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ELECTROMAGNETIC DIFFRACTION BY

WEDGE SHAPED OBSTACLES

by

Anthony David Rawlins, M.Sc.

University of Surrey

September, 1972
This thesis is divided into two distinct parts.

Part I, consisting of chapters 1 to 3, deals with the determination of the far field when an E-polarised electromagnetic plane wave is incident on an imperfectly conducting rectangular cylinder. In chapter 1 the field is determined when an E-polarised plane wave is incident on an imperfectly conducting right-angle wedge. In chapter 2, the results of chapter 1 and Keller's method of geometrical diffraction are used to determine the far field when an E-polarised plane wave is incident on an imperfectly conducting rectangular cylinder. Chapter 3 deals with certain singular directions for which the results of chapter 2 are no longer valid.

Part II, consisting of chapters 4 to 7, deals with the determination of the electromagnetic field when an electromagnetic wave is incident on a dielectric wedge. In chapter 4 a basic integral equation and an iteration scheme for its solution, subject to certain restrictions on the refractive index of the wedge, is derived for incidence by an E or H-polarised field on an arbitrary angle dielectric wedge. Chapter 5 determines the electromagnetic field when an E-polarised plane wave is incident on a right-angle dielectric wedge whose refractive index \(= n\) is such that \(1 < |n| < \sqrt{2}\). In chapter 6 the electromagnetic far field is determined, when an arbitrary angle dielectric wedge whose refractive index is such that \(n \leq 1\), is illuminated by an E-polarised plane wave. The method of approach is different to that used in chapters 4 and 5, and uses Kontorovich-Lebedev
transforms. In chapter 7 the field near to the tip of an arbitrary angle dielectric wedge of arbitrary refractive index, when illuminated by a line source of E or H-polarisation, is determined.
The author wishes to express his thanks to Professor W.E. Williams for his encouragement and valuable criticism during the preparation of this thesis. He would also like to express his gratitude to the Institution of Electrical Engineers for their financial support during the final year of research.
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PART I

ELECTROMAGNETIC DIFFRACTION BY AN IMPERFECTLY

CONDUCTING RECTANGULAR CYLINDER
INTRODUCTION TO PART I

The problem considered is the calculation of the diffracted far field, produced by an electromagnetic plane wave diffracted by an imperfectly conducting rectangular cylinder. The sides of the cylinder are large compared to the wavelength of the incident electromagnetic wave. The generators of the cylinder are infinite, and from the symmetry of the situation, and the polarisation of the incident plane wave, we are dealing with a two dimensional problem.

In chapter one the canonical boundary value problem of the diffraction of an electromagnetic plane wave incident on an imperfectly conducting right angle corner is solved. From this solution the angular variation of the far field for a single corner can be determined.

In the second chapter the results of chapter one, in conjunction with Keller's geometrical theory of diffraction and the theory of multiple diffraction, are applied to the problem outlined in the first paragraph. These various methods can be used to determine the effect of the four corners, which comprise the cylinder, on the diffracted far field.

In certain angular directions the far field varies rapidly for small angular variation and Keller's method seems to be no longer applicable. In the final chapter an alternative approach, using Green's theorem, is adopted to determine the far field in such angular directions.
CHAPTER 1

We propose to solve the boundary value problem related to the physical problem of diffraction of an E-polarised plane wave incident on an imperfectly conducting right angle wedge.

For large values of the conductivity Jones and Pidduck [1] showed that for an E-polarised incident plane wave the electric field component on the wedge is proportional to its normal derivative; the constant of proportionality involving the complex refractive index, Born and Wolf [2] of a thin surface skin of the wedge material. The electric field components satisfy impedance boundary conditions on the wedge surfaces.

The first exact solution to diffraction by an imperfectly conducting wedge of arbitrary angle was obtained by Malyuzhinets (1950) [3] and later independently by Senior (1959) [4] and Williams (1959) [5]. The solutions obtained by the above authors are represented in an implicit form in terms of solutions of a special functional equation, the latter can only be solved with difficulty, and the resulting expressions are rather complicated. Jones and Pidduck [1] obtained a perturbation solution in inverse powers of the conductivity. The expressions obtained are complicated and no substantial reduction of the results was obtained.

Felsen [6] tried to simplify the work of Jones and Pidduck [1], but his results yielded a solution similar to that obtained, many years before, by Raman and Krishnan [7]. The solution obtained by the latter authors was empirically derived, and the solution did not obey the reciprocity condition.
Williams [8] published a short paper indicating how one could obtain simply an explicit solution for a boundary value problem, similar to the above mentioned boundary value problem, when one is considering wedge angles of multiples of π/2. The method used relies heavily on the commutativity of the cartesian coordinate differential operators. We shall apply this method, with a slight modification, to the problem of diffraction of an E-polarised plane wave by a right-angled wedge whose faces are imperfectly conducting; the conductivity of each face being different.

**Formulation of the boundary value problem**

The wedge is assumed to be defined by the surfaces y = 0, x > 0 and x = 0, y < 0 and polar coordinates (ρ,θ) are defined by x = ρ cos θ, y = ρ sin θ. The case when the only component of the incident electric field is that parallel to the z-axis will be considered. We shall assume that the incident field is given by

\[ u_0 = e^{-i[ωt+kρ \cos(θ-θ_0)]} \]

\[ 0 \leq θ_0 \leq \frac{3π}{2}. \]

If \( u = e^{-iωt} \) denotes the total electric intensity parallel to the z-axis then Maxwell's equations give,

\[ (v^2+k^2)u = 0, \quad 0 \leq θ \leq \frac{3π}{2}, \]

where \( v^2 = \frac{∂^2}{∂x^2} + \frac{∂^2}{∂y^2} \), and \( k^2 = \varepsilon_0 μ_0 ω^2 \).

In order to obtain a unique solution it is necessary to specify the behaviour of \( u \) for both large and small values of \( ρ \). For small values of \( ρ \) the electric field must satisfy the edge condition, i.e. \( u \) is assumed to be bounded although
grad $u$ may be unbounded. For large values of $\rho$ the field excluding the incident and the reflected plane waves must satisfy Sommerfeld's radiation condition for outgoing waves, i.e.,

$$\lim_{\rho \to \infty} \rho^{\frac{1}{2}} (z \frac{\partial}{\partial z} (u - u_g) - ik(u - u_g)) = 0$$

where $u_g$ represents the geometrical optics terms.

The boundary conditions appropriate to the present problem are given by,

$$\frac{\partial u}{\partial y} + \lambda_1 u = 0, \quad (y = 0, \ x > 0), \quad (1.2)$$

$$\frac{\partial u}{\partial x} - \lambda_2 u = 0, \quad (x = 0, \ y < 0). \quad (1.3)$$

The boundary conditions defined by the expressions (1.2) and (1.3) are the so-called impedance boundary conditions. The parameters $\lambda_1$ and $\lambda_2$ are dependent on the conductivity of the faces for which the boundary condition applies.

Grunberg [9] and Senior [10] have studied this boundary condition and obtain from Maxwell's equation,

$$\lambda_1 = ik \frac{\mu_o}{\mu_1} n_1, \quad \lambda_2 = ik \frac{\mu_o}{\mu_2} n_2,$$

where

$$n_s = \sqrt{\frac{\mu_s}{\mu_0} \left( \frac{\varepsilon_s}{\varepsilon_0} - \frac{i \sigma_s}{\omega \varepsilon_0} \right)}, \quad s = 1, 2, \quad (1.4)$$

is the complex refractive index of a small skin depth of the $s^{th}$ face; and $\mu_s$, $\varepsilon_s$, and $\sigma_s$ are the permeability,
permittivity and conductivity respectively, of the material comprising the skin of the appropriate face s. It is shown in [9] and [10], that the impedance boundary condition is valid provided \(|N_s| >> 1\). \(|N_s|\) must not be too large however, because then the effect of conductivity will be negligible. For the non-ferrous metals the impedance boundary condition is best realised in the visible light spectrum as pointed out by Jones and Pidduck [1]. It is worth noting that the theoretical value for \(N_s\) given by the expression (1.4) gives a much higher value than the actual refractive index for wavelengths smaller than 1mm. This is because conductivity is probably a function of frequency; for this reason Jones and Pidduck's derivation of the impedance boundary condition is more realistic because they only consider the magnitude of \(N_s\).

Solution of the boundary value problem

A solution \(v\) of the equation (1.1) is now defined by means of the relationship

\[ u = \left( \frac{\partial}{\partial x} + \lambda_2 \right) v, \quad (1.5) \]

so that from equations (1.1) to (1.5)

\[ \frac{\partial v}{\partial y} + \lambda_1 v = A e^{-\lambda_2 x}, \quad (y = 0, x > 0), \quad (1.6) \]

\[ v = B e^{i(\lambda_2^2 + k^2)^{\frac{1}{2}} y} + C e^{-i(\lambda_2^2 + k^2)^{\frac{1}{2}} y}, \quad (x = 0, y < 0). \quad (1.7) \]
For outgoing waves $B = 0$; and at the origin $(x = y = 0)$ $v$
and its derivatives must be continuous so that $C$ and $A$ are
related by

$$A = \lambda_1 - i(\lambda_2^2 + k^2)^{\frac{1}{2}} C. \quad (1.8)$$

Hence

$$\frac{\partial v}{\partial y} + \lambda_1 v = (\lambda_1 - i(\lambda_2^2 + k^2)^{\frac{1}{2}}) C e^{-\lambda_2 x}, \quad (y = 0, x > 0), \quad (1.9)$$

$$v = C e^{-i(\lambda_2^2 + k^2)^{\frac{1}{2}} y}, \quad (x = 0, y < 0). \quad (1.10)$$

We now simplify the work which follows by introducing the
complex Brewster angle $\varphi_t$ and $\theta_t$ defined by

$$\lambda_1 = -i k \cos \varphi_t, \quad (\lambda_2^2 + k^2)^{\frac{1}{2}} = -k \sin \varphi_t, \quad (1.11)$$

$$\lambda_2 = -i k \cos \theta_t, \quad (\lambda_2^2 + k^2)^{\frac{1}{2}} = -k \sin \theta_t,$$

or

$$\theta_t = -\frac{\pi}{2} + \text{arsinh} \frac{\lambda_2}{k}, \quad \varphi_t = -\frac{\pi}{2} + \text{arsinh} \frac{\lambda_1}{k}.$$

Using the expressions (1.11) the equations (1.9) and (1.10)
become

$$\frac{\partial v}{\partial y} - i k \cos \varphi_t v = i k (\sin \theta_t - \cos \varphi_t) C e^{i k x \cos \vartheta_t}, \quad \theta = 0, \quad (1.12)$$

$$v = C e^{i k y \sin \varphi_t}, \quad \varphi = \frac{3\pi}{2}, \quad (1.13)$$
We now define a new auxiliary function \( \psi(\rho, \theta, \theta_0) \) which satisfies the following conditions:

\[
(v^2 + k^2) \psi = 0, \quad 0 < \theta < \frac{3\pi}{2},
\]

\[
\psi = 0 \quad \text{for} \quad \theta = \frac{3\pi}{2}, \quad (1.14)
\]

\[
\frac{\partial \psi}{\partial \gamma} - i k \cos \phi_t \psi = 0, \quad \text{for} \quad \theta = 0. \quad (1.15)
\]

\( \psi \) may be also written as

\[
\psi(\rho, \theta, \theta_0) = e^{-i k \rho \cos(\theta - \theta_0)} \psi_d(\rho, \theta, \theta_0), \quad (1.16)
\]

where \( \psi_d \) represents a diffracted field.

Substituting (1.16) into (1.14) and (1.15) gives

\[
\frac{\partial \psi_d}{\partial \gamma} - i k \cos \phi_t \psi_d = i k (\sin \theta_0 \cos \phi_t) e^{-i k x \cos \theta_0}, \quad \text{for} \quad \theta = 0, \quad (1.17)
\]

\[
\psi_d = e^{-i k y \sin \theta_0}, \quad \text{for} \quad \theta = \frac{3\pi}{2}. \quad (1.18)
\]

On comparing equations (1.12) and (1.13) with equations (1.17) and (1.18), it can be seen that \( \psi \) can be related to \( \psi_d \) by replacing \( \theta_0 \) by the complex angle \( \pi + \theta_0 \) in (1.17) and (1.18). Thus a particular solution for \( \psi \) is

\[
\psi = -C \psi_d(\rho, \theta, \pi + \theta_0). \]

In order to obtain the general solution it is necessary to
add a solution satisfying equations (1.12) and (1.13) with 
\( C = 0 \); clearly such a solution is \( \phi(\rho, \theta, \theta_0) \). Thus a
general solution for \( v \) is given by

\[
v = \phi(\rho, \theta, \theta_0) - C\phi_d(\rho, \theta, \pi + \theta_0).
\]

This latter solution has also to be such that it provides
the appropriate plane wave behaviour in \( u \) at infinity.

Thus the final expression for \( u \) is

\[
u = \left( \frac{\partial}{\partial x} - ik \cos \theta_0 \right) \left\{ -\frac{\phi(\rho, \theta, \theta_0)}{ik(\cos \theta_0 + \cos \theta')} - C\phi_d(\rho, \theta, \pi + \theta_0) \right\}.
\]

Thus \( u \) is determined to within the auxiliary function \( \phi \) and
the unknown constant \( C \). The unknown constant \( C \) can be
determined from the requirement that \( u \) must satisfy the edge
corition. Thus we need to know the solution to the
boundary value problem for \( \phi \):

\[
(v^2 + k^2)\phi = 0, \quad 0 \leq \theta \leq \frac{3\pi}{2},
\]

\[
\phi = 0, \quad \text{for} \quad \theta = \frac{3\pi}{2},
\]

\[
\frac{\partial \phi}{\partial y} - ik \cos \theta \phi = 0, \quad \text{for} \quad \theta = 0,
\]

\[
\phi = e^{-ik\rho \cos(\theta - \theta_0)} + \phi_d.
\]

\( \phi \) satisfies the edge condition and the outgoing wave
condition.
A solution $\chi$ of equation (1.20) is now defined by means of the relationship

$$\phi = \left( \frac{\partial}{\partial y} + ik \cos \phi \right) \chi,$$

(1.23)

so that from equations (1.20) to (1.23) we obtain

$$\chi = A e^{-ikx \sin \phi} + B e^{ikx \sin \phi}, \text{ for } \theta = 0,$$

$$\chi = C e^{-iky \cos \phi}, \text{ for } \theta = \frac{3\pi}{2}.$$

The conditions of continuity at the origin and outgoing waves at infinity give

$$\chi = A e^{-ikx \sin \phi}, \text{ for } \theta = 0,$$

(1.24)

$$\chi = A e^{-iky \cos \phi}, \text{ for } \theta = \frac{3\pi}{2}.$$

We now define a new auxiliary function $\psi(p, \theta_0)$ which satisfies the following conditions:

$$(\nabla^2 + k^2) \psi = 0, \quad 0 \leq \theta \leq \frac{3\pi}{2},$$

$$\psi = 0, \text{ for } \theta = 0 \text{ and } \theta = \frac{3\pi}{2},$$

(1.25)

where 

$$\psi = e^{-ik p \cos(\theta - \theta_0)} + \psi_d.$$
Hence from (1.25) and (1.26)

\[ \psi_d = -e^{-ikx \cos \theta_o}, \quad \text{for } \theta = 0, \tag{1.27} \]
\[ \psi_d = -e^{-iky \sin \theta_o}, \quad \text{for } \theta = \frac{3\pi}{2}. \]

Comparing equations (1.24) with (1.27) it follows that an appropriate solution for \( \psi \) is \( -A' \psi_d(\rho, \theta, \frac{\pi}{2}, \phi) \). In order to obtain a general solution it is necessary to add a solution satisfying equations (1.24) with \( A' = 0 \). This latter solution has also to be such that it provides the appropriate plane wave behaviour for \( \phi \) at infinity. Thus the final expression for \( \phi \) is

\[ \phi = \left( \frac{\partial}{\partial \nu} + ik \cos \phi \right) \left[ -\frac{\psi(\rho, \theta, \phi)}{ik(\sin \theta_o \cos \phi)} - A' \psi_d(\rho, \theta, \frac{\pi}{2}, \phi) \right]. \tag{1.28} \]

Thus \( \phi \) is determined to within the auxiliary function \( \psi \) and the unknown constant \( A' \). The unknown constant \( A' \) can be determined from \( \psi \) and the requirement that \( \phi \) satisfy the edge condition. The boundary value problem for \( \psi \) corresponds to the physical problem of diffraction by an \( \vec{E} \)-polarised plane wave by a perfectly conducting right angle wedge. This latter problem has been solved by a number of authors and the expressions for \( \psi \) and \( \psi_d \) are:

\[ \psi = \frac{8}{3} \sum_{n=1}^{\infty} e^{\frac{12n\pi}{3} \rho^2} J_{2n} \left( kp \right) \sin \frac{2n\theta}{3} \sin \frac{2n\theta_o}{3}, \tag{1.29} \]

Macdonald [11],
where, in the latter integral, the path of integration $S(\theta)$ is the path of steepest descent. The asymptotic values for $\psi$ and $\Phi_d$ are given from (1.29) and (1.30) by

$$\psi \sim \frac{2\sin \frac{\pi}{3} \left( \frac{ke}{2} \right)^{2/3}}{\Gamma(1+2/3)} \sin \frac{2\theta}{3} \sin \frac{2\theta}{3}, \text{ as } \rho \to 0,$$

$$\Phi_d \sim -2e^{i\pi/4} K(\theta,\theta_o) \frac{\sqrt{\rho}}{\sqrt{k}}, \text{ as } \rho \to \infty,$$

where

$$K(\theta,\theta_o) = \left( \frac{\sin \frac{1}{3}(\theta - \theta_o)}{\sin(\theta - \theta_o)} - \frac{\sin \frac{1}{3}(\theta + \theta_o)}{\sin(\theta + \theta_o)} \right).$$

The constant $A'$ may now be determined from the edge condition i.e. $\psi$ must remain finite as $\rho \to 0$. Substituting

$$\Phi_d(\rho,\theta,\frac{\pi}{2} - \nu_t^e) = \frac{\sin \theta}{2} \cos(\frac{\pi}{2} + \nu_t^e)$$

and then replacing $\psi$ by the asymptotic expression (1.31) it is seen that for the edge condition to be satisfied

$$A' = -\frac{\sin 2\theta_o/3}{ik(\sin \theta_o - \cos \nu_t^e) \sin \frac{2}{3}(\pi/2 - \nu_t^e)}.$$  

Having found $A'$ we now proceed to determine $C$, by applying
the edge condition to the expression (1.19). We thus require the form of \( \phi(\rho, \theta, \theta_0) \) as \( \rho \to 0 \). Substituting (1.29) into (1.28) gives after carrying out the various differentiations,

\[
\phi(\rho, \theta, \theta_0) = \frac{ae^{\frac{i\pi}{3}}(\frac{b}{3})^3 \cos^3 \frac{\theta}{3} \left[ \sin \frac{\pi}{3} \left( \frac{\tau_2 - \eta_2}{3} \right) - \sin \frac{\pi}{3} \left( \frac{\tau_2 - \eta_2}{3} \right) \sin^2 \frac{\eta_2}{3} \right] \rho^\frac{\eta_2}{3} }{3\Gamma(\eta_2^2)ik (\sin \theta_0 - \cos \theta_0) \sin \frac{\pi}{3}(\theta + \theta_0)}
\]

as \( \rho \to 0 \).

Substituting this expression into (1.19) and applying the edge condition yields

\[
C = \frac{\sin \frac{\pi}{3} \left( \frac{\tau_2 - \eta_2}{3} \right) - \sin \frac{\pi}{3} \left( \frac{\tau_2 - \eta_2}{3} \right) \sin^2 \frac{\eta_2}{3} [\sin (\pi + \theta_0) - \cos \theta_0] }{ik (\cos \theta_0 + \cos \theta_0) [\sin (\pi + \theta_0) \sin \frac{\pi}{3}(\theta + \theta_0) - \sin \frac{\pi}{3}(\theta + \theta_0) \sin \frac{\pi}{3}(\theta + \theta_0)]}
\]

(1.34)

Thus we have solved completely the boundary value problem for \( u \).

An expression for the field in terms of an infinite series of Bessel functions can now be obtained; this series is useful for computation when \( kp \) is not large. Firstly we represent \( u_d(\rho, \theta, \theta_0) \) and \( \phi_d(\rho, \theta, \theta_0) \) in terms of total field quantities \( \phi(\rho, \theta, \theta_0) \) and \( \psi(\rho, \theta, \theta_0) \) respectively, and not diffracted fields. This can be achieved by noting that

\[
\left( \frac{\partial}{\partial x} - ik \cos \theta_0 \right) e^{-ikp \cos(\theta - (\pi + \theta_0))} = 0,
\]
\[
\left( \frac{\partial}{\partial y} + i k \cos \theta_t \right) e^{-i k \rho \cos(\theta - (\pi/2 - \nu t))} = 0,
\]
and since \( \phi(p, \theta, \theta_0) = e^{-i k \rho \cos(\theta - \theta_0)} + \phi_d(p, \theta, \theta_0) \),

\[\psi(p, \theta, \theta_0) = e^{-i k \rho \cos(\theta - \theta_0)} + \psi_d(p, \theta, \theta_0), \text{ then} \]

\[
\left( \frac{\partial}{\partial x} - i k \cos \theta_t \right) \phi_d(p, \theta, \pi + \theta_t) = \left( \frac{\partial}{\partial x} - i k \cos \theta_t \right) \phi(p, \theta, \pi + \theta_t),
\]

\[
\left( \frac{\partial}{\partial y} + i k \cos \theta_t \right) \psi_d(p, \theta, \frac{\pi}{2} - \nu t) = \left( \frac{\partial}{\partial y} + i k \cos \theta_t \right) \psi(p, \theta, \frac{\pi}{2} - \nu t).
\]

Thus (1.28) and (1.19) become on using the above identities

\[
\phi(p, \theta, \theta_0) = \left( \frac{\partial}{\partial y} + i k \cos \theta_t \right) \left\{ - \frac{\phi(p, \theta, \theta_0)}{i k (\sin \theta_0 - \cos \theta_0)} - A \psi(p, \theta, \frac{\pi}{2} - \nu t) \right\},
\]

(1.35)

\[
u(p, \theta, \theta_0) = \left( \frac{\partial}{\partial x} - i k \cos \theta_t \right) \left\{ - \frac{\phi(p, \theta, \theta_0)}{i k (\cos \theta_0 + \cos \theta_0)} - C \phi(p, \theta, \pi + \theta_t) \right\}.
\]

(1.36)

Substituting (1.29) into (1.35) and carrying out the differential operations, and then substituting the resulting relation for \( \phi(p, \theta, \theta_0) \) into (1.36) we obtain, after some manipulation,

\[
u(p, \theta, \theta_0) = \frac{8}{i k^2 (\sin \theta_0 - \cos \theta_t) (\cos \theta_0 + \cos \theta_t) \sin \frac{\theta}{2}(\pi/2 - \nu t)}.
\]
\[
\begin{align*}
&\left( \cos \theta \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \theta \frac{\partial}{\partial \theta} - i k \cos \theta \right) \left( \sin \theta \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \theta \frac{\partial}{\partial \theta} + i k \cos \theta \right), \\
&\sum_{n=1}^{\infty} e^{2\pi i n \theta} \frac{\partial}{\partial \theta} \left( k \rho \sin \frac{2\pi n \theta}{3} \right).
\end{align*}
\]

\[
\begin{align*}
&\left\{ \left( \sin \frac{2\pi n \theta}{3} - \sin \frac{2\pi n - \theta}{3} \right) \left( \sin \frac{2\pi n - \theta}{3} - \sin \frac{2\pi n + \theta}{3} \right) \\
&\left. - \left( \sin \frac{2\pi n + \theta}{3} - \sin \frac{2\pi n - \theta}{3} \right) \left( \sin \frac{2\pi n - \theta}{3} - \sin \frac{2\pi n + \theta}{3} \right) \right\}.
\end{align*}
\]

where \( \theta_1 = \frac{\pi}{2} - \psi_t \), \( \theta_2 = \pi + \theta_t \).

The expression (1.37) is suitable for regions near to the wedge tip, but at distant regions from the tip the field is composed of plane waves and a diffracted field. We shall now obtain an integral representation of the diffracted field suitable for computation of the field for \( k \rho \) large.

Rewriting (1.28) in terms of the diffracted field \( \psi_d(\rho, \theta, \theta_0) \) and a plane wave terms gives

\[
\phi(\rho, \theta, \theta_0) = \left( \frac{\partial}{\partial \rho} + i k \cos \phi_0 \right) \left\{ - \frac{-i k \cos(\theta - \theta_0)}{i k(\sin \theta - \cos \theta_0)} \right. \\
- \frac{\psi_d(\rho, \theta, \theta_0)}{i k(\sin \theta - \cos \theta_0)} - \lambda \psi_d(\rho, \theta, \frac{\pi}{2} - \psi_0) \right\}.
\]

The first term gives rise to plane waves and is therefore dispensed with since only the diffracted field \( \psi_d(\rho, \theta, \theta_0) \) is
required, thus

\[ \psi_d(\rho, \theta, \phi) = \left( \sin \theta \frac{\partial}{\partial \rho} + \frac{1}{\rho} \frac{\partial}{\partial \theta} + i k \cos \theta \right) \left[ \frac{Y_d(\rho, \theta, \phi)}{i k (\sin \theta_0 - \cos \phi)} \right]. \]

(1.38)

In the integral representation of \( \psi_d(\rho, \theta, \theta_0) \), see (1.30) \( \rho \) and \( \theta \) only occur in the term \( e^{ik \rho \cos(\gamma-\theta)} \), and thus

\[ \left( \sin \theta \frac{\partial}{\partial \rho} + \frac{1}{\rho} \frac{\partial}{\partial \theta} + i k \cos \theta \right) e^{ik \rho \cos(\gamma-\theta)} = \frac{ik}{i k (\sin \gamma + \cos \phi)} e^{ik \rho \cos(\gamma-\theta)}. \]

(1.39)

Substituting (1.30) into (1.38) and using (1.39) gives

\[ \psi_d(\rho, \theta, \phi) = \frac{-1}{\sqrt{3 \pi}} \int \frac{\sin \frac{2 \theta}{3} \sin \frac{3 \gamma}{3} (\sin \gamma \cos \phi) e^{ik \rho \cos(\gamma-\theta)}}{(\sin \theta_0 - \cos \phi)} \]

\[ \left. \left[ \frac{1}{(\cos \frac{2}{3} (\gamma-\theta_0) - \cos \frac{3 \gamma}{3})(\cos \frac{2}{3} (\gamma + \theta_0) - \cos \frac{3 \gamma}{3})} - \frac{1}{(\cos \frac{2}{3} (\gamma-\theta_0) - \cos \frac{3 \gamma}{3})(\cos \frac{2}{3} (\gamma + \theta_0) - \cos \frac{3 \gamma}{3})} \right] d\gamma \right), \]

(1.40)

where differentiation under the integral sign is valid since the integral representation of \( \psi_d(\rho, \theta, \theta_0) \) is uniformly convergent.

Similarly from (1.19)

\[ u_d(\rho, \theta, \phi) = \left( \cos \frac{2 \theta}{3} - \frac{1}{\rho} \sin \frac{3 \theta}{3} - i k \cos \theta \right) \left[ \frac{\phi_d(\rho, \theta, \phi)}{i k (\cos \theta_0 + \cos \phi)} \right]. \]

(1.41)
Again in $\phi_d(\rho, \theta, \theta_0)$, (1.40) $\rho$ and $\theta$ only occur in the exponential $e^{ik\rho \cos(\gamma-\theta)}$ so that

$$
\left( \cos \theta \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \theta \frac{\partial}{\partial \theta} - ik \cos \theta \right)e^{ik\rho \cos(\gamma-\theta)} = ik(\cos^2 \theta - \cos \theta) e^{ik\rho \cos(\gamma-\theta)}.
$$

Thus after some manipulation

$$
u_d(\rho, \theta, \theta_0) = \frac{1}{\sqrt{\pi}i} \int \frac{(\cos \gamma - \cos \theta_0)(\sin \gamma + \cos \theta_0) e^{ik\rho \cos(\gamma-\theta)}}{(\cos \theta_0 + \cos \theta_0)(\sin \theta_0 - \cos \theta_0)(\sin \theta_0^2 - \sin \theta_0^2 \sin \theta_0^2)} \sum_{n=0}^{2} \left( \sin \frac{\theta}{3} \sin \frac{\theta_0 n}{3} - \sin \frac{\theta}{3} \sin \frac{\theta_0 (n+1)}{3} \sin \frac{\theta_0 n}{3} \right) K(\gamma, \theta_{n+2}) d\gamma,
$$

(1.42)

where $\theta_{n+2} = \theta_n$, and $\theta_1$, $\theta_2$ and $K(\alpha, \beta)$ have already been defined, see (1.32).

**Calculation of the diffraction coefficient**

A straightforward application of the method of steepest descent to (1.42) gives, assuming that no poles of the integrand occur near the saddle point $\gamma = 0$,

$$
u_d(\rho, \theta, \theta_0) = D(\theta, \theta_0) \frac{e^{ik\rho}}{\sqrt{\rho}} + O(k^{-3/2}),
$$

(1.43)

where the 'diffraction coefficient' $D(\theta, \theta_0)$ is given by
\[ D(\theta, \phi) = \frac{(\cos \theta - \cos \phi_0)(\sin \theta + \cos \phi_0)}{\sqrt{\cos \theta + \cos \phi_0}} \left( \frac{2e^{in\phi}}{\sqrt{\cos \theta + \cos \phi_0}} \right) \]

\[ \frac{1}{\cos \theta_0 + \cos \phi_0} \left( \sin \theta_0 - \cos \phi_0 \right) \left( \sin \theta_0 + \cos \phi_0 \right) \left( \sin \frac{2\theta_0}{3} - \sin \frac{2\phi_0}{3} \right) \]

\[ \sum_{n=0}^{\infty} \left( \sin \frac{2\theta_0}{3} \sin \frac{2\phi_0}{3} + \sin \frac{2\theta_0}{3} \sin \frac{2\phi_0}{3} \right) K(\theta, \theta + n) \]  \hspace{1cm} (1.44)

In the specific problem to which we are going to apply the diffraction coefficient both faces of the wedge will have the same impedance; and therefore we may set \( \theta_0 = \theta_0 \) in the expressions (1.42) and (1.44), this gives, after some trigonometric manipulation,

\[ u_4(\beta, \theta) = \frac{1}{\sqrt{3\pi i}} \int \frac{(\cos \gamma - \cos \theta)(\sin \gamma + \cos \theta)}{(\cos \theta + \cos \phi_0)(\sin \theta_0 - \cos \phi_0)} \cdot \frac{1}{\cos \theta_0 + \cos \phi_0} \left( \frac{2e^{in\phi}}{\sqrt{\cos \theta + \cos \phi_0}} \right) \left( \sin \frac{2\theta_0}{3} - \sin \frac{2\phi_0}{3} \right) \]

\[ \frac{1}{\cos \frac{2\theta_0}{3} - \cos \frac{2\phi_0}{3}} \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \cos \theta \right) \frac{k \cos (\gamma - \theta)}{\cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \cos \theta} \frac{\sin \frac{2\theta_0}{3} \sin \frac{2\phi_0}{3}}{\cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \cos \theta} \right) \]

\[ (1.45) \]

\[ D(\theta, \phi) = \]

\[ \frac{2e^{in\phi}}{\sqrt{\cos \theta + \cos \phi_0}} \left( \sin \theta_0 - \cos \phi_0 \right) \left( \sin \theta_0 + \cos \phi_0 \right) \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} + \frac{1}{2} \right) \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \frac{1}{2} \right) \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} + \frac{1}{2} \right) \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \frac{1}{2} \right) \]

\[ \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} + \frac{1}{2} \right) \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \frac{1}{2} \right) \left( \cos \frac{2\theta_0}{3} \cos \frac{2\phi_0}{3} - \frac{1}{2} \right) \]

\[ (1.46) \]

An important property of the diffraction coefficient (1.46)
is that
\[
D(\theta, \theta_0) = D\left(\frac{3\pi}{2} - \theta, \frac{3\pi}{2} - \theta_0\right),
\]
which means that the angle of incidence \(\theta_0\) and the angle of observation \(\theta\) can be measured from either face of the wedge provided they are both measured from the same datum face.

Other properties of \(D(\theta, \theta_0)\) which will be required later are
\[
D(0, \theta_0) = D(\theta, 0) = 0,
\]
\[
\frac{\partial D(\theta, \theta_0)}{\partial \theta} = D(\theta, \theta_0) = \frac{4e^{im\pi}(1 - \cos \theta_0)\cos \theta_0 \left(\cos \frac{4\theta_0}{3} - \cos \frac{4\theta_0}{3} - \frac{1}{2} - \cos \frac{4\theta_0}{3}\right) \sin 2\theta_0}{3 \sqrt{\pi} k \left(\cos \theta_0 + \cos \theta_0\right) \left(\sin \frac{2 \theta_0}{3} - \cos \theta_0\right) \left(\cos \frac{2 \theta_0}{3} + \frac{1}{2}\right)^2},
\]
\[
\frac{\partial D(\theta, \theta_0)}{\partial \theta_0} = -D(\theta, \theta_0) = \frac{4e^{im\pi}(\cos \theta - \cos \theta_0)(\sin \theta + \cos \theta_0)(1 - \cos \frac{4\theta_0}{3})(\cos \frac{2 \theta_0}{3} + \frac{1}{2} - \cos \frac{4\theta_0}{3}) \sin 2\theta_0}{3 \sqrt{\pi} k \left(\cos \theta_0 + \cos \theta_0\right) \left(\sin \frac{2 \theta_0}{3} - \cos \theta_0\right) \left(\cos \frac{2 \theta_0}{3} + \frac{1}{2}\right)^2},
\]
\[
D_{\theta 0}(0, \theta_0) = \frac{8e^{im\pi}(1 - \cos \theta_0)\cos \theta_0 \left(\cos \frac{4\theta_0}{3} - \cos \frac{4\theta_0}{3} - \frac{1}{2} - \cos \frac{4\theta_0}{3}\right) \sin 2\theta_0}{3 \sqrt{\pi} k \left(\cos \theta_0 + \cos \theta_0\right) \left(\sin \frac{2 \theta_0}{3} - \cos \theta_0\right) \left(\cos \frac{2 \theta_0}{3} + \frac{1}{2}\right)^2},
\]
\[
D_{\theta 0}(\theta, 0) = -D_{\theta 0}(0, \theta_0) = \frac{8e^{im\pi}(\cos \theta - \cos \theta_0)(\sin \theta + \cos \theta_0)(1 - \cos \frac{4\theta_0}{3})(\cos \frac{2 \theta_0}{3} + \frac{1}{2} - \cos \frac{4\theta_0}{3}) \sin 2\theta_0}{3 \sqrt{\pi} k \left(\cos \theta_0 + \cos \theta_0\right) \left(\sin \frac{2 \theta_0}{3} - \cos \theta_0\right) \left(\cos \frac{2 \theta_0}{3} + \frac{1}{2}\right)^2}.
\]

Rewriting the integral (1.45) in a more convenient form gives
\[ u_d(r, \theta) = \frac{1}{2\pi \sqrt{1 - \cos \gamma}} \int \mathcal{D}(\gamma, \theta_0) \left\{ \frac{1}{\cos \frac{\gamma}{3} - \cos \frac{\gamma}{3}(\gamma - \theta_0)} - \frac{1}{\cos \frac{\gamma}{3} - \cos \frac{\gamma}{3}(\gamma + \theta_0)} \right\} e^{ik \cos(\gamma \theta)} \, d\gamma, \]

(1.53)

where

\[ \mathcal{D}(\gamma, \theta_0) = \frac{(\cos \gamma - \cos \theta_0)(\sin \gamma + \cos \theta_0)(\cos \frac{1}{3} \gamma - \cos \frac{1}{3} \theta_0)(2\cos \frac{2}{3} \theta_0 \cos \frac{2}{3} \gamma + \frac{1}{2} - \cos \frac{5}{3} \theta_0)}{(\cos \theta_0 + \cos \theta_0)(\sin \theta_0 - \cos \theta_0)(\cos \frac{2}{3} \gamma - \cos \frac{2}{3} \theta_0)(\cos \frac{1}{3} \gamma - \cos \frac{1}{3} \theta_0)} \]

(1.54)

The reason for writing \( u_d(r, \theta) \) in this form, is that, complex poles involving \( \theta \) only appear in \( \mathcal{D}(\gamma, \theta_0) \). These poles never occur in the range of \( \gamma \) which we shall be concerned with. The physical reason for the latter statement is that no surface waves are excited or supported on the wedge faces under consideration.

