unbiased TPRS input.

Fig. 6.4.3 Crosscorrelation function for the linear system.
If a nonlinearity is placed at the system output, it might be expected that the response and crosscorrelation functions would be distorted. The theory predicts, and experiments show, that this is not necessarily true of the crosscorrelation function. It is known that all even order autocorrelation functions of TPRS are zero for all lags; this means that all the terms containing even order crosscorrelation functions are zero. It has been shown that, for a Wiener type system, with \( u(t) \) and \( v(t) \) representing the nonlinear element input and output signals

\[
\phi_{xy}(\tau) = \beta_1(m)\phi_{xu}(\tau) + \beta_2(m)\phi_{xu^2}(\tau) + \sum_{i=3}^{n} \beta_i(m)\phi_{xui}(\tau)
\]  

(6.4.1)

Consider now Fig. 6.4.4 representing a Wiener type system containing a nonlinear element of order six, with the odd order nonlinear coefficients zero.

![Diagram](image)

Fig. 6.4.4 Nonlinear system excited with TPRS

Using the correlation properties of TPRS equation (6.4.1) becomes

\[
\phi_{xy}(\tau) = \beta_1(m)\phi_{xu}(\tau) + \beta_3(m)\phi_{xu^3}(\tau) + \beta_5(m)\phi_{xu^5}(\tau)
\]  

(6.4.2)

If reference is made to the form taken by the \( \beta_i(m) \) coefficients, it can be seen that when the input is unbiased, for the system shown in Fig. 6.4.4, then \( \beta_3(m) = \beta_5(m) = 0 \), and \( \beta_1(m) = 2 \).
Therefore

\[ \phi_{xy}(\tau) = 2\phi_{xu}(\tau) = 2g_A(\tau) \]  

(6.4.3)

and the crosscorrelation function is proportional to the impulse response function of the linear part of the system. This fact is derived very simply mathematically but is not obvious when one compares the response functions obtained without and with the nonlinear element, shown respectively in Fig. 6.4.5 and Fig. 6.4.2. The differences are pronounced; for the nonlinear case a definite bias appears at the output, and the inverse repeat property no longer applies. However, the crosscorrelation function in the nonlinear case, shown in Fig. 6.4.6, is identical, apart from a gain of two, to that obtained for the linear system, shown in Fig. 6.4.3.

These results should be compared to the ones obtained using PRBS, injected to a system with a nonlinearity of the same type (even nonlinearity), but less severe as it involves only a second order nonlinear term. The difference is due to the nature of TPRS, this latter signal having zero even order autocorrelation functions. This property does not apply to biased TPRS. The theoretical study given in Chapter 3 resulted in the derivation of the expression for the crosscorrelation function given in (6.4.1). When a bias is superimposed on the input, then the third and fifth order \( g(m) \) coefficients are nonzero. Although the even order ones do not affect the crosscorrelation function, due to the zero even order autocorrelation functions, distortions arise due to the odd order ones, which can clearly be seen in Fig. 6.4.7. This shows the crosscorrelation function across the nonlinear system illustrated in Fig. 6.4.4, when the input bias is one volt. Note that the inverse repeat property is retained. In summarising this section, it is concluded that TPRS signals have definite advantages over PRBS whether a bias is superimposed or not, but that care is needed in analysing results and predicting system behaviour. The theory shows, and experiments confirm, that when a bias is superimposed on the TPRS inputs to a nonlinear system, the presence of odd order kernels will affect the response even if the nonlinearity only contains even order coefficients. Conversely, if the the system contains only odd order nonlinear coefficients, and the inputs are biased, then even order kernels will be present in the system output functional expansion. The crosscorrelation function between the system output and the TPRS input is however always unaffected by even order kernels, due to the correlation properties of TPRS.
Fig. 6.4.5 Nonlinear system response to unbiased TPRS.

Fig. 6.4.6 Crosscorrelation function for the nonlinear system. Unbiased TPRS input.
Estimates of the system impulse response obtained using TPRS rather than PRBS are always more accurate if the system is nonlinear. The price is longer testing and computation times since twice the length of PRBS inputs are needed to comply with the requisites of the system's time constants.

Fig. 6.4.7 Crosscorrelation function for the nonlinear system. Biased TPRS input.
6.5 A procedure for the identification of the nonlinear system gain function

A method has been developed to enable the identification of a system's gain as a function of its input bias. The method assumes that the nonlinearity can be described by a polynomial equation in terms of its input bias. The system is assumed to be separable, within the definition given in Chapter 3, and be of the form (L-PNL-L) as shown in Fig. 6.5.1.

\[
\begin{array}{cccc}
 & K_A & \rightarrow & g_A(t) & \rightarrow & N(u) & \rightarrow & K_B & \rightarrow & g_B(t) \\
x(t) & & u(t) & & v(t) & & y(t)
\end{array}
\]

Fig. 6.5.1 General (L-PNL-L) nonlinear system.

The nonlinear element input output relationship is

\[
v(t) = N[u(t)] = \sum_{i=0}^{n} a_i [u(t)]^i
\]  

(6.5.1)

It has been shown in Chapter 5 that when the input is a biased maximal length sequence of amplitude A, then

\[
\phi_{xy}(\tau) = K_2 \sum_{i=1}^{n} b_i^{(m)} A^i D_i(\tau)
\]  

(6.5.2)

where

\[
D_i(\tau) = \frac{1}{K_1} \int_0^\lambda g_2(\lambda) \phi_{xu}(\tau-\lambda) d\lambda
\]  

(6.5.3)
From the theory in Chapter 3

\[ \beta_0(m) = \frac{n}{i=0} \alpha_i(K_m^i) \quad (6.5.4) \]

\[ = N(K_m^i) \quad (6.5.5) \]

The \( \beta_i(m) \) coefficients are related by recursive formulae;

\[ \beta_i(m) = \frac{1}{i!} \frac{d^i}{dm^i} \beta_0(m) = \frac{1}{i!} \frac{d^i}{dm^i} N(K_m^i) \quad (6.5.6) \]

The coefficient \( \beta_1(m) \) is of particular significance.

\[ \beta_1(m) = \frac{d}{dm} N(K_m^i) = \frac{d}{dm} \beta_0(m) \quad (6.5.7) \]

Equation (6.5.7) states that \( \beta_1(m) \) is the slope with respect to the system input bias of the nonlinear gain function, existing between the system input, and the output from the nonlinear element. It follows that the overall system (small signal) gain is \( K_2 \) times this value. This overall gain is a function of input bias. Denoting it by the symbol \( T(m) \), let

\[ T(m) = K_2 \beta_1(m) \quad T'(m) = \sigma^2 T(m) \quad (6.5.8) \]

For the purpose of this section, let \( \chi(t) = Az_3(t) \), where \( z_3(t) \) is the unit amplitude TPRS, and \( A \) the base amplitude of \( x(t) \). The crosscorrelation function in (6.5.2) can be written in terms of \( T(m) \).

\[ \phi_{XY}(\tau) = \sum_{i=0}^{n} A^i D_1(\tau) \cdot \frac{1}{(i-1)!} \frac{d^{i-1}}{dm^{i-1}} T(m) \quad (6.5.9) \]

This equation can be expanded, and using the fact that

\[ D_1(\tau) = \sigma^2 \int g_2(\lambda)g_1(\tau-\lambda)d\lambda \]

\[ = \sigma^2 g(\tau) \quad (6.5.10) \]
obtain

\[
\phi_{XY}(\tau) = A^2 T'(m)g(\tau) + \sum_{i=2}^{n} A^i D_i(\tau) \frac{1}{(i-1)!} \frac{d^{i-1}}{dm^{i-1}} T(m) \tag{6.5.11}
\]

The (small signal) gain \( T(m) \) appears as a multiplier of the impulse response function \( g(\tau) \). If it is possible to obtain this, then gain identification is achieved. The terms within the summation sign in (6.5.11) are the ones which corrupt the estimate of \( T(m) \). By using TPRS inputs their number is immediately halved.

\[
\phi_{XY}(\tau) = A^2 T'(m)g(\tau) + \sum_{i=3,5,7}^{n} A^i D_i(\tau) \frac{1}{(i-1)!} \frac{d^{i-1}}{dm^{i-1}} T(m) \tag{6.5.12}
\]

If the period of the TPRS is \( 2T \), then integrating (6.5.12) in the range \( 0 \) to \( T \) yields

\[
\psi = A^2 T'(m) + \sum_{i=3,5,7}^{n} A^i G_i \frac{d^{i-1}}{dm^{i-1}} T(m) \tag{6.5.13}
\]

where

\[
\psi = \int_{0}^{T} \phi_{XY}(\tau)d\tau \tag{6.5.14}
\]

\[
G_i = \frac{1}{(i-1)!} \int_{0}^{T} D_i(\tau)d\tau \tag{6.5.15}
\]

Equations (6.5.13) to (6.5.15) state that if the terms \( G_i \) and derivatives of \( T(m) \) decrease sufficiently fast, then a good estimate of \( T(m) \) is obtained by carrying out the correlation test and integrating the resulting function. This procedure yields the value of the small signal gain at one operating point. The use of the three-level sequence has enabled the elimination of half the distortion terms. Unfortunately it is every alternate term which is eliminated, and not the first half. Experience shows that the terms become smaller as \( 'i' \) increases, and so greater advantage would be obtained by eliminating the first \( \left( \frac{n}{2} \right) \) alone. It is possible however to eliminate more terms by using a procedure of multiple testing, at the same operating point, but with a different
amplitude TPRS. This is described later, but first the terms $G_i$ are examined more closely.

The $G_i$ terms are dependant on the dynamics of the linear elements, and the value of $i$. They are independent of the actual nonlinearity in the system. They can therefore be considered as weighting coefficients, acting on the derivatives of $T(m)$. In fact they can be shown to be most heavily dependant on the dynamics of the element preceeding the nonlinearity.

$$G_i = \frac{1}{(i-1)!} \int_0^T D_i(\tau) d\tau$$  \hspace{1cm} (6.5.16)

$$= \frac{1}{(i-1)!} \int_0^T \frac{1}{K_1} \int g_2(\lambda) \phi_{Z_3} u_i(\tau-\lambda) d\lambda d\tau$$  \hspace{1cm} (6.5.17)

The integral with respect to $\tau$ can be brought inside the convolution integral yielding

$$G_i = \frac{1}{(i-1)!} \frac{1}{K_1} \int g_2(\lambda) \int_0^T \phi_{Z_3} u_i(\tau-\lambda) d\tau d\lambda$$  \hspace{1cm} (6.5.18)

The crosscorrelation function $\phi_{Z_3} u_i(\tau)$ is an inverse repeat function of period $T$. Define a function $f(\lambda)$ as

$$f_i(\lambda) = \int_0^T \phi_{Z_3} u_i(\tau-\lambda) d\tau$$  \hspace{1cm} (6.5.19)

Then using the inverse repeat property of $\phi_{Z_3} u_i(\tau)$

$$f_i(\lambda) = \int_{\lambda T}^{T} \phi_{Z_3} u_i(\tau) d\tau - \int_0^{\lambda T} \phi_{Z_3} u_i(\tau) d\tau$$  \hspace{1cm} (6.5.20)

Let the maximum of $|f_i(\lambda)|$ occurs a value of $\lambda = \lambda_m$, for which

$$|f_i(\lambda_m)| > |f_i(\lambda)| \text{ all } \lambda \neq \lambda_m$$  \hspace{1cm} (6.5.21)
To obtain a conservative estimate of the order of magnitude of the $G_i$ functions, it is sufficient to use $f_i(\lambda_m)$ in (6.5.18). For convergence studies the estimate of $G_i$ given by (6.5.22) is considered sufficient.

$$
\tilde{G}_i = \frac{1}{(i-1)!} \left| f_i(\lambda_m) \right| \int g_i^2(\lambda) d\lambda
$$

(6.5.22)

$$
= \frac{1}{(i-1)!} \left| f_i(\lambda_m) \right|
$$

(6.5.23)

The order of magnitude of the $G_i$ terms, and in consequence the convergence of the series for the $\psi$ function given in (6.5.13), depends on the magnitude of the term $f_i(\lambda_m)$, and this depends mostly on the element with impulse response $g_i(t)$. Experiments were carried out on many different types of linear systems to determine how $\tilde{G}_i$ varies with $i$. Some of the results are shown in Fig. 6.5.2. To obtain these, values of $f_i(\lambda_m)$ were found, and were then substituted into (6.5.23). The results obtained are very encouraging since they show very rapid convergence. Magnitudes are plotted rather than actual values to help with the logarithmic type representation. All values are actually positive except for the ones for $i = 9$. These are however slightly suspect, as extremely large numbers were involved in their computation. It is felt that machine error can be attributable to these negative terms. The data would tend to indicate that the convergence is not as fast for systems which are highly oscillatory. However it is felt that a great deal more research could be carried out in this area, possibly to find more precise relationships between the linear system characteristics, such as damping, and the $G_i$ terms. Such information would be useful because it would then be possible to write the cross-correlation function only in terms of derivatives of gain; identification methods could then be designed to find $T(m)$ exactly. However, for the purpose of this work, the results obtained were considered to be sufficient to support the assumption of rapid $G_i$ convergence.

The estimate of the gain function $T(m)$ for any single operating point, by the method of a single correlation test (yielding $\psi$), is in error by an amount which greatly depends on the amplitude $A$ and that of the derivatives of $T(m)$. As will be shown, in many cases the estimate obtained with the single test is quite good. If however a better estimate is desired the following procedure can be adopted.
Convergence modes of the characteristic Volterra $|\tilde{G}_\ell|$ constants as a function of the order $i$.

- System with impulse response $g_A(t)$
- System with impulse response $g_A(t) \ast g_B(t)$
- Low pass filter (4th order)

Fig. 6.5.2
Keeping the input signal at a fixed operating point, a TPRS test is performed with the amplitude of the input random signal denoted by \( A \), say. As before the system output can be crosscorrelated with the random input, and the resulting crosscorrelation function integrated over its half-period. Let the value obtained be denoted by \( \psi_1 \). This procedure can then be repeated, but with a different amplitude TPRS. If this amplitude is denoted by \( B \) where

\[
B = R \cdot A \tag{6.5.24}
\]

\( R \) is then the ratio of the two amplitudes. Let the new value of the integrated crosscorrelation function be denoted by \( \psi_2 \). It is possible to write two equations.

\[
\psi_1 = A^2 \cdot T'(m) + \sum_{i=3,5,7}^n A^i G_i \frac{d^{i-1}}{dm} [T(m)] \tag{6.5.25}
\]

\[
\psi_2 = R^2 A^2 \cdot T'(m) + \sum_{i=3,5,7}^n R^i A^i G_i \frac{d^{i-1}}{dm} [T(m)] \tag{6.5.26}
\]

For notational convenience let

\[
v_i(m) \equiv v_i = G_i \frac{d^{i-1}}{dm} [T(m)] \quad i \geq 1 \tag{6.5.27}
\]

Combining the two equations (6.5.25) and (6.5.26) in matrix form and using (6.5.27) yields

\[
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} =
\begin{bmatrix}
A^2 & A^4 \\
R^2 A^2 & R^4 A^4
\end{bmatrix}
\begin{bmatrix}
T'(m) \\
v_3
\end{bmatrix} +
\begin{bmatrix}
A^6 v_5 + A^8 v_7 + \ldots \\
R^6 A^6 v_5 + R^8 A^8 v_7 + \ldots
\end{bmatrix} \tag{6.5.28}
\]

Assuming that the terms \( v_5 \) and higher are negligible and defining \( H \) as

\[
H =
\begin{bmatrix}
A^2 & A^4 \\
R^2 A^2 & R^4 A^4
\end{bmatrix} \tag{6.5.29}
\]
then

\[ H^{-1} = \frac{1}{A^4 R^4 (R^2 - 1)} \begin{bmatrix} A^2 R^4 & -A^2 \\ -R^2 & 1 \end{bmatrix} \]  

(6.5.30)

Premultiplying (6.5.28) by \( H^{-1} \) yields finally

\[ \begin{bmatrix} T'(m) \\ v_3 \end{bmatrix} = \frac{1}{A^4 R^4 (R^2 - 1)} \begin{bmatrix} A^2 R^4 & -A^2 \\ -R^2 & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \]  

(6.5.31)

or

\[ T(m) = \frac{1}{A^4 R^4 (R^2 - 1)} \begin{bmatrix} R^4 & \psi_1 - \psi_2 \end{bmatrix} \frac{1}{\sigma_{v_3}^2} \]  

(6.5.32)

The estimate of \( T(m) \) which is obtained by the use of (6.5.32) is unaffected by nonlinear terms up to and including the fourth order one, and is therefore more accurate than that obtained by writing \( T'(m) = \psi \). The procedure of eliminating the third order nonlinear term can be extended to higher order ones, but computation and testing times increase accordingly. It was found that satisfactory results can be expected with the single amplitude test. The repeated amplitude constant mean level test improves this, but it is not considered worthwhile to go beyond two tests per operating point. The repeated mean level test is necessary when there is a strong third order nonlinear effect, over the range of influence of the test signal.

