EFFECTS OF VARIABLE INERTIA ON
THE TORSIONAL VIBRATIONS OF
MARINE DIESEL ENGINE SYSTEMS

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The analysis of torsional vibrations in the running gear of reciprocating engine systems is normally carried out by neglecting the variation in inertia torques of the system arising from the motion of the reciprocating parts. When the variable inertia effect is allowed for the equation of motion taking into account the effect is non-linear. Assuming small displacements, the equation can be linearized to predict important characteristics of the motion.

Such an equation when solved by numerical methods using a digital computer predicts the regions of instability and the manner in which the amplitude and frequency vary with the speed of rotation of the engine. The responses of the system show a modulation of amplitude and frequency at definite rotational speeds. The occurrence of such a modulation in amplitude and frequency is established by use of the process given by Wentzel, Kramers, Brillouin, and Jeffreys generally known as the WKBJ approximation. The method of variation of parameters and the WKBJ approximation are used to determine the time responses of the system as an independent verification of the numerical methods although this method is applicable only in a limited range of the engine speed for the specific ratio of the equivalent inertia of the reciprocating parts to the total equivalent inertia of the system.

The first order term in the equation, the forcing term which represents the outer impulse from the reciprocating parts and the variable part of the elastic term are investigated for their effect on the waveforms of the responses of the system.
Further investigations of the effect of external excitations on the characteristics of the motion are carried out which explain the cause of failure in some multi-cylinder engines due to secondary resonance. Various other characteristics of the variable inertia system and the complete solutions of the equation of motion including damping are explored in the present work. Theoretical results are compared with solutions of the equation obtained from an analogue computer.

A discussion of results obtained by measurements on an engine in service with suspected secondary resonance and some actual cases of crankshaft failures in practice is included.
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NOMENCLATURE

$A_{\text{max}}$ maximum amplitude of $\gamma$

$a$ crank radius

$I_A$ moment of inertia of flywheel A

$I$ moment of inertia of the rotating parts

$I_m = I + \frac{1}{2} Ma^2$

$M$ mass of reciprocating parts

$r$ ratio of the angular velocity $\omega$ of crankshaft to the natural frequency $\omega_n$ of the system neglecting variable inertia effects

$\eta = \frac{\frac{1}{2} Ma^2}{I + \frac{1}{2} Ma^2}$

$\theta$ angular displacement of crank $E$ from top dead centre

$\theta_A$ angular displacement of flywheel $A$

$\mu$ torsional stiffness of crankshaft

$\gamma$ displacement of torsional motion

$\omega$ angular velocity of crankshaft

$\omega_n$ natural frequency of the system neglecting variable inertia effects

$\omega_I = \left[ \frac{\mu}{I + \frac{1}{2} Ma^2} \right]^\frac{1}{2}$

$\omega_I$ frequency of response at maximum amplitude
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CHAPTER 1

INTRODUCTION

1.1. General Review of Torsional Vibration Effects

Torsional vibration of shafting is that phenomenon which is manifested by a repeated twisting and untwisting of the shaft. The motion of each section of the shaft due to this type of vibration is an angular one, the shaft vibrating about some mean position. In the case of reciprocating engines, it may be caused by the torques which arise from the uneven turning effort. Hence the crankshaft of an internal combustion engine does not rotate with a steady angular velocity as it would do if it were the main shaft of a turbine. A torsional vibration is superimposed on the normal steady state rotation which can be of large amplitude when the exciting frequency coincides with the natural frequency of the crankshaft-connecting rod system. Thus the study of torsional vibration is a major design problem.

The calculation of torsional vibration in crankshafts of reciprocating engines is of great importance in modern designs. Even, in an earlier stage of development, when the peak cylinder pressures and accelerating forces were less torsional vibrations were the cause of a number of crankshaft and other failures. In marine service numerous broken crankshafts, propeller shafts, coupling bolts and propeller blades can be ascribed to torsional vibration.

During recent years the development of large marine diesel engines has had a tendency to go towards higher gas pressure in the cylinder and subsequently increased forces in the crank-connecting rod mechanism.
The primary cause of some crankshaft and other failures in multi-cylinder diesel engines is considered to be the phenomenon of secondary resonance which arises in the system from the variation in inertia. The purpose of the present investigation is to study the phenomenon of secondary resonance.

The crankshaft is a complex structure containing stress discontinuities and is loaded in a complex manner. Factors which contribute to failures of crankshafts are generally grouped under the two main headings which are

(a) Additional load due to
   (i) Torsional oscillation.
   (ii) Mal-alignment due to excessive wear-down of bearings.
   (iii) Axial vibration.

and (b)

   (i) Defective forgings.
   (ii) Faulty shrink fits.

In the early part of the 20th century a number of unexplained fractures occurred in the crankshafts of the main engines of motor ships. On investigation it was found that the failures were due to torsional vibrations. It was also shown that the same form of vibration was present in the diesel engines of generator sets resulting in crankshaft failures by fatigue.

Lewis stated that diesel engines and other forms of reciprocating internal combustion engines were far more prone to torsional vibration than reciprocating steam engines. This was due to the compression stroke and the higher initial pressure, both of which factors produced far greater irregularities in the turning effort of the engine than occurred
with steam. After dealing with the importance of the subject, Lewis gave methods of calculating the frequencies and the maximum amplitudes of torsional vibrations. He also gave the magnitude of the exciting impulses due to the inertia forces and the pressure forces of a 4-stroke cycle diesel engine. The impulses due to the pressure forces were obtained by harmonic analysis from a set of indicator cards. Those due to the inertia forces were obtained by calculation.

Since the confirmation of the seriousness of neglecting the existence of the phenomenon of torsional vibration in the shafting of reciprocating diesel engines, as evidenced by the many crankshaft failures, consideration of this type of vibration became a design problem. Remarkable success followed in the case of the diesel engine because that made possible the avoidance of failure due to torsional vibration by changes at the design stage.

Porter gave a general treatment of the problem of torsional vibration of crankshafts. A method was developed for calculating the amplitudes of vibration over the entire speed range of the engine. Application of this method was made to three typical diesel - engine installations. The material presented by him was the result of a study to determine accurately the range of vibrations and the speeds at which they occurred. The data also gave the results of a study on the effect of elastic hysteresis and other damping forces in determining the maximum vibration at the various critical speeds.

Kjaer presented a method of determining whether torsional vibrations would be set up in an engine system. He demonstrated how to calculate the magnitude of the vibrations and showed how they would
act on the different parts of the engine and how they would affect the performance.

Dorey dealt with defects most frequently encountered in the forgings of marine shafting, crankshafts, pistons, connecting rods and crossheads. The section on propeller shafting dealt mainly with the question of corrosion fatigue and also included statistics related to broken shafting. In the cases of crankshafts, various factors contributed to failure, such as methods of manufacture, heat treatment, stress raisers, etc. The remarks were particularly concerned with oil engine crankshafts. Having discussed the methods of manufacture and heat treatment of large crankshaft forgings some necessary comments from the design standpoint were given. Actual cases of defects in crankshafts from service were also presented. Several cases referred to shafts in which stresses set up due to torsional vibrations had contributed towards ultimate failure. Further sections dealt with cases encountered in piston rods, connecting rods and crossheads, while in the final section the causes and suggested means for prevention of breakages in bolts were summarized.

Dorey explained various causes of crankshaft failures and several important conclusions based on thorough investigations of such failures from service were outlined. These investigations contributed greatly to the knowledge of the strength of shafting. In his investigations, some crankshaft failures indicated the possibility of stresses arising in addition to those due to piston load and torsional vibration. He arrived at an interesting conclusion that torsional vibration could be a major disturbing force for the axial vibrations under certain
circumstances and therefore decided that where the torsional natural frequency were equal to or close to the axial vibration resonance, extreme care should be taken to eliminate even small torsional criticals from the running range of speed of the engine. According to his studies many fractures of crank-pins indicated that fatigue had been caused by bending stresses where such stresses were caused by mal-alignment of the main bearings. He also emphasized that in the case of large sizes of crankshafts, the built-up type should be preferred to the semi-built or solid-forged type since sound forgings could be obtained. He also expressed a desire for continued effort to improve shrink fits.

Engine torque curves which depend upon the combined effects of gas pressures, inertia forces and weights were analysed by Porter. He gave tables for the harmonic coefficients of the torque due to inertia and weight effects for a wide range of crank-to-connecting rod ratios. Families of indicator diagrams, representative of various types of engines, were shown and tables of the harmonic coefficients of the resulting torques were given. The types of engines considered were 2-stroke cycle gas, semi-diesel, single-acting diesel and double acting diesel.

Various kinds of crankshaft failures due to torsional or bending stresses, faulty forgings or casting defects were illustrated by Milton in his unpublished paper "Interesting Investigations". He also dealt with the vibrations in the gearing and the occurrence of "gear hammer" in marine systems and discussed the use of hydraulic and magnetic couplings to eliminate the effect of "gear hammer".

For the calculation of the free and forced vibrations of torsional systems with small damping, the method of Holzer is widely used. It
consists of determining the shapes and frequencies of the free vibrations, neglecting damping. An extension of Holzer's method was given by Den Hartog and Li\textsuperscript{8} for the case of damped systems of discreet as well as of uniformly distributed inertias and flexibilities. Forced vibrations of such systems were also investigated in the above paper.

Of the many questions which the marine engineer is called upon to answer, one of the most complex is the problem of what constitutes a safe torsional vibration stress. Dorey\textsuperscript{9} described such a basis which provided a consistent "yardstick" by which to assess the torsional vibration characteristics of marine propelling and auxiliary installations. The recommended limits in this paper were the results of many years of experience gained by Lloyd's Register of Shipping in the investigation of torsional vibration stresses. The stresses were nominal values calculated on sections of parallel shafting neglecting stress raisers and could be compared with the stresses derived from actual measurement of vibration amplitudes by a torsiograph or other equivalent means.

Draminsky\textsuperscript{10} carried out experimental work with a single-cylinder oil engine in which the bearing pressures could be readily calculated and the significant variables, such as bearing dimensions, magnitude of forcing impulses, oil viscosity, etc., varied to enable the fullest possible information to be obtained about the damping influences. The engine was driven by external power, and excited into torsional vibrations by a spring-loaded cam disc to obtain approximate formulae for pre-calculation of the damping in any given case. The damping of the moving parts was investigated separately. It was found that the damping was almost entirely due to hysteresis in the crankshaft and oil damping
due to lateral shaft movements in the main and crankpin bearings which was directly proportional to the bearing clearance. Formulae for the total damping in multi-cylinder engines were also given and compared with the measured damping of oil engines in service.

For torsional systems where the damping is derived chiefly from extrinsic sources of appreciable magnitude, the normal Holzer method cannot be applied. Examples of such systems are those employing dampers and those in which the power transmission is affected by hydraulic or electrical couplings. Den Hartog and Li showed that by an extension of Holzer's conventional method it was possible to treat systems which have large extrinsic damping forces. In the case of a system with unequal discrete masses the method would permit the determination of the damped natural frequencies and the amplitudes of the various masses, the modal shapes, where there was a forcing torque applied to one end of the system. By utilizing this extended type of damped Holzer table and by introducing into it, in complex fashion the harmonic forcing torques at the various masses Spaetgens found that the forced-vibration characteristics of a viscoously damped system of any configuration could be determined completely. Since such a Holzer table contained the damping effects as well as the forcing effects, it could be used for the determination of the absolute amplitudes at all forcing frequencies, including the damped natural frequency, and the peak amplitude thus found indicated the value of the damped natural frequency.

Thomson, Scott and Moir carried out experimental investigations on shrink-fit effects on full-scale crankshaft webs and ancillary
studies of the behaviour of simple ring-and-plug arrangements and model two-pin web assemblies. These small-scale tests allowed such variables as surface conditions and fit allowances to be more fully and systematically examined. The range of surface finish used was from superfine to dry-scrape standards with mating surface conditions from lubricated to chemically dry and thus it covered a wide variation of fit allowances. The investigation was confined to the static effects of shrink fits.

Bunyan considered certain aspects of vibration problems in ships and made some design stage recommendations.

Apart from failures due to torsional vibration or defective material, the most frequent type of crankshaft failure was a bending fatigue fracture across a crankweb and in such cases the cause usually lay in weakness of the crankwebs accentuated by excessively abrupt changes of section at the junctions of the crankpins, journals and crankwebs, or excessive and uneven wear of journal bearings resulting in severe alternating bending stresses. It was necessary to make sure that the form and dimensions of the crankwebs were suitable for carrying the total combined stress induced in the crankpin or journal fillets by gas pressure, inertia loading, and transmitted torque neglecting any supplementary stress due to torsional vibration.

Ker Wilson gave consideration to the fatigue strength of crankshafts in the light of the above observations.

Bisshopp in his analysis extended similar results previously obtained by Den Hartog and Li where the remainder torque was calculated at one end of a homogeneous system. Comparative computations made there
with complex Holzer tables showed agreement with results obtained from the theory of distributed systems. The general boundary value problem, including the response to an externally applied torque at any section was solved by Bisshopp and he also analysed the special case of a homogeneous engine system with a flywheel at one end. The theory was illustrated with comparative computations using complex Holzer tables for such a system with external torque applied at a section remote from the ends.

De Pieri gave a brief historical review of the torsional vibration work carried out in the past and the contribution made by Fiat to the progress of the marine diesel engine.

Pemberton dealt with cases of failures which generally occurred in marine machinery and investigated an important factor which caused failure of crankshafts. He cited the cases of engines which were running on boiler oil and due to wear on piston and scraper rings combustion gases had escaped into the crankcase. The damage had developed within a period of two years. The oil in the crankcase was found to be acidic and had a high sulphur and water content. The oil purifiers had also become severely corroded. Such cases emphasized the importance of maintaining lubricating oil in a clean and acid-free condition and in preventing combustion gases from entering crankcases.

Atkinson and Jackson described some cases of failures of the Doxford 750-mm engine crankshafts. They established the causes and gave the remedies which were successfully applied for the proper functioning of the crankshafts. The original 750-mm six-cylinder engine crankshaft had the following important features which were of
prime concern:

(a) The journals and crankpins had the fillets recessed into the side crankwebs in order to provide adequate bearing length and greater elasticity to the crankwebs therefore giving a reduction of stress in the middle of the web by spreading the loads and deflections. The recessed fillets were directly opposite each other on two sides of the web.

(b) The crankshaft was manufactured in two pieces, with the Doxford type flexible coupling inserted between the halves.

(c) The engine was operating on the flank of the one-node critical.

(d) There was a transverse oil hole, drilled through the crankweb for the lubrication of the side connecting rod bottom end bearing.

A crack was first discovered in the fillet of number three forward side crankpin. The fracture of the side crankweb suggested that probably it had started from the transverse oil hole and the practice of drilling the hole through the web was abandoned. Since there was no apparent explanation of the crack, the torsional vibration characteristics were rechecked and further improved by reducing the diameter of the intermediate shaft and thereby reducing the one-node frequency and a prohibited speed range was introduced below the full load speed. However after a period of time, the same crankshaft had a crack in an exactly similar position in number four aft side crankpin. This showed that torsional characteristics were not the reason of the failures. Further investigations to analyse the cause of failures were conducted and the following factors were examined.
(1) Forgings (2) Misalignment; (3) Recessed fillets and (4) Influence of flexible coupling.

It was shown that in the manufacture of the crankweb pieces, the centre of the billet formed the underside of the side crankpin fillet (where fractures had started) and this was where inferior material was concentrated. The method of forging was changed so that the fillet location was of the better material. It was also found that the recessed fillets of a crankpin on one side and of a journal on the other side of a crankweb must not be directly opposite each other. If this condition was allowed it could lead to a concentration of bending stresses in the centre of the crankweb. Especially where misalignment of the crankshaft was present, the flexible coupling at the centre of a large crankshaft could introduce high stresses in the adjacent side crankpins and therefore flexible couplings were replaced by rigid couplings.

Jackson described the ideas and objects in the development of the new Doxford engine. The main objects of the new engine were to achieve reduced length and weight and to remedy the defects and difficulties which had been experienced with some of the earlier Doxford engines.

The crankshaft of the new engine was made much more rigid with large diameter short length bearings and this together with the reduction of the weights of the rotating and reciprocating parts gave a high natural frequency of torsional vibration of the engine and thus the torsional vibration characteristics, from an operational point of view, were greatly improved. The stresses in the crankshaft were reduced by the large diameter of the pins and journals and the large
overlap of the fillets of the pins and journals reduced the stresses in the side crankwebs.

Andersson et al. gave experimental and theoretical tests to determine the value of the stresses occurring in the crankshaft of large marine diesel engines arising from gas pressure, piston forces, misalignment, and torsional and axial vibration. The effect of different fillet types on the stress distribution in a crankshaft was investigated by taking static stress measurements, using small wire strain gauges.

Sarstan showed that with the advent of modern computers it was possible to make complex calculations for torsional vibration characteristics of marine diesel engine installations. Torsional vibration problems for general systems admitting both internal and external viscous friction were formulated and adapted to programmed machine computation. It was shown that the introduction of complex numbers led to a tabulation entirely analogous to the Holzer undamped method and the resulting basic computer programme was described. This procedure always gave correct results even where the commonly used undamped approximations became inaccurate.

Orbeck gave the theory for forced damped vibrations of multi-mass systems and the main features of the forced vibration Holzer table method of obtaining numerical solutions were described. A computer programme developed for the numerical calculations was given. The crankshaft with running gear was considered as the equivalent torsional
vibration system with stiffnesses, inertias and damping constants. Each of the harmonic components of the applied torque was treated separately and the magnitudes and the phase angles of the vibration amplitudes and torques throughout the system were determined. These calculations were based on the forced vibration Holzer table for multi-mass systems and damping was taken into account. Therefore the calculations were valid for resonance as well as for flank conditions.

Bradshaw stated that, in the case of the single branch systems, the first attempts at a solution involved the construction of a "complex Holzer table" with a great deal of arithmetic and that the method could not easily be applied to multi-branch systems. An alternative formulation of the problem based on finite-element method for the analysis of structures was evolved and was equally suitable for multi-branch systems as for single branch systems. The availability of the digital computer, together with the appropriate numerical techniques made the rigorous solution of the basic equations of motion of torsional vibration of the shafting of even the most complicated diesel engine installations a practical possibility.

Ker Wilson stated that the earliest studies of vibration as an aid to industrial development were undertaken towards the end of the nineteenth century at a time when the marine-engineering industry was confronted by a number of difficult vibration problems. These included the sympathetic response of propeller shafting to pulsating torques from engines and propellers which sometimes resulted in costly failures, even to the extent that some vessels of the period acquired the reputation of being notorious "crankshaft smashers". Large number
of crankshaft failures from the effects of torsional vibration threatened the development of the diesel engine in its early stages of development. Torsional vibration could also cause excessive wear of gearing assemblies used for driving camshafts and could result in interference with valve timing thus causing a rough engine which required excessive maintenance. Ker Wilson gave examples of a number of problems faced by him in his associations with marine machinery and the remedies applied to overcome them. An interesting case of the failure of a side connecting rod, which fractured without warning at about one-third of the length along the shank from the crosshead end, was illustrated and the cause of this fracture was shown to be due to torsional vibrations in the crankshaft.

Fig. 1.1. shows the torsional vibration cracks in the side crank-pin of a marine engine system which occurred because of a severe seventh-order torsional-vibration critical being at or near resonance with the service speed.

1.2. Effects of Variable Inertia

The analysis of torsional vibrations in the running gear of reciprocating engine systems is normally carried out by neglecting the variation in inertia torques of the system arising from the motion of the reciprocating parts. Such a simplified system does not reproduce the exact dynamic characteristics of the actual system, since the effective inertia of each slider crank mechanism varies about a mean value in relation to the position of the crank. For many practical cases, however, where the mass of the reciprocating parts is small or the engine is of a medium power output the simplified theory is adequate. Modern marine diesel engines are not only increasing in size.
Fig. 1.1  Torsional vibration cracks in side crankpin
but the output required per cylinder is also going up. Hence the sim-
plified theory, where the variable inertia is neglected, is not likely
to explain adequately the behaviour of a system in torsional vibration.

Porter used a variant of Kryloff and Bogoliuboff's method to
analyse the periodic vibrations of a non-linear two-degree-of-freedom
system which was an idealization of the dynamic system of a two-cylinder
in-line reciprocating engine. It was shown that there were two principal
critical speed ranges associated with each normal mode of the system
within which periodic harmonic or subharmonic vibrations of large
amplitude could occur as a result of variable inertia excitation.

In both types of critical speed ranges such characteristically
non-linear phenomena as oscillation hysteresis and jumps in amplitude
and phase were found to occur if the damping was light. Furthermore,
if the damping was very light, regimes of shock-excited motions were
shown to exist whose amplitudes were greater than those which occurred
spontaneously under normal conditions of excitation. The existence
of shock-excited vibration of the latter type in a single -degree-of-
freedom variable-inertia system was demonstrated experimentally by
Brook.

Goldsbrough examined the vibratory motion of a six-cylinder
four-stroke cycle, single-acting diesel engine driving an elastic shaft
with a load at the end assuming that only the longer part of the shaft
was flexible and that all the rest of the material were perfectly rigid.
Parts of the shaft between the various cranks should be considered
as elastic but those portions were, however, usually short and stiff,
and compared with the longer part of the shaft, were considered rigid.
Oscillations about the steady state rotation were found to be of two types, analogous to free and forced vibrations respectively. The type of vibration produced was in general very complex and could not be said to consist at any speed of a simple vibration of definite period. It was obvious from the equations representing the motion that forced vibrations did not arise entirely from the pressure of the expanding gases on the piston. They were also produced by the incomplete balance of the reciprocating parts. The positions of the critical speeds were approximately those found by the usual methods.

In view of the interesting results obtained by the analysis and the importance of torsional vibrations in engineering practice, Goldsbrough carried out a theoretical and experimental study to examine the effect of the reciprocating parts in producing or modifying the vibrations. He allowed for variable inertia in a single degree of freedom system, that is to say a single cylinder engine, the gas pressure in the cylinder being omitted.

The equation describing the motion of the system is non-linear but can be linearized following the method of small oscillations. The equation in its linear form is:

\[(1 - \eta \cos 2\tau)\ddot{y} + 2\eta \sin 2\tau \dot{y}' + (\frac{1}{\tau^2} + 2\eta \cos 2\tau)y = -\eta \sin 2\tau \] (1.1)

Goldsbrough in his investigations arrived at the following conclusions:

(1) In the vicinity of \( r = 1 \), the vibrations are unstable and are shown experimentally to be large in amplitude. The frequency of vibration is once per revolution of the shaft.

(2) At a speed corresponding to \( r = \frac{1}{2} \), the vibrations are again...
unstable but over a smaller range. The frequency of vibration is twice per revolution of the shaft.

(3) At still lower speeds there may be other ranges of instability of much smaller extent. They are not observed in the experiment.

The present analysis confirms conclusions (1) and (2) that the instability range in the vicinity of \( r = 1 \) is of considerable extent compared to that of \( r = \frac{1}{2} \). No other ranges of instability are found however for speeds corresponding to \( r = \frac{1}{3}, \frac{1}{4} \), etc. which is contradictory to Goldsbrough's theoretical analysis but in agreement with his experimental results.

Gregory\(^{29}\) constructed approximate solutions of the non-linear equation to predict amplitudes for idealized single cylinder engines and confirmed his theory by experimental tests. An interesting feature of his analysis was that under certain circumstances the amplitude of oscillation was predicted to be multiple-valued function of the speed of rotation. He also derived expressions for characteristic numbers in free motion for equation (1.1) without the forcing term, by the use of methods in which Mathieu's and Hill's equations are discussed with the help of infinite continued fractions. These characteristic numbers defined the unstable regions.

In the present analysis the regions of instability for equation (1.1) in the vicinity of \( r = 1 \) and \( \frac{1}{2} \) have been determined making use of the modified Euler's method. The results are found to be in close agreement with the work of Gregory.

With reference to variable inertia effects Archer\(^{30}\) added further information by citing typical examples of crankshaft failures in large
ten-cylinder and twelve-cylinder marine engines after the ships had been in service for up to ten years. The measured stress values of certain orders were found to be up to five times larger than those calculated by normal methods.

1.3. Secondary Resonance

In recent years several cases of marine crankshaft failures are attributed to the phenomenon of secondary resonance, that is to say, the possibility of an nth order critical of small equilibrium amplitude occurring at or near resonance with the service speed being excited by large resultant engine excitations of order \((n - 2)\) and \((n + 2)\).

Draminsky\(^{31}\) explained this phenomenon by the use of a non-linear theory and stated that it was large second order variations in inertia which were responsible for some failures in crankshafts of multi-cylinder engines. He derived the equation of vibratory motion for the crankshaft of an engine and by the use of certain approximations omitted the variable part, \(2\eta \cos 2\tau\), from the elastic coefficient of equation (1.1).

Draminsky\(^{32,33}\) further explained, making use of a single-cylinder engine with a heavy flywheel and gas pressure being omitted, that if \(\omega_n = \omega_0\) and the system was executing natural or free vibrations of phase velocity \(\omega_n\) the waveform would be composed of a main component of order \(n\) and two smaller components of order \((n - 2)\) and \((n + 2)\). He added that an external excitation in resonance with one of the secondary components of the free vibration of the system was capable
of exciting all components. This excitation would supply energy to
the system through the component with which it was in resonance
thereby evoking a response from all three components of order \((n - 2)\),
\(n\), and \((n + 2)\) since none could exist independently of the others.

It was suggested that in practice secondary resonance in
torsional vibration occurred only for resonance with the lower-order
secondary component, \((n - 2)\), and not the higher-order component
\((n + 2)\) and a method was described to calculate the nth order stress
in the shaft including this effect.

With these facts in view, a ten-cylinder, two-stroke cycle,
single acting engine is examined by the author for secondary resonance
effects. It has a 2-node, 9th order vibration nearly in resonance with
the service speed and all the circumstances lead to the conclusion
that the 9th order torsional vibration will be magnified by the 7th
order in secondary resonance. The torsiograph records taken at the
front end of the engine and the stress records taken on the intermediate
shaft at 114 r.p.m and 115 r.p.m are shown in Fig. 1.2 and Fig. 1.3
respectively.

In these Figs. the stress records are given above the horizontal
line and below the line are shown the torsiograph records. The recor-
dds clearly show the presence of the 9th order vibration near the service
speed. The 9th order measured stress obtained by harmonic analysis
of stress records taken on the intermediate shaft and the corresponding
theoretical conventional stress are compared. It is found that the
measured stress is only slightly higher than the calculated value.
Draminsky's calculation predicts higher 9th order stress than calculated
Fig. 1.2. Stress and vibration pick up records at 114 r.p.m.

Fig. 1.3. Stress and vibration pick up records at 115 r.p.m.
by conventional methods which fail to appear in the measurements in this particular case. These practical measurements are more in agreement with the present theoretical work which shows that a finite solution is possible and that the actual amplitude may also depend on the phase angle of the applied torque component. The system however will not be in a truly resonant condition except when $\eta$ is low and in the vicinity of 0.1.

From the work carried out in the past it is clear that the behaviour of the variable inertia system is far from understood and no definite information exists which can be used to predict positively the possibility of failure due to the effect. That establishes the need for further work on this aspect of torsional vibrations. With the availability of computers it is possible to get a complete solution of the equation which forms the theoretical basis of the present investigation.

Various terms of the equation are investigated for their effect on the waveforms of the responses of the system and it is found that all the terms in the equation have particular effects on the solutions. No term should be omitted as it is done by Draminsky, without a careful appraisal of its importance at a particular value of $r$ or $\eta$.