**Calculation of the far field for regions where the diffraction coefficient becomes infinite**

For certain values of \( \theta \) and \( \theta_0 \), the poles of the integrand in (1.53) approach near to, and can exist at, the saddle point. In such a situation the normal saddle point method breaks down, and the expression (1.46) obtained for the diffraction coefficient is no longer valid. The situation described above corresponds to the physical situation when the field is observed near to the geometrical optics boundaries.

We shall now use the method of Oberhettinger [13] to derive expressions for the diffracted field which are uniformly
valid near the geometrical optics boundaries. Before applying the method of Oberhettinger the diffracted field integral representation (1.53) must be represented in terms of Laplace type integrals.

Letting \( \theta + w = \gamma \), and then \( w = iw \) in the equation (1.53) gives

\[
\psi (\rho, \theta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{D}(\theta+iw, \theta_0) \left\{ \frac{1}{\cos \frac{2\pi}{3} - \cos \frac{2\pi}{3}(iw+\theta-\theta_0)} \right\} e^{ikp\cosh \omega} d\omega
\]

The two integrals in the expression (1.55) are of the general form

\[
P(\rho, \theta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathcal{D}(\theta+iw, \theta_0) e^{ikp\cosh \omega} d\omega}{(\cos \frac{2\pi}{3} - \cos \frac{2\pi}{3}(iw+\theta))}
\]

where \( \phi = \theta \pm \theta_0 \).

Some manipulation enables one to rewrite (1.56) in the form

\[
P(\rho, \theta) = -\frac{i}{6\pi} \sin \frac{2\pi}{3}(\pi-\psi) \left[ \mathcal{D}_0(\theta, \theta_0, \omega) e^{ikp\cosh \omega} d\omega \right]
\]

\[
- \frac{i}{6\pi} \sin \frac{2\pi}{3}(\psi+\pi) \left[ \mathcal{D}_0(\theta, \theta_0, \omega) e^{ikp\cosh \omega} d\omega \right]
\]

\[
- \frac{i}{6\pi} \int_{0}^{\infty} \frac{\mathcal{D}_0(\theta, \theta_0, \omega) \sinh 2\omega/3 e^{ikp\cosh \omega} d\omega}{\cosh 2\omega/3 - \cos 2\pi/3(\pi-\psi)}
\]
\[ + \frac{i}{6\pi} \int_0^\infty \left( D_0(\theta, \theta_0, \omega) \sin \frac{\omega}{2} e^{ikp \omega} \right) d\omega \]

(1.57)

where

\[ D_0(\theta, \theta_0, \omega) = D(\theta + i\omega, \theta_0) + D(\theta - i\omega, \theta_0) , \]
\[ D_0(\theta, \theta_0, \omega) = D(\theta + i\omega, \theta_0) - D(\theta - i\omega, \theta_0) . \]

Thus the field \( u_0(r, \theta) \) can now be written in the form

\[ u_0(\rho, \theta) = - \left[ \{ I(\pi - \theta + \theta_0) + J(\pi - \theta + \theta_0) \} - \{ I(\pi + \theta - \theta_0) - J(\pi + \theta - \theta_0) \} \right] \]
\[ + \left\{ I(\pi - \theta - \theta_0) + J(\pi - \theta - \theta_0) \right\} + \left\{ I(\pi + \theta + \theta_0) - J(\pi + \theta + \theta_0) \right\} , \]

(1.58)

where

\[ I(6) = \frac{\sin \frac{2\pi}{3}}{6\pi} \int_0^\infty \left( D_0(\theta, \theta_0, \omega) \sin \frac{\omega}{2} e^{ikp \omega} \right) d\omega \]

(1.59)

\[ J(6) = \frac{i}{6\pi} \int_0^\infty \left( D_0(\theta, \theta_0, \omega) \sin \frac{\omega}{2} e^{ikp \omega} \right) d\omega \]

(1.60)

\[ I(6) = - I(-6) , \quad J(6) = J(-6) \]

(1.61)

\[ I(6 + 3\pi n) = I(6) , \quad J(6 + 3\pi n) = J(6) , \quad n = 0, 1, 2, \ldots \]

To evaluate (1.59) and (1.60) asymptotically for large \( kp \) it is necessary to expand the numerator, excluding the exponential, in terms of \( \sin \frac{\omega}{2} \). This is achieved by means of Burmann's theorem, Whittaker and Watson [14], which gives
\[ \mathcal{D}(\theta + iw, \theta_0) = \sum_{n=0}^{\infty} B_n \left( \frac{\text{sh} \frac{w}{2}}{2} \right)^n, \]

where \( B_0 = \mathcal{D}(\theta, \theta_0) \). The remaining coefficients up to \( B_3 \) are given in Appendix 1A, since they are of a complicated form. From the relationship between \( \mathcal{D}(\theta - iw, \theta_0) \), \( \mathcal{D}_o(\theta, \theta_0, w) \) and \( \mathcal{D}_e(\theta, \theta_0, w) \) and the expansion, Erdelyi [15],

\[ \text{sh} \frac{w}{2} = 2v \text{sh} \frac{w}{2} \cdot 2^{-v} \left( v + \frac{1}{2}, \frac{1}{2} - v, \frac{3}{2}; -\text{sh}^2 \frac{w}{2} \right), \]

we obtain

\[ \mathcal{D}(\theta, \theta_0, w) = 2 \sum_{n=0}^{\infty} B_{2n} \left( \frac{\text{sh} \frac{w}{2}}{2} \right)^{2n}, \tag{1.62} \]

and

\[ i\text{sh} \frac{w}{2} \mathcal{D}_o(\theta, \theta_0, w) = 4\text{sh} \frac{w}{2} \mathcal{D}_o'(\theta, \theta_0, w), \tag{1.63} \]

where

\[ \mathcal{D}_o'(\theta, \theta_0, w) = \sum_{n=0}^{\infty} B_{2n} \left( \frac{\text{sh} \frac{w}{2}}{2} \right)^{2n}, \tag{1.64} \]

and the relationships between the coefficients \( B_{2n} \) and \( B_{2n+1} \), for \( n = 0, 1, 2, \ldots \) are given in Appendix 1A. The above expansions are valid in the region \( |\text{sh} \frac{w}{2}| < 1 \), and, since the saddle point of the integrals (1.59) and (1.60) occur at \( w = 0 \), are in a suitable form for the asymptotic evaluation of the integrals.

**Uniform asymptotic expansion for \( I(\delta) \)**

We shall now obtain the asymptotic expansion for the integral (1.59) for arbitrary \( \delta \) and \( k \) large. Let \( \text{ch} w = 1 + t \)
in (1.59) so that

$$I(\delta) = e^{ik\rho} \int_0^\infty k(t) e^{ik\rho t} \frac{dt}{\sqrt{t}},$$  \hspace{1cm} (1.65)$$

where

$$k(t) = \sin \frac{2\delta}{2} \cdot \frac{(t+2)^{-3/2}}{3\pi} \frac{D_\delta(\theta, \theta_0, t)}{[\cosh(1+t+t^2+t^2) - \cos \frac{2\delta}{2}]}$$

and from (1.62)

$$D_\delta(\theta, \theta_0, t) = 2 \sum_{n=0}^{\infty} B_{2n} \frac{(t)^n}{(2^n)},$$  \hspace{1cm} (1.67)$$

The expression (1.66) can be expanded in an infinite series in $t$, about the origin, and the series will be analytic in the region $|t| < |t_0|$, where $t_0 = -2 \sin^2 \delta/2$ is the nearest pole to the origin.

Thus

$$k(t) = \sum_{n=0}^{\infty} C_n t^n, \quad \text{where} \quad C_n = \frac{1}{n!} \left[ \frac{d^n}{dt^n} k(t) \right]_{t=0},$$

$$|t| < |t_0|.$$  \hspace{1cm} (1.68)$$

The first coefficient $C_0$ is given by

$$C_0 = \frac{D(\theta, \theta_0)}{\sqrt{2} \frac{3\pi}{2} \cot \frac{\delta}{2}}.$$

and $C_1$ and $C_2$ are given in Appendix 1A. To obtain a uniform asymptotic expansion for $I(\delta)$, which is valid for $\delta \to 0$, it is necessary to remove the pole $t = t_0$ from the expression for $k(t)$. Thus we define $k^*(t)$ by
\[ K^*(t) = K(t) - \frac{b^{-1}}{(t-t_o)} , \quad (1.69) \]

where \( K^*(t) \) is analytic at \( t = t_o \), and from (1.66)

\[ b_{-1} = \lim_{t \to t_o} (t-t_o)K(t) = \frac{\Re e(0,0_o,-2 \sin^2 \delta/2) \sin \delta/2}{\sqrt{2\pi}} . \quad (1.70) \]

From (1.69) it can be seen that \( K^*(t) \) has a larger radius of analyticity than \( K(t) \), this is because we have removed the singularity, \( t_o \), nearest to the origin, from \( K(t) \). Denoting by \( t_1 \), the singularity of \( K^*(t) \) nearest to the origin, then \( |t_1| > |t_o| \); and thus we can expand \( K^*(t) \) about the origin, the expansion being analytic for \( |t| < |t_1| \). Thus

\[ K^*(t) = \sum_{n=0}^{\infty} d_n t^n , \text{where } d_n = \frac{1}{n!} \left[ \frac{d^n}{dt^n} K^*(t) \right]_{t=0} , \quad (1.71) \]

for \( |t| < |t_1| \).

Hence substituting \( K(t) \) given by (1.69) into (1.65) we obtain

\[ I(\delta) = e^{ikp} b_{-1} \int_0^\infty \frac{e^{ikpt} dt}{\sqrt{t(t-t_o)}} + e^{ikp} \int_0^\infty K^*(t) e^{ikpt} \frac{dt}{\sqrt{t}} . \quad (1.72) \]

To evaluate the first integral of the expression (1.72) the following formula, which can be obtained from Erdelyi [15] page 219 formula 34 and page 221 formula 52, is required

\[ \int_0^\infty \frac{e^{ix} dx}{x^\nu(x+b)} = \frac{\Gamma(1-\nu)}{b^\nu} e^{-iab} \Gamma(\nu,-iab) , \quad (1.73) \]

\[ a > 0 , \quad |\text{Re } \nu| < 1 , \quad |\text{ang } b| < \pi . \]
Thus for the particular value \( v = \frac{1}{2} \)

\[
\int_0^\infty \frac{e^{iax}}{\sqrt{x(x+a)}} \, dx = \frac{\sqrt{\pi}}{\sqrt{b}} \, e^{-iba} \, \Gamma\left(\frac{1}{2}, -iba\right). \tag{1.74}
\]

The incomplete gamma function can be expressed in terms of the Fresnel integral. Thus,

\[
\int_0^\infty \frac{e^{iax}}{\sqrt{x(x+a)}} \, dx = 2 \sqrt{\frac{\pi}{b}} \, e^{-(ab+\pi/4)} \, S(\sqrt{ab}), \tag{1.75}
\]

where

\[
S(u) = \int_u^\infty e^{it^2} \, dt, \tag{1.76}
\]

is the Fresnel integral.

The second integral of the expression (1.72) can be evaluated by a straightforward application of Watson's Lemma; and noting that

\[
d_n = \frac{1}{n!} \left[ \frac{d^n}{dt^n} K^*(t) \right]_{t=0} = \frac{1}{n!} \left[ \frac{d^n}{dt^n} \left( K(t) - \frac{b-1}{t-t_0} \right) \right]_{t=0},
\]

\[
= \frac{1}{n!} \left[ \frac{d^n}{dt^n} K(t) \right]_{t=0} - \frac{b-1}{(-t_0)^{n+1}},
\]

\[
= c_n \left( -1 \right)^n 2^{-\frac{n+1}{2}} D_e(\theta, \theta_0, -2 \sin^2 \delta/2) \frac{2^{n+1}}{2\pi(\sin \delta/2)^{2n+1}}, \tag{1.77}
\]

we obtain.
\[ I(\delta) = \frac{1}{\sqrt{\pi}} D_0(\theta, \theta_0, -2 \sin^2 \delta/2) \cos \frac{\pi}{4} \left( \sqrt{2k\rho} \right) \]
\[ \times \left\{ \left( \sqrt{2k\rho} \right) \left( \sin \frac{\delta}{2} \right) \right\} \]

\[ + \frac{i (k\rho + n\rho) \omega}{\sqrt{k\rho}} \sum_{n=0}^{\infty} \left[ C_n - \frac{(-1)^n D_0(\theta, \theta_0, -2 \sin^2 \delta/2)}{2\pi \sqrt{2^{n+1} \sin^{2n+1} \left( \frac{\delta}{2} \right)}} \right] \left[ \frac{\Gamma(n+\frac{1}{2})}{(kr)^n} \right] \]

which is the uniform asymptotic series for \( I(\delta) \).

**Uniform asymptotic expansion for \( J(\delta) \)**

Making the change of variable \( chw = 1 + t \) in the integral representation (1.60) gives

\[ J(\delta) = e^{ik\rho} \int_0^{\infty} t^\frac{1}{2} M(t) e^{ik\rho t} dt, \quad (1.79) \]

where

\[ M(t) = \frac{(t+2)^{-\frac{3}{2}} D_0'(\theta, \theta_0, t)}{3\pi [\cosh \left( \frac{2}{3} \log(1+t+(t^2+2t)^{\frac{3}{2}}) \right) - \cos \frac{2\delta}{3}]} , \quad (1.80) \]

and from (1.64)

\[ D_0'(\theta, \theta_0, t) = \sum_{n=0}^{\infty} B_{2n}(\frac{t}{2})^n . \]

The expression (1.80) can be expanded in an infinite series of \( t \), about the origin, which will be analytic in the region \( |t| < |t_0| \), where \( t_0 = -2 \sin^2 \delta/2 \) is the nearest pole to the origin. Thus

\[ M(t) = \sum_{n=0}^{\infty} c_n t^n , \quad \text{where} \quad c_n = \frac{1}{n!} \left[ \frac{d^n}{dt^n} M(t) \right]_{t=0} \]

\[ |t| < |t_0| , \quad (1.81) \]
where the first few coefficients are given in Appendix 1A.

To obtain a uniformly valid asymptotic expansion for $J(\theta)$ it is necessary to isolate the pole 'to', by defining a new function $M^*(t)$ by

$$M(t) = \frac{b}{(t-t_0)^{\nu}} + M^*(t), \quad (1.82)$$

where

$$b = \lim_{t \to t_0} M(t) = \frac{B_0'(\theta, \theta_0, -2 \sin^2 \delta/2 \sin \alpha/2)}{\sqrt{2 \pi} \sin^2 \delta/3}, \quad (1.83)$$

Thus (1.60) can be written in the form

$$J(\theta) = b e^{i k \rho} \int_0^\infty \frac{t^{1/2} e^{i k t}}{(t-t_0)^\nu} + e^{i k \rho} \int_0^\infty t^{1/2} M^*(t) e^{i k \rho} dt, \quad (1.84)$$

where

$$M^*(t) = \sum_{n=0}^\infty d_n t^n, \quad d_n = \frac{1}{n!} \left[ \frac{d^n}{dt^n} M^*(t) \right]_{t=0}, \quad (1.85)$$

The first integral in the expression (1.84) is evaluated by means of the formula (1.73), with $\nu = -\frac{1}{2}$, i.e.

$$\int_0^\infty x^{1/2} e^{iax} \frac{e^{ixt}}{(x+b)^{1/2}} dx = \frac{\Gamma(3/2)}{b^{1/2}} e^{-ia \theta} \Gamma(-\frac{1}{2}, -ia \theta). \quad (1.86)$$

By means of the recurrence relationship for the incomplete gamma function the integral (1.86) can be expressed in terms of the Fresnel integral. Thus
The second integral of the expression (1.84) is evaluated by a straightforward application of Watson's Lemma. Thus

\[
J(\theta) = \int_{0}^{\infty} \frac{e^{ix}}{(x+b)^{2}} \, dx = \sqrt{\pi b} \, e^{-iab} \left\{ -\Gamma\left(\frac{1}{2}, -iab\right) + (-iab)^{-\frac{1}{2}} \, e^{iab} \right\},
\]

\[
= \sqrt{\frac{\pi}{a}} \, e^{\frac{1}{4}} - 2^{\frac{1}{2}} \pi b^{-\frac{1}{2}} \, e^{-iab} \, e^{-\frac{1}{4}} \, B(\sqrt{\pi b}) \quad (1.87)
\]

Having determined the uniform asymptotic expansions for \( I(\theta) \) and \( J(\theta) \) we shall now determine the expressions for particular values of \( \theta \).

\( \theta \) not near zero

In this situation the arguments of the Fresnel integrals are large and we can use the asymptotic expansion for these integrals, viz.

\[
S(1x!) \sim \frac{e^{ix^{2}} \, e^{ix}/2}{2\pi \, 1x!} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2}) \, (-i)^{n}}{x^{2n}} \quad \text{as} \quad 1x! \to \infty. \quad (1.89)
\]
Thus expression (1.78) becomes after using (1.89)

\[ I(\delta) \sim \frac{e^{i(kp+n/4)}}{\sqrt{kp}} \sum_{n=0}^{\infty} \frac{c_n \Gamma(n+\frac{3}{2}) i^n}{(kp)^n}, \quad (1.90) \]

\[ \sim \frac{D(\theta,\theta_0)}{3/2kp} e^{i(kp+n/4)} \cot \frac{\delta}{3} + o(kp^{-3/2}). \quad (1.91) \]

Rewriting (1.89) as

\[ S(|x|) = \frac{e^{ix^2+n/2}}{2|x|} - \frac{e^{ix^2-n}}{2/\pi |x|^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+3/2)(-i)^n}{x^{2n}}, \quad (1.92) \]

and substituting this expression into (1.88) yields

\[ J(\delta) = \frac{e^{i(kp+3\pi/4)}}{(kp)^{3/2}} \sum_{n=0}^{\infty} \frac{c_n i^n \Gamma(n+3/2)}{(kp)^n}, \quad (1.93) \]

\[ J(\delta) \sim 0 + o(kp^{-3/2}). \quad (1.94) \]

Substituting the expressions (1.91) and (1.94) into the expression (1.58) gives the expression (1.43), where \( D(\theta,\theta_0) \) is given by (1.46), in accordance with expectation.

\( \delta \) near zero

For small \( \delta \) the arguments of the Fresnel integrals are also small, and a suitable asymptotic expansion for \( S(|x|) \) must be found. For \( |x| \to 0 \)

\[ S(|x|) = \int_{|x|}^{\infty} e^{it^2} dt = \int_{0}^{\infty} e^{it^2} dt - \int_{0}^{|x|} e^{it^2} dt = \]

\[ = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}} - \int_{0}^{|x|} e^{it^2} dt. \quad (1.95) \]
For $|x| \ll 1$, the series expansions for $e^{it^2}$ is uniformly convergent for all $t \leq |x|$, and we may therefore integrate term by term the series expansion for $e^{it^2}$ in (1.95).

Thus

$$S(|x|) \sim \frac{\sqrt{\pi}}{2} e^{\frac{ix^2}{4}} - |x| \sum_{n=0}^{\infty} \frac{(ix^2)^n}{n!(2n+1)} , \quad |x| \to 0.$$  

(1.96)

By means of the following formulae

$$\csc \frac{\theta}{2} = \frac{3}{4} \csc \frac{\theta}{2} + \frac{21}{216} \sin \frac{\theta}{2} + O(\sin^3 \frac{\theta}{2}) ,$$

$$\cos \frac{\theta}{3} = 1 - \frac{2}{9} \sin^2 \frac{\theta}{2} - \frac{16}{243} \sin^4 \frac{\theta}{2} + O(\sin^6 \frac{\theta}{2}) ,$$

$$\sin \frac{\theta}{3} = \frac{2}{3} \sin \frac{\theta}{2} + \frac{10}{162} \sin^3 \frac{\theta}{2} + \frac{154}{3888} \sin^5 \frac{\theta}{2} + O(\sin^7 \frac{\theta}{2}) ,$$

$$\csc \frac{\theta}{3} = \frac{3}{2} \csc \frac{\theta}{2} - \frac{15}{108} \sin \frac{\theta}{2} + \frac{543}{11664} \sin^3 \frac{\theta}{2} + O(\sin^5 \frac{\theta}{2}) ,$$

$$\cot \frac{\theta}{3} = \frac{3}{2} \cot \frac{\theta}{2} - \frac{51}{108} \sin \frac{\theta}{2} + \frac{1335}{11664} \sin^3 \frac{\theta}{2} + O(\sin^5 \frac{\theta}{2}) ,$$

which are obtained from Erdelyi [15],

$$\sin \nu z = 2 \nu \sin \frac{\nu}{2} \mathcal{F}_1 \left( \frac{1}{2}, \frac{1}{2} \nu \mid \frac{1}{2} ; \sin^2 \frac{z}{2} \right) ,$$

$$\cos \nu z = 2 \mathcal{F}_1 \left( \nu \mid - \frac{1}{2} ; \sin^2 \frac{z}{2} \right) ,$$

and the expression (1.96), we may expand (1.78) and (1.88).
in terms of $\sin \delta/2$. The manipulations have to be carried out with care, and this is a rather tedious task. The parts which would be singular if $\delta = 0$ cancel each other out and we finally obtain

\[
I(\delta) \sim B(\theta, \phi) e^{i(k_\rho - \pi/4)} \left[ \frac{i}{2} e^{i\pi/4} - \sqrt{2k_\rho} |\sin \delta/2| \right] - \frac{i}{3} \left( \sqrt{2k_\rho} |\sin \delta/2| \right)^3 \ldots + \frac{e^{i(k_\rho + \pi/4)}}{2\sqrt{2m_\rho}} \left( B_2 - \frac{17}{54} B_2 + B_4 \right) \sin \delta/2
\]

\[
+ \mathcal{O}\left( \frac{\sin^2 \delta/2}{(k_\rho)^{3/2}} \right) + \mathcal{O}\left( \sin^2 \delta/2 \right).
\]

Similarly

\[
J(\delta) = \frac{3}{4} B_0 e^{-i(k_\rho + \pi/4)} - \frac{3}{2} B_1 \sin \delta/2 e^{i(k_\rho - \pi/4)} \left[ \frac{i}{2} e^{i\pi/4} - \sqrt{2k_\rho} |\sin \delta/2| \right] - \frac{i}{3} \left( \sqrt{2k_\rho} |\sin \delta/2| \right)^3 \ldots ]
\]

\[
- \frac{e^{i(k_\rho - \pi/4)}}{2\sqrt{2m_\rho} k_\rho} \left( \frac{3}{4} B_2 - \frac{17}{54} B_2 + B_4 \right) + \mathcal{O}\left( \sin^2 \delta/2 \right) + \mathcal{O}\left( \frac{\sin \delta/2}{(k_\rho)^{3/2}} \right).
\]
\[ \delta = \psi + \delta' \] where \( \delta' \) is small and \( \psi \) is a fixed angle

The arguments of the Fresnel integrals will be large and we can therefore substitute (1.89) into (1.78) and (1.88) and setting \( \delta = \psi + \delta' \) we obtain after some manipulation of the resulting expressions

\[
I(\psi \pm \delta') = \frac{\mathcal{D}(\theta, \beta) e^{i(kp+\pi/4)}}{3\sqrt{2\pi k'}} \coth \psi \frac{e^{i(kp+\pi/4)}}{3} \mp \frac{2\mathcal{D}(\theta, \beta) e^{i(kp+\pi/4)}}{9\sqrt{2\pi k'} k' \sin^2 \psi/3}

- \frac{i \mathcal{D}(\theta, \beta) e^{i(kp+\pi/4)}}{24\sqrt{2\pi} (kp)^{3/2}} \coth \psi \frac{1}{3} \left(1 + \frac{\delta}{9\sin^2 \psi/3}\right)

+ \frac{i B_1 e^{i(kp+\pi/4)}}{12\sqrt{2\pi k'} kp} \cos \psi \frac{1}{3} + o \left(\frac{\sin \psi/2}{(kp)^{5/2}}\right) \quad (1.99)

\[
J(\psi \pm \delta') = -\frac{B_1 e^{i(kp-\pi/4)}}{12\sqrt{2\pi k'} kp} (\csc \psi/3)^2 + o \left(\frac{\sin \psi/2}{(kp)^{5/2}}\right) \quad (1.100)
\]
An E-polarised plane wave

\[ u_i(p) = e^{-ikr \sin(\theta + \phi_0)} \tag{2.1} \]

where \( x = r \cos \theta, \ y = r \sin \theta \), is incident on an imperfectly conducting rectangular cylinder: \( |x| \leq a, \ |y| \leq b, \ -\infty < z < \infty \); see figure 1. The permeability, permittivity, and conductivity of the cylinder are \( \mu, \varepsilon \) and \( \sigma \) respectively; and the complex refractive index of the cylinder material is given by

\[ N = \sqrt{\frac{\mu}{\mu_0} \left( \frac{\varepsilon}{\varepsilon_0} - \frac{1}{\omega \mu_0 \varepsilon_0} \right)}. \]

The sides of the cylinder are assumed to be large compared to the incident wave wavelength (i.e., \( kb > 1 \))
The problem under consideration is that of finding the field at the point \( P(r, \theta) \), \( 0 \leq \theta < 2\pi \), where \( r \) is large compared to the dimensions of the cylinder, i.e. the diffracted far field. Clearly from the symmetry of the problem we need only consider an angle of incidence within the range \( \frac{\pi}{2} \leq \theta_o \leq \pi \). In particular we shall obtain the total scattering cross section, (which is the sum of the absorption and scattering cross section, Jones [17]) for oblique incidence \( \left( \frac{\pi}{2} < \theta_o < \pi \right) \). At "grazing incidence" \( (\theta_o = \pi \text{ or } \pi/2) \) the expression obtained for the total scattering cross section at oblique incidence becomes infinite, indicating that the method used to obtain the total scattering cross section is no longer valid. We defer this situation to the next chapter where the total scattering cross section at grazing incidence is obtained by an alternative method.

To obtain the diffracted far field we shall need to use the results of chapter 1 in conjunction with Keller's theory of geometrical diffraction [18]. Since the scatterer has more than one corner the effect of multiple diffractions will have to be considered, and thus an outline of Keller's theory will be given with relevance to the problem under consideration.

**Keller's geometrical theory of diffraction and multiple diffraction**

According to Keller's geometrical theory of diffraction the diffracted field \( u_d(P) \) at a point \( P \) is equal to the sum of the fields on all rays through \( P \):
Here $u_j(P)$ is the diffracted field on the $j^{th}$ such ray, and if this is an $m$-fold diffracted ray then

$$u_j(P) = \frac{e^{-iks_j}}{km^{2}} \sum_{n=0}^{\infty} \left( \frac{ik}{n} \right)^{n} A_{jn}(P)$$

where $k(=2\pi/\lambda)$ is the propagation constant, $s_j$ the arc length along the ray, and the function $A_{jn}$ depends on the geometry and material of the diffracting object. For a rectangular cylinder all the diffracted rays are produced by wedges of $90^\circ$ angle. Hence the inclusion of higher $A_{jn}$ ($n = 1,$ $2,$ ....) in the expansion (2.3) involves the use of more terms in the asymptotic solution of the wedge diffraction problem. The calculation of the diffraction coefficients corresponding to these higher-order terms necessitates the solution of the wedge problem for nonplane-wave incidence. However, as is shown in the work of Zitron and Karp [19], the relevant nonplane waves are expressible in terms of linear combinations of plane waves and their derivatives. Thus the diffraction coefficients are easily found; and therefore it is possible to calculate the off shadow far fields corresponding to wedge excitations which are not shadow boundary fields. In order to calculate diffraction coefficients corresponding to shadow boundary fields we show that these fields too are expressible in terms of plane waves and their derivatives.
From (1.43), where $D(\theta, \theta_0)$ is given by (1.46), applying a formula due to Zitron and Karp [19], we obtain the field on and near to the ray determined by the point $P_1(r_1, \theta_1)$. More precisely, we obtain the asymptotic expansion of the diffracted field, $u_d(E_1; E_1)$, at any point $E_1 = E_1(\xi, \eta)$ in the neighbourhood of the point $P_1$, see figure 2.

$$\rho_1 = [(r_1 + \xi_1)^2 + \eta_1^2]^{\frac{1}{2}},$$

$$\delta = \tan^{-1} \frac{\eta}{r_1 + \xi}.$$

Figure 2

$$u_d(E_1; E_1) = \frac{ikr_1}{\sqrt{r_1}} \left[ D(\theta_1, \theta_{01}) p(o) + \left[ \frac{D(\theta_1, \theta_{01})p''(o)}{2ik} - \frac{D_0(\theta_1, \theta_{01})p'(o)}{ik} + D(1)(\theta_1, \theta_{01})p(o) \right] \frac{1}{r_1} + O(kr_1^{-2}) \right],$$

(2.4)

where
\[ D^{(1)}(\theta_1, \theta_{o1}) = (2ik)^{-1} \left[ \frac{1}{4} p(\theta_1, \theta_{o1}) + D_{\theta\theta}(\theta_1, \theta_{o1}) \right], \]

and 
\[ p(\psi) = e^{ik(\xi \cos \psi - \eta \sin \psi)}. \quad (2.5) \]

It will be noticed that (2.4) is expressed in terms of \( p(\omega) \) and derivatives of \( p(\omega) \), where \( p(\psi) \) is a plane wave expression. Since (2.4), which may be continued to any desired order of \( \frac{1}{r_{12}} \), is linear in \( p(\psi) \) and its derivatives the fields resulting from successive interactions are readily derived. Thus the expression for the diffracted field when an incident plane wave strikes \( E_1 \) and the resulting diffracted field, given by (2.4), hits the second wedge \( E_2 \), see figure 3, is given by

\[
\begin{align*}
\psi_d(E_2; E_1, E_2) &= \frac{e^{ikr_{12}}}{\sqrt{r_{12}}} \left\{ D(\theta_{12}, \theta_{o1}) u_d(E_2; E_2) \\
+ (2ik)^{-1} u_d''(E_2; E_2) D(\theta_{12}, \theta_{o1}) -(ik)^{-1} u_d'(E_2; E_2) D_{\theta}(\theta_{12}, \theta_{o1}) \\
+ u_d(E_2; E_2) D^{(1)}(\theta_{12}, \theta_{o1}) + O[\alpha r_{12}^{-2}] \right\}. \quad (2.6)
\end{align*}
\]

In expression (2.6), \( (r_{12}, \theta_{12}) \) are the polar coordinates of \( E_2 \) with respect to \( E_1 \), see figure 3; \( u_d(E_2; E_2) \) is given by (2.4) with the subscript 2 in place of 1, and the primes on this function indicate derivatives with respect to \( \pm \theta_{o2} \) depending respectively on whether or not \( \psi \) and \( \theta_{o2} \) have the same orientation.
$\psi$ and $\theta_{02}$ as shown above are measured from the same face $E_1E_2$ and have therefore the same orientation.

**Figure 3**

Equation (2.4) is obtained assuming oblique incidence. Whenever the incident rays are parallel to a side (i.e. when $\theta_{01} = \frac{\pi}{2}$) the value of the field given by (2.4) must be divided by two. This normalisation is due to the coalescing of the incident and reflected field at this angle of incidence.

The expressions (2.4) and (2.6) have been obtained under the assumption that the poles of the integrand do not reside near to the saddle point in the integral (1.53). As a result these expressions are not valid near to the shadow or specular boundaries, see figure 4, indeed the expressions (2.4) and (2.6) become infinitely large near these boundaries.
To determine the field near to the above mentioned boundaries it is necessary to use the asymptotic expansions (1.97) to (1.100); and in order to account for the effects of multiple diffraction these results must be expressed in terms of the plane wave $p(\psi) = e^{ik(\cos \psi - \eta \sin \psi)}$, and its derivatives. Thus introducing the coordinate system

$$\delta = \tan^{-1} \left( \frac{\eta}{x+\xi} \right), \quad \rho = \left[ (x+\xi)^2 + \eta^2 \right]^{\frac{1}{2}}$$

(2.7)
where $\delta$ is defined as being positive for the situation shown in figure 5, the asymptotic expressions (1.97) to (1.100) become

$$I(\delta) = D(\theta, \phi) \left[ sgn \delta \ e^{i(kr + \delta)} \left\{ \frac{1}{2} - \frac{\frac{i\pi}{4}}{\sqrt{\pi}} \left( \frac{k^2}{4kr^3} \right) \right\} - \frac{17{k^2}{\delta e^{i(kr + \delta)}}}{216\sqrt{2\pi}^6 (kr)^{3/2}} \right]$$

$$+ \frac{B_2 \ e^{i(kr + \delta)}}{4\sqrt{2\pi}^2 (kr)^{3/2}} + O \left[ (kr)^{-5/2} \right], \quad (2.8)$$

$$J(\delta) = \frac{B_0' \ e^{i(kr + \delta)}}{\sqrt{\pi}} \left[ \frac{3e^{i\varphi}}{4\sqrt{2kr^3}} - \frac{3e^{i\varphi}}{4kr^3} \left\{ \frac{e^{i\varphi} - 1}{\varphi} \right\} \right]$$

$$- \frac{3e^{i\varphi} \ i(k^2 \gamma^2 - i k \gamma \delta)}{8\sqrt{2} (kr)^{3/2}} - \frac{e^{i\varphi}}{24kr^3 (kr)^{3/2}} \left\{ \frac{3B_2'}{8} - \frac{17B_0'}{144i} \right\} e^{i(kr + \delta)}$$

$$+ O \left[ (kr)^{-5/2} \right], \quad (2.9)$$

$$I(\psi \delta) = \frac{D(\theta, \phi) e^{i(kr + \delta)}}{3 \sqrt{2\pi} kr^3 \left[ \cot \psi \left( \frac{3}{2} + \frac{1}{2} \left( \frac{ik^2 \gamma^2}{2} \right) \right) + \left( \frac{ik^2 \gamma^2}{3} \right) \cot \psi \right]}$$

$$\frac{i D(\theta, \phi) e^{i(kr + \delta)}}{3 \sqrt{2\pi} kr^3 \left[ \frac{1 + \frac{8}{9} \cos \frac{\psi}{3} \frac{1}{2} \right]} + \frac{B_2 i e^{i(kr + \delta)}}{12 \sqrt{2\pi}^6 (kr)^{3/2}} \left[ \left( \frac{kr}{2} \right)^{-5/2} \right], \quad (2.10)$$
Expressing the above formulae in terms of the plane wave function $p(\psi)$ gives for $\eta > 0$

$$I(\psi) = D(\theta_0, 0) \left[ \frac{e^{ikr} p(\psi)}{2} - \frac{e^{ikr} e^{ikr}}{\sqrt{\pi}} \left\{ \frac{i p'(\psi) + \sqrt{2} \left[ p'(\psi) + \frac{p''(\psi)}{2} \right]}{12 (kr)^{3/2}} \right\} \right]$$

$$- \frac{17 e^{ikr} e^{im\psi}}{2\pi (kr)^{3/2}} + \frac{B_2 e^{ikr} e^{im\psi}}{4 (kr)^{3/2}} p'(\psi) + O((kr)^{-5/2})$$

$$J(\psi) = \frac{B_0' e^{ikr}}{\sqrt{\pi}} \left\{ \frac{3 e^{im\psi}}{4 \sqrt{2kr}} \right\}$$

$$- \frac{54 e^{ikr} B_2' e^{im\psi}}{2 \pi (kr)^{3/2}} p(\psi) + O((kr)^{-3/2})$$

$$J(\psi') = -\frac{B_0' e^{ikr} e^{im\psi}}{12 \sqrt{2\pi (kr)^{3/2}}} \cos \psi + O((kr)^{-5/2})$$

$$I(\psi') = \frac{D(\theta_0, 0) e^{ikr+im\psi}}{3 \sqrt{2\pi (kr)^{3/2}}} \left( \cot \psi + \frac{8 i p'(\psi) \csc \psi}{24 kr} \right)$$

$$- \frac{\cot \psi^2}{\sqrt{2\pi (kr)^{3/2}}} \left\{ 1 + \frac{8}{9} \cot^2 \psi \right\} i p''(\psi) + O((kr)^{-5/2})$$

$$+ \frac{B_2 e^{im\psi}}{12 \sqrt{2\pi (kr)^{3/2}}} \cot \psi p(\psi) + O((kr)^{-5/2})$$

$$+ \frac{B_2 e^{im\psi}}{12 \sqrt{2\pi (kr)^{3/2}}} \cot \psi \frac{p(\psi)}{3} + O((kr)^{-5/2})$$

$$+ \frac{B_2 e^{im\psi}}{12 \sqrt{2\pi (kr)^{3/2}}} \cot \psi \frac{p(\psi)}{3} + O((kr)^{-5/2})$$

$$+ \frac{B_2 e^{im\psi}}{12 \sqrt{2\pi (kr)^{3/2}}} \cot \psi \frac{p(\psi)}{3} + O((kr)^{-5/2})$$
where
\[ p(o) = e^{ik\xi}, \quad p'(o) = -ik\eta e^{ik\xi}, \quad p''(o) = -(k^2\eta^2 + ik\xi)e^{ik\xi}, \]
\[ \bar{p}''(\pi) = (ik\xi - k^2\eta^2)e^{ik\xi}, \quad p''(o) = (-3k^2\eta^2 + ik\eta + ik^3\eta^3)e^{ik\xi}. \]

The expressions (2.12) to (2.15) will be required in the next chapter and the dominant terms of (2.8) to (2.11) will be required when dealing with the field near the shadow or specular boundaries.