In Fig. 6.5.3 is shown a very simplified flow diagram explaining the general procedure of identifying a typical system gain function. Under program control, a computer sends out sets of TPRS excitation signals with different amplitudes, and samples the system response. Then follows the arithmetic work outlined by formulae in this section culminating in an estimate of \( T(m) \) for that particular operating point. The computer then increments the system input bias and repeats the cycle. The range of bias values, the step between tests, denoted \( \Delta m \) in Fig. 6.5.3, sample rates and other parameters are externally set prior to the start of the experiment. Such a procedure was used to obtain all the identification results presented in this thesis. Some of the details of the software is given in Chapter 2.
Fig. 6.5.3 Flow diagram for the procedure used to identify the nonlinear gain function $T(m)$. 
The choice of the amplitude ratio $R$ cannot be completely arbitrary. Although it has not been possible to obtain precise rules to select an optimum value, the analysis which follows gives insight into the problem and rough guide lines in the selection.

6.6 Error analysis

The systems considered so far have been assumed to be noise free. This restriction is now removed.

Fig. 6.6.1 Additive noise corruption of system output.

If $y(t)$ is the system output, then the observed output $q(t)$ can be written

$$q(t) = y(t) + n(t)$$  \hspace{1cm} (6.6.1)

where $n(t)$ is the noise signal. If this is biased, with mean level $m_n$ and dynamic component $\bar{n}(t)$ then

$$q(t) = y(t) + m_n + \bar{n}(t)$$  \hspace{1cm} (6.6.2)

The crosscorrelation function between the dynamic component of the input and the output becomes

$$\phi_{Xq}(\tau) = \phi_{XY}(\tau) + \phi_{X\bar{n}}(\tau)$$  \hspace{1cm} (6.6.3)

If the input is a maximal length sequence of amplitude $A$ then

$$\phi_{Xq}(\tau) = \phi_{XY}(\tau) + A \phi_{Z\bar{n}}(\tau)$$  \hspace{1cm} (6.6.4)
Let \( m_\phi \) represent the integral
\[
m_\phi = \int_0^T \phi z n(t) dt
\]  
(6.6.5)

It is possible now to rewrite the matrix equation (6.5.28) including the term due to the noise.

\[
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = \begin{bmatrix}
T'(m) \\
V_3
\end{bmatrix} + \begin{bmatrix}
A^6 v_5 + A^8 v_7 + \ldots \\
R^6 A^6 v_5 + R^8 A^8 v_7 + \ldots
\end{bmatrix} + \begin{bmatrix}
A \\
R A
\end{bmatrix} m_\phi
\]  
(6.6.6)

As before the equation is premultiplied by \( H^{-1} \), yielding

\[
H^{-1} \begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = \begin{bmatrix}
T'(m) \\
V_3
\end{bmatrix} + H^{-1} \begin{bmatrix}
A^6 v_5 + A^8 v_7 + \ldots \\
R^6 A^6 v_5 + R^8 A^8 v_7 + \ldots
\end{bmatrix} + H^{-1} \begin{bmatrix}
A \\
R A
\end{bmatrix} m_\phi
\]  
(6.6.7)

This may be rewritten as

\[
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix} = \begin{bmatrix}
T'(m) \\
V_3
\end{bmatrix} + \begin{bmatrix}
e_1 + e_3 \\
e_2 + e_4
\end{bmatrix}
\]  
(6.6.8)

and thus

\[
f_1 = T'(m) + e_1 + e_3
\]  
(6.6.9)

where the terms \( e_1 \) to \( e_4 \) and \( f_1, f_2 \) can be obtained by expanding the matrix multiplications in (6.6.7). In particular, \( f_1 \) is the estimate of \( T'(m) \) and \( e_1, e_3 \) are the errors in this estimate due to higher order nonlinear terms and noise respectively. The errors are now considered separately.

### 6.6.1 Errors due to nonlinear terms

Matrix multiplication yields

\[
e_1 = -[A^4 R^2 v_5 + A^6 (R^2 + R^4) v_7 + A^8 (R^2 + R^4 + R^6) v_9 + \ldots]
\]  
(6.6.10)
It will be assumed that the amplitude $A$ is the smallest one which is feasible from noise and quantisation considerations so that $R$ must be greater than one. It is possible to write an expression for the error obtained by carrying out a single test, not eliminating the term containing $v_3$. Denoting this error by $e_0$

$$e_0 = A^2v_3 + A^4v_5 + A^6v_7 + A^8v_9 + \ldots \tag{6.6.11}$$

Note that if $A$ could be made very small, both $e_0$ and $e_1$, would tend to zero, and the identification would be free from errors due to the nonlinearity. Comparing the two error equations shows that the repeated mean level test has the effect of eliminating the term containing $v_3$ and of changing the polarity of all the terms containing $v_5$, $v_7$, and so on. Further, since $R > 1$, the magnitude of these terms is increased. It might be expected that as $R$ is increased these terms become so large that the beneficial effect of eliminating the third order one is completely offset. However this limit is rarely reached in practice. Experiments have shown that for values of $R$ up to four, the repeated mean level test is always advantageous. Higher values of $R$ are impractical and pointless. Nevertheless this theory does show that there is a limit to which the repeated mean level tests can be taken. For convenience a function is now defined which measures the improvement which is obtained with the repeated mean level test as opposed to the single one. Denoting it by $\Delta e(m)$, it is given by

$$\Delta e(m) = |e_0| - |e_1| \tag{6.6.12}$$

$$= A^2v_3 - \{ A^4(R^2-1)v_5 + A^6(R^4+R^2-1)v_7 + \ldots \} \tag{6.6.13}$$

This function is dependent of bias $m$, since the coefficients $v_i$ are functions of $m$. When $\Delta e(m)$ is positive, the repeated mean level test is advantageous. As pointed out, this is nearly always the case, but (6.6.13) does show that the improvement function decreases with amplitude $A$. From these arguments it can be concluded that $R$ should be as close to one as possible. The lower limit of $R$ is then imposed by the finite word length of the computer used to control the experiment and analyse the data.

Only the odd order $v_i$ terms appear in the expression for the errors; these contain only odd order derivatives of the nonlinear system d.c. input output function. If this is symmetric about zero then it implies that
\[ v_i(m) = v_i(-m) \quad (6.6.14) \]

and therefore

\[ \Delta e(m) = \Delta e(-m) \quad (6.6.15) \]

The term \( v_i^{(m)} \) contains the \( i^{\text{th}} \) derivative of the d.c. input-output function multiplied by a constant, \( G_i \), which itself rapidly becomes smaller as \( i \) increases. Some experimental evidence of this is illustrated in Fig. 6.5.2 For symmetric nonlinearities, within the relatively small range of influence of the test signals, the first and third order terms of the input-output polynomial predominate over the fifth and higher order ones. Although the complete nonlinear characteristic may need, say, a 15\(^{\text{th}}\) order polynomial for its adequate description, within the range of influence of the test signal it is unlikely that polynomials over fifth order are needed, except near highly nonlinear regions. These arguments imply that the error which is obtained in the estimation of gain with the repeated mean level test is rarely large. If, for the repeated mean level test, the polynomial representing the nonlinearity over the range of influence of the test signal is of order 5 or less, the error in the estimated small signal gain is always zero. If it is of order 7 the error is a weak quadratic function of operating point. On the other hand, with the single mean level test, if the polynomial is of order 5 the error is a quadratic function of the operating point. Experimental evidence is given in this chapter.

These arguments imply that the higher \( v_i^m \) terms are so small that even if they are multiplied by powers of \( R \), these are rarely so large that they impact significantly the improvement derived from eliminating the third order term. In any case \( R \) is always kept small to limit the range of influence of the test signal. In the next section it will be shown that \( R \) cannot be made as small as the computer resolution will allow because other factors limit its lower value. A point of interest which arises from this discussion is that the errors are mostly due to the third derivative of the nonlinear input-output function. This suggests that it is only worthwhile conducting a repeated mean level test near regions where the gain function is changing rapidly. Therefore, rather than carrying out such accurate tests over the full range of the nonlinear characteristic, it is quicker to carry out the single level one, finding the regions where the gain changes rapidly, and then going back over those regions to carry out a
second test, improving the results. This is especially true for characteristics which are nonlinear over only portions of their total range.

6.6.2 Errors due to noise

The error due to noise denoted by \( e_3 \) in (6.6.9) is evaluated by matrix multiplication, resulting in the expression

\[
e_3 = \frac{m_\phi}{A} \left[ \frac{R^3 - 1}{R^3 - R} \right]
\]

(6.6.16)

The function \( e_3 \) is shown in Fig. 6.6.2. It is asymptotic to the value \( (m_\phi/A) \), this being the error due to noise with the single amplitude test. Therefore it must be concluded that the repeated mean level test has the adverse effect of always amplifying errors due to additive noise.

When \( R = 1 \), the noise error is 1.5 times that obtained with the single amplitude test. When \( R = 2 \) this figure is reduced to 7/6. The convergence to the value \( m_\phi/A \) is hence quite rapid, but nevertheless the noise effects are never reduced by the two level test. To reduce to a minimum \( e_3 \), \( R \) should be as large as possible.
There is however another source of error which cannot be predicted easily in mathematical terms. This stems from quantisation and rounding effects. It was found experimentally that as the value of $R$ approaches one, the error curve shown in Fig. 6.6.2 deviates from its predicted shape. In fact it takes the form of that shown in Fig. 6.6.3, which was obtained experimentally.

![Graph](image)

**Fig. 6.6.3** Experimental variation of error due to noise as a function of amplitude ratio $R$.

A physical explanation of this is that as the two amplitudes get closer together, the responses from the system become less distinguishable from each other due to quantisation in the A/D and D/A converters, and to the build up of rounding errors. To find a measure of this effect experiments were carried out on simulated linear systems; to obtain better estimates, rms values of hundreds of tests were obtained. These have resulted in the data presented in Fig. 6.6.4, which shows how the rms error in estimated gain ($\varepsilon_q$) varies with base amplitude 'A' and with ratio 'R'. $\varepsilon_q$ is the actual rms error in estimated gain due to the quantisation effects. Only values of $R$ greater than one have been investigated, for the considerations mentioned earlier.
Fig. 6.6.4 Variation of rms error due to quantisation as a function of amplitude ratio $R$, for fixed base amplitude $A$.

The units of amplitude are computer or 12-bit converter units (see Chapter 2). The experimental results obtained are particular to the computer and the software used as well as the precision of the A/D equipment. Similar curves can however be obtained experimentally using the repeated mean level test technique on a known linear system. By comparing such data with that obtained in noisy conditions it appears that $\epsilon_q$ is approximately linearly related to $m$.

The errors due to the nonlinear terms, those due to noise and quantisation have been analysed. It is now possible to make some general conclusions regarding which value of $R$ produces the best results.
6.7 The choice of an optimal second amplitude for the repeated mean level test

In the absence of noise, and assuming no quantisation effects, it would not be necessary to carry out a repeated mean level test, because it would be possible to test with minute amplitudes, and the resulting estimates of the small signal system gain would be very accurate. In the presence of noise, the smallest practical test amplitude is assumed to have been found and it is denoted by A. Noise can be cyclic, random or both. Random noise, if stationary, can be reduced by averaging and so can cyclic noise provided certain rules are followed. These are investigated later in this chapter. However noise effects, especially due to random noise, cannot be eliminated completely and hence $m_\phi$ will be finite.

If a repeated mean level test is necessary the theory and experiment have shown that the errors due to nonlinearity increase with R, and those due to noise decrease with R. In particular, when R is very close to one, with small base amplitudes, noise errors add to those due to quantisation. Illustrating these points in Fig. 6.7.1 which shows results from a simulation run on a system containing a ninth order nonlinearity. The rms error in the identified small signal gain is shown plotted against R.

![Fig. 6.7.1 Effect of amplitude ratio R on the rms error in estimated nonlinear gain T(m).]
To calculate the rms error, ten tests were carried out over a portion of the system's operating range. From the figure it is seen that the optimum choice of $R$ is approximately 1.5. However, the figure is only useful in demonstrating a trend, and not of general use. Each system will have a different optimum value of $R$. A method of finding an approximation to the optimum is to obtain first the curves relating $\varepsilon_q$ to $R$ and $A$; the value of $A$ will be fixed by the environment. It is then a matter of choosing a value of $R$ to yield a value of $\varepsilon_q$ smaller than a desired value. It must be remembered however that $\varepsilon_q$ is often only a small part of the contributing error. The quantity $m_\phi$ can be determined sometimes, but more often than not the noise cannot be measured directly.

For the experiments described in this thesis, good values of $R$ were between 1.2 and 1.7, with base amplitudes of the order of $1/20^{th}$ of the system's operating range. This has included tests on severely nonlinear systems, with signal-to-noise ratios never below three. Other situations, however, can be envisaged when the noise is highly correlated with the input signals giving large values of $m_\phi$. This might necessitate larger values of $R$ to reduce the noise amplification effect of the repeated mean level test. It is significant that $m_\phi$ is not proportional to the signal to noise ratio; it depends rather on the average degree of correlation between the dynamic components of system input signal and the noise.

6.8 Identification of the gain characteristic of system containing polynomial nonlinearities \( (L-PNL-L) \)

Tests were carried out on systems with imbedded polynomial type nonlinearity, the systems being either computer simulated or hard-wired electrical networks. The nonlinearities tested were mostly symmetric although the method is not restricted to this case. The effect of random noise added to the output response of the system was also studied.
6.8.1 Identification with noise free system

The system shown above was tested using 80 bit, three-level sequences. This allows 40 bits to define the impulse response, which implies a coarse resolution of the crosscorrelation function. The reason for choosing a short sequence length is that computation times are shortened. The testing time however remains unaltered, as this is chosen independently of the sequence length. The correlation process used for these tests was not optimised to take into account the digital nature of the input signal. This is because the software was designed as a research tool, to be as flexible as possible. Thus, for an N bit correlation, approximately $N^2$ multiplications are needed. In the computer used, the multiplication operation is the most computationally expensive. Cutting the sequence length N drastically reduces the processing times due to the $N^2$ term.

In addition, with three level sequences the choice of sequence lengths is limited to 80, 242 or 728. Other sequence lengths exist but they are not very practical. For this work only $N = 80$ or $N = 242$ were used and the latter only when accuracy was important. The computational savings in using $N = 80$ as compared to $N = 242$ is of the order of ten to one. This advantage is less pronounced if FFT or FWT procedures are used. Such methods of obtaining faster throughputs are briefly discussed in Chapter 9.

Having identified $T(m)$, the d.c. input-output curve is obtained by its integration with respect to $m$. This was done for the example given in Fig. 6.8.2 and the results are given in Fig. 6.8.3. For the single level
test, the predicted outputs are greater than the actual, while for the two level test they are slightly smaller. This is due to the polarity change of the fifth and higher order error terms which is a characteristic of the identification method. The integration of the gain function $T(m)$ produces the dynamic component of the input-output curve, but it is necessary to make some assumptions about the nonlinear characteristic to obtain the complete curve. Mathematically this is equivalent to choosing the constant of integration. If the nonlinear characteristic is symmetric, then with zero steady state input, the output will be zero. This means $\alpha_0 = 0$, and the integrated characteristic must pass through zero. The value of the constant of integration, which must be added to the integrated $T(m)$ curve, is therefore fixed. In cases when it cannot be assumed that $\alpha_0 = 0$, some other assumption is necessary, dictated by the nature of the system being identified.