The responses of the equation at different speeds of rotation are determined and the harmonic analysis of these solutions contradicts Draminsky's deduction that with the presence of the $n$th order main component in the waveform, when the system executes natural vibrations, only $(n - 2)$ and $(n + 2)$ order secondary components are obtained. The present analysis indicates that the orders of the components have to
be determined from the harmonic analysis of the waveform solution of the equation of motion at definite speed of rotation.

The effects of external excitations acting on the single cylinder engine system are investigated which offer an explanation as to why secondary resonance contributes to the crankshaft failures only in some multi-cylinder engines and not in others.

Various other characteristics of the variable inertia system and the complete solutions of the equation of motion including damping and the external excitations are explored in the present work.
2.1. Introduction

In recent years several cases of fractures in the crankshafts of large marine engines occurred after the engines had been in service for up to ten years. After repairs, measurements taken in some cases indicated torsional vibration stresses at service speed up to five times greater than expected. These crankshaft failures are thought to be due to the existence of large second order variations in inertia. It should be noted that a crankshaft-connecting rod system is a vibrating system with varying inertia. The total effective inertia of the crank assembly varies twice per revolution due to variation of the inertia of the reciprocating parts. In the above case of crankshaft failures an external excitation of order \( n \) is in direct or nearly in direct resonance with the service speed of the engine but the resulting equilibrium amplitude is small. Due to the second order variation in inertia of the system the \( n \)th order vibration at or near the service speed is said to have been magnified by the external excitations of order \((n - 2)\) and \((n + 2)\) in secondary resonance depending on their equilibrium amplitudes. It is suggested by Draminsky that in practice secondary resonance occurs only for resonance with the lower-order secondary component \((n - 2)\). The equilibrium amplitude of the external excitation of order \((n - 2)\) is normally of no great significance since its corresponding critical speed is much higher than the service speed of the engine. Failure has not occurred in all cases of engines in service. In view of this it is considered essential to carry out further investigations on the effects of variable inertia.
in reciprocating engines.

A new ten-cylinder, two stroke cycle engine is examined by the author on its trial run for secondary resonance effects. The engine has the following specifications:

- Number of cylinders = 10
- Bore = 900 mm
- Stroke = 1550 mm
- b.h.p = 29000
- r.p.m = 122
- m.i.p = 169.5 lb/in²
- Mean effective pressure = 154.3 lb/in²
- One node natural frequency = 299 c.p.m
- Two node natural frequency = 1024 c.p.m
- Firing order = 1-9-5-6-2-10-4-3-8-7

The particulars of the equivalent dynamic system are given in Table 2.1.

2.2. Conventional Stress Calculations

The following conventional analysis based on Lloyd's Register of Shipping recommendations is carried out to predict the vibration characteristics of the engine system prior to the trials. The particulars of the dynamic system of the engine neglecting the effects of variable inertia are given in Table 2.1, together with the other relevant data for use in subsequent calculations. The natural torsional frequencies of the installation together with their associated modal characteristics are calculated by use of the Holzer tabulation technique and this tabular scheme is used as the basis for predicting the positions of resonant criticals within, and in the proximity of,
<table>
<thead>
<tr>
<th>S.No</th>
<th>Mass</th>
<th>Inertia</th>
<th>Flexibility</th>
<th>Diameter of shafting</th>
<th>Section modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Kg.cm.sec²</td>
<td>x 10⁻¹⁰ rad/kg.cm</td>
<td>cm</td>
<td>in³</td>
</tr>
<tr>
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<td>70.00</td>
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<td>70.00</td>
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</tr>
<tr>
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<td>70.00</td>
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</tr>
<tr>
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<td>0.766</td>
<td>70.00</td>
<td>4109.8164</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
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<td>—</td>
<td>69.40</td>
<td>4005.0391</td>
</tr>
</tbody>
</table>
the operating speed range. The method of calculation is based on a simplified semi-empirical treatment of the system as a whole, excitation being applied in the form of a vectorial summation of engine exciting torques for each harmonic order. The effects of various sources of damping are expressed as a combined dynamic magnifier.

2.3.1. Definition of Symbols

Each torsional mode of vibration, with a natural frequency $F_{o.p.m}$, is excited by the harmonic components of the applied torque, the order number of each harmonic being denoted by $n$. The numerous critical speeds associated with these harmonic orders of excitation are given by

$$N_0 = \frac{F}{n} \text{ r.p.m}$$

The information about the vibratory torques in the shaft sections of the equivalent system when the amplitude at the reference mass is 1 radian is given in the natural frequency tabulation. The stress in each shaft for one radian amplitude at the reference mass can therefore be obtained by dividing these torques by the torsional moduli of the various shaft cross-sections. These stresses are called the specific stresses since they are the values for unit amplitude at the reference mass. Therefore,

$$f_{s} = M \Theta_0 f$$

where

$M = \text{dynamic magnifier for the whole system for the mode of vibration under consideration}$,

$$\Theta_0 = \frac{\varepsilon A aY\Delta}{\omega_n^2 \ddot{x}(I\Delta^2)} \text{ rad}^2,$$
\[ T_m = \text{resultant nth order harmonic component of tangential effort at each crankpin, expressed per unit area of piston,} \]

\[ A = \text{area of each piston,} \]

\[ a = \text{crank radius,} \]

\[ \overrightarrow{\sum A} = \text{phase-vector sum for all engine cylinders, derived from the relative amplitudes given in the frequency Table 2.2 and Table 2.5 and the nth order phase angle diagram,} \]

\[ \omega_n = \text{natural frequency of the mode of vibration under consideration, in rad/sec,} \]

\[ I = \text{inertia,} \]

\[ I(I\Delta^2) = \text{arithmetic sum of the products of the polar moments of inertia of the masses and the squares of the ordinates at the masses on the relevant normal elastic curve, obtained from the natural frequency table, for the whole system,} \]

\[ f = \text{specific stress, lb/in}^2 \text{ per radian amplitude at the reference mass for the mode of vibration under consideration,} \]

\[ f_s = \text{shear stress amplitude, for the mode of vibration under consideration, lb/in}^2 . \]

2.3.2. \textbf{Dynamic Magnifiers}

(a) The method described above for calculating magnitudes of vibration at critical speeds involves the use of a dynamic magnifier applied to the system as a whole. The more important sources of damping
are determined individually. The partial dynamic magnifiers for these separate sources of damping are then combined on an empirical basis to give an overall magnifier for the system.

The dynamic magnifier associated with the effects of damping arising within the engine, \( M_E \), is expressed as a function of \( \Theta_o \),

\[
M_E = 3.8 \Theta_o^{\frac{1}{4}}
\]

In practice, the value of \( M_E \) so obtained is limited to a maximum of 50 .

(b) Propeller damping, which assumes importance in the case of shafting modes of vibration of propulsion machinery, is taken into account by the use of a propeller dynamic magnifier, \( M_P \), determined from the following formula:

\[
M_P = \frac{\frac{1}{2} \sum (I \Delta^2) N_s^3 n}{18 \times 10^6 \frac{H \Delta_p^2}{N}} = M_K n
\]

where

\[
M_K = \frac{\frac{1}{2} \sum (I \Delta^2) N_s^3}{18 \times 10^6 H \Delta_p^2}
\]

\( \sum (I \Delta^2) = \) summation of \((I \Delta^2)\) terms for the whole system, lb.in.sec²,

\( N_s = \) maximum continuous engine speed, r.p.m,

\( H = \) rated brake horsepower at the maximum continuous engine speed,

\( \Delta_p = \) relative modal amplitude at the propeller, rad,

\( n = \) order number.
(c) The overall dynamic magnifier, M, for the system as a whole is obtained by combining the magnifiers, estimated for the individual sources of damping, in the following empirical manner:

\[ M = \left[ \left( \frac{1}{M_e} \right)^2 + \left( \frac{1}{N_p} \right)^2 \right]^{-\frac{1}{2}} \]

The procedure outlined above provides means for predicting magnitudes of vibration at resonant criticals in the operating speed range.

2.3.3. Flank Stresses

The dynamic magnifier for the flank condition is calculated from the formula:

Flank magnifier (FM) = \[ \left[ \left( 1 - \left( \frac{N}{N_c} \right)^2 \right)^2 + \left( \frac{N}{N_c} \right)^2 \cdot \frac{1}{M_e^2} \right]^{-\frac{1}{2}} \]

where

FM = flank magnifier
M = overall dynamic magnifier
\( \theta_o \) = equilibrium amplitude at resonance
\( \theta_{OF} \) = equilibrium amplitude determined by use of new value of harmonic component of tangential effort at flank speed.

It should be noted that the ratio of \( \theta_{OF} \) to \( \theta_o \) is equal to the ratio of harmonic component of tangential effort at flank speed to harmonic component of tangential effort at resonance speed. The harmonic components of tangential effort are related to mean indicated pressure (M.I.P) which is determined from the relation

\[ \text{M.I.P} = \text{(rated mean indicated pressure)} \times \left( \frac{N_o}{N_s} \right)^2 + \text{MFP} \]
where \( \text{MFP} = (\text{rated mean indicated pressure}) - (\text{mean effective pressure}) \)

\( N_c \) = critical speed

\( N_s \) = service speed

This relation is not applied to critical speeds higher than the service speed. At these higher speeds the mean indicated pressure used to determine the components of tangential effort is the rated pressure.

2.4. Calculation of Stress according to Draminsky

The calculations in the Table 2.2 to Table 2.7 are carried out neglecting the second order variation in inertia which occurs in the crank-connecting rod system due to the variation in inertia of the reciprocating parts. Allowing for these variations in inertia the vibratory motion of the system can be represented by the equation (1,1). Making use of this equation Draminsky\(^{32,33}\) explained that if \( \omega_n = n\omega \) and the system was executing natural vibrations of phase velocity \( \omega_n \) the waveform would be composed of a principal component of order \( n \) and two secondary components of order \( (n - 2) \) and \( (n + 2) \). Draminsky further added that the free state secondary components of order \( (n - 2) \) and \( (n + 2) \), could be excited by external excitations of one of these orders when the principal vibration of order \( n \) was at or near resonance provided the system had large second order inertia variation. All three components were present in the free vibration of the system and when an external excitation of order \( n \) was in resonance with the principal vibratory component of free motion, external excitations of order \( (n - 2) \) and \( (n + 2) \) could be considered in secondary resonance since they put energy into the system although to a much less degree
than the external excitation of order \( n \). It was also suggested that
in practice secondary resonance in torsional vibration occurred only
for resonance with the lower-order secondary component \( (n-2) \).

He developed a method of calculation\(^{31}\) based on a non-linear
time to investigate the magnification of the principal torsional
vibration stress when the external excitation of order \( n \) was in direct
or nearly in direct resonance and the external excitation of order
\( (n-2) \) was in secondary resonance. The external excitation of order
\( (n-2) \) became significant because of large variation of inertia in
the system which was considered unimportant in the conventional
procedure of calculations since the corresponding critical speed
would be situated well above the service speed.

The engine under investigation has 2-node mode 9th order vibration
almost in resonance with the service speed and the geometric resultant
of the ten torque impulses of the 9th order is very small. The proce-
dure of secondary resonance calculations incorporating the theory
of fictive forces is used to determine the 2-node mode 9th order
resonant stress. According to Draminsky the stress at the service
speed will occur due to the direct resonance of the 9th order and
the secondary resonance of the 7th order external excitations.

**Second order vector sum of \( \eta \beta^2 \) (see Fig. 2.1)**

Vertical component = \( 0.3 + 0.3 \times 0.99475 + \cos 72^\circ (0.24679 \\
+ 0.82624 + 0.25686 + 0.81598) \times 0.3 - \\
\cos 36^\circ (0.54721 + 0.05303 + 0.53513 + \\
0.04808) \times 0.3 \\
= 0.51364 \)
Fig. 2.1. (a) Crank arrangement

Fig. 2.1. (b) Second order vector diagram
Horizontal component = \( \cos 18^\circ (0.25686 + 0.81598 - 0.24679 - 0.82624) \times 0.3 + \cos 54^\circ (0.54721 + 0.05303 - 0.53513 - 0.04808) \times 0.3 \)

\[ = -0.01606 \]

\[ \sum \eta \Delta^2 = \sqrt{(0.51364)^2 + (0.01606)^2} \]

\[ = 0.5139 \]

For two node mode vibration form \( \beta = \frac{\text{Resultant} (\eta \Delta^2)}{\sum \eta_{rel} \Delta^2} = 0.0894 \)

The calculations for the factors used to determine the coefficient of total mass variation (\( \beta \)) in 2-node mode are given in Table 2.8.

The 2-node mode direct resonance equilibrium amplitude of 9th order,

\[ \theta_{09} = 15.66 \times 10^{-6} \text{ rad.} \]

The 2-node mode equilibrium amplitude of the 7th order

\[ \theta_{07} = 223.67 \times 10^{-6} \text{ rad.} \]

Factor \( k = \frac{\beta n (n-2)}{8 (n-1)} = \frac{0.0894 \times 9 \times 7}{8 \times 8} \)

\[ = 0.088 \]

The 9th order fictive amplitude,

\[ b_9 = k \times \theta_{07} \]

\[ = 19.68 \times 10^{-6} \]

Neglecting phase and other vibration forms, the total 9th order amplitude,

\[ \theta_{0T} = (15.66 + 19.68) \times 10^{-6} \]

\[ = 35.34 \times 10^{-6} \]
The 9th order stress in no. 6 shaft using damping coefficient $\rho = 0.02$ that is the dynamic magnifier of 50 is given by

$$\frac{T_n}{Z} \times \frac{0^T}{\rho} = \frac{3875.5 \times 10^6 \times 35.34 \times 10^{-6}}{4109.815 \times 0.02} \times 2.2 \times 2.94$$

$$= \pm 1445 \text{ lb/in}^2$$

where $T_n = 3875.5 \times 10^6 \text{ kg/cm}$ (see Table 2.5 for torque in shaft no. 6)

$Z = 4109.816 \text{ in}^3$ (see Table 2.1 for section modulus of shaft no. 6)

The variation of the 2-node mode 9th order stress in the crankshaft calculated by damped-forced vibration tabulation is shown in Fig. 2.2 with a resonant stress of $\pm 640 \text{ lb/in}^2$. The dynamic magnifier method also gives the same value of stress in the crankshaft (see Table 2.6). The corresponding stress worked out from a Draminsky calculation based on non-linear theory would be $\pm 1445 \text{ lb/in}^2$ pointing to the strong possibility of large 9th order torsional vibrations being excited due to secondary resonance.

The conventional flank stress calculations for the 2-node mode 5th, 7th, and 9th harmonic orders at service speed (see Table 2.7) give the following results:

<table>
<thead>
<tr>
<th>Order</th>
<th>Stress ($\pm \text{lb/in}^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-node 9th</td>
<td>90</td>
</tr>
<tr>
<td>2-node 7th</td>
<td>600</td>
</tr>
<tr>
<td>2-node 5th</td>
<td>765</td>
</tr>
<tr>
<td>Arithmetical sum</td>
<td>1455</td>
</tr>
</tbody>
</table>

* The calculations by the damped-forced vibration tabulation method are carried out by use of the computer programme provided by Lloyd's Register of Shipping, London.
Fig. 2.2  2-Node 9th order stress in crankshaft
It is by coincidence\textsuperscript{30} that the Draminsky 9th order calculated stress agrees closely in value with the arithmetical sum of the above flank stresses. The calculations are carried out by two methods based on different considerations and also in Draminsky's case, the 9th order stress would predominate.
<table>
<thead>
<tr>
<th>Mass no.</th>
<th>$\Delta$</th>
<th>$\Sigma I_2^2 \Delta$</th>
<th>$\Delta^2$</th>
<th>Specific stress ($f$)</th>
</tr>
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<tr>
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<td>1.00032</td>
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$\Sigma I_2^2 = 2073342.000 \text{ kg.cm.sec}^2$

$\Sigma I_2^2 = 1799560.000 \text{ lb.in.sec}^2$

$K_k = 3.007$

$M_p = K_k \times n$
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<th>$N_C/N_S$</th>
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<th>Phase-vector sum $\Sigma A$</th>
<th>$T_{m/2}$</th>
<th>Equilibrium amplitude $x 10^{-6}$ $\theta_0$ rad.</th>
<th>$M_E$</th>
<th>$M_P$</th>
<th>$M$</th>
<th>Maximum relative stress in intermediate shaft $M \theta_o f$ lb/in$^2$</th>
<th>Permissible vibration stress lb/in$^2$</th>
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<td>( T_m/2 ) At service speed lb/in(^2)</td>
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<td>Resonant stress in intermediate shaft lb/in(^2)</td>
<td>Flank stress lb/in(^2)</td>
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TABLE 2.5  TWO NODE NATURAL FREQUENCY TABULATION

Two node frequency \[= 1024 \text{ c.p.m.} \]
\[\omega_n^2 = 11500.6\]

<table>
<thead>
<tr>
<th>Mass no.</th>
<th>(\Delta) rad.</th>
<th>(\Sigma I\omega^2\Delta) (\times 10^6\text{ kg.cm.})</th>
<th>(I\Lambda^2) kg.cm.sec(^2)</th>
<th>Specific stress ((f)) (\times 10^6) lb/in(^2)</th>
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\[\Sigma I\Lambda^2 = 578804.000 \text{ kg.cm.sec}^2\]
\[\Sigma I\Lambda^2 = 502374.063 \text{ lb.in.sec}^2\]

Propeller damping is negligible
### Table 2.6: Stresses at Two Node Resonant Conditions

<table>
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<th>Order no.</th>
<th>( N_C ) r.p.m</th>
<th>( N_C/N_S )</th>
<th>MIP ( \text{lb/in}^2 )</th>
<th>Phase-vector ( \sum \Sigma A \text{rad} )</th>
<th>Equilibrium amplitude ( T_m/2 \text{lb/in}^2 \times 10^{-6} )</th>
<th>( M_E )</th>
<th>( M )</th>
<th>Maximum relative stress in shaft no. 6 ( \text{lb/in}^2 )</th>
<th>Permissible vibration stress ( \text{lb/in}^2 )</th>
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<tbody>
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<td>2.798</td>
<td>169.5</td>
<td>3.69</td>
<td>20.01</td>
<td>769.72</td>
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<td>22.8</td>
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<td>( T_{m/2} ) at service speed ( \text{lb/in}^2 )</td>
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<td>Flank stress ( \text{lb/in}^2 )</td>
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CHAPTER 3

EXPERIMENTAL MEASUREMENTS ON THE ENGINE

During the conventional stress calculations and the calculation of 2-node 9th order stress according to Draminsky, it is felt desirable to verify by measurement on the engine whether or not secondary resonance exists. The block diagram of the monitoring system used to measure torsional vibrations is shown in Fig. 3.1.

3.1. Description of Monitoring System

(a) Torsional vibration pick up

The torsional pick up unit (Consolidated Electrodynamics Corporation type 9-102) is an electromagnetic, self-energising unit and is fixed at the front end of the engine crankshaft. The pick up mounted on to the shaft under investigation is shown in Fig. 3.2. The torsigraph is based on the seismic principle and the seismic assembly is mounted on precision ball-bearings, assembled without end play, and is coupled to the casing by weak springs to provide a low natural frequency of the flywheel/spring system. The relative motion is damped by silicon fluid which also acts as a lubricant and has the advantage of considerably reduced sensitivity to temperature variations. These features ensure that the response of the instrument to axial and transverse vibration is negligible, while the torsional response shows excellent amplitude linearity. The output is taken through solid-silver slip-rings and silver-graphite brushes.

The natural frequency of the seismic system determines the
lowest frequency which the instrument is capable of recording with reasonable accuracy. The natural frequency of the mass-spring system should be not more than half the frequency of the lowest significant harmonic component of the applied motion. The upper frequency limit is determined by the characteristics of the pick up unit and its associated amplifying and recording equipment.

The principal operational characteristics are as follows:

Open circuit instantaneous output $9 \times \dot{\theta}$ mv

where $\dot{\theta}$ = instantaneous velocity deg./sec

Natural frequency of (Flywheel/Spring) system 3 c.p.m

Temperature range 0 to 150°F

Frequency response 10 to 1000 c.p.m (usable down to 4 c.p.m)

Amplitude response $\pm 0.05$ to $\pm 2^\circ$

Damping is by silicon fluid at 64% of critical damping.

(b) 6-Channel integrating amplifier

The amplifier unit is manufactured by Lloyd's Register of Shipping and is designed to provide a high degree of mechanical and electrical stability in the hostile environments to which it would be subjected in use at sea.

An electromagnetic pick up unit produces an output voltage which is directly proportional to the relative angular velocity between the armature and the magnetic assemblies. Therefore the voltage developed by the pick up is proportional to the product of frequency and amplitude. This implies that an oscillogram of the
Fig. 3.1  Block diagram of monitoring system for measurement of torsional vibration
Fig. 3.2. Torsional vibration pick-up fixed at the front end of the engine

Fig. 3.3. Set of recording instruments forming part of the overall monitoring system
resultant voltage would not correspond precisely to the applied
motion in waveform and amplitude. Also, if the total motion consists
of several harmonic components the high-frequency components will
be recorded in a considerably magnified form. To overcome this
difficulty an integrating circuit is used to produce the waveform
whose ordinates are proportional to displacement. Its action depends
on the fact that the voltage across the terminals of a condenser
is proportional to the time integral of the current flowing into
the condenser.

Thus if the current flowing into the condenser is made propo-
rtional to the output voltage from the pick-up unit the amplifier
voltage across the condenser will produce a record of vibratory
displacements and hence an electrical integrating circuit is essen-
tially a resistance-capacitance filter. The unit is battery operated
to eliminate extraneous effects from changes in supply line voltage.
The amplifier contains the normal controls for the alteration of
gain, calibration etc. Within the amplifier is contained a fixed
frequency stable oscillator the output of which could at any time
be substituted for that of the transducer in order to provide a
check on the calibration (that is on change of amplifier gain or
galvanometer sensitivity in the ultra violet recorder). Such tests
are carried out during the test and no changes in gain of the moni-
oring system are observed.

(c)  **Recording unit**

An ultra violet galvanometer recorder fitted with moving coil
galvanometers having a natural frequency of 250 c.p.s is used in the
monitoring system. The galvanometer assemblies are damped with silicon fluid, adjusted to provide about 64% of the critical damping. This enables a flat frequency response to be obtained to within 60% of the natural frequency (that is from 0 to 150 c.p.s). The galvanometers with an undamped natural frequency of 250 c.p.s are selected from the consideration that the natural frequency of the galvanometers should be no higher than is necessary to cover the frequency range over which measurements are required since the amplitude sensitivity diminishes significantly with increase of natural frequency.

An electric current passing through the galvanometer coil produces rotation of the coil, and hence the mirror mounted on it, proportional to the characteristics of the incoming signal. These movements are magnified by an optical arm and are recorded on direct print-out ultra-violet photo-sensitive paper as a wave-trace of varying amplitude displayed on a time-base. The time-base is provided by transporting the paper chart in a direction at right-angles to the oscillations of the light spot. Signals from a torsional vibration pick-up fixed on the front end of the crankshaft and signals derived from strain gauges based on the intermediate shaft are displayed on the recorder. The recorder also carries a half second time marker and shaft revolution marker.

(d) Installation of strain gauges and slip ring unit on intermediate shaft

The maximum tensile and compressive strains of a shaft in torsion lie along 45° helices on the shaft surface. Because of this two pairs of strain gauges fixed diametrically opposite on the shaft
surface are connected in the form of a wheatstone-bridge circuit. This system results in automatic temperature compensation for all gauges and the effects of any axial and bending strains are eliminated. For elimination of the nontorsional strains, considerable care must be exercised in mounting the gauges at precisely 45° to the shaft axis. To achieve the correct orientation, the gauges are encapsulated in a mixture of araldite, plasticizer and hardener. The bridge output is four times the strain of one gauge since four active gauges are used. The gauges are of Micro Measurements. The resistance of each gauge is 120 ohm and the gauge factor is 2.095. The gauges are fixed to the shaft using Philips strain gauge cement type PR 9244/04. The output from the strain gauge bridge is carried via a slip ring unit to the recording equipment.

(e) **Slip rings and brush gear**

The slip rings consist of re-inforced neoprene rubber strip 1/8 in. thick and approximately 3 in. wide to which shim brass strips are fixed. The cement is applied on the inside surface of the strip and wrapped around the intermediate shaft. The brass strips are soldered in an overlapping joint and filed to give a fine smooth finish. The brushes are of silver graphite and the pressure between the slip rings and brushes is adjusted for optimum noise free performance. Fig. 3.3 shows the recording instruments which form part of the overall monitoring system. The arrangement of slip rings and brush gear on the intermediate shaft are shown in Fig. 3.4 and Fig. 3.5.
Fig. 3.4 Arrangement of slip rings and brush gear used for measurements on the intermediate shaft of the engine

Fig. 3.5 Arrangement of slip rings and brush gear used for measurements on the intermediate shaft of the engine
A Sperry Massachusetts Institute of Technology vibrometer calibrator making use of the Hooke's joint mechanism is used for the calibration of the torsional vibration pick up. The Hooke's joint calibrator is usually preferred for torsiograph calibration since it enables distortion due to the inertia of the members of the torsiograph transmission system and its associated control springs to be taken into account since it is provided with driving and driven shafts of adequate diameter to avoid shaft twists due to the reaction torque of the torsiograph inertias particularly at high frequencies. The Hooke's joint is of interest in the present discussion. When the input shaft to a single universal joint rotates at substantially uniform angular velocity the output shaft executes almost pure sinusoidal torsional vibrations having a frequency equal to twice the r.p.m of the input shaft provided that the angle between the input and output shafts does not exceed about 10°. The output from a Hooke's joint assembly therefore provides a means for calibrating instruments used for measuring torsional vibration. The amplitude of the superimposed torsional vibration varies from zero, when the input and output shafts are in line, to a value which increases as the angle between the shafts is increased.

3.2. Derivation of the Hooke's Joint Calibration formula

From the general theory of the Hooke's joint, the relationship between the angular motion \( \lambda \) of the driving shaft and the corresponding angular motion \( \beta \) of the driven shaft is given by

\[
\tan \lambda = \cos \alpha \tan \beta
\]
where \( \alpha \) = angle between the driving and the driven shafts.