**Diffraction field for oblique incidence**

We now derive the far field expression for the diffracted field in all regions around the cylinder for oblique incidence except in certain specified directions. To obtain the results in terms of the coordinates \((r, \theta)\) of the observation point \(P\) we use asymptotic approximations to the length and the angles of the various rays from the edges of the rectangle to the observation point \(p(r, \theta)\).

From figure 6 on applying the cosine rule we get for large \(r\)

\[ r_{\perp} \sim r - \ell \cos \theta \cos \beta - \ell \sin \theta \sin \beta + O(r^{-2}). \]

![Figure 6](image)
but $x_1 = \ell \cos \beta$ and $y_1 = \ell \sin \beta$, where $x_1$ and $y_1$ are the coordinates of the point on the cylinder. Hence

$$ r_1 = r - x_1 \cos \theta - y_1 \sin \theta. $$

(2.16)

The diffracted rays being dealt with are those from the corners of the rectangle $E_1(= r_1)$, $E_2(= r_2)$, $E_3(= r_3)$ and $E_4(= r_4)$ so that

$$ r_1 = r + a \cos \theta + b \sin \theta + O(r^{-2}), 
$$

$$ r_2 = r - a \cos \theta + b \sin \theta + O(r^{-2}), 
$$

$$ r_3 = r - a \cos \theta - b \sin \theta + O(r^{-2}), 
$$

$$ r_4 = r + a \cos \theta - b \sin \theta + O(r^{-2}). 
$$

(2.17)

The angles of the various rays from the corners of the rectangle to the observation point $P$, $E_1(= \theta_1)$, $E_2(= \theta_2)$, $E_3(= \theta_3)$, and $E_4(= \theta_4)$, are measured as shown in figure 7; and for large $r$ are asymptotic to:

$$ \theta_1 \sim \theta - \frac{\pi}{2}, \quad (\text{mod } 2\pi), 
$$

$$ \theta_2 \sim \frac{\pi}{2} - \theta, \quad (\text{mod } 2\pi), 
$$

$$ \theta_3 \sim \frac{\pi}{2} + \theta, \quad (\text{mod } 2\pi), 
$$

$$ \theta_4 \sim \theta, \quad (\text{mod } 2\pi). 
$$

(2.18)
In figure 7 \( r_1 \) is a broken line because where \( P \) is depicted no diffracted rays come from corner \( E_1 \).

We shall now calculate the multiply-diffracted fields at the corners of the rectangle by means of formulae (2.4) and (2.6). An \( n \)-tuply diffracted field at the corner \( E_s \) \( (s = 1, 2, 3, 4) \) will be denoted by \( u_n(E_s) \). The notation \( h(o) \) and \( v(o) \) will be used in place of \( p(o) \), given in (2.4); to distinguish between plane waves striking the same corner on the horizontal or vertical side respectively, see figure 8.

\[ \begin{align*}
&h(o) \\
&\rightarrow \\
&\uparrow \\
&v(o)
\end{align*} \]

Figure 8

For \( \frac{\pi}{2} < \theta_o < \pi \), \( E_4 \) is in the shadow, and the only singly
diffracted ray reaching $E_1$ is produced by the incident field striking $E_2$, see figure 9. Thus from the expression (2.4)

$$u_1(E_1) = U_d(E_1, E_2) = u_i(E_2) \frac{e^{i2\pi k}}{2a} \left\{ D(\theta_j, \frac{3\pi}{2} - \theta_0) \right\}$$

$$+ \frac{(2ik)^{-1}}{2a} \left\{ h''(\theta) J(\theta_j, \frac{3\pi}{2} - \theta_0) - (ik)^{-1} h'(\theta) D(\theta_j, \frac{3\pi}{2} - \theta_0) + h(\theta) D(\theta_j, \frac{3\pi}{2} - \theta_0) \right\}$$

$$+ O\left\{ (2ka)^{-2} \right\} \right\}$$

which, because of (1.48) to (1.52) reduces to

$$u_1(E_1) = -\frac{U_i(E_2) e^{i2\pi k}}{2a} \frac{k}{i} \left\{ D(\theta_j, \frac{3\pi}{2} - \theta_0) \left( h'(\theta) - \frac{h(\theta)}{\cos \theta_0} \right) \right\} + O\left\{ (kd)^{-5/2} \right\}$$

(2.20)

The singly diffracted fields at the three remaining corners are obtained in a similar manner, and are given by

$$u_1(E_2) = -\frac{U_i(E_2) e^{i2\pi k}}{2a} \frac{k}{i} \left\{ D(\theta_j, \frac{3\pi}{2} - \theta_0) \left( h'(\theta) - \frac{h(\theta)}{\cos \theta_0} \right) \right\} + O\left\{ (kd)^{-5/2} \right\}$$

$$-\frac{U_i(E_2) e^{i2\pi k}}{2a} \frac{k}{i} \left\{ D(\theta_j, \frac{3\pi}{2} - \theta_0) \left( h'(\theta) - \frac{h(\theta)}{\cos \theta_0} \right) \right\} + O\left\{ (kd)^{-5/2} \right\}$$

(2.21)

$$u_1(E_3) = -\frac{U_i(E_3) e^{i2\pi k}}{2a} \frac{k}{i} \left\{ D(\theta_j, \frac{3\pi}{2} - \theta_0) \left( h'(\theta) - \frac{h(\theta)}{\cos \theta_0} \right) \right\} + O\left\{ (kd)^{-5/2} \right\}$$

(2.22)
Doubly diffracted fields are produced when singly diffracted rays are incident on a corner. These fields can be determined by using the expression (2.6). After the first diffraction the resulting singly diffracted rays will be incident on the second corner along the sides of the rectangular cylinder and thus it is necessary to divide the expression (2.6) by two. In the following calculations only terms up to the order \((kd)^{-3/2}\), where \(d\) is a typical dimension representing 'a' or 'b'.

The doubly diffracted field at the corner \(E_1\) is given by the rays shown in figure 9.

\[
\begin{align*}
\mathbf{u}_2(E_1) &= \mathbf{u}_d(E_1;E_1,E_2) + \mathbf{u}_d(E_1;E_1,E_4) + \mathbf{u}_d(E_1;E_3,E_2) + \mathbf{u}_d(E_1;E_3,E_4)
\end{align*}
\]

Figure 9
Thus

\[ u_2(E_1) = \frac{\iiota_i(E_1) e^{i4abk}}{4a} D(0, 0 - 2\pi, 0) h(0) \]

\[ + \frac{\iiota_i(E_2) e^{i2(a+b)k}}{4ab} D(\pi, 0 - \theta, 0) D(0, \pi \alpha, 0) h(0) \]

\[ + \frac{\iiota_i(E_3) e^{i2(a+b)k}}{4\sqrt{a}b} D(0, \pi + \theta, 0) D(0, \pi \alpha, 0) h(0) \]

\[ + O[(ka)^{-5/2}] \]  

(2.24)

and from the expression (1.48) the above expression reduces to

\[ u_2(E_1) = O(d^{-5/2}) \]

Similarly it can be shown that the second order diffracted field at the other corners is \( O(d^{-5/2}) \), and thus

\[ u_2(E_j) = O(d^{-5/2}) , \quad j = 1, 2, 3, 4. \]

Hence up to \( O(d^{-5/2}) \) the first order diffracted field gives the field at each corner of the cylinder. The above method can of course be extended, enabling one to determine the field at any corner to any desired order of accuracy.

However this involves much more work, since the multiplicity of rays as they travel round the faces of the cylinder requires one to deal with protracted expressions. We will only consider terms up to \( O(d^{-3/2}) \).
In order to find the total field at the edges we must add the incident field at those edges which are illuminated by the incident field. Thus the total field \( u(E_j) \) at corner

\[
\begin{align*}
E_1 \text{ is } u(E_1) &= u_1(E_1) + u_1(E_1) + O[(kd)^{-5/2}] , \\
E_2 \text{ is } u(E_2) &= u_1(E_2) + u_1(E_2) + O[(kd)^{-5/2}] , \\
E_3 \text{ is } u(E_3) &= u_1(E_3) + u_1(E_3) + O[(kd)^{-5/2}] , \\
E_4 \text{ is } u(E_4) &= u_1(E_4) + O[(kd)^{-5/2}] .
\end{align*}
\]

(2.25)

The above expressions are in terms of plane waves and on hitting the corner at which they are calculated they will give rise to diffracted rays which we shall assume are observed a long way from the cylinder at a point \( P \).

According to Keller's geometrical theory of diffraction the total diffracted field at \( P \) is the sum of the fields on all the rays through \( P \). Clearly for \( 0 < \theta < 2\pi \) in certain ranges of \( \theta \) some of the corners of the cylinder will produce no diffracted rays at \( P \). Hence we will consider specific ranges of \( \theta \).

For \( 0 < \theta < \frac{\pi}{2} \) the first order diffracted fields at \( P \) are produced by the incident field diffracted by the corners \( E_2 \) and \( E_3 \).

\[
\begin{align*}
\frac{u_1(P)}{\sqrt{r_1}} &= \frac{u_i(E_2) e^{ik_0 r_2} D(\frac{\pi}{2} - \theta, \theta_2)}{\sqrt{r_1}} + \frac{u_i(E_3) e^{ik_0 r_3} D(\frac{\pi}{2} + \theta, \pi - \theta_3)}{\sqrt{r_3}} \\
&+ O[(kd)^{-3/2}] \\
&\quad (2.26)
\end{align*}
\]
This expression above is valid provided that the point \( P \) is not near to the specular direction \( \theta = \theta_0 - \frac{\pi}{2} \), in which case the diffraction coefficients become infinite. The second order diffraction contribution at \( P \), from the corners \( E_2, E_3 \) and \( E_4 \) is

\[
u_2(P) = -\frac{ekr^2}{2\sqrt{r^2}} \left[ \frac{u_i(E)}{ik(2a)^{3/2}} \left\{ D_0(\theta, \theta_0) \left\{ D_0(\pi+\theta, \theta) - D(\pi+\theta, \theta_0) \right\} \right\} \right]
\]

\[
+ \frac{u_i(E_3) e^{i2\alpha k}}{ik(2b)^{3/2}} \left\{ D_0(\theta, \pi-\theta_0) \left\{ D_0(\pi+\theta, \theta) - D(\pi+\theta, \theta_0) \right\} \right\} \right] + O[(k\ell)^3]
\]

\[
- \frac{ekr^3}{2\sqrt{r^2}} \left[ \frac{u_i(E_3) e^{i2\alpha k}}{ik(2a)^{3/2}} \left\{ D_0(\theta_0, \theta_0) \left\{ D_0(\theta, \theta_0) - D(\theta, \theta_0) \right\} \right\} \right] + O[(k\ell)^3]
\]

\[
- \frac{ekr^4}{2\sqrt{r^2}} \left[ \frac{u_i(E_3) e^{i2\alpha k}}{ik(2a)^{3/2}} \left\{ D_0(\theta_0, \theta_0+\pi) \left\{ D_0(\theta, \theta_0) - D(\theta, \theta_0) \right\} \right\} \right] + O[(k\ell)^3]
\]

\[
+ \frac{u_i(E_1) e^{i2\alpha k}}{ik(2b)^{3/2}} \left\{ D_0(\theta, 2\pi-\theta_0) \left\{ D_0(3\pi-\theta, \theta_0) - D(3\pi-\theta, \theta_0) \right\} \right\} \right] + O[(k\ell)^3].
\]

Converting the above expression into terms involving \( r, \theta \), and \( \theta_0 \), and noting that

\[
u_d(P) = u_1(P) + u_2(P),
\]

\[
D(\alpha, \theta) = 0, \quad u_4(E_4) = 0,
\]
$u_1(E_1) = e^{i k(a \sin \theta_0 + b \cos \theta_0)}$, $u_1(E_2) = e^{-i k(a \sin \theta_0 + b \cos \theta_0)}$, $u_1(E_3) = e^{-i k(a \sin \theta_0 + b \cos \theta_0)}$, gives

For $0 < \theta < \pi/2$, and $\theta \neq \theta_0 - \pi/2$,

$$u_1(P) = \frac{e^{i k \rho}}{\sqrt{\pi}^1} \left[ e^{D(\pi - \theta, 0)} e^{i k[a(2 \sin \theta - \cos \theta) + b(\cos \theta + \sin \theta)]} + e^{D(\pi + \theta, \pi - \theta)} e^{i k[a(2 \sin \theta - \cos \theta) + b(\cos \theta + \sin \theta)]} \right]$$

$$- \frac{1}{2ik} \left[ \frac{D_\theta(0, 0 - \pi/2) D_\theta(0, \pi/2 - \theta, 0)}{(2a)^{3/2}} + \frac{D_\theta(0, \theta) D_\theta(\theta + \pi/2, 0)}{(2b)^{3/2}} + \frac{i k[a(2 \sin \theta - \cos \theta) + b(2 \cos \theta + \sin \theta)]}{(2b)^{3/2}} + \frac{D_\theta(0, 0 - \pi/2) D_\theta(0, \pi/2 - \theta, 0)}{(2a)^{3/2}} + \frac{D_\theta(0, \theta) D_\theta(\theta + \pi/2, 0)}{(2b)^{3/2}} + \frac{i k[a(2 \sin \theta - \cos \theta) + b(2 \cos \theta + \sin \theta)]}{(2b)^{3/2}} \right]$$

$$+ O \left[ (k \rho)^{-3/2} \right] + O \left[ (kr)^{-3/2} \right] \quad (2.28)$$

The expressions for the diffraction coefficients in (2.28)
are given by (1.46) to (1.50). By an analogous procedure we obtain for the remaining quadrants.

For \( \pi/2 < \theta < \pi \) and \( \theta \neq 3\pi/2 - \theta_0 \)

\[
u_\alpha (p) = \frac{ikr}{\sqrt{p}} \left[ \begin{array}{c}
\mathcal{D} \left( \frac{\pi}{2} + \theta_0 - \pi \right) + e \mathcal{D} \left( \theta - \frac{3\pi}{2}, 2\pi - \theta_0 \right) \\
\end{array} \right]
\]

\[
- \frac{1}{2ik} \left[ \begin{array}{c}
\mathcal{D}_0 (\theta_0, \theta) \mathcal{D}_0 (\pi/2 + \theta_0, 0) e^{ik[-a \sin \theta_0 + \cos \theta + b(\cos \theta_0 + \sin \theta)]} \\
\end{array} \right]
\]

\[
+ \frac{D_\theta (0, 2\pi - \theta_0) D_\theta (3\pi/2 - \theta_0, 0) e^{ik[a(2 - \sin \theta_0 + \cos \theta) - b(\cos \theta_0 + \sin \theta)]}}{(2b)^{3/2}}
\]

\[
+ \mathcal{O} \left[ (kd)^{-3/2} \right] + \mathcal{O} \left[ (kr)^{-3/2} \right],
\]

(2.29)

For \( \pi < \theta < 3\pi/2 \) and \( \theta \neq \theta_0 + \pi/2 \)

\[
u_\alpha (p) = \frac{ikr}{\sqrt{p}} \left[ \begin{array}{c}
\mathcal{D} \left( \frac{\pi}{2} + \theta_0 - \pi \right) + e \mathcal{D} \left( \theta - \frac{3\pi}{2}, 2\pi - \theta_0 \right) \\
\end{array} \right]
\]

\[
- \frac{1}{2ik} \left[ \begin{array}{c}
\mathcal{D}_0 (\theta_0, \theta) \mathcal{D}_0 (\pi/2 + \theta_0, 0) e^{ik[-a \sin \theta_0 + \cos \theta + b(\cos \theta_0 + \sin \theta)]} \\
\end{array} \right]
\]

\[
+ \frac{D_\theta (0, 2\pi - \theta_0) D_\theta (3\pi/2 - \theta_0, 0) e^{ik[a(2 - \sin \theta_0 + \cos \theta) - b(\cos \theta_0 + \sin \theta)]}}{(2b)^{3/2}}
\]

\[
+ \mathcal{O} \left[ (kd)^{-3/2} \right] + \mathcal{O} \left[ (kr)^{-3/2} \right],
\]
\[
- \frac{1}{2ik} \left[ D_\theta (0, \frac{3\pi}{2} - \theta, 0) D_\theta (2\pi - \theta, 0) e \right. \\
\left. \frac{i k [a(2 - \sin \theta + \cos \theta) + b(\cos \theta + \sin \theta)]}{(2a)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, 0 - \pi/2) D_\theta (0, \pi/2, 0) e \left[ i k [a(2 + \sin \theta - \cos \theta) + b(\cos \theta + \sin \theta)] \right]}{(2a)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, \pi - \theta) D_\theta (5\pi/2 - \theta, 0) e \left[ i k [a(\sin\theta+\cos\theta)+b(\cos\theta+\sin\theta)] \right]}{(2b)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, 0 + \pi/2) D_\theta (0, \pi/2, 0) e \left[ i k [a(2 - \sin \theta + \cos \theta) - b(\cos \theta + \sin \theta)] \right]}{(2a)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, 2\pi - \theta) D_\theta (3\pi/2 - \theta, 0) e \left[ i k [a(\sin\theta+\cos\theta)+b(\cos\theta+\sin\theta)] \right]}{(2b)^{3/2}} \right. \\
\left. + \frac{O[(k_d)^{-5/2}]}{+} + \frac{O[(k_d)^{-3/2}]}{(2.30)} \right]
\]

For \(3\pi/2 < \theta < 2\pi\)

\[
u_d (\mathbf{p}) = \frac{e}{\sqrt{F}} \left[ \left[ \left[ i k [a(\sin \theta + \cos \theta) + b(\cos \theta + \sin \theta)] \right] + \left[ i k [a(\sin \theta + \cos \theta) + b(\cos \theta + \sin \theta)] \right] + \left[ i k [a(\sin \theta + \cos \theta) + b(\cos \theta + \sin \theta)] \right] + \left[ i k [a(\sin \theta + \cos \theta) + b(\cos \theta + \sin \theta)] \right] \right] \right]
\]

\[
- \frac{1}{2ik} \left[ \frac{D_\theta (0, 3\pi/2 - \theta, 0) D_\theta (2\pi - \theta, 0) e \left[ i k [a(2 - \sin \theta + \cos \theta) + b(\cos \theta + \sin \theta)] \right]}{(2a)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, \pi - \theta) D_\theta (5\pi/2 - \theta, 0) e \left[ i k [a(\sin\theta+\cos\theta)+b(\cos\theta+\sin\theta)] \right]}{(2a)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, 0 + \pi/2) D_\theta (0, \pi/2, 0) e \left[ i k [a(2 - \sin \theta + \cos \theta) - b(\cos \theta + \sin \theta)] \right]}{(2a)^{3/2}} \right. \\
\left. + \frac{D_\theta (0, 2\pi - \theta) D_\theta (3\pi/2 - \theta, 0) e \left[ i k [a(\sin\theta+\cos\theta)+b(\cos\theta+\sin\theta)] \right]}{(2a)^{3/2}} \right. \\
\left. + \frac{O[(k_d)^{-5/2}]}{+} + \frac{O[(k_d)^{-3/2}]}{(2.30)} \right]
\]
\[
+ \frac{D_0(\alpha, \beta_0 - \pi/4)}{(2\alpha)^{3/2}} D_0(\theta, \pi, 0) e^{i[k/a(2 + \sin \theta_0 - \cos \theta) + b(\cos \theta_0 + \sin \theta)]} \\
+ \frac{D_0(\alpha, \pi - \theta_0)}{(2b)^{3/2}} D_0(\theta - \pi/2, \pi/2, 0) e^{i[k[-a(\sin \theta_0 + \cos \theta) + b(2 + \cos \theta_0 - \sin \theta)]]} \\
+ \frac{D_0(\alpha, \theta_0)}{(2b)^{3/2}} D_0(\theta - 3\pi/2, \pi) e^{i[k[-a(\sin \theta_0 + \cos \theta) + b(2 + \cos \theta_0 - \sin \theta)]]} \\
+ O[(k\alpha)^{3/2}] + O[(kr)^{-3/2}] \\
\] (2.31)

The first order diffracted field is now determined for points close to the specular and shadow boundaries, for oblique incidence. These expressions will replace the first square brackets of the expressions (2.28) to (2.31), which become infinite on these boundaries. From the expression for the field along the shadow boundary the total scattering cross section can be obtained.

**Diffracted field near the specular boundaries**

Consider the situation where the field along the specular direction \( \theta = \theta_0 - \pi/2, \pi/2 < \theta_0 < \pi \) is required. We must distinguish here between two cases, (i), (ii), depending on whether or not the observation point \( P \) is in the geometrical optics reflection zone, see figure 10 and figure 11.
case (i)
Figure 10

case (ii)
Figure 11
case (i) \( \theta_0 = \pi/2 < \tan^{-1} b/a \)

The diffracted field from the corner \( E_2 \) due to the incident plane wave, is given, with the geometry of figure 12, and formulae (1.58) and (1.61) by

\[
\begin{align*}
\mathbf{u}(P) &= \mathbf{u}_1(E_2) \left[ - I(2\pi - 2\theta_0 + \delta_2) + I(-2\theta_0 + \delta_2) + I(-\pi + \delta_2) \\
&\quad - I(\delta_2) + J(2\pi - 2\theta_0 + \delta_2) + J(2\theta_0 - \delta_2) + J(-\pi + \delta_2) - J(\delta_2) \right]. \\
\end{align*}
\]

(2.32)

\( \delta_2 \) shown is positive, and the coordinate of \( P \) with respect to face \( E_1E_2 \) is \( P = P(p_2, \frac{\pi}{2} + \theta_0 - \delta_2) \).

Figure 12

Using the asymptotic formulae (2.18) to (2.11), (2.32) becomes asymptotic to:

\[
\begin{align*}
\mathbf{u}(P) &= e^{ikr - ie2\pi k \sin \theta_0} \left\{ \left( \frac{\sin \theta_0 + \cos \theta_0}{\cos \theta_0 - \sin \theta_0} \right) \sgn \delta_2 \left\{ -\frac{i}{2} + e^{-i\pi k |\eta_1|} \right\} \\
&\quad - \frac{e^{-i\pi k}}{3\sqrt{2\pi kr}} \left( \frac{1 + 2\sqrt{3}}{\sqrt{3} \cos \theta_0 + 1} \right) \right\} + \mathcal{O}(kr^{-2}) \\
&\quad + 3B'_0(\eta_2 + \theta_0, 3\pi - \theta_0) e^{i\pi k} + \mathcal{O}(kr^{-2}) \\
&\quad \text{(2.33)}
\end{align*}
\]
where the following relations have been used,

\[
\frac{\pi}{2} + \theta_p^0, \frac{3\pi}{2} - \theta_p^0 = \frac{\sin \theta_p^0 + \cos \theta_p^0}{\cos \theta_p^0 - \sin \theta_p^0} \quad u_i(E_2) e^{i k r} = e^{i k r - \frac{2 \pi}{\lambda} \delta_3 \sin \theta_p^0}
\]

Similarly for the diffracted rays from the corner \( E_2 \), using the geometry of figure 13 gives

\[
u(P) = e^{i k r - \frac{2 \pi}{\lambda} \delta_3 \sin \theta_p^0} \left( \frac{\sin \theta_p^0 + \cos \theta_p^0}{\cos \theta_p^0 - \sin \theta_p^0} \right) - \frac{1}{2} + \frac{e^{i \eta \nu_k / \eta_2}}{\sqrt{2 \pi k r}} \right) + \mathcal{O}(kr^{-3/2}) \]

\[ u(P) = \frac{e^{i \eta \nu_k}}{3 \sqrt{2 \pi k r}} \left( \frac{1}{\sqrt{3}} + \frac{2 \sqrt{3}}{2 \cos \theta_p^0 + 1} \right) - \frac{3 B'(\theta_p^0, \pi - \theta_p^0) e^{i \eta \nu_k}}{4 \sqrt{2 \pi k r}} \]  

(2.34)

\[ \delta_3 \] shown is positive, and the coordinate of \( P \) with to face \( E_2 E_3 \) is \( P(\rho_3, \theta_p^0 - \delta_3) \).

**Figure 13**

Thus the field at a point \( P \) inside the geometrical reflection zone, see figure 14, is given by the sum of the contribution
from corner $E_2$ and $E_2'$. Adding expressions (2.33) and (2.34) together gives,

$$u(P) = e^{ikr - i2k\sin \theta_0} \left[ 1 - \frac{e^{-i\pi/4}}{\sqrt{2\pi kr}} e^{i\pi/4} \left( \frac{1}{\sqrt{3}} + \frac{2\sqrt{3}}{2 \cos \theta_0 - 1} \right) \right] x$$

$$\times \left( \frac{\sin \theta_0 + \cos \theta_0}{\cos \theta_0 - \sin \theta_0} \right) - \frac{i e^{i\pi/4}}{2 \sqrt{2\pi kr}} \left\{ B_1(\eta_2 - \eta_0, \delta_2 - \theta_0) - B_1(\theta_0, \pi - \theta_0) \right\} + O[(kr)^{-3/4}]$$

(2.35)

$\eta_2 = b \sin \theta_0 + a \cos \theta_0$

$\eta_2' = b \sin \theta_0 - a \cos \theta_0$

$\delta_2 > 0, \quad \delta_2 < 0$

![Diagram](image)

Figure 14

Replacing the first square bracket of the expression (2.28) by (2.35) and replacing $\theta$ by $\theta_0 - \pi/2$ in the remaining part of the expression we obtain:
For $\theta = \theta_0 = \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$, where $\theta_0 = \frac{\pi}{2} < \tan^{-1}\frac{b}{a}$

\[
\begin{align*}
u(P) &= \left(\frac{\sin\theta_0 + \cos\theta_0}{\cos\theta_0 - \sin\theta_0}\right)e^{i k_2 \alpha x} \sin\theta_0 e^{ikr} \left[\frac{e^{-i z k \sin\theta_0}}{\sqrt{2\pi}} \left[2k \sin\theta_0 + 2i \left(\frac{1}{\sqrt{3}} + \frac{2 - \sqrt{3}}{2 \cos\theta_0 + 1}\right)\right]\right] \\
\times \left(\frac{\sin\theta_0 + \cos\theta_0}{\cos\theta_0 - \sin\theta_0}\right) + \frac{1}{2} \left[ B_1 \left(\theta_0, \pi - \theta_0\right) - B_1 \left(\pi + \theta_0, \frac{3\pi}{2} - \theta_0\right)\right] \\
- \frac{1}{2i k} \left[ \frac{e^{i2\alpha k}}{(2a)^{\frac{3}{2}}} D_0 \left(\theta_0, \pi - \theta_0\right) D_0 \left(\pi + \theta_0, 0\right) + \frac{e^{-i2\alpha k}}{(2b)^{\frac{3}{2}}} D_0 \left(\theta_0, \pi - \theta_0\right) D_0 \left(\pi + \theta_0, 0\right)\right] \\
+ \frac{e^{i2\alpha k}}{(2a)^{\frac{3}{2}}} D_0 \left(\theta_0, \pi - \theta_0\right) D_0 \left(\pi + \theta_0, 0\right) + \frac{e^{-i2\alpha k}}{(2b)^{\frac{3}{2}}} D_0 \left(\theta_0, \pi - \theta_0\right) D_0 \left(\pi + \theta_0, 0\right)\right] \\
+ \mathcal{O}[k\alpha]^{-\frac{3}{2}}] + \mathcal{O}[k\alpha]^{-3}\right]. \quad (2.36)
\end{align*}
\]

For case (ii) where $\theta_0 = \frac{\pi}{2} > \tan^{-1}\frac{b}{a}$ the expression for the far field is the same as above except for the first term (reflected wave) which is not included.

By symmetry we can obtain the far field in the remaining specular direction $\theta = \theta_0 + \frac{\pi}{2}$ by interchanging 'a' and 'b' and substituting $\frac{3\pi}{2} - \theta_0$ for $\theta_0$ in the expression (2.36). As before we have to consider two situations depending on whether the point P is inside or outside the geometrical optics reflection zone, see figure 15. These situations correspond to $\theta_0 = \frac{\pi}{2} < \tan^{-1}\frac{b}{a}$.
For \( \theta = \theta_o + \frac{\pi}{2} \) and \( \frac{\pi}{2} < \theta_o < \pi \), where \( \theta_o - \frac{\pi}{2} > \tan^{-1} \frac{b}{a} \)

\[
 u(P) = \left( \frac{\cos \theta_b - \cos \theta_b}{\cos \theta_b + \cos \theta_b} \right)^{i k (r+2b \cos \theta_o)}
 + \frac{e^{i k r}}{\sqrt{r}} \left\{ - \frac{e}{\sqrt{2mr}} \left( 1 - 2 i \frac{a \cos \theta_b + \frac{2 i}{3} \left( \frac{1}{\sqrt{3}} + \frac{2\sqrt{3}}{3} \cos \frac{2\pi}{3} \right) \right) \right\} \left( \frac{\cos \theta_b - \cos \theta_o}{\cos \theta_b + \cos \theta_b} \right)
 + \frac{1}{2} \left[ B_1 \left( \frac{3\pi}{2} - \theta_o, \theta_o - \pi \right) - B_1 \left( 2\pi - \theta_o, \theta_o \right) \right]

- \frac{1}{2ik} \left[ \frac{e^{-i2h\theta}}{(2b)^{3/2}} D_\theta \left( \bar{a}, \pi - \theta_o \right) D_\theta \left( 4\pi - \theta_o, 0 \right) + \frac{e^{-i2k\theta}}{(2a)^{3/2}} D_\theta \left( \bar{a}, \pi - \theta_o \right) D_\theta \left( 4\pi - \theta_o, 0 \right) \right]

(2.37)

For \( \theta_o - \frac{\pi}{2} < \tan^{-1} \frac{b}{a} \) the expression for the far field is exactly the same as (2.37) with the first term (reflected wave) dropped.
Forward diffracted field

To obtain the total scattering cross section for oblique incidence we need to determine the diffracted far field in the shadow of the cylinder, when $\theta = \frac{3\pi}{2} - \theta_o$, see figure 16.
The leading term in the square bracket in the expression (2.29) becomes infinite when the observation point $P$ is along this ray. In order to calculate this first order diffraction term, it is necessary to use the asymptotic expressions which take into account the situation where geometrical optic poles approach the saddle point. The forward far field due to the scattering of the incident plane wave by the corner $E_1$ is given, using the convention shown in figure 17 by

$$u(P) = u_1(E_1)[1 - I(\delta_1) + I(\pi + \delta_1) - I(2\pi - 2\theta_0 + \delta_1)]$$

$$+ I(\pi - 2\theta_0 + \delta_1) - J(\pi + \delta_1) + J(\delta_1) + J(2\pi - 2\theta_0 + \delta_1)$$

$$= J(\pi - 2\theta_0 + \delta_1), \quad (2.38)$$

where we have used formulae (1.58) and (1.61). Substituting the asymptotic expansions (2.8) to (2.11) into (2.38) gives

$$u_d(P) = u_i(E_i) \left[ D(\pi - \theta_0, 2\pi - \theta_0) \left[ \sin \delta_1 \epsilon \left\{ \frac{1}{2} + \frac{e^{-\gamma_1 \epsilon}}{\sqrt{2\pi kr_1}} \right\} \right] \right.$$
Using the results

\[ u_1(E_1) e^{i k(r+s_1)} = e^{i k r}, D(\pi - \theta_o, 2\pi - \theta_o) = 1 \]

gives

\[ u_d(\mathbf{p}) = e^{i kr} \left[ \frac{\epsilon_{\eta h}^2}{2} \left( 1 + \frac{2 k \eta_1}{\sqrt{2 \pi kr}} \right) + \frac{e^{i \eta h}}{3 \sqrt{2 \pi kr}} \left( \frac{1}{\sqrt{2 \pi kr}} + \frac{2 - \sqrt{3}}{2 \cos^2 \theta_o + 1} \right) \right] \]

\[ - \frac{3 e^{i \eta h} B_0 \left( 2\pi - \theta_o, 2\pi - \theta_o \right)}{4 \sqrt{2 \pi kr}} + O \left[ \left( kr \right)^{-\eta \epsilon} \right]. \]

(2.40)

\( \delta_1 \) is positive as shown.