Fig. 6.8.2 Identification of system gain function $T(m)$ with single and repeated mean level tests. $R = 1.5$. 
Fig. 6.8.2 shows the gain function $T(m)$ for the system illustrated in Fig. 6.8.1, together with the estimated values using single and repeated mean level testing. The difference between the two sets of estimated data represents $\Delta e(m)$, the improvement function defined in section (6.6.1). As predicted $\Delta e(m)$ is greater for the larger values of $|m|$, the magnitude of the operating point. Further, the error in the estimate of $T(m)$ obtained with the repeated mean level test is a constant over the whole of the test range, while that for the single level test increases approximately quadratically with $|m|$. This is a severe test of the method as the amplitude of the three level sequence, $A$, is one quarter of the total test range. This means that the range of influence of the test signal is half the total test range.

Fig. 6.8.3 Identification of system d.c. input-output function, with single and repeated mean level tests.
6.8.2 Identification in the presence of Gaussian noise

To test the performance of the identification algorithm described in section (6.6.5) under controlled random noise conditions, Gaussian noise was added to the system output. The test arrangement is shown in Fig. 6.8.4 and the crosscorrelation function between noise and input signal in Fig. 6.8.5.

![Diagram of system corrupted by Gaussian noise](image)

Fig. 6.8.4 Non-linear (L-PNL-L) system corrupted by Gaussian noise.

![Crosscorrelation function between TPRS and noise process](image)

Fig. 6.8.5 Crosscorrelation function between the TPRS and the noise process.

The estimated gain curves using single and repeated mean level testing are shown in Fig. 6.8.6 and Fig. 6.8.7 respectively. If a moving average process is operated on this data, the resulting experimental curves approach the ones obtained with no noise. The presence of the noise has not prevented the elimination of the effects of the cubic error term. The much smaller effects due to the fifth and seventh order terms are masked.
by the noise process. The noise effects appear more pronounced for the two level test which is as predicted in section (6.6.2).

The estimates of the d.c. input-output relationships are shown in Fig. 6.8.8. The integration process used to derive these from the \( T(m) \) curves, has almost completely eliminated the effect of the high frequency variations on the gain curves, but some bias remains. In particular, for the two level test a slight positive bias is present throughout the range.

The results obtained are good especially considering that the test conditions were severe. The signal to noise ratio was 5, the nonlinearity a strong seventh order polynomial throughout the operating range, and the amplitude of the dynamic part of the test signal was half the total test range.

The repeated mean level test has been demonstrated to achieve good results even in the presence of noise; some of the characteristics of the method, like that of noise effect amplification, can be seen in the estimated gain curves. Fig. 6.8.7.
Fig. 6.8.6 Identification of $T(m)$ function with the SLT in the presence of additive output Gaussian noise.

Fig. 6.8.7 Identification of the $T(m)$ function with the RMLT in the presence of additive output Gaussian noise - ($R = 1.5$).
Fig. 6.8.8 Identification of the system d.c. input-output function with single and repeated mean level tests, in the presence of output additive Gaussian noise.
6.9 Cyclic noise

The effects of random unbiased noise on repetitive signals are often reduced using averaging methods which rely on the properties of ergodic signals. When the noise is not ergodic, and in particular is cyclic, averaging produces unpredictable and biased estimates. To overcome this problem a detailed analysis was carried out and a method found to eliminate such errors, based upon a combination of the averaging process and the choice of the period of the repetitive signal. The method is thus confined to those situations when the period can be changed. A very common situation is that of a system fed with a repetitive test signal and the output is corrupted by noise prior to its observation. The period of the test signal is often chosen to be longer than the system's settling time. There is no stringent limit as to how much longer it should be and in any case it will be shown that the variations in the period which are required are small.

![Diagram](image)

**Fig. 6.9.1** System corrupted by cyclic noise.

In this section the cyclic noise period is considered to be smaller than that of the system input signal. This restriction can be removed, but in most cases such an assumption is correct. When the noise period is longer than that of the system input, other specialised techniques exist for its reduction, based on drift removal schemes.
Consider Fig. 6.9.1. Let the period of \( y(t) \), the repetitive output from the system, be \( T_s \) and the period of the cyclic noise process, \( n(t) \), be \( T_n \). Then

\[
y(t) = y(t + K T_s) \quad ; \quad K \text{ integer } \geq 0
\]

\[
n(t) = n(t + K T_n)
\]  

(6.9.1)

If \( Z(t) \) is the observed output, for additive noise

\[
Z(t) = y(t) + n(t)
\]  

(6.9.2)

The periods \( T_s \) and \( T_n \) can be related by

\[
T_s = K T_n + \Delta \quad ; \quad K \text{ integer } \geq 0
\]

\[
\Delta < T_n
\]  

(6.9.3)

If \( \hat{y}(t) \) is the estimate of \( y(t) \) obtained by averaging \( N \) successive observed output sequences \( Z(t) \) then

\[
\hat{y}(t) = \frac{1}{N} \sum_{r=0}^{N-1} Z(t + r T_s)
\]  

(6.9.4)

but \( Z(t) \) can be written in terms of \( y(t) \) from (6.9.1), and \( T_s \) is related to \( T_n \) thus

\[
\hat{y}(t) = y(t) + \frac{1}{N} \sum_{r=0}^{N-1} n(t + r \Delta + r K T_n)
\]  

(6.9.5)

which on using the recursive relationship in (6.9.3) becomes

\[
\hat{y}(t) = y(t) + \frac{1}{N} \sum_{r=0}^{N-1} n(t + r \Delta)
\]  

(6.9.6)

The estimate of \( y(t) \) is therefore independent on the period of the cyclic noise \( T_n \), but is related to \( \Delta \).

The idealised case of \( n(t) \) being a sinusoidal process is considered first.
In this case the noise process is described by

\[ n(t) = A \exp[jw(t)] \quad ; \quad j = \sqrt{-1} \]  

(6.9.7)

where \( A \) is the amplitude of the sinusoidal noise process and \( w \) its frequency.

\[ w = \frac{2\pi}{T_k} \]  

(6.9.8)

If \( n(t) \) is substituted into the expression for \( \xi(t) \) then

\[ \xi(t) = y(t) + \frac{1}{N} \sum_{r=0}^{N-1} \exp[j2\pi r \frac{\Delta}{T_k}] \cdot A \exp[jw(t)] \]

\[ = y(t) + B_1 A \exp[jw(t)] \]  

(6.9.9)

where

\[ B_1 = \frac{1}{N} \sum_{r=0}^{N-1} \exp[j2\pi r \frac{\Delta}{T_k}] \quad ; \quad \xi \leq 1 \]  

(6.9.10)

\[ = \frac{1}{N} \sum_{r=0}^{N-1} \exp[j2\pi r \frac{\Delta}{T_k}] \quad ; \quad \xi \leq 1 \]  

(6.9.11)

The magnitude of \( B_1 \) depends both on average number \( N \) and time magnitude \( \Delta \). For certain combinations of these variables \( B_1 \) is zero, implying that the estimated system output after averaging, is equal to the actual output. This can be seen by putting \( B_1 = 0 \) in (6.9.9); it implies that the effects due to the sinusoidal noise process have been completely eliminated.

It can be shown that \( B_1 \) has zero values only when \( N \), the average number, is a power of two and when \( \Delta \) satisfies

\[ \frac{\Delta}{T_k} \bigg|_{B_1 = 0} = \frac{1 + 2K_1}{2(N-1)} \quad ; \quad K_1 = 0, 1, \ldots, N-2 \]  

(6.9.12)

where

\[ N = 2^p \quad ; \quad p \text{ integer} \]  

(6.9.13)
Examples of the dependence of $B_1$ on $N$ and $\Delta$ are given in Fig. 6.9.2. $B_1$ is shown plotted for fixed values of $N$, as a function of normalised times $(\Delta/T_n)$. It can be seen that $B_1$ is zero when the conditions given by (6.9.12) are satisfied. The curves are also of interest since they show that if a large number of averages are taken then, unless $(\Delta/T_n)$ is zero or one, the reduction in noise from averaging is large even if not optimal, whatever value of $(\Delta/T_n)$ is chosen.

As an example of the use of this expression, consider the case of a system being excited with a signal $x(t)$ of period 100 ms. (The output $y(t)$ will also have the same period), and the output be corrupted by 'mains' noise, with period $(T_n = 20$ ms). To eliminate this corruption by averaging without making some modifications to the input is impossible, since $\Delta$ is zero; when $\Delta$ is zero $B_1$ is always one and the amplitude of the noise is unreduced. Assume that the period of $x(t)$ is flexible, even if by small amounts. To eliminate the noise in 2 averages (6.9.12) as well as Fig. 6.9.2 reveals that $(\Delta/T_n)$ must be 0.5. This means that the period of $x(t)$ must be 110 ms. If this variation of the period of $x(t)$ is unacceptably large, a greater number of averages are required. With four averages, the zeros of $B_1$ occur at $(\Delta/T_n) = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{5}{6}$ so that an acceptable value of $\Delta$ is 3.33 ms; the required period of $x(t)$ therefore becomes 103.33 ms.
Fig. 6.9.2(a) Variation of $B_1$ with normalized time ($\Delta/T_n$).

$N = 2$

$N = 4$
Fig. 6.9.2(b) Variation of $B_1$ with normalized time ($\Delta/T_n$).
The more general case of finite band width cyclic noise is now considered. Let this noise be expressable as

\[ n(t) = \sum_{i=1}^{M} A_i \exp[ji\omega_0 t] \quad ; \quad M \text{ finite integer} \quad (6.9.14) \]

where \( \omega_0 \) is the noise fundamental frequency in radians.

\[ \omega_0 = \frac{2\pi}{T_n} \quad (6.9.15) \]

As before \( T_n \) is the period of the cyclic noise. The expression for \( \varepsilon(t) \), the estimate of \( y(t) \) after \( N \) averages, can be deduced from (6.9.6) and that for the noise (6.9.14);

\[
\varepsilon(t) = y(t) + \frac{1}{N} \sum_{r=0}^{N-1} \sum_{i=1}^{M} A_i \exp[ji\omega_0 t + jir\omega_0] \\
= y(t) + \sum_{i=1}^{M} A_i \exp[ji\omega_0 t] \cdot \frac{1}{N} \sum_{r=0}^{N-1} \exp[jir\omega_0] 
\]

(6.9.16)

This can be rewritten as

\[
\varepsilon(t) = y(t) + \sum_{i=1}^{M} B_i A_i \exp[ji\omega_0 t] 
\]

(6.9.17)

(6.9.18)

where

\[
B_i = \frac{1}{N} \sum_{r=0}^{N-1} \exp[jir\omega_0] \quad ; \quad \varepsilon 1 
\]

(6.9.19)

\[
= \frac{1}{N} \sum_{r=0}^{N-1} \exp[j2\pi r \frac{i\Delta}{T_n}] \quad ; \quad \varepsilon 1 
\]

(6.9.20)

Expressions (6.9.18) and (6.9.20) imply that the estimate of \( y(t) \) after \( N \) averages is corrupted by a noise signal with the same frequency components as those contained in the original noise, but with different component amplitudes. The amplitude of the \( i^{TH} \) noise harmonic after averaging is \( (B_i A_i) \), as compared to \( A_i \) prior to averaging. Since \( B_i \leq 1 \) the
worst case is that averaging which leaves the noise unaffected. The expression for \( B_i \) becomes that for \( B_1 \) with \( i = 1 \). However \( B_i \) has 'i' times more zero values in the range \( 0 \leq \frac{\Delta T_n}{T_n} < 1 \) than \( B_1 \), for the same \( N \). It can be shown that

\[
\frac{\Delta}{T_n} \bigg|_{B_i = 0} = \frac{1 + 2K_i}{2(N-1)i} ; \quad K_i = 0, 1, \ldots, i(N-1)-1
\] (6.9.21)

For the noise to be completely eliminated by averaging, a value of \( \frac{\Delta T_n}{T_n} \) must be found which makes all the \( B_i \) terms zero.

It is sufficient to find which values of \( \frac{\Delta T_n}{T_n} \) making \( B_1 \) zero also make the harmonic amplitudes \( B_i \), zero. To do this the relationships in (6.9.12) and (6.9.21) must be equated. From (6.9.12) and (6.9.21)

\[
\frac{\Delta}{T_n} \bigg|_{B_i = 0} = \frac{1 + 2K_i}{2(N-1)} ; \quad K_i = 0, 1, \ldots, N-2
\] (6.9.22)

\[
\frac{\Delta}{T_n} \bigg|_{B_i = 0} = \frac{1 + 2K_i}{2(N-1)i} ; \quad K_i = 0, 1, \ldots, i(N-1)-1
\] (6.9.23)

The conditions

\[
\frac{\Delta}{T_n} \bigg|_{B_i = 0} \quad \text{and} \quad \frac{\Delta}{T_n} \bigg|_{B_i = 0}
\]

occur simultaneously when

\[
\frac{1 + 2K_i}{2(N-1)} = \frac{1 + 2K_i}{2(N-1)i}
\] (6.9.24)

which implies that

\[
K_i = iK_1 + \left( \frac{i-1}{2} \right) ; \quad K_i \leq i(N-1)-1 \quad \text{and} \quad K_i \leq (N-2) \] (6.9.25)

Expression (6.9.25) states that the value of \( \frac{\Delta T_n}{T_n} \) making \( B_1 \) zero also makes \( B_i \) zero, provided \( i \) satisfies (6.9.25). Since \( K_1 \) and \( K_i \) must be integer, an immediate conclusion is that \( i \) must be odd. There are however no values of \( i \) which infringe the range restriction of \( K_i \), provided \( K_1 \) is smaller than \( (N-2) \). From these arguments follows the important conclusion that the choice of \( \frac{\Delta T_n}{T_n} \) which eliminates the fundamental noise frequency,
(making \( B_1 \) zero), also eliminates all the odd harmonics of the fundamental (making all \( B_1 \) zero, 'i' odd).

By using exactly the same procedure, but using a value of \( \Delta/T_n \) equal to 0.5 times that which makes \( B_1 = 0 \), it is easily shown that it is possible to eliminate all the even noise harmonics. The method of eliminating selectively all the even or all the odd noise harmonics from the sampled data record can be extended to the elimination of all noise effects.

Let \( \hat{y}_e(t) \) and \( \hat{y}_o(t) \) represent the estimates of the signal \( y(t) \) corrupted respectively by the even and odd noise harmonics \( e(t) \) and \( o(t) \). Then

\[
\hat{y}_e(t) = y(t) + e(t) \tag{6.9.26}
\]

\[
\hat{y}_o(t) = y(t) + o(t) \tag{6.9.27}
\]

If the original noise process is \( n(t) \) then

\[
n(t) = e(t) + o(t) \tag{6.9.28}
\]

It is possible also to obtain an estimate of \( y(t) \) containing all the noise terms. This is given by the symbol \( Z(t) \), and from (6.9.2)

\[
Z(t) = y(t) + n(t) \tag{6.9.29}
\]

\[
= y(t) + e(t) + o(t) \tag{6.9.30}
\]

From (6.9.26) and (6.9.27)

\[
\hat{y}_e(t) + \hat{y}_o(t) = 2y(t) + e(t) + o(t) \tag{6.9.31}
\]

It follows that

\[
y(t) = \hat{y}_e(t) + \hat{y}_o(t) - Z(t) \tag{6.9.32}
\]

Therefore from the estimates of \( y(t) \) containing selectively only the even and odd harmonics of the cyclic noise process, and from that containing all the noise effects, can be determined the uncorrupted signal \( y(t) \). It must
6.9.1 Elimination of cyclic noise effects from the crosscorrelation function by the use of inverse repeat signals

Consider Fig. 6.9.3. The system output $y(t)$ is additively corrupted by a noise process $n(t)$. The observed output is $Z(t)$

$$Z(t) = y(t) + n(t) \quad (6.9.33)$$

The crosscorrelation function between the dynamic component of the input and the observed output is

$$\phi_{xz}(\tau) = \phi_{xy}(\tau) + \phi_{xn}(\tau) \quad (6.9.34)$$

It is the purpose of this discussion to show how $\phi_{xn}(\tau)$ can be minimised. Accordingly this term is considered separately. If $T_g = 2T$ is the period of the input $x(t)$ then

$$\phi_{xn}(\tau) = \frac{1}{2T} \int_0^{2T} n(t)x(t+\tau)dt \quad (6.9.35)$$

Expression (6.9.35) is written with the assumption that $x(t)$ is a repetitive signal with period $2T$. If $x(t)$ is also an inverse repeat signal then

$$x(t) = -x(t+T) \quad (6.9.36)$$
be pointed out that this procedure is designed to eliminate cyclic noise effects only; the averaging process reduces the errors due to random noise as a useful side effect of the method. For environments where random as well as cyclic noise are strong, cyclic noise reduction parameter \( \Delta/T_n \) should be chosen in conjunction with large values of \( N \), the average number. Furthermore it must be stressed that these procedures will not eliminate noise bias effects.