Differentiating both sides of equation (3.1) with respect to time, and rearranging terms gives

\[
\frac{d\beta}{d\lambda} = \frac{\cos \alpha}{(1 - \sin^2 \alpha \cos^2 \lambda)}
\]

(3.2)

If \( H = \frac{\cos \alpha}{(1 - \frac{1}{2} \sin^2 \alpha)} \) and \( s = \frac{\sin^2 \alpha}{(2 - \sin^2 \alpha)} \)

then equation (3.2) can be expressed as

\[
\frac{d\beta}{d\lambda} = H (1 - s \cos 2\lambda)^{-1}
\]

(3.3)

Writing equation (3.3) in terms of multiples of \( \lambda \) by the use of de Moivre's theorem we get

\[
\frac{d\beta}{d\lambda} = H \left[ (1 + \frac{s^2}{2} + \frac{3s^4}{8} + \ldots) + (s + \frac{3s^3}{4} + \ldots) \cos 2\lambda \right.
\]

\[
+ \left( \frac{s^2}{2} + \frac{s^4}{2} + \ldots \right) \cos 4\lambda + \left( \frac{3s^3}{4} + \ldots \right) \cos 6\lambda
\]

\[
+ \left( \frac{s^4}{8} + \ldots \right) \cos 8\lambda \right]
\]

(3.4)

From equation (3.2a)

\[
\sin^2 \alpha = \frac{2s}{(1 + s)}
\]

hence

\[
H = (1 - s^2)^{\frac{1}{2}}
\]
Expanding the above expression for $H$ by the binomial theorem

$$H = (1 - \frac{s^2}{2} - \frac{s^4}{8} - \cdots)$$  \hspace{1cm} (3.5)

Substituting equation (3.5) into (3.4), and rearranging the terms,

$$\frac{d\theta}{d\lambda} = \left[ 1 + s \left( 1 + \frac{s^2}{4} + \cdots \right) \cos 2\lambda + 
\quad s^2 \left( \frac{1}{2} + \frac{s^2}{4} + \cdots \right) \cos 4\lambda + \frac{s^4}{4} \left( \frac{1}{4} + \cdots \right) \cos 6\lambda
\quad + \frac{s^6}{8} \left( \frac{1}{8} + \cdots \right) \cos 8\lambda + \cdots \right]$$  \hspace{1cm} (3.6)

Limiting the range of $\alpha$ to within $30^\circ$, the value of $s$ is less than 0.15 and hence $s^3$ and higher powers of $s$ can be neglected. Equation (3.6) then reduces to

$$\frac{d\theta}{d\lambda} = \left[ 1 + s \cos 2\lambda + \frac{s^2}{2} \cos 4\lambda \right]$$  \hspace{1cm} (3.7)

The amplitude of torsional vibration of output shaft is obtained by integrating equation (3.7) and subtracting the steady component $\lambda$ from the fluctuating component $\beta$, giving the following equation

$$\beta - \lambda = \left[ \left( -\frac{s^2}{2} \right) \sin 2\lambda + \left( -\frac{s^4}{8} \right) \sin 4\lambda \right] \text{ rad.}$$  \hspace{1cm} (3.8)

The 4th order amplitude is under 1% of the 2nd order amplitude when the latter is about $41^\circ$, increasing to under 3.5% when the
2nd order amplitude is $\pm 4^\circ$. The amplitudes to be recorded by a torsiograph rarely exceed $\pm 2^\circ$ and are usually nearer $\pm 1^\circ$ or less and hence an $\alpha$ of $15 - 20^\circ$ will fulfil most practical requirements. The 4th order harmonic component can be safely disregarded giving the amplitude of the 2nd-order vibration as

$$\lambda(2) = (\beta - \lambda) = \pm \frac{\theta}{2}$$ (3.9)

A torsional vibration pick-up does not produce any output for uniform rotation, but only indicates deviations from the steady condition. When fitted to a Hooke's joint calibrator the pick-up will indicate only $\lambda(2)$ (the relative deviation) with a frequency equal to twice the speed of rotation of the pick-up shaft. Fig. 3.6 gives the graph between the index setting ($\alpha$) and the resulting torsional amplitude in degrees.

3.3 Calibration of Torsional Vibration Pick-up

The complete arrangement of the calibration system is shown in Fig. 3.7. The shaft carrying the torsional vibration pick-up is supported in the bearing bracket which is arranged to swing about the centre located perpendicularly below the Hooke's joint. The shaft carrying the pick-up can thus be set at various angles to the driving shaft in order to obtain any desired value of vibration amplitude. To reduce cyclical variation a heavy flywheel is fitted on the driving shaft. The drive is taken from the motor through a gear box via a belt and pulley arrangement to the driving shaft.

The test procedure is to spin the flywheel up to the speed
Fig. 3.6. Calibrator index setting, and resulting torsional amplitude
Fig. 3.7  The arrangement used for calibration of torsional vibration pick-up
(1000 r.p.m in this case) which corresponds to the highest torsional vibration frequency (2000 c.p.m) for which the calibration is required. The motor is then declutched from the flywheel and when the flywheel is running down in speed a record of the integrated output of the transducer is taken on the ultra violet recorder. Revolution marks are also recorded on the film. To facilitate analysis the fine gain control on the amplifier is adjusted so that a convenient trace width corresponding to the angular displacement (say 2 cm. for \( \frac{\pi}{8} \) double amplitude) is obtained. The fine gain control is then locked in position and is not disturbed thereafter. A signal at a suitable level from the calibration oscillator within the amplifier is then injected into the amplifier and the trace is observed. A check of the trace produced by the injection of the same signal at any time during the tests will indicate any change in sensitivity. The calibration curve, Fig. 3.8, is plotted from the readings taken. The inertia of the rotating parts is sufficient to enable records to be taken at regular but falling intervals of shaft revolutions. Calibration curves are obtained in the calibration tests carried out before and after the trial.

3.4. Calibration of Strain Gauges

The output of the strain gauge amplifier is an electrical signal whose magnitude depends on the strain to which the gauge is subjected and some means of judging its absolute magnitude must be provided. The calibration method used involves producing a known change in resistance of the strain gauge by means of resistors connected in parallel with one of the active gauges as shown in Fig. 3.9.
Fig. 3.8. Calibration of torsional vibration pick-up
Fig. 3.9. Calibration technique for measuring dynamic strain
Putting a suitable shunt resistor across the strain gauge by use of a switch on the amplifier results in a change of resistance of the shunted arm of the bridge circuit which appears in the form of a step on the record. With a knowledge of the amplifier gain setting, gauge resistance, gauge factor and shunt resistance the strain and consequently stress corresponding to the height of the step on the record can be determined.

Letting

\[ R_C = \text{calibrating shunt resistance} \]
\[ R_g = \text{strain gauge resistance} \]
\[ K_s = \text{gauge factor} \]
\[ S_t = \text{strain} \]

the change in resistance \( \Delta R \) is given by

\[ \Delta R = R_g - \frac{R_C \times R_g}{R_C + R_g} \]

and thus

\[ \Delta R = \frac{R_g^2}{R_C + R_g} \]  \hspace{1cm} (3.10)

Combining equation (3.10) with \( S_t = \frac{\Delta L}{L} = \frac{1}{K_s} \times \frac{\Delta R}{R} \)

gives

\[ S_t = \frac{1}{K_s} \left[ \frac{R_g}{R_C + R_g} \right] \approx \frac{R_g}{K_s \times R_C} \]  \hspace{1cm} (3.11)

since \( R_C \) is large compared to \( R_g \).

When the strain gauges are fixed on a shaft as already described to measure the torsional vibration characteristics, the strain
measured by the strain gauge in the direction of one of the normal stresses, say tensile, is composed of the direct strain due to tensile stress and the component of lateral strain due to the compressive normal stress. Since compressive normal stress is of equal magnitude but of opposite sign to tensile stress it follows that the torsional stress corresponding to the measured strain, \( e \), is given by the following expression

\[
\sigma_s = \frac{e E}{1+\mu}
\]

where \( E \) = Young's modulus and \( \mu \) = Poisson's ratio

Therefore the torsional stress \( \sigma_s \) = \[
\frac{R_s \times E}{K_s \times R_C \times (1 + \mu)}
\] (3.12)

The calibration referred to above was carried out with the stationary shaft before and after the trial. It will be appreciated that during the test it is necessary to re-adjust the bridge balance control (resistive) to back off the direct current output corresponding to the steady torque in order to operate the amplifier at a particular sensitivity. Different calibration settings can be obtained by use of the attenuators on the amplifiers. On both the integrating amplifier and the strain gauge amplifier the attenuators used are of the same type. Changing the setting from \( 'B_s' \) to \( 'C_s' \) for example, will reduce the trace deflection by half. In other words there is a factor of 2 between adjacent steps on the attenuator with setting \( 'A_s' \) being the most sensitive.
Sensitivity setting "E"

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<th>Parameter</th>
<th>Value</th>
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<td>Number of active gauges</td>
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<tr>
<td>Resistance of each strain gauge</td>
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<tr>
<td>Shunt resistance R_C</td>
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</tr>
<tr>
<td>Step deflection on the record</td>
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</tr>
<tr>
<td>µ</td>
<td>0.3</td>
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<tr>
<td>K_S</td>
<td>2.095</td>
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</tbody>
</table>

Torsional stress \( f_s \) = \( \frac{R_e x E}{K_S x R_C x (1 + \mu)} \) = \( \frac{120 x 30 x 10^6}{2.095 x 10^6 x 1.3 x 4} \)

= \pm 165 \text{ lb/in}^2 \text{ for 2.1 mm}

= \pm 78 \text{ lb/in}^2 \text{ per mm}

3.5. **Carrier Wave Strain Indicating System**

An audio-frequency electronic oscillator is used to obtain a carrier wave signal. The oscillator output is modulated by the strain signal from the bridge circuit. The alternating voltage is amplified in the 6-channel strain amplifier and presented for indication on an ultra violet recorder. The system is shown in Fig. 3.10 and the method of calibration is already described.

3.6. **Experimental Results**

The stress records taken on the intermediate shaft and the amplitude records obtained from the torsional pick-up fixed at the front end of the ten-cylinder, two stroke cycle engine are examined.
for 2-node mode 9th order vibration. The actual records are enlarged five times for their harmonic analysis which is carried on with the use of D-mac plotter. Two consecutive stress and amplitude cycles at each speed are analysed and the averages of those values determined. The results of the analysis are given in Table 3.1 and Table 3.2. In Table 3.3 are given the theoretical results calculated from damped-forced frequency tabulation for 2-node mode 9th order torsional amplitude at front end of the engine and stress in intermediate shaft. The 2-node mode 9th order measured stress obtained by harmonic analysis of stress records taken on the intermediate shaft and the corresponding stress calculated from the damped-forced vibration tabulation are given in Fig. 3.11. The measured stress at resonance is only slightly higher than the calculated value.

The 2-node mode 9th order measured torsional single amplitude at the front end of the engine and the corresponding torsional amplitude calculated from the damped-forced vibration tabulation are plotted against engine speed in Fig. 3.12. It is interesting to note on examination of Fig. 3.11 and Fig. 3.12 that the variation of 2-node 9th order stress in the intermediate shaft against engine speed and the variation of 2-node 9th order torsional amplitude at the front end of the engine against speed follow a similar pattern and it is clear that in this case stress magnification due to secondary resonance fails to appear despite the conditions pointing to the strong possibility of large 9th order torsional vibrations being excited.

From the above observations it is concluded that further theoretical work on the basic equation representing the linear motion is required.
Fig. 3.10. An A.C. powered bridge system for indicating dynamic strains
Fig. 3.11. Stress in intermediate shaft plotted against r.p.m for the 2-node 9th order torsional vibration

- experimental
- theoretical
Fig. 3.12. Amplitude at front end of the engine plotted against r.p.m for the 2-node 9th order torsional vibration

--- experimental

--- theoretical
TABLE 3.1. TWO NODE NINTH ORDER STRESS IN INTERMEDIATE SHAFT

\[ f_s = \pm 78 \text{ lb/in}^2 \text{ per mm of double amplitude} \]

Magnification of records = 5

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<tr>
<th>Shaft speed</th>
<th>Magnified single amplitude by harmonic analysis</th>
<th>Actual single amplitude</th>
<th>Double amplitude</th>
<th>2-Node 9th order stress in intermediate shaft</th>
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<tr>
<td>mm</td>
<td>mm</td>
<td>mm</td>
<td>mm</td>
<td>1b/in^2</td>
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<tr>
<td>r.p.m</td>
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<tr>
<td>110.0</td>
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<td>( \pm 118.56 )</td>
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<td>3.52</td>
<td>( \pm 274.0 )</td>
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<td>( \pm 243.0 )</td>
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<td>2.58</td>
<td>( \pm 201.24 )</td>
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<td>( \pm 124.0 )</td>
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<tr>
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<td>1.42</td>
<td>( \pm 110.0 )</td>
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<tr>
<td>118.5</td>
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<td>1.42</td>
<td>( \pm 110.0 )</td>
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<tr>
<td>120.0</td>
<td>1.90</td>
<td>0.38</td>
<td>0.76</td>
<td>( \pm 60.0 )</td>
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TABLE 3.2  TWO NODE NINTH ORDER TORSIONAL AMPLITUDE AT FRONT END
OF THE ENGINE CRANKSHAFT

Calibration of torsional vibration pick-up at setting 'C'.
20 mm double amplitude (d.a) of records = 0.5 deg. (d.a)
1 mm (d.a) = 0.025 deg. (d.a)

<table>
<thead>
<tr>
<th>Shaft speed</th>
<th>Magnified single amplitude by harmonic analysis (mm)</th>
<th>Actual single amplitude (mm)</th>
<th>Double amplitude (mm)</th>
<th>2-Node 9th order torsional amplitude (d.a)</th>
<th>2-Node 9th order torsional amplitude (single amplitude) x 10^{-3} rad.</th>
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</thead>
<tbody>
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<td>r.p.m</td>
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<td></td>
<td></td>
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</tr>
<tr>
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<td>0.81</td>
<td>1.62</td>
<td>0.0402</td>
<td>0.0201</td>
</tr>
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<td>2.54</td>
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## TABLE 3.3
THEORETICAL RESULTS CALCULATED FROM DAMPED-FORCED FREQUENCY TABULATION FOR TWO NODE MODE NINTH ORDER TORSIONAL AMPLITUDE AT FRONT END OF THE ENGINE AND STRESS IN INTERMEDIATE SHAFT

<table>
<thead>
<tr>
<th>Shaft speed r.p.m</th>
<th>2-Node 9th order torsional amplitude (single amplitude) at front end of the engine $\times 10^{-3}$ rad.</th>
<th>2-Node 9th order stress in intermediate shaft $1\text{b/in}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>110.0</td>
<td>0.212</td>
<td>$\pm 65$</td>
</tr>
<tr>
<td>110.5</td>
<td>0.243</td>
<td>$\pm 74$</td>
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<td>111.0</td>
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<td>$\pm 86$</td>
</tr>
<tr>
<td>111.5</td>
<td>0.337</td>
<td>$\pm 102$</td>
</tr>
<tr>
<td>112.0</td>
<td>0.410</td>
<td>$\pm 124$</td>
</tr>
<tr>
<td>112.5</td>
<td>0.509</td>
<td>$\pm 155$</td>
</tr>
<tr>
<td>113.0</td>
<td>0.636</td>
<td>$\pm 193$</td>
</tr>
<tr>
<td>113.5</td>
<td>0.752</td>
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<td>114.0</td>
<td>0.767</td>
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</tr>
<tr>
<td>114.5</td>
<td>0.669</td>
<td>$\pm 203$</td>
</tr>
<tr>
<td>115.0</td>
<td>0.546</td>
<td>$\pm 166$</td>
</tr>
<tr>
<td>115.5</td>
<td>0.447</td>
<td>$\pm 135$</td>
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<tr>
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<td>0.374</td>
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<td>119.5</td>
<td>0.171</td>
<td>$\pm 52$</td>
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<tr>
<td>120.0</td>
<td>0.159</td>
<td>$\pm 48$</td>
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CHAPTER 4

4.1. Introduction

From the work carried out in the past it is clear that the behaviour of the variable inertia system is far from understood. Draminsky in his analysis has shown the method of treating a multi-cylinder engine as a single mass system to calculate the stresses due to secondary resonance. To understand the phenomenon of secondary resonance and the behaviour of a variable inertia system more clearly a single cylinder engine system is used in the present work to simplify the analysis and also because it is possible to extend it to develop a simple and convenient method of stress calculations in multi-cylinder engines.

4.2. Derivation of the Equation of Motion for a Single Cylinder Engine System

Fig. 4.1 shows diagramatically a single cylinder reciprocating engine driving a heavy flywheel A of moment of inertia $I_A$. The crank radius is ''a'' and the connecting-rod to crank ratio is assumed to be large such that the reciprocating mass moves with simple harmonic motion. Dividing the mass of the connecting-rod into two lumped masses one at the crankpin C and the other at the piston crosshead assembly B, the total mass of the reciprocating parts is denoted by $M$ and the total moment of inertia of the rotating parts by $I$.

Let the angular displacement of the crank from the position where the reciprocating mass is farthest from the shaft be $\theta$
corresponding to some instant of time \( t \) and the rotation of the flywheel from the same datum at this instant of the configuration be \( \Theta_1 \). Since the inertia of the flywheel \( A \) is very large, its angular velocity, \( \omega \), can be assumed to be constant.

The kinetic energy \( T \) of the system is thus given by:

\[
T = \frac{1}{2} \dot{\theta}^2 (I + \frac{1}{2} M a^2 - \frac{1}{2} M a^2 \cos 2\Theta) + \frac{1}{2} I_A \omega^2
\]  \hspace{1cm} (4.1)

The potential energy \( V \) of the system is:

\[
V = \frac{1}{2} \mu (\Theta - \Theta_1)^2
\] \hspace{1cm} (4.2)

The equation of Lagrange for the co-ordinate \( \Theta \) is:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\Theta}} \right) - \frac{\partial T}{\partial \Theta} + \frac{\partial V}{\partial \Theta} = 0
\] \hspace{1cm} (4.3)

Application of equation (4.3) to equations (4.1) and (4.2) gives:

\[
\frac{d}{dt} \left[ \dot{\Theta} (I + \frac{1}{2} M a^2 - \frac{1}{2} M a^2 \cos 2\Theta) \right] - \frac{1}{2} M a^2 \dot{\Theta}^2 \sin 2\Theta
+ \mu (\Theta - \Theta_1) = 0
\] \hspace{1cm} (4.4)

which reduces to:

\[
\dot{\Theta} (I + \frac{1}{2} M a^2 - \frac{1}{2} M a^2 \cos 2\Theta) + \frac{1}{2} M a^2 \dot{\Theta}^2 \sin 2\Theta
+ \mu (\Theta - \Theta_1) = 0
\] \hspace{1cm} (4.5)

Letting:
\[ \eta = \frac{\frac{1}{2} M a^2}{I + \frac{1}{2} M a^2}, \quad \frac{1}{x^2} = \frac{\frac{1}{2}}{\omega^2(I + \frac{1}{2} M a^2)} \] (4.6)

and \[ \theta_1 = \omega t, \quad \theta = \omega t + \gamma \] (4.6a)

Making use of the relationships (4.6) and (4.6a) in equation (4.5) and neglecting second order terms gives:

\[
\ddot{\gamma}(1 - \eta \cos 2\omega t) + 2\omega \eta \dot{\gamma} \sin 2\omega t + \gamma \left(\frac{\omega^2}{x^2} + 2\omega^2 \eta \cos 2\omega t\right) = -\omega^2 \eta \sin 2\omega t \] (4.7)

Equation (4.7) is further simplified by changing the independent variable to \( \tau = \omega t \), and letting dashes represent differentiation with respect to \( \tau \),

\[
\gamma''(1 - \eta \cos 2\tau) + 2\eta \gamma' \sin 2\tau + \gamma \left(-\frac{1}{x^2} + 2\eta \cos 2\tau\right) = -\eta \sin 2\tau \] (4.8)

Equation (4.8) is a second order linear differential equation with single-valued variable coefficients and can be expressed in the functional form:

\[ \gamma'' = F(\tau, \gamma, \dot{\gamma}) \] (4.9)

Equation (4.8) shows that forced vibrations do not arise entirely from the pressure of the expanding gases on the piston. They are also produced by the action of the reciprocating parts.
4.3. Solution of the Equation of Motion by the Modified Euler's Method

A differential equation of the second order of the form of equation (4.8) may be expressed as:

\[ \frac{d^2 \gamma}{dT^2} = f(T, \gamma, \frac{d\gamma}{dT}) \]  \hspace{1cm} (4.10)

The solution of equation (4.10) can be written as:

\[ \gamma = F(T) \]  \hspace{1cm} (4.11)

which also can be represented graphically as shown in Fig. 4.2.

Given the starting values \( \gamma_0 \) and \( \left( \frac{d\gamma}{dT} \right)_0 \) and assuming that \( \gamma \) is linear for a small range of \( T \),

\[ \Delta \gamma \approx \Delta T \tan \alpha \approx \left( \frac{d\gamma}{dT} \right)_0 \Delta T \]  \hspace{1cm} (4.12)

so that

\[ \gamma_1 \approx \gamma_0 + \left( \frac{d\gamma}{dT} \right)_0 \Delta T \]  \hspace{1cm} (4.13)

Hence, the values of \( \gamma \) corresponding to \( T_2, T_3 \), etc. are

\[ \gamma_i \approx \gamma_{i-1} + \left( \frac{d\gamma}{dT} \right)_{i-1} \Delta T, \quad i = 2, 3, \text{ etc.} \]  \hspace{1cm} (4.14)

Similarly, the values of \( \left( \frac{d\gamma}{dT} \right) \) can be written as

\[ \left( \frac{d\gamma}{dT} \right)_i \approx \left( \frac{d\gamma}{dT} \right)_{i-1} + \left( \frac{d^2 \gamma}{dT^2} \right)_{i-1} \Delta T \]  \hspace{1cm} (4.15)

It is clear from Fig. 4.2 that the computed \( \gamma \) values will deviate
farther and farther from the true $\gamma$ values provided that the curvature of the graph does not change.

To allow for this deviation an improved method of iteration can be used. The value of $\gamma_1$ determined from equation (4.13) is represented by the point $K$ in Fig. 4.3. If PL is drawn parallel to the tangent at $N$, then $\gamma_1$ computed by using the slope at $N$ is given by point $L$. This value can be improved by taking the average of the slopes at the points $P$ and $N$ on $\gamma = F(\Gamma)$, corresponding to $\gamma_0$ and $\gamma_1$ as:

$$\gamma_1^{(2)} = \gamma_0 + \frac{\left(\frac{d\gamma}{dT}\right)_0 + \left(\frac{d\gamma}{dT}\right)_1}{2} \Delta T$$  (4.16)

where $\left(\frac{d\gamma}{dT}\right)_1$ determined from equation (4.15) is assumed to represent the slope at point $N$. Equation (4.16) thus gives a better value for $\gamma_1$ somewhere midway between points $L$ and $K$. This process can be repeated until two successive values of $\gamma_1$, say steps $j-1$ and $j$ agree with each other to the desired accuracy. Equation (4.16) can then be written for the $j$th iteration as:

$$\gamma_1^{(j)} = \gamma_0 + \frac{\left(\frac{d\gamma}{dT}\right)_0 + \left(\frac{d\gamma}{dT}\right)_{j-1}}{2} \Delta T$$  (4.17)

Using a similar iteration process for $\frac{d\gamma}{dT}$, the modified Euler's equations for the $i$th step can in general be written as:
4.4. Solution of Equation of Motion by the Runge-Kutta Method

The solution of differential equations of the form of equation (4.9) can be obtained by use of the Runge-Kutta method. The form of the Runge-Kutta equations obtained by the method of successive substitution and using the Taylor series expansion is:

\[
\begin{align*}
\gamma_1' &= \gamma_{1-1}' + \frac{\gamma_{1-1}'' + \gamma_{1}''(j-1)}{2} \Delta T \\
\gamma_1(j) &= \gamma_{1-1} + \frac{\gamma_{1-1}'' + \gamma_{1}'(j)}{2} \Delta T \\
\end{align*}
\]

(4.18)

Given the initial conditions of \( \gamma_0 \) and \( \gamma'_0 \) at \( T = 0 \), the increments in the functions for the first step are:

\[
\begin{align*}
\Delta \gamma_0 &= \Delta T \left[ \gamma'_0 + \frac{1}{6} (K_1 + K_2 + K_3) \right] \quad (4.20 \ a) \\
\Delta \gamma'_0 &= \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4) \quad (4.20 \ b)
\end{align*}
\]
The values of $\gamma$ and $\gamma'$ at the end of the first step are calculated from

$$\gamma_1 = \gamma_o + \Delta \gamma_o \quad (4.21a)$$

$$\gamma'_1 = \gamma'_o + \Delta \gamma'_o \quad (4.21b)$$

The process given in equations (4.20) and (4.21) can be repeated for further steps and the general expressions for $\gamma$ and $\gamma'$ can be written as:

$$\gamma_{i+1} = \gamma_i + \Delta T \left[ \gamma'_i + \frac{1}{6} (K_i + K_2 + K_3) \right] \quad (4.22a)$$

$$\gamma'_{i+1} = \gamma'_i + \frac{1}{6} (K_i + 2K_2 + 2K_3 + K_4) \quad (4.22b)$$

4.5. **Method of Simultaneous Equations**

The solutions of equation (4.8) can also be determined by the method of simultaneous equations. The following steps are used to calculate the increments in the functions $\gamma$ and $\gamma'$.

$$K_1 = \Delta T \cdot \gamma'_1 ,$$

$$K_2 = \Delta T \cdot (\gamma'_1 + \frac{1}{2} ) ,$$

$$K_3 = \Delta T \cdot (\gamma'_1 + \frac{1}{2} ) ,$$

$$K_4 = \Delta T \cdot (\gamma'_1 + \frac{1}{3} ) ,$$

The increment in $\gamma$ is given by the expression

$$\Delta \gamma = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$
The increment in $y'$ is then calculated from

$$\Delta y'_1 = \frac{1}{6} (l_1 + 2l_2 + 2l_3 + 1_4)$$

The general equations for the $(i+1)$th step can be written as:

$$y_{i+1} = y_i + \frac{1}{6} (K_i + 2K_2 + 2K_3 + K_4)$$

$$y'_{i+1} = y'_i + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

It may be noted here that the accuracy of these methods depends on the step length used. The three numerical processes, namely, the modified Euler's equations (4.18), the Runge-Kutta equations (4.22) and expressions (4.23) of the simultaneous equations method are programmed in Algol language for the solution of equation (4.8) on a digital computer. The three methods are used as a basis of comparison with each other in order to confirm the correctness and accuracy of the results obtained. Having verified the validity of the numerical methods further solutions of the equation of motion are obtained.
using only the modified Euler's procedure with other values of \( \eta \)
and \( r \).

The solution of equation (4.8) with the initial conditions
\( y_0 = 1 \) and \( y'_0 = 0 \) at \( T = 0 \), obtained by the modified Euler's method
for \( r = 0.06 \) and \( \eta = 0.544 \), is shown in Fig. 4.4. The results obtained
by the Runge-Kutta process are also given in Fig. 4.4. In both
cases one hundred steps over one cycle of the natural frequency of
the linear system are used. It can be seen that the modified Euler's
method shows close agreement with the Runge-Kutta process. Results
obtained by the use of simultaneous equations method are not presented.

4.6. Application to a Typical Marine Engine

The following data for a typical single cylinder engine is
used to determine the response of the system at different speeds
of rotation.

Equivalent inertia of rotating and
reciprocating parts, \((I + \frac{1}{2} M a^2) = 93934 \text{ lb.in sec}^2\)

Inertia of reciprocating parts
\((-\frac{1}{2} M a^2) = 51100 \text{ lb.in sec}^2\)

Natural frequency of the system \(f_n = 71.4 \text{ rad/sec}\)
\(\eta = 0.544\)

The response of the system is determined using an ICT 1905 F
computer for the different speeds of the engine using equations
(4.18) with initial conditions as:
\( y_0 = 1 \) at \( T = 0 \) and \( y'_0 = 0 \) at \( T = 0 \).
Using the modified Euler's method the responses of the system for different ratios of angular velocity of the shaft to the natural frequency of the system are given in Fig. 4.5 to Fig. 4.19. Fig. 4.5 to Fig. 4.13 show the responses of the system corresponding to $r = 0.04, 0.06, 0.08, 0.4, 0.5, 0.6, 0.7, 0.8 \text{ and } 0.9$ respectively. Fig. 4.14 and Fig. 4.15 demonstrate the growth of amplitude to very high values in the critical speed ranges at $r = 0.55$ and $r = 1.0$. The responses for $r = 2, 4, 6 \text{ and } 8$ are given in Fig. 4.16 to Fig. 4.19.