Figure 17

The contribution to the far field by the corner \( E_2 \), using the geometry of figure 18, and the formulae used to determine
the contribution from $E_1$, is given by

$$u_6(P) = e^{ikr} \left[ s_{9} \eta_{9} \left\{ \frac{1}{2} + \frac{e^{-ik/2}}{\sqrt{2\pi kr}} \right\} ^{3 \sqrt{2\pi kr}} \left\{ \frac{1 + \frac{2}{\sqrt{3} \cos \theta_0}}{2 \cos \theta_0 + 1} \right\} \right]$$

$$+ \frac{3 e^{-ik/2} B_{6}^0 (2\pi - \theta_0, \pi - \theta_0)}{4 \sqrt{2\pi kr'}} + O \left[ (kr)^{-3/2} \right].$$

(2.41)

Figure 18

$\delta_3$ is positive as shown; and the coordinate of $P$ with respect to the face $E_2 E_3$ is $P(r_3, 2\pi - \theta_0 - \delta_3)$

Adding the contribution to the far field from the corners $E_1$ and $E_3$ for an observation point $P$ as shown in figure 19 gives

$$u(P) = e^{ikr} + \frac{e^{ikr}}{\sqrt{\pi}} \left\{ \frac{-e^{-ik/2}}{\sqrt{2\pi kr}} \right\}^{2k(b \sin \theta_0 - a \cos \theta_0)} +$$
\[ \frac{2\sqrt{3}}{3\sqrt{2\pi}} \left( \frac{1}{3} + \frac{2}{2\cos^2\frac{\theta}{2} + 1} \right) + \frac{2i}{2\sqrt{2\pi}} \left( B_1(\pi-\phi, 2\pi-\phi) - B_1(\pi-\phi, \pi-\phi) \right) \]

\[ + \mathcal{O}[\sqrt{kr}] \]

(2.42)

\[ \eta_1' = -a \cos \theta_0 + b \sin \theta_0 \]

\[ \eta_3' = -a \cos \theta_0 + b \sin \theta_0 \]

\[ \delta_1 < 0, \quad \delta_3 < 0 \]

Figure 19

The first term of the expression (2.42) represents the incident plane wave in the direction we are observing, that is,

\[ u_i = e^{-ikr \sin(\theta + \theta_0)} = e^{-ikr \sin \frac{\theta_0}{2}} = e^{ikr} \]

Thus the diffracted field is given by \( u_d(P) = u(P) - u_i \),
\[ u_d(p) = e^{i(kr - \pi/4)} \left\{ -2k(b \sin \theta_o - a \cos \theta_o) \right. \]
\[ + \frac{24}{\sqrt{3}} \left( \frac{1}{3} + \frac{2\theta}{4\theta} \right) + \frac{(B_1(\pi - \theta_o, 2\pi - \theta_o) - B_1(2\pi - \theta_o, \pi - \theta_o))}{2} \]
\[ + o(kr^{-3/2}) \]  \hspace{1cm} (2.43) \]

Replacing the first square bracket of the expression (2.29) by (2.43) and replacing \( \theta \) by \( \frac{3\pi}{2} - \theta_o \), in the remaining part of the expression (2.29) gives

For \( \frac{\pi}{2} < \theta_o < \pi \) and \( \theta = \frac{3\pi}{2} - \theta_o \)

\[ u_d(p) = \frac{e^{-i\theta_0}}{\sqrt{\beta}} \left\{ -\frac{e^{-i\theta_0}}{\sqrt{2\pi k^3}} 2k(b \sin \theta_o - a \cos \theta_o) + \frac{2\sqrt{\beta}}{3\sqrt{2\pi k^3}} \left( \frac{1}{3} + \frac{2}{2\cos^2 \theta_o + 1} \right) \right. \]
\[ + \frac{e^{-i\theta_0}}{2\sqrt{2\pi k^3}} \left\{ B_1(\pi - \theta_o, \pi - \theta_o) - B_1(2\pi - \theta_o, \pi - \theta_o) \right\} \]
\[ - \frac{1}{2ik} \left\{ i2\beta \left( 1 + \cos \theta_o \right) \right. \]
\[ + \frac{e}{(2\alpha)^{3/2}} \left\{ D_0(0, \theta_o) D_0(\theta_o, 0) + D_0(0, 2\pi - \theta_o) D_0(\theta_o, 0) \right\} \]
\[ + O \left[ (kr)^{-3/2} \right] \]  \hspace{1cm} (2.44) \]

Total scattering cross section for large complex refractive index \( N \)

We shall now derive the total scattering cross section
from the expression (2.44) for the situation when the complex refractive index of the material comprising the cylinder (usually metal), is such that \(|N| \gg 1\) where \(N = n(1 + ik)\), is the complex refractive index. Clearly the diffraction coefficients which appear in (2.44) are of such a complicated form that to express (2.44) in terms of its real and complex components would result in a very long formula; which would be of little use in computation. We exploit the fact that \(|N| \gg 1\) and expand (2.44) out in terms of inverse powers of \(N\). To accomplish this we note that since

\[
\cos \theta_t = \frac{-N_{\mu_0}}{\mu}, \quad \text{and} \quad \sin \theta_t = -\sqrt{1 - \left(\frac{N_{\mu_0}}{\mu}\right)^2}
\]

then

\[
\cos \frac{4\theta_t}{3} = \left(-\frac{2N_{\mu_0}/\mu}{2}\right)^{4/3} \left[ 1 + O\left(\frac{1}{N}\right) \right],
\]

\[
\sin \frac{4\theta_t}{3} = \left(-\frac{2N_{\mu_0}/\mu}{2}\right)^{4/3} \left[ 1 + O\left(\frac{1}{N^2}\right) \right], \quad N > 1.
\]

Expanding the expressions (1.49) and (1.50) in terms of \(\cos \theta_t\), \(\cos 4\theta_t/3\) and \(\sin 4\theta_t/3\), and then substituting the expressions (2.45), for these trigonometric terms we obtain eventually

\[
P_\theta (0, \theta) = \frac{k e^{i\nu \mu}}{3 \sqrt{6\pi k^3}} \left(\cos 2\theta / 3 + \frac{1}{x}\right)^2 \left[ 1 + \mu (\sin \theta / \cos \theta - 1) \right] \left(\frac{1}{N N_{\mu_0}}\right) + O\left(\frac{1}{N^2}\right)
\]

\[
P_\theta (0, \theta) = \frac{k e^{i\nu \mu}}{3 \sqrt{6\pi k^3}} \left(\cos 2\theta / 3 + \frac{1}{x}\right)^2 \left[ 1 + \mu (\sin \theta / \cos \theta - 1) \right] \left(\frac{1}{N N_{\mu_0}}\right) + O\left(\frac{1}{N^2}\right)
\]
also from appendix A(i) we obtain

\[ B_1(\kappa - \theta_0, 2\pi - \theta_0) - B_1(2\pi - \theta_0, \kappa - \theta_0) = -\frac{4i \cos \theta_0}{N\mu_0/\mu} + O\left(\frac{1}{N^{3/2}}\right). \quad (2.47) \]

\[ = \frac{4i}{N\mu_0/\mu} \left(\cos \theta_0 - \sin \theta_0\right) + O\left(\frac{1}{N^{3/2}}\right) \]

Substituting (2.46) and (2.47) into (2.44) gives

\[ u_d(r, \frac{3\pi}{2} - \theta_0) = \frac{E}{\sqrt{2\pi} k\rho} \left[ -2k(b \sin \theta_0 - a \cos \theta_0) + \frac{2i}{3} \left(\frac{1}{\sqrt{3}} + \frac{2\sqrt{3}}{2 \cos^2 \theta_0 + 1}\right) \right] \]

\[ - \frac{4}{27/\pi} \left\{ \frac{E(2\pi - \theta_0) E(\theta_0)}{(kb)^{3/2}} \left[ \cos \left[ 2ka(1 - \sin \theta_0) + \frac{\pi}{4} \right] + i \sin \left[ 2ka(1 - \sin \theta_0) + \frac{\pi}{4} \right] \right] \right\} \]

\[ + \frac{E(\theta_0 + \pi_0) E(3\pi_0 - \theta_0)}{(ka)^{3/2}} \left[ \cos \left[ 2ka(1 - \sin \theta_0) + \frac{\pi}{4} \right] + i \sin \left[ 2ka(1 - \sin \theta_0) + \frac{\pi}{4} \right] \right] \]

\[ + \frac{1 - iK}{2\sin \theta_0} \left\{ + 2i \left[ \cos \theta_0 - \frac{B}{M \cos \left(1 + k^2\right)} \right] \left(\frac{E(2\pi - \theta_0) E(\theta_0)}{(kb)^{3/2}} \right) \right\} \]

\[ + i \sin \left[ 2kb(1 - \cos \theta_0) + \frac{\pi + i/4}{4} \right] + \frac{E(\theta_0 + \pi_0) E(3\pi_0 - \theta_0)}{(ka)^{3/2}} \left[ \cos \left[ 2ka(1 - \sin \theta_0) + \frac{\pi}{4} \right] + i \sin \left[ 2ka(1 - \sin \theta_0) + \frac{\pi}{4} \right] \right] \]

\[ + \cos \left[ \frac{i}{2} \right] + O\left(\left(\frac{k\rho}{R}\right)^{3/2} + O\left(\frac{1}{N^{3/2}}\right)\right) \]

\[ + O\left[ (k\rho)^{-3/2}\right] \quad (2.48) \]

where \[ E(0) = \frac{\sin \frac{2\theta}{3}}{\left(\cos \frac{2\theta}{3} + 1\right)^2}. \quad (2.49) \]
The total scattering cross-section $\sigma$ for the cylinder can be found by using the scattering cross-section theorem of Jones [17]. Applied to the present case, this states that if

$$u_d = u + O(kr^{-3/2})$$

then

$$\sigma = \frac{2}{k} \text{Re} \left[ \frac{e^{i(kr-\pi/4)}}{\sqrt{2\pi k^3}} u \right].$$

Thus from (2.48)

$$\sigma = 4\left( b \sin \theta_o - a \cos \theta_o \right) + \frac{E}{27k^4\pi} \left[ \frac{E(3\pi/2-\theta_0)E(\theta_0+\pi/2)}{(ka)^{3/2}} \cos[2ka(1-\sin \theta_0)+\pi/4] \right]$$

$$+ \frac{E(2\pi-\theta_0)E(\theta_0)}{(kb)^{3/2}} \cos[2kb(1+\cos \theta_0)+\pi/4] \right]$$

$$+ \frac{1}{\mu \Omega \left( 1 + \frac{1}{k} \right)} \left[ \frac{4 \text{J}(1-\sin \theta_o)}{k} + \frac{16}{27k^4\pi} \left[ \frac{E(2\pi-\theta_0)E(\theta_0+1)}{(ka)^{3/2}} \right] \right]$$

$$+ k \sin \left[ 2kb(1+\cos \theta_0)+\pi/4 \right] \right] + \frac{E(\theta_0+\pi/2)E(3\pi/2+\theta_0)}{(ka)^{3/2}} \left[ \cos[2ka(1-\sin \theta_0)+\pi/4] \right] + O\left( \frac{1}{N^{4/3}} \right)$$

$$+ O\left[ (kb)^{-5/2} \right].$$

(2.50)
In the limit as $N \rightarrow \infty$ the above expression reduces to the scattering cross-section for a perfectly conducting cylinder which agrees with Morse [20]. These results are only valid for oblique incidence $\frac{\pi}{2} < \theta_o < \pi$. In the next chapter we shall obtain the total scattering cross-section for the situation when $\theta_o = \pi/2$ or $\pi$. 
In this chapter we shall obtain an expression for the forward scattered field and the total scattering cross-section for grazing incidence \( (\theta_0 = \pi \text{ or } \pi/2) \). In this case the expressions (2.48) and (2.50) become infinite and the previous methods used are no longer suitable. The breakdown is due to the fact that near the grazing incidence shadow boundaries, which run along the sides of the cylinder faces, the asymptotic form of the far field changes rapidly for slight angular deviation from these boundaries. In using the plane wave expansion formula of Zitrin and Karp [19] for the far field it was necessary to evaluate angular derivatives of the asymptotic expressions for the far field. These asymptotic expressions were obtained by asymptotically evaluating the field integrals. The rapid variation of the asymptotic field representations prevents the angular derivatives from being calculated directly from the asymptotic expressions, hence an alternative method has to be sought.

The starting point of this alternative approach is Green's theorem in two dimension, i.e.

\[
\begin{align*}
    u_{So}(r,\theta) &= u(r,\theta) - u_i(r,\theta) \\
    &= \frac{i}{4} \oint \left[ u \frac{\partial}{\partial n} H_0^{(1)}(kR) - H_0^{(1)}(kR) \frac{\partial u}{\partial n} \right] ds.
\end{align*}
\]

Here \( u_{So}(r,\theta) \) is the scattered field at the point \((r,\theta)\) and hence is the total field \( u(r,\theta) \) minus the incident field \( u_i(r,\theta) \). The field quantity \( u \) which appears in the integrand...
of the equation (3.1), is the total field on the surface of the cylinder; \( n \) is the outer normal from the sides of the cylinder, and \( R(s, x) \) the distance from the point \( s \) on the cylinder to the observation point \( P \), see figure 20.

\[ R \sim r - x \cos \theta - y \sin \theta \] for \( R, r \), large

**Figure 20**

In order to determine the asymptotic representation of the integral (3.1) for \( r \) large compared to the lengths \( 2a \) and \( 2b \), the following well known asymptotic representations are required.

\[ H_0^{(1)}(kR) = \left( \frac{2}{\pi kR} \right)^{1/2} e^{i(kR - \pi/4)} + O((kR)^{-3/2}) \]  \hspace{1cm} (3.2)

\[ \frac{\partial H_0^{(1)}(kR)}{\partial n} = \pm ik \left\{ \begin{array}{c} \cos \theta \\ \sin \theta \end{array} \right\} \left( \frac{2}{\pi kR} \right)^{1/2} e^{i(kR - \pi/4)} + O((kR)^{-3/2}) \] \hspace{1cm} (3.3)

where the \( \pm \) sign depends on the direction of the normal and \( \{\cos \theta, \sin \theta\} \) the face from which the normal protrudes. Using the
convention that the faces $E_1E_2$, $E_2E_3$, $E_3E_4$, $E_4E_1$ are
identified by the parameter $j$ ($j = 1,2,3,4$ respectively)
the expression (3.3) can be re-written as

$$\frac{\partial H^{(1)}(kR)}{\partial n} = i k \sin[\theta \cdot (1-j)\frac{\pi}{2}] \left(\frac{2}{nkR}\right)^{\frac{3}{2}} e^{i(kR-n/4)} + O(kR)^{-3/2).} (3.4)$$

Substituting (3.2) and (3.4) into (3.1) gives

$$u_j(x,\theta) = e^{i(kx-n/4)} \sum_{j=1}^{2d} \exp[-ik(x_j\cos\theta + y_j\sin\theta)]$$

$$x \int \left[ -k u_{j,m} \sin[\theta \cdot (1-j)\frac{\pi}{2}] - i \frac{\partial u_{j,m}}{\partial n} \right] e^{i k \rho_j \cos[\theta \cdot (1-j)\frac{\pi}{2}] d\rho_j}$$

$$+ O[ (kR)^{-3/2} ] , \quad (3.5)$$

where $\rho_j$, ($j = 1,2,3,4$), is the distance measured from the
vertex $E_j$ towards the vertex $E_{j+1}$; see figure 21, $2d_j$ is
the length of the side $E_jE_{j+1}$; and $(x_j, y_j)$ are the
coordinates of $E_j$; the field quantity $u_j, j+1$ is the field
on the side $E_jE_{j+1}$.

```
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure_21}
\caption{Figure 21}
\end{figure}
```
The boundary condition on the cylinder faces is given by

$$\frac{\partial u}{\partial n} = i k \cos \theta_v u,$$  \hspace{1cm} (3.6)

which on substitution into (3.5) gives

$$u_s(r, \theta) = \frac{1}{2 \sqrt{2 \pi r r}} \sum_{j=1}^{\infty} e^{i(kr - \pi n)} \left[ -i k(x_j \cos \theta + y_j \sin \theta) \right]$$

$$\times \left\{ \sin \left[ \frac{\theta + (1-j)\pi}{2} \right] \cos \theta \right\} \int_{0}^{2\pi} u_{ij} \exp \left\{ -i k \rho_j \cos \left[ \frac{\theta + (1-j)\pi}{2} \right] \right\} d\rho_j.$$  \hspace{1cm} (3.7)

We shall now apply equation (3.7) to determine the forward scattered field when \( \theta = \pi/2 \) and the incident field runs along the sides, i.e. \( \theta_0 = \pi \), see figure 22.

---

**Figure 22**
Substituting into the expression (3.7) the appropriate values of $\theta$ and $\theta_0$ gives

$$u_s(x, \theta) = -\frac{k e^{i(kr-n\psi_k)}}{2 \sqrt{2\pi kr}} \sum_{j=1}^{24} e^{-iky_j} \left[ \sin((z-j)\pi - \cos \theta \rho) \right] u_{ij,1} e^{-ik\rho \cos(z-j)\pi}$$

$$+ O \left[ (kr)^{-3/2} \right].$$

(3.8)

Expanded out (3.8) gives

$$u_s(x, \theta) = -\frac{k e^{i(kr-n\psi_k)}}{2 \sqrt{2\pi kr}} \left\{ e^{i\rho \cos \theta} \sum_{j=1}^{24} u_{ij,1} e^{-ik\rho \cos(z-j)\pi} \right\}$$

$$+ O \left[ (kr)^{-3/2} \right].$$

(3.9)

Calculation of the field at the corners of the cylinder

It is now necessary to evaluate the field quantities under the integral signs of the expression (3.9). To achieve this the field at each corner of the cylinder must first be calculated. The field at the corners $E_1$ and $E_2$ due to direct illumination by the incident wave is $u_i(E_1) = u_i(E_2) = e^{-ikb}$. It is required to determine the field at the corners $E_3$ and $E_4$ after the incident wave has been diffracted by the corners $E_2$ and $E_1$ respectively. We shall also need to consider the field at $E_1$ produced by
diffraction of the incident wave by $E_2$ and vice-versa. As in previous computations, only terms up to order $(kd)^{-3/2}$ will be retained.

The field at $E_3$ caused by the diffraction of the incident wave from $E_2$ can be obtained by referring to figure 23 and using the expression (1.58).

As shown in figure 23 the coordinate system $(\xi, \eta)$ has been located at the corner $E_3$ and $P(\xi, \eta)$ is a point in the neighbourhood of $E_3$ such that $\eta > 0$. The total field at the corner $E_3$ is defined by $u_1(E_3) = u_1(P) + u_d(P,E_2)$; and from expression (1.58)

$$u_d(P,E_2) = \frac{e^{-ikb}}{2} \left[ -2I(8) + I(2π+8) + I(-2π+8) + 2J(8) \right] .$$

The factor of one half arises from the coalescing of the
incident and reflected wave along $E_x E_x$. As $\delta \to 0$, by using
the expressions (2.12) to (2.15) we obtain after simplifying
and noting that $u^{(p)}(p) = e^{i \gamma_k b} v(0)$.

$$u_1(E_3) = u_1(E_3) + u_2(E_3, E_2) = \frac{e^{\frac{i k b \pi}{4}}}{2 \sqrt{k b}}$$

$$\left[ v'(0) \left( 1 + \frac{\sqrt{\pi} e^{i \pi/4}}{2 \sqrt{k b} \cos \theta_t} \right) \right]$$

$$- \frac{v(0)}{\cos \theta_t} + O((k b)^{-3/2}). \quad (3.10)$$

From the symmetry of figure 23: the field at $E_4$ due to
diffraction by $E_1$ of the incident plane wave, is also given
by (3.10), i.e. $u_1(E_3) = u_1(E_4)$. Consistent with our
definition for $u_1(E_3)$ the only single diffracted field
reaching $E_1$ comes from $E_2$. Hence using the formula (2.4)
gives, on using the fact that $D(0, \pi/2) = 0$,

$$u_1(E_1) = -\frac{e^{i k b + i 2 k a}}{i k (2a)^{3/2}} h'(0) v(0, \pi/2) + O((k a)^{-5/2}) \quad (3.11)$$

also by symmetry $u_1(E_1) = u_1(E_2)$. The second order
diffracted fields at these corners, ($E_1$ and $E_2$) need not be
considered because they are of $O((k d)^{-2})$.

Calculation of the field on the cylinder faces

Having found the vertex fields we can now proceed to
calculate the field on any face of the cylinder. The
notation \[ u_{mn}/u_i(E_j) \] will be used to denote the component of \( u_{mn} \) contributed by the vertex field \( u_i(E_j) \).

**Determination of \( u_{2,4} \)**

When the incident plane wave illuminates the corner \( E_2 \), the diffracted field can be represented as a series of plane waves; these plane waves will now illuminate the corner \( E_3 \). We shall thus require the response of the corner \( E_3 \) to a plane wave. From figure 24

![Figure 24](image)

The field along \( \overline{E_3E_4} \) due to a plane wave hitting \( E_3 \) at an angle \( \theta_0 \) is given by \( u_d(\rho_3, 3\pi/2) \) where \( u_d \) is given by the expression (1.53). To find the field along the face \( \overline{E_2E_4} \) after the incident field has been diffracted at \( E_2 \) and then subsequently \( E_3 \) we substitute the above field \( u_d(\rho_3, 3\pi/2) \) into (3.10) and allow \( \theta_0 \to 0 \). Hence
\[
[ u_{34} | \psi_1 (E_3) ] = \frac{e^{ikb+in/4}}{2\sqrt{ikb}} \left[ \frac{\partial u_d}{\partial \theta_0} \left\{ 1 + \frac{\sqrt{n} e^{i\pi/4}}{2\sqrt{kb} \cos \theta_t} \right\} \right. \\
\left. - \frac{u_d (\rho_3, 3\pi/2)}{\cos \theta_t} \right]_{\theta_0 \to 0} + O((kd)^{-3/2}). \quad (3.12)
\]

By symmetry the contribution to the surface field \( u_{34} \), by the incident wave after first being diffracted by corner \( E_1 \) and then by the corner \( E_4 \), i.e. \([ u_{34} | \psi_1 (E_4) ]\) is the same as (3.12) except that \( \rho_3 \) is replaced by \( 2a - \rho_3 \), see figure 25.

\[
[ u_{34} | \psi_1 (E_4) ] = \frac{e^{ikb+in/4}}{2\sqrt{ikb}} \left[ \frac{\partial u_d}{\partial \theta_0} (2a - \rho_3, 3\pi/2) \left\{ 1 + \frac{\sqrt{n} e^{i\pi/4}}{2\sqrt{kb} \cos \theta_t} \right\} \right. \\
\left. - \frac{u_d (2a - \rho_3, 3\pi/2)}{\cos \theta_t} \right]_{\theta_0 \to 0} + O((kd)^{-3/2}). \quad (3.13)
\]

Figure 25

The total field at the point \( P \) on the face \( \overline{E_3 E_4} \) is now given by adding (3.12) and (3.13) together, see figure 26. Thus

\[
u_{34} (P) = \left[ u_{34} | \psi_1 (E_4) \right] + \left[ u_{34} | \psi_1 (E_3) \right].
\]
From figure 27 the field at a point P along the face $E_2E_3$ contributed by the incident wave being diffracted by corner $E_1$ is given by the expression

$$[u_{23}|u_d(E_2)] = u_d(\rho_2, 0) e^{-ikb}.$$  

(3.15)

$$\theta_o = \pi$$
Figure 27

The contribution to $u_{23}$ from the incident field diffracted by corner $E_2$ and then subsequently by $E_3$, i.e. $[u_{23} | u_1(E_3)]$, can be calculated by considering figure 28, and using the expression (3.10).

$$\begin{align*}
\left[ \begin{array}{c}
u_{23} \\ u_1(E_3)
\end{array} \right] &= \frac{e^{i k b + i \gamma}}{2 \sqrt{\pi h b}} \left[ \frac{\partial u_1}{\partial \theta_0} \frac{(2b - \rho_1 \gamma)}{2 \sqrt{h b \cos \theta_0}} \right] \\
&\quad + O \left[ (kd)^{-3/2} \right].
\end{align*}$$

(3.16)
The total field along $E_2,E_3$ due to diffraction at corners $E_2$ and $E_3$ is the sum of (3.15) and (3.16)

$$u_{2,3} = u_a(\rho_2,0) e^{-i k b} + \frac{e^{-i k b \cos \theta_v}}{2 \sqrt{\pi} \sqrt{kb}} \left[ \frac{\partial u_1(\rho_2-\rho_2,0)}{\partial \theta_v} \left\{ 1 + \frac{\sqrt{\pi} e^{\frac{1}{2} i k b \cos \theta_v}}{2 \sqrt{\pi} \sqrt{kb} \cos \theta_v} \right\} \right] - \frac{u_a(2b-\rho_2,0)}{\cos \theta_v} + O[(kd)^{\gamma_0}]$$

(3.17)

The various contributions to $u_{2,3}$ are shown in figure 29.

![Figure 29](image)

**Figure 29**

*Determination of $u_{4,1}$*

From symmetry the field on the face $E_2,E_1$ is exactly the same except that the field quantities will be in terms of $\rho_4$ instead of $\rho_2$. To obtain the appropriate field we must replace $\rho_2$ in the expression (3.17) by $2b-\rho_4$, since for any particular value of $y$, $\rho_2 + \rho_4 = 2b$. Thus
\[ u_{1,4} = u_4(xb - yb, 0) e^{-ikb} + \frac{e^{i k b + im \theta}}{2 \sqrt{\pi k b}} \left\{ \frac{\partial u_4(\rho, y)}{\partial \theta} \left[ 1 + \sqrt{\pi} e^{i m \theta} \right] \right\} \]

\[ + \frac{u_4(\rho, y)}{cos \theta} + O[(kd)^{-2}] \]

(3.18)

**Determination of \( u_{1,2} \)**

From figure 30 the field along the face \( \overline{E_1E_2} \) due to diffraction of the incident wave by the corner \( E_1 \) is given by

\[ [u_{12} | u_{1}(E_1)] = e^{-ikb}u_4(x_1, 0). \]

(3.19)

\[ \theta_o = \pi/2 \]

The contribution to \( u_{12} \) from the incident field first being diffracted by \( E_2 \), and the resulting field being diffracted by \( E_1 \) is given by, see figure 31,
using the formula (3.11) and noting the direction of the diffracted rays in figure 31. Thus

\[ u_{12} | u_1(E_2) | = \frac{e^{ik(2a-b)}}{ik(2a)^{3/2}} D_\theta(0, \pi/2) \frac{\partial}{\partial \theta_0} u_d(2a-\rho_1,0) \cdot (3.20) \]

Thus the total contribution caused by the incident wave illuminating corner \( E_1 \), see figure 32, is given by

\[ e^{-ikb} u_d(\rho_1,0) + \frac{e^{ik(2a-b)}}{ik(2a)^{3/2}} D_\theta(0, \pi/2) \frac{\partial}{\partial \theta_0} u_d(2a-\rho_1,0) \cdot (3.21) \]

\[ \theta_0 = \pi/2 \]

\[ \theta_0 \to 0 \]
From symmetry the contribution to $u_{12}$ of the incident ray illuminating $E_2$, see figure 33, is given by replacing $\rho_1$ by $2a-\rho_1$ in (3.21), and is

$$e^{-ikb} u_d(2a-\rho_1) + \frac{s\imath k(2a-b)}{ik(2a)^{3/2}} p_\theta(0,\pi/2) \frac{\partial u_d}{\partial \theta_0}(\rho_1,0). \quad (3.22)$$

$$\theta_0 = \pi/2 \quad \quad \quad \theta_0 \rightarrow 0$$

$$[u_{11}|u_{11}(E_1)] + [u_{12}|u_{12}(E_1)]$$

**Figure 33**

Finally the cylinder face $E_1E_2$ will experience direct illumination by the incident plane wave, which, because of the impedance boundary condition, gives rise to the geometrical optics field contribution, see figure 34

$$[u_{12}|u_{12}^b] = -\frac{2s\imath k b}{\cos \theta_t}. \quad (3.23)$$

**Figure 34**
Thus the total field on the face $\mathbf{E}_1 \mathbf{E}_2$ is given by adding the expressions (3.21), (3.22) and (3.23) together giving

$$u_{1,2} = e^{-ikb} \left\{ u_4(r,0) + u_4(2a-r,0) \right\}_{\theta_0 = \pi/2} - \frac{2e^{-ikb}}{\cos \theta_c}$$

$$+ \frac{e^{ik(2a-b)}}{ik(2a)^{3/2}} \left[ \frac{\partial u_4(r,0)}{\partial \theta_0} + \frac{\partial u_4(2a-r,0)}{\partial \theta_0} \right]_{\theta_0 \to 0}$$

$$+ O \left[ (kd)^{-5/2} \right].$$

(3.24)

Substituting the expressions (3.14), (3.17), (3.18) and (3.24) into the equation (3.9); and then making a change of the variable of integration where appropriate, to bring similar terms under a common range of integration, we obtain eventually

$$u_3 \left( r, \frac{\pi}{2} \right) = \frac{-k e^{i(kr - \pi/4)}}{2 \sqrt{2 \pi k^2 r^3}} \left\{ \frac{4a(1 - \cos \theta_c)}{\cos \theta_c} \right\}$$

$$- \cos \theta_c e^{i\pi k} \left( 1 + \frac{e^{i\pi k}}{2 \cos \theta_c \sqrt{ikb}} \right) \left[ \frac{\partial u_4(r,0)}{\partial \theta_0} e^{ik\rho_3} d\rho_3 \right]_{\theta_0 \to 0}$$

$$- \frac{1}{k e^{i\pi k}} \int_{\theta_0 \to 0} u_4(r,0) e^{ik\rho_3} d\rho_3 - 2 \cos \theta_c \int_{\theta_0 \to \pi} u_4(r,0) e^{ik\rho_2} d\rho_2$$
The expressions appearing in the equation (3.25) are in fact double integrals, because $u_d(r, \theta)$ is given by (1.53), i.e.

$$u_d(r, \theta) = \frac{1}{2\pi i \sqrt{3}} \int_{S(\theta)} D(\psi, \theta_0) \left\{ \frac{1}{\cos \frac{2\pi}{3} - \cos \frac{2}{3}(\gamma - \theta_0)} \right\} e^{ik\cos(\gamma - \theta_0)} d\gamma,$$

where

$$D(\gamma, \theta_0) = \frac{(\cos\gamma - \cos\theta_0)(\sin\gamma + \cos\theta_0)(\cos\frac{4\theta_0}{3} - \cos\frac{2\theta_0}{3})(2\cos\frac{2\theta_0}{3} + \cos\theta_0 + \frac{1}{2} - \cos\frac{4\theta_0}{3})}{(\cos\theta_0 + \cos\theta_0)(\sin\theta_0 - \cos\theta_0)(\cos\frac{4\theta_0}{3} - \cos\frac{2\theta_0}{3})(2\cos\frac{2\theta_0}{3} + \cos\theta_0 + \frac{1}{2} - \cos\frac{4\theta_0}{3})}.$$
\[
\begin{align*}
\phi_\alpha(r, \pi/2) &= -\frac{k e^{i(kr-\mu_\alpha)}}{2\sqrt{2\pi kr}} \left\{ -\frac{4\phi_\alpha(1-\cos \Theta_\alpha)}{\cos \Theta_\alpha} \\
&\quad - \frac{\cos \Theta_\alpha e^{im_\alpha}}{\sqrt{\pi}kb} \left( 1 + \frac{\sqrt{\pi}e^{im_\alpha}}{2\cos \Theta_\alpha \sqrt{kb}} \right) \right\}^{2b} \left( \frac{\partial \phi_\alpha(\rho, \Theta_\alpha)}{\partial \theta_\alpha} \right)_{\theta_\alpha \to 0} \\
&\quad - 2\cos \Theta_\alpha \int_0^{2b} \frac{\partial \phi_\alpha(\rho_2, \Theta_\alpha)}{\partial \theta_\alpha} \, d\rho_2 + 2(1-\cos \Theta_\alpha) \int_0^{2a} \frac{\partial \phi_\alpha(\rho_3, \Theta_\alpha)}{\partial \theta_\alpha} \, d\rho_3 \\
&\quad - \frac{(1+\cos \Theta_\alpha) e^{im_\alpha}}{\sqrt{\pi}kb} \left( 1 + \frac{\sqrt{\pi}e^{im_\alpha}}{2\cos \Theta_\alpha \sqrt{kb}} \right) \right\}^{2a} \left( \frac{\partial \phi_\alpha(\rho_3, \Theta_\alpha)}{\partial \theta_\alpha} \right)_{\Theta_\alpha \to \pi/2} \\
&\quad + O\left[ (kr)^{-3/2} \right] \left\} + O\left[ (kr)^{-3/2} \right].
\end{align*}
\]

For the evaluation of the integrals appearing in the expression (3.26), it will be necessary to express \(u_d(\rho, \Theta)\) given by (1.53) in the alternative form:

\[
u_d(\rho, \Theta) = \frac{1}{2\pi i} \int_{S(\Theta)} \left\{ (F(\gamma-\pi)-F(\gamma+\pi)) \right\} e^{ik\rho \cos(\gamma-\Theta)} \, d\gamma, \quad (3.27)
\]

where

\[
\begin{align*}
F(\chi) &= -\frac{2(\cos \chi + \cos \Theta_\alpha)(\cos \Theta_\alpha - \sin \chi) \sin^2 \Theta_\alpha/3}{3(\cos \Theta_\alpha + \cos \Theta_\beta)(\sin \Theta_\alpha - \cos \Theta_\beta)} \\
&\times \left\{ \left( \frac{\cos 2\chi/3 + \cos 2\Theta_\alpha/3}{\left( \left( \frac{\cos 2\chi}{3} \right)^2 - \left( \frac{\cos 2\Theta_\beta}{3} \right)^2 \right) \cos 2\Theta_\alpha/3 - \cos 2\Theta_\beta/3} \right)^2 \right\}.
\end{align*}
\]

\[
(3.28)
\]
By a straightforward change of the variables of integration it can be shown that for any \( G(x) \) which renders the following integral convergent:

\[
\frac{1}{2\pi i} \int_{S(\theta)} \left\{ G(z-\pi) - G(z+\pi) \right\} e^{ikp\cos(y-v)} \, dz = \frac{1}{2\pi i} \left\{ \int_{S(\pi)} + \int_{S(-\pi)} \right\} G(\theta + a) e^{-ikp\cos\alpha} \, d\alpha.
\]

(3.29)

Finally we will require the following result which is proved in the Appendix 3A. For \( p = 0 \) in (3.29) we have that

\[
\begin{align*}
\oint_{\gamma_1} G(\alpha + \theta) \, d\alpha &= iG(R) \\
\oint_{\gamma_2} G(\alpha + \theta) \, d\alpha &= -iG(-R)
\end{align*}
\]

\[ R = i\infty; \theta \text{ finite} \]  

(3.30)

Provided \( G(\pm R) \rightarrow \text{constant as } R \rightarrow i\infty \).