One practical difficulty which is encountered in the implementation of the method to eliminate all the noise harmonics, lies in the fact that \( \xi(t) \) and \( \xi_0(t) \) must be obtained from a process corrupted by cyclic noise of known period and equal starting phase. The period of the cyclic noise can be easily determined but the restriction of equal noise starting phases for the estimation of \( \xi(t) \) and \( \xi_0(t) \) necessitates their simultaneous sampling at two slightly different sampling rates. This means that two clocks must be used in conjunction with the A/D equipment. An alternative is to obtain \( \xi(t) \) first, say, and then \( \xi_0(t) \), starting the sampling for this latter estimate an integer number of noise periods later. This still requires two clocks.

The complete elimination of all the cyclic noise effects is thus possible, and is easily implemented provided two clocks are available in conjunction with the sampling equipment. Programmable hardware clocks are not very expensive, nevertheless this is a limitation of the noise elimination technique. It must be pointed out however that in many cases the elimination of either all even or all odd noise harmonics is sufficient.

In the next section it will be shown how the effects of cyclic noise can be completely eliminated from the crosscorrelation function when the system inputs are inverse repeat signals. This has particular relevance to the identification of nonlinear noisy systems.
and

\[ \phi_{\chi n}(\tau) = \frac{1}{2T} \int_{0}^{T} n(t) \chi(t+\tau) dt + \frac{1}{2T} \int_{0}^{2T} n(t) \chi(t+\tau) dt \] (6.9.37)

\[ = \frac{1}{2T} \int_{0}^{T} n(t) \chi(t+\tau) dt + \frac{1}{2T} \int_{0}^{T} n(t+T) \chi(t+\tau+T) dt \] (6.9.38)

\[ = \frac{1}{2T} \int_{0}^{T} [n(t) - n(t+T)] \chi(t+\tau) dt \] (6.9.39)

The inverse repeat property of \( \chi(t) \) has enabled the crosscorrelation function \( \phi_{\chi n}(\tau) \) to be written in terms of the term \( D \) given by

\[ D = n(t) - n(t+T) \] (6.9.40)

The noise crosscorrelation component \( \phi_{\chi n}(\tau) \) will be zero when \( D = 0 \). The noise is assumed to be cyclic with period \( T_n \) where, as before

\[ T_s = 2T = cT_n + \Delta \quad ; \quad c \text{ integer} \] (6.9.41)

If (6.9.41) is substituted into (6.9.40) then

\[ D = n(t) - n(t + \frac{cT_n}{2} + \frac{\Delta}{2}) \] (6.9.42)

There are two cases of interest; \( 'c' \) even or \( 'c' \) odd.

**Case 1 : \( 'c' \) even**

If \( 'c' \) is even then \( \frac{cT_n}{2} \) is an integer and (6.9.42) becomes

\[ D = n(t) - n(t + \frac{\Delta}{2}) \] (6.9.43)

**Case 2 : \( 'c' \) odd**

If \( 'c' \) is odd then it can be written as \( (c_o + 1) \) where \( 'c_o' \) is even and (6.9.42) becomes
\[ D = n(t) - n(t + \frac{c \frac{T}{2} n}{2} + \frac{T}{2} + \frac{\Delta}{2}) \] (6.9.44)

\[ = n(t) - n(t + \frac{T}{2} + \frac{\Delta}{2}) \] (6.9.45)

When \( D \) is zero, \( \phi_{\chi_n}(t) \) is zero and the effects due to cyclic noise are completely eliminated. When \( c \) is even, the condition making \( D \) zero is \( \Delta = 0 \). When \( c \) is odd \( D \) is zero when \( \Delta = T_n \). These conditions are deduced by finding the solutions of (6.9.42) and (6.9.44) for which \( D = 0 \). The two solutions can be seen in fact to be equivalent. The two conditions can be combined and a simple rule deduced to obtain total cyclic noise elimination. The method is to choose the test signal to be an exact even multiple of the noise period, the signal being also of the inverse repeat type. Note that all noise bias effects are also eliminated.

6.10 Summary

This Chapter has demonstrated the principles outlined theoretically in Chapter 3, giving experimental proof of the properties of nonlinear systems as derived by application of the modified Volterra functional analysis. The theory of an identification scheme has then been outlined and a detailed error analysis carried out to predict its limitations. Experimental evidence is given to demonstrate the performance of the identification scheme when applied to idealised noise free systems as well as to real systems in the presence of additive output random noise. The identification results obtained indicate that the method is quite powerful, being able to handle systems containing nonlinear elements causing significant distortions to their inputs. Theory has then been developed to produce testing procedures enabling the effects of cyclic noise to be eliminated from the estimates of output sample records and crosscorrelation functions. The theory developed to obtain uncorrupted crosscorrelation functions is particularly applicable to the nonlinear identification schemes developed in this Chapter.
CHAPTER 7

TEST PROCEDURES FOR SYSTEMS CONTAINING DISCRETE NONLINEAR ELEMENTS
7.1 Introduction

In this chapter the identification algorithms developed in Chapter 6 for systems containing polynomial type nonlinearities are applied to systems containing discrete ones. Using the theory developed in Chapter 5, which links the statistical describing function (Gaussian inputs) to the coefficients of the modified Volterra series, it is shown that the identification performance is improved by whitening the input signal prior to its injection into the system.

The conventions used to annotate diagrams and graphs are identical to those used and described in Chapter 6.

7.2 Effect of amplitude on the resolution of discontinuous nonlinearities

Sometimes discontinuous nonlinearities occur imbedded in linear systems. If these nonlinearities significantly distort their input signals then their identification does not present great problems since the methods already described can be applied with reasonable results. Difficulties arise however when the nonlinear effects are confined to small portions of the range traversed by the dynamic component of the input signal, or when the boundaries of the nonlinear regions are required accurately. In such cases the resolution of the identification procedures becomes very important.

An intuitive approach to justify the use of identification methods developed for systems containing polynomial type nonlinearities to systems containing discontinuous ones, is to consider that the nonlinearity can be approximated by a polynomial within the range of influence of its input. However, if the nonlinear effect is concentrated in a small region of this range, and further if it is discontinuous in nature, the approximating polynomial has to be of very high order, and this means loss of resolution, whatever identification technique is used. The link which was found between the crosscorrelation function and the statistical describing function provides some justification for the application of the methods used in the analysis of smooth nonlinearities to the discontinuous ones. This, however, raises an interesting philosophical point.

Consider a system containing a dead-band nonlinearity, and the problem
of identifying the region of zero gain. The value of the gain over that region is obviously zero. All the derivative terms are also zero, i.e.

\[
\frac{d^i}{dm^i} T(m) = 0 \tag{7.2.1}
\]

where \(T(m)\) is the small signal system gain at input bias \('m'\). The cross-correlation function is given by (see Chapter 6)

\[
\phi_{xy}(\tau) = \sum_{i=1}^{n} \beta_i(m) A^i \frac{1}{i!} \frac{d^i}{dm^i} T(m) G_i \tag{7.2.2}
\]

where the meanings of the symbols is as before. The nonlinearity order can be extended to infinity, so that sufficient terms are included in the series to adequately describe the discontinuous nonlinearity. However, from (7.2.1) all the terms are zero, and the predicted crosscorrelation function is zero for all lags. This does not conform to reality if the signal input to the dead-band has an amplitude greater than the dead-band range; this is a very common situation. It can be concluded therefore that even an infinite series does not describe the discontinuity adequately enough. What is disturbing is that if the dead-band region is small, then \(\phi_{xy}(\tau)\) approaches that obtained with no nonlinearity, but the predicted \(\phi_{xy}(\tau)\) function is still zero. This problem can partly be overcome by applying the theory outlined in Chapter 3, where it is shown how the derivatives in the functional equations can be replaced by finite difference relationships. The finite difference relationships involve terms like \(T[m + i(dm)]\) which can be non-zero even if \(T(m)\) is zero. It is felt however that further research into this area could yield greater insight into the problem.

To find the resolution limits of the correlation identification method several experiments were carried out on simulated systems. The test signals were, as usual, 3-level signals; single amplitude tests were carried out to find the resolution limits of the procedure. Identification tests were then carried out using the double amplitude, or repeated mean level, method showing the kind of improvements possible, and the results are given later in this chapter.

Experiments were carried out with different types of nonlinearities.
The systems tested were of the general (L-NL-L) form. Firstly, the nonlinear element was of the gain change type as shown in Fig. 7.2.1, where the width W was kept constant and the gains $Q_1$ and $Q_2$ variable.

![Fig. 7.2.1 Definition of gain-change nonlinear element parameters.](image)

To obtain a measure of the identification resolution, it was decided to use as a yardstick the error in the estimate of the system gain about null. If the estimated gain is denoted by $P$, then the error is $(Q_1 - P)$. If this is zero, however, it does not imply zero identification error throughout the range, but experiments showed that in such cases the whole characteristic is identified with small errors. An alternative would have been to use the error in the estimate of gain when the nonlinearity input bias is $\pm W/2$; see Fig. 7.2.1.

The experiment configuration is shown in Fig. 7.2.2.

![Fig.7.2.2 Nonlinear system excited with TPRS.](image)
Fig. 7.2.3 Effect of TPRS amplitude on the identification of the gain function $T(m)$ of a (L-NL-L) system containing a dead-band nonlinear element; $W = Iv., \; Q_1 = 0, \; Q_2 = 1.$

(5) TPRS amplitude 0.25 volts
(4) TPRS amplitude 0.50 volts
(3) TPRS amplitude 0.75 volts
(2) TPRS amplitude 1.50 volts
(1) TPRS amplitude 2.25 volts
Fig. 7.2.4 Effect of TPRS amplitude on the identification of the d.c. input-output function of a (L-NL-L) system containing a dead-band nonlinear element; $W = 1V$, $Q_1 = 0$, $Q_2 = 1$. 

(1) TPRS amplitude 0.25 volts
(2) TPRS amplitude 0.5 volts
(3) TPRS amplitude 1.0 volts
7.2.1 Resolution with dead-band nonlinearity

In Fig. 7.2.3 and Fig. 7.2.4 are shown estimates of the system gain and d.c. input-output functions, obtained with the single amplitude correlation method with different values of TPRS amplitude 'A*'. As expected the identification errors increase with 'A'. Experiments showed that doubling the amplitude 'A' produces the same accuracy loss as halving the dead-band range W.

The error in the estimate of gain about null given by \( \frac{Q_1 - P}{A/W} \) was plotted as a function of the ratio \( \frac{A}{W} \). The result is shown in Fig. 7.2.5. Note that since \( Q_1 \) is zero in this case, the curve shown is really a plot of \(-P\) as a function of \( \frac{A}{W} \). The error curve indicates that when \( W \geq 1.6A \) the identification of the gain about null is exact. The physical explanation for this lies in the nature of the linear system preceeding the nonlinear one. Although the dynamic range of the TPRS test signal is 2A, the actual dynamic range of the signal entering the nonlinear element is of the order of 1.6A; this means that if the dead band width is greater than 1.6A, when the nonlinear element input is unbiased its output is always zero. The identified gain is, in such cases, exact.

![Fig. 7.2.5 Error in estimate of the gain of a (L-NL-L) system about null; the nonlinear element is a dead-band of width 1 volt; W = 1v., Q1 = 0, Q2 = 1.](image)
Using the simulation routines, experiments were carried out to find how the error \( (Q_1 - P) \) varies with amplitude 'A', gains \( Q_1 \) and \( Q_2 \) and range \( W \). Examples of the gain estimates obtained to test resolution on systems containing a gain change nonlinearity are shown in Fig. 7.2.6. They bear considerable resemblance in form to those obtained for the dead-band.

(1) TPRS amplitude 0.5 volts
(2) TPRS amplitude 1.0 volts
(3) TPRS amplitude 1.5 volts
(4) TPRS amplitude 2.0 volts

\[
T(m) = \begin{cases} 4 & \text{for } A = 0 \\ 3 & \text{for } 0 < A < 1 \\ 2 & \text{for } 1 \leq A < 2 \\ 1 & \text{for } A \geq 2 \end{cases}
\]

\[\begin{array}{c}
\text{IB} \\
1 \quad 1.5 \\
0.5 \\
1 \\
2 \\
3 \\
4 \\
\end{array}\]

\[W = 1v, \ Q_1 = 1, \ Q_2 = 4\]

Fig. 7.2.6 Effect of TPRS amplitude on the identification of \( T(m) \) function of a (L-NL-L) system containing a gain change nonlinear element.

In Fig. 7.2.7 are shown plots of error \( (Q_1 - P) \) as functions of gain \( Q_1 \), the gain \( Q_2 \) being fixed at the value of one. These are by way of example, showing a typical data set. Numerous other tests on the same type of nonlinear system, but with different parameters, were carried out.

The results of these experiments can be summarised by the relationship

\[
\text{Error} = (Q_1 - P) = (Q_1 - Q_2)K_1 \frac{2A}{W} f(v)
\]

where \( f(x) \) is a function of the form shown in Fig. 7.2.2 and where \( f(v) \) was found to be a constant depending on the system input characteristics as
Fig. 7.2.7 Effect of intermediate gain $Q_1$ and amplitude of system input TPRS on resolution of gain change nonlinearity, imbedded in (L-NL-L) system; $W = 1v., Q_1$ = variable, $Q_2 = 1$. 
well as the impulse response of the system preceding the nonlinearity. In particular if the system is all-pass, \( f(v) \) tends to 1. For the linear system used in conjunction with the experiments reported here, \( f(v) \) was found to be 0.81. From the equation (7.2.3) it can be seen that the error is proportional to the gain difference \( (Q_1 - Q_2) \); further it can be seen that the error is zero if \( W > 2A f(v) \).

When \( f(v) \) is smaller than one, the dynamics of the linear element can thus be thought of as increasing resolution.

7.2.3 Preload type nonlinearities. Effects of input amplitude on the identification resolution.

In Fig. 7.2.8 are shown a typical preload nonlinearity, and its gain function. The impulse in the gain function about null is a theoretical representation. However experimental results show that it is not unrealistic to represent the gain at that point in such a way. In Fig. 7.2.9 are shown for a range of input TPRS amplitudes, the estimated gain functions of a (L-NL-L) system, the nonlinear element being a preload as shown in Fig. 7.2.8. The results show that as the amplitude of the TPRS input becomes small, which is equivalent to small variance, the estimated gain
Fig. 7.2.9 Effect of TPRS amplitude $A$ on the identification of the gain function $T(m)$ of a (L-NL-L) system containing a preload nonlinear element; $H = 0.5$ volts, $Q = 1$. 

$A = 0.5$ volts $\quad A = 1.0$ volts

$A = 1.5$ volts $\quad A = 2.0$ volts
Fig. 7.2.10  Effect of TPRS amplitude $A$ on the identification of the d.c. input-output function of a (L-NL-L) system containing a preload nonlinear element. $H = 0.5$ volts, $Q = 1$. 

$A = 0.5$ volts

$A = 1.0$ volts

$A = 1.5$ volts

$A = 2.0$ volts
functions approach the idealised one shown in Fig. 7.2.8. Single amplitude correlation tests were used to obtain these results.