4.7. Analysis of Waveforms

The method of waveform analysis given by Manley enables the frequencies, amplitudes and phases of the principal components of a complex wave to be determined speedily and with sufficient accuracy for practical purposes. Since the majority of recorded waveforms contain only two or three principal components, they can be analysed by constructing the auxiliary envelope lines. The waveform records usually obtained in practice and the theoretical waveforms obtained from the analysis of equation (4.8), are the resultant motions due to the superposition of several harmonic waves of different frequencies, amplitudes and phases. A typical but fairly common type of waveform is shown in Fig. 4.16. It consists mainly of a high frequency ripple with a low frequency surge. If the higher frequency is more than about four times the lower frequency, there is no difficulty in determining the amplitudes by drawing the wave envelope. To obtain a complete understanding of the problem at the later stage when dealing with forced torsional vibration the
waveforms are analysed directly on the digital computer.

Fig. 4.5 to Fig. 4.7, Fig. 4.9 and Fig. 4.10 and also at the higher speeds of rotation Fig. 4.17 and Fig. 4.18 show the presence of beats which is an indication that the waveform is mainly composed of two sinusoidal waves which do not differ appreciably in frequency and according to the methods of analysis of Manley for waveforms when the phenomenon of beats is exhibited it follows that

(a) the resultant waveform has the same apparent frequency as the major component (that is the component with the greater amplitude) and its amplitude varies between the sum and the difference of the component amplitudes, the beat frequency being the difference between the frequencies of the components.

(b) the separation of successive peaks (crests or troughs) at the bulge and at the waist of the beat determines whether the minor component is of higher or lower frequency than the major. If the peak separation at the bulge is greater than at the waist, the frequency of the minor component is less than that of the major; and if the peak separation at the bulge is less than that at the waist, the minor component is of higher frequency than the major.

The results of the analyses of the complex waveforms based on the above method are given in Table 4.1 for \( \eta = 0.544 \). Table 4.2, Table 4.3 and Table 4.4 give the results of the similar analyses carried out for \( \eta = 0.3, 0.42 \) and 0.8 respectively.

The frequency of a cycle of the response envelope as given
in these tables, is approximately twice the speed of rotation in the range of \( r \) given by \( 0.02 \leq r < 0.2 \). Above this range of \( r \) no correlation is observed between the frequency of one cycle of the response envelope and the speed of rotation. It may be noted here that a cycle of the response envelope is defined as the number of vibration cycles within a range of time containing one complete pattern of the response which then repeats itself. For some ratios of \( r \) the waveform analysis is not given in the tables, due to the complexity of the responses obtained. One such complex waveform is given in Fig. 4.19.

Fig. 4.20 to Fig. 4.23 show the variation of maximum amplitude \( A_{\text{max}} \) against frequency ratio \( r \) for \( \eta = 0.544, 0.3, 0.42 \) and 0.8 respectively. For \( r \neq 1 \) and \( r = \frac{1}{2} \) the pairs of broken vertical lines bound the regions of instability containing the two critical speed ranges in which the amplitude of vibration grows infinitely large.

It may be recollected here that the results obtained from equation (4.8) are true only for small displacements. For large amplitudes of vibration in the unstable regions, the nonlinearities become predominant and the solutions in the unstable regions would be different from those predicted by the linear equation (4.8). But an important outcome of this analysis is that it shows the regions in which the amplitudes of vibration are small and the corresponding waveforms for speeds outside the unstable regions have been predicted.

Fig. 4.24 to Fig. 4.27 give the variation of the ratio \( \omega/\omega_1 \) over the range of \( r \) from 0.02 to 10 for the values of \( \eta = 0.544 \).
0.3, 0.42 and 0.8 respectively. Horizontal portions of the curves show the regions of instability clearly bringing out the fact that the ratio $\omega/\omega_1$ remains constant at $\frac{1}{2}$ and 1 corresponding to unstable regions for $r \approx \frac{1}{2}$ and $r \approx 1$ respectively. In other words, for the unstable region at $r \approx 1$, one cycle of the vibration occurs during each revolution of the shaft and for the unstable range at $r \approx \frac{1}{2}$, the frequency of vibration is twice the revolutions of the shaft.

Fig. 4.28 to Fig. 4.32 show the solutions of equation (4.8) for $\eta = 0.3$ at $r = 0.06, 0.08, 0.1, 0.25, 0.333$ respectively. Also the waveforms on both sides of the two instability regions are shown for $\eta = 0.3$ at $r = 0.5, 0.504, 0.52, 0.54, 0.56, 0.58, 0.92, 0.94, 0.96, 0.962, 1.15, 1.2, 1.4, 1.6 and 1.8$ from Fig. 4.33 to Fig. 4.47 which show the variations of frequency and the manner in which the amplitudes build up near these regions.

It may be noted here that the solutions are unstable for $r$ varying from 0.507 to 0.513 and then again from 0.963 to 1.132 when $\eta = 0.3$.

A typical waveform solution for $\eta = 0.8$ at $r = 0.3$ is given in Fig. 4.48.

The visual analyses of the waveforms for $\eta = 0.544$ and $\eta = 0.3$ given in Table 4.1 and Table 4.2 and the review of the solutions given in Fig. 4.5 to Fig. 4.7 and in Fig. 4.28 to Fig. 4.30 show the existence of beats for all speeds defined by $r < 0.2$ and that there are two beats in one revolution of the shaft. It can also be seen that as the amplitude and instantaneous frequency fluctuate, the maximum amplitude and the maximum apparent frequency of one oscillation of the solution occur together, as do minimum values for these quantities.
For $\eta = 0.544$ and $r = 0.06$, 0.08 and 0.1 the respective waveforms are composed of 18th and 20th order, 14th and 16th order, 12th and 14th order harmonic components, whereas at the corresponding values of $r$ and $\eta = 0.3$ the waveforms are composed of 16th and 18th order, 12th and 14th order, 10th and 12th order harmonic components, which is contradictory to Draminsky's theory. Hence it is thought desirable to confirm these results obtained by visual analyses based on Manley's method and to carry out a complete harmonic analysis of these solutions on the digital computer.

4.8. **Harmonic Analysis**

The harmonic analyses of the waveforms are programmed in Algol language on the digital computer as shown in appendix 6. A large number of ordinates at equal intervals are taken in all the cases of the solutions under investigation and the analysis can be considered accurate up to the 20th harmonic. The analysis will give harmonic number and for each harmonic the sine component, cosine component, resultant magnitude, the phase of each harmonic in radians and the phase in degrees.

Since there are two beats in one revolution of the shaft at all speeds for $r < 0.2$, the order number is twice the harmonic number. Table 4.5 to Table 4.7 give the harmonic analysis results for $\eta = 0.544$ at $r = 0.06$, 0.08 and 0.1 and Table 4.8 to Table 4.10 show such an analysis of the solutions at the corresponding speeds of rotation for $\eta = 0.3$. The waveform solutions for $\eta = 0.3$ at $r = 0.25$ and 0.333 as shown in Fig. 4.31 and Fig. 4.32 show a modulation of amplitude and frequency over a long period of time and, therefore, to get a rough approximation of the orders of the components present
the harmonic analysis is carried out over the period of one revolution of the shaft. The order number in these cases will be equal to the harmonic number and the analysis is presented in Table 4.11 and Table 4.12.

Fig. 4.49 to Fig. 4.53 show the solutions of the equation of motion at low values of \( \eta = 0.2, 0.15, 0.1, 0.08 \) and 0.05 at \( r=0.1 \).

The harmonic analysis results for these waveforms are presented in Table 4.13 to Table 4.17.

4.9. Some Interesting Observations on the Harmonic Analysis Results

Table 4.7 shows that when \( \eta = 0.544 \) and \( r = 0.1 \) the waveform is composed of a principal component of order 10 and other components of order 8, 12, 14, 16 and 18 and that the secondary components are present in a significant proportion. Similarly Table 4.9 shows that when \( \eta = 0.3 \) and \( r = 0.08 \) the waveform is composed of a principal component of order 12 and secondary components of order 10, 14 and 16 which is contradictory to the theory put forward by Draminsky\(^{32,33}\) that if \( \omega_n = n \omega \), and the system is executing natural or free vibrations of phase velocity \( \omega_n \), the waveform is composed of a principal component of order \( n \) and two smaller components of order \( (n - 2) \) and \( (n + 2) \).

A perusal of the present analysis from Table 4.5 to Table 4.17 suggests that the orders of the harmonic components present in the vibratory motion of the system should be determined by the harmonic analysis of the solution of the equation of motion (4.8) for definite speed of rotation and the specific value of \( \eta \) of the system. Table 4.11 and Table 4.12 show that when \( r = 0.25 \) and \( \eta = 0.3 \) the waveform is composed of the 4th order as a predominant component and for \( \eta = 0.3 \) at \( r = 0.333 \) the main component in the solution is 3rd order. Table 4.13
to Table 4.17 indicate that as the value of \( \eta \) decreases the magnitude of the principal component in the waveform increases and the secondary components become smaller in size.

4.10. Stability

One important feature of the solutions of the equation (4,8) is its stability. If for certain values of the ratio \( r \) the solution for \( \gamma \) grows without bound as \( T \) increases then the solution is said to be unstable. A solution is defined as being stable if it tends to zero or remains bounded as \( T \to +\infty \).

Gregory has derived expressions for characteristic numbers in free motion for equation (4.8) without the forcing term, by the use of methods in which Mathieu's and Hill's equations are discussed with the help of infinite continued fractions. These characteristic numbers define the unstable regions.

In the present analysis the boundaries between stable and unstable regions are determined for equation (4.8) making use of the modified Euler's method for \( \eta \) ranging from 0 to 0.8. Characteristic numbers calculated by Gregory for \( \eta = 0 \) to 0.6 and the results obtained from the modified Euler's method are given in Table 4.18 and Table 4.19. The results obtained are also plotted in Fig. 4.54 and Fig. 4.55, which divide the plane \((r - \eta)\) into stable and unstable regions. For the range of \( \eta \) varying from 0 to 0.6, it can be seen that the results obtained by the modified Euler's method are in close agreement with Gregory's results.
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<th>Speed of rotation $\omega$ rad/sec</th>
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<th>Frequency of response $\omega_I$ rad/sec</th>
<th>Ratio $\omega/\omega_I$</th>
<th>Frequency of one cycle of response envelope ($f$) rad/sec</th>
<th>Waveform analysis/cycle of response envelope</th>
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*Table 4.4: Analysis of response of the system when \( \eta = 0.8 \)*
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<th>Resultant component</th>
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<th>Phase (deg.)</th>
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</tbody>
</table>

Order no. = 2 x Harmonic no.
### TABLE 4.9 HARMONIC ANALYSIS RESULTS WHEN $\eta = 0.3$ AND $r = 0.08$

| Harmonic no. | Sine component | Cosine component | Resultant $|\text{rad.}|$ | Phase $|\text{deg.}|$ |
|--------------|----------------|-----------------|-----------------|-----------------|
| 1            | $2.2780 \times 10^{-3}$ | $-1.2832 \times 10^{-2}$ | $1.3033 \times 10^{-2}$ | $1.3951$ | $79.903$ |
| 2            | $6.6233 \times 10^{-4}$ | $-1.3590 \times 10^{-2}$ | $1.3606 \times 10^{-2}$ | $1.5221$ | $87.176$ |
| 3            | $2.6457 \times 10^{-3}$ | $-1.8797 \times 10^{-2}$ | $1.8983 \times 10^{-2}$ | $1.4309$ | $81.956$ |
| 4            | $1.8573 \times 10^{-2}$ | $3.1350 \times 10^{-2}$ | $3.6439 \times 10^{-2}$ | $1.0359$ | $59.333$ |
| 5            | $1.0684 \times 10^{-1}$ | $-3.8117 \times 10^{-1}$ | $3.9586 \times 10^{-1}$ | $1.2975$ | $74.313$ |
| 6            | $1.4195 \times 10^{-1}$ | $6.9110 \times 10^{-1}$ | $7.0552 \times 10^{-1}$ | $1.3682$ | $78.362$ |
| 7            | $4.6219 \times 10^{-2}$ | $3.9629 \times 10^{-1}$ | $3.9898 \times 10^{-1}$ | $1.4546$ | $83.315$ |
| 8            | $6.3900 \times 10^{-3}$ | $1.4711 \times 10^{-1}$ | $1.4725 \times 10^{-1}$ | $1.5273$ | $87.479$ |
| 9            | $8.3521 \times 10^{-5}$ | $5.3191 \times 10^{-2}$ | $5.3191 \times 10^{-2}$ | $1.5692$ | $89.875$ |
| 10           | $1.0689 \times 10^{-4}$ | $2.3357 \times 10^{-2}$ | $2.3357 \times 10^{-2}$ | $1.5662$ | $89.703$ |
| 11           | $4.1874 \times 10^{-4}$ | $1.3366 \times 10^{-2}$ | $1.3372 \times 10^{-2}$ | $1.5394$ | $88.171$ |
| 12           | $5.0660 \times 10^{-4}$ | $9.3012 \times 10^{-3}$ | $9.3150 \times 10^{-3}$ | $1.5363$ | $88.849$ |
| 13           | $4.8267 \times 10^{-4}$ | $7.1988 \times 10^{-3}$ | $7.2150 \times 10^{-3}$ | $1.5038$ | $86.131$ |
| 14           | $4.2472 \times 10^{-4}$ | $5.9206 \times 10^{-3}$ | $5.9358 \times 10^{-3}$ | $1.4991$ | $85.863$ |
| 15           | $3.5806 \times 10^{-4}$ | $5.0364 \times 10^{-3}$ | $5.0459 \times 10^{-3}$ | $1.4998$ | $85.900$ |
| 16           | $3.0722 \times 10^{-4}$ | $4.3963 \times 10^{-3}$ | $4.4071 \times 10^{-3}$ | $1.5010$ | $85.969$ |
| 17           | $2.4748 \times 10^{-4}$ | $3.9052 \times 10^{-3}$ | $3.9130 \times 10^{-3}$ | $1.5075$ | $86.340$ |
| 18           | $2.0387 \times 10^{-4}$ | $3.5113 \times 10^{-3}$ | $3.5172 \times 10^{-3}$ | $1.5128$ | $86.643$ |
| 19           | $1.5556 \times 10^{-4}$ | $3.2021 \times 10^{-3}$ | $3.2059 \times 10^{-3}$ | $1.5222$ | $87.185$ |
| 20           | $1.0393 \times 10^{-4}$ | $2.9566 \times 10^{-3}$ | $2.9586 \times 10^{-3}$ | $1.5334$ | $87.827$ |

Order no. = 2 x Harmonic no.
<table>
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<tr>
<th>Harmonic no.</th>
<th>Sine component</th>
<th>Cosine component</th>
<th>Resultant</th>
<th>Phase (rad.)</th>
<th>Phase (deg.)</th>
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<tbody>
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<tr>
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Order no. = 2 x Harmonic no.
TABLE 4.11  HARMONIC ANALYSIS RESULTS WHEN $\eta = 0.3$ AND $r = 0.25$

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<th>Harmonic no.</th>
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<th>Cosine component</th>
<th>Resultant</th>
<th>Phase rad.</th>
<th>Phase deg.</th>
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<td>$1.3782 \times 10^{-2}$</td>
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<td>$1.2131 \times 10^{-1}$</td>
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Order no. = Harmonic no.
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<th>Cosine component</th>
<th>Resultant</th>
<th>Phase rad.</th>
<th>Phase deg.</th>
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Order no. = Harmonic no.
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<th>Cosine component</th>
<th>Resultant</th>
<th>Phase rad.</th>
<th>Phase deg.</th>
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Order no. = 2 x Harmonic no.
## TABLE 4.14  HARMONIC ANALYSIS RESULTS WHEN $\eta = 0.15$ AND $r = 0.1$

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<th>Harmonic no.</th>
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<th>Cosine component</th>
<th>Resultant</th>
<th>Phase rad.</th>
<th>Phase deg.</th>
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Order no. = 2 x Harmonic no.
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<th>Cosine component</th>
<th>Resultant</th>
<th>Phase rad.</th>
<th>Phase deg.</th>
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Order no. = 2 x Harmonic no.
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<th>Cosine component</th>
<th>Resultant</th>
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<th>Phase deg.</th>
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Fig. 4.1  Diagramatic arrangement of engine running gear
Solution $F(T)$ in $\gamma \sim T$ plane

Fig. 4.2 Solution $F(T)$ in $\gamma \sim T$ plane
Fig. 4.2 Modification of solution at $T = T_1$
Fig. 4.4 Comparison between theoretical waveforms for

\[ r = 0.06 \text{ and } \eta = 0.544 \]

- Modified Euler's Method
- Runge-Kutta Method
Fig. 4-5 Theoretical waveform relationship of $\gamma \sim t$ for

$\alpha = 0.04$ and $\eta = 0.544$
Fig. 4.6  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 0.06$ and $\eta = 0.544$
Fig. 4.7  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 0.08$ and $\eta = 0.544$
Fig. 4.8 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.4$ and $\eta = 0.544$
Fig. 4.9 Theoretical waveform relationship of $\gamma \sim t$ for 
\[ r = 0.5 \text{ and } \eta = 0.544 \]
Fig. 4.10 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.6$ and $\eta = 0.544$. 
Fig. 4.11  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.7$ and $\eta = 0.544$
Fig. 4.12  Theoretical waveform relationship of $\gamma \sim t$ for 
$r = 0.8$ and $\eta = 0.544$
Fig. 4.13 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.9$ and $\eta = 0.544$
Fig. 4.14 Theoretical waveform relationship of $y \sim t$ in the unstable region corresponding to $r = \frac{1}{2}$ for $r = 0.55$ and $\eta = 0.544$
Fig. 4.15 Theoretical waveform relationship of $\gamma \sim t$ in the unstable region corresponding to $r \approx 1$ for $r = 1.0$ and $\eta = 0.544$
Fig. 4.16 Theoretical waveform relationship of $\gamma \sim t$ for $r = 2.0$ and $\eta = 0.544$
Fig. 4.17  Theoretical waveform relationship of $\gamma \sim t$ for $r = 4.0$ and $\eta = 0.544$
Fig. 4.18 Theoretical waveform relationship of $\gamma \sim t$ for $r = 6.0$ and $\eta = 0.544$. 
Fig. 4.19  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 8.0$ and $\eta = 0.544$
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Figure 4.21: Maximum amplitude $A_{\text{max}}$ of the response versus $x$ for $\eta = 0.5$.
Fig. 4.22 Maximum amplitude $A_{\text{max}}$ of the response versus $r$ for $\eta = 0.42$
Fig. 4.23 Maximum amplitude $A_{\text{max}}$ of the response versus $r$ for $\eta = 0.8$
Fig. 4.24  Relationship between $\omega/\omega_1$ and $r$ for $\eta = 0.544$
Fig. 4.25 Relationship between $\omega/\omega_1$ and $r$ for $\eta = 0.3$
Fig. 4.26 Relationship between $\omega/\omega_1$ and $r$ for $\eta = 0.42$
Fig. 4.27  Relationship between ω/ω₀ and r for η = 0.8
Fig. 4.28  Theoretical waveform relationship of $y \sim t$ for $r = 0.06$ and $\eta = 0.3$
Fig. 4.29  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 0.08$ and $\eta = 0.3$
Fig. 4.30  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 0.1$ and $\eta = 0.3$
Fig. 4.31  Theoretical waveform relationship of $\gamma \sim t$ for

$\alpha = 0.25$ and $\eta = 0.3$
Fig. 4.32  Theoretical waveform relationship of $\gamma \sim t$ for

   $r = 0.333$ and $\eta = 0.3$
Fig. 4.35 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.52$ and $\eta = 0.3$
Fig. 4.36  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 0.54$ and $\eta = 0.3$
Fig. 4.37  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 0.56$ and $\eta = 0.3$
Fig. 4.38 Theoretical waveform relationship of $\gamma \sim t$ for $x = 0.58$ and $\eta = 0.3$
Fig. 4.39  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.92$ and $\eta = 0.3$
Fig. 4.41 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.96$ and $\eta = 0.3$
Fig. 4.42 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.962$ and $\eta = 0.3$
Fig. 4.43  Theoretical waveform relationship of $\gamma \sim t$ for $\sigma = 1.15$ and $\eta = 0.3$
Fig. 4.45  Theoretical waveform relationship of $\gamma \sim t$ for $r = 1.4$ and $\eta = 0.3$
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Fig. 4.49 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.1$ and $\eta = 0.2$
Fig. 4.50 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.1$ and $\eta = 0.15$. 
Fig. 4.51 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.1$ and $\eta = 0.1$
Fig. 4.52  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.1$ and $\eta = 0.08$
Fig. 4.53 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.1$ and $\eta = 0.05$
FIG. 4-51. Stability diagram (η - r) for the region r = 1.
5.1. Introduction

Draminsky method is widely used for obtaining the effects of secondary resonance. There is some considerable doubt, however, about the validity of the results obtained by this process. In certain cases calculations of secondary resonance confirm the practical measurements. In other cases secondary resonance is predicted when it is not present in the actual system. Draminsky in his analysis has omitted the variable part, $2\eta \cos 2\tau$, from the elastic term of equation (4.8) which appears when the equation is derived from energy considerations. Hence it was decided to study the effect of different terms of the equation of motion on its solutions which would explain the above anomaly.

5.2. Effects of Different Terms of Equation of Motion (4.8) on the Solutions

Taking equation (4.8) in its simplest form with all the variable coefficients omitted gives

$$\frac{d^2\gamma}{dt^2} + \frac{\omega_n^2}{\omega^2} \gamma = -\eta \sin 2\tau \quad (5.1)$$

and since $\tau = \omega t$,

$$\frac{d^2\gamma}{dt^2} + \omega_n^2 \gamma = -\eta \omega^2 \sin 2\omega t \quad (5.2)$$

To obtain a general solution assume an equation of the form
\[ \frac{d^2\gamma}{dt^2} + \omega_n^2 \gamma = -\eta \omega^2 (\cos 2\omega t + 1 \sin 2\omega t) \]  \hspace{1cm} (5.3)

and, thus,

\[ \frac{d^2\gamma}{dt^2} + \omega_n^2 \gamma = -\eta \omega^2 e^{2\omega t} \]  \hspace{1cm} (5.4)

The imaginary part of the solution of equation (5.4) is the one required. Taking a trial solution of the form

\[ \gamma = A e^{2\omega t} \]

then,

\[ \frac{d\gamma}{dt} = 2\omega A e^{2\omega t} \]  \hspace{1cm} (5.5)

and

\[ \frac{d^2\gamma}{dt^2} = (2\omega)^2 A e^{2\omega t} = -4\omega^2 A e^{2\omega t} \]

Substituting equations (5.5) into equation (5.4) we obtain

\[ -4\omega^2 A e^{2\omega t} + \omega_n^2 A e^{2\omega t} = -\eta \omega^2 e^{2\omega t} \]

and thus

\[ A = \frac{-\eta \omega^2}{(\omega_n^2 - 4\omega^2)} \]  \hspace{1cm} (5.6)

Hence,

\[ \gamma = \frac{-\eta \omega^2}{(\omega_n^2 - 4\omega^2)} e^{2\omega t} \]  \hspace{1cm} (5.7)

The imaginary part of this solution is the required solution of the form

\[ \gamma = \frac{-\eta \omega^2}{(\omega_n^2 - 4\omega^2)} \sin 2\omega t \]
and thus
\[ \gamma = \frac{-\eta \omega^2}{(\frac{\omega}{\omega_n} - 4) \omega^2} \sin 2\omega t \] (5.8)

Equation (5.8) shows that the solution has a simple harmonic form but the resonance will occur when \( \omega/\omega_n = 0.5 \). Further terms are now added with equation (5.1) and the solutions are obtained by use of modified Euler's method.

Writing equation (4.8) in the form neglecting first order term and the variable coefficient of the second order term we have

\[ \gamma'' + \left( \frac{\omega^2}{\omega_n^2} + 2\eta \cos 2\tau \right) \gamma = -\eta \sin 2\omega t \] (5.9)

Fig. 5.1 and Fig. 5.2 show the solutions of the above equation when \( \eta = 0.544 \) and \( r = 0.06 \) and 0.5 respectively. At low values of \( r = 0.06 \) the solution is simple harmonic and Fig. 5.2 demonstrates the occurrence of a resonance at \( r = 0.5 \).

Omitting only the variable coefficient of the second order term equation (4.8) can be written as

\[ \gamma'' + 2\eta \gamma' \sin 2\tau + \gamma \left( \frac{1}{r^2} + 2\eta \cos 2\tau \right) = -\eta \sin 2\tau \] (5.10)

The solutions of equation (5.10) for \( \eta = 0.544 \) when \( r = 0.06, 0.5 \) and 0.6 are shown in Fig. 5.3 to Fig. 5.5. The solutions show a modulation of amplitude and frequency but different from the corresponding solutions of the complete equation (4.8).

Further investigations are carried out on equation (5.1) adding to it the variable coefficient with the second order term.
The equation is now of the form

\[(1 - \eta \cos 2T) \frac{\partial^2 \gamma}{\partial t^2} + \frac{\omega_n^2}{\omega_0^2} \gamma = -\eta \sin 2T \quad (5.11)\]

and various solutions are obtained by the modified Euler's method. The solutions of equation (5.11) when \(\eta = 0.544\) for \(r = 0.04, 0.08, 0.1, 0.6\) and 2.0 are given in Fig. 5.6 to Fig. 5.10 and the results of the analysis of the waveforms of the solutions are presented in Table 5.1. Fig. 5.6 and Fig. 5.7 show the phenomenon of beats at low values of ratio \(r\) with the peaks being wider at the bulge and narrow at the waist. The solution (Fig. 5.8) at \(r = 0.1\) is a typical complex waveform. Fig. 5.11 shows the variation of maximum amplitude \(A_{\text{max}}\) against frequency ratio \(r\). Fig. 5.12 gives the variation of the ratio \(\omega/\omega_1\) over the range of \(r = 0.02\) to 10. The values of the maximum amplitude \(A_{\text{max}}\) are in general larger than those obtained by solution of the full equation of motion (4.8) but the range of instability at \(r \approx 0.5\) is much narrower.

5.3. Effect of First Order Term

It has been stated by Draminsky in his analysis that variation in inertia must be accompanied by a 'Coriolis' effect because the mass variation can not take place except by moving the mass to another place in the vibrating system. This fact gives rise to the first order term in equation (4.8). By omitting the first order term, the equation can be written as follows, namely

\[(1 - \eta \cos 2T) \frac{\partial^2 \gamma}{\partial t^2} + \left(\frac{1}{r^2} + 2\eta \cos 2T\right) \gamma = -\eta \sin 2T \quad (5.12)\]
Fig. 5.13 to Fig. 5.17 show the time responses obtained from equation (5.12) corresponding to $r = 0.04, 0.06, 0.08, 0.1$ and $0.5$ for $\eta = 0.544$. This shows that at the lower speeds of rotation, the maximum amplitude and minimum apparent frequency of one oscillation of the solution occur together as do the minimum amplitude and maximum frequency. The shape of the response at $r = 0.5$ is also distorted compared to the beatform in the corresponding solution of equation (4.8).

Fig. 5.18 shows the variation of maximum amplitude against frequency ratio $r$. The pair of broken vertical lines bound the region of instability for $r < 0.5$. The solutions for $r > 0.83$ are found to be unstable. The time responses are investigated in the range of $r = 0.02$ to $r = 10$.