The contours appropriate to the formulae (3.29) and (3.30) are shown in figure 35. Provided the contours of integration terminate within the shaded regions the exponential term of the integrand ensures uniform convergence.
Consider the integral,

\[ I = \int_0^{2\pi} u_d(\rho_2, \theta) \, d\rho_2 \tag{3.31} \]

which appears in (3.26) for the particular case of \( \theta = 0 \), \( \theta_0 = \frac{\pi}{2} \). Substituting the contour integral representation for \( u_d(\rho_2, \theta) \), by means of (3.27) to (3.29), will give

\[ I = \frac{1}{2\pi i} \int_0^{2\pi} \left( \int_{\gamma_1} + \int_{\gamma_2} \right) e^{-ik\rho_2 \cos \alpha} F(\alpha + \theta) \, d\rho_2. \tag{3.32} \]
It is now possible to integrate under the integral sign by virtue of the uniform convergence of the complex contour integral. Thus

\[ I = \frac{1}{2\pi i} \left\{ \int_{S(n)} + \int_{S(-n)} \right\} \frac{f(\alpha+\theta)}{(-1k \cos \alpha)} \left[ e^{-2ika \cos \alpha} - 1 \right] d\alpha. \tag{3.33} \]

In the expression (3.33) we now convert the integrals involving \(e^{-2ika \cos \alpha}\) back into a single integral along the path \(S(\theta)\) by means of the expression (3.29), giving

\[ I = \frac{1}{2\pi i} \left\{ \int_{S(n)} + \int_{S(-n)} \right\} \frac{F(\alpha+\theta)}{ik \cos \alpha} \]

\[ + \frac{1}{2\pi i \sqrt{3}} \int_{S(\theta)} \frac{ID(\gamma, \eta)}{ik \cos (\gamma-\theta)} \left\{ \cos \frac{2\pi}{3} - \cos \frac{2}{3}(\gamma-\theta) \right\} \cos \frac{2\pi}{3} - \cos \frac{2}{3}(\eta \theta) \right\} e^{i2ak \cos (\gamma-\theta)} d\gamma \]

\[ \tag{3.34} \]

Substituting in the particular values \(\theta = 0, \theta_0 = \pi/2\) gives

\[ I = \frac{1}{2\pi i} \left\{ \int_{S(n)} + \int_{S(-n)} \right\} \frac{F(\alpha)}{ik \cos \alpha} \]

\[ + \frac{1}{2\pi i \sqrt{3}} \int_{S(\theta)} \frac{ID(\gamma, \eta)}{ik \cos \gamma} \left\{ \cos \frac{2\pi}{3} - \cos \frac{2}{3}(\gamma-\eta) \right\} \cos \frac{2\pi}{3} - \cos \frac{2}{3}(\gamma \eta) \right\} e^{i2ak \cos \gamma} d\gamma \]

\[ \tag{3.35} \]
The evaluation of the last integral of the expression (3.35) can be achieved by a direct application of the ordinary saddle point method, (k\alpha \gg 1); because no poles of the integrand lie in the vicinity of the saddle point. Thus

\[ \frac{1}{2\pi i} \int_{S(\alpha)}^{S(-\alpha)} \frac{d\alpha}{\sqrt{3} ik \cos \alpha} \left\{ \frac{1}{\cos \frac{2\pi}{3} - \cos \frac{2\pi}{3} \alpha} - \frac{1}{\cos \frac{2\pi}{3} - \cos \frac{2\pi}{3} (\alpha + \pi/2)} \right\} e^{i2\alpha \cos \gamma} d\gamma \]

\[ \sim -2 \frac{\text{Im}(\alpha+i\pi/2)}{3 ik \sqrt{2\pi} (2k\alpha)^{3/2} \cos \theta} + O \left[ (k\alpha)^{-5/2} \right], \]

\[ \sim O \left[ (k\alpha)^{-3/2} \right]. \quad (3.36) \]

We now evaluate the remaining integral of the expression (3.35), i.e.

\[ \frac{1}{2\pi i} \left\{ \int_{S(\alpha)}^{S(-\alpha)} + \int_{\alpha=0}^{\alpha=\pi/2} \right\} \frac{F(\alpha)}{ik \cos \alpha} \, d\alpha, \]

where

\[ \frac{F(\alpha)}{ik \cos \alpha} \bigg| \begin{array}{c} \theta = 0 \\ \theta = \pi/2 \end{array} = \frac{(\cos \alpha + \cos \theta_k) (\cos \theta_k - \sin \alpha)}{\sqrt{3} (1 - \cos \theta_k) \cos \theta_k ik \cos \alpha} \]

\[ \left| \left( \frac{\cos 2\alpha/3 + \pi/2}{\cos 2\alpha/3 + \cos \theta_k} \right) - \frac{1}{\cos 2\alpha/3 - \cos \theta_k/3} \right|. \quad (3.37) \]
Clearly,
\[
\frac{F(\pm i\infty)}{\cos(\pm i\infty)} = 0
\]
so that, by virtue of the result (3.30), the contours of steepest descent, \( S(\pi) \) and \( S(-\pi) \) can be joined by the contours \( \gamma_1 \) and \( \gamma_2 \) to form a closed loop. Since no surface wave poles, which would be complex, are enclosed by this closed contour we can deform \( S(\pi) \) and \( S(-\pi) \) to take up the straight line contours shown in figure 36. This closed contour will be denoted by \( C \).

Hence
\[
\frac{1}{2\pi i} \left\{ \int_{S(\pi)} + \int_{S(-\pi)} \right\} \frac{F(\alpha)}{ik \cos \alpha} d\alpha = \frac{1}{2\pi i} \int_{C} \frac{F(\alpha)}{ik \cos \alpha} d\alpha
\]
The only poles enclosed by $C$ are the poles along the real axis between $(-\pi,\pi)$. These poles occur at the values of $\alpha$ for which
\[
\cos \alpha = 0, \quad \text{i.e. } \alpha = \pm \pi/2;
\]
\[
\cos \frac{2\alpha}{3} = \cos \frac{\pi}{2} = 0, \quad \text{i.e. } \alpha = \pm \pi/2.
\]
Thus single and double poles occur in the expression (3.37) at $\alpha = \pm \pi/2$. By means of the formulae of Appendix 3B the sum of residues of the function (3.37) are given by
\[
\sum_{\alpha=\pm \pi/2} \text{Res} \frac{F(\alpha)}{ik \cos \alpha} = -\frac{2}{i k(1-\cos \theta_0)} \left\{ \frac{1}{\cos \theta_0} + \frac{1}{3\sqrt{3} (\sqrt{3} - (\cos 2\theta_0)^2)} \right\} + \mathcal{O}[a^4].
\]

(3.39)

Thus
\[
\int_0^{2\alpha} \frac{u_l(\rho_0, \theta)}{\theta_0, \pi/2} d\rho_0
\]
\[
= \frac{2}{i k(1-\cos \theta_0)} \left\{ \frac{1}{\cos \theta_0} + \frac{1}{3\sqrt{3} (\sqrt{3} - (\cos 2\theta_0)^2)} \right\} + \mathcal{O}[a^4].
\]

(3.40)

We now consider the integral:
\[
I = \int_0^{2\alpha} \frac{\partial u_0}{\partial \theta_0}(\rho_2, \pi/2) d\rho_2.
\]
\[
\theta_0 = 0
\]

After differentiating under the integral sign of the integral
representation for $u_d(\rho_3, \theta)$, this being permissible because the integral converges uniformly, we obtain

$$
\frac{\partial u_d(\rho_3, \frac{\pi}{2})}{\partial \theta_0} \bigg|_{\theta_0=0} = \frac{1}{2\pi i} \left[ \frac{\Im D(y, \theta_0) \sin 2\gamma/3}{3\sqrt{3}} \right] e^{i k \rho_3 \cos (\gamma - \frac{2\pi}{3})} \left( \frac{\cos \frac{2\pi}{3} - \cos \frac{2\gamma}{3}}{\cos \frac{2\pi}{3} - \cos \frac{2\gamma}{3}} \right)^2
$$

Transforming the above contour, as before, to take up the two contours $S(\pi)$ and $S(-\pi)$ and integrating with respect to $\rho_3$ gives,

$$
I = \int_0^{2\pi} \frac{1}{2\pi i} \left[ \int_{S(\pi)} + \int_{S(-\pi)} \right] \frac{\partial F(\alpha + \frac{3\pi}{2})}{\partial \theta_0} \bigg|_{\theta_0=0} e^{i k \rho_3 \cos \alpha} d\alpha d\rho_3,
$$

where

$$
\frac{\partial F(\alpha + \frac{3\pi}{2})}{\partial \theta_0} \bigg|_{\theta_0=0} = \frac{4 (\sin \alpha + \cos \theta_e)(\cos \theta_e + \cos \alpha)}{9 (1 + \cos \theta_e) \cos \theta_e}
$$

$$
I = \int_{S(\pi)} - \int_{S(-\pi)} \frac{4 \left( 1 - \cos \frac{2\alpha}{3} \right)}{\left( \cos \frac{2\alpha}{3} - \cos \frac{2\theta_e}{3} \right)^2} + \frac{1}{\left( \cos \frac{2\alpha}{3} + 1 \right)} \frac{1}{2\pi i} \left[ \int \frac{\partial F(\alpha + \frac{3\pi}{2})}{\partial \theta_0} \bigg|_{\theta_0=0} e^{i k \cos \alpha} \right]
$$

Integrating with respect to $\rho_3$ gives, after re-arranging the integrals as before

$$
I = \frac{1}{2\pi i} \int \frac{\partial F(\alpha + \frac{3\pi}{2})}{\partial \theta_0} \bigg|_{\theta_0=0} e^{i k \cos \alpha} \frac{1}{2\pi i} \left[ \int \frac{4 \left( 1 - \cos \frac{2\alpha}{3} \right)}{\left( \cos \frac{2\alpha}{3} - \cos \frac{2\theta_e}{3} \right)^2} + \frac{1}{\left( \cos \frac{2\alpha}{3} + 1 \right)} \right]
$$

$$
+ \frac{1}{2\pi i} \int \frac{\partial D(y, \theta_0) \sin 2\gamma/3}{3\sqrt{3}} e^{i k \rho_3 \cos (\gamma - \frac{2\pi}{3})} \left( \frac{\cos \frac{2\pi}{3} - \cos \frac{2\gamma}{3}}{\cos \frac{2\pi}{3} - \cos \frac{2\gamma}{3}} \right)^2 d\gamma.
$$
The second integral in the expression (3.42) can be evaluated directly by the normal saddle point method; it is found to be of order $O((ka)^{-3/2})$. The evaluation of the first integral of the expression (3.42) is achieved by summing the residues of $\frac{\partial F}{\partial \theta_0}(\alpha+3\pi/2)/ik\cos \alpha$ enclosed by $C$. The only poles that occur in the interval $|\alpha| < \pi$ are those for which $\cos \alpha = 0$, i.e., $\alpha = \pm \pi/2$, since

$$\frac{\partial F}{\partial \theta_0}(\alpha+3\pi/2) = 4(\sin \delta + \cos \theta_0)(\cos \theta_0 + \cos \alpha)$$

$$+ ik \cos \alpha$$

The sum of the residues is easily found to be

$$- \frac{8}{9ik(\cos \theta_0 + 1)} \left\{ \frac{1}{2(\frac{7}{4} - (\cos 2\theta_0/3)^2)} + \frac{2}{3} \right\}.$$

Thus

$$\int_0^{2\alpha} \frac{\partial u_d}{\partial \theta_0}(\rho_3, \frac{3\pi}{2}) d\rho_3 = \frac{8}{9ik(\cos \theta_0 + 1)} \left\{ \frac{1}{2(\frac{7}{4} - (\cos 2\theta_0/3)^2)} + \frac{2}{3} \right\}$$

$$+ O[(ka)^{-3/2}] \quad (3.42)$$

Considering the integral

$$I = \int_0^{2b} \frac{\partial u_d}{\partial \theta_0}(\rho_3, 0) e^{ik\rho_3} d\rho_3,$$

$$\theta_0 \rightarrow 0$$
we obtain, following a similar procedure as before

\[ I = - \frac{1}{2\pi i} \int_\gamma \frac{2F(\alpha)}{\partial \theta_0 \theta_0 \to 0} i k (1 - \cos \alpha) \, d\alpha \]

\[ + \frac{1}{2\pi i} \int_\gamma \frac{4kD_y(v_0) \sin \gamma/3 \, e^{2kb(1 + \cos \gamma)} \, dy}{3\sqrt{3} \, ik(1 - \cos \gamma)(\cos^2 \gamma - \cos 2\gamma)^2} \]  \hspace{1cm} (3.43)

The saddle point of the second integral in the above expression occurs at \( \gamma = 0 \); and since the poles of the integrand are not near the saddle point, a straight forward application of the saddle point method shows that this integral is of order \( O(kb^{-3/2}) \).

As before the first integral in the expression (3.43) is equal to the sum of the residues at the poles of

\[ \frac{2F(\alpha)}{\partial \theta_0 \theta_0 \to 0} i k (1 - \cos \alpha) \, g(1 - \cos \alpha)(1 + \cos \theta_0 \cos \theta t) \]

\[ \times \left\{ \frac{\cos 2\alpha + \cos \theta t}{\left( \frac{2\cos \theta t}{3} \right) - \cos 2\theta t} \right\} \left( \frac{1}{\cos \frac{\alpha}{3} - 1} \right) \]  \hspace{1cm} (3.44)

which lie within |\( \alpha | < \pi \). These poles occur at the values of \( \alpha \) for which

\[ \cos \alpha - 1 = -2 \sin \frac{\alpha}{2} = 0, \]

and

\[ \cos \frac{2\alpha}{3} - 1 = -2 \sin^2 \frac{\alpha}{3} = 0, \]

thus the poles all occur at \( \alpha = 0 \). Rewriting the expression (3.44) in the form
from which we can see that the first term has a double pole at \( \alpha = 0 \), and the second term a fourth order pole at \( \alpha = 0 \). The combined residue is given by

\[
\frac{16}{9ik[1 - (\cos \frac{\theta^2}{3})^2]} - \frac{5}{27ik \cos \theta_t} - \frac{2}{ik(1 + \cos \theta_t) \cos \theta_t}.
\]

Thus

\[
\int_0^{2b} \frac{16}{9ik[1 - (\cos \frac{\theta^2}{3})^2]} e^{ik\rho_4 d\rho_3} - \frac{5}{27ik \cos \theta_t} - \frac{2}{ik(1 + \cos \theta_t) \cos \theta_t}.
\]

Finally we will evaluate the remaining integral in the expression (3.26), i.e.

\[
I = \int_0^{2b} u_d(\rho_2, 0) e^{-ik\rho_2 d\rho_2}.
\]

The integral representation of \( u_d(\rho_2, 0)_{\theta_0=\pi} \) has a pole at the saddle point and thus to be able to perform the \( \rho \) integration it is necessary to indent the steepest descent contour \( S(\theta) \) at the saddle point. This ensures uniform convergence and thus we may interchange the order of integration.
After substituting in the integral representation for 
\( u_d(p_2,0) \) into (3.46) and carrying out the \( p_2 \) integration
the result can be put in the form

\[
I = \frac{1}{2\pi i} \int_C \frac{F(\alpha)}{ik(1+\cos \alpha)} \, d\alpha
\]

\[
+ \frac{1}{2\pi i \sqrt{3} i k} \int_{S(\theta)} \frac{\Gamma(\gamma, \pi)}{(\cos \gamma - 1)} \left\{ \frac{1}{\cos^2 \frac{\pi}{3} - \cos^2 \frac{\gamma-\pi}{3}} - \frac{1}{\cos^2 \frac{\pi}{3} - \cos^2 (\gamma+\pi)} \right\} e^{i2\beta k (\cos \gamma - 1)} \, d\gamma
\]

(3.47)

To evaluate the second integral of the expression (3.47) it
is required to expand the integrand in terms of \( \sin \gamma/2 \),
which gives
\[
\frac{1}{2\pi i \sqrt{3} i k} \int_{S(\epsilon)} \left\{ \frac{\frac{1}{\epsilon^2} e^{i2\beta k (\cos \gamma - 1)} (\cos \frac{2\pi}{3} - \cos \frac{2\pi}{3} (\gamma - \pi))}{\cos \frac{2\pi}{3} - \cos \frac{2\pi}{3} (\gamma + \pi)} \right\} \frac{d\gamma}{\sqrt{3} i k} 
\]

= \frac{1}{2\pi i \sqrt{3} i k} \int_{S(\epsilon)} e^{i2\beta k (\cos \gamma - 1)} \sum_{n=-3}^{\infty} \alpha_n (\sin \gamma)^n d\gamma,

(3.48)

where \( \alpha_{-3} = \frac{1}{2} \), \( \alpha_{-2} = \frac{\sqrt{3}}{\cos \theta_t} \),

\[
\alpha_{-1} = \sqrt{3} \left\{ \frac{1}{\cos \theta_t - 1} + \frac{4}{9 (\cos \frac{4\theta_t}{3} + \frac{1}{2})} + \frac{32 (1 + \frac{1}{2} \cos \theta_t)}{9 (\frac{1}{2} + \cos \theta_t)^2} \right\} + \frac{5\sqrt{3}}{36},
\]

\[
\alpha_0 = \frac{1}{2\sqrt{3}} \left\{ \frac{12}{\cos \theta_t - 1} - \frac{4}{9 (\cos \frac{4\theta_t}{3} + \frac{1}{2})} + \frac{32 (1 + \frac{1}{2} \cos \theta_t)}{9 (\frac{1}{2} + \cos \theta_t)^2} \right\} - \frac{3}{\cos \theta_t} + \frac{12}{\cos^3 \theta_t} \right\} + \frac{5\sqrt{3}}{18 \cos \theta_t}.
\]

(3.49)

It is not necessary to consider higher order terms for the accuracy we are interested in. Making the substitution \( \cos \gamma - 1 = is^2 \), so that

\[
\sin \frac{\gamma}{2} = \frac{e^{-i\pi/4}}{\sqrt{2}} s, \quad \frac{dy}{ds} = \sqrt{2} e^{-i\pi/4} \left( 1 - \frac{i s^2}{4} - \frac{3 s^4}{32} + \cdots \right),
\]

\[1s^2/2 < 1\]

the integral (3.48) can be written as

\[
\frac{1}{2\pi i \sqrt{3} i k} \int_{-\infty}^{\infty} e^{-2 \beta k s^2} \sum_{n=-3}^{\infty} q_n s^n ds,
\]

(3.50)
where
\[ q_{-3} = 4i\alpha_{-3}, \quad q_{-2} = 2\sqrt[4]{e^{i\pi/4}} \alpha_{-2}, \]
\[ q_{-1} = 2(\alpha_{-1} + \frac{\alpha_{-2}}{2}), \quad q_{0} = \sqrt{2} e^{-i\pi/4}(\alpha_{0} + \frac{\alpha_{-2}}{2}). \]

(3.51)

Interchanging the summation and integration sign in (3.50), (by virtue of Watson's Lemma), gives integrals of the form
\[
\int_{-\infty}^{\infty} s e^{-2ks^2} ds,
\]
which can be easily evaluated for positive or negative \( n \), i.e. for \( n \) odd
\[
\int_{-\infty}^{\infty} s e^{-2ks^2} ds = 0,
\]
for \( |n| (= 0, 2, 4, \ldots) \) even
\[
\int_{-\infty}^{\infty} s^ne^{-2ks^2} ds = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left(2kd\right)^{n+1/2}}.
\]

Thus the integral (3.48) is asymptotic to
\[
-\frac{1}{ik} \left[ \frac{ik\sqrt{k}e^{-i\pi/4}}{\sqrt{3} \sqrt{w}} \alpha_{-z} + \frac{1}{\sqrt{3} 2 \sqrt{w} k d^2} \left(\alpha_{0} + \frac{\alpha_{-2}}{2}\right) e^{i\pi/4} \right] + O[(kd)^{-3/2}]
\]
(3.52)

We now evaluate the remaining integral of the expression (3.47),
\[
\frac{1}{2\pi i} \int_{C} \frac{F(\alpha) d\alpha}{ik(1+\cos \alpha)},
\]
(3.53)
where

\[
F(\alpha) = -\frac{(\cos \alpha + \cos \theta_0)(\cos \theta_0 - \sin \alpha)}{ik(1 + \cos \alpha)}
\]

\[
\times \frac{\left(\cos \sqrt[3]{2} \alpha - \frac{1}{2}\right)}{\left(\cos \frac{2\alpha}{3} - \cos \frac{2\theta_0}{3}\right) - \left(\cos \frac{2\alpha}{3} + \frac{1}{2}\right)}
\]

(3.54)

The poles of \(\frac{F(\alpha)}{ik(1 + \cos \alpha)}\), for \(1\alpha < \pi\) occur at \(\alpha = \pm \pi\); which means they are on the contour \(C\), which is indented as shown in figure 38.

![Figure 38](image_url)

The integral (3.58) is thus equal to
\[ \frac{1}{2\pi i} \int_C \frac{F(\alpha)}{ik(1+\cos \alpha)} \, d\alpha = \left\{ \frac{\text{Res}(-\pi) + \text{Res}(\pi)}{2} \right\} \frac{F(\alpha)}{ik(1+\cos \alpha)} \]

and after some computation with the help of the formula of Appendix 3B for the evaluation of the residues at the second and third order poles which appear at \( \alpha = \pm \pi \), see (3.54) we obtain

\[ \frac{1}{2\pi i} \int_C \frac{F(\alpha)}{ik(1+\cos \alpha)} \, d\alpha = \frac{2}{3\sqrt{3}ik \cos \theta_t} + \frac{2}{\sqrt{3}ik \cos \theta_t \left[ \frac{1}{4} - \left( \cos \frac{2\theta_t}{3} \right)^2 \right]}. \]

(3.55)

Hence

\[ \int_{0}^{2b} \bar{u}(\rho_0) e^{-ik\rho_0^2} d\rho_0 = -\frac{1}{ik} \left[ -4\sqrt{2} \frac{e^{i\theta_t}}{\sqrt{1 - \cos \theta_t}} \cdot \frac{3}{\sqrt{3} \cos \theta_t} \right. \]

\[ - \frac{2}{\sqrt{3} \cos \theta_t \left( \frac{1}{4} - \left( \cos \frac{2\theta_t}{3} \right)^2 \right)} \cdot \frac{e^{i\theta_t}}{2\sqrt{3} \left| \frac{1}{a^2} \left( \frac{1}{4} + \frac{1}{2} \cos \frac{4\theta_t}{3} \right) \right|} \cdot \frac{12}{\cos \theta_t - 1} \]

\[ - \frac{4\sqrt{2}}{9} \left( \frac{1}{4} + \frac{1}{2} \cos \frac{4\theta_t}{3} \right) + \frac{32 \left( -\frac{1}{2} - \frac{1}{2} \cos \frac{4\theta_t}{3} \right)}{9 \left( \frac{1}{4} + \frac{1}{2} \cos \frac{4\theta_t}{3} \right)^2} + \frac{12}{\cos^3 \theta_t} \]

\[ + \frac{5\sqrt{3}}{18 \cos \theta_t} + O((kd)^{3/2}) \]

(3.56)

Thus the expression for the total scattered field is given by substituting the values of the integrals (3.40), (3.42), (3.45) and (3.56) into the expression (3.26), which gives
\[ u_e (r, \frac{\pi}{2}) = -\frac{k e^{i(kr - \pi k)}}{2 \sqrt{2\pi k r}} \left\{ -\frac{1 - e (1 - \cos \theta_e)}{\cos \theta_e} \right. \]

\[ - \frac{\cos \theta_e e^{imk}}{\sqrt{\pi k b}} \left\{ \frac{1 + \sqrt{\pi} e^{imk}}{2 \cos \theta_e \sqrt{k b}} \right\} \left\{ \frac{16}{9 i k (1 - (\cos 2 \theta_e)^2) \cos \theta_e} \right. \]

\[ + \frac{5}{27 i k \cos \theta_e} \left\{ - \frac{2}{i k (1 + \cos \theta_e) \cos \theta_e} \right\} \]

\[ + \frac{2 \cos \theta_e}{\cos^3 \theta_e} \left\{ \frac{1}{i k} \left\{ \frac{12}{\cos \theta_e} - \frac{2}{3 \sqrt{3} \cos \theta_e} \right\} + \frac{2}{\sqrt{3} \cos \theta_e} \right\} \left\{ \frac{4}{9 \left( \cos 4 \theta_e + \frac{1}{2} \right)} + \frac{32}{9 \left( \frac{1}{2} + \cos 4 \theta_e \frac{1}{3} \right)^2} \right\} \right. \]

\[ + \frac{12}{\cos^3 \theta_e} \left\{ + \frac{5 \sqrt{3}}{18 \cos \theta_e} \right\} \right\} \}

\[ + 2 \left( 1 - \cos \theta_e \right) \left\{ \frac{2}{i k (1 - \cos \theta_e)} \left\{ \frac{1}{\cos \theta_e} + \frac{1}{3 \sqrt{3}} + \frac{1}{\sqrt{3} \left( \frac{1}{4} - (\cos 2 \theta_e)^2 \right)} \right\} \right. \]

\[ - \frac{(1 + \cos \theta_e) e^{imk}}{\sqrt{\pi k b}} \left\{ \frac{1 + \sqrt{\pi} e^{imk}}{2 \cos \theta_e \sqrt{k b}} \right\} \left\{ \frac{8}{9 i k (1 + \cos \theta_e)} \left\{ \frac{1}{2 \left( \frac{1}{4} - (\cos 2 \theta_e)^2 \right)} \right. \right. \]

\[ + \frac{2}{3} \left\} \right\} + O \left[ (kr)^{-\frac{1}{2}} \right] \right\} + O \left[ (kr)^{-\frac{3}{2}} \right]. \]
expressing the above expression in a more tractable form by expanding everything out in inverse powers of $\cos \theta_t$,
(where in the limit for a perfectly conducting cylinder $\cos \theta_t \to \infty$), we obtain,

$$u_g(x,\pi) = \frac{ke^{i(kr-\pi/4)}}{2\sqrt{2\pi kr}} \left\{ 4a - \frac{19e^{-i\pi/4}}{18k\sqrt{kr}} + \frac{8\sqrt{k_0}e^{i\pi/4}}{\sqrt{\pi}k} \right\}$$

$$- \frac{1}{\cos \theta_t} \left\{ 4a - \frac{4}{ik} + \frac{7}{18k^2b} + \frac{4e^{-i\pi/4}}{\sqrt{\pi}kbk} \right\} + O[(\cos \theta_t)^2] + O[(kb)^{-1}] + O[(k)]^3.$$

Substituting the complex refractive index $N (= n(1+ik))$ for $\cos \theta_t$ and applying the cross section theorem Jones [17], gives

$$\sigma = 4a - \frac{19}{18k/2\pi kb} + \frac{8\sqrt{k_0}}{\sqrt{2\pi}k}$$

$$+ \frac{u}{\mu_0(1+k^2)} \left[ 4a + \frac{4k}{k} + \frac{7}{18k^2b} + \frac{(1-k)^4}{\sqrt{2\pi}k^4b^2} \right]$$

$$+ O(n^{-2}) + O \left[ \left( \frac{k_0}{k} \right)^{-3/2} \right].$$

In the limit as $n \to \infty$ the above expression reduces to the scattering cross section for a perfectly conducting cylinder; which agrees with the expression obtained by Morse [20].
Conclusion

The method used in the final chapter could be applied to the problem of perfect conductivity, and would involve more straightforward calculations than those used by Morse; whose method was found to be unsuitable for the problem we considered. The method used in chapter 3 can be used to obtain the field in any direction although the work involved would be greater than using Keller's method, when it is applicable.

Finally it is worth mentioning that the methods used for this particular problem can be carried over to solve the simpler problem of diffraction by a thick half plane whose sides are imperfectly conducting.
Appendix 1A

(i)

From Burman's theorem, see Whittaker and Watson [14], we can expand \( \mathcal{D}(\theta+i\omega,\theta_0) \) in terms of \( \text{sh} \frac{\omega}{2} \) thus

\[
\mathcal{D}(\theta+i\omega,\theta_0) = \sum_{n=0}^{\infty} B_n(\text{sh} \frac{\omega}{2})^n
\]

where

\[
B_0 = \mathcal{D}(\theta,\theta_0),
\]

\[
B_1 = 2i \mathcal{D}(\theta,\theta_0) \left[ \frac{\cos \theta}{\sin \theta + \cos \theta} - \frac{\sin \theta}{\cos \theta - \cos \theta} \right]
\]

\[
+ \frac{4}{3} \cos 2\theta_0 \frac{\sin \theta}{\sin \theta + \cos \theta} - \cos \theta_0 \frac{\sin \theta}{\cos \theta - \cos \theta}
\]

\[
+ \frac{4}{3} \left[ \frac{\sin \frac{\theta}{3}(\theta + \theta_0)}{\cos \frac{\theta}{3}(\theta + \theta_0) - \cos \frac{\theta_0}{3}} + \frac{\sin \frac{\theta}{3}(\theta - \theta_0)}{\cos \frac{\theta}{3}(\theta - \theta_0) - \cos \frac{\theta_0}{3}} \right]
\]

\[
= 2i \mathcal{D}(\theta,\theta_0) \left[ \theta, \theta_0 \right],
\]

\[
B_2 = -2 \mathcal{D}(\theta,\theta_0) \left[ \left[ \theta, \theta_0 \right]^2 + \frac{\partial}{\partial \theta} \left[ \theta, \theta_0 \right] \right],
\]

\[
B_3 = -\frac{1}{3} \mathcal{D}(\theta,\theta_0) \left[ \left[ \theta, \theta_0 \right] + \frac{4}{3} \left[ \theta, \theta_0 \right]^3 + \frac{12}{3} \left[ \theta, \theta_0 \right] \frac{\partial}{\partial \theta} \left[ \theta, \theta_0 \right] \right]
\]

\[
+ 4 \frac{\partial^2}{\partial \theta^2} \left[ \theta, \theta_0 \right],
\]

where

\[
\frac{\partial}{\partial \theta} \left[ \theta, \theta_0 \right] = -\frac{\sin \theta}{\sin \theta + \cos \theta} - \frac{\cos \theta}{\cos \theta - \cos \theta_0} + \frac{\cos \theta}{\cos \theta - \cos \theta_0}
\]

\[
= \frac{\sin^2 \theta}{\cos \theta - \cos \theta_0} + \frac{2 \cos 2\theta_0 / 3 \cos 2\theta / 3}{2 \cos 2\theta_0 \cos 2\theta + \frac{1}{3} \cos 4\theta_0}
\]
\[ -\frac{\sin \frac{4\theta}{3}}{\cos \frac{4\theta}{3} - \cos \frac{4\pi}{3}} \]

\[ + \frac{16}{9} \frac{\left( \cos \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)} \]

\[ + \frac{16}{9} \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)} \]

\[ + \frac{16}{9} \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]

\[ + \frac{\left( \sin \frac{4\theta}{3} \right)^2}{\cos \frac{4\theta}{3} \left( \frac{4\theta}{3} - \cos \frac{4\pi}{3} \right)^2} \]
For the series (1.68) the first few coefficients are given by

\[ B_0' = \frac{2}{3} B_1, \quad B_2' = \frac{2}{3} B_3 + \frac{7}{27} B_4, \]

\[ B_4' = \frac{2}{3} B_5 - \frac{91}{486} i B_1 + \frac{7}{27} i B_3. \]

(iii)

For the series (1.68) the first few coefficients are given by

\[ C_0 = \frac{M(\theta, \theta_0)}{\sqrt{2} 3\pi} \cot\left(\frac{\delta}{3}\right), \]

\[ C_1 = \frac{B_2}{6\pi/2} \cot\left(\frac{\delta}{3}\right) - \frac{M(\theta, \theta_0)}{12\pi/2} \cot\left(\frac{\delta}{3}\right) \left(1 + \frac{8}{3} (\sin\frac{\delta}{3})^{-2}\right), \]

\[ C_2 = \frac{B_4}{12\pi/2} \cot\left(\frac{\delta}{3}\right) - \frac{B_2}{24\pi/2} \cot\left(\frac{\delta}{3}\right) \left(1 + \frac{8}{3} (\sin\frac{\delta}{3})^{-2}\right) + \frac{M(\theta, \theta_0)}{6\pi/2} \cot\left(\frac{\delta}{3}\right) \left(\frac{3}{8} + \frac{74}{243} (\sin\frac{\delta}{3})^{-2} + \frac{16}{31} (\sin\frac{\delta}{3})^{-4}\right), \]

where the B coefficients are given in part (i).

For the series (1.81) the first few coefficients are given by

(iv)

\[ C_0 = \frac{1 B_1}{9\pi/2 (\sin\frac{\delta}{3})^2}, \]

\[ C_1 = \frac{1}{36\pi/2 (\sin\frac{\delta}{3})^2} \left\{2 B_2 = \frac{2}{9} (1+4(\sin\frac{\delta}{3})^{-2}) B_1\right\}, \]

where the B coefficients are given in part (i).
Consider the integrals

\[ \frac{1}{2\pi i} \int_{\gamma_1} e^{-ik\rho \cos \alpha} G(\alpha+\theta) d\alpha \quad (3A.1) \]

\[ \frac{1}{2\pi i} \int_{\gamma_2} e^{-ik\rho \cos \alpha} G(\alpha+\theta) d\alpha \quad (3A.2) \]

where the contours \( \gamma_1 \) and \( \gamma_2 \) are shown in figure 39.

The shaded regions are where \( e^{-ik\rho \cos \alpha} \to 0 \) for \( k\rho \) real and positive.
The function $G(\alpha)$ is assumed to be an analytic function which is regular in the regions $|\text{Im}\alpha| > N_0$, where $N_0$ is a sufficiently large positive number. We also assume that the paths of integration $\gamma_1, \gamma_2$ lie within the regions $|\text{Im}\alpha| > N_0$; and thus they can be displaced, as $\theta$ varies, without capturing any poles of the integrands. We ensure that the end points of the paths of integration always lie in the shaded regions; this ensures uniform convergence of the integrals $(3A.1)$ and $(3A.2)$ for $\rho > 0, -\infty < \theta < \infty$. It will be shown that for $\rho = 0$ the expressions $(3A.1)$ and $(3A.2)$ are also bounded at this point.

Since $G(\alpha)$ is regular for $|\text{Im}\alpha| > N_0$, then it follows that in any band of finite width $A < \text{Re}\alpha < B$ the function $G(\alpha)$ tends to a constant limit as $\text{Im}\alpha \to \pm \infty$. These limits will be denoted by $G(i\infty)$ and $G(-i\infty)$ respectively. Since they do not depend on $\theta$ it is sufficient to carry out the calculation with $\theta = 0$. Deforming the contour of integration $\gamma_1$ to take up the path shown in figure 40, the integral $(3A.1)$ can be written as,

$$
\frac{1}{2\pi i} \int_{i\alpha - \frac{3\pi}{2}}^{i\alpha + \frac{3\pi}{2}} e^{-ik\rho \cos \alpha} G(\alpha) d\alpha + \frac{1}{2\pi i} \int_{i\alpha - \frac{3\pi}{2}}^{i\alpha + \frac{3\pi}{2}} e^{-ik\rho \cos \alpha} G(\alpha) d\alpha
$$

$$
+ \frac{1}{2\pi i} \int_{i\alpha + \frac{3\pi}{2}}^{i\alpha + \frac{3\pi}{2}} e^{-ik\rho \cos \alpha} G(\alpha) d\alpha . \quad (3A.3)
$$
For a sufficiently large $a$, $a > N_0$, it is possible to assume that $G(\alpha) = G(i\infty)$ and we can remove this term from under the integral signs. After doing this, by virtue of the periodicity of the first and the third integrals of the expression (3A.3) these cancel each other out. The remaining second integral (and therefore the entire integral over the contour $\gamma_1$) becomes equal to $-iG(i\infty)$ in the limit as $\rho = 0$, $a \to \infty$.

By an analogous procedure on the lower path of integration $\gamma_2$, one obtains the result that it is equal to $iG(-i\infty)$. Thus

$$\quad \int_{\gamma_1} G(\alpha+\delta)\,d\alpha = -iG(i\infty)$$

$$\int_{\gamma_2} G(\alpha+\delta)\,d\alpha = iG(-i\infty).$$
Consider the function

\[ H(z) = \frac{f(z)}{g(z)} \]

where \( g(z) \) has multiple poles, say at \( z = a \), which cannot be separated explicitly from \( g(z) \); and \( f(a) \neq 0 \). The residue of an \( m \)th order pole is given by the well known formula

\[ \text{Res}(H(z))_{z=a} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[ \frac{f(z)}{g(z)} (z-a)^m \right]_{z=a}. \]

By expanding \( f(z) \) and \( g(z) \) in their Taylor series at the point \( z = a \) one can obtain after some manipulation the following results:

\( m = 1 \) (simple pole)

\[ \text{Res}(H(z))_{z=a} = \frac{f(a)}{g'(a)} \]

\( m = 2 \) (double pole)

\[ \text{Res}(H(z))_{z=a} = \frac{2f'(a)}{g''(a)} - \frac{2f(a)g'''(a)}{3[g''(a)]^2} \]

\( m = 3 \)

\[ \text{Res}(H(z))_{z=a} = \frac{3f''(a)}{g'''(a)} - \frac{3f(a)g^{(4)}(a)}{10[g''(a)]^2} - \frac{3g^{(iv)}(a)f'(a)}{2[g'''(a)]^2} \]

\[ + \frac{3f(a)g^{(iv)}(a)}{8[g''(a)]^2}. \]
\[ m = 4 \]

\[
\text{Res}[\Pi(z)]_{z=a} = \frac{5f(a)g^V(a)}{25(g^{IV}(a))^2} - \frac{4f(a)g^{VII}(a)}{35(g^{IV}(a))^2} - \frac{24f(a)g^{V}(a)}{125(g^{IV}(a))^4} \\
+ \frac{4f^{'''}(a)}{g^{IV}(a)} + \frac{24f'(a)g^{V}(a)}{25(g^{IV}(a))^3} - \frac{12f'(a)g^{V}(a)}{15(g^{IV}(a))^2} \\
- \frac{12f^{''}(a)g^{V}(a)}{5(g^{IV}(a))^2}
\]
References


PART II

ELECTROMAGNETIC DIFFRACTION BY

A DIELECTRIC WEDGE
INTRODUCTION TO PART II

The study of electromagnetic diffraction by dielectric wedges is of particular interest to the theory of dielectric waveguide matching, Nefyodov [1], Lee and Mittra [2]; the theory of resonators in the optical and radio-frequency ranges Kurilko [3]; radio propagation over the Earth Clemmow [4]; and in radar for the effect of scattering by dielectric randomes and antennae, Tricoles and Rope [5]. Analogous problems to that of the dielectric wedge appear in the fields of acoustics and elasticity Kraut [6].