The measurement of the identification resolution cannot be accomplished in the same way as for gain change nonlinearities. This is because $W$ is now zero, and because the theoretical gain about null is infinity. Thus it was decided to use the width of the estimated gain function, measured at the point where the spike begins, as an estimate of the error. Calling this '$\delta$', then from Fig. 7.2.9 it is seen that $\delta$ is approximately proportional to $A$. Of interest is the fact that in all the cases shown, the spike area, shown shaded (see figure) is, to a good approximation, equal to 1. This corresponds to the magnitude of the 'preload' $H$. This fact was confirmed by tests carried out for other values of $H$. These tests have demonstrated the fact that with a smaller nonlinear element input variance, the identification is more precise, providing there are no external factors influencing the test data, such as noise.

In Fig. 7.2.10 are shown the integrated versions of the gain functions. Since an integrated delta function is a step function it is obvious that the closer the estimated gain curve approximates to a delta function superimposed on a constant value, the better will be the estimated nonlinear element d.c. input-output function.

The action of the preload nonlinearity is very closely connected to that of the ideal relay. In fact, the action of the preload nonlinearity is equivalent to that of an ideal relay and an amplifier in parallel. The block diagram shown overleaf, in Fig. 7.2.11, illustrates this principle.
The statistical describing functions of the relay and the preload have a mathematical equivalence to Fig. 7.2.11. Many other commonly occurring discontinuous elements can be 'decomposed' as shown above.

The identification of systems containing a relay is examined next. The results are directly applicable to systems with preload, using the arguments just outlined and, further, they give insight into a method of obtaining optimal identification results using the correlation method.

7.2.4 Relay type nonlinearities. Effect of nonlinear element input variance on identification resolution.

The statistical describing function for the relay nonlinearity shown in Fig. 7.2.11 can be shown to be given by

\[ f_K = 2H p_u (m_u) \]  
\[ f_m = \frac{H}{m_u} \text{erf} \left( \frac{m_u}{\sigma_u/2} \right) \]
where \( u(t) \) is the nonlinearity input signal, \( m_u \) the bias, and \( p_u(x) \) the probability density function (p.d.f.) of \( u(t) \). The describing function constants \( f^K \) and \( f^m \) have been defined in Chapter 4. The expression for \( f^K \) is particularly interesting as it states that the gain which effectively acts on the dynamic component of the nonlinearity input, is proportional to the probability of that input having a value equal to its bias. This means that if \( f^K \) is obtained for all possible input bias values, the result will be a map of the probability density function of the signal at the nonlinearity input. It was shown that when the system input is a Gaussian process the integral of the crosscorrelation function is proportional to the statistical describing function; when the system contains a relay it is therefore also proportional to the p.d.f. of the nonlinearity input, implying that the estimated gain function \( T(m) \) has a shape resembling the characteristic bell function of the normal distribution. If the system input variance is decreased, then the variance of the nonlinearity input will also be decreased, resulting in a more peaked p.d.f., and also estimated gain function. In the limit with 'zero' input variance, the estimated gain function will tend to the impulsive shape shown in Fig. 7.2.8 implying exact identification. However such a limit is not possible in practice due to noise considerations.

When the input to the nonlinearity is not normally distributed, but can still be considered to be a random function, the statistical describing function representation of the nonlinearity is probably sub-optimal since it is unlikely that such a random function will possess the separability property [12]. Further, the integral of the system crosscorrelation function will no longer be proportional to the statistical describing function.

Experiments were carried out to find qualitative relationships between the statistical properties of the nonlinear element input, and the estimated gain function. A three-level sequence was injected into a system consisting of a low-pass filter followed by the relay. The filter cutoff frequency was varied and an identification carried out for each setting. The p.d.f. of the signal entering the relay was obtained using a digital instrument (Hewlett-Packard type 3721 A). The experimental set up is shown in Fig. 7.2.12.
Fig. 7.2.12 Experimental arrangement used to investigate the effect of band-limiting TPRS on its p.d.f., and on the identification resolution of a relay when the filtered TPRS is used as test signal.

Linear systems have the property that they tend to normalise their input signals. The three-level sequence is used as an approximation to band limited white noise; in the frequency domain, the approximation is quite good providing the linear system into which the TPRS is injected has a band width smaller than that of the TPRS itself. The output from the linear system will tend, under such conditions, to be normally distributed. The power spectrum and p.d.f. of a three-level sequence are shown overleaf, in Fig. 7.2.13. The p.d.f. of the TPRS signal is seen to be completely unrelated to that of a normally distributed random variable. The p.d.f. of the signal entering the nonlinear element is however different from that of the TPRS, due to the action of the low pass filter, this element tending to normalize its inputs.
Fig. 7.2.13 Power spectrum and p.d.f. of TPRS.

Considering the experiment configuration shown in Fig. 7.2.12, the first low-pass filter has thus the effect of normalising the p.d.f. of the three-level sequence, at the same time rendering it a better approximation to band-limited white noise. This may be realised by imagining the effect of a low-pass operation on the spectral density function shown in Fig. 7.2.13. The second low-pass filter is used to prevent aliasing. The nonlinear element generates harmonics from its input spectrum, and thus, an anti-aliasing filter is essential especially when the nonlinear element appears at the output of the system.

In Fig. 7.2.14 and Fig. 7.2.15 are shown the p.d.f.'s of the signal at the nonlinearity input, for different filter cut off frequencies, while Fig. 7.2.16 and Fig. 7.2.17 contain respectively estimates of the system gain function, and their integrated counterparts. These latter results were obtained using the single amplitude correlation method. These curves are interesting because they show both how filtering affects the p.d.f. of the three-level signals, and also show that the estimate of the gain function is affected very significantly by the statistical properties of the signal entering the nonlinearity.
Fig. 7.2.14 Probability density functions of low-pass filtered TPRS. $f_c$ = filter cut off frequency. The TPRS signal is unbiased, with amplitude 1 volt.
Fig. 7.2.15 Probability density functions of low-pass filtered TPRS. $f_c$ = filter cut off frequency in Hz. The TPRS signal is unbiased, with amplitude 1 volt.
When the cut off frequency $f_c$ is very small compared to the TPRS clock frequency ($f_T = \frac{1}{\Delta t} = 1.67 \text{ KHz}$) which is $1.67 \text{ KHz}$ in this case, then the p.d.f. of the signal entering the nonlinear element shows the presence of a finite number of independently phased, sinusoidal random processes. This point is demonstrated by Fig. 7.2.14 showing the p.d.f.'s of the three-level sequence filtered with cut off frequencies of 50 and 20 Hz, drawn to a large horizontal scale. As $f_c$ is increased, (see Fig. 7.2.15), the p.d.f. takes a form which more closely approximates that of a normally distributed random variable. This is expected since, when $f_c$ is smaller than the half-power frequency, the spectrum of the three-level sequence is fairly flat; increasing $f_c$ within this range has the effect of producing a signal with a larger number of independent random variables which, as predicted by the 'Central Limit Theorem', tends to produce a normally distributed process. The flat spectrum in itself is not important towards the production of a normal process, but it does mean that when the process approximates to the normal one it also approximates to band limited white noise. When the cut off frequency is further increased, the shape of the TPRS power spectral density function begins to affect the signal, imposing on it a deterministic imprint. The autocorrelation function becomes more like the theoretical one of band limited white noise, but with a finite period, and the p.d.f., distorts away from that of a normally distributed variable; the three spikes appear, and the other components decrease in magnitude. Eventually the filter cut off is so high that most of the three-level sequence is passed, and the p.d.f. then approximates closely to that of the unfiltered three-level signal. The p.d.f. data given in Fig. 7.2.15 however does not contain examples of TPRS filtered with $f_c > 830 \text{ Hz}$. However the three spikes are very evident for cut off frequencies greater than $750 \text{ Hz}$. The TPRS clock frequency $f_T$ is $1.67 \text{ KHz}$.

Estimates of the system gain and d.c. input-output functions are shown in Fig. 7.2.16 and Fig. 7.2.17. Comparing them to the exact relationships shown in Fig. 7.2.8 these results tend to suggest that as the cut off frequency is decreased further and further the estimates become monotonically better. A measure of the resolution of identification of the ideal relay can be defined as the distance $\delta_0$ corresponding to the range over which the identified relay characteristic is in error by more than 5%. See Fig. 7.2.18 for a diagramatic explanation of $\delta_0$. 
Identification of the gain function of a \((L-NL)\) system containing a relay nonlinear element. The linear element is a low-pass filter with variable cut off frequency \(f_c\). The identification performance is compared for different \(f_c\) values.
Fig. 7.2.17 Identification of the system d.o. input-output function of a (L-NL) system containing a relay nonlinear element. The linear element is a low-pass filter with variable cut off frequency $f_c$. The identification performance is compared for different $f_c$ values.
Perfect identification is achieved when $\delta_0 = 0$. Applying this resolution measure to the data in Fig. 7.2.17, showing identifications of an ideal relay characteristic as a function of system input amplitude, yields the graph shown in Fig. 7.2.19. This quantitatively suggests that as the amplitude of the input TPRS is reduced, the identification of the relay characteristic becomes more accurate. This is a misleading conclusion in the presence of noise; when the system is simulated and noise free the lower limit of the input amplitude is imposed by resolution considerations.

![Diagram](image)

**Fig. 7.2.18 Definition of the 5% resolution measure.**

The estimated gain functions are interesting, since considerable resemblance can be found between them and the corresponding p.d.f.'s shown in Fig. 7.2.15. The similarity is greatest when the filtered three-level sequence most closely approximates to a normally distributed random variable. Among the experimental records shown, this corresponds to the case of a cut off frequency of 200 Hz. The action of the low-pass filter is predominantly to band limit the spectrum of the three-level sequence. This results in the changes of the p.d.f. which have been qualitatively explained. However, one very important side effect of the filtering operation, is that the variance of the output signal is decreased; in particular, for unbiased signals, the power of the signal is decreased.
Fig. 7.2.19 Identification performance of correlation method, applied to a (L-NL.) system, the nonlinear element being an ideal relay. The 5% resolution measure is used.
The signal entering the nonlinear element has a smaller variance as the filter cut off frequency is reduced. Therefore quite apart from effects due to the form taken by the nonlinear element input signal p.d.f., the results of the identification procedure are bound to be improved by the fact that the variance is smaller. However, as the variance of the signal input to the nonlinear element becomes smaller, so does that of the system output; this means that the output signal to noise ratio deteriorates, and the improvements which would otherwise have been obtained due to the smaller variances, are offset due to the greater effects of noise. What is required is to find how the shape of the nonlinear element input p.d.f. is related to the identification performance, independently of the signal variance.

7.2.5 Variance equalisation of filtered TPRS

An investigation was carried out to find relationships between filter output variance and its cut off frequency, with a TPRS input. Theoretical relationships for idealised filters were compared with experimental results.

Consider Fig. 7.2.20; $K_f$ is an ideal gain element, and $H(s)$ the transfer function of an ideal low-pass filter with cut off frequency $f_c$. If the system input is a TPRS, its power spectral density is

$$S_{xx}(f) = \frac{2}{\pi} \frac{B}{N} \sum_{K=1,3,5} b^2 \delta \left( \frac{K}{N\Delta t} - f \right)$$  (7.2.6)
where \( N \) is the sequence bit length, \((\frac{1}{\Delta t})\) the sequence clock frequency and if \( A \) is the TPRS amplitude then

\[
B = A^2 \frac{2}{3} \left( \frac{N + 1}{N} \right)
\]

\[
h_k = \frac{\sin(\pi K/N)}{(\pi K/N)}
\]

The input sequence is unbiased.

The variance of the three-level sequence is given by

\[
\sigma_x^2 = \frac{1}{\pi} \int_0^\infty S_{xx}(f) df
\]

(7.2.8)

Similarly that of the signal entering the filter is

\[
\sigma_u^2 = \frac{K_f^2}{\pi} \int_0^\infty S_{xx}(f) df
\]

(7.2.9)

The variance of the ideal filter output, with cut off frequency \( f_c \) is given by

\[
\sigma_y^2 = \frac{K_f^2}{\pi} \int_0^{f_c} S_{xx}(f) df
\]

(7.2.10)

It is useful to find the relationship between \( \sigma_x^2 \) and \( \sigma_y^2 \) keeping \( K_f \) equal to one, varying the cut off frequency \( f_c \), and the relationship between the cut off frequency and \( K_f \), to maintain \( \sigma_y^2 \) constant.

\[
\frac{\sigma_x^2}{\sigma_y^2} = \frac{K_f}{f_c} \int_0^{f_c} S_{xx}(f) df \leq 1
\]

(7.2.11)

An example of the relationship between power transfer ratio \( \sigma_y^2/\sigma_x^2 \) as a function of \( f_c/f_T \) for unity gain \( K_f \) is shown in Fig. 7.2.21. The theoretical curve is shown together with experimental ones, derived using
Fig. 7.2.21 Relationships between power transfer ratio (of low-pass filters) and normalized cut-off frequency, for TPRS inputs.

- Ideal filter
- 8th order filter
- 4th order filter
Fig. 7.2.22 Relationship between low-pass filter gain $K_f$ and its normalized cut off frequency, required to maintain constant, the output variance of the filter when driven by TPRS.
two analogue filters, one of fourth and one of eighth order. The curves in Fig. 7.2.22 show, for the same test conditions, the variation of $K_f$ needed to maintain the filter output variance constant as the normalized cut off frequency is changed, both for experimental as well as the theoretical cases. Several other experiments were carried out showing that the curves are practically invariant with sequence length and clock frequency (1/Δt).

7.2.6 Identification resolution as a function of nonlinear element input p.d.f. with variance equalisation

Using the relationships shown graphically in Fig. 7.2.22 it was possible to carry out experiments to determine the influence of the filter cut off, independently of nonlinear element input variance, on the resolution of identification of the relay nonlinearity. In other words the experiments described earlier, the results of which are given in Fig. 7.2.16 and Fig. 7.2.17, are repeated for the same variation of cut off frequencies, but with the added constraint of constant variance at the nonlinearity input. The experimental set up is shown in Fig. 7.2.23. The gain of the low pass filter is adjusted in such a way that as the filter cut off frequency is varied, the low pass filter output variance remains constant.

Fig. 7.2.23 Experimental arrangement used to investigate the effect of band-limiting TPRS on its p.d.f., and on the identification resolution of a relay when the filtered TPRS is used as test signal.
Fig. 7.2.24 Probability density functions of low-pass filtered and 'variance equalised' TPRS. $f_c$ = filter cut off frequency in Hz. The TPRS signal is unbiased with amplitude 1 volt.
A demonstration of the effect of low-pass filtering with output variance equalisation are given by the p.d.f.'s shown in Fig. 7.2.24. The filter output variance was kept constant by varying the filter gain $K_f$. The appropriate values of $K_f$ to be used were obtained from the experimental curves shown in Fig. 7.2.22. The p.d.f.'s obtained are in fact similar to the ones shown in Fig. 7.2.15, but appear different because the gain $K_f$ has amplified the horizontal scale. As a result the presence of independently phased sinusoidal components is much more evident as the cut off frequency is decreased.

Identification tests were carried out with the different filter cut offs and constant nonlinearity input variance, the results of which are shown in Fig. 7.2.26 and Fig. 7.2.27. These results show how, for constant nonlinearity input variance, the cut off frequency affects the resolution of the identification method. The integrated gain curves, showing the system d.c. input-output relationship, gives a better indication of the identification performance. To measure quantitatively the resolution of identification, the 5% criterion is used; this was defined in section (7.2.4). The graph shown overleaf is obtained by applying the resolution measure to Fig. 7.2.27. Note that perfect identification is obtained when $\delta = 0$. These curves suggest that as the cut-off frequency is reduced, the performance improves up to a critical value, given in this case by $f_c = 200$ Hz, beyond which it drops rapidly. This is a very important point especially as it was found that such behaviour occurs with all the discontinuous nonlinearities which were tested. Further, the optimum cut off frequency was found to be that for which the filter output p.d.f. most closely resembles that of a normally distributed variable. It was found that, with TPRS inputs, the cut off frequency of a low-pass filter which results in a normally distributed output, lies in the range

$$\frac{1}{5\Delta t} < f_c < \frac{1}{9\Delta t}$$
Fig. 7.2.25 Identification of the gain function of a (L-NL) system containing a relay nonlinear element. The linear element is a low-pass filter with variable gain and cut off frequency $f_c$. The identification performance is compared for different $f_c$ values, the filter output variance being kept constant by gain adjustment.
Fig. 7.2.26 Identification of the d.c. input-output function of a (L-NL) system containing a relay nonlinear element. The linear element is low-pass filter with variable gain and cut-off frequency $f_c$. The identification performance is compared for different $f_c$ values, the filter output variance being kept constant by gain adjustment.
Fig. 7.2.27 Identification performance of correlation method in conjunction with TEPES prenormalisation filtering, applied to a (L-NL) system, the nonlinear element being an ideal relay. The 5% resolution measure is used.
7.2.7 Optimised identification procedure

Using the experimental evidence outlined in the last section it is possible to optimise the nonlinear gain function identification procedure, for the general case of a \((L-NL-NL)\) system. It has been indicated that when the input to the nonlinearity is normally distributed, the identification performance is optimal. The best way of ensuring this is to low-pass filter the three-level sequence prior to its injection into the system. The filter requirements are a gain such that its output variance is just large enough to provide the desired output signal-to-noise ratio, and a cut-off frequency such that the filter output p.d.f. is as 'normal' as possible. If the system itself contains linear elements prior to the nonlinear ones, then these will tend to further normalise the input signals. The generalised experimental set up is as shown in Fig. 7.2.28.