Fig. 5.19 gives the variation of the ratio of the shaft speed to the apparent frequency of response at maximum amplitude ($\omega / \omega_1$) over the range of $r = 0.02$ to $r = 10$. Horizontal portions of the curve show the regions of instability clearly bringing out the fact that the ratio $\omega / \omega_1$ remains constant at $0.5$ and $1$ corresponding to region for $r < 0.5$ and $r > 0.83$. Fig. 5.20 and Fig. 5.21 divide the plane $(r - \eta)$ into stable and unstable regions. This also shows that the presence of the first order term has a stabilising effect on the solutions for the higher values of $r$. The results of the analysis of the waveforms at different values of ratio $r$ are given in Table 5.2.

5.4 Effect of Impulse Term $-\eta \sin 2\pi$

The forcing term in equation (4.8) represents an impulse arising from the variable inertia due to the reciprocating parts. The equation
(1 - η cos 2T) γ'' + 2η γ' sin 2T + (\frac{1}{r^2} + 2η cos 2T) γ = 0 \quad (5.13)

which corresponds to equation (4.8) with the forcing term omitted represents the free motion of the system. The responses determined from equation (5.13) for r = 0.06 and 0.5 are shown in Fig. 5.22 and Fig. 5.23 respectively. Comparing Figs. 5.22 and 5.23 with corresponding solutions of equation (4.8), Fig. 4.6 and Fig. 4.9, it can be seen that the response at r = 0.06 shows no change whereas at the value of r = 0.5 the response is different for the two cases and a reduction in the maximum value of the amplitude is noted. It can be concluded that in the lower range of r, the forcing term which arises due to the incomplete balance of the reciprocating parts has no effect on the response of the system but at higher values of r, namely r = 0.5, the amplitude and frequency are modified.

5.5. **Effect of Neglecting the Variable Part, 2η cos 2T, from the Elastic Term**

Draminsky in his analysis of the effects of variable inertia has omitted the term, 2η cos 2T, from the equation of motion. The form of the equation used by him neglecting the effect of damping can be written as follows, namely

(1 - η cos 2T) γ'' + 2η sin 2T γ' + (\frac{1}{r^2}) γ = - η sin 2T \quad (5.14)

The time responses determined from equation (5.14) for r = 0.06, 0.08, 0.5 and 6.0 when η = 0.544 are shown from Fig. 5.24 to Fig. 5.27 respectively. Comparing Figs. 5.24 and 5.25 with corresponding solutions
of equation (4.8), Fig. 4.6 and Fig. 4.7, it can be seen that the responses at low values of \( r = 0.06 \) and 0.08 show no change. The solution at \( r = 0.5 \) retains the beat form but the amplitudes become large in size. The beat form disappears from the solution of equation (5.14) at \( r = 6.0 \) and the amplitude is again larger in size (compare Fig. 5.26 with Fig. 4.9 and Fig. 5.27 with Fig. 4.18). Table 5.3 gives the results of the waveform analysis at different values of the ratio \( r \).

Fig. 5.28 shows the variation of maximum amplitude against frequency ratio \( r \) and it is found that the solutions for the ratio \( r \gtrsim 0.5 \) are stable although the amplitudes grow quite large compared to those in the vicinity of this region.

Fig. 5.29 shows the variation of the ratio \( (\omega/\omega_1) \) over the range of \( r = 0.1 \) to \( r = 10 \) and Fig. 5.30 divides the plane \( (r - \eta) \) into stable and unstable regions for \( r \gtrsim 1 \). The instability region does not extend indefinitely when \( \eta = 0.8 \) as it is found to do for equation (4.8). A study of these solutions, therefore, indicates that although they do not change at lower values of \( r < 0.2 \), the theoretical amplitudes of the waveforms, at values of \( r = 0.5, 6.0 \) and greater are comparatively larger in size. The shapes of the waveforms are not greatly modified but nevertheless still affected by the omission of the term from the elastic coefficient of the equation as it is derived from energy considerations.

5.6. Effect of the Presence of a Constant Term on the Right Hand Side of Equation (4.8)

The gas pressure tangential effort diagram when resolved would
lead to a constant term in addition to the other harmonic components. Hence the effects of the addition of a constant term on the right hand side of equation (4.8) are investigated. For this purpose the equation can be written as

\[(1 - \eta \cos 2T) \gamma'' + 2\eta \sin 2T \gamma' + \left(\frac{1}{r^2} + 2\eta \cos 2T\right) \gamma = N_1 - \eta \sin 2T \quad (5.15)\]

Fig. 5.31 to Fig. 5.34 show the solutions of equation (5.15) when \(N_1 = 1\) and \(r = 0.06, 0.3, 0.6\) and 0.8 respectively. Fig. 5.35 to Fig. 5.38 show the corresponding waveforms for \(N_1 = 2\). On comparison of Fig. 5.31 with Fig. 5.35, it can be seen that at low values of \(r = 0.06\), the solutions do not change and they are similar to that given by equation (4.8) in Fig. 4.6. At higher values of \(r = 0.3, 0.6\) and 0.8 the comparison of respective solutions, when \(N_1 = 1\) to those when \(N_1 = 2\), shows that the vibratory motion is not symmetrical about the reference axis and the deviations increase with the increase in the values of \(N_1\) but no change is observed in the size of the instability regions.

From the analysis of the different terms of equation (4.8) for their effects on the solutions, it can be concluded that all the terms in the equation have an important bearing on the vibratory motion.
### TABLE 5.1 ANALYSIS OF SOLUTIONS OF EQUATION (5.11) FOR $\eta = 0.544$

<table>
<thead>
<tr>
<th>Ratio $r$</th>
<th>Speed of rotation $\omega$ rad/sec</th>
<th>Maximum amplitude $A_{\text{max}}$</th>
<th>Frequency of response $\omega_1$ rad/sec</th>
<th>Ratio $\omega/\omega_1$</th>
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</thead>
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### Table 5.3: Analysis of Waveform Solutions Omitting the Variable Term, $2\eta \cos 2t$, from the Elastic Coefficient of the Equation of Motion for $\eta = 0.54$.

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<th>Ratio $r$</th>
<th>Speed of rotation $\omega$ rad/sec</th>
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<th>Frequency of response $\omega_I$ rad/sec</th>
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<td>$A_{\text{max}}$</td>
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Fig. 5.1 Theoretical waveform relationship of $\gamma(t)$ for $r = 0.06$ and $\eta = 0.54$, neglecting the first order term and the variable part in the coefficient of the second order term.
Fig. 5.2 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.5$ and $\eta = 0.544$ neglecting the first order term and the variable part in the coefficient of the second order term.
Fig. 5.3  Theoretical waveform relationship of $y \sim t$ for $r = 0.06$ and $\eta = 0.544$
neglecting the variable part in the coefficient of the second order term
Fig. 5.4  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.5$ and $\eta = 0.544$ neglecting the variable part in the coefficient of the second order term.
Fig. 5.5  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.6$ and $\eta = 0.544$ neglecting the variable part in the coefficient of the second order term
Fig. 5.6 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.04$ and $\eta = 0.544$

neglecting the first order term and the variable part in the coefficient of the elastic term
Fig. 5.7 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.08$ and $\eta = 0.544$, neglecting the first order term and the variable part in the coefficient of the elastic term.
Fig. 5.8  Theoretical waveform relationship of γ ~ t for r = 0.1 and η = 0.544, neglecting the first order term and the variable part in the coefficient of the elastic term.
Fig. 5.9  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.6$ and $q = 0.544$ neglecting the first order term and the variable part in the coefficient of the elastic term
Fig. 5.10 Theoretical waveform relationship of $\gamma(t)$ for $r = 2.0$ and $\eta = 0.544$, neglecting the first order term and the variable part in the coefficient of the elastic term.
Fig. 5.11 Maximum amplitude $A_{\text{max}}$ of the response versus $r$ for $\eta = 0.544$ neglecting the first order term and the variable part in the coefficient of the elastic term.
Fig. 5.13  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.04$ and $\eta = 0.544$

neglecting the first order term
Fig. 5.14 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.06$ and $\eta = 0.544$

neglecting the first order term
Fig. 5.15  Theoretical waveform relationship of \( Y \sim t \) for \( r = 0.08 \) and \( n = 0.544 \), neglecting the first order term.
Fig. 5.16  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.1$ and $\eta = 0.544$

neglecting the first order term
Fig. 5.17  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.5$ and $\eta = 0.544$

neglecting the first order term
Fig. 5.18 Maximum amplitude $A_{\text{max}}$ of the response versus $r$
for $\eta = 0.544$ neglecting the first order term
Fig. 5.19 Relationship between $\omega/\omega_x$ and $r$ for $\eta = 0.544$ neglecting the first order term
Fig. 5.20 Stability diagram ($\eta$--$r$) for the region $r = \frac{1}{2}$, neglecting the first order term.
Fig. 5.21 Stability diagram ($\eta - r$) for the region real neglecting the first order term
Fig. 5.22 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.06$ and $\eta = 0.544$

when the forcing term $-\eta \sin 2T$ is omitted
Fig. 5.23 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.5$ and $\eta = 0.544$

when the forcing term $- \eta \sin 2T$ is omitted
Fig. 5.24  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.06$ and $\eta = 0.544$

when the variable part, $2\eta \cos 2T$, from the elastic term is omitted
Fig. 5.25 Theoretical waveform relationship of $\gamma - t$ for $\gamma = 0.08$ and $\eta = 0.54$.

when the variable part, $2n \cos 2T$, from the elastic term is omitted.
Fig. 5.26 Theoretical waveform relationship of $\gamma \sim t$ for $r=0.5$ and $\eta=0.544$ neglecting $2\eta\cos 2T$ from the elastic term.
Fig. 5.27  Theoretical waveform relationship of $\gamma \sim t$ for $r = 6.0$
and $\eta = 0.544$ neglecting $2\eta \cos 2T$ from the elastic term
Fig. 5.28 Maximum amplitude $A_{\text{max}}$ of the response versus $r$ for $\eta = 0.544$ neglecting $2\eta \cos 2\pi f$ from the elastic term.
Fig. 5.29  Relationship between $\omega/\omega_i$ and $r$ for $\eta = 0.544$ neglecting $2\eta \cos 2T$ from the elastic term.
Fig. 5.30  Stability diagram (η–r) for the region r=1
neglecting $2\eta \cos 2\theta$ from the elastic term
Fig. 5.31  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.06$, $\eta = 0.544$ and $N_1 = 1$. 
Fig. 5.32 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.3$, $\eta = 0.544$ and $N_1 = 1$. 
Fig. 5.33  Theoretical waveform relationship of $\gamma(t)$ for $r = 0.6$, $\eta = 0.544$ and $N_t = 1$
Fig. 5.34 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.8$, $\eta = 0.544$ and $N_1 = 1$
Fig. 5.35  Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.06$, $\eta = 0.544$, and $N_1 = 2$
Fig. 5.36  Theoretical waveform relationship of $y \sim t$ for $z = 0.3$, $\eta = 0.5$, $\xi$ and $N_1 = 2$. 
Fig. 5.37 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.6$, $\eta = 0.544$ and $N_1 = 2$
Fig. 5.38 Theoretical waveform relationship of $y \sim t$ for $r = 0.8$, $\eta = 0.544$ and $N_1 = 2$
6.1. Introduction

The solutions of equation of motion (4.8) obtained by use of numerical processes may sometimes be misleading. Hence it is thought to be desirable to obtain time responses from some analytical process as an independent verification of the numerical methods. Therefore the solutions are determined by use of the WKBJ approximation and the method of variation of parameters\(^{41}\).

6.2. Method of Variation of Parameters

Once the complementary function of a linear differential equation is known, the method of variation of parameters can be used to find the additional parts of the solution namely the particular integral. If the method is applied to a higher order differential equation it is found simpler to reduce the equation to a set of simultaneous first order equations. The principle involved is that the particular integral of a linear equation can be written as the product of one or more parts of the complementary function and one or more functions of the independent variable. Take the case of two first order equations represented as

\[
\begin{align*}
\frac{dx}{dt} &= f_1 (X, Y, t) + \psi_1 (X, Y, t) \\
\frac{dy}{dt} &= f_2 (X, Y, t) + \psi_2 (X, Y, t)
\end{align*}
\]  

(6.1)
where $X$ and $Y$ are the dependent variables and $t$ is an independent variable. Function $\psi$ represents the forcing function of the equation for the determination of the particular integral.

The solutions of the equations

$$\frac{dX}{dt} = f_1(X, Y, t) \quad (6.2)$$

$$\frac{dY}{dt} = f_2(X, Y, t)$$

can be written as

$$X = F_1(A_1, A_2, t) \quad (6.3)$$

$$Y = F_2(A_1, A_2, t)$$

where $A_1$ and $A_2$ are arbitrary constants and should be determined from the initial conditions of the system. Equations (6.3) are known as the generating solution.

Now to extend the procedure to determine the particular integral by use of the variation of parameters, the quantities $A_1$ and $A_2$ are no more supposed to be constants but are allowed to vary with respect to an independent variable $t$. Equations (6.3) with the varying parameters when substituted in equation (6.1) yield

$$\frac{dX}{dt} = \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial A_1} \frac{dA_1}{dt} + \frac{\partial F_1}{\partial A_2} \frac{dA_2}{dt} = f_1(F_1, F_2, t) + \psi_1(F_1, F_2, t) \quad (6.4)$$

$$\frac{dY}{dt} = \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial A_1} \frac{dA_1}{dt} + \frac{\partial F_2}{\partial A_2} \frac{dA_2}{dt} = f_2(F_1, F_2, t) + \psi_2(F_1, F_2, t)$$
The solutions $F_1$ and $F_2$ are determined so that

$$\frac{\partial F_1}{\partial t} = f_1 (F_1, F_2, t)$$

$$\frac{\partial F_2}{\partial t} = f_2 (F_1, F_2, t)$$

and hence,

$$\frac{\partial F_1}{\partial A_1} \frac{dA_1}{dt} + \frac{\partial F_1}{\partial A_2} \frac{dA_2}{dt} = \Psi_1 (F_1, F_2, t)$$

$$\frac{\partial F_2}{\partial A_1} \frac{dA_1}{dt} + \frac{\partial F_2}{\partial A_2} \frac{dA_2}{dt} = \Psi_2 (F_1, F_2, t)$$

(6.5)

The two simultaneous algebraic equations (6.5) give the expressions

$$\frac{dA_1}{dt} = \frac{(\partial F_2/\partial A_2) \Psi_1 (F_1, F_2, t) - (\partial F_1/\partial A_2) \Psi_2 (F_1, F_2, t)}{(\partial F_1/\partial A_1) (\partial F_2/\partial A_2) - (\partial F_1/\partial A_2) (\partial F_2/\partial A_1)}$$

$$\frac{dA_2}{dt} = \frac{(\partial F_1/\partial A_1) \Psi_2 (F_1, F_2, t) - (\partial F_2/\partial A_1) \Psi_1 (F_1, F_2, t)}{(\partial F_1/\partial A_1) (\partial F_2/\partial A_2) - (\partial F_1/\partial A_2) (\partial F_2/\partial A_1)}$$

(6.6)

Therefore,

$$A_1 (t) = \frac{(\partial F_2/\partial A_2) \Psi_1 (F_1, F_2, t) - (\partial F_1/\partial A_2) \Psi_2 (F_1, F_2, t)}{(\partial F_1/\partial A_1) (\partial F_2/\partial A_2) - (\partial F_1/\partial A_2) (\partial F_2/\partial A_1)} dt$$

$$A_2 (t) = \frac{(\partial F_1/\partial A_1) \Psi_2 (F_1, F_2, t) - (\partial F_2/\partial A_1) \Psi_1 (F_1, F_2, t)}{(\partial F_1/\partial A_1) (\partial F_2/\partial A_2) - (\partial F_1/\partial A_2) (\partial F_2/\partial A_1)} dt$$

(6.7)

Substitution of equations (6.7) into the generating solution of equations (6.3) gives the final solution.
This method of variation of parameters can be applied for the determination of the particular integral to the equation of the form

\[ y'' + 2U(T) y' + V^2(T) y = F(T) \quad (6.8) \]

where dashes represent differentiation with respect to the independent variable \( T \). Equation (6.8) can be reduced into two simultaneous first-order equations

\[ y' = x \quad (6.9) \]
\[ x' = -2U(T)x - V^2(T)y + F(T) \]

The complementary function may be written as

\[ y = Ay_1 + By_2 \quad (6.10) \]
and,

\[ x = Ay_1' + By_2' \quad (6.11) \]

where \( A \) and \( B \) are arbitrary constants. Now allowing the quantities \( A \) and \( B \) to vary with respect to \( T \) and using the relationships (6.10) and (6.11), the first of the equation (6.9) gives

\[ A y_1' + y_1 A' + B y_2' + y_2 B' = A y_1' + B y_2' \]

and therefore, \( A y_1' + B y_2' = 0 \) \quad (6.12)
The second part of the relations (6.9) yields

\[ A \frac{\gamma_1''}{\gamma_1} + B \frac{\gamma_2''}{\gamma_2} + A' \frac{\gamma_1'}{\gamma_1} + B' \frac{\gamma_2'}{\gamma_2} = -2U(T) - V^2(T) \gamma + F(T) \]

which gives

\[ A' \frac{\gamma_1'}{\gamma_1} + B' \frac{\gamma_2'}{\gamma_2} = F(T) \]  

(6.13)

since,

\[ A \frac{\gamma_1''}{\gamma_1} + B \frac{\gamma_2''}{\gamma_2} = -2U(T) - V^2(T) \gamma \]

By solving the equations (6.12) and (6.13), the expressions for \( A' \) and \( B' \) are

\[ A' = \frac{-F}{\sqrt{\gamma_1 \gamma_2' - \gamma_1' \gamma_2}} \]

(6.14)

\[ B' = \frac{F}{\sqrt{\gamma_1 \gamma_2' - \gamma_1' \gamma_2}} \]

These relations may be integrated to give values for \( A(T) \) and \( B(T) \) which on substitution in equation (6.10) would give the complete solution.

6.3. The WKBJ Approximation Method

Considering a function \( \gamma \) of the form

\[ \gamma = \left[ G(T) \right]^{-\frac{1}{2}} \left\{ A_1 \exp \left[ i \Phi(T) \right] + A_2 \exp \left[ -i \Phi(T) \right] \right\} \]

(6.15)

where \( \Phi(T) = \int G(T) \, dT \), the independent variable is \( T \) and \( i \) is the imaginary unit. Differentiating the relation (6.15) twice it can be shown to satisfy the equation

\[ \gamma'' + \left[ g^2 + \frac{g''}{2g} - \frac{3}{4} \left( \frac{g'}{g} \right)^2 \right] \gamma = 0 \]

(6.16)
If
\[
\left| \frac{G^2}{2G} \right| \ll \sqrt{\frac{G''}{2G} - \frac{3}{4} \left( \frac{G'}{G} \right)^2} \tag{6.17}
\]
equation (6.15) can represent an approximate solution of the equation of the form
\[
\gamma'' + G^2(T) \gamma = 0 \tag{6.18}
\]
Equation (6.15) can be written in the convenient form
\[
\gamma = \left[ G(T) \right]^{-\frac{1}{2}} \left[ A \cos \phi(T) + B \sin \phi(T) \right] \tag{6.19}
\]
if \( G^2 \) has a relatively large mean value about which only small variations take place so that \( G \) and \( \phi \) are both real.

6.4. Application of the WKBJ Approximation and Variation of Parameters Method to the Solutions of Equation (4.8)

Rewriting equation (4.8) in the form
\[
\gamma'' + 2 \left[ \frac{\eta \sin 2T}{1 - \eta \cos 2T} \right] \gamma' + \left[ \frac{\lambda + 2\eta \cos 2T}{1 - \eta \cos 2T} \right] \gamma = \frac{-\eta \sin 2T}{1 - \eta \cos 2T} \tag{6.20}
\]
where \( \lambda = \frac{1}{r^2} \),

let
\[
U(T) = \frac{\eta \sin 2T}{1 - \eta \cos 2T} , \quad V^2(T) = \frac{\lambda + 2\eta \cos 2T}{1 - \eta \cos 2T}
\]
and
\[
F(T) = \frac{-\eta \sin 2T}{1 - \eta \cos 2T}
\]
Equation (6.20) then reduces to
\[ y'' + 2 U(T) y' + V^2(T) y = F(T) \quad (6.21) \]

Changing the dependent variable from \( y \) to \( y \) through the relationship

\[ y = y \exp \left(- \int U(T) \, dT \right) \]

it becomes,

\[ y'' + \left[ V^2(T) - U^2(T) - U(T) \right] y = F(T) \exp \left[ \int U(T) \, dT \right] \quad (6.22) \]

or,

\[ y'' + \left[ \frac{\lambda + 2\eta \cos 2T}{1 - \eta \cos 2T} - \left( \frac{\eta \sin 2T}{1 - \eta \cos 2T} \right)^2 - \frac{(2\eta \cos 2T - \eta^2)^2}{(1 - \eta \cos 2T)^2} \right] y \]

\[ = F(T) \exp \left[ \frac{1}{2} \log_e (1 - \eta \cos 2T) \right] \quad (6.23) \]

Since,

\[ U(T) = \frac{(2\eta \cos 2T - \eta^2)}{(1 - \eta \cos 2T)^2} \]

and

\[ \int U(T) \, dT = \frac{1}{2} \log_e (1 - \eta \cos 2T) \]

equation (6.23) on simplification reduces to

\[ y'' + \left[ \frac{\lambda - \lambda \eta \cos 2T + \eta^2 \sin^2 2T}{(1 - \eta \cos 2T)^2} \right] y \]

\[ = F(T) \exp \left[ \frac{1}{2} \log_e (1 - \eta \cos 2T) \right] \quad (6.24) \]

Substituting \[ \frac{\lambda - \lambda \eta \cos 2T + \eta^2 \sin^2 2T}{(1 - \eta \cos 2T)^2} = g^2(T) \]
equation (6.24) can be expressed as

\[ y'' + G^2(T) y = F(T) \exp \left[ \frac{1}{2} \log_e (1 - \eta \cos 2T) \right] \]  \hspace{1cm} (6.25)

For the complementary function, equation (6.25) is written in the form

\[ y'' + G^2(T) y = 0 \]  \hspace{1cm} (6.26)

Referring to equation (6.17) if the condition

\[ \left| g^2 \right| \gg \left| \frac{g''}{2G} - \frac{3}{4} \left( \frac{g'}{G} \right)^2 \right| \]  is satisfied then the complementary function for the equation (6.26) from the WKBJ approximation is

\[ y = \left[ G(T) \right]^{-\frac{1}{2}} \left[ A \cos \Phi(T) + B \sin \Phi(T) \right] \]

where \( \Phi(T) = \int G(T) \, dT \)

Therefore,

\[ \gamma = [G(T)]^{-\frac{1}{2}} \left[ A \cos \Phi(T) + B \sin \Phi(T) \right] \exp \left[ - \frac{1}{2} \log_e (1 - \eta \cos 2T) \right] \]  \hspace{1cm} (6.27)

Now the expression for \( G^2 \) is

\[ G^2(T) = \frac{\lambda - \lambda \eta \cos 2T + \eta^2 \sin^2 2T}{(1 - \eta \cos 2T)^2} \]  \hspace{1cm} (6.28)
\[ g' = \frac{(a^2)'}{2a} \]  \hspace{1cm} (6.29)

\[ g'' = -\frac{1}{2} \left[ \frac{(a^2)''}{a} - \frac{[(a^2)']^2}{2a^3} \right] \]  \hspace{1cm} (6.30)

\[ (a^2)' = \left[ \sin 4T \left( 2\eta^2 \lambda + 2\eta^2 + 2\eta^4 \right) - 4\eta^3 \sin 4T \cos 2T - 2\eta^2 \sin 2T \right. \]

\[-4\eta^3 \sin^2 2T - \eta^3 \lambda \sin 4T \cos 2T \]  \hspace{1cm} (6.31)

\[ (a^2)'' = \left[ \left\{ 8\eta^2 \cos 4T \left( 1 + \lambda + \eta^2 \right) + 8\eta^3 \sin 2T \sin 4T - 16\eta^3 \cos 2T \cos 4T \right. \right. \]

\[-4\eta \cos 2T - 24\eta^3 \sin^2 2T \cos 2T + 2\eta^3 \lambda \sin 2T \sin 4T \]  \hspace{1cm} (6.32)

\[-4\eta^3 \lambda \cos 2T \cos 4T \right] \right/ (1 - \eta \cos 2T)^4 \right] \]

\[ - \left[ \left\{ 2\eta^2 \sin 4T \left( 1 + \lambda + \eta^2 \right) - 4\eta^3 \sin 4T \cos 2T - 2\eta^2 \sin 2T \right. \right. \]

\[-4\eta^3 \sin^2 2T - \eta^3 \lambda \sin 4T \cos 2T \right] \right\} (8\eta \sin 2T)/ \]

\[ (1 - \eta \cos 2T)^5 \]  \hspace{1cm} (6.32)

Making use of the expressions (6.28) to (6.32) a computer programme in Algol language is written for the condition (6.17) and the range of \( r \) in which the above method is applicable is determined for the specific value of \( \eta = 0.544 \) (see appendix 4). It is found that the condition (6.17) is satisfied for low values of \( r \leq 0.08 \).

Writing the complementary solution (6.27) in the form
\[ \gamma = A \gamma_1 + B \gamma_2 \] (6.33)

where

\[ \gamma_1 = G^{-\frac{1}{2}} \left[ \cos\phi \exp\left\{ -\frac{1}{2} \log_e (1 - \eta \cos 2\tau) \right\} \right] \]

and

\[ \gamma_2 = G^{-\frac{1}{2}} \left[ \sin\phi \exp\left\{ -\frac{1}{2} \log_e (1 - \eta \cos 2\tau) \right\} \right] \]

the method of variation of parameters can be applied by allowing the quantities A and B to vary with \( \tau \). Differentiating \( \gamma_1 \) and \( \gamma_2 \) gives

\[ \gamma'_1 = -G^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \log_e (1 - \eta \cos 2\tau) \right\} \left[ \frac{\eta \sin 2\tau \cos\phi}{(1 - \eta \cos 2\tau)} \right] \]

\[ + \phi' \sin\phi + \frac{(g^2)'}{4g^2} \cos\phi \] (6.34)

\[ \gamma'_2 = -G^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \log_e (1 - \eta \cos 2\tau) \right\} \left[ \frac{\eta \sin 2\tau \sin\phi}{(1 - \eta \cos 2\tau)} \right] \]

\[ - \phi' \cos\phi + \frac{(g^2)'}{4g^2} \sin\phi \] (6.35)

Rewriting the expressions (6.14)

\[ A' = \frac{-F \gamma_2}{\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2} \]

\[ B' = \frac{F \gamma_1}{\gamma_1 \gamma'_2 - \gamma'_1 \gamma_2} \]

and integrating, the values of A(\( \tau \)) and B(\( \tau \)) are given by
\[ A(T) = -\int \frac{F \gamma_2}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \, dT + C_1 \]  

(6.36)

and

\[ B(T) = \int \frac{F \gamma_1}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \, dT + C_2 \]

Substituting the equations (6.36) in equation (6.33) would give the solution of equation (6.20) so that

\[ \gamma = \left( -\int \frac{F \gamma_2}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \, dT + C_1 \right) \gamma_1 + \left( \int \frac{F \gamma_1}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \, dT + C_2 \right) \gamma_2 \]  

(6.37)

where \( C_1 \) and \( C_2 \) are arbitrary constants which should be determined from the initial conditions

\[ \gamma = 1 \quad \text{when} \quad T = 0 \]

\[ \gamma' = 0 \quad \text{when} \quad T = 0 \]

Thus,

\[ C_1 = \left( \frac{\lambda}{1 - \eta} \right) \frac{1}{4} \exp \left\{ \frac{1}{2} \log_e (1 - \eta) \right\} \]

and

\[ C_2 = 0 \]

Therefore the complete solution of equation (4.8) is

\[ \gamma = \left[ -\int \frac{F \gamma_2}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \, dT + \left( \frac{\lambda}{1 - \eta} \right) \frac{1}{4} \exp \left\{ \frac{1}{2} \log_e (1 - \eta) \right\} \right] \gamma_1 \]

\[ + \left[ \int \frac{F \gamma_1}{\gamma_1 \gamma_2' - \gamma_1' \gamma_2} \, dT \right] \gamma_2 \]  

(6.38)

Making use of Simpson's rule for integration equation (6.38) is
programmed in Algol language for the solution of equation (4.8) on a digital computer (see appendix 5).