The object of the present work is to produce approximate expressions, useful to physicists and engineers for the diffraacted or near field, for certain wedge angles and with certain limitations on the material constants of the dielectric wedge.

Chapter 5 deals with the problems of diffraction by a right angle dielectric wedge whose refractive index (= n) is limited to a certain range of values. This problem has been solved theoretically by a number of authors, Radlow [7], Plonus and Kuo [8], Latz [9], Kurilko [10] and Kraut and Lehman [11]. Radlow, Plonus and Kuo, and Kraut and Lehman, solve the problem by a generalisation of the function-theoretical method of Wiener and Hopf from one to two complex variables. Their final result ends up as a double integral the integrand of which involves Wiener Hopf split functions which themselves are expressed in terms of complicated integrals. They do not simplify their final results and as Plonus and Kuo [8] state the solutions are too complicated to be of practical use. It should also be mentioned that Kraut
and Lehman [11] claim that Radlow's [7] and Plonus and Kuo's [8] solutions are incorrect. Kurilko [10] obtains a solution in the form of a rather complicated system of Fredholm integral equations which are amenable to numerical computation. His approach is an extension of the normal one variable Wiener-Hopf method; but his analysis involves rather tricky analytic continuation and his work is not well known. Latz's [9] final result ends up as an infinite system of Hilbert singular integrals which he states are suitable for numerical computation but he does not actually obtain any explicit result of practical use. The mathematical analysis in chapter 5 is not as complex as that given by many of the above mentioned authors but the end results are conceptually more related to the physical problem. The results obtained are more explicit than those obtained by these authors.

In chapter 4 a general integral equation for an infinite dielectric wedge of arbitrary angle and refractive index is derived. In chapter 4 a standard perturbation technique is carried out by means of a Neumann series solution of the integral equation for restricted values of the refractive index. The first term of the Neumann series is evaluated in an explicit closed form for the diffracted field in all space. This corresponds to the Rayleigh-Gans-Born approximation Jones [12]. Sternberg [13] has used this perturbation technique to determine the field diffracted by cylindrical dielectric objects; it would also seem (see Bouwkamp [14]) that Karp and Solfrey [15] have used the method to solve the problem of a dielectric wedge placed on a perfectly conducting
Infinite plane.

In chapter 6 the first term of the Neumann series solution is obtained for an arbitrary angle dielectric wedge, when the refractive index is near unity. The approach is different to that used in chapter 5 and relies on the Kontorovich Lebedev transform and certain properties of this transform discovered by Smith [16], in his investigation into neutron diffusion in wedge shaped regions.

In chapter 7 the field at the edge of a dielectric wedge of arbitrary angle and refractive index is obtained. The field at the edge of a dielectric wedge is of some importance when dealing with dielectrically loaded waveguides. If the field transmitted down the waveguide is too strong there is a possibility that the edge of the dielectric wedge will pit and crack due to overheating caused by the intense field at the edge. Meixner [17] obtains the variation of the field in the region of the wedge tip as a function of the radial distance, but he does not obtain the angular variation of the field.
CHAPTER 4

In this chapter the scalar integral equation for the two dimensional problem of diffraction of an electromagnetic wave by a dielectric wedge of arbitrary angle will be derived.

It is assumed that all space (= V) is filled with two contiguous dielectric media, V_v and V_d, where V = V_v U V_d. With the coordinate system x = r cos θ, y = r sin θ, the dielectric wedge shaped prism comprising V_d is defined by 
\[ \{ V_d : 0 \leq r < \infty, 0 \leq \theta \leq \alpha, \ |z| < \infty \} \] 
and its complement 
\[ \{ V_v : 0 \leq r < \infty, \alpha < \theta < 2\pi, \ |z| < \infty \} \], see figure 41. In the
region \( V_d \) the constant permittivity, constant permeability, and constant conductivity are denoted by \( \varepsilon_d \), \( \mu_d \) and \( \sigma_d \) respectively; similarly these constants will be denoted by \( \varepsilon_v \), \( \mu_v \) and \( \sigma_v \) in the region \( V_v \). It will be assumed that the relative dielectric constant of \( V_d \) is greater than or equal to unity, i.e. \( \varepsilon_d/\varepsilon_v \geq 1 \).

If the harmonic time dependence of the electromagnetic fields is chosen thus:

\[
E(r,t) = \mathbf{E}(r)e^{-i\omega t},
\]

\[
H(r,t) = \mathbf{H}(r)e^{-i\omega t},
\]

Maxwell's equations yield

\[
\nabla \times \mathbf{H} + i\omega \mathbf{E} = 0,
\]

\[
\nabla \times \mathbf{E} - i\omega \mu \mathbf{H} = 0,
\]

where

\[
\varepsilon = \varepsilon_d + \frac{i\sigma_d}{\omega}, \quad \mu = \mu_d \text{ in } V_d,
\]

and

\[
\varepsilon = \varepsilon_v + \frac{i\sigma_v}{\omega}, \quad \mu = \mu_v \text{ in } V_v.
\]

It is assumed that a plane wave \((E_1(r), H_1(r))\) exists in the region \( V_v \). If the incident field is now polarised such that \( E_1(r) = E_1(x,y)k \) or \( H_1(r) = H_1(x,y)k \) then Maxwell's equations (4.2) reduce to
\[ \begin{align*}
E_x &= 0, \quad E_y = 0, \quad E_z = E_z, \quad \text{or} \\
H_x &= 0, \quad H_y = 0, \quad H_z = H_z, \quad \text{where } \mathbf{k} \text{ is the unit vector parallel to the } z \text{-axis.}
\end{align*} \tag{4.5} \]

The equations (4.5) state that if the propagation vector of the primary wave is parallel to the dielectric edge, the polarisation of the secondary waves will remain the same as that of the primary wave. As no surface charge and current can exist on the dielectric surface, the tangential components of the electric and magnetic intensity are continuous across the dielectric surface, Jones [12]. These constitute the boundary condition on the dielectric interface.

The Maxwell equations (4.5) and the above mentioned boundary condition reduce the three dimensional problem to the following two dimensional scalar problem.
In the region $S_d$ defined by $\{ S_d : 0 \leq r < \infty, 0 \leq \theta \leq \alpha \}$ and bounded by $C + C_1$, see figure 42,

$$(v^2 + k_{d}^2)u = 0, \quad (4.6)$$

In the region $S_v$ defined by $\{ S_v : 0 \leq r < \infty, \alpha \leq \theta \leq 2\pi \}$ and bounded by $C + C_2$

$$(v^2 + k_{v}^2)(u - u_o) = 0 \quad (4.7)$$

where $u$ is the total field in $S_v$; and $u_o$ is the incident field which is assumed to only exist in $S_v$ and such that it satisfies

$$(v^2 + k_{v}^2)u_o = 0. \quad (4.8)$$

On the common boundary $C$ the following boundary conditions hold

$$u(x, y) \epsilon S_v = u(x, y) \epsilon S_d, \quad \frac{\partial u}{\partial n}(x, y) \epsilon S_v = \tau \frac{\partial u}{\partial n}(x, y) \epsilon S_d \quad (4.9)$$

where

$$\tau = \left\{ \begin{array}{cl} \frac{\mu_v}{\mu_d} & \text{for } u = \left\{ \begin{array}{c} E_z \\ H_z \end{array} \right\} \\
\frac{\varepsilon_v}{\varepsilon_d} & \end{array} \right. \quad (4.10)$$

and $n$ is an appropriate normal to $C$.

In the expressions (4.5) to (4.10) $k_v$ and $k_d$ are the complex propagation constants in $V_v$ and $V_d$ respectively, and are given by
or
\[ k_v = \sqrt{\frac{\sqrt{\varepsilon_v^2 + \sigma_v^2/\omega^2} + \varepsilon_v}{2}} + i \sqrt{\frac{\sqrt{\varepsilon_v^2 + \sigma_v^2/\omega^2} - \varepsilon_v}{2}} \] (4.11)

or
\[ k_d = \sqrt{\frac{\sqrt{\varepsilon_d^2 + \sigma_d^2/\omega^2} + \varepsilon_d}{2}} + i \sqrt{\frac{\sqrt{\varepsilon_d^2 + \sigma_d^2/\omega^2} - \varepsilon_d}{2}} \] (4.12)

For \( \varepsilon_d/\varepsilon_v > 1, \sigma_d/\sigma_v > 1 \) then
\[ \text{Im} \ k_d > \text{Im} \ k_v > 0, \quad \text{Re} \ k_d > \text{Re} \ k_v > 0. \]

A small conductivity parameter \((\sigma_v, \sigma_d)\) is introduced to ensure convergence during the ensuing analytic manipulations. In the final result we can assume \( \sigma_v = 0, \sigma_d = 0 \). To simplify the work which follows it will be assumed that

\[ k_d = \delta k_v, \quad \delta > 1 \] (4.13)
i.e. \( k_d = 1 k_v e^{i\xi} \), \( k_v = 1 k_v e^{i\xi} \), \( \xi > 0 \), where \( \xi, \delta \) are absolute constants.

At the vertex of the wedges we assume the field obeys the "edge condition" Jones [12].

\[ \text{grad} \ u = 0(x^{-\beta_v}), \quad \text{grad} \ u = 0(x^{-\beta_d}) \]

\[ u = 0(1) \quad x \to 0, \] (4.14)
where \( 0 < \beta_v, \beta_d < 1 \) and \( \beta_v \) corresponds to \( r \in S_v \), and \( \beta_d \)
corresponds to \( r \in S_d \).

The final boundary condition is the behaviour of the field at infinity, i.e. the radiation condition. A radiation condition corresponds to the physical requirement that in a non-absorbing medium the field at infinity must behave as an outgoing wave. In a non-absorbing medium, (conductivity parameter zero) which has a real propagation constant \( k \), Sommerfeld showed that for a time variation \( e^{-i\omega t} \) outgoing waves are assured in a two dimensional domain if

\[
\lim_{r \to \infty} r^\alpha \left( \frac{\partial u}{\partial r} - ik u \right) = 0 \quad \text{Im} k = 0.
\]  
(4.15)

If the medium has slight conductivity present then

\[
\text{Im} k > 0,
\]  
(4.16)

and this ensures that the field decays as \( r \to \infty \). The condition (4.16) is the radiation for an absorbing medium.

When dealing with two infinite contiguous regions with different propagation constants it is necessary to have two radiation conditions. Thus

In \( S_v \)

\[
\lim_{r \to \infty} r^\alpha \left( \frac{\partial u}{\partial r} - ik_v u \right) = 0
\]  
(4.18)

\[
\text{Im} k_v > 0.
\]  
(4.17)

In \( S_d \)

\[
\lim_{r \to \infty} r^\alpha \left( \frac{\partial u}{\partial r} - ik_d u \right) = 0
\]  
(4.18)

\[
\text{Im} k_d > 0.
\]  
(4.19)
\[
\lim_{r \to \infty} r^{\frac{3}{2}} \left( \frac{\partial u}{\partial r} - i k_3 u \right) = 0, \quad (4.20)
\]
\[
\text{Im} k_3 = 0
\]

(4.17) and (4.19) are ensured by the choice of root sign in (4.11) and (4.12).

The integral equation can now be derived having defined the problem. In two-dimensional space it is well known that the Green's function \( G(\mathbf{r}, \mathbf{r}') \) for two-dimensional space is given by

\[
G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} H_0^{(1)}(k r - r'), \quad (4.21)
\]

where \( |\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2} \),

and \( \text{Im} k > 0 \); also

\[
(v^2 + k^2)G = 5(x-x')5(y-y'). \quad (4.22)
\]

Green's theorem in two dimensions is given by

\[
\iint_S (\partial^2 \psi + \psi \partial^2 \phi) \, ds = - \oint_\ell (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) \, dl. \quad (4.24)
\]

where \( \phi \) and \( \psi \) are functions of position over a region \( S \), which is bounded by the closed curve \( \ell \), see figure 43.
Over this surface $S$ and on the boundary it is assumed that $\phi, \psi$ and the first and second derivatives of $\phi$ and $\psi$ are continuous.

Consider the situation when the point $(x', y')$ lies in the region $S_d$. From the conditions imposed on $u$ we can let $u = \phi$ in Green's theorem (4.24); however, the Green's function $G(\xi, \xi')$ has a discontinuous first derivative at $(x', y')$ and so it is not possible to apply Green's theorem with $\psi = G$ unless the small region about $(x', y')$ is cut out.

Thus constructing a small circular region about the point $(x', y')$ of area $\Omega$ and circumference $\Sigma$, see figure 44, and applying Green's theorem, we obtain

$$\iint_{S_d-\Omega} (uv^2 G - Gv^2u) \, ds = -\int_C \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) \, dl. \quad (4.25)$$

In the region $S_d-\Omega$ equations (4.6) and (4.23) give

$$uv^2 G - Gv^2u = (k_d^2 - k^2)uG.$$
which on substituting into (4.25) yields

\[ \int (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) d\ell = (k^2 - k^n_0^2) \int_{\Sigma_{d-\Omega}} u d\ell - \int_{\Sigma_{d-\Omega}} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) d\ell . \]

The integral on the left hand side of the expression (4.26) is now considered for the situation when the small area about \((x',y')\) shrinks to zero.

The integral on the left hand side of the expression (4.26) is now considered for the situation when the small area about \((x',y')\) shrinks to zero.

Figure 45

For small \(r_o\)

\[ H_0^{(1)}(kr_o) \sim \frac{2i}{\pi} \log r_o, \]

\[ \frac{\partial H_0^{(1)}}{\partial r_o} \sim \frac{2i}{\pi r_o}, \]

and on the circumference \(\Sigma, d\ell = r_o d\theta\); thus

\[ \int_{\Sigma} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) d\ell. \]

Clearly as \(r_o \rightarrow 0\) since \(\frac{\partial u(r)}{\partial n}\) is finite at \((x',y')\) then
Thus (4.26) can now be written as

\[ u(x', y') = \left( k^2 - k_d^2 \right) \iint u \delta s - \int \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\ell \]

\[ (x', y') \in S_d \]

\[ - \int \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\ell \]  \hspace{1cm} (4.27)

We now consider the integral

\[ \int \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\ell \]

\[ c_1 \]

Figure 46

On the boundary \( c_1 \)

\[ \frac{\partial}{\partial n} = -\frac{\partial}{\partial r}, \quad H_0^{(1)}(kr-x)^4 \sim H_0^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}, \]

\[ \frac{\partial}{\partial n} H_0^{(1)}(kr) \sim -ik \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}, \quad d\ell = rd\theta, \]
and thus

\[ \lim_{r \to \infty} \int_{c_1} = \frac{e^{-ikr}}{4\pi} \sqrt{\frac{2}{\pi k}} \int_{0}^{\pi} r \frac{\partial u}{\partial r} (\partial u - ik) e^{ikr} \, d\theta = A \int_{0}^{\pi} r \frac{\partial u}{\partial r} (\partial u - ik) e^{ikr} \, d\theta . \]

Since \( \text{Im} k > 0 \) then as \( r \to \infty \)

\[ r \frac{\partial u}{\partial r} (\partial u - ik) e^{ikr} \to 0 , \quad (4.28) \]

so that

\[ \lim_{r \to \infty} \int_{c_1} = 0 . \]

Hence (4.27) can now be written as

\[ u(x', y') = (k^2 - k_d^2) \int_{S_d} uGds - \int_{S_d} (u \frac{\partial G}{\partial n} + G \frac{\partial u}{\partial n}) \, d\ell , \quad (x', y') \in S_d \]

\[ (x', y') \in S_d \quad S_d \quad c \quad (4.29) \]

which states that the field at a point in the domain \( S_d \) is equal to the surface integral of the product \( uG \) over \( S_d \) minus the integral over the common boundary \( c \) between \( S_d \) and \( S_v \).

By an analogous procedure for a point in the region \( S_v \), see figure 47, one obtains,

\[ u(x', y') = (k^2 - k_v^2) \int_{S_v} uGds + \int_{S_v} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) \, d\ell . \quad (4.30) \]

\[ (x', y') \in S_v \quad S_v \quad c \]
The difference in the sign in the second integral on the right hand side of the expressions (4.29) and (4.30) is due to the fact that the normal into \( S_v \) is opposite to that into \( S_a \). The contribution to \( u(x',y') \), when \((x',y') \in S_v\), from the infinite circular sector \( C_2 \) is given by

\[
\oint_{C_2} \frac{1}{\alpha} \left( \frac{\partial u}{\partial x} - i ku \right) e^{iku \theta} d\ell = B \int_{C_2} \frac{2\pi}{\alpha} r^{\frac{1}{2}} e^{iku \theta} d\ell \to 0, \quad (4.31)
\]

as \( r \to \infty \) because \( \text{Im} k > 0 \).

By adding the expressions (4.29) and (4.30) the sum of the left hand side will give \( u(x',y') \) for all space, i.e., \( S_u \cup S_v \). Thus
\[ u(x',y') = (k^2 - k_d^2) \int_{S_d} u \, ds + (k^2 - k_v^2) \int_{S_v} u \, ds \]

\[ + \int_{C} \left[ \left( \frac{\partial G}{\partial n} \right) (x,y) \right]_{S_v} \, ds + \left[ \left( \frac{\partial G}{\partial n} \right) (x,y) \right]_{S_v} \, ds \]

(4.32)

where \((x,y) \in S_v\) means that the quantities inside the bracket approach the values \((x,y)\) on \(C\) from the region \(S_v\), and \((x,y) \in S_d\) means that the quantities inside the bracket approach the values of \((x,y)\) on \(C\) from the region \(S_d\). The edge condition (4.14) ensures that the integrals evaluated in the region of the origin are bounded.

Applying the boundary conditions (4.9) to the expression (4.32) gives

\[ u(x',y') = (k^2 - k_d^2) \int_{S_d} u \, ds + (k^2 - k_v^2) \int_{S_v} u \, ds \]

\[ + (1 - \tau) \int_{C} G \left( \frac{\partial u}{\partial n} \right) (x,y) \, ds \]  

(4.33)

or

\[ u(x',y') = (k^2 - k_d^2) \int_{S_d} u \, ds + (k^2 - k_v^2) \int_{S_v} u \, ds \]

\[ + (1 - \frac{1}{4}) \int_{C} G \left( \frac{\partial u}{\partial n} \right) (x,y) \, ds \]  

(4.34)

The conditions of continuity of the field and its normal derivative across \(C\) are automatically built into the
expressions (4.32), (4.33) and (4.34). As a check applying \( v'^2 \) to (4.33) assuming \((x',y') \in S_d \) gives

\[
v'^2 u(x',y') = (k^2 - k_d^2) \iint u(-k^2 G + \delta(x-x') \delta(y-y')) ds_d + (k^2 - k_v^2) \iint u(-k^2) ds_v + (1-\tau) \int (-k^2 G) \left( \frac{\partial u}{\partial n}(x,y) \right) ds_c - (1-\tau) f\left(-k^2 G\right) \left( \frac{\partial u}{\partial n}(x',y') \right),
\]

\[
+ (k^2 - k_v^2) u(x',y'),
\]

\[
= -k^2 u(x',y') + (k^2 - k_d^2) u(x',y') = -k^2(x',y'),
\]

as expected.

In the region \( S_v \) there exists a primary plane wave source, \( u_o \), and thus for points in this region the field \( u \) can be replaced by \( u - u_o \) by virtue of (4.7) and (4.8).

This expedient is introduced in order that the geometrical optics and diffracted field can be easily distinguished. Thus (4.33) and (4.34) become

\[
u(x',y') = \{u_o(x',y')H(\theta-\alpha) + (k^2 - k_v^2) \iint u_o G ds_v \} + (k^2 - k_d^2) \iint u ds_d + (k^2 - k_v^2) \iint u ds_v + (1-\tau) \int G \left( \frac{\partial u}{\partial n}(x,y) \right) ds_c,
\]

(4.35)
\[ u(x', y') = \left\{ u_0(x', y')H(\theta - \alpha) + (k_v^2 - k_d^2) \iint_{S_v} u_0 G ds \right\} \\
- \left(1 - \frac{1}{\tau}\right) \int \left. G \left(\frac{\partial u_0}{\partial n}\right)\right|_{(x, y) \in S_v} \frac{d\ell}{\ell} \right\} \\
+ (k_v^2 - k_d^2) \iint_{S_d} u G ds + (k_v^2 - k_d^2) \iint_{S_v} u G ds + \left(1 - \frac{1}{\tau}\right) \int G \left(\frac{\partial u}{\partial n}\right)_{(x, y) \in S_v} \frac{d\ell}{\ell} \right\} \\
\left(4.36\right) \]

where \( H(x) = 1 \) for \( x > 0 \), \( H(x) = 0 \) for \( x < 0 \).

For the specific problem of diffraction of an E-polarised wave by a perfect dielectric with slight conductivity \( \mu_v = \mu_d \), and therefore \( \tau = 1 \), so that (4.35) and (4.36) reduce to

\[ u(x', y') = \left\{ u_0(x', y')H(\theta - \alpha) + (k_v^2 - k_d^2) \iint_{S_v} u_0 G ds \right\} \\
+ (k_v^2 - k_d^2) \iint_{S_d} u G ds + (k_v^2 - k_d^2) \iint_{S_v} u G ds \right. \\
\left. + \left(1 - \frac{1}{\tau}\right) \int G \left(\frac{\partial u}{\partial n}\right)_{(x, y) \in S_v} \frac{d\ell}{\ell} \right\} \\
\left(4.37\right) \]

The term in the curly bracket of (4.37) is a known function. The expression (4.37) can be written in the operational form

\[ u = T[u] \right. \\
\left. \left(4.38\right) \]

where

\[ T[u] = U_0 + (k_v^2 - k_d^2) \iint_{S_d} u G ds + (k_v^2 - k_d^2) \iint_{S_v} u G ds \right. \\
\left. + \left(1 - \frac{1}{\tau}\right) \int G \left(\frac{\partial u}{\partial n}\right)_{(x, y) \in S_v} \frac{d\ell}{\ell} \right\} \\
U_0 = \left\{ u_0(x', y')H(\theta - \alpha) + (k_v^2 - k_d^2) \iint_{S_v} u_0 G ds \right\} .
A Neumann series solution to (4.38) is obtained by the iteration scheme

\[ u_{n+1} = T[u_n], \quad u_0 = U_0, \quad (4.39) \]

so that

\[ u_1 = T[U_0], \]
\[ u_2 = T^2[U_0], \]
\[ \ldots \]
\[ u_{n+1} = T^n[U_0]. \]

It is now necessary to prove that the solution given by the iteration scheme exists and is unique, and what constraints must be put on \( k_v, k_d, k \).

**Theorem 1.**

If \( U_0(x',y') \in L_2(\infty, \infty) \) i.e.

\[ \|U_0\|_{L_2} = \left( \int_{\infty}^{\infty} \int_{\infty}^{\infty} |U_0(x',y')|^2 \, dx' \, dy' \right)^{\frac{1}{2}} < \infty \]

and

\[ 0 < \int_{\infty}^{\infty} \int_{\infty}^{\infty} G(x,y,x',y') \, dx' \, dy' < \infty \]

then there exists a value of \( \lambda \) \( (\max(1k^2-k_d^2,1k_v^2-k_d^2)) \) with \( 0 < \|\lambda\| < \infty \), such that the function \( u(x',y') \) satisfying the integral equation (4.37) has a bounded \( L_2 \) norm, i.e.

\[ \|u(x',y')\| < \infty. \]

**Proof:**

From (4.37)
\[ \|u(x', y')\|_2 = \left| \|u_0(x', y')\| + (k^2 - k_0^2) \int_{S_v} uGdx dy + (k^2 - k_0^2) \right| \]

\[ \leq \left| \|u_0\|_2 + \lambda \right| \left| \int_{S_v} uGdx dy + \int_{S_d} uGdx dy \right|_2 \]

\[ = \left| \|u_0\|_2 + \lambda \right| \left| \int_{S_v} uGdx dy \right|_2 \]

\[ = \left| \|u_0\|_2 + \lambda \right| \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uGdx dy \right)^2 dx' dy' \right\}^{\frac{1}{2}} \]

\[ \leq \left| \|u_0\|_2 + \lambda \right| \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^2(x, y, x', y') dx' dy' \right] dx dy \right\}^{\frac{1}{2}} \]

by Schwartz's inequality,

\[ \leq \left| \|u_0\|_2 + \lambda \right| \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^2(x, y, x', y') dx' dy' \right\}^{\frac{1}{2}} \left| \|u\|_2 \right|, \]

and since \( G \) is symmetrical in \((x, y) (x', y')\) then

\[ \|u\|_2 \leq \left| \|u_0\|_2 + \lambda \right| \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^2 dx dy \right\} \|u\|_2 , \]

or

\[ \|u\|_2 \leq \frac{\|u_0\|_2}{1 - \left| \lambda \right| \int_{-\infty}^{\infty} Gdx dy} \]

Thus the choice

\[ \left| \lambda \right| \int_{-\infty}^{\infty} Gdx dy < 1 \]
ensures that $\|u\|_2 < \infty$.

Theorem 2.
For $0 < |\lambda| < \left[ \int\int G dx dy \right]^{-1}$ and $u(x',y')$ in the complete normed linear space $L_2$, 

$$Tu = u(x',y') + (k^2 - k_d^2) \int\int u G dx dy + (k^2 - k_v^2) \int\int v G dx dy$$

is a contraction mapping with respect to the $L_2$ norm. Hence the integral equation 

$$Tu = u$$

has one and only one fixed point which belongs to $L_2$. This fixed point (solution of the integral equation) is the limit of successive approximations converging in $L_2$ norm to $u$.

Proof:

Let $u(x',y')$ and $v(x',y')$ belong to $L_2$ function space; then 

$$\|Tu -Tv\|_2 = \left\| (k^2 - k_d^2) \int\int (u - v) G dx dy + (k^2 - k_v^2) \int\int (u - v) G dx dy \right\|_2$$

$$\leq |\lambda| \left\| \int\int (u - v) G dx dy \right\|_2$$

$$\leq |\lambda| \left\| \int\int G dx dy \right\| \|u - v\|_2$$

by Schwartz's inequality,
which shows that $T\alpha$ is a contraction mapping with respect to the $L_2$ norm provided

$$\|\lambda\| \left| \iint G \, dx \, dy \right| < 1.$$  \hspace{1cm} (4.40)

The remainder of the theorem is a consequence of Banach's fixed point theorem and the fact that the normed function space $L_2$ is complete.

From the way $V_0(x', y')$ is defined assuming $\text{Im} \, k > 0$, it is easily shown that $\|V_0\| < \infty$. The integral in the inequality (4.40) is now evaluated to give an explicit range for $\|\lambda\|$.

$$\left| \iint G \, dx \, dy \right| = \left| \frac{i}{4} \iint \mathcal{H}_0^{(1)} (k \sqrt{(x-x')^2 + (y-y')^2}) \, dx \, dy \right|$$

and using the well known result, Morse and Feshbach [18], page 823,

$$\mathcal{H}_0^{(1)} (k \sqrt{x-x'} \sqrt{y-y'}) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{e^{i u (x-x') + i v (y-y')}}{u^2 + v^2 - k^2} \, du \, dv,$$

gives

$$\left| \iint G \, dx \, dy \right| = \left| \iint \left( \int_{-\infty}^{\infty} \frac{e^{-i u \cdot x - i v \cdot y}}{4 \pi^2} \, dx \, dy \right) \frac{e^{i u \cdot x' - i v \cdot y'}}{u^2 + v^2 - k^2} \, du \, dv \right|,$$

$$= \left| \iint \delta(u) \delta(v) \frac{e^{i u \cdot x' - i v \cdot y'}}{u^2 + v^2 - k^2} \, du \, dv \right| = \frac{1}{|k^2|}.$$
Hence from (4.40) the solution by iteration exists and is unique for
\[ \frac{|\lambda|}{|k|^2} < 1. \]  

(4.41)

Up to now \( k \) is arbitrary except that \( \text{Im} k > 0 \), however the integral equation (4.37), and consequently the iteration calculations, can be considerably simplified by choosing a specific value for \( k \). Thus setting \( k = k_v \) and restricting \((x',y') \in S_v\) then (4.37) becomes
\[ u(x',y') = u_o(x',y') + (k_v^2 - k_d^2) \int_{S_d} u G ds, \quad (4.42) \]
\((x',y') \in S_v, \)
and the inequality (4.41) becomes
\[ |1 - n^2| < 1 \Rightarrow |n| < \sqrt{2} \]
(4.43)

where \( n = k_d/k_v \) is the complex refractive index of the dielectric wedge. Since by hypothesis \( |n| > 1 \) then the range of \( |n| \) for iterative solution of (4.42) is \( 1 < |n| < \sqrt{2} \). Using the iteration scheme the Rayleigh-Gans-Born approximation for the field in \( S_v \) up to the order of \( (k_v^2 - k_d^2) \) is given from (4.42) as
\[ u(x',y') = u_o(x',y') + (k_v^2 - k_d^2) \int_{S_d} u_o G ds + O[(k_v^2 - k_d^2)^2] \]
\((x',y') \in S_v. \)
(4.44)

To obtain the field in \( S_d \) let \( k = k_d \) and restrict
(x', y') ∈ S_d so that (4.37) becomes

$$u(x', y') = (k_v^2 - k_d^2) \int_{S_v} u_o \, ds + (k_v^2 - k_d^2) \int_{S_d} u \, ds \quad (4.45)$$

and the inequality (4.41) becomes

$$|l| = 1/n^2 < 1 = |n| > \frac{1}{\sqrt{2}},$$

and since |n| > 1 then the range of n for an iterative solution of (4.45) is $1 \leq |n| < \infty$. Using the iteration scheme the Rayleigh-Gans-Born approximation for the field in $S_d$ up to the order of $(k_v^2 - k_d^2)$ is given from (4.45) as

$$u(x', y') = (k_v^2 - k_d^2) \int_{S_v} u_o \, ds + O[(k_d^2 - k_v^2)^2] \quad (4.46)$$

$$(x', y') \in S_d.$$

It will be noticed that by the choice of k for the region $S_v$ and $S_d$ the Green's function appearing in (4.42) will involve $k_v$, and in (4.45) will involve $k_d$, so that on carrying out the iterations the diffracted far fields in $S_d$ and $S_v$ will have the correct propagation constants giving outgoing waves at infinity.
The results of chapter 4 will now be used to obtain the diffracted field when an E-polarised plane wave is incident on a right angle dielectric wedge ($\alpha = \pi/2$).

The incident plane wave is given by

$$u_0 = E^i_{z}(x,y) = e^{i k y (x \cos \theta_o + y \sin \theta_o)} , \quad 0 < \theta_o < \pi/2.$$  

Figure 48

The Rayleigh-Gans-Born approximation of the field in $S_\nu$ is given by (4.42) in this case as

$$E^i_{z}(x,y) = E^i_{z}(x,y) + \frac{i (k^2_0 - k^2_v)}{k_v} \int_{S_\nu} E^i_{z}(x',y') (x-x')^2 + (y-y')^2 \, dx' dy'$$

$$\quad \quad + O \left[ (k^2_0 - k^2_v)^2 \right] , \quad (5.1)$$
where \( u = E_z \) and \( G \) is given by (4.21) with \( k = k_v \).

More explicitly (5.1) can be written as

\[
E_z(x,y) = \mathcal{E}^{i k_v (x \cos \theta_0 + y \sin \theta_0)} \cdot \left\{ \frac{i k_v (k_v^2 - k_a^2)}{4 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i k_v \sqrt{(x-x')^2 + (y-y')^2}}}{\sqrt{k_v^2 - \kappa^2}} \, dx' \, dy' \right\}

+ O\left[ (k_v^2 - k_a^2)^2 \right],
\]

where the integration is carried out over the first quadrant.

The evaluation of (5.2) is achieved by using the well known integral representation of the Hankel function, to wit

\[
H_0^{(1)}(k_v \sqrt{(x-x')^2 + (y-y')^2}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i k_v \sqrt{x^2 + y^2}}}{\sqrt{k_v^2 - \kappa^2}} \, d\kappa.
\]

In the work which follows the square root will be defined by

\[
\sqrt{k_v^2 - \kappa^2} \vert_{\kappa = 0} = k_v \text{ or } \text{Im} \sqrt{k_v^2 - \kappa^2} > 0.
\]

Substituting (5.3) and formally interchanging the order of integration one obtains

\[
E_z(x,y) = \mathcal{E}^{i k_v (x \cos \theta_0 + y \sin \theta_0)} \cdot \left\{ \frac{i k_v (k_v^2 - k_a^2)}{4 \pi} \int_{-\infty}^{\infty} \frac{e^{-i k_v \sqrt{(x-x')^2 + (y-y')^2}}}{\sqrt{k_v^2 - \kappa^2}} \, dx' \, dy' \right\}

+ O\left[ (k_v^2 - k_a^2)^2 \right].
\]

The presence of the term \( |y-y'| \) in the expression (5.4) requires that one consider the observation point \((x,y) \in E_v^y\) in each quadrant separately.
In the domain $S_d$ the integration must be carried over two regions, one where $y' > y$ and the other where $y' < y$, see figure 49. Hence (5.4) becomes

$$E(x; y) = e^{ik_0(x\cos\theta_0 + y\sin\theta_0)} + i(k_0^2 - k_x^2) \left[ \int_0^{\infty} e^{i\nu x} \frac{e^{i\nu y}}{\sqrt{k_0^2 - \nu^2}} d\nu \right]$$

$$-i\sqrt{k_0^2 - y'^2} \int_0^{y'} e^{i(k_0\sin\theta_0 - \sqrt{k_0^2 - y'^2})y'} dy' \int_0^{\infty} e^{i(k_0\cos\theta_0 - \nu)x'} dx'$$

$$+ i\sqrt{k_0^2 - y'^2} \int_y^{\infty} e^{i(k_0\sin\theta_0 + \sqrt{k_0^2 - y'^2})y'} dy' \int_0^{\infty} e^{i(k_0\cos\theta_0 - \nu)x'} dx'$$

$$+ \mathcal{O}[(k_0^2 - k_x^2)^2],$$

for $y > 0$, $x < 0$ (5.5)
Carrying out the $x'$ and $y'$ integration; noting that since
\[ \text{Im} \ k_v > 0, \ \text{Im} \sqrt{k_v^2 - v^2} > 0 \text{ and } \cos \theta_o > 0, \text{ since } \theta_o > 0, \]
$0 < \theta_o < \frac{\pi}{2}$, then convergence at the infinite limits is assured, the term in the square brackets becomes

\[
\left[ \frac{1}{(v + k_v \cos \theta_o)(v - k_v \cos \theta_o)^2} \right]
\]

\[
= -2 \int_{-\infty}^{\infty} e^{i k_v \sin \theta_o \frac{y}{\sqrt{v^2 - k^2}} + i k_v \cos \theta_o \frac{x}{\sqrt{v^2 - k^2}}} \frac{e^{i k_v \left(\frac{-yx + \sqrt{(v^2 - k^2)k^2}}{\sqrt{v^2 - k^2}}\right)}}{\sqrt{k^2 - v^2} \left(k_v \sin \theta_o - \sqrt{k^2 - v^2}\right)} e^{i k_v \left(\frac{-yx + \sqrt{(v^2 - k^2)k^2}}{\sqrt{v^2 - k^2}}\right)} \frac{dy}{\sqrt{k^2 - v^2}}
\]

\[ (5.6) \]

$y > 0, \ x < 0$.