When the crosscorrelation is carried out, it can be performed between the preconditioned input \(x(t)\), and the output, but account has to be taken of the extra gain \(K_p\). However such a procedure is advantageous since the signal \(x(t)\) is digital, and special crosscorrelation algorithms can be employed involving only additions.
Concluding this section, it can be stated that the correlation method, although conceptually simple, needs careful application to obtain the best possible results. It has been shown how test signal preconditioning can improve the estimates, applied to systems containing discrete nonlinearities. Equivalent improvements are to be expected if the nonlinearities are smooth. In the next sections it is shown that further improvement in resolution can be achieved using the two amplitude test method explained in Chapter 6, but first are given results of identification tests carried out on the system shown in Fig. 7.2.29, with different prenormalisation filter cut-off frequencies; the system contains a dead band nonlinearity of width 0.5 volts.

\[
\begin{align*}
\text{Dead-band width} &= 0.5 \text{ volts} \\
K_f & \rightarrow h_f(\tau) \rightarrow x(t) \rightarrow K_A \rightarrow g_A(t) \rightarrow y(t) \rightarrow h(\tau) \\
\text{Fig. 7.2.29 Experimental arrangement for the identification of the } T(m) \text{ function of a } (L-NL) \text{ system, containing a dead-band nonlinear element.}
\end{align*}
\]

The identifications of the \( T(m) \) functions are shown in Fig. 7.2.30, the results being presented in that particular form so that easy comparison can be made between them. The pattern is, as before, improvements as \( f_c \) is reduced from large values until a limit is reached, below which the identifications become less accurate.

The accuracy of the identifications is to a certain extent subjective. The exact \( T(m) \) curve is (a); the estimates most closely approximating to this are those given in (g) or (h), corresponding to cut-off frequencies of the order \( \frac{1}{8\pi t} \). These results demonstrate that the effect of the prenormalising filter is beneficial even when another linear element is interposed between the filter output and the nonlinearity; in fact, as mentioned earlier, theory predicts that such an arrangement produces even
Fig. 7.2.30 Effect of prenormalisation filter cut off frequency with variance equalisation on the identification of the $T(m)$ function of a (L-NL) system containing a dead-band nonlinear element; dead-band width = 0.5 volts.
better results due to the added normalization process due to the second linear element.

It has been shown how optimum results may be achieved with single mean level testing; in the next section it will be demonstrated that further improved identification performance can be achieved with the repeated mean level test. This will lead to the conclusion that a combination of the repeated mean level test, and prenormalization filtering, yields even more accurate identifications.

7.3 Repeated mean level testing and systems with discontinuous nonlinearities

Experiments showed that the method developed in Chapter 6 for the repeated mean level test applied to systems with polynomial nonlinearities is in fact equally well applicable to systems with discontinuous ones. When this is considered in relation to the arguments given in section (7.2) it is perhaps surprising. However definite improvements can be obtained, as can be seen by the example in Fig. 7.3.2. The smooth line represents the single amplitude and the broken line the double amplitude test estimate of a dead-band characteristic. The second amplitude is 1.33 times larger than the first. The improvement is considerable; Fig. 7.3.3 shows the integrated gain characteristics, giving an even better indication of this, the functions being smoothed by the integration process. Details of the experiment configuration are shown in Fig. 7.3.1.

Another example of the resolution improvement is shown in Fig. 7.3.4 showing the effect of a repeated mean level test on the same system as considered above, but with much smaller TPRS amplitudes. The improvements are such that almost perfect identification can be claimed. In both cases the nonlinearities were imbedded between linear systems of unity gain; the signal preconditioning described in section (7.2.7) was not used.
Fig. 7.3.1 Experimental arrangement for the RMLT identification of \((L-NL-L)\) system containing a dead-band element; dead-band width = 1.0 volts.

Fig. 7.3.2 Identification of \(T(m)\) function of \((L-NL-L)\) system containing a dead-band nonlinear element, by SLT and RMLT methods. TPRS base amplitude, 1 volt. Dead-band width 1 volt.
Fig. 7.3.3 Identification of the d.c. input-output function of (L-NL-L) system containing a dead-band nonlinear element by SLT and RMLT methods. TPRS base amplitude, 1 volt. Dead-band width, 1 volt.
Fig. 7.3.4 Identification of the d.c. input-output function of a (L-NL-L) system containing a dead-band nonlinear element by SLT and RMLT methods. TPRS base amplitude, 0.5 volts; dead-band width, 1 volt.
7.4 Signal preconditioning and repeated mean level testing; combined method to achieve high resolution

The repeated mean level test theory is applicable to Wiener (L-NL) or Hammerstein (NL-L) as well as to the general (L-NL-L) type systems. The preconditioning of the TPRS prior to its injection into the system has the effect of changing the identification of a Hammerstein type system into the identification of a general (L-NL-L) one, but leaves the basic experiment configuration unaltered if the system is already of Wiener or general type. This is simply due to the fact that the preconditioning is carried out by a linear filter, which combines with any other linear element prior to the nonlinear one, in making up a single linear system. The repeated mean level test technique can thus be applied in conjunction with TPRS preconditioning and the improvements due to the two methods must be basically additive. This was confirmed by experiments. An example is provided by Fig. 7.4.1 which shows the effect of carrying out a repeated mean level test identification on a system containing a dead band nonlinear element. The actual experiment configuration is shown in Fig. 7.3.1; the second amplitude for the TPRS correlation procedure was 1.33 times greater than the base one; \( R = 1.33 \). The preconditioning filter cut-off frequency was varied and the estimated gain curve obtained and plotted in Fig. 7.4.1 for each setting. The dotted lines represent the identified gain functions obtained with the single level test, while the continuous ones are the result of applying the repeated mean level test. The improvements are obvious.
Fig. 7.4.1 Effect of prenormalisation cut off frequency with variance equalisation on the identification of the $T(m)$ function of a (L-NL) system containing a dead-band nonlinear element. Comparison of SLT and RMLT methods. Dead-band width, 0.5 volts.
This chapter has demonstrated that the identification methods developed for systems containing smooth nonlinearities can equally well be applied to systems containing discrete ones. The resolution of identification was demonstrated to be a function of nonlinear element input variance as well as p.d.f. shape. It was shown that by passing a TPRS signal through a low-pass filter with a cut-off frequency $f_c$ which lies in the range

$$\frac{1}{5\Delta t} < f_c < \frac{1}{9\Delta t}$$

where $(\frac{1}{\Delta t})$ is the TPRS clock frequency, then the output from the filter is a reasonable approximation to a normally distributed random variable. It was further shown that when the input to the nonlinear element is normally distributed, then its identification, via the correlation methods described in Chapter 6, is improved significantly. It is therefore suggested that the procedure of normalising the TPRS signal prior to its injection into the system under test, should be always adopted. To compensate for the fact that the filtering operation removes power from the TPRS signal, adequate gain should be incorporated into the preconditioning, or normalising, filter. To further improve the resolution of identification, it was shown that the method of repeated mean level testing, whereby two correlation tests are carried out for each input bias setting, can be used even when discontinuous nonlinearities occur within the system.

The procedures outlined in this chapter to improve resolution by preconditioning the input TPRS are applicable in fact to the identification of systems with smooth nonlinearities; the method was however discovered testing discrete nonlinearity systems as in such cases the improvements which can be obtained are far more evident, and due to the particular characteristics of the ideal relay.
CHAPTER 8

IDENTIFICATION OF THE GAIN AND FRICTION CHARACTERISTICS OF A HYDRAULIC SERVODRIVE SYSTEM
8.1 Introduction

In this Chapter some of the identification procedures already described are applied to the testing of an electrohydraulic servodrive system. In particular the input current vs. output speed d.c. characteristic is identified, and then this data applied to a fast method, developed by the author, for the determination of the system viscous and Coulomb friction values. The results obtained are compared to those derived via quasi-steady state techniques. It is shown that the presence of drift in the system parameters, caused mostly by temperature effects, render the results of quasi-steady state tests difficult to interpret, and prone to errors.

8.2 Description of servo-system

An electrohydraulic servodrive is a marriage of a flow control servo valve, a rotary hydraulic motor, and an instrumentation package that contains electrical transducers to measure performance.

Servodrives are intended for use in closed loop velocity or position control systems. The system which was tested was, however, in open loop, but this is not a limitation of the test procedure.

A functional block diagram of the system is shown in Fig. 8.2.1, including the location of the computer used in the testing. As can be seen the motor is bi-directional; its speed is a function of oil flow as well as loading conditions. The oil flow is controlled by the electrohydraulic servo valve; for the purpose of the experiments described the servo valve was actuated by an electrical signal generated within the computer (mostly biased TPRS signals), and fed through a D/A converter and a servo-amplifier to provide current amplification.

The servo valve itself consists of a pilot operated closed centre four way sliding spool valve with a slight underlap, giving an output flow to a constant load, which is essentially proportional to the electrical input current. This is a two-stage servo valve having a double nozzle flapper valve as its first stage and the spool valve as its second stage. The electrical actuating device at the input is a permanent magnet electrical torque motor. An internal mechanical feedback exists, in the form of a
CV1 Cross port relief valves
CV2 Anti-cavitation valves
RV

$P_s$ = supply pressure
$P_R$ = return pressure

Fig. 8.2.1 Servodrive function schematic showing computer location.
flexure tube. The servo valve ports high pressure oil to an axial roller vane hydraulic motor. The load torque acting on the motor produces an essentially proportional differential pressure \( \Delta P_L \) across the motor ports. Detailed specifications on the Dowty electrohydraulic spool valve and Hartman motor can be found in the appendix at the end of this chapter. Of importance to the discussion which follows is the fact that this valve-motor combination, constitutes a highly responsive, fast acting system.

The purpose of the tests was to establish the relationship between input current to the torque motor, motor speed, and differential load pressure so that the gain of the servodrive could be established and the frictional losses identified. The idealized gain relationship is linear, but valve lapping, leakage, friction and oil temperature effects combine to cause slight nonlinear distortions. A theoretical study of the relationships existing between differential pressure \( \Delta P_L \) and torque output culminated in the derivation of a method which enabled the determination of the friction characteristics of the servodrive.

The fluid torque available at the motor is ideally \( \Delta P_L D_m \) where \( D_m \) is the motor displacement/radian. This torque less friction torques acts on the load, to provide acceleration. The motor load equation thus becomes

\[
\Delta P_L D_m = J \frac{d\omega}{dt} + B\omega + F_c \text{sgn}(\omega)
\]  

(8.2.1)

where

\( J = \) inertia of motor and all rotating parts
\( B = \) motor viscous friction coefficient
\( F_c = \) Coulomb friction coefficient
\( \omega = \) motor angular velocity

The equation thus states that the torque supplied by the motor on the load acts partially in accelerating the load and partially in overcoming viscous and Coulomb friction effects. At constant velocity the theoretical plot of \( \Delta P_L \) vs. \( \omega \) is as shown in Fig. 8.2.1, illustrating the friction terms. In the load equation, no provision is made for static friction. Such friction is recognised to exist, but it cannot be easily modelled directly in a differential equation.
The theoretical relationship between motor speed and applied torque is depicted in Fig. 8.3.1. The relationship between input current and motor output speed can be approximately determined using the simple procedure of inputting a steady current and measuring the output speed, repeating this process by slowly sweeping the whole range of possible inputs, and plotting the results. The same procedure can also be carried out to find the relationship between input steady current, and differential pressure $\Delta P_L$. In the absence of friction terms equation (8.2.1) reveals that $\Delta P_L$ is zero for a constant input current. This is because a constant current will produce a constant motor speed $\omega$, and then $\frac{d\omega}{dt} = 0$. This is not the case of the system being tested indicating significant friction effects.

8.3 The d.c. characteristics of the servo drive, obtained using quasi-steady state testing

Plots of input current vs. output speed and differential pressure $\Delta P_L$ were obtained by the procedure mentioned above. Several practical difficulties were however encountered due to the fact the motor speed is only approximately constant for constant input current. The variations in motor speed are due to the fact that its displacement changes with shaft angle. This is due to changes of effective radius and centre-of-pressure location of the pressurized power elements of the motor as the shaft rotates. These displacement variations are usually associated with motor valving used to create unidirectional shaft rotation for constant input flow.
Fig. 8.3.1 Typical ripple on $\Delta P_L$ present for constant values of valve input current.
The resulting motor speed (and output torque) variations are called 'ripple'.

Ripple can be effectively reduced by provision of a velocity feedback, but only within the servo valve band width. An example of the ripple is provided by Fig. 8.3.1 showing variation of differential pressure $\Delta P_L$ with time, for constant values of input current. The example serves to demonstrate several points. Firstly, and most evident, is the fact that $\Delta P_L$ is non-zero, implying the existence of friction effects. Secondly, it reveals the presence of servo valve null-shift in the particular valve tested. This represents the level of input current required to restore valve null, and the figure shows its existence by the fact that $\Delta P_L$ is different for forward and reverse directions (positive and negative input current). The speed of the motor can be roughly gauged from the figure by the period of the motor ripple; the shorter periods correspond to the greater speeds.

Due to the presence of the ripple, the differential pressure and output shaft speed signals must be arranged in such a way as to allow for the variation in ripple frequency with shaft speed. This requires that the testing time for each setting of input current be inversely proportional to the magnitude of that current. This can be realised by considering that even if the average of the motor speed or differential pressure is taken over only one revolution of the motor shaft at, say, 100 rpm, the testing time is 10 ms.; when the speed is 10 rpm the averaging time is 100 ms. Theoretically, the testing time will reach infinity as the speed approaches zero. This is a very real consideration since it is at the low speeds that most of the motor anomalies have their greatest effects, and therefore it is at the low speeds that the motor characteristics should be measured.

In Fig. 8.3.2, 8.3.3 and 8.3.4 are shown the results of the steady state tests, their aim being to find the relationships between input current, differential pressure $\Delta P_L$ and output speed $\omega$. The tests were carried out by progressively incrementing the input current up to its maximum, then decrementing it to zero, changing its polarity and again incrementing it to its maximum, and then reducing it again to zero. Thus a full cycle was carried out. The purpose of such procedure was to reveal any hysteresis characteristic; indeed the results show the presence of significant hysteresis, predominant about the null input region. This hysteresis effect is evident on all the characteristics shown, and is attributed to the combined motor and valve friction. The three characteristics, shown in
Although these are theoretically steady state tests, drift in important parameters such as oil temperature is such that they should strictly-speak be termed quasi-steady state tests.

8.3.1 Input current vs. output speed. Steady state test. (Fig. 8.3.2)

Fig. 8.3.2 to 8.3.4 are now discussed separately.

The ideal relationship relating input steady current to output speed is a straight line passing through the origin, with a slope corresponding to the system gain. In many practical cases, this characteristic will show saturation effects. An estimate of the actual input current vs. output speed characteristic, for the servodrive, is shown in Fig. 8.3.2. It reveals hysteresis as already mentioned as well as a slight saturation effect. Coupled with the hysteresis is also a gain change, about null. Both effects are attributed to friction. The slope of the curve is an estimate of the
As already noted, in the absence of friction the differential pressure across an incertially loaded motor would be zero for constant input current. The motor ripple itself causes some differential pressure oscillations, as shown in Fig. 8.3.1, but its average would be zero. The differential pressure curve shown in Fig. 8.3.3 is thus the result of friction effects.