Fig. 6.1 to Fig. 6.4 show the time responses of equation (4.8) using the WKBJ approximation and variation of parameters method for \( r = 0.02, 0.06, 0.08 \) and 0.3 respectively. Comparing Fig. 6.2 with Fig. 4.6 and Fig. 6.3 with Fig. 4.7 it can be seen that the solutions are in close agreement with those found from the numerical analysis methods. If at some point of the \( \gamma \sim t \) relationship there is an abrupt change in \( G^2(t) \) it may give values of \( G \) so large that the inequality (6.17) is not justified. Occurrence of such a situation can be clearly seen in the solution of the equation of motion for \( r = 0.3 \) and \( \eta = 0.544 \) (Fig. 6.4).

The theoretical solutions show a modulation of amplitude and frequency which can be explained from the WKBJ approximation. As it has already been shown (see section 5.4) that the Impulse term \( -\eta \sin 2T \) has no effect on the responses for low values of \( r \), the solution (6.27) may be written in the form

\[
\gamma = A \left[ \exp \left( -\frac{1}{2} \log_v \left( 1 - \eta \cos 2T \right) \right) \left\{ G(T) \right\}^{-\frac{1}{2}} \right] \\
\cos \left[ \int \frac{(\lambda - \lambda \eta \cos 2T + \eta^2 \sin^2 2T)\frac{1}{2}}{(1 - \eta \cos 2T)} \, dT + \Theta_0 \right] \quad (6.39)
\]

where \( A \) and \( \Theta_0 \) are arbitrary constants.

This solution represents the time responses of \( \gamma \) in which a modulation of both amplitude and frequency is seen to occur. The region of \( r \) in which equation (6.39) is valid can be determined for specific values of \( \eta \) by applying the test of inequality (6.17).
Fig. 6.1  Theoretical waveform relationship of \( \gamma \sim t \) for \( r=0.02 \) and \( \eta = 0.544 \) determined by use of the method of variation of parameters and the WKBJ approximation
Fig. 6.2 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.06$ and $\eta = 0.544$ determined by use of the method of variation of parameters and the WKBJ approximation.
Fig. 6.3 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.08$ and $\eta = 0.544$
determined by use of the method of variation of parameters and the
WKBJ approximation
Fig. 6.4 Theoretical waveform relationship of $\gamma \sim t$ for $r = 0.3$ and $\eta = 0.544$ determined by use of the method of variation of parameters and the WKBJ approximation.
The solutions of equation (4.8) obtained by numerical analysis for low values of \( r \leq 0.08 \) are compared with those found from the WKBJ method of analysis and are seen to be in close agreement. It is thought desirable to confirm the numerical solutions for the full range of \( r \) under investigation and also to compare the solutions for the effect of different terms of the equation of motion with those obtained by use of an analogue - computer.

Fig. 7.1 shows the unscaled computer flow diagram of the arrangement for the solution of equation (4.8) which includes division networks. The symbols in the diagram follow conventional notation. Analogue computer arrangements involving division circuits are troublesome in practice since they exhibit a tendency towards instability. In this case it is found necessary to inhibit the instability by introducing a feedback capacitor across the high gain amplifier. Such an addition modifies the high - frequency response of the amplifier but no practical disadvantage arises in its use.

The solutions at \( r = 0.04, 0.06, 0.08, 0.5, 0.6, 0.55, 1.0 \) and \( 6.0 \) for \( \eta = 0.544 \) are shown from Fig. 7.2 to Fig. 7.9. These time responses show close agreement with corresponding solutions obtained by use of a digital computer given in Fig. 4.5 to Fig. 4.7, Fig. 4.9, Fig. 4.10, Fig. 4.14, Fig. 4.15 and Fig. 4.18.
Fig. 7.10 and Fig. 7.11 give the solutions of the equation:

\[(1 - \eta \cos 2\tau) \gamma'' + \left(\frac{1}{x^2} + 2\eta \cos 2\tau\right) \gamma = 0\]  \hspace{1cm} (7.1)

at \(r = 0.5\) and 0.6 for \(\eta = 0.544\) and it is found that the general behaviour of the system is modified but nevertheless still affected by the variable inertia.

The effect of the first order term is also investigated by modifying the analogue computer arrangement of Fig. 7.1 to give the solutions of equation (5.12). Fig. 7.12 to Fig. 7.16 give the time responses for \(r = 0.04, 0.06, 0.5, 0.6\) and 6.0. Fig. 7.16 shows that the solution is unstable at \(r = 6.0\) and from further comparison of the time responses of Fig. 7.12 to Fig. 7.14 with corresponding numerical analysis results of Fig. 5.13, Fig. 5.14 and Fig. 5.17 it can be seen that the analogue computer results confirm the accuracy of the numerical analysis.
Fig. 7.1 Analogue computer arrangement for the solution of equation (4.8)
Fig. 7.4  Analogue solution for the waveform relationship of $y \sim t$ for $r = 0.05$ and $\eta = 0.5\mu$
Fig. 7.6 Analogous solution for the waveform relationship of $\gamma \sim t$ for $r = 0.5$ and $\eta = 0.544$. 
Fig. 7.7 Analogue solution for the waveform relationship of $\gamma \sim t$ in the unstable region corresponding to $r = \frac{1}{2}$ for $r = 0.55$ and $\eta = 0.544$. 
Fig. 7.8 Analogue solution for the waveform relationship of $\gamma \sim t$ in the unstable region corresponding to $r = 1.0$ for $r = 1.0$ and $\eta = 0.544$. 
Fig. 7.9 Analogue solution for the waveform relationship of $\gamma \sim t$ for $r = 6.0$ and $\eta = 0.544$
Fig. 7.10. Analogue solution for the waveform relationship of $\gamma = t$
for $r = 0.5$ and $\eta = 0.5$ when the first order term and
the forcing term $\eta \sin 2T$ are omitted.
Fig. 7.11 Analogue solution for the waveform relationship of $\gamma(t)$ for $r = 0.6$ and $\eta = 0.24$ when the first order term and the forcing term $-\eta \sin 2\pi T$ are omitted.
Fig. 7.15 Analogue solution for the waveform relationship of $\gamma \sim t$

for $r = 0.6$ and $\eta = 0.544$ neglecting the first order term
Fig. 7.16
Analogue solution for the waveform relationship of $y/t$
for $r = 6.0$ and $n = 0.54$, neglecting the first order term.
CHAPTER 8

EFFECT OF FORCING TERMS

Representative values for the harmonic components of the gas pressure tangential effort in internal combustion engines are given by Ker Wilson (34). The various harmonic excitations are calculated from the corresponding tangential effort and the following single cylinder engine data:

Equivalent inertia of rotating and reciprocating parts \( I_m = I + \frac{1}{2} M a^2 \) = 97697 kg cm sec\(^2\)

Bore diameter = 90 cm

Stroke = 155 cm

m.i.p = 169.5 lb/in\(^2\)

b.h.p = 2900

\( \eta = 0.3 \)

The complete equation including the cylinder impulse \( K \) of order \( n \) periods per revolution is therefore

\[
I_m (1 - \eta \cos 2\omega t) \frac{d^2 y}{dt^2} + 2I_m \eta \sin 2\omega t \frac{dy}{dt} + \left( \frac{\omega^2}{r^2} + 2\omega^2 \eta \cos 2\omega t \right) I_m y
\]

\[
= -\eta I_m \omega^2 \sin 2\omega t + K \sin (n\omega t + \alpha) \quad (8.1)
\]

Since the engine is of the two-stroke cycle type, the harmonic analysis of the tangential effort contains all the integer orders, that is to say \( n = 1, 2, 3, 4 \) etc. The responses of the system with each order...
of excitation from 1 to 4 are determined from equation (8.1) making use of the modified Euler's method for the range of \( r = 0.02 \) to \( r = 10.0 \).

It is of interest to note that the responses are exactly similar to the case of equation (4.8), except when the waveform solution of equation (4.8), contains a component of motion of the same order as the external excitation. The second order cosine component of the excitation also shows similar results, but the second order sinusoidal impulse arising from the incomplete balance of reciprocating parts has a significant effect at higher speeds for \( r > 0.2 \) as shown previously. Table 4.11 shows that when \( r = \frac{1}{4} \) and \( \eta = 0.3 \) the waveform solution of equation (4.8) is composed of 4th order as a predominant component and for \( \eta = 0.3 \) at \( r = \frac{1}{3} \). Table 4.12 shows that the main component in the solution is of 3rd order. The waveform solutions of equation (4.8) for \( \eta = 0.3 \) when \( r = \frac{1}{3} \) and \( \frac{1}{4} \) are given in Fig. 8.1 and Fig. 8.2 respectively. The responses from equation (8.1) are determined for \( n = 3 \) and 4 allowing the variation of \( \alpha \) from 0 to \( 2\pi \) in steps of \( \frac{\pi}{4} \).

Solutions for each case at \( \alpha = \frac{3\pi}{2} \) are given in Fig. 8.3 and Fig. 8.4. The maximum values of the amplitudes determined from such responses for the full range of \( \alpha \) are plotted in Figs. 8.5 and 8.6 for \( n = 3 \) and 4 with respect to the maximum amplitude (±1) when no forcing term is acting on the system.

Further investigations are carried out to study the effects of the higher order terms of the tangential effort arising from the gas pressure in the cylinder on the behaviour of the system. The harmonic analysis of Table 4.10 implies that when \( \eta = 0.3 \) and \( r = \frac{1}{10} \) the time response of equation (4.8) is composed of 10th order as the principal
component and two small components of order 8 and 12. Fig. 8.7 to Fig. 8.9 show for example the modulated waveform solutions obtained from equation (8.1) for $\eta = 0.3$ when $n = 10$ and $\alpha = \frac{6\pi}{4}$, $n = 8$ and $\alpha = \frac{\pi}{2}$, $n = 12$ and $\alpha = \frac{6\pi}{4}$ respectively. Such time responses are determined for $n = 8$, 10 and 12 for $\alpha$ varying from 0 to $2\pi$ and the values of the maximum amplitudes, $A_{\text{max}}$, from these responses are plotted in Fig. 8.10. Similarly Fig. 8.11 to Fig. 8.13 show the time responses of equation (8.1) when $\eta = 0.2$ for $n = 10$ and $\alpha = \frac{6\pi}{4}$, $n = 8$ and $\alpha = \frac{\pi}{2}$, $n = 12$ and $\alpha = \frac{6\pi}{4}$ respectively. Fig. 8.14 gives the variation of $A_{\text{max}}$ with respect to the phase angle $\alpha$ for $\eta = 0.2$.

Another set of such investigations is presented for $\eta = 0.1$. At this low value of $\eta = 0.1$ the responses show that resonance for $n = 8$ and 12 occurs only in certain regions of $\alpha$. Fig. 8.15 to Fig. 8.17 show the resonance effect due to external excitations of order $n = 10$ and $\alpha = \frac{7\pi}{4}$, $n = 8$ and $\alpha = \frac{\pi}{4}$, $n = 12$ and $\alpha = \frac{5\pi}{4}$ respectively. The regions of $\alpha$ when the external excitations of order 8 and 12 show resonance effect have been marked in Fig. 8.18 indicating also that, if $n = 10$, resonance occurs at all values of the phase angle.

The magnitude of the harmonic components of the tangential effort diagram is taken as 100 lb/in$^2$ for all values of $n = 8$, 10 and 12 and twice that of the normal size when $n = 3$ and 4. The reason of assuming higher values is to get the clear picture of the behaviour of the system.

From Fig. 8.10 it can be seen that when $r = \frac{1}{10}$ and $\eta = 0.3$ the amplitude is increased by the application of externally applied torques of frequency 10 times engine speed, 8 times engine
speed and 12 times engine speed but it is not a significant increase. A similar effect is shown in Figs. 8.5 and 8.6 when the external excitations of order \( n = 3 \) and 4 act on the system for \( r = \frac{1}{3} \) and \( \frac{1}{4} \) respectively. On comparison of Fig. 8.10 with Fig. 8.14 it can be seen that as the value of \( \eta \) decreases the size of amplitudes increases and the gain in amplitude depends on the order number and the phase angle of the external excitation.

For \( r = \frac{1}{10} \) when \( \eta = 0.3 \) and 0.2 a truly resonant condition is not excited by the higher order torque components but for \( \eta = 0.1 \) and \( r = \frac{1}{10} \) the external excitation of order 10 excites a truly resonant condition at all values of \( \alpha \) changing from 0 to \( 2\pi \) as would be expected if there were no variation in inertia. Such resonant conditions with 8th order and 12th order externally applied torques are observed only in the limited ranges of phase angle as shown in Fig. 8.18. Thus Draminsky's observations are not found to be true in general except for the cases when the values of \( \eta \) are low and in the vicinity of 0.1.

This seems to be the reason why dangerous vibrations evoked due to secondary resonance have been observed only in certain cases of large marine diesel engines including units with 9, 10 and 12 cylinders which has undoubtedly been the cause of crankshaft failures. In such cases the reduction of a multi-mass vibration system to an equivalent single-mass system generally gives the factor \( \eta \) of the order of 0.1 in equation (8.1) and the working speeds of the engines are such that \( r < \frac{1}{5} \).
It is evident from the preceding discussion that an external excitation of the same order as one of the secondary components of the free vibrations of the system, for the range under investigation, $r < 0.2$, is capable of exciting all components and the system will be in true resonance only in a limited range of the phase angle $\alpha$. For this reason secondary resonance effects may be found in some engines and not in others.
Fig. 8.1 Theoretical waveform relationship of \( \gamma \sim t \) for

\[ r = 1/3 \quad \text{and} \quad \eta = 0.3 \]
Fig. 8.2 Theoretical waveform relationship of $\gamma \sim t$ for

$r = 1/4$ and $\eta = 0.3$
Fig. 8.3 Theoretical waveform relationship of $\gamma \sim t$ for

$r = 1/3$, $q = 0.3$, $n = 3$ and $\alpha = 3\pi/2$
Fig. 8.4  Theoretical waveform relationship of $\gamma \sim t$ for

$r = 1/4$, $n = 0.3$, $n = 4$ and $\alpha = 3\pi/2$
Fig. 8.5 Maximum amplitude $A_{\text{max}}$ of the response versus phase angle $\alpha$ for $r = 1/3$, $\eta = 0.3$ and $n = 3$.
Fig. 8.6  Maximum amplitude $A_{\text{max}}$ of the response versus phase angle $\alpha$ for $r = 1/4$, $\eta = 0.3$ and $n = 4$. 
Fig. 3.7 Theoretical waveform relationship of $\gamma \sim t$ for

$r = 1/10$, $\eta = 0.3$, $n = 10$ and $\alpha = 6\pi/4$
Fig. 8.9 Theoretical waveform relationship of $\gamma \sim t$ for $r = 1/10$, $\eta = 0.3$, $n = 12$ and $\alpha = 6\pi/4$
Fig. 8.10 Maximum amplitude $A_{\text{max}}$ of the response versus phase angle $\alpha$ for $r = 1/10$, $\eta = 0.3$ and $n = 8$, 10 and 12
Fig. 8.11 Theoretical waveform relationship of $y \sim t$ for

$r = 1/10$, $\eta = 0.2$, $n = 10$ and $\alpha = 6\pi/4$
Fig. 8.12 Theoretical waveform relationship of $\gamma \sim t$ for

\[ r = 1/10, \quad \eta = 0.2, \quad n = 8 \text{ and } \alpha = \pi/2 \]
Fig. 8.13 Theoretical waveform relationship of $\gamma \sim t$ for

$r = 1/10$, $\eta = 0.2$, $n = 12$ and $\alpha = 6\pi/4$
Fig. 8.14 Maximum amplitude $A_{\text{max}}$ of the response versus phase angle $\alpha$ for $r = \frac{1}{10}$, $\eta = 0.2$ and $n = 8, 10$ and 12.
Fig. 8.18  Regions of true resonance for $r = 1/10$, $\eta = 0.1$ and $n = 8$, 10 and 12
EFFECTS OF DAMPING

9.1. Derivation of the Equation of Motion

The equation of Lagrange for the coordinate \( \theta \) is

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = Q_\theta
\]

(9.1)

where \( Q_\theta \) includes both conservative and nonconservative forces. One can distinguish between the conservative force \( Q_{\text{C}} \) which can be derived from the potential energy expression \( V \), and the nonconservative force \( Q_{\text{nc}} \), which may include internal dissipative forces or any external forces and is not derived from a potential function, so that

\[
Q_\theta = Q_{\text{C}} + Q_{\text{nc}}
\]

(9.2)

Hence,

\[
Q_{\text{C}} = - \frac{\partial V}{\partial \theta} = - \mu (\theta - \theta_1)
\]

(9.3)

and

\[
Q_{\text{nc}} = - \rho \dot{\theta}
\]

The kinetic energy, \( T \), is given by

\[
T = \frac{1}{2} \dot{\theta}^2 (I + \frac{1}{2} Ma^2 - \frac{1}{2} Ma^2 \cos 2\theta) + \frac{1}{2} I_A \omega^2
\]

(9.4)

Substituting the relations (9.3) and (9.4) in equation (9.1), it reduces to
\[ \ddot{\theta} \left( I + \frac{1}{2} M a^2 - \frac{1}{2} M a^2 \cos 2\theta \right) + \frac{1}{2} M a^2 \dot{\theta}^2 \sin 2\theta + \rho \dot{\theta} + \mu (\theta - \theta_1) = 0 \quad (9.5) \]

Making use of the equations

\[ \eta = \frac{\frac{1}{2} M a^2}{I + \frac{1}{2} M a^2}, \quad \frac{1}{r^2} = \frac{\mu}{\omega^2 (I + \frac{1}{2} M a^2)} \quad (9.6) \]

and

\[ 2 c \omega_n = \frac{\rho}{I + \frac{1}{2} M a^2} \]

where \( 2 c \) is dimensionless damping coefficient, equation (9.5) becomes

\[ \ddot{\theta} (I - \eta \cos 2\theta) + \eta \dot{\theta}^2 \sin 2\theta + 2 c \omega_n \dot{\theta} + \omega_n^2 (\theta - \theta_1) = 0 \quad (9.7) \]

Since \( \theta_1 = \omega t \) and \( \theta = \omega t + \gamma \quad (9.8) \)

equation (9.7) simplifies to

\[ \ddot{\gamma} \left\{ 1 - \eta \cos 2(\omega t + \gamma) \right\} + \eta (\omega + \dot{\gamma})^2 \sin 2(\omega t + \gamma) + 2 c \omega_n (\omega + \dot{\gamma}) + \omega_n^2 \gamma = 0 \quad (9.9) \]

Equation (9.9) is further simplified by changing the independent variable to \( T = \omega t \) and letting dashes represent differentiation with respect to \( T \).
Neglecting the second order terms equation (9.10) can be linearised into the form

\[
\gamma' \left( 1 - \eta \cos (2\gamma + 2\eta) \right) + \eta \left( 1 + \gamma \right)^2 \sin (2\gamma + 2\eta) + \frac{2c}{r^2} (1 + \gamma)
\]

\[+ \frac{\gamma}{r^2} = 0 \quad (9.10)
\]

Making use of the modified Euler's relations (4.18), equation (9.11) is programmed in Algol language on the digital computer for its solution with the initial conditions

\[\gamma_o = l \quad \text{at} \quad T = 0 \quad \text{and} \quad \gamma'_o = 0 \quad \text{at} \quad T = 0
\]

The boundaries between stable and unstable regions are determined from equation (9.11) for different values of the coefficient \(c\) making use of the modified Euler's method for \(\eta\) ranging from 0 to 0.8. Fig. 9.1 and Fig. 9.2 divide the plane \((r - \eta)\) into stable and unstable regions in the vicinity of \(r = 1\) and \(r = 0.5\) respectively. This analysis shows that the stabilising effect of damping turns out to be much more pronounced when \(r \approx 0.5\) than when \(r \approx 1\) and also that if the value of the damping factor \(c\) is increased the ranges of instability become narrow and it increases the range of \(\eta\) in which the motion is always stable. These results closely agree with
Gregory's observations that when \( r \approx 1 \) the motion is always stable for the values of \( \eta \) between 0 to 4\( \pi \) and when \( r \approx 0.5 \) the range of \( \eta \) for stable motion is defined by \( \eta = 0 \) to \( \eta = 4\sqrt{2}\pi \).

Using the modified Euler's method the steady state time responses of the system determined from equation (9.11) for \( \eta = 0.3 \) and \( c = 0.008 \) are shown in Fig. 9.3 to Fig. 9.23 corresponding to \( r = 0.06, 0.08, 0.1, 0.2, 0.3, 0.4, 0.5, 0.532, 0.56, 0.6, 0.65, 0.7, 0.8, 0.9, 0.95, 1.5, 2, 4, 6, 8 \) and 10 respectively. Fig. 9.24 shows the growth of amplitude to a very high value in the critical speed range at \( r = 1.0 \). The analysis of these theoretical records is given in Table 9.1. Table 9.2 shows the analysis of similar theoretical records for \( \eta = 0.544 \).

The above results show that the steady state motion of the system is not always symmetrical about the reference axis and that there are two vibration cycles in one revolution of the shaft for all values of \( r \) below the unstable region \( r \approx 1 \) when \( \eta = 0.544 \) and in the range of \( r \leq 0.532 \) when \( \eta = 0.3 \). The time responses at higher values of \( r \) namely \( r = 2, 4, 6 \) etc. are repetitive complex waveforms. The variations of maximum negative and maximum positive amplitudes for \( \eta = 0.3 \) and \( \eta = 0.544 \) against speed ratio \( r \) varying from 0.06 to 10.0 are given in Fig. 9.25 to Fig. 9.28. The pairs of broken vertical lines on these figures show the region of instability for \( r \approx 1 \).

9.2. **Effect of Forcing Terms Including Damping**

The equation of motion, including the effect of damping and the externally applied torques, can be written as
The steady state time responses of the system defined by equation (9.12) are given in Fig. 9.29 to Fig. 9.36 for \( n = 2 \), \( r = 0.1 \), \( \eta = 0.3 \) and \( \alpha \) varying from 0 to \( 2\pi \) in steps of \( \pi/4 \). The investigation of the effect of the external excitation of order 2 is carried out since the steady state solution of equation (9.11) contained the vibrations such that two vibration cycles occurred in each revolution of the shaft. As the waveform solutions are not symmetrical about the reference axis, the variations of the maximum positive amplitude and the maximum negative amplitude with respect to phase angle \( \alpha \) are given in Fig. 9.37 and Fig. 9.38 respectively.

Further study of the solutions of equation (9.12) is also carried out for \( n = 10 \), \( r = 0.1 \) and \( \eta = 0.3 \) for the complete range of \( \alpha \) between 0 and \( 2\pi \). Fig. 9.39 shows one such time response of the system at \( \alpha = \pi \). Other factors remaining the same, the variation in the values of the phase angle produces no change and the steady state solutions are similar at all values of \( \alpha \) varying from 0 to \( 2\pi \). These solutions show the presence of beats and there are two beats in one revolution of the shaft.