The first integral of the expression (5.6) can be evaluated exactly and in fact represents the reflected field. The second integral has the familiar form of the diffracted field, and can only be evaluated asymptotically. Evaluation of the integral

\[
I = \int_{-\infty}^{\infty} \frac{e^{i k_v x}}{(v + k_v \cos \theta_o)(v - k_v \cos \theta_o)^2}, \ x < 0. \tag{5.7}
\]

From the form of (5.7) Jordan's Lemma can be applied directly and the real axis of integration can be closed by an infinite semi-circle below the axis $\text{Im} \ v = 0$. The only singularities that occur in the $v$-plane are a double pole at $v = k_v \cos \theta_o$ and a simple pole at $v = -k_v \cos \theta_o$; see figure 50.

Remembering that $\text{Im} k_v > 0$ and $0 < \theta_o < \frac{\pi}{2}$ the only singularity that is enclosed by the closed contour is $v = -k_v \cos \theta_o$, and
an application of Cauchy's residue theorem gives:

\[ I = \frac{\frac{\partial}{\partial z} e^{ikx \cos \theta_o}}{2k_x \cos^2 \theta_o} \]  

(5.8)

\[ \int_{L_{\mp}} L_{mp} e^{-ikx \cos \theta_o} \, \mathrm{d}y \]

\[ \, \mathrm{d}y \]

\[ \frac{\partial}{\partial z} e^{ikx \cos \theta_o} \]

\[ \int_{-\infty}^{\infty} \frac{(ky \sin \theta_o + \sqrt{k^2 - v^2}) e^{i(yx + \sqrt{k^2 - v^2} y)}}{(y - kv \cos \theta_o)^2 (y + kv \cos \theta_o) \sqrt{k^2 - v^2}} \, \mathrm{d}y \]

\[ \, \mathrm{d}y \]

\[ \int_{-\infty}^{\infty} \frac{(ky \sin \theta_o + \sqrt{k^2 - v^2}) e^{i(yx + \sqrt{k^2 - v^2} y)}}{(y - kv \cos \theta_o)^2 (y + kv \cos \theta_o) \sqrt{k^2 - v^2}} \, \mathrm{d}y \]

(5.9)

by using the identity

\[ (k_y \sin \theta_o - \sqrt{k^2 - v^2})(k_y \sin \theta_o + \sqrt{k^2 - v^2}) = v^2 - k_y^2 \cos^2 \theta_o \]

On the Riemann sheet we are using \( \text{Im} \sqrt{k^2 - v^2} > 0 \) and therefore the numerator of (5.9) does not vanish on this sheet since
Thus the integral (5.9) has branch points at \( \nu = \pm k_v \), a simple pole at \( \nu = -k_v \cos \theta_0 \), and a double pole at \( \nu = k_v \cos \theta_0 \), see figure 51.

Figure 51

The approximate evaluation of the integral (5.9) can be carried out by means of the saddle point method Laouwerier [19] for \( kr \gg 1 \). A more general type of integral than (5.9) is asymptotically evaluated in appendix 5A and this result can be used to evaluate (5.9). In the process of asymptotically evaluating (5.9) the real path of integration is vertically displaced such that it runs through the saddle point, which in (5.9) is \( \nu_s = k_v \cos \theta \), see appendix 5A. The saddle point lies below the axis \( \text{Im} \nu = 0 \), because \( \text{Im} k_v > 0 \) and \( \frac{\pi}{2} \leq \theta < \pi \). In shifting the real axis of integration so that it runs through the saddle point there is a possibility that a pole can be captured and also the pole can be in the vicinity of the saddle point \( \nu_s \). Since the saddle point lies below \( \text{Im} \nu = 0 \) the only singularity that can be captured is the
simple pole \( v_p = -k_v \cos \theta_o \). Two possible situations occur, see figure 52:

**Case (1)** \( \text{Im}(v_p - v_g) e^{-i\varepsilon/2} > 0 \Rightarrow -(\cos \theta_o + \cos \theta) > 0 \Rightarrow \theta + \theta_o > \pi \)

**Case (2)** \( \text{Im}(v_p - v_g) e^{-i\varepsilon/2} < 0 \Rightarrow -(\cos \theta_o + \cos \theta) < 0 \Rightarrow \theta + \theta_o < \pi \)

see appendix 5A and 5B.

![Diagram of cases](image)

**Figure 52**

Case (1) \( \theta + \theta_o > \pi \)

In shifting the real contour of integration the pole 
\(-k_v \cos \theta_o\) is captured and therefore its residue contribution must be included. Thus

\[
\int_{-\infty}^{\infty} \frac{e^{i(vy + \sqrt{k_v^2 - y^2})}}{\sqrt{k_v^2 - y^2} (k_v \sin \theta_o - \sqrt{k_v^2 - y^2})(y - k_v \cos \theta_o)} dy
\]

\[
= -\frac{i e^{ik_v(x \cos \theta_o - y \sin \theta_o)}}{k_v \cos^2 \theta_o} + \sqrt{2\pi} \left[ \frac{2G}{k_v} \right] F(0) e^{i(k_v + \pi/4)}
\]

\[
\left[ \frac{2G}{k_v} \right] F(0) e^{i(k_v + \pi/4)} \right]
\]

(5.10)
where \( Q = - \sqrt{2k_r r} \frac{\cos(\theta + \theta_0)/2 \cos(\theta - \theta_0)/2}{\sin \theta} \),

and \( \sqrt{k_r^2 - \nu_s^2} = k_r \sin \theta \) since \( \text{Im} \sqrt{k_r^2 - \nu^2} > 0 \).

Case (2) \( \theta + \theta_0 < \pi \)

No pole is captured in deforming the contour of integration and thus from (5A.12)

\[
\int_{-\infty}^{\infty} \frac{e^{i(yx + \sqrt{k_r^2 - \nu^2} y)}}{\sqrt{k_r^2 - \nu^2}(k_r \sin \theta_0 - \sqrt{k_r^2 - \nu^2})(\nu - k_r \cos \theta_0)} \, d\nu
\]

\[
\sim \int \frac{2\pi [-2Q] F(-\nu) e^{i(k_r^2 + \nu^2/4)}}{\sqrt{k_r \rho} k_r^2 (\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)}
\]

Hence (5.6) becomes

\[
[] = \left[ 2\pi \frac{2QF(\nu)}{\sqrt{k_r \rho} k_r^2 (\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)} \right]
\]

\[
\theta + \theta_0 < \pi
\]

\[
= \frac{-i k_r(x \cos \theta_0 - y \sin \theta_0)}{k_r \cos^2 \theta_0} + \frac{2\pi [-2Q] F(-\nu) e^{i(k_r^2 + \nu^2/4)}}{\sqrt{k_r \rho} k_r^2 (\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)}
\]

\[
\theta + \theta_0 > \pi.
\]

Substituting (5.12) and (5.13) into (5.5) gives

Case (1) \( \theta + \theta_0 > \pi, \frac{\pi}{2} \leq \theta \leq \pi \),

\[
E_z = e^{i(k_r(x \cos \theta_0 + y \sin \theta_0))} + \frac{i(n^2 - 1) 2Q F(\nu) e^{i(k_r^2 + \nu^2/4)}}{2\sqrt{2\pi k_r \rho} (\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)} + O[(n^2 - 1)^2]
\]

\[
(5.14)
\]
Case (2) $\theta + \theta_0 < \pi$, $\frac{\pi}{2} \leq \theta \leq \pi$

$$\bar{E}_z = e^{ikr(x\cos \theta_c + y\sin \theta_c)} - \frac{(n^2-1) e^{-ikr(x\cos \theta_c - y\sin \theta_c)}}{4\cos^2 \theta_c} \cdot$$

$$+ \frac{i(n^2-1) e^{ikr(x\cos \theta_c + y\sin \theta_c)}}{2 \sqrt{2\pi kr}} \frac{-2 \zeta e^{iQ}}{(\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)} + o[(n^2-1)^2].$$

(5.15)

The second term in the expression (5.14) corresponds to the reflected field, which one would expect in this region, see figure 53.

![Figure 53](image)

Figure 53

$(x, y)$ in the third and fourth quadrant; $\pi \leq \theta \leq 2\pi$

In the domain $S_q$ the integration can be carried out in one operation, there being no necessity to break up the $y'$ integration because everywhere in $S_q \ y' > y$. Hence (5.4) gives
\[ E_z(x, y) = e^{ikr(x\cos\theta + y\sin\theta)} \]

\[ + \frac{i}{k^2 - k_e^2} \left[ \int_{-\infty}^{\infty} \frac{e^{-i(vx + \sqrt{k_r^2 - v^2}y)}}{\sqrt{k_r^2 - v^2}} dv \int_{-\infty}^{\infty} e^{-i\sqrt{k_r^2 - v^2}} dy' \right] \]

\[ + O\left[(k_r^2 - k_e^2)^2\right]. \]  

(5.16)

Carrying out the \( x' \) and \( y' \) integration; noting that since
\[ \text{Im}k_v > 0, \text{Im}\sqrt{k_r^2 - v^2} > 0 \] and \( 0 < \theta_0 < \frac{\pi}{2} \) then convergence at the infinite limits is assured, the term in the square brackets is equal to

\[ \left[ \begin{array}{c} \int_{-\infty}^{\infty} \frac{e^{-i(vx - \sqrt{k_r^2 - v^2}y)}}{\sqrt{k_r^2 - v^2}} dv \\ \int_{-\infty}^{\infty} e^{-i\sqrt{k_r^2 - v^2}} dy' \end{array} \right] \]

\[ = \frac{e^{i(vx - \sqrt{k_r^2 - v^2}y)}}{\sqrt{k_r^2 - v^2}(y - k_v\cos\theta_0)(k_v\sin\theta_0 + \sqrt{k_r^2 - v^2})}. \]  

(5.17)

\[ -\infty < x < \infty, \quad y < 0. \]

This integral can be evaluated by a straightforward application of the result in appendix 5A. Before this result can be applied the relative position of the poles and saddle point \( (v_p = k_v\cos\theta) \) must be considered. The only pole that occurs in the \( v \)-plane, which appears in the integral (5.17) is \( v_p = k_v\cos\theta_0 \). The term \( (k_v\sin\theta_0 + \sqrt{k_r^2 - v^2})^{-1} \) is regular throughout the chosen Riemann surface since \( \text{Im}\sqrt{k_r^2 - v^2} > 0 \) and \( \text{Im}k_v\sin\theta_0 \) > 0.

The pole \( v_p = k_v\cos\theta_0 \) lies above the line \( \text{Im}v = 0 \) in the third quadrant \( \pi < \theta < 3\pi/2 \) and therefore \( \text{Im}(k_v\cos\theta) < 0 \).
Thus in shifting the real axis to make it run through $v_s = k_v \cos \theta$ the pole $v_p$ is not captured. In the fourth quadrant the saddle point can lie below $\text{Im} v = 0$ because $\frac{3\pi}{2} < \theta < 2\pi$, and therefore there is a possibility that the pole can be captured or come close to the saddle point in the asymptotic evaluation of $(5.17)$. Applying the results $(5A.11)$ and $(5A.12)$ directly to $(5.17)$ gives the two cases

**case (1)** $\text{Im}(v_p - v_s) e^{-i\varepsilon/2} > 0 \Rightarrow (\cos \theta_o - \cos \theta) > 0 \Rightarrow \theta + \theta_o < 2\pi$

**case (2)** $\text{Im}(v_p - v_s) e^{-i\varepsilon/2} < 0 \Rightarrow (\cos \theta_o - \cos \theta) > 0 \Rightarrow \theta + \theta_o > 2\pi$

i.e.

**case (1)**

$$\int_{-\infty}^{\infty} \frac{e^{i(-v + \sqrt{k_v^2 - v^2})y}}{\sqrt{k_v^2 - v^2} (v - k_v \cos \theta_o) (k_v \sin \theta_o + \sqrt{k_v^2 - v^2})} dy$$

$$\sim \int_{-\infty}^{2\pi} 2 \frac{\Omega \Phi(y) e^{i(k_v r + \pi/4)}}{k_v \sqrt{\sin \theta_o - \sin \theta} (\cos \theta_o - \cos \theta)}$$

**N.B.** $\sqrt{k_v^2 - \nu^2} = k_v \sin \theta$, since $\text{Im} \int_{-\infty}^{\infty} k_v^2 - \nu^2 > 0$.

**case (2)**

$$\int_{-\infty}^{\infty} \frac{e^{i(-v + \sqrt{k_v^2 - v^2})y}}{\sqrt{k_v^2 - v^2} (v - k_v \cos \theta_o) (k_v \sin \theta_o + \sqrt{k_v^2 - v^2})} dy$$

$$\sim \int_{-\infty}^{2\pi} -\frac{2 \Omega \Phi(y) e^{i(k_v r + \pi/4)}}{k_v \sqrt{\sin \theta_o - \sin \theta} (\cos \theta_o - \cos \theta)}$$

Substituting $(5.18)$ and $(5.19)$ into $(5.16)$ gives for
\[ \pi \leq \theta < 2\pi; \quad 0 < \theta_0 < \pi/2, \quad \theta + \theta_0 < 2\pi, \]

\[ E_z(x,y) = e^{i(k(x\cos\theta_0 + y\sin\theta_0))} + \frac{i(n^2-1)\cdot2\cdot Q\cdot F(\theta) \cdot e^{i(kr + \pi n z)\cdot2}}{2\cdot\sqrt{2\pi kr \cdot r^2 \cdot (\sin\theta_0 - \sin\theta)(\cos\theta_0 - \cos\theta)}} + O[(n^2-1)^2] \]  \hspace{1cm} (5.20)

\[ \pi \leq \theta < 2\pi; \quad 0 < \theta_0 < \pi/2, \quad \theta + \theta_0 > 2\pi, \]

\[ E_z(x,y) = e^{i(k(x\cos\theta_0 + y\sin\theta_0))} + \frac{(1-n^2)\cdot e^{i(k(x\cos\theta_0 - y\sin\theta_0))}}{4\sin^2\theta_0} + \frac{i(n^2-1)\cdot2\cdot Q\cdot F(\theta) \cdot e^{i(kr + \pi n z)\cdot2}}{2\cdot\sqrt{2\pi kr \cdot r^2 \cdot (\sin\theta_0 - \sin\theta)(\cos\theta_0 - \cos\theta)}} + O[(n^2-1)^2] \]  \hspace{1cm} (5.21)

The geometrical optics terms in the expressions (5.15) and (5.21) represent the incident and reflected waves from the wedge faces. The amplitude of the reflected waves correspond to the Fresnel coefficients when expanded to the order \((n^2-1)\). Thus

\[ \frac{k_r \sin\theta_0 - \sqrt{k_d^2 - k_r^2 \cos^2\theta_0}}{k_r \sin\theta_0 + \sqrt{k_d^2 - k_r^2 \cos^2\theta_0}} = \frac{(1-n^2)}{4\sin^2\theta_0} + O[(n^2-1)^2], \]

\[ \frac{k_r \cos\theta_0 - \sqrt{k_d^2 - k_r^2 \sin^2\theta_0}}{k_r \cos\theta_0 + \sqrt{k_d^2 - k_r^2 \sin^2\theta_0}} = \frac{(1-n^2)}{4\cos^2\theta_0} + O[(n^2-1)^2]. \]

Having found the field outside the dielectric wedge we now proceed to find the field inside the dielectric wedge.
The appropriate integral equation for this region is given by (4.45), and the Rayleigh-Gans-Born approximation is given by (4.46), i.e.

\[ E_z(x,y) = \frac{i}{\varepsilon_0} \int_{S_V} E_z(x',y') H_0^1(kd \sqrt{(x-x')^2 + (y-y')^2}) \, dx' \, dy' \]

+ \mathcal{O} \left[ (k_y^2 - k_x^2) \right], \quad (5.22)

where the integration is carried over the second, third and fourth quadrant. The expression (5.22) gives the correct propagation constant for the region \( S_d \). Unfortunately if the incident plane wave used up to the present, i.e.

\[ E_i(x',y') = e^{ik_V(x' \cos \theta_o + y' \sin \theta_o)} 0 < \theta_o < \pi/2 \]

is substituted into the expression (5.22) the resulting integrals become divergent since \( \text{Im} \, k_V > 0 \) and over the region \( S_V x' \) and \( y' \) can become infinitely large and negative. The reason for this is that the incident plane wave being used up to the present decays in amplitude for \( x', y' > 0 \) and increases in amplitude for \( x', y' < 0 \). In order to overcome the divergent aspect of the incident plane wave in the region \( S_V \) an alternative incident plane wave is introduced which monotonically decays for \( \text{Im} \, k_V > 0 \) when \( |x'|, |y'| \to \infty \), but which closely resembles the original plane wave at the boundary of the dielectric wedge. Clearly an obvious choice would be a line source located say at \((x_o,y_o)\) in \( S_V \), but the resulting integrals become too difficult to carry out. The representation for the plane
wave fulfilling the requirements already set out is:

$$E^z = e^{-ik\nu (|x-x_0| \cos \theta_0 + |y-y_0| \sin \theta_0)} e^{ik\nu (x \cos \theta_0 + y \sin \theta_0)},$$

(5.23)

where $x_0, y_0 < 0$.

In the region $x > x_0$, $y > y_0$ then

$$E^z = e^{-ik\nu (x_0 \cos \theta_0 + y_0 \sin \theta_0)} e^{ik\nu (x \cos \theta_0 + y \sin \theta_0)},$$

and thus the original plane wave is multiplied by an amplitude factor $e^{-ik\nu (x_0 \cos \theta_0 + y_0 \sin \theta_0)}$. After carrying out the
integrations using the plane wave representation above (to correlate the results for the fields already calculated for unit amplitude incidence), the resulting expressions are divided by $e^{-ik_y(x_0 \cos \theta_0 + y_0 \sin \theta_0)}$ and $x_0, y_0 \to -\infty$. The incident plane wave (5.23) could have been used for calculating the field in $S_y$, and on dividing the end result by $e^{-ik_y(x_0 \cos \theta_0 + y_0 \sin \theta_0)}$ and allowing $x_0, y_0 \to -\infty$ the same results as before would be obtained.

Substituting the integral representation of the Hankel function (5.3) and the plane wave representation (5.23) into (5.22) gives

$$E_z(x, y) = \frac{i(k_x - k_y)}{4\pi} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{k_d^2 - y^2}} \int_{S_y} e^{i(y(x - x') + \sqrt{k_d^2 - y^2} |y - y'|)}$$

$$\times \frac{e^{-ik((x' - x_0) \cos \theta_0 + (y' - y_0) \sin \theta_0)}}{ik((x' - x_0) \cos \theta_0 + (y' - y_0) \sin \theta_0)} dx' dy'$$

$$+ O[(k_d - k_y)^2],$$

(5.24)

where $\text{Im} \sqrt{k_d^2 - y^2} > 0$. Breaking up the range of integration over $S_y$, see figure 55, the integral (5.24) can be written

$$E_z(x, y) = \frac{i(k_x - k_y)}{4\pi} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{k_d^2 - y^2}} \left[ \right.$$

$$\left. e^{i\sqrt{k_d^2 - y^2} y'} \left\{ e^{-ik \cos \theta_0 x_0} \int_{-\infty}^{x_0} e^{-i(k \cos \theta_0 + \nu) x'} dx' + e^{ik \cos \theta_0 x_0} \int_{x_0}^{0} e^{-i(k \cos \theta_0 - \nu) x'} dx' \right\} \times$$

$$\times e^{-ik \sin \theta_0 y_0} \left\{ e^{-i\sqrt{k_d^2 - y^2} y} \int_{-\infty}^{y} e^{-i(k \sin \theta_0 - \sqrt{k_d^2 - y^2} y') dy'} + e^{i\sqrt{k_d^2 - y^2} y} \int_{y}^{\infty} e^{i(k \sin \theta_0 + \sqrt{k_d^2 - y^2} y')} dy' \right\} + \right.$$
Carrying out the integration, convergence at the infinite limits being assured since $\text{Im} k_v > 0, \text{Im} k_d > 0, \text{Im} \sqrt{k_d^2 - \nu^2} > 0$, one obtains after re-arranging the resulting expression
The first integral of the above expression can be calculated by an application of Jordan's Lemma and Cauchy's residue theorem, by closing the contour by an infinite semi-circle in the upper v-plane. The only pole enclosed by this contour for \( \text{Im} k_v > 0, \text{Im} k_d > 0, k_d = \delta k_v, \delta > 0 \) is \( \nu_p = \sqrt{k_v^2 - k_d^2 \sin^2 \theta_o} \), the integrand being regular at \( \nu = k_v \cos \theta_o \). Hence

\[
E(x, y) = \frac{(k_v - k_d) e^{-ik_v(x \cos \theta_o + y \sin \theta_o)}}{4 \pi i} \int_{-\infty}^{\infty} \frac{e^{iux}}{(k \cos \theta_o - u) \left( u^2 - k_d^2 - k_v^2 \sin^2 \theta_o \right)} \left( 1 - \frac{e^{i(k_v \cos \theta_o - u)x_0}}{2k_v \cos \theta_o} \right) \, du
\]

\[
= \frac{2 \pi i e^{ik_v y \sin \theta_o}}{\sqrt{k_d^2 - k_v^2 \sin^2 \theta_o} \left( k \cos \theta_o - \sqrt{k_d^2 - k_v^2 \sin^2 \theta_o} \right)} \left( 1 - \frac{e^{i(k_v \cos \theta_o - \sqrt{k_d^2 - k_v^2 \sin^2 \theta_o})}}{2k_v \cos \theta_o} \right),
\]
\[
\frac{2\pi i}{\gamma_p(k_o - k_v)} e^{i(k_o \cos \theta_v - \gamma_p)x_0} \left( 1 - \frac{e^{-2k_o \cos \theta_v}}{(k_o \cos \theta_v + \gamma_p')} \right),
\]

(5.27)

where \( \gamma_p' = \sqrt{k_o^2 - k_v^2 \sin^2 \theta_v} \).

The second integral in the expression (5.26) is evaluated asymptotically using the results of appendix 5A. Before we can apply the above mentioned result the relative positions of the poles (there are two in this integral \( \gamma_p' = \sqrt{k_o^2 - k_v^2 \sin^2 \theta_v} \) and \( \gamma_p'' = k_v \cos \theta_v \)) and the saddle point \( = k_d \cos \theta \) must be determined, see figure 56. The saddle point lies in the upper \( \nu \)-plane. From the conditions on \( k_v, k_d, \) and \( \theta_v \) the following inequality holds

\[ |k_d| > |k_v^2 - k_o^2 \sin^2 \theta_v| > |k_v \cos \theta_v| \]

Figure 56
Depending on the position of the saddle point \( v_s = k d \cos \theta_0 \) the pole will or will not be captured on deforming the real path of integration to take up the curve of steepest descent.

There are three cases to be considered.

**Case (1)** \[ \text{Im}(v'_p - v_s) e^{i \theta/2} > 0 \Rightarrow |kd| \cos \theta_0 > |kd|^2 \cos \theta > 0 \]

**Case (2)** \[
\begin{cases}
\text{Im}(v'_p - v_s) e^{i \theta/2} < 0 \Rightarrow \sqrt{|kd|^2 - |kd|^2 \sin^2 \theta_0} > |kd| \cos \theta > |kd| \cos \theta_0 \\
\text{Im}(v'_p - v_s) e^{i \theta/2} > 0
\end{cases}
\tag{5.28}
\]

**Case (3)** \[ \text{Im}(v'_p - v_s) e^{i \theta/2} < 0 \Rightarrow |kd| > |kd| \cos \theta > \sqrt{|kd|^2 - |kd|^2 \sin^2 \theta_0} \]

**Case (1)** \( \cos \theta_0 > \delta \cos \theta \)

In this case no poles are captured; a straightforward application of the result (5.11) gives

\[
\begin{align*}
&\int_{-\infty}^{\infty} \frac{e^{i(\nu_x + \sqrt{\nu^2 - z^2})^y}}{(k_r \sin \theta_0 - \sqrt{k_r^2 - \nu^2})(\nu - k_r \cos \theta_0)} \left( \frac{i(k_r(\nu_x \cos \theta_0 + y_x \sin \theta_0) - \nu y_x - \sqrt{\nu^2 - \nu^2})}{4 \nu^2 \cos \theta_0 \sin \theta_0} \right) \, d\nu \\
&\sim \sqrt{\frac{2\pi}{k_d}} \frac{e^{i(k_d(\nu + \nu_0))}}{k_d \nu} \frac{2 \nu F(\nu)}{(k_r \cos \theta_0 - k_d \cos \theta)(k_r \sin \theta_0 - k_d \sin \theta)} + O \left( \frac{e^{i(k_d(\nu - r_0 - k_d \nu_0))}}{\sqrt{|r - r_0|}} \right) \\
&\tag{5.29}
\end{align*}
\]

where \( r_0 = \sqrt{x_0^2 + y_0^2} \), and
\[ Q'' = \sqrt{\frac{k_v \cos \theta_o - k_r \cos \theta}{2k_d \sin \theta}}. \]

Case (2) \[ \sqrt{\delta^2 - \sin^2 \theta_o} > \delta \cos \theta > \cos \theta_o \]

In this case the only pole captured is \( \nu'' = k_v \cos \theta_o \), however there is a possibility that the saddle point will be close to either \( \nu'_p \) or \( \nu''_p \). To overcome this difficulty the poles occurring in the integral are separated by partial fractions and two separate integrals are asymptotically evaluated. Thus since

\[
\frac{1}{(\nu - k_v \cos \theta_o)(k_v \sin \theta_o - \sqrt{k_d - \nu^2})} = \frac{1}{(\nu - k_r \cos \theta_o)(k_r \sin \theta_o - \sqrt{k_d - k_r \cos^2 \theta_o})}
\]

then

\[
\int_{-\infty}^{\infty} \frac{e^{i(\nu z + \sqrt{k_d - \nu^2} y)}}{(\nu - k_v \cos \theta_o)(k_v \sin \theta_o - \sqrt{k_d - \nu^2})} \left( \frac{-i[k_r \cos \theta_o + \sqrt{k_d - \nu^2} y_o]}{k_r \cos \theta_o + \nu} \right) \left( 1 - \frac{e^{-4k_r \cos \theta_o \sin \theta_o \nu}}{(k_r \cos \theta_o + \nu)(k_r \sin \theta_o + \sqrt{k_d - \nu^2})} \right) d\nu
\]

\[
= \frac{1}{(k_r \sin \theta_o - \sqrt{k_d - k_r \cos^2 \theta_o})} \int_{-\infty}^{\infty} \frac{e^{i(\nu z + \sqrt{k_d - \nu^2} y)}}{(\nu - k_v \cos \theta_o)(\sqrt{k_d - \nu^2})} \left( \frac{-i[k_r \cos \theta_o + \sqrt{k_d - \nu^2} y_o]}{k_r \cos \theta_o + \nu} \right) \left( 1 - \frac{e^{-4k_r \cos \theta_o \sin \theta_o \nu}}{(k_r \cos \theta_o + \nu)(k_r \sin \theta_o + \sqrt{k_d - \nu^2})} \right) d\nu
\]
In the last calculation the result (5A.12) of appendix 5A was used for the first integral and the result (5A.11) of appendix 5A was used for the second integral.
Case (3) \( 5 \cos \theta > \sqrt{5^2 - \sin^2 \theta_0} \)

In this case the two poles are captured in letting the path of integration take up the curve of steepest descent. The only pole that can approach the saddle point is

\[ \nu_p = \sqrt{k_r^2 - k_v^2 \sin^2 \theta_0}. \]

Thus using the result (5A.12) of appendix 5A gives

\[
\int_{-\infty}^{\infty} \frac{e^{i(\nu x + \sqrt{k_r^2 - k_v^2} y)}}{(k_r \sin \theta_0 - \sqrt{k_r^2 - k_v^2})(\nu - k_r \cos \theta_0) \sqrt{k_r^2 - k_v^2}} \left( \frac{-i [k_r x_0 + \nu x_0 + \sqrt{k_r^2 - k_v^2} y_0]}{\sqrt{k_r^2 - k_v^2} (\nu - k_r \cos \theta_0 + \nu_p)} \right) d\nu
\]

\[
\sim \frac{i(k_r \sin \theta_0 + \sqrt{k_r^2 - k_r \sin \theta_0} x)}{\nu_p' (k_r^2 - k_r)} \left( 1 - \frac{-i [k_r \cos \theta_0 - \nu_p] x_0}{\nu_p' (k_r^2 - k_r)} \right)
\]

\[
+ \frac{i(k_r \cos \theta_0 + \nu_p y)}{\nu_p (k_r^2 - k_r)} \left( 1 - \frac{-i [k_r \sin \theta_0 - \nu_p] y_0}{\nu_p (k_r \sin \theta_0 + \nu_p)} \right)
\]

\[
+ \sqrt{\frac{2\pi}{k_d r}} \frac{i(k_d r - r_0)}{\sqrt{\frac{k_d r}{k_r \cos \theta_0 - k_r \cos \theta})(k_r \sin \theta_0 - k_r \sin \theta)} - 2 \frac{\alpha'}{\alpha'} + O\left( \frac{\epsilon}{\sqrt{r - r_0}} \right)
\]

(5.31)

It can be seen that the first term of the above expression will cancel the pole contribution from the integral (5.27). Substituting the results (5.27), (5.29), (5.30) and (5.31) into (5.26) dividing the resulting expression by
\[ e^{-ik_v(x_0 \cos \theta_0 + y_0 \sin \theta_0)} \], (the amplitude of the incident wave at the wedge faces) and letting \( x_0, y_0 \rightarrow -\infty \) one finally obtains for the field in the first quadrant \( 0 < \theta_0 < \pi/2 \), \( 0 < \theta < \pi/2 \).

For \( \cos \theta_0 > \delta \cos \theta \),

\[ E_z(x, y) \sim \frac{(k \cos \theta_0 + y_0)}{2} e^{i(y_0'x + k \cos \theta_0 y_0)} \]

\[ + \frac{i e^{i(kd + \pi/2)}}{2 \sqrt{2\pi kd r^3}} \frac{2 \Omega'' F(\Omega'')}{(k \cos \theta_0 - \sqrt{k^2 - k_0^2 \cos^2 \theta_0})(k \sin \theta_0 - k_0 \sin \theta)} + O[(1-n^2)^2], \]

(5.32)

For \( \sqrt{\delta^2 - \sin^2 \theta_0} > \delta \cos \theta > \cos \theta_0 \),

\[ E_z(x, y) \sim \frac{(k \sin \theta_0 + y_0)}{2} e^{i(k \cos \theta_0 x + y_0 y_0)} \]

\[ + \frac{i e^{i(kd + \pi/2)}}{2 \sqrt{2\pi kd r^3}} \frac{k^2(1-n^2) - 2 \Omega'' F(\Omega'')}{2(k \sin \theta_0 - \sqrt{k^2 - k_0^2 \cos^2 \theta_0})(k \cos \theta_0 - k_0 \cos \theta)} \]

\[ + \frac{i e^{i(kd + \pi/2)}}{2 \sqrt{2\pi kd r^3}} \frac{(k \sin \theta_0 - \sqrt{k^2 - k_0^2 \cos^2 \theta_0})(k \cos \theta_0 - k_0 \cos \theta)}{(k \sin \theta_0 - k_0 \sin \theta)} + O[(n^2 - 1)^2], \]

(5.33)
For $\delta \cos \theta > \sqrt{\delta^2 - \sin^2 \theta}$,

$$E_z(x, y) = \frac{(k_v \sin \theta_o + y)}{2\nu} e^{i(k_r \cos \theta x + y)}$$

$$+ \frac{e^{i(k_r \cos \theta x + y)}}{2\sqrt{2\pi k_r \nu}} (k_r(1-\nu) - 2\alpha \mathcal{E}(-\alpha'))$$

Thus the field in the dielectric wedge has been determined.

Assuming the imaginary part of the propagation constants $k_d, k_v$ is very small, i.e. $k_v \sim |k_v|$, $k_d \sim |k_d|$ then the inequalities (5.28) can be written as

$$k_v \cos \theta_o > k_d \cos \theta > 0$$

$$\sqrt{k_d^2 - k_v^2 \sin^2 \theta_o} > k_d \cos \theta > k_v \cos \theta_o$$ (5.35)

$$k_d > k_d \cos \theta > \sqrt{k_d^2 - k_v^2 \sin^2 \theta_o}$$

which correspond to Snell's law for the refraction at the two faces of the wedge.

The geometrical optics terms in the expressions (5.32) to (5.34) represent the refracted rays and although the amplitude coefficients do not seem to be the Fresnel coefficients they do agree with the Fresnel coefficients up to the order $O[\nu^2 - 1]$. This can easily be seen by noting that
\[
\nu_p = k_v \cos \theta_0 \left[ 1 + \frac{(n^2-1)}{2 \cos^2 \theta_0} \right] + \mathcal{O}(n^2-1)^2, \\
\nu_q = k_v \sin \theta_0 \left[ 1 + \frac{(n^2-1)}{2 \sin^2 \theta_0} \right] + \mathcal{O}(n^2-1)^2, \\
\]
which gives

\[
\frac{k_v \cos \theta_0 + \nu_p}{2 \nu_p} = 1 + \frac{(1-n^2)}{4 \cos^2 \theta_0} + \mathcal{O}(n^2-1)^2, \quad (5.36) \\
\frac{k_v \sin \theta_0 + \nu_q}{2 \nu_q} = 1 + \frac{(1-n^2)}{4 \sin^2 \theta_0} + \mathcal{O}(n^2-1)^2. \quad (5.37)
\]

Expanding the appropriate Fresnel coefficients, i.e.

\[
\frac{2k_v \cos \theta_0}{k_v \cos \theta_0 + \nu_p} = 1 - \frac{(n^2-1)}{4 \cos^2 \theta_0} + \mathcal{O}(n^2-1)^2, \quad (5.38) \\
\frac{2k_v \sin \theta_0}{k_v \sin \theta_0 + \nu_q} = 1 - \frac{(n^2-1)}{4 \sin^2 \theta_0} + \mathcal{O}(n^2-1)^2, \quad (5.39)
\]

it is seen that up to the order \((n^2-1)\) these agree with (5.36) and (5.37).

Provided the argument of the Fresnel integrals is not small the result

\[
F(Q) \sim \frac{i}{2Q} + \mathcal{O}(Q^{-2}), \quad |Q| \to \infty,
\]
can be used and the fields in the entire \((x,y)\) plane can be put in tabular form, see table I and figure 57.