The motor torque equation is

\[
\text{Torque} = \Delta P_L \cdot D_m = J \frac{d\omega}{dt} + B \omega + F_c \text{sgn}(\omega) \tag{8.3.1}
\]
It is known that for a steady input current, the average output speed can be approximately written as

\[ \omega = K(m_i) i \]  

(8.3.2)

where \( K(m_i) \) is a gain function, depending on input bias \( m_i \), and \( \omega \) and \( i \) are the output average speed and input current respectively. Substitution of (8.3.2) into (8.3.1), and allowing for the fact that \( \frac{d\omega}{dt} = 0 \)

\[ \Delta P_L \frac{D_m}{D_m} = B \frac{K(m_i)}{D_m} i + F_c \frac{K(m_i)}{D_m} \text{sgn}(\omega) \]  

(8.3.3)

\[ \Delta P_L = B \frac{K(m_i)}{D_m} i + \frac{F_c}{D_m} \frac{K(m_i)}{D_m} \text{sgn}(\omega) \]  

(8.3.4)

Plotting differential pressure \( \Delta P_L \) as a function of input current \( i \), as in Fig. 8.3.3, is thus equivalent to (approximately) plotting the RHS of (8.3.4) as a function of \( i \). That the plot is only an approximation to the RHS of (8.3.4) is obvious by inspection of the graph (in Fig. 8.3.3). The equation does not cater for hysteresis effects and, furthermore, it implies a symmetric characteristic. The offset is due to the servo valve null shift.

By obtaining a graph relating output speed to differential pressure \( \Delta P_L \), it is possible from a knowledge of the motor displacement \( D_m \) to estimate the viscous and Coulomb friction coefficients \( B \) and \( F_c \). The procedure to obtain such estimates is now described.

8.3.3 Output speed vs. differential pressure. Steady state test. (Fig. 8.3.4)

The graph of motor output speed vs. differential pressure can be obtained by cross-plotting from Fig. 8.3.2 and Fig. 8.3.3; for each setting of input current a value of differential pressure and output speed has already been obtained. The result of each procedure is shown in Fig. 8.3.4. The plot is an equivalent to the theoretical curve shown in Fig. 8.2.1, the former having been derived from measurements on the actual servodrive. The differences are self evident, namely small hysteresis and offset effects. Nevertheless, from the knowledge of the motor displacement \( D_m \),
estimates of the motor viscous and Coulomb friction coefficients can be made. The viscous friction coefficient \( B \) can be estimated by finding the average slope of the linear part of the characteristic, (see Fig. 8.2.1). This was done and a value of \( B \) estimated at .25 in.lb/rpm. The Coulomb friction coefficient \( F_c \) can be estimated from the distance \( d_o \) (see Fig. 8.3.4) using the relationship

\[
d_o = \frac{2F_c}{D_m}
\]  

(8.3.5)

from which \( F_c \) can be calculated. The estimated value of \( F_c \), from Fig. 8.3.4, is 186 in-lbs; this is the Coulomb friction coefficient. The values of \( F_c \) and \( B \) given here were calculated from knowledge of the motor displacement, which is equal to 2.51 in\(^3\)/rev.

Inspection of the curve showing motor speed as a function of differential pressure (Fig. 8.3.4) shows that the viscous friction coefficient \( B \) is in fact quite variable over the range of possible speeds and further, that there is considerable difference between the variations of \( B \) for forward and reverse directions. The value of \( B \) given above as 0.25 psi/rpm is an average, useful only as an order of magnitude. The variations in \( B \) are attributable to oil temperature fluctuations during the course of the 'steady state' tests. The oil temperature is to a certain extent a function of motor speed, and therefore it was not possible to keep it constant.

All the pressure curves show an apparent static friction effect, revealed by the fact that close to zero speed, the differential pressure actually increases.

The data presented in this section has enabled the estimation of the viscous and Coulomb friction magnitudes present in the servodrive, quantities which are of paramount importance for design as well as fault-finding applications. The method of obtaining these quantities is however lengthy, even if simple. Due to the ripple in the motor making the averaging necessary, the testing time to obtain the curves was 2\( \frac{1}{2} \) hours. Though this already is obviously a long time, it does not include the time taken to allow the servodrive system to reach steady conditions. Oil temperature influences
the system significantly since oil viscosity is a function of temperature. It was necessary therefore to allow the oil temperature to reach a steady state prior to the start of the actual tests, and this took more than 2 hours. The overall experiment time was thus in excess of 4½ hours. In the next section the theoretical correlation techniques developed in this thesis are used to obtain equivalent data in 5 minutes.

Fig. 8.3.4 Steady state output speed vs. differential pressure characteristic.

8.4 Motor load equation and correlation theory

The motor load equation

$$\Delta P_L(t) = \frac{J}{D_m} \frac{d}{dt} [\omega(t)] + \frac{B}{D_m} \omega(t) + \frac{F}{D_m} \text{sgn}[\omega(t)]$$  \hspace{1cm} (8.4.1)

Let $J' = \frac{J}{D_m}$ \hspace{1cm} $B' = \frac{B}{D_m}$ \hspace{1cm} $F' = \frac{F}{D_m}$  \hspace{1cm} (8.4.2)
If both sides of (8.4.1) are multiplied by \( i(t+\tau) \), where \( i \) is the unbiased component of the servodrive actuating current then

\[
\Delta P_L(t) \ i(t+\tau) = J' \ \frac{d}{dt} [\omega(t)] \ i(t+\tau) + B' \ \omega(t) \ i(t+\tau) + F_c' \ \text{sgn} [\omega(t)] \ i(t+\tau)
\]

(8.4.3)

If \( i(t) \), the servodrive driving current is made to be a repetitive signal, with bias \( m_i \) and period \( T \) then

\[
i(t) = m_i + i(t)
\]

(8.4.4)

\[
i(t) = i(t+T)
\]

(8.4.5)

Integrating and averaging (8.4.3) over the period of the driving signal then the following correlation equation is obtained.

\[
\phi_{i\Delta}(\tau) = J' \ \phi_{i\omega}(\tau) + B' \ \phi_{i\omega}(\tau) + F_c' \ \phi_{is}(\tau)
\]

(8.4.6)

where

\[
\phi_{i\Delta}(\tau) = \frac{1}{T} \int_0^T \Delta P_L(t) \ i(t+\tau) dt
\]

(8.4.7)

\[
\phi_{i\omega}(\tau) = \frac{1}{T} \int_0^T \frac{d}{dt} [\omega(t)] \ i(t+\tau) dt
\]

(8.4.8)

\[
\phi_{i\omega}(\tau) = \frac{1}{T} \int_0^T \omega(t) \ i(t+\tau) dt
\]

(8.4.9)

\[
\phi_{is}(\tau) = \frac{1}{T} \int_0^T \text{sgn} [\omega(t)] \ i(t+\tau) dt
\]

(8.4.10)

These crosscorrelation functions are functions of bias \( m_i \) since differential pressure \( \Delta P_L \) and motor speed \( \omega \) are functions of \( m_i \). If both sides of (8.4.6) are integrated over the period of the crosscorrelation function, from \( 0 \) to \( T \), and denoting
The integration carried out in (8.4.12) is effected using the properties of ergodic functions. In equation (8.4.16) note that the quantities $I_{\lambda}(m_i)$, for any value of input current bias $\nu$.

Consider now Fig. 8.4.1. This represents a functional diagram of the servo system; this is assumed to be composed of an equivalent set of linear dynamics and a nonlinear gain, function of input bias. The input to the servodrive is the actuating current $i$ and the output the motor speed $\omega$. The quantity $\dot{\omega}$ (or $\frac{d\omega}{dt}$) is produced by a fictitious differentiator, and the function $\text{sgn}(\omega)$, by an ideal relay of unit amplitude. The following facts have already been established in Chapter 7. If $\phi_{\omega\omega}(t)$ is the cross-correlation function between the input and the output of the relay, defined as

$$\phi_{\omega\omega}(t) = \frac{1}{T} \int_0^T \text{sgn}[\omega(t)] \omega(t+\tau)d\tau$$

then equation (8.4.6) reduces to

$$I_{\lambda}(m_i) = J' V(m_i) + B' I_{\lambda}(m_i) + F' I_{\lambda}(m_i)$$

The integration carried out in (8.4.12) is effected using the properties of ergodic functions. In equation (8.4.16) note that the quantities $I_{\lambda}(m_i)$, $V(m_i)$ and $I_{\lambda}(m_i)$ are directly obtainable from a single correlation test for any value of input current bias $m_i$.
then if \( \omega(t) \) is biased and repetitive with period \( T \),

\[
\int_0^\infty \phi_\omega(t) \, dt = I_{\omega_0}^\omega(m_\omega) 
\]

(8.4.18)

where \( m_\omega \) is the bias of \( \omega \). The function of \( I_{\omega_0}^\omega(m_\omega) \) has a shape which approximates the form shown in Fig. 8.4.2. The area underneath this function is equal to unity, independently of the p.d.f. of \( \omega(t) \). This was shown experimentally in Chapter 7. By referring to Fig. 8.4.1 it can be seen that, to a good approximation

\[
I_{\omega_0}^\omega(m_\omega_1) I_{\omega_1}^\omega(m_\omega_1) = I_{is_1}^\omega(m_\omega_1) \quad (8.4.19)
\]

where \( m_\omega_1 \) is the servomotor actuating current bias.
It follows that

\[ I_{i\Delta}(m_i) = J' V(m_i) + B' I_{i\omega}(m_i) + F' I_{i\omega}(m_i) I_{\omega\omega}(m) \]  \hspace{1cm} (8.4.20)

and thus

\[ I_{i\Delta}(m_i) = J' V(m_i) + [B' + F' I_{\omega\omega}(m_i)] I_{i\omega}(m_i) \]  \hspace{1cm} (8.4.21)

Note that \( I_{\omega\omega}(m) \) is zero for large \( |m| \).

In equation (8.4.20) the quantities \( I_{i\Delta}(m_i) \), \( J' \), \( V(m_i) \), and the function \( I_{i\omega}(m_i) \) are all measurable. Further, the integral of \( I_{\omega\omega}(m) \) is known to be unity. The remaining unknowns in (8.4.20) are \( B' \) and \( F' \). These can be easily determined, following the procedure outlined in the following sections.

8.4.1 Determination of the gain function \( K(m_i) \) for the servomotor

The servomotor gain function \( K(m_i) \) was found using the combined prenormalization and repeated mean level test procedures outlined in Chapters 6 and 7. The servodrive system has a bandwidth of approximately 25 Hz. The bandwidth of the TPRS input was set at 100 Hz with prenormalization filter cut off frequency set at 40 Hz. This ensures that the signal actually entering the servodrive is normally distributed (see Chapter 7). The sample rate was set at 300 Hz, and the testing signals biased TPRS. The amplitude of the TPRS was 1/40th of the servo valve input range. The second amplitude, for the repeated mean level test, was 1.5 times the base amplitude.

The integrated, identified \( K(m_i) \) function is shown in Fig. 8.4.3; this can be interpreted as being the estimated instantaneous d.c. input-output function of the servomotor. It is similar to the function identified via the steady state input tests, shown in Fig. 8.3.2., but the effects of the hysteresis are smoothed out. This is due to the nature of the correlation identification procedure used which tends to average effects which are due to multivalued nonlinearities. The gain function nevertheless shows the incumbent saturation effect, as well as the gain reduction near the null position. This is due to friction effects in the servo valve.
Fig. 8.4.3 Integrated identified $K(m_1)$ function: this represents the estimated servomotor steady state input current output speed characteristic.

8.4.2 Determination of the viscous and Coulomb friction coefficients $B$ and $F_c$

Equation (8.4.21) is repeated for ease of reference.

$$I_{i\Delta}(m_1) = J' V(m_1) + [B' + F' \frac{C}{\omega} \omega \omega] I_{i\omega}(m_1) \quad (8.4.22)$$

The function $I_{i\omega}(m_1)$ has already been obtained. The values of $B'$ and $F'$ are now easily determined by the following procedure. First obtain the functions $I_{i\Delta}(m_1)$ and $[-J' V(m_1)]$. This can be done easily by carrying out correlation tests over the range of operating points $m_1$. Note that it is required to know the inertia of the motor rotating parts to find $J'$. The function $I_{i\Delta}(m_1)$ is shown plotted in Fig. 8.4.4, and the function $[I_{i\Delta}(m_1) - J' V(m_1)]$ in Fig. 8.4.5. This latter function can be written,
Fig. 8.4.4 Input current vs. $I_i \Delta (m_i)$ characteristic.

Fig. 8.4.5 Input current vs. $[I_i \Delta (m_i) - J'V(m_i)]$ characteristic.
The function $K(m_i)$ is known. It is therefore possible to obtain the function $I_{i\omega}(m_i)$ since $I_{i\omega}(m_i) = \sigma_i^2 K(m_i)$.

$$
\left[ \frac{I_{i\Lambda}(m_i) - J' V(m_i)}{I_{i\omega}(m_i)} \right] = B' + F'_c I_{\omega S}(m) \quad (8.4.24)
$$

Calling this new function $\zeta(m_\omega)$ then

$$
\zeta(m_\omega) = B' + F'_c I_{\omega S}(m) \quad (8.4.25)
$$

From a plot of $\zeta(m_\omega)$ the values of $B$ and $F_c$ can be determined. A direct method of obtaining the function $\zeta(m_\omega)$ is to plot $\zeta(m_\omega)$ as a function of speed $\omega$, the average output speed for average input current $m_i$. This is shown plotted in Fig. 8.4.6. From this value of $B$ can be obtained directly by noting that $I_{\omega S}(m_\omega)$ is zero for large $m_i$ (or $m_\omega$).
The contribution of $I_{\omega S}(m_\omega)$ is in fact evident from the figure, corresponding to the peak around the null. Having obtained $B$, $F_c$ can also be obtained, from the figure by noting that the area under the spike, shown shaded, is equal to $2F_c/D$. This fact follows from theory already derived.

The estimated values of $B$ and $F_c$ by this method are

$$B = 0.24 \text{ in}-\text{lb/rpm}.$$  
$$F_c = 188 \text{ in}-\text{lb}.$$  

These can be contrasted with the values obtained with the steady state tests,

$$B = 0.25 \text{ in}-\text{lb/rpm}.$$  
$$F_c = 186 \text{ in}-\text{lb}.$$  

The value of viscous friction $B$ obtained via the correlation method is steady throughout the range of inputs. This contrasts with the variations of $B$ which are obtained with the steady state test. It must be concluded that the variation in system parameters, such as oil temperature, during the course of the 'steady state' test cause significant changes in the effective viscous friction. The viscous friction coefficient is proportional to oil viscosity; oil viscosity is in turn a function of oil temperature. In any case, over the $2^{1/2}$ hour testing time some small variations in oil temperature probably did occur.

Although the correlation method might appear complicated, the steps outlined by the equations (8.4.23) through to (8.4.25) can be automated. The interpretation of the $\zeta(m_\omega)$ function can be slightly difficult due to the integration which needs to be carried out to obtain $F_c$. The integration is very much simplified with the aid of a digital computer.

The time taken to obtain the estimates of $B$ and $F_c$ was approximately $1/4$ hour. The actual testing time was however under 5 minutes, the rest of the time being taken with the plotting operations and interpretation. These operations could be automated to further reduce the estimation times. Of
importance is the fact that since the testing time is so short, the tests can be carried out even if the servodrive system has not reached its steady state conditions, i.e. even if the oil temperature has not been established. The values of $B$ and $F_c$ might be temperature dependent, and in that case series of tests carried out to investigate such dependancies. The correlation procedure is evidently far superior to the traditional one for obtaining estimates of the servodrive friction characteristics.