The combined effect of the 2nd order and 10th order external excitations on the motion of the system including the effects of damping
can be determined from the solutions of the equation written as follows, namely

\[ I_m (1 - \eta \cos 2\omega t) \frac{d^2 \gamma}{dt^2} + I_m \omega (2 \eta \sin 2\omega t + \frac{2c}{r}) \frac{d\gamma}{dt} \]

\[ + \left( \frac{\omega^2}{r^2} + 2 \omega^2 \eta \cos 2\omega t \right) I_m \gamma = -\frac{2c}{r} \omega^2 I_m - \eta \omega^2 I_m \sin 2\omega t \]

\[ \frac{2a}{t} \sin (2\omega t + \alpha) + \frac{2a}{t} \sin (10\omega t + \alpha) \]

\[ \frac{\omega^2}{r^2} + 2 \omega^2 \eta \cos 2\omega t \]

The solutions of the above equation when \( r = 0.1, \ \eta = 0.3 \) for \( \alpha = 0 \), \( \pi/4 \) and \( 3\pi/4 \) are shown in Fig. 9.40 to Fig. 9.42 respectively. Again the beat form of the solutions is retained and there are two beats in each revolution of the shaft. The maximum amplitude and the maximum apparent frequency of one oscillation of the solution occur together as do the minimum values of these quantities. The solutions of equation (9.13) are only slightly modified compared to those given by equation (9.12) for \( n = 10 \).
### TABLE 9.1 ANALYSIS OF RESPONSE OF THE SYSTEM INCLUDING DAMPING FOR $\eta = 0.3$

<table>
<thead>
<tr>
<th>Ratio $r$</th>
<th>Speed of rotation $\omega$ rad/sec</th>
<th>Time for one revolution of shaft sec</th>
<th>Time for one vibration cycle sec</th>
<th>Maximum negative amplitude</th>
<th>Maximum positive amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>5.4</td>
<td>1.163</td>
<td>0.582</td>
<td>$2.05 \times 10^{-3}$</td>
<td>$1.40 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.08</td>
<td>7.2</td>
<td>0.870</td>
<td>0.435</td>
<td>$3.25 \times 10^{-3}$</td>
<td>$6.98 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.10</td>
<td>9.0</td>
<td>0.70</td>
<td>0.350</td>
<td>$4.72 \times 10^{-3}$</td>
<td>$1.55 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.20</td>
<td>18.0</td>
<td>0.349</td>
<td>0.174</td>
<td>$1.88 \times 10^{-2}$</td>
<td>$1.26 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.30</td>
<td>27.0</td>
<td>0.232</td>
<td>0.116</td>
<td>$5.29 \times 10^{-2}$</td>
<td>$4.53 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.40</td>
<td>36.0</td>
<td>0.174</td>
<td>0.087</td>
<td>$1.40 \times 10^{-1}$</td>
<td>$1.36 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.50</td>
<td>45.0</td>
<td>0.138</td>
<td>0.069</td>
<td>1.57</td>
<td>1.63</td>
</tr>
<tr>
<td>0.532</td>
<td>47.88</td>
<td>0.132</td>
<td>0.066</td>
<td>1.10</td>
<td>1.10</td>
</tr>
<tr>
<td>0.56</td>
<td>50.4</td>
<td>0.125</td>
<td>-</td>
<td>$5.40 \times 10^{-1}$</td>
<td>$5.22 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.60</td>
<td>54.0</td>
<td>0.116</td>
<td>-</td>
<td>$3.50 \times 10^{-1}$</td>
<td>$3.25 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.65</td>
<td>58.5</td>
<td>0.107</td>
<td>-</td>
<td>$2.54 \times 10^{-1}$</td>
<td>$2.40 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.70</td>
<td>63.0</td>
<td>0.099</td>
<td>-</td>
<td>$2.10 \times 10^{-1}$</td>
<td>$1.94 \times 10^{-1}$</td>
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</table>

(Cont.)
<table>
<thead>
<tr>
<th>Ratio ( r )</th>
<th>Speed of rotation ( \omega ) ( \text{rad/sec} )</th>
<th>Time for one revolution of shaft ( \text{sec} )</th>
<th>Time for one vibration cycle ( \text{sec} )</th>
<th>Maximum negative amplitude</th>
<th>Maximum positive amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>72.0</td>
<td>0.086</td>
<td>-</td>
<td>1.87x10^{-1}</td>
<td>1.53x10^{-1}</td>
</tr>
<tr>
<td>0.90</td>
<td>81.0</td>
<td>0.077</td>
<td>-</td>
<td>1.82x10^{-1}</td>
<td>1.50x10^{-1}</td>
</tr>
<tr>
<td>0.95</td>
<td>85.5</td>
<td>0.073</td>
<td>-</td>
<td>1.84x10^{-1}</td>
<td>1.50x10^{-1}</td>
</tr>
<tr>
<td>1.0</td>
<td>90.0</td>
<td>0.070</td>
<td>-</td>
<td>UNSTABLE</td>
<td>UNSTABLE</td>
</tr>
<tr>
<td>1.5</td>
<td>135.0</td>
<td>0.047</td>
<td>-</td>
<td>1.82x10^{-1}</td>
<td>1.35x10^{-1}</td>
</tr>
<tr>
<td>2.0</td>
<td>180.0</td>
<td>-</td>
<td>-</td>
<td>2.02x10^{-1}</td>
<td>1.40x10^{-1}</td>
</tr>
<tr>
<td>4.0</td>
<td>360.0</td>
<td>-</td>
<td>-</td>
<td>2.38x10^{-1}</td>
<td>1.72x10^{-1}</td>
</tr>
<tr>
<td>6.0</td>
<td>540.0</td>
<td>-</td>
<td>-</td>
<td>2.75x10^{-1}</td>
<td>2.02x10^{-1}</td>
</tr>
<tr>
<td>8.0</td>
<td>720.0</td>
<td>-</td>
<td>-</td>
<td>2.63x10^{-1}</td>
<td>1.86x10^{-1}</td>
</tr>
<tr>
<td>10.0</td>
<td>900.0</td>
<td>-</td>
<td>-</td>
<td>2.03x10^{-1}</td>
<td>1.41x10^{-1}</td>
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### TABLE 9.2  ANALYSIS OF RESPONSE OF THE SYSTEM INCLUDING DAMPING FOR $\eta = 0.544$

<table>
<thead>
<tr>
<th>Ratio $r$</th>
<th>Speed of rotation $\omega$ (rad/sec)</th>
<th>Frequency of one vibration cycle (rad/sec)</th>
<th>Maximum negative amplitude</th>
<th>Maximum positive amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>5.4</td>
<td>10.8</td>
<td>$2.95 \times 10^{-3}$</td>
<td>$1.03 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.08</td>
<td>7.2</td>
<td>14.4</td>
<td>$4.84 \times 10^{-3}$</td>
<td>$2.29 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.10</td>
<td>9.0</td>
<td>18.0</td>
<td>$7.25 \times 10^{-3}$</td>
<td>$4.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.20</td>
<td>18.0</td>
<td>36.0</td>
<td>$3.38 \times 10^{-2}$</td>
<td>$2.84 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.30</td>
<td>27.0</td>
<td>54.0</td>
<td>$1.12 \times 10^{-1}$</td>
<td>$1.12 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.40</td>
<td>36.0</td>
<td>72.0</td>
<td>$2.0 \times 10^{-1}$</td>
<td>$2.0 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.50</td>
<td>45.0</td>
<td>90.0</td>
<td>$8.1 \times 10^{-1}$</td>
<td>$8.3 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.532</td>
<td>47.88</td>
<td>95.75</td>
<td>$3.20$</td>
<td>$3.80$</td>
</tr>
<tr>
<td>0.60</td>
<td>54.0</td>
<td>108.0</td>
<td>$9.80 \times 10^{-1}$</td>
<td>$9.80 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.70</td>
<td>63.0</td>
<td>126.0</td>
<td>$4.20 \times 10^{-1}$</td>
<td>$4.20 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

(Cont.)
<table>
<thead>
<tr>
<th>Ratio (r)</th>
<th>Speed of rotation (ω) rad/sec</th>
<th>Frequency of one vibration cycle (rad/sec)</th>
<th>Maximum negative amplitude</th>
<th>Maximum positive amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>72.0</td>
<td>144.0</td>
<td>3.1x10^-1</td>
<td>3.1x10^-1</td>
</tr>
<tr>
<td>0.90</td>
<td>81.0</td>
<td>162.0</td>
<td>2.4x10^-1</td>
<td>2.4x10^-1</td>
</tr>
<tr>
<td>1.0</td>
<td>90.0</td>
<td>90.0</td>
<td>UNSTABLE</td>
<td>UNSTABLE</td>
</tr>
<tr>
<td>2.0</td>
<td>180.0</td>
<td>-</td>
<td>2.4x10^-1</td>
<td>1.8x10^-1</td>
</tr>
<tr>
<td>4.0</td>
<td>360.0</td>
<td>-</td>
<td>2.4x10^-1</td>
<td>2.1x10^-1</td>
</tr>
<tr>
<td>6.0</td>
<td>540.0</td>
<td>-</td>
<td>3.0x10^-1</td>
<td>3.2x10^-1</td>
</tr>
<tr>
<td>8.0</td>
<td>720.0</td>
<td>-</td>
<td>2.0x10^-1</td>
<td>1.8x10^-1</td>
</tr>
<tr>
<td>10.0</td>
<td>900.0</td>
<td>-</td>
<td>2.6x10^-1</td>
<td>2.4x10^-1</td>
</tr>
</tbody>
</table>
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$0.5$ 

$0.05$ 

$y$ 

$t$
Fig. 9.25 Maximum negative amplitude $A_{\text{max}}$ of the response versus $r$
for $\eta = 0.3$ and $c = 0.008$
Fig. 9.26 Maximum positive amplitude $A_{\text{max}}$ of the response versus $r$

for $\eta = 0.3$ and $c = 0.008$
Fig. 9.27 Maximum negative amplitude $A_{\text{max}}$ of the response versus $r$

for $\eta = 0.544$ and $c = 0.008$
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for $\eta = 0.544$ and $c = 0.008$
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\[
\text{Theoretical waveform relationship of } Y \sim t \text{ for } r = 1/10, \quad \eta = 0.3, \quad c = 0.008, \quad n = 2 \text{ and } \alpha = 3\pi/4. 
\]
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Fig. 9.33 Theoretical waveform relationship of $Y(t)$ for $r = 1/10$, $q = 0.3$, $c = 0.008$, $n = 2$, and $a = \pi$. 
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CHAPTER 10

SUMMARY OF RESULTS WITH DISCUSSION

Cases of failures of several large marine engine units with 9, 10 or 12 cylinders in practice are reported to have occurred due to secondary resonance. Draminsky in his analysis has shown using the example of a single cylinder engine system that large second order variation in inertia can undoubtedly be the cause of such crankshaft failures.

A new ten cylinder, two stroke cycle engine which was being put into service is investigated for such an effect. The conventional stress calculations for one node and two node modes are shown from Table 2.1 to Table 2.7. The stress values are found well within the safe limits defined by Lloyd's Register of Shipping. The 2-node 9th order resonant stress in the crankshaft calculated by a damped forced vibration tabulation (Fig. 2.2) is found to be the same (+ 640 lb/in²) as determined from the dynamic magnifier method (Table 2.6). It can also be seen from Table 2.6 that the 2-node 9th order critical of small equilibrium amplitude is nearly in resonance with the service speed and the 2-node 7th order critical has a much greater equilibrium amplitude. The critical speed corresponding to this larger harmonic torque, being well above the service speed, is normally disregarded. The 2-node 9th order resonant stress in the crankshaft worked out from a Draminsky calculation based on non-linear theory is + 1445 lb/in². According to the concept of the secondary resonance phenomenon to date it appears that large torsional vibrations
of 9th order at the service speed can be evoked by the 2-node geometric resultant of the ten torque impulses of 7th order.

The arithmetic sum of the 2-node 5th, 7th and 9th order flank stresses at the service speed, calculated by the conventional method as given in Table 2.7, is ± 1455 lb/in² which closely agrees in value with the 2-node 9th order stress in the crankshaft calculated from Draminsky's theory. The above agreement in the values of 2-node 9th order Draminsky stress and the arithmetic sum of the 5th, 7th and 9th order flank stresses at the service speed is more coincidental since the calculations are carried out by two methods based entirely on different considerations and also, in the case of Draminsky's theory, the 2-node 9th order would predominate.

The 2-node 9th order measured stress obtained by harmonic analysis of stress records taken on the Intermediate shaft (see Table 3.1) and the corresponding stress calculated from a damped - forced vibration tabulation are plotted in Fig. 3.11 for comparison. Similar curves are plotted in Fig. 3.12 for comparison between the 2-node 9th order measured torsional single amplitude at the front end of the engine and the corresponding torsional amplitude obtained from a damped forced vibration tabulation. An examination of Fig. 3.11 and Fig. 3.12 shows that the variation of the 2-node 9th order stress in the Intermediate shaft and the variation of the 2-node 9th order torsional amplitude at the front end of the engine follow similar patterns with respect to the corresponding theoretical results and that in this case stress magnification due to secondary resonance fails to appear despite all the conditions pointing to the strong possibility of large 9th order torsional vibrations being excited. In recent years
however several cases of suspected secondary resonance have been found in torsional vibrations in crankshaft systems of multi-cylinder diesel engines referred to by Draminsky and Archer. Therefore it was concluded that further theoretical work was necessary to clearly understand the behaviour of the variable inertia system.

The equation of motion (4.8) representing the mathematical model of a single cylinder engine system allowing for variable inertia is analysed by numerical analysis processes namely the modified Euler's method, Runge-Kutta method and the method of simultaneous equations. The results obtained by the modified Euler's method and Runge-Kutta method for the solution of equation (4.8) when \( r = 0.06 \) and \( \eta = 0.544 \) are compared in Fig. 4.4 and they show close agreement. The results of the modified Euler's method are further confirmed by the method of simultaneous equations. Therefore in all the following analyses of the equations of motion, the modified Euler's method is used.

The analyses of the theoretical responses at different values of the speed ratio \( r \) varying from 0.02 to 10 carried out by the use of Manley's method are shown from Table 4.1 to Table 4.4 for \( \eta = 0.544, 0.3, 0.42 \) and 0.8 respectively. In Table 4.1 the waveform analyses at the speed ratio \( r = 0.2, 0.4, 0.8, 8 \) and 10 are not given due to the complexity of the responses obtained. Such complex waveforms are given in Fig. 4.8, Fig. 4.12 and Fig. 4.19 for \( r = 0.4, 0.8 \) and 8 respectively. The frequency of a cycle of the response envelope as given in Table 4.1 to Table 4.4, is twice the speed of rotation in the range of \( r \) varying from 0.02 < \( r < 0.2 \). Above this range of \( r \) no
correlation is observed between the frequency of one cycle of the response envelope and the speed of rotation.

Visual analysis of the waveforms by Manley's method was carried out to develop some understanding of these time responses, but it is considered desirable to study the theoretical solutions by use of the Fourier's analysis carried out on the digital computer. In Table 4.5 to Table 4.17 are given the harmonic analysis results for the time responses at different values of \( \eta \) and \( r \). Table 4.7 shows that when \( \eta = 0.544 \) and \( r = 0.1 \) the waveform is composed of a main component of order 10 and the other components of order 8, 12, 14, 16 and 18 and that the secondary components are present in a significant proportion. Similarly Table 4.9 shows that when \( \eta = 0.3 \) and \( r = 0.08 \) the waveform is composed of a main component of order 12 and small components of order 10, 14 and 16 which is contradictory to the theory put forward by Draminsky that if \( \omega_n = n \omega \), and the system is executing natural or free vibrations of phase velocity \( \omega_n \) the waveform is composed of a principal component of order \( n \) and two small components of order \( (n - 1) \) and \( (n + 1) \). A careful study of the present analysis from Table 4.5 to Table 4.10 suggests that the orders of the harmonic components present in the vibratory motion of the system should be determined by the harmonic analysis of the solution of the equation of motion (4.8) for a definite speed of rotation and the specific value of \( \eta \) of the system.

Table 4.13 to Table 4.17 show that for a constant value of \( r = 0.1 \) the magnitude of the principal component in the waveform increases and the secondary components become smaller in size with the decrease
in the value of \( \eta \).

The boundaries of the unstable regions when \( r = 1 \) and \( \frac{1}{2} \) are determined for equation (4.8) making use of the modified Euler's method for \( \eta \) ranging from 0 to 0.8 as given in Fig. 4.54 and Fig. 4.55 respectively, (see also Table 4.18 and Table 4.19) and the results are in close agreement with Gregory's results calculated for \( \eta = 0 \) to \( \eta = 0.6 \).

Fig. 4.20 to Fig. 4.23 show the variation of maximum amplitude \( A_{\max} \) against frequency ratio \( r \) for \( \eta = 0.544, 0.3, 0.42 \) and 0.8 respectively and the regions of instability for \( r = 1 \) and \( \frac{1}{2} \) are marked on these graphs. Goldsbrough and Gregory have shown theoretically in their analyses that at still lower speeds \( r = 1/3, 1/4 \) etc. there are other ranges of instability of much smaller extent. They were not observed in the experimental investigations carried out by Goldsbrough. In the present analysis no such unstable regions are found for speeds corresponding to \( r = 1/3, 1/4 \) etc. (see solutions in Fig. 4.31 and Fig. 4.32) which is contradictory to the above mentioned theoretical prediction but is in agreement with Goldsbrough's experimental results.

Fig. 4.24 to Fig. 4.27 give the variation of the ratio of the shaft speed to the apparent frequency at maximum amplitude over the range of \( r \) varying from 0.02 to 10. Horizontal portions of the curves show the regions of instability clearly bringing out the fact that the ratio \( \omega/\omega_1 \) remains constant at \( \frac{1}{2} \) and 1 corresponding to respective unstable regions at \( r = \frac{1}{2} \) and \( r = 1 \).

Equation (4.8) is further analysed to investigate the effect
of each term on the solutions. This was felt necessary in view of
the omission of the variable part, \(2\eta \cos 2\gamma\), from the elastic term
of the equation by Draminsky. This term appears in the equation when
it is derived from energy considerations. Analysis of the effect of
the omission of the above term shows that the solutions at lower
values of \(r < 0.2\) do not change but they are nevertheless modified
at higher values of \(r = 0.5, 6.0\) etc., as can be seen from Fig. 5.24
to Fig. 5.27. Fig. 5.28 shows that the solutions are stable for the
values of \(r \approx \frac{1}{2}\) although the amplitudes grow large compared to the
solutions of the complete equation in the vicinity of this speed ratio.
The variation of ratio \((\omega/\omega_\gamma)\) is given in Fig. 5.29 over the range
of \(r = 0.1\) to \(r = 10\). The ratio \(\omega/\omega_\gamma\) is found to be equal to \(1/2\)
and \(1\) in the regions of \(r \approx \frac{1}{2}\) and \(1\) respectively.

Similarly the investigations carried out for the effect of first
order term are presented in Table 5.2, Fig. 5.18 and Fig. 5.19.
In the absence of this term from equation \((4.8)\), the solutions become
unstable at higher values of \(r\). Neglecting the impulse term, \(-\eta \sin 2\gamma\),
the solutions at low values of \(r < 0.2\) are unaffected whereas at higher
speed ratios the impulse term has a significant effect as shown in
Fig. 5.22 and Fig. 5.23.

The gas pressure tangential effort, when harmonically analysed,
contains a constant term and sine and cosine components. Hence the
presence of a constant term on the right hand side of equation \((4.8)\)
is investigated for its effect on the responses obtained by solution
of equation \((5.15)\). On comparison of Fig. 5.31 with Fig. 5.35 it
can be seen that the solutions are similar to those given by equation \((4.8)\)
for low values of the speed ratios $r < 0.2$ but the vibratory patterns at higher values of $r$ show deviations from the reference axis as can be seen from Fig. 5.32 to Fig. 5.34 and Fig. 5.36 to Fig. 5.38. The term by term analysis of equation (4.8) shows that all the terms have their effect on the time responses and no part of the equation should therefore be omitted in obtaining the solutions.

Use of the analytical processes namely the WKBJ approximation and variation of parameters method to find solutions of equation (4.8) is made, only, for low values of the speed ratio $r \leq 0.08$ when $\eta = 0.544$ since the WKBJ approximation would give solution only if the condition of expression (6.17) is satisfied for a specific combination of $\eta$ and $r$.

The occurrence of the modulation in amplitude and frequency in the solutions is explained from equation (6.39) which shows the time response of $\gamma$ in which a modulation of both amplitude and frequency is seen to occur. The region of $r$ in which equation (6.39) is valid can be determined for a specific value of $\eta$ by applying the test of inequality (6.17).

A comparison of the analogue computer results of Fig. 7.2 to Fig. 7.9 for equation (4.8) with the corresponding solutions of Fig. 4.5 to Fig. 4.7, Fig. 4.9, Fig. 4.10, Fig. 4.14, Fig. 4.15 and Fig. 4.18 obtained by use of a digital computer and further comparison of solutions given in Fig. 7.12 to Fig. 7.14 for equation (5.12) with corresponding digital computer results of Fig. 5.13, Fig. 5.14 and Fig. 5.17 confirm the accuracy of the numerical analysis method used.

The values of the maximum amplitudes ($A_{\text{max}}$) determined from
the time responses obtained from equation (8.1) for \( n = 8, 10 \) and 12 at \( r = 0.1 \) and phase angle \( \alpha \) varying from 0 to \( 2\pi \) in steps of \( \pi/4 \) are shown in Fig. 8.10 for \( \eta = 0.3 \), Fig. 8.14 for \( \eta = 0.2 \) and Fig. 8.18 for \( \eta = 0.1 \). On comparing Fig. 8.10 with Fig. 8.14 it can be seen that as the value of \( \eta \) decreases the size of amplitudes increases and the gain in amplitude depends on the order number and the phase angle of the external excitation. Fig. 8.18 shows that an external harmonic torque of order 10 excites a truly resonant condition when \( \eta = 0.1 \) at all values of the phase angle whereas 8th order and 12th order show resonant conditions only under certain circumstances.

Fig. 9.1 and Fig. 9.2 divide the plane \((r \sim \eta)\) into stable and unstable regions obtained from the solutions of equation (9.11) in the vicinity of \( r \simeq 1 \) and \( r \simeq \frac{1}{2} \) respectively. This shows that as the value of \( \alpha \) is increased the ranges of instability become narrow. The results closely agree with Gregory's observations that when \( r \simeq 1 \) the motion is always stable for the values of \( \eta \) between 0 to \( 4\alpha \) and when \( r \simeq \frac{1}{2} \) the range of \( \eta \) for stable motion is defined by \( \eta = 0 \) to \( \eta = 4\sqrt{2} \alpha \).

From Fig. 9.25 to Fig. 9.28 are shown the variations of maximum negative and maximum positive amplitudes against the speed ratio \( r \) which clearly indicate that the steady state responses of equation (9.11) including the effect of damping are not symmetrical about the reference axis. There are two vibration cycles in one revolution of the shaft for all values of \( r \) below the unstable region \( r \simeq 1 \) when \( \eta = 0.544 \) and in the range of \( r \leq 0.532 \) when \( \eta = 0.3 \).
Fig. 9.37 and Fig. 9.38 show the effect of a second order externally applied torque on the single cylinder engine system including damping and it may be noted that the steady state responses have small amplitudes compared to those obtained in the time responses of the system being excited by the 10th order impulse.

Equation (9.13) represents the motion of a single cylinder engine system with both externally applied torques of order 2 and 10 acting together. The solutions of this equation for $r = 0.1$ and $\eta = 0.3$ are given from Fig. 9.40 to Fig. 9.42 when $\alpha = 0$, $\pi/4$ and $3\pi/4$ respectively. Comparing these solutions with the time response of the system represented by equation (9.12) when $n = 10$ (Fig. 9.39), it is seen that the solutions of equation (9.13) are only slightly modified compared to those of equation (9.12) due to the presence of very small second order amplitudes.

Draminsky has given a method, based on a non-linear theory, to calculate the stresses of certain orders in multi-cylinder diesel engines allowing for the effect of secondary resonance. He has attributed the phenomenon of secondary resonance to the existence of large variable inertia in the systems. But his calculations do not always lead to positive results which is confirmed by the investigations presented in this thesis carried out on the case of a suspected resonance. The measured stress in this case is only slightly higher than the theoretical value.

In the light of the above observations the vibratory behaviour of the crank-connecting rod system including the effect of variable inertia has been studied in detail. It shows that in the range of
r < 0.2 the external excitations of the same orders as the components of motion in the solution of equation (4.8) do not put the crank-connecting rod system into a resonant condition at values of η = 0.3 and 0.2, although motions are magnified at corresponding values of phase angle α with decrease in the value of η from 0.3 to 0.2. The system vibrates in true resonance due to the effect of the external excitation of the same order as the principal component of the motion when η = 0.1 but the true resonance for secondary components exists only within certain limited regions of phase angle α (see Figs. 8.10, 8.14 and 8.18). It can be seen that the factors η and α are very important parameters in the solution of the equation of motion.

Draminsky in his analysis has also given a method of reducing a multi-mass system into an equivalent single degree of freedom system to simplify the problem. In the opinion of the author further work can be done by obtaining data of actual cases of failures of ten and twelve cylinder engines from practice together with their tangential effort diagrams. Then by reducing the multi-mass system into an equivalent single degree of freedom system the solutions of the equation of motion can be studied by the methods suggested in the present work. This may lead to the prediction of the size of magnification of the stress related to the values of η, α, r and K parameters. It will also be interesting to study in some greater detail the effect of variation in damping factor, c, on the waveforms of steady state solutions of the equation of motion.
Draminsky\textsuperscript{31,33} and Archer\textsuperscript{30} have cited, from practice, examples of crankshaft failures in large ten cylinder and twelve cylinder marine diesel engines due to the effect of secondary resonance. In these cases torsional vibrations of the 8th order or 9th order have been observed which cannot be explained by the simple vibration theory commonly used for the practical calculation of torsional vibration in crankshaft systems. The measurements taken on the engines which failed in practice show that the measured stresses of either the 8th or 9th order are three to five times greater than would normally have been predicted. Draminsky has also given a method of calculation commonly adopted in industry to determine the effect of secondary resonance due to large variable inertia in the system.

The measurements carried out by the author on a ten cylinder two-stroke cycle engine with suspected secondary resonance show that no theoretical method existed which could positively indicate the possibility of failure due to such an effect and that the behaviour of a system with large variable inertia was far from understood.

The following torsional effects are derived from the analysis of the response of the system representing a single-cylinder engine allowing for variable inertia but neglecting the effect of damping and external excitations:

The theoretical time responses examined in the range of $\eta$ defined as $0.05 \leq \eta \leq 0.8$ at definite speeds of rotation corresponding to $r < 0.2$ and at some higher rotational speeds exhibit the presence of
beats. For the range of $r < 0.2$ the frequency of one cycle of the response envelope is found to be twice the speed of rotation. The solutions show a modulation of amplitude and instantaneous frequency and the maximum amplitude and maximum apparent frequency of one oscillation of the solution occur together as do the minimum values of these same quantities. The presence of beats at some speeds of rotation for $r > 0.2$ shows no correlation between the frequency of one cycle of the response envelope and the speed of rotation for example when $r = 0.6, 4$ and $6$.

Two principal critical speed ranges occur in a single - degree-of - freedom variable inertia system at approximately $r = 1$ and $1/2$. The limits of the regions of instability depend upon the value of $\eta$ that is the ratio of the equivalent inertia of the reciprocating parts to the total equivalent moment of inertia of the system. The unstable range which occurs at $r \approx 1$ becomes very large for heavy pistons with substantial inertia variations and one cycle of the vibration occurs during each revolution of the shaft. The other range of instability is comparatively small at $r \approx 1/2$ and the frequency of vibration is twice the revolutions of the shaft. No such instability regions are found when $r \approx 1/3, 1/4$ etc.

The analyses of the solutions for $r = 0.1$ and $\eta \leq 0.1$ show that the waveforms contain a principal component of order $n$ and two main secondary components of order $(n - 2)$ and $(n + 2)$ but this is not the case in general and also that for a constant value of $r = 0.1$ the magnitude of the principal component in the waveform increases and the secondary components become smaller in size with the decrease
in the value of $\eta$.

The omission of the variable part, $2\eta \cos 2T$, from the elastic term of the equation shows that the solutions at lower values of $r < 0.2$ do not change but they are nevertheless modified at higher values of $r$. Although amplitudes grow large for the values of $r \approx 1/2$ but the solutions are stable.

The absence of the first order term from the equation of motion increases the theoretical amplitudes of the responses in the stable regions. At lower speeds of rotation for $r < 0.2$ the maximum amplitude and minimum apparent frequency of one oscillation occur together as do the minimum amplitude and maximum frequency which is the reverse effect to that exhibited by the complete solution. A region of instability exists for $r \approx 1/2$ and the solutions are found to be unstable for all values of $r > 0.83$ when $\eta = 0.544$. The ratio $\omega/\omega_1$ remains constant at $1/2$ and $1$ corresponding to unstable region $r \approx 1/2$ and $r > 0.83$ respectively.

The study of the time responses omitting the impulse term, $-\eta \sin 2T$, shows that although the solutions are not affected at lower values of $r < 0.2$ the theoretical amplitudes and frequencies at higher values of $r$ are modified. The solutions of the equation of motion neglecting the time varying coefficient of the second order term show a modulation of amplitude and frequency but different from the corresponding solutions of the complete equation.

The addition of a constant term on the right hand side of the equation of motion shows no change in the time responses for $r < 0.2$ but at higher values of $r$ the vibratory motion is not symmetrical.
about the reference axis and the deviations become larger with the increase in the value of the constant term. The addition of the constant term has no effect on the size of the unstable regions. All the terms of the equation have their effect on the time responses and no part of the equation should therefore be omitted in obtaining the solutions.

The WKBJ analytical method and the analogue computer results confirm the accuracy of the numerical analysis.

The orders of the harmonic components of motion through which the energy can be transferred to the system from external excitations of the same orders can be determined from the harmonic analysis of the waveform at the speed of rotation for specific value of $\eta$ of the system. The visual analysis of the waveforms carried out by Manley's method is found to be inaccurate but is useful to the extent of getting rough approximations.

**Effect of Forcing Terms**

The solutions of the equation of motion representing a single cylinder engine system with external excitations of order $n = 1, 3, 4$ and a cosine component of the second order as forcing terms are exactly similar to those obtained for free vibration of the system, except when the waveform solution of the basic equation of motion contains a component of motion of the same order as the external excitation. The second order sinusoidal impulse arising from the reciprocating parts has a significant effect at higher speeds of rotation for $r > 0.2$.