**Figure 57**

**TABLE I**

<table>
<thead>
<tr>
<th>REGION</th>
<th>FIELD ( E_x(x,y) = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1)) ( k_d &gt; k_d \cos \theta &gt; \sqrt{k_d^2 - k_v^2 \sin^2 \theta_o} )</td>
<td>( T_1 + D_d )</td>
</tr>
<tr>
<td>((2)) ( \sqrt{k_d^2 - k_v^2 \sin^2 \theta_o} &gt; k_d \cos \theta &gt; k_v \cos \theta_o )</td>
<td>( T_1 + T_2 + D_d )</td>
</tr>
<tr>
<td>((3)) ( k_v \cos \theta_o &gt; k_d \cos \theta &gt; 0 )</td>
<td>( T_2 + D_d )</td>
</tr>
<tr>
<td>((4)) ( \pi/2 &lt; \theta &lt; \pi - \theta_o )</td>
<td>( I + R_1 + D_v )</td>
</tr>
<tr>
<td>((5)) ( \pi - \theta_o &lt; \theta &lt; 2\pi - \theta_o )</td>
<td>( I + D_v )</td>
</tr>
<tr>
<td>((6)) ( 2\pi - \theta_o &lt; \theta &lt; 2\pi )</td>
<td>( I + R_2 + D_v )</td>
</tr>
</tbody>
</table>
where

\[ T_1 = \frac{(\sin \theta_0 + \sqrt{n^2 - \cos^2 \theta_0})}{2 \sqrt{n^2 - \cos^2 \theta_0}} e^{ikr(x \cos \theta_0 + \sqrt{n^2 - \cos^2 \theta_0} y)} \]

\[ T_2 = \frac{(\cos \theta_0 + \sqrt{n^2 - \sin^2 \theta_0})}{2 \sqrt{n^2 - \sin^2 \theta_0}} e^{ikr(x \sqrt{n^2 - \sin^2 \theta_0} + \sin \theta_0 y)} \]

\[ R_1 = \frac{(1-n^2)}{1} e^{-ikr(x \cos \theta_0 - y \sin \theta_0)} \]

\[ R_2 = \frac{(1-n^2)}{1} e^{ikr(x \cos \theta_0 - y \sin \theta_0)} \]

\[ D_d = \frac{\hat{e} e^{ikr(x+\pi/4)}}{2(1-n^2)} \frac{(1-n^2)}{2 \sqrt{2\pi k r^4} (k r \cos \theta_0 - k d \cos \theta)(k r \sin \theta_0 - k d \sin \theta)} \]

\[ D_v = \frac{\hat{e} e^{ikr(x+\pi/4)}}{2(1-n^2)} \frac{(1-n^2)}{2 \sqrt{2\pi k r^4} (\cos \theta_0 - \cos \theta)(\sin \theta_0 - \sin \theta)} \]

\[ I = e^{ikr(x \cos \theta_0 + y \sin \theta_0)} \]

**Field at the wedge tip**

In the expression (5.4) putting \( x = 0, y = 0 \), the field at the wedge tip is given by the expression

\[ F_2(0, 0) = 1 + \frac{i(k^2 - k_0^2)}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{i\left[(k r \cos \theta y + \sqrt{k_0^2 - y^2} + k r \sin \theta) y^2\right]} dx^2 dy^2 \]

\[ + \alpha \left[(k^2 - k_0^2)^2\right], \]
\[ I = 1 + \frac{i(k_d^2 - k_v^2)}{4\pi} \int_0^\infty \frac{dv}{\sqrt{k_v^2 - v^2} (\nu + k_v \cos \theta_0) (\sqrt{k_v^2 - v^2} + k_r \sin \theta_0)}, \tag{5.40} \]

where

\[ I = -\int_0^\infty \frac{dv}{\sqrt{k_v^2 - v^2} (\nu + k_v \cos \theta_0) (\sqrt{k_v^2 - v^2} + k_r \sin \theta_0)} \]

\[ = \int_0^\infty \frac{(\sqrt{k_v^2 - v^2} - k_r \sin \theta_0)}{\sqrt{k_v^2 - v^2} (\nu^2 - k_v^2 \cos^2 \theta_0)(\nu + k_v \cos \theta_0)} \frac{dv}{\sqrt{k_v^2 - v^2} (\nu^2 - k_v^2 \cos^2 \theta_0)(\nu + k_v \cos \theta_0)} \]

\[ = \int_0^\infty \frac{dv}{\nu^2 - k_v^2 \cos^2 \theta_0)(\nu + k_v \cos \theta_0)} - k_r \sin \theta_0 \int_0^\infty \frac{dv}{\nu^2 - k_v^2 \cos^2 \theta_0)(\nu + k_v \cos \theta_0)} \]

\[ \tag{5.41} \]

The location of the poles and branch points are shown in figure 58 below.
The first integral of the expression (5.41) has no branch points and a straightforward application of the Cauchy residue theorem gives

$$\int_{-\infty}^{\infty} \frac{dy}{(y^2 - k_v^2 \cos^2 \theta_o)(y + k_v \cos \theta_o)} = \frac{\pi i}{2k_v^2 \cos^2 \theta_o}.$$ \hspace{1cm} (5.42)

The remaining integral of the expression (5.41) is now evaluated, i.e.

$$I_1 = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{k_v^2 - y^2}(y - k_v \cos \theta_o)(y + k_v \cos \theta_o)^2}.$$ \hspace{1cm} (5.43)

Applying Cauchy's theorem to the integral $I_1$ around the contour $C$ shown in figure 59 gives

$$\oint_C \frac{dy}{\sqrt{k_v^2 - y^2}(y - k_v \cos \theta_o)(y + k_v \cos \theta_o)} = \frac{\pi i}{2k_v^2 \cos^2 \theta_o \sin \theta_o}.$$
The contribution from the infinite circular part of the contour is zero and thus

\[ I_1 = \frac{\pi i}{2 k_v \cos^3 \theta_0 \sin \theta_0} - \frac{\omega e^{i\phi}}{k_v} \int \frac{d\nu}{\sqrt{\nu^2 - k_v^2} (\nu - k_v \cos \theta_0) (\nu + k_v \cos \theta_0)^2} \]

where integration of the second integral on the right hand side is carried out over both sides of the branch cut. Thus

\[ I_1 = \frac{\pi i}{2 k_v \cos^3 \theta_0 \sin \theta_0} + \frac{\omega e^{i\phi}}{2 i k_v} \int \frac{d\nu}{\sqrt{\nu^2 - k_v^2} (\nu - k_v \cos \theta_0) (\nu + k_v \cos \theta_0)^2} \]

\[ = \frac{\pi i}{2 k_v^2 \cos^2 \theta_0 \sin \theta_0} + \frac{\omega e^{i\phi}}{2 i} \int \frac{d\nu}{\sqrt{\nu^2 - k_v^2} (\nu - k_v \cos \theta_0) (\nu + k_v \cos \theta_0)^2} \]

\[ = \frac{\pi i}{2 k_v^2 \cos^2 \theta_0 \sin \theta_0} + \frac{\omega e^{i\phi}}{2 i} \int \frac{d\nu}{\sqrt{\nu^2 - k_v^2} \left( \frac{1}{2 k_v \cos \theta_0 (\nu - k_v \cos \theta_0)} - \frac{1}{2 k_v \cos \theta_0 (\nu + k_v \cos \theta_0)^2} \right)} \]
The first integral of the latter expression can be evaluated by using the standard result

\[
\int \frac{dv}{(\nu^2 - a^2)\sqrt{\nu^2 - 1}} = \frac{1}{a\sqrt{1-a^2}} \sin^{-1}\left(\frac{\sqrt{\nu^2 - 1}}{\sqrt{\nu^2 - a^2}}\right), \quad a^2 < 1,
\]

(5.45)

see Pettit Bois [20] page 48, last formula on page.

The second integral can be put into standard form by letting \( \nu = \sinh x \) which gives

\[
\int_{\nu}^{\infty} \frac{dv}{\sqrt{\nu^2 - 1}} = \int_{0}^{\infty} \frac{dx}{(\sinh x + \cos \Theta_0)^2} = \frac{1}{\sin^2 \Theta_0} - \frac{\Theta_0 \cos \Theta_0}{\sin^2 \Theta_0}
\]

(5.46)

see Gradshteyn and Ryzhik [21] page 345 sec. 3.514 formula No. 3. Hence after substituting the limits into (5.45) and using \( \cos^{-1}x + \sin^{-1}x = \pi/2 \) (5.44) becomes

\[
I_1 = \frac{\pi i k}{2k' \cos^2 \Theta_0 \sin \Theta_0} + \frac{1}{i k' \cos \Theta_0} \left\{ \frac{(\pi x_0 - \Theta_0)}{\cos \Theta_0 \sin \Theta_0} \right. \\
- \left. \frac{1}{\sin^2 \Theta_0} + \frac{\Theta_0 \cos \Theta_0}{\sin^2 \Theta_0} \right\}
\]

(5.47)
Substituting (5.42) and (5.47) into (5.41) gives
\[
I = -\frac{1}{ik} \left[ \left( \frac{\theta_0}{\sin^2 \theta_0} - \frac{\cos \theta_0}{\sin \theta_0} \right) + \left( \frac{\pi/2-\theta_0}{\cos^2 \theta_0} - \frac{\sin \theta_0}{\cos \theta_0} \right) \right],
\]
and therefore
\[
E_z(0,0) = \left[ 1 - \frac{(n^2-1)}{4\pi} \left( \left( \frac{\theta_0}{\sin \theta_0} - \frac{\cos \theta_0}{\sin \theta_0} \right) + \left( \frac{\pi/2-\theta_0}{\cos \theta_0} - \frac{\sin \theta_0}{\cos \theta_0} \right) \right) \right]
+ O[(n^2-1)^2]. \tag{5.48}
\]

Kraut and Lehman [11] only obtain the field at the wedge tip in explicit form and this they leave as the following integral
\[
E_z(0,0) = \left[ 1 - \frac{(n^2-1)}{2\pi} \int_0^\infty \frac{\xi d\xi}{(1+\xi^2)^2} \left( \frac{\sin \theta_0}{\xi^2 + \cos \theta_0 \xi} + \frac{\cos \theta_0}{\xi^2 + \sin \theta_0 \xi} \right) \right]
+ O[(n^2-1)^2]. \tag{5.49}
\]

In fact this integral can be evaluated by making the substitution \( \xi^2 = x \) and using the results from Gradshteyn and Ryzhik [21] page 78 sec. 2.246 formula No. 4. On carrying out this (5.49) is found to be equal to (5.48).

Having found an expression for \( E_z(x,y) \) up to the order \( O(n^2-1) \) the next obvious step is to use the iteration scheme (4.39) to find the expression for \( E_z(x,y) \) up to the order \( O(n^2-1)^2 \) and thereafter up to \( O(n^2-1)^m \). However, although integral expressions for \( E_z(x,y) \) can be obtained the explicit
evaluation, even for the terms $O(n^2-1)^2$, are quite difficult if not impossible. To illustrate this point an attempt will be made to evaluate the field $E_z(x,y)$ in the third quadrant up to the order $O(n^2-1)^2$. Thus substituting the first order approximation for $E_z(x,y)$ for the third quadrant (given by the expression in region (5) of Table I) into the integral expression (5.1) where $E^i_z$ is replaced by the first order field expression in region (5) gives, noting that $y' > y$

$$E^i_z(x,y) = E^i_z(x,y) + i\frac{k^2 - k^2_r}{4\pi} \int_{-\infty}^{\infty} \left[ \frac{e^{-i(\nu_x + \sqrt{k^2_r - \nu^2} y)}}{\sqrt{k^2_r - \nu^2}} \right] dv$$

where $x' = r' \cos \theta'$, $y' = r' \sin \theta'$.

The first integral has already been evaluated in determining the first order approximation and thus

$$E_z(x,y) = E^i_z(x,y) + \frac{i(k_r r + i\eta_k)}{2\sqrt{2\pi k} r'(\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)}

- \frac{i(n^2-1)^2 k^2 r \sin \eta_k}{8\pi \sqrt{2\pi k}} \int_{-\infty}^{\infty} \left[ \frac{e^{-i(\nu_x + \sqrt{k^2_r - \nu^2} y)}}{\sqrt{k^2_r - \nu^2}} \right] dv \int_{-\infty}^{\infty} \frac{e^{i(k_r r' + \nu x' + \sqrt{k^2_r - \nu^2} y')}}{\sqrt{\nu^2 - \nu'^2}} \right] dx'dy'$$

$$+ O[(n^2-1)^3] .$$
Replacing \( x', y' \), by their equivalent polar coordinates in the above integral gives

\[
E_z(x, y) = E_z^i(x, y) = \frac{\epsilon i(k_0 r + \eta i)}{2\sqrt{2\pi r'}(\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)}
\]

\[
- \frac{i(n^2-1)^2 k_0^2 e^{i\pi n}}{16\pi \sqrt{2k_0}} \int_{-\infty}^{\infty} \frac{e^{i(\nu x - r_x' - \nu^2 y)}}{\sqrt{k_0^2 - \nu^2}} d\nu \int_0^{\theta_0} \frac{\epsilon_r' (k_0 + \nu \cos \theta')}{(\sin \theta_0 - \sin \theta')(\cos \theta_0 - \cos \theta')} \, d\theta' .
\]

The \( r' \) integration can be carried out directly, convergence being assured since \( \text{Im} k_0 > 0 \), \( \text{Im} k_0 - \nu > 0 \) and \( 0 < \nu' \leq \pi/2 \). Thus

\[
E_z(x, y) = E_z(x, y) = \frac{\epsilon i(k_0 r + \eta i)}{2\sqrt{2\pi r'}(\sin \theta_0 - \sin \theta)(\cos \theta_0 - \cos \theta)}
\]

\[
+ \frac{i(n^2-1)^2 k_0^2 e^{i\pi n}}{16\pi \sqrt{2k_0}} \int_{-\infty}^{\infty} \frac{e^{i(\nu x - r_x' - \nu^2 y)}}{\sqrt{k_0^2 - \nu^2}} \, d\nu ,
\]

(5.50)

where

\[
I(\theta_0, \nu) = \int_0^{\theta_0} \frac{d\theta'}{(\sin \theta_0 - \sin \theta')(\cos \theta_0 - \cos \theta')(k_0 - \nu \cos \theta' + \sqrt{k_0^2 - \nu^2 \sin \theta'})^{3/2}}
\]

This integral \( I(\theta_0, \nu) \) does not seem to be tractable.

Re-arranging the order of integration in the integral of the
expression (5.50) gives

\[ K = \int_{-\infty}^{\infty} \frac{e^{i(\nu x - \sqrt{k_r^2 - \nu^2} y)}}{\sqrt{k_r^2 - \nu^2}} I(\Theta_0, \nu) d\nu \]

\[ = \begin{cases} \frac{\pi}{2} \frac{d\nu'}{(\sin \Theta_0 - \sin \Theta')(\cos \Theta_0 - \cos \Theta')} \left\{ \frac{\sqrt{\nu + \sqrt{k_r^2 - \nu^2} (k_r - \nu \cos \Theta' + k_r^2 - \nu^2 \sin \Theta')^{3/2}}} {\sqrt{k_r^2 - \nu^2}} \right\} \end{cases} \]

(5.51)

The normal saddle point method is now applied to the integral in the curly bracket. Since \( \text{Im} k_r > 0, \text{Im} \sqrt{k_r^2 - \nu^2} > 0 \),

\( 0 < \Theta' < \pi/2 \), the term \( (k_r - \nu \cos \Theta' + \sqrt{k_r^2 - \nu^2 \sin \Theta'})^{3/2} \) has no singularities in the lower \( \nu \) half plane, except a branch point at \( \nu = -k_r \) which presents no trouble. The saddle point occurs at \( \nu = k_r \cos \Theta \) and since \( \pi < \Theta < \frac{3\pi}{2} \) the saddle point occurs in the lower half plane. Thus in applying the saddle point method no poles are captured. Hence

\[ K \sim \sqrt{\frac{2\pi}{k_r^2}} e^{i(k_r x - \pi/4)} e^{i(k_r x - \pi/4)} I(\Theta_0, k_r \cos \Theta). \]

Therefore \( \pi < \Theta < \frac{3\pi}{2} \).

\[ E_z(x, y) = e^{i(k_r x \cos \Theta_0 + y \sin \Theta_0)} \]

\[ \frac{e^{i(k_r x - \pi/4)}}{2\sqrt{2\pi k_r}} \left\{ \frac{(n^2 - 1)}{(\sin \Theta_0 - \sin \Theta)(\cos \Theta_0 - \cos \Theta)} \right\} \]
\[ + \left( \frac{n^2-1}{4} \right)^2 \frac{\pi}{2} \int_{0}^{\theta'} \frac{d\theta'}{(\sin \theta_0 \sin \theta')(\cos \theta_0 \cos \theta')(\cos(\theta-\theta')-1)} \]

\[ + O[(n^2-1)^3] . \quad (5.52) \]

The expression for \( E_z(x,y) \) in the remaining quadrants of \( S_v \) will differ from the above expression in the geometrical optic terms only. The amplitude of the geometrical optics terms will represent the Fresnel coefficients when expanded out to the order \( O[(n^2-1)^2] \). The expression with the curly brackets in (5.52) represents the diffracted field in the entire region \( S_v \). Clearly higher order approximations for \( E_z(x,y) \) will involve far more difficult multiple integrations with respect to \( \theta' \), which could only be handled effectively by numerical methods.
CHAPTER 6

Consider two contiguous dielectric wedges occupying the regions defined by, \( S_v : 0 \leq r < \infty, \mid z \mid < \infty, -\beta \leq \theta \leq \beta, \mid \beta \mid > 0 \) and \( S_d : 0 \leq r < \infty, \mid z \mid < \infty, -\alpha \leq \phi \leq \alpha, \mid \alpha \mid > 0 \); of material constants \( \mu_v, \varepsilon_v, \sigma_v \) and \( \mu_d, \varepsilon_d, \sigma_d \) respectively. The propagation constant in the region \( S_v \) is assumed to be \( k \), and in the region \( S_d \) is \( kn \) where \( n > 1 \). Two sets of cylindrical coordinates are used, namely \((r, \theta, z)\) and \((r, \phi, z)\); the former refers to points in \( S_v \) and the latter to points within \( S_d \). In accordance with the convention of measuring the angles shown in the figure it follows that \(-\beta \leq \theta \leq \beta, -\alpha \leq \phi \leq \alpha, \) and \( \alpha + \beta = \pi \). A time variation \( e^{j\omega t} \) is assumed throughout the work which follows, and the dielectrics will have slight conductivity, i.e. \( \text{Im} k > 0 \). An incident plane wave

\[
u_i = e^{-ikr \cos(\theta - \theta_0)} \quad -\beta < \theta_0 < \beta
\]

in the region \( S_v \) is assumed to be either \( E_z \) or \( H_z \) polarised so that the three dimensional problem reduces to a two dimensional scalar problem.
The scalar problem can be formulated thus:

(I) \((v_d^2 + k^2 n^2)u_d = 0\) \(-\alpha \leq \phi \leq \alpha\),

(6.1)

where \(v_d^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\).

\((v_v^2 + k^2)(u_0 - u_1) = 0\) \(-\beta \leq \theta \leq \beta\)

(6.2)

or \((v_v^2 + k^2)u = 0\),

where \(v_v^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\).

\(u_d\) is the total scalar field in \(S_d\) and \(u_v\) is the total scalar field in \(S_v\). The field in \(S_v\) is represented in terms of the incident plane wave thus

\(u_v = e^{-ikr \cos(\theta - \theta_0)} + u\).

(6.3)

(II)

\(u_d\) and \(u\) must satisfy the Sommerfeld radiation condition for \(\text{Im} k = 0\), i.e.

\[\lim_{r \to \infty} r^\frac{3}{2} \left( \frac{\partial u_d}{\partial r} - ikn u_d \right) = 0\] in \(S_d\),

\[\lim_{r \to \infty} r^\frac{3}{2} \left( \frac{\partial u}{\partial r} - iku \right) = 0\] in \(S_v\).

(6.4)

For \(\text{Im} k > 0\) the fields must decay as \(r \to \infty\).
(III)

\( v, u = 0(1) \), \( \nabla v = 0(r^{-\gamma_v}) \), \( 0 < \gamma_v < 1 \); \( (6.5) \)

(IV)

The fields must satisfy the following boundary condition

\[
\begin{align*}
\frac{\partial u_v}{\partial \theta} &\bigg|_{\theta=\beta} = \frac{\partial u_d}{\partial \theta} \bigg|_{\theta=-\alpha}, & \frac{\partial u_v}{\partial \phi} &\bigg|_{\phi=\beta} = \tau \frac{\partial u_d}{\partial \phi} \bigg|_{\phi=-\alpha}, \\
\frac{\partial u_v}{\partial \theta} &\bigg|_{\theta=-\beta} = \frac{\partial u_d}{\partial \theta} \bigg|_{\theta=\alpha}, & \frac{\partial u_v}{\partial \phi} &\bigg|_{\phi=-\beta} = \tau \frac{\partial u_d}{\partial \phi} \bigg|_{\phi=\alpha},
\end{align*}
\]

(6.6)

where

\[
\tau = \begin{cases} \frac{\mu_v}{\mu_d} & \text{for } u_i = \begin{bmatrix} E_z \\ H_z \end{bmatrix} \text{ polarisation.} \\
\frac{\varepsilon_v}{\varepsilon_d} & \text{for } u_i = \begin{bmatrix} E_z \\ H_z \end{bmatrix} \text{ polarisation.} \end{cases}
\]

The scalar problem set out in (I) - (IV) is non-separable and thus an integral solution is considered. Clearly a solution to the equations (6.1) and (6.2) and satisfying (6.4) and (6.5) is

\[
\begin{align*}
u_v &= \int_{-i\infty}^{i\infty} (c(v)\cos \psi+d(v)\sin \psi)H_{1}^{(1)}(kr)dv, \\
u &= \int_{-i\infty}^{i\infty} (a(v)\cos \psi+b(v)\sin \psi)H_{1}^{(1)}(kr)dv.
\end{align*}
\]

(6.7)
Lebedev [22], however this is not in the most convenient form for the manipulation which ensues on substituting (6.7) into the boundary conditions (6.6) to calculate $a(v), b(v), c(v)$ and $d(v)$. A technique originally used by Oberhettinger [23] and successfully used by a number of authors is now introduced, namely $k$ is replaced by $ik$ in the above formulated problem. This is equivalent to solving the analogous diffusion problem. After solving the diffusion problem and representing this solution in a convenient form $\kappa$ is replaced by $-ik$ to give the solution to the original diffraction problem.

Thus letting $k$ be replaced by $ik$ the equations (6.1) and (6.2) reduce to

\[
\begin{align*}
(v^2 - k^2 n^2)u_d &= 0 \quad -\alpha \leq \phi \leq \alpha, \\
(v^2 - k^2)u &= 0 \quad -\beta \leq \theta \leq \beta,
\end{align*}
\]

\[u_v = e^{ikr \cos(\theta - \theta_0)} + u.
\]

The boundary conditions (6.6) remain unchanged. At infinity the radiation condition (6.4) is now replaced by the requirement that the field decays exponentially to zero; and the edge condition remains the same as before. Clearly a solution of this diffusion problem is given by

\[
\begin{align*}
u_v &= \frac{2}{n} \int_0^\infty \{A(v) e^{ikr \cos(\theta - \theta_0)} + B(v) e^{ikr \sin(\theta - \theta_0)} \} K_{1/2}(nr) dv, \\
u_d &= \frac{2}{n} \int_0^\infty \{C(v) e^{ikr \cos(\theta - \theta_0)} + D(v) e^{ikr \sin(\theta - \theta_0)} \} K_{1/2}(nr) dv,
\end{align*}
\]

(6.9)
where in (6.9) the result, Gradshteyn and Ryzhik [21] page 773 sec. 6.795, formula No. 1,

\[
K_r \cos(\theta - \theta_0) = \frac{2}{\pi} \int_0^\infty \text{chv}(\pi - \theta - \theta_0) K_{1\nu}(kr) dv, \quad (6.11)
\]

has been used; \( K_{1\nu}(kr) \) is the modified Bessel function of the third kind of imaginary order.

Substituting the expressions (6.9) and (6.10) into the boundary conditions (6.6) and using the fact that

\[
\frac{\partial u}{\partial \theta} |_{\theta > \theta_0} = \frac{2}{\pi} \int_0^\infty \{-\text{chv}(\pi - \theta_0 + \theta) + C(\nu)\text{chv} + D(\nu)\text{chv}\theta\} K_{1\nu}(kr) dv ,
\]

\[
\frac{\partial u}{\partial \theta} |_{\theta < \theta_0} = \frac{2}{\pi} \int_0^\infty \{\text{chv}(\pi - \theta_0 + \theta) + C(\nu)\text{chv} + D(\nu)\text{chv}\theta\} K_{1\nu}(kr) dv ,
\]

gives

\[
\int_0^\infty \{\text{chv}(\pi - \theta_0 + \theta_0) + C(\nu)\text{chv} + D(\nu)\text{chv}\theta\} K_{1\nu}(kr) dv
\]

\[
= \int_0^\infty \{A(\nu)\text{chv} - B(\nu)\text{shv}\} K_{1\nu}(kr) dv , \quad (6.12)
\]

\[
\int_0^\infty \{\text{chv}(\pi - \theta_0 + \theta_0) + C(\nu)\text{chv} - D(\nu)\text{shv}\theta\} K_{1\nu}(kr) dv
\]

\[
= \int_0^\infty \{A(\nu)chv + B(\nu)shv\} K_{1\nu}(kr) dv , \quad (6.13)
\]

\[
\int_0^\infty \{-\text{shv}(\pi - \theta_0 + \theta_0) + C(\nu)\text{shv} + D(\nu)\text{chv}\theta\} v K_{1\nu}(kr) dv
\]

\[
= \int_0^\infty \{-A(\nu)shv + B(\nu)chv\} v K_{1\nu}(kr) dv , \quad (6.14)
\]
\[
\int_0^\infty \{\text{shv}(\pi-\beta_{\cdot \circ})-C(\nu)\text{shv}\beta+D(\nu)\text{chv}\beta\}vK_{i\nu}(kr)dv \\
= \tau \int_0^\infty \{A(\nu)\text{shv}\alpha+B(\nu)\text{chv}\alpha\}vK_{i\nu}(knr)dv . \quad (6.15)
\]

Multiplying (6.12), (6.13), (6.14) and (6.15) by \(\frac{K_{i\sigma}(knr)}{r}\) and integrating over \(r\) from 0 to \(\infty\), and then using the well-known orthogonality property

\[
\int_0^\infty K_{i\nu}(br)K_{i\sigma}(br) \frac{dr}{r} = \frac{\pi^2\delta(\sigma,\nu)}{2\cosh\sigma\pi} , \quad (6.16)
\]

where \(\delta(\sigma,\nu)\) is the delta function, gives

\[
A(\sigma)\text{chv}\alpha-B(\sigma)\text{shc}\alpha = \frac{2\cosh\sigma\pi}{\pi^2} \int_0^\infty (\text{chv}(\pi-\beta_{\cdot \circ})+C(\nu)\text{chv}\beta \\
+ D(\nu)\text{shv}\beta)K_{i\nu}(kr)K_{i\sigma}(knr) \frac{dr}{r} dv , \quad (6.17)
\]

\[
A(\sigma)\text{chv}\alpha+B(\sigma)\text{shc}\alpha = \frac{2\cosh\sigma\pi}{\pi^2} \int_0^\infty (\text{chv}(\pi-\beta_{\cdot \circ})+C(\nu)\text{chv}\beta \\
- D(\nu)\text{shv}\beta)K_{i\nu}(kr)K_{i\sigma}(knr) \frac{dr}{r} dv , \quad (6.18)
\]

\[
-A(\sigma)\text{shc}\alpha+B(\sigma)\text{chv}\alpha = \frac{2\cosh\sigma\pi}{\pi^2} \int_0^\infty (-\text{shv}(\pi-\beta_{\cdot \circ})+C(\nu)\text{shv}\beta \\
+ D(\nu)\text{chv}\beta)vK_{i\nu}(kr)K_{i\sigma}(knr) \frac{dr}{r} dv , \quad (6.19)
\]

\[
A(\sigma)\text{shc}\alpha+B(\sigma)\text{chv}\alpha = \frac{2\cosh\sigma\pi}{\pi^2} \int_0^\infty (\text{shv}(\pi-\beta_{\cdot \circ})-C(\nu)\text{shv}\beta \\
+ D(\nu)\text{chv}\beta)vK_{i\nu}(kr)K_{i\sigma}(knr) \frac{dr}{r} dv . \quad (6.20)
\]
From (6.17) and (6.18)

\[
A(\sigma) = \frac{2\text{sh}(\pi)}{\pi^2 \text{sh} \alpha} \int_0^\infty (\text{sh} \alpha \text{csh} \theta + 0(\nu) \text{ch} \beta) K_{1v}(kr) K_{1\sigma}(\text{kr}) \frac{dr}{r} \, d\nu ,
\]

(6.21)

\[
B(\sigma) = \frac{2\text{sh}(\pi)}{\pi^2 \text{sh} \alpha} \int_0^\infty (-\text{sh} \alpha \text{csh} \theta - D(\nu) \text{sh} \beta) K_{1v}(kr) K_{1\sigma}(\text{kr}) \frac{dr}{r} \, d\nu .
\]

(6.22)

From (6.19) and (6.20)

\[
A(\sigma) = \frac{2\text{sh}(\pi)}{\pi^2 \text{sh} \alpha} \int_0^\infty (\text{sh} \alpha \text{csh} \theta - 0(\nu) \text{sh} \beta) v K_{1v}(kr) K_{1\sigma}(\text{kr}) \frac{dr}{r} \, d\nu ,
\]

(6.23)

\[
B(\sigma) = \frac{2\text{sh}(\pi)}{\pi^2 \text{sh} \alpha} \int_0^\infty (-\text{sh} \alpha \text{csh} \theta + D(\nu) \text{ch} \beta) v K_{1v}(kr) K_{1\sigma}(\text{kr}) \frac{dr}{r} \, d\nu .
\]

(6.24)

Equating (6.21) to (6.23) and (6.22) to (6.24) gives

\[
\int_0^\infty \left[ (\text{sh} \alpha \text{sh} \alpha - \text{sh} \alpha \text{sh} \alpha) \text{csh} \theta + 0(\nu) [\text{sh} \beta \text{sh} \alpha ]
\right.
\]
\[
+ \sigma \text{ch} \beta \text{sh} \alpha] K_{1v}(kr) K_{1\sigma}(\text{kr}) \frac{dr}{r} \, d\nu = 0 ,
\]

(6.25)

and

\[
\int_0^\infty \left[ (\text{sh} \alpha \text{sh} \alpha - \text{sh} \alpha \text{sh} \alpha) \text{sh} \theta + D(\nu) [\text{sh} \beta \text{sh} \beta]
\right.
\]
\[
+ \sigma \text{ch} \beta \text{sh} \beta] K_{1v}(kr) K_{1\sigma}(\text{kr}) \frac{dr}{r} \, d\nu = 0 .
\]

(6.26)

By an exactly analogous procedure, if (6.12), (6.13), (6.14)
and \((6.15)\) are multiplied by \(\frac{K_{10}(kr)}{r}\) and integrated over \(r\) from 0 to \(\infty\) on using \((6.16)\) one obtains eventually

\[
\frac{2}{\pi^2} \int_0^\infty A(v) [\text{chv} \cos \beta + v \text{shv} \sin \beta] K_{i\nu}(knr) K_{10}(kr) \frac{dr}{r} dv
= \text{ch} \theta_0 ,
\]

(6.27)

and

\[
\frac{2}{\pi^2} \int_0^\infty B(v) [\text{shv} \cos \beta + v \text{chv} \sin \beta] K_{i\nu}(knr) K_{10}(kr) \frac{dr}{r} dv
= -\text{sh} \theta_0 .
\]

(6.28)

The equations \((6.25)\), \((6.26)\), \((6.27)\) and \((6.28)\) will now be reduced to Fredholm integral equations of the second kind by exploiting various operational properties of Bessel functions of complex order. In particular the following remarkable result derived by Smith [16], and proved in appendix 6A, will be used

\[
\int_0^\infty K_{i\nu}(knr) K_{10}(knr) \frac{dr}{r} = \frac{\pi^2 \cos(\sigma \text{log} n)}{2 \sigma \text{sh} \sigma \pi} \left[ \delta(\sigma \nu + \sigma \nu) + \delta(\sigma \nu - \sigma \nu) \right]
\]

\[
+ \left(1 - \frac{1}{\nu^2}\right) \frac{\pi \nu^{-i\sigma}}{2} \Gamma\left(1 + i(\nu \sigma \nu) \right) \frac{1 + i(\sigma \nu)}{2} \frac{2 \Gamma(1 - i(\nu \sigma))}{2 \nu \pi \nu - 2 \nu \pi \sigma}
\]

(6.29)

\(\sigma > 0, \ n > 1\).

For the particular case which will be considered here
\(\sigma > 0, \ \nu > 0\) and thus \((6.29)\) will reduce to
Applying the result (6.30) to (6.25) and (6.26) gives

\[ C(\sigma) \cos(\sigma \log n) = \frac{(1 - \gamma) \text{sh} \sigma \alpha \text{sh} \alpha \text{ch} \sigma \theta_0 \cos(\sigma \log n)}{\text{sh} \sigma \beta \text{ch} \sigma \alpha + \text{ch} \sigma \beta \text{sh} \sigma \alpha} \]

\[ + \frac{(1 - \gamma^2) \text{sh} \sigma \theta}{2(\text{sh} \sigma \alpha \text{ch} \sigma \alpha + \text{ch} \sigma \beta \text{sh} \sigma \alpha)} \int_0^\infty \left[ \text{ch} \text{ch} \text{ch} \alpha \beta - \text{sh} \text{sh} \alpha \beta \right] \text{ch} \sigma \theta_0 \]

\[ + C(n) \left[ \text{sh} \text{sh} \alpha \beta + \sigma \text{ch} \text{ch} \alpha \beta \right] \left\{ L(\sigma, \nu) \right\} d\nu, \quad (6.31) \]

and

\[ D(\sigma) \cos(\sigma \log n) = \frac{(1 - \gamma) \text{ch} \sigma \alpha \text{sh} \alpha \text{sh} \sigma \theta_0 \cos(\sigma \log n)}{\text{sh} \sigma \beta \text{ch} \sigma \alpha + \gamma \text{ch} \sigma \beta \text{sh} \sigma \alpha} \]

\[ + \frac{(1 - \gamma^2) \text{sh} \sigma \theta}{2(\text{sh} \sigma \alpha \text{ch} \sigma \alpha + \gamma \text{ch} \sigma \beta \text{sh} \sigma \alpha)} \int_0^\infty \left[ \gamma \text{ch} \text{sh} \text{sh} \alpha \beta - \text{sh} \text{sh} \alpha \beta \right] \text{sh} \sigma \theta_0 \]

\[ + D(n) \left[ \text{sh} \text{sh} \alpha \beta + \sigma \gamma \text{ch} \text{ch} \alpha \beta \right] \left\{ L(\sigma, \nu) \right\} d\nu, \quad (6.32) \]

where
Similarly applying the result (6.30) to (6.27) and (6.28) gives

\[ A(\sigma) \cos(\sigma \log n) = \frac{\cosh \Phi \sinh \omega}{\cosh \Phi + \cosh \omega} \]

\[ + \frac{(1 - h^2) \sinh \omega}{2 (\cosh \Phi + \cosh \omega)} \int_0^\infty A(\nu) \left[ \sigma \cosh \Phi + \nu \cosh \omega \cosh \beta \right] M(\nu, \omega) \, d\nu, \]

(6.34)

and

\[ B(\sigma) \cos(\sigma \log n) = \frac{-\sinh \Phi \sinh \omega}{\sinh \Phi + \sinh \omega} \]

\[ + \frac{(1 - h^2) \sinh \omega}{2 (\sinh \Phi + \sinh \omega)} \int_0^\infty B(\nu) \left[ \sigma \sinh \Phi + \nu \cosh \omega \cosh \beta \right] M(\nu, \omega) \, d\nu, \]

(6.35)

where

\[ M(\sigma, \nu) = I(\nu, \sigma). \]

The integral equations (6.31), (6.32), (6.34) and (6.35) are Fredholm integral equations of the second kind; and they are too difficult to solve exactly. Thus some simplifying assumptions are made and a perturbation solution is obtained.
in the form of a Neumann series. For the specific problem of an $E_z$-polarised incident plane wave with $\mu_\nu = \mu_d$, i.e., a perfect dielectric, then $\tau = 1$ and (6.31), (6.32), (6.34) and (6.35) reduce to

$$
A(\sigma) \cos(\sigma \log n) = \frac{\cos \theta_0}{2} \int_0^\infty A(\nu) \left[ \sigma