8.5 Summary

A servodrive hydraulic system has been tested to obtain its static characteristics. Traditional methods have been used to obtain estimates of its viscous and Coulomb friction characteristics. These methods have been shown to suffer due to the long time taken for the tests due to the ripple which exists, inherent in rotating vane motors. The long testing times are a disadvantage also due to drifting of system parameters during the experiment. The system is particularly sensitive to oil temperature changes.

A novel method is then developed to obtain the motor static characteristics quickly and accurately, based upon a combination of the identification procedures outlined in previous chapters, and a correlation differential equation. Application of the properties of ergodic processes, enables its interpretation in terms of easily measured quantities. The procedure yields results which closely agree with those obtained by the much slower steady state method, but with a parameter variance which is much smaller. This is interpreted as a measure of the accuracy of the method as the effect of slow drift in parameter values (for example, due to temperature) are effectively eliminated.
The data obtained from the manufacture of the servo valve are listed as follows:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moog Series 76 Flow Control Servovalve</td>
<td>Type 76-134.</td>
</tr>
<tr>
<td>Rated Flow (at 1000 Psi Pressure drop)</td>
<td>38.5 in³/sec.</td>
</tr>
<tr>
<td>Rated Differential Current</td>
<td>15 ma.</td>
</tr>
<tr>
<td>Coil Resistance</td>
<td>200 Ω ± 12%.</td>
</tr>
<tr>
<td>Coil Inductance</td>
<td>0.8 Henry (approx.).</td>
</tr>
<tr>
<td>Supply Pressure</td>
<td>100 psi - 3000 Psi.</td>
</tr>
<tr>
<td>Flow Gain Nonlinearity</td>
<td>Vary from 50% to 200% of its rated valve for ±5% of rated current from null.</td>
</tr>
<tr>
<td>Null Pressure Gain</td>
<td>Normally greater than 30% of supply pressure for 1% of rated current and can be up to 80%.</td>
</tr>
<tr>
<td>Maximum Null Shift</td>
<td>2% for 40°C Temp. Change 2% for supply pressure change from 80% to 110%. 2% for back pressure change from 0 to 20% of supply pressure.</td>
</tr>
<tr>
<td>Hysteresis</td>
<td>Less than 3% of rated current. (&lt;.45 ma)</td>
</tr>
<tr>
<td>Null Leakage</td>
<td>Less than 3% of rated flow.</td>
</tr>
<tr>
<td>Resolution</td>
<td>Less than 0.5% of rated signal without dither. If dither is used peak to peak amplitude less than 20% of rated current is recommended.</td>
</tr>
<tr>
<td>Torque Motor - Spool Valve Data (Series 4551).</td>
<td></td>
</tr>
<tr>
<td>Torque Motor Gain</td>
<td>0.02 lb-in/ma.</td>
</tr>
<tr>
<td>Natural Frequency Armature Flapper</td>
<td>600 Hz.</td>
</tr>
</tbody>
</table>
Damping Ratio 0.5.
Armature Flapper Stiffness 90 lb-in/inch.
Nozzle Flapper Flow Coefficient 140 in³/sec/inch.
Feedback Wire Stiffness 6.5 lb-in/in.
Spool End Area 0.077 in².
Maximum Flapper Displacement 0.0012 ins.
Maximum Spool Displacement 0.02 ins.

The Manufacture Data for the Axial Roller Vane motor is listed as follows:

Manufacture (WSI Washington Scientific Industries Ltd.).
Model No. 25 (Dowty Package No. 4554-134).
Displacement 2.5 in³/rev.
Inertia 0.0679 in lb sec²
Torque 0.397 lb-in/psi
Volume of Oil under Compression 1.25 in³.
Maximum Continuous Pressure Rating 2000 psi.
Maximum Intermittent Pressure Rating 3000 psi.
Theoretical Power at Continuous Rated pressure 18.9 hp
Theoretical Input Flow 16.2 gal/min.
Motor leakage Characteristic 0.8 in³/sec/1500 psi.
Temp. Range -40°C to +70°C.
The objectives of the research described in this thesis are wide ranging, covering a theoretical treatment of the Volterra series representation of nonlinear systems, identification, computer organisation and simulation. It is outlined how a minicomputer system can be organised to become an efficient system research tool. Then theoretical developments are described, relating to the analysis of nonlinear systems, from a functional point of view. A modified form of Volterra analysis is introduced, its properties discussed, and then this theory applied to the analysis of nonlinear systems with pseudo-random and Gaussian inputs. In Chapters 6 and 7, an identification method is developed and applied with success to systems containing smooth or discontinuous nonlinearities. An optimum identification procedure is suggested based upon the experimental evidence given. Some of these methods are applied in Chapter 8 to the identification of the d.c. input-output and friction characteristics of a hydraulic servodrive.

The minicomputer proved invaluable throughout the research period as the software developed enabled new experiment configurations to be tested quickly, and established techniques to be implemented without difficulty or errors. Some limitation was imposed by the core size (32 KB), even if all the programs were coded directly in Assembler. It is felt that a minimum core size, to enable a user to comfortably manipulate data and carry out very complex arithmetic routines in Assembler, is approximately 48 KB for a 16-bit machine. Such a core size is sufficient to carry out operations such as a two-dimensioned FFT on a 1024-bit data array, without the need for overlay programming.

The Resident Support Program (RSP) played a central role throughout this project, enabling some of the tasks more suited to main-frame machines, to be easily accomplished with the minicomputer.

The theoretical developments described in Chapter 3 are regarded as being of major importance to the study and analysis of separable single-valued nonlinear systems, from a functional point of view. The conventional functional method of analyzing such nonlinear systems is by means of the Volterra series, the kernels of which are formed by products.
of non-unity gain impulse response functions. The terms of the conventional functional series are multidimensional convolution operations between such kernels and the biased signals. The modified Volterra functional series, developed by the author, introduces a degree of normalization and standardisation in the expressions for output and correlation functions. It makes it possible to predict the convergence properties of the series for the output of a system when a known nonlinearity exists. This is because the Volterra series is expressed as a summation of terms which are constituted by derivatives of the nonlinear element instantaneous input-output function, weighted by coefficients which are dependent only on the order of the nonlinear operation being described. Further, the analysis of the nonlinear systems by the modified Volterra series enables the derivation of expressions for the system crosscorrelation function, which are constituted by summations of nth order autocorrelation functions of the dynamic component of the system input. This is important because in some cases the high order statistical properties of signals are known; when bias exists superimposed on such signals those properties no longer apply. Such is the case of inverse repeat and Gaussian signals. Maximal length sequences which are also inverse repeat signals are convenient system test signals, and it is demonstrated how TPRS in particular has useful properties.

Quite apart from the simplifications which can be made to the expressions for output and correlation functions when the system inputs are restricted to a particular class of signals, such as pseudo-random sequences, the modified Volterra functional analysis enables explicit expressions to be derived for output and correlation functions of nonlinear feedback systems with biased inputs. Such derivations would be extremely difficult with conventional analysis.

Further research in this area could concentrate on multivariable systems, or systems containing several nonlinear elements. Such extensions would be useful, finding application in identification of multivariable systems. However it is felt that a more interesting problem is posed by the study of the multidimensional autocorrelation functions of restricted classes of signals. The standardisation in the Volterra series is due to the fact that the terms of the series are constituted by multidimensional convolution operations between unity gain impulse response functions and unbiased signals, weighted by derivatives of the nonlinear element instantaneous input-output function. These
convolutions are shown to be related to the signal entering the nonlinear element. The statistical properties of this signal play a central role in the system behaviour. Methods have been briefly examined by the author to determine the multidimensional autocorrelation functions of TPRS. These are based on transforming the TPRS signal from linear time to dyadic time domain, obtaining the Walsh Transform of the TPRS. Having obtained this Transform, the multidimensional autocorrelation functions of TPRS can be evaluated, the only arithmetic operation required being addition. A knowledge of such autocorrelation functions would mean that an expression for the crosscorrelation function across the nonlinear system could be found, simplified in the same way as the equivalent one for Gaussian inputs. Furthermore it would lead to enhanced identification performance, as briefly mentioned in Chapter 6.

A knowledge of the high order autocorrelation functions of the system input enables, for example, the expression for the system crosscorrelation function to be explicitly determined. Knowing the p.d.f. of the nonlinear element input, and the bivariate p.d.f. between the nonlinear element input and its output, also enables such expression to be determined. The bivariate p.d.f. approach to the determination of the crosscorrelation function is more efficient; it involves integrations of two dimensional functions, each integral being completely determinate once the bivariate p.d.f. is known. This is shown in Chapter 3.

Determination of the crosscorrelation function by means of the functional series containing the higher order autocorrelation functions is however more difficult because such autocorrelation functions are seldom known. The p.d.f. approach to the solution of the problem appears to have additional advantages since knowledge of the p.d.f. is the first step in the study of the effects of nonlinear element input statistics on identification resolution.

It is demonstrated in Chapter 7 that when the input to the nonlinear element is normally distributed, the identification performance is at an optimum. It is the author's belief that this is not a unique solution; indeed it seems likely that providing the nonlinear element input signal is separable, (the definition of separability is given in Chapter 6), then the identification performance is at a 'local' optimum. This can be shown to correspond to a minimum of the integral of the crosscorrelation function across the nonlinear system. Research could be carried out to find analytical
expressions relating the p.d.f. at the output of a linear system to that at its input, in particular when the latter is that of TPRS. Much work has already been reported in this area, which is normally described under the heading of filtered Poisson processes. However a more unified approach is needed. The next step would be the formulation of a dynamic programming problem, writing the integral of the crosscorrelation function across a nonlinear system as the cost function, and allowing, for example, the bandwidth of the linear system preceeding the nonlinearity to be the variable parameter. Such work was outside the scope of this thesis, but it is felt that it would give great insight into the mechanisms by which the statistical properties of the signals involved are changed by the various elements in the system. As far as the transformations which are undergone by the probability density functions, the greatest problems are those posed by the linear elements. The theory of nonlinear, amplitude, transformations of the p.d.f. shape is already well established.

Chapter 4 deals with the analysis of separable nonlinear systems with Gaussian biased inputs. The chapter concentrates on the derivation of very generalized expressions demonstrating how bias, power and variance are inter-related, and obtaining expressions for output autocorrelation functions and system crosscorrelation functions. The modified Volterra series is shown to be simply related to the biased, random input, Describing Function, and supports the arguments for the use of identification procedures obtained with the assumption of smooth nonlinear elements, to systems containing discrete ones. The exact analysis of nonlinear systems with Gaussian inputs is shown to become very complicated when feedback exists around the nonlinearity. It is doubtful whether such exact analysis would be useful in practice, and only an indication is given as to the possible method of attack of such problems.

Maximal length sequences as system inputs are considered in Chapter 5. TPRS and PRBS are examined in detail. Even if the form taken by the high order TPRS statistics are unknown, considerable simplifications are made to the generalised formulae of Chapter 3 by restricting the nonlinear system inputs to such signals. It is shown how discrete pseudo-random signals can be treated using the modified Volterra functional expansion, demonstrating that considerable simplifications are possible to the generalised expressions of Chapter 3; particular expressions are also derived to treat the special case of Hammerstein type systems. Of particular importance is the analysis carried out applicable to nonlinear systems with PRBS inputs which,
quantitatively, explains many of the effects reported in the literature. It is shown that PRBS has considerable disadvantages compared to, say, TPRS, due to its inherently biased nature and due to the fact that it does not have the inverse repeat property. It is further shown that the method of 'inverse repeat PRBS testing' is inherently prone to error. The quantitative derivation of expressions showing the magnitude of such errors is straightforward, due to the particular nature of the modified Volterra functional treatment used.

Chapters 3 to 5 are thus devoted to the development of an analytical approach to the study of the separable, single valued class of nonlinear systems. In Chapter 6 some of the more important theoretical concepts are supported by means of experimental results. Then the modified Volterra functional series is used in the derivation of a simple and effective procedure for the identification of the nonlinear part of systems, namely the small signal gain function. The procedure is limited to systems containing single valued and lagless nonlinear elements. Nevertheless such a procedure is very useful as many physical systems which are nonlinear fit such requirements. The identification procedure itself is developed for systems containing smooth nonlinear elements, but in Chapter 7 it is shown that it can equally well be applied to systems containing discrete nonlinearities. The identification performance can be improved by carrying out an extended test and the advantages of this are discussed. The errors involved are examined analytically as well as experimentally, special attention being paid to the effects of noise. To reduce the effects of cyclic noise a novel method is developed which is easily implemented and can be used in conjunction with this or any other identification procedure which is based upon the concept of system excitation with a repetitive test signal.

Testing discrete nonlinear systems, it was discovered that the identification procedure can be optimised by preconditioning the TPRS signal prior to its injection into the system. The improvements gained by such a process are shown to be considerable. The preconditioning involved is a low-pass filtering operation on the TPRS, with appropriate amplification to maintain the required system input variance. The amount of filtering involved is that which most closely transforms the TPRS signal into a Gaussian random variable. Precise guidelines are given to help with the choice of filter gain and cut off frequency.
The identification procedure used as outlined in this thesis is based upon correlation methods. Though the relevant programs were coded in Assembler, computation times are not insignificant. They are considered to be long or short depending on the type of system being tested, and in particular on the time required for system excitation which is in turn dependent on the system settling time. If the systems tested are mechanical and noisy, requiring many averaging runs, then the computation times will be short in comparison with the data acquisition time. As an example, a typical correlation of, say, 256 data points takes approximately 0.5 seconds. However, if electronic equipment is tested, then such computation times are usually long.

Methods have been considered for speeding up the computation time but not implemented since they are outside the scope of this work. They revolve around two main ideas. The first one is to render the correlation process more efficient by removing all multiplications and replacing them with gated additions. This is possible since in all cases one of the variables is a discrete signal (TPRS). A hardware correlator could even be constructed naturally being a very efficient method as far as time is concerned. An alternative to this which has been considered is the use of FFT or FWT routines in place of the correlation ones; this is probably more difficult since relationships need to be found between the integral of the cross-correlation function and the cross-spectral density function. The use of Walsh Transforms, rather than Fourier Transforms, for such work would speed the computation even further, but interpretation of the results from dyadic domain is quite difficult and considerable problems are envisaged. In conclusion it is felt that the hardware correlator would be a much better solution since its implementation is sure to yield good results. The Transform methods are more interesting, but also more complex.

The identification procedures outlined in Chapters 6 and 7 were applied to the testing of a hydraulic servomotor; the description of the tests involved are given in Chapter 8. It is shown how the application of simple correlation equations, in conjunction with test data, can be used to find the static characteristics of the servodrive quickly and accurately. The experimentation time required for the conventional quasi-steady state test was of the order of 3 hours, due to the presence of ripple at the system output, necessitating long averaging times. Equivalent results obtained by
means of the correlation technique were obtained in approximately 5 minutes. Furthermore it was shown that such results are more accurate because of the presence of drift in some system parameters, (such as oil temperature), during the course of the quasi-steady state test. The correlation tests used are based on the identification procedures already described, as well as a novel method, described in Chapter 8, of interpreting the load-torque relationships for the servodrive system.
10.1 Conclusions

It has been shown how the Volterra series analysis of separable single-valued nonlinear systems can be modified to enable the properties of such systems to be examined more easily. Using the modified Volterra analysis approach, several explicit expressions have been derived which quantitatively show the effects of system input bias on the output properties of nonlinear systems. In particular the cases of Gaussian and discrete pseudo-random inputs have been examined in detail. Furthermore it has been possible to obtain explicit expressions for system output bias variance, auto and cross-correlation functions when the system inputs are normally distributed. A relationship has also been found linking the random input biased Describing Function to the modified Volterra functional analysis.

The analysis carried out for the case of discrete pseudo-random system inputs enabled the quantitative prediction of effects which have been noted by other workers in the field, but could not be explicitly analyzed with conventional functional analysis.

The modified Volterra functional analysis has then been applied to the problem of nonlinear system gain identification with good correlation between theory and experimental data. A mini computer based test procedure involving random signal generation, data acquisition and signal processing has been described which has allowed automated identification procedures to be explored. Tests on both simulated systems within the computer and a practical hydraulic servo-system have been carried out, and the results presented in the thesis. The identification procedure is shown to be robust, yielding good results even in the presence of noise. Analytical expressions have been derived to predict the effects and to reduce or eliminate the errors due to cyclic or random noise.
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