At lower values of $r$, for example, when $r = 0.1$ and $\eta = 0.3$
the amplitude is increased by the application of externally applied
torques of frequency 8 times engine speed, 10 times engine speed
and 12 times engine speed, but it is not a significant increase.
It is shown that the increase in amplitude is dependent on the phase
angle of the external excitation and that when the value of $\eta$ is
decreased to 0.2 the amplitudes become larger at the corresponding
values of the phase angles. A truly resonant condition is excited
at $r = 0.1$ and $\eta = 0.1$ for all values of the phase of the 10th order
harmonic component of the tangential effort diagram whereas true
resonance for 8th order and 12th order external excitations exists
only within a limited range of their phase.

Effect of Damping

The ranges of instability both at $r \approx 1$ and $1/2$ become narrow
and the values of $\eta$ below which the instability regions disappear
increase with the increase in the damping factor, $c$. For values
of $\eta$ above the datums defined by different damping factors the regions
of instability are predicted. In the unstable region when $r \approx 1$,
one vibration occurs in one revolution of shaft and for the region
when $r \approx 1/2$ two vibrations occur in every revolution. Steady state
time responses of the equation of motion including the effect of
damping are not symmetrical about the reference axis and for the
lower values of $r < 0.2$ there are always two vibrations in one revolution
of the shaft.

Effect of Forcing Terms including Damping

The steady state solution of the equation of motion including
damping and the external excitation of frequency 10 times the engine
speed shows the presence of beats at \( r = 0.1 \) and that there are two beats in one revolution of the shaft. The 2nd order external excitation acting on the system along with the 10th order shows only slight modification of the beatform solutions.
APPENDIX 1

Computer Programme for the Solution of Equation (4.8) Using the Modified Euler's Method

```
'LIBRARY'(ED, SUBGROUPPLOT, SUBRoutines)
'LIBRARY'(ED, SUBGROUPSRT, SUBRoutines)
BEGIN
REAL'TAU, INTAU, EPS, LAM1, OMEGA, LAMY,
RATIO, OMEGA, B2, B3, GAM1, GAM2, GAM3, GAM4,
INTEGER'I, H, J, P, X,
COMMENT WITH SIMPLE MODIFICATIONS IN THIS PROGRAM, THE
MODIFIED EULER'S METHOD CAN BE USED TO INVESTIGATE DIFFERENT
PROPERTIES OF THE VARIABLE INERTIA SYSTEM INCLUDING THE
EFFECT OF DAMPING AND THE EFFECT OF FORCING TERMS
ARRAY'GAM1[1:1600], GAM2[1:1600], GAM3[1:1600], GAM4[1:1600], TIME[1:1600];
ARRAY'GAM5[1:800], GAM6[1:800], GAM7[1:800], GAM8[1:800], TIME[1:800];
ARRAY'TITLE[1:5], INTEGER'
PROCEDURE'HGPTAPE&P, BCD, IS, IG, IR, J; INTEGER'IS, IG, IR, L;
ARRAY'BCD', EXTERNAL';
PROCEDURE'STARR(A, N, S); ARRAY'A; INTEGER'N, STRING'S;
EXTERNAL'
PROCEDURE'HGPLOTT(X, Y, IC, L); VALUE'X, Y; REAL'X, Y;
INTEGER'IC, L', EXTERNAL';
PROCEDURE'HGPSCURVT(X, Y, N, L, YOP, YFP); REAL'YOP, YFP;
INTEGER'N, L; ARRAY'X, Y', EXTERNAL';
PROCEDURE'ORH(X); INTEGER'X', EXTERNAL'; ON(9);
STARR(TITLE, ', (''PLOTTERMS!''));
WRITE(text, '(''A''(''C''(1))''));
HGPTAPE(0, TITLE, '1, 1, 1');
WRITE(text, '(''B''(''C''(1))''));
STARR(TITLE, ', (''PICTURE.ONE!''));
WRITE(text, '(''C''(''C''1))''));
HGPTAPE(1, TITLE, '0, 0, 0');
WRITE(text, '(''D''(''C''(1))''));
HGPLOTT(0, 0, 0, 0, 14, 1);
WRITE(text, '(''E''(''C''(1))''));
J:=1;
SELECTINPUT(3);
P:=READ;
'FOR'X;:=1'STEP'2'UNTIL'P'DO'
(Cont.)
```
APPENDIX 1 (Cont.)

'BEGIN'
RATIO:=READ;
EPS:=READ;
M:=READ;
OMEGA:=714;
LAM:=OMEGA/2;
GAM[1]:=1;
GAM[2]:=0;
TIM[1]:=0;
OMEGA:=RATIO*OMEGA;
LAM1:=LAM/OMEGA^2;
GAMDD[1]:=(LAM1-2.0*EPS)/(1.0-EPs);
GAMD2:=0;
'COMMENT' CHANGE INTAU AS REQUIRED
INTAU:=2.0*0.0628/OMEGA;
INTAU:=2.0*0.0628/OMEGA;
NEWLINE(1);
WRITETEXT(('"FREQUENCY\%RATIO\%OMEGA\%EPS"'));
NEWLINE(1);
PRINT(RATIO, 3, 3);
PRINT(OMEGA, 3, 3);
PRINT(EPS, 3, 3);
NEWLINE(1);
WRITETEXT(('"TIME\%ACCELERATION\%VELOCITY\%DISTANCE"'));
NEWLINE(1);
'FOR' I:=2 'STEP' 1 'UNTIL' I'M' DO!
'BEGIN' TAM:=INTAU*(I-1)*OMEGA;
TIM[1]:=INTAU*(I-1);
D:=(EPS*SIN(2.0*TAU))/
(1-EPS*COS(2.0*TAU))
D2:=(2.0*EPS*SIN(2.0*TAU))/
(1.0-EPS*COS(2.0*TAU))
D3:=(2.0*EPS*COS(2.0*TAU)-LAM1)/(1.0-EPS*COS(2.0*TAU))
LABEL:
GAMDD1:=GAMDD[1-1]+(GAMDD[1-1]+GAMD2)*0.5*INTAU*OMEGA;
GAM1:=GAM1[1-1]+(GAM1[1-1]+GAM1)*0.5*INTAU*OMEGA;
GAMD1:=GAMD1*2*GAM1*BS+B;
GAMD2:=GAMD1+GAMD1*0.5*INTAU*OMEGA;
GAM2:=GAM2[1-1]+(GAM2[1-1]+GAMD2)*0.5*INTAU*OMEGA;
GAMDD2:=GAMDD2*B2*GAM2*BS+B;
'IF' ABS(GAMD1-GAMD2) 'LE' 0.00001
APPENDIX 1 (Cont.)

'THEN' 'BEGIN' 'IF' 'ABS(GAM1-GAM2) LE 0.00001
'THEN' 'BEGIN' 'GAM1:=GAM2;
GAMD[1]:=GAMD2;
GAMDP[1]:=GAMDP2;
'END' 'ELSE' 'GOTO' 'LABEL:
PRINT(TIM[1], 0, 8);
PRINT(GAMDP[1], 0, 8);
PRINT(GAMD[1], 0, 8);
PRINT(GAM[1], 0, 8);
NEWLINE(1);
'END';
'END';
'FOR' 'I' := 1 'STEP' '1' 'UNTIL' 'INH' 'DO:
'BEGIN';
GAM(1) := 2.0 + GAM(I);
TIM(1) := 5.0 * TIM(I);
'END';
Y := J * 10.0;
HGPLUTT(0, 0, Y, 0, 4);
WRITETEXT('I F(<C>)!!)');
J := J + 1;
HGPSCURVE(TIM, GAM, H, 0, 0, 0, 0, 0);
WRITETEXT('I G(<C>)!!)');
HGPLUTT(0, 0, 0, 0, 3, 0);
HGPLUTT(0, 0, 0, 0, 5);
WRITETEXT('I H(<C>)!!)');
HGPFAPEP(TITLE, 0, 0, 0, 0);
WRITETEXT('I K(<C>)!!)');
'END';
'END';
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APPENDIX 2 (Cont.)

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
*(\text{GAMDI} + K1 \times 0.5)\]

\[
(-2.0 \times \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)) = \text{LAM1})/
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
*(\text{GAMII} \times \text{INTAU} \times \Omega \times 0.5 + \text{GAMIII} + \text{INTAU} \times \Omega \times 0.125 \times K1)\]

\[
(-\text{EPS} \times \sin(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))/
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
K3 = \text{INTAU} \times \Omega \times 0.5 \times (\text{EPS} \times \sin(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
*(\text{GAMDI} + K2 \times 0.5)\]

\[
(-2.0 \times \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)) = \text{LAM1})/
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
*(\text{GAMII} \times \text{INTAU} \times \Omega \times 0.5 + \text{GAMIII} + \text{INTAU} \times \Omega \times 0.125 \times K1)\]

\[
(-\text{EPS} \times \sin(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))/
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
K4 = \text{INTAU} \times \Omega \times 0.5 \times (\text{EPS} \times \sin(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU} \times 0.5)))
\]

\[
*(\text{GAMDI} \times K3)\]

\[
(-2.0 \times \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU})) = \text{LAM1})/
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU})))
\]

\[
*(\text{GAMII} \times \text{INTAU} \times \Omega \times \text{GAMIII} + \text{INTAU} \times \Omega \times 0.5 \times K3)\]

\[
(-\text{EPS} \times \sin(2.0 \times \Omega \times (\tau + \text{INTAU})))/
\]

\[
(1 - \text{EPS} \times \cos(2.0 \times \Omega \times (\tau + \text{INTAU})))
\]

\[
\text{GAMII} \times 1.1666 \times (K1 + 2.0 \times K2 + 0.0 \times K3 \times K4) ;
\]

\[
\text{GAMII} + 1.1666 \times (\text{GAMIII} + 0.1666 \times (K1 + K2 + K3)) ;
\]

\[
\text{PRINT} (\tau, 0, 8) ;
\]

\[
\text{PRINT} (\text{GAMII}, 0, 8) ;
\]

\[
\text{PRINT} (\text{GAMIII}, 0, 8) ;
\]

\[
\text{NEWLINE} (1)
\]

'END';

'GOTO' RESTART;

'END';
APPENDIX 3

Computer Programme for the Solution of Equation (4.8) Using the Method of Simultaneous Equations

'BEGIN'
'REAL TAU, INTAU, EPS, LAM1, OMEGA, K1, K2, K3, K4, L1, L2, L3, L4, LAM, RATIO, OMEGA1
'INTEGER I1;
'ARRAY GAM[0:800], GAMD[0:800];
EPS := 0.544;
OMEGA := 71.4;
LAM := OMEGA/2;
GAM[0] := 1.0;
GAMD[0] := 0.0;
RATIO := 0.0;
RECAL: RATIO := RATIO + 0.01;
OMEGA := RATIO + OMEGA;
LAM1 := LAM / (OMEGA + 2);
INTAU := (0.0628 / OMEGA);
NEWLINE(1);
WRITE TEXT('('FREQUENCY%RATIO%OMEGA)');
PRINT(RATIO, 3.3);
PRINT(OMEGA, 5.5);
NEWLINE(1);
WRITE TEXT('('TIME%VELOCITY%DISTANCE)');
NEWLINE(1);
'FOR I1 := 0 STEP 1 UNTIL 800 DO
'BEGIN'
TAU := INTAU + I1;
K1 := INTAU + OMEGA * GAM[1];
L1 := INTAU + OMEGA * ((-2.0*EPS*SIN(2.0*OMEGA*TAU)) / (1.0 - EPS*COS(2.0*OMEGA*TAU)) * GAM[1] + (-2.0*EPS*COS(2.0*OMEGA*TAU) + LAM1) / (1.0 - EPS*COS(2.0*OMEGA*TAU)) * GAM[1] + (-EPS*SIN(2.0*OMEGA*TAU)) / (1.0 - EPS*COS(2.0*OMEGA*TAU)));
K2 := INTAU + OMEGA * (GAMD[1] + I1 * 0.5);
L2 := INTAU + OMEGA * ((-2.0*EPS*SIN(2.0*OMEGA*(TAU + INTAU*0.5)) / (1.0 - EPS*COS(2.0*OMEGA*(TAU + INTAU*0.5)) * GAMD[1] + I1 * 0.5) + (-2.0*EPS*COS(2.0*OMEGA*(TAU + INTAU*0.5)) - LAM1) / (1.0 - EPS*COS(2.0*OMEGA*(TAU + INTAU*0.5)) * GAM[1] + K1 * 0.5) + (-EPS*SIN(2.0*OMEGA*(TAU + INTAU*0.5)) / (1.0 - EPS*COS(2.0*OMEGA*(TAU + INTAU*0.5))));
APPENDIX 3 (Cont.)

K3 := INTAU*OMEGA*(GAMD[I]*L2*0.5);
L3 := INTAU*OMEGA*((-2.0*EPS*SIN(2.0*OMEGA*(TAU+INTAU*0.5)))/
(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU*0.5)))*(GAMD[I]*0.5*L2)
+(-2.0*EPS*COS(2.0*OMEGA*(TAU+INTAU*0.5)))*LAM1)/
(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU*0.5)))*LAM1)
+(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU*0.5)))*LAM1)/
(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU*0.5)))*LAM1)

K4 := INTAU*OMEGA*(GAMD[I]*L3);
L4 := INTAU*OMEGA*((-2.0*EPS*SIN(2.0*OMEGA*(TAU+INTAU)))/
(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU)))*(GAMD[I]*L3)
+(-2.0*EPS*COS(2.0*OMEGA*(TAU+INTAU)))*LAM1)/
(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU)))*LAM1)
*(GAM[1]+K3)*
*(-EPS*SIN(2.0*OMEGA*(TAU+INTAU)))/
(1.0- EPS*COS(2.0*OMEGA*(TAU+INTAU)))*
GAM[I+1] := GAM[1]+0.1666*(K1+2.0*K2+2.0*K3+K4);
GAMD[I+1] := GAMD[I]+0.1666*(L1+2.0*L2+2.0*L3+L4);
PRINT(TAU,0,8);
PRINT(GAMD[I],0,8);
PRINT(GAM[1],0,8);
NEWLINE(1);
'END';
'IF RATIO.LE 0.4 THEN' GOTO'RECAL;
'END';
Computer Programme for the Test of Inequality in the WKBJ Approximation

'BEGIN'
'REAL TAU,INTAU, EPS, LAM1, OMEGA, LAM, GI, GID, GIDD, GIS, FGI, FR,
J1, JID, J1D;
'INTEGER I;
EPS:=0.544;
OMEGA:=35.75;
LAM:=0(1.4)^2;
LAM1:=LAM/(OMEGA)^2;
INTAU:=0.01;
WRITE TEXT('"FINAL%%RATIO"');
NEWLINE(1);
'FOR I:=1 STEP 1 UNTIL 150 DO1:
'BEGIN'
TAU:=INTAU*I;
J1:=(LAM1-LAM)*EPS*COS(2*OMEGA*TAU)+EPS^2*(SIN(2*OMEGA*TAU)^2)/
((1-EPS*COS(2*OMEGA*TAU))^2);
G1:=SORT(J1);
JID:=((((1-EPS*COS(2*OMEGA*TAU)^2)*2*LAM1*EPS*SIN(2*OMEGA*TAU)+
+2*EPS^2*SIN(2*OMEGA*TAU))/((LAM1-LAM)*EPS*COS(2*OMEGA*TAU))
+SIN(2*EPS^2*SIN(2*OMEGA*TAU)-2*EPS^2*SIN(4*OMEGA*TAU)))
)/(1-EPS*COS(2*OMEGA*TAU)$4));
JIDD:=(((1-LAM1*EPS^2)*2*EPS^2*COS(4*OMEGA*TAU)+
B*EPS^3*SIN(2*OMEGA*TAU)+SIN(4*OMEGA*TAU))$4)
-16*EPS^3*COS(2*OMEGA*TAU)+COS(4*OMEGA*TAU)-4*EPS*LAM1*
COS(2*OMEGA*TAU)-2*EPS^3*SIN(2*OMEGA*TAU)+2*COS(2*OMEGA*TAU)
+2*EPS^3*LAM1*SIN(2*OMEGA*TAU)+SIN(4*OMEGA*TAU)+4*EPS^3*LAM1*
COS(2*OMEGA*TAU)+COS(4*OMEGA*TAU))/((1-EPS*COS(2*OMEGA*TAU))^4)
-((1+LAM1*EPS^2)^2*EPS^2*SIN(4*OMEGA*TAU))
+S*EPS^3*SIN(4*OMEGA*TAU)+COS(2*OMEGA*TAU)-4*EPS^3*LAM1*
SIN(2*OMEGA*TAU)+4*EPS^3*SIN(2*OMEGA*TAU)$4)
-3*EPS^3*LAM1*SIN(4*OMEGA*TAU)+COS(2*OMEGA*TAU)-2*EPS^3*LAM1*
SIN(2*OMEGA*TAU)+4*EPS^3*SIN(2*OMEGA*TAU))
/(1-EPS*COS(2*OMEGA*TAU)^5));
GID:=JID/(2*G1);
GID0:=0.5*((J1D/61)-(J1D/2)/(2*G1^3));
GIS:=ABS(J1);
APPENDIX 5

Computer Programme for the Solution of Equation (4.8) Using the Method of Variation of Parameters and the WKBJ Approximation

```
LIBRARY (FD, SURGROUPSLOT, SUBROUTINES)
LIBRARY (ED, SURGROUPSRF7, SUBROUTINES)
BEGIN
REAL TAU, TIP, INTAU, EPS, LAM1, OMEGA, LAM. RATIO, OMEGNY, Y, TWOP1,
X1, X2, F1, P, 13, C1;
INTEGER I, M, J, P, S;
ARRAY GAM[I:800], TIM[I:800];
ARRAY TITLE[I:3]; INTEGER N;
PROCEDURE HGTAPE(L, BCD, IS, IG, IR); INTEGER IS, IG, IR, L;
ARRAY BCD; EXTERNAL;
PROCEDURE STRARR(A, N, S); ARRAY A!! INTEGER N; STRING S;
EXTERNAL;
PROCEDURE HGPLLOT(X, Y, IC, L); INTEGER X, Y;
EXTERNAL IC, L; INTEGER IC, L; INTEGER X, Y;
PROCEDURE HGPSCURVEF(X, Y, N, L, YOP, YFP); REAL YOP, YFP;
INTEGER N, L; ARRAY X, Y; EXTERNAL;
PROCEDURE ON(X); INTEGER X; EXTERNAL; ON;
STRARR (TITLE, N, "('PLOTEROP')");
WRITE TEXT("('A' ('C')")");
HGTAPE(0, TITLE, 1, 1, 1);
WRITE TEXT("('B' ('C')")");
STRARR (TITLE, N, "('PICTUREXONE')");
WRITE TEXT("('C' ('C')")");
HGTAPE(1, TITLE, 0, 0, 0);
WRITE TEXT("('D' ('C')")");
HGPLLOT(0, 0, 0, 0, 14, 1);
WRITE TEXT("('E' ('C')")");
J := 1;
BEGIN
REAL 'PROCEDURE' D1SIM3(F, X, A, B, EPS);
VALUE A, B, EPS;
REAL F, X, A, B, EPS;
BEGIN 'COMMENT' CACM ALGORITHM 182, NON RECURSIVE ADAPTION OF 145.
ADAPTIVE SIMPSONS RULE;
INTEGER LVL;
SWITCH RETURN := R1, R2, R3, RECUR, UP;
```
'REAL' 'ARRAY' DX, EPS, P, X2, X3, F2, F3, F4, FBP, EST2, EST3[1:30],
PVAL[1:30], 1:31;
'INTEGER' 'ARRAY' RTRN[1:30];
'REAL' ABSAREA, EST, FA, FM, FB, DA, SX, EST1, SUM, F1;
'COMMENT' THE PARAMETER SETUP FOR THE INITIAL CALL;
LVL:=0; ABSAREA:= EST:=1.0;
DA:=B-A;
X:=(A+B)/2.0; FM:=-4.0*F;
X:=B; FB:=F;
RECUR:
LVL:=LVL+1; DX[LVL]:=DA/3.0;
SX:=DA/18.0; X:=(A+DX[LVL])/2.0; F1:=4.0*F;
X:=X2[LVL]:=(A+DX[LVL]); F2[LVL]:=F;
X:=X3[LVL]:=X2[LVL]+DX[LVL]; F3[LVL]:=F;
EPS[LVL]:=EPS; X:=X3[LVL]+DX[LVL]/2.0; FA[LVL]:=4.0*F;
FM[LVL]:=FM; EST1:=(FA+F1+F2[LVL])*SX;
FBP[LVL]:=FB; EST2[LVL]:=(F2[LVL]+F3[LVL]+FM)*SX;
EST3[LVL]:=(F3[LVL]+F4[LVL]+FB)*SX;
SUM:=EST1+EST2[LVL]+EST3[LVL];
ABSAREA:=ABSAREA-ABS(EST)+ABS(EST1)+ABS(EST2[LVL])+ABS(EST3[LVL]);
'IF' ('ABS(EST-SUM)' 'LE' EPS[LVL]*ABSAREA 'AND' (EST # 1.0))
'OR' ('LVL 'GE' 30)) 'THEN'
'BEGIN' 'COMMENT' DONE ON THIS LEVEL;
UP:
LVL:=LVL-1;
PVAL[LVL, RTRN[LVL]]:=SUM;
'GOTO' RETURN[RTRN[LVL]]1
'END';
RTRN[LVL]:=1; DA:=DX[LVL]; FM:=F1;
FB:=F2[LVL]; EPS:=EPS[LVL]/1.7;
EST:=EST1;
'GOTO' RECUR;
R1:
RTRN[LVL]:=2; DA:=DX[LVL]; FA:=F2[LVL];
FM:=FM[LVL]; FR:=F3[LVL]; EPS:=EPS[LVL]/1.7;
EST:=EST2[LVL]; A:=X2[LVL];
'GOTO' RECUR;
R2:
RTRN[LVL]:=3; DA:=DX[LVL]; FA:=F3[LVL];
FM:=F4[LVL]; FB:=FB[LVL]; EPS:=EPS[LVL]/1.7;
APPENDIX 5 (Cont.)

EST:=%EST3(LVL): A:=%X3[LVL];
'GOTO' RECUR:
R3:
SUM:=PV[LVL,1]+PV[LVL,2]+PV[LVL,3];
'IF' LVL>1 'THEN' 'GOTO' UP;
DISM3:=SUM
'END' DISM3;
SELECT INPUT(3);
P:=READ;
'FOR' S:=1 'STEP' 1 'UNTIL' P'DO'
'BEGIN'
RATIO:=READ;
EPS:=READ;
M:=READ;
OMEG:=7.14;
LAM:=OMEG+2;
TIM[1]:=0;
GAN[1]:=1.0;
OMEGA:=RATIO*OMEG;
LAM1:=LAM/OMEGA;
'COMMENT! CHANGE INTAU AS REQUIRED
INTAU:=0.0428/OMEGA;
INTAU:=2.0+0.0628/OMEGA;
NEWLINE(1);
WRITE TEXT('('%FREQUENCY%RATIO%OMEGA%EPS)'');
NEWLINE(1);
PRINT(RATIO,3,3);
PRINT(OMEGA,5,5);
PRINT(EPS,5,5);
NEWLINE(1);
NEWLINE(1);
WRITE TEXT('('%TIME%OMEGA%DISPLACEMENT%')');
NEWLINE(1);
TWOP:=8*ARCTAN(1);
'FOR' I:=2 'STEP' 1 'UNTIL' M'DO'
'BEGIN'
TIP:=INTAU*(1-I)*OMEGA;
TIP:=TIP-TWOP*ENTER(TIP/TWOP+0.5);
TIM[1]:=INTAU+(I-1);
FI:=DISM3*SORT((LAM1-LAM1*EPS+COS(2.0*TAU))/

APPENDIX 5 (Cont.)

Computing Unit
APPENDIX 5 (Cont.)

\[
\begin{align*}
\text{TAU} & = 2 \times \text{EPS} \times \text{SIN}(4 \times \text{TAU}) - (\text{LAM1} \times \text{LAM1} \times \text{EPS} \times \text{COS}(2 \times \text{TAU}) + \\
& \text{EPS} \times (\text{SIN}(2 \times \text{TAU}) \times 2) \times (4 \times \text{EPS} \times \text{SIN}(2 \times \text{TAU}) = 2 \times \text{EPS} \times 2 \times \text{SIN}(4 \times \\
\text{TAU}))
\end{align*}
\]

/ \[(1 - \text{EPS} \times \text{COS}(2 \times \text{TAU}))) ^ {\times 4}\) / / (4 \times (\text{LAM1} \times \text{LAM1} \times \text{EPS} \times \text{COS} \\
(2 \times \text{TAU}) + \text{EPS} \times \text{EPS} \times \text{SIN}(2 \times \text{TAU}) \times 2)
\]

/ / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU}))) \times 2\) / (4 \times (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU})))

/ / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU}))) \times 2\) / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU})))

/ / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU}))) \times 2\) / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU})))

/ / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU}))) \times 2\) / (1 - \text{EPS} \times \text{COS}(2 \times \text{TAU})))
```plaintext
2*TAU+EPS*EPS*(SIN(2*TAU)+2)/
((1- EPS*COS(2*TAU))+2));
,TAU,0.TIP,0.1,8-03);
GAM[1] := (12+C1)+X1+13*X2;
PRINT(TIM[1],0,0);
PRINT(GAM[1],0,0);
NEWLINE(1):
'END';
'FOR' I := 1 'STEP' 1 'UNTIL' M 'DO';
'REGIN' GAM[1 ] := 2.0*GAM[1 ];
TIM[1] := 5.0*TIM[1 ];
'END';
Y := J+10.0;
HPLOUT(0.0,Y,0,4);
WRITE TEXT('F(''C)'')1);  
HPSCURVE(TIM, GAM,M,0,0,0,0,0);
WRITE TEXT('G(''C)'')1);  
HPLOUT(0.0,0.0,3,0);
HPLOUT(0.0,0.0,0,5);  
WRITE TEXT('H(''C)'')1);  
HPPTAPE(2,TITIE,0.0,0.0);
WRITE TEXT('K(''C)'')1);  
'END';
'END';
'END';
```
APPENDIX 6

Computer Programme for Harmonic Analysis of Waveforms

'BEGIN' 'INTEGER' N,K;
N:=READ;
K:=READ;
'BEGIN'
'REAL' 'ARRAY' F[0:N];
'REAL' SUM, SUM1, C, S, FO, XP, PHIR, PHID, A, B, R, VC, VS;
'INTEGER' P, I;
'FOR' P:=0 'STEP' 1 'UNTIL' N 'DO'
F[I]:=READ;
FO:=(FO+F[I])/2;
NEWLINE(1);
WRITE('H%NO%%%SIN%%%COS%%%RES%%%PHI%RAD%%%PHI%DEG');
NEWLINE(1);
'FOR' I:=1 'STEP' 1 'UNTIL' K 'DO'
'SUM1:=0.0;
'SUM2:=0.0;
'FOR' P:=1 'STEP' 1 'UNTIL' N-1 'DO'
'BEGIN'
XP:=P/N;
C:=COS(6.285714*XP*1);
S:=SIN(6.285714*XP*1);
VC:=F[I]*C;
VS:=F[I]*S;
SUM:=SUM+VC;
SUM1:=SUM1+VS;
'END';
A:=2*(SUM1)/N;
B:=2*(FO+SUM2)/N;
R:=SQR(A*2+B*2);
PHIR:=ARCTAN(B/A);
PHID:=PHIR*180/3.1428;
PRINT(1,2,0);
PRINT(A,0,4);
PRINT(B,0,4);
APPENDIX 6 (Cont.)

PRINT(R, 0, 4);
PRINT(PHIR, 0, 5);
PRINT(PHID, 0, 5);
NEWLINE(1);
'END';
'END';
'END';
'END';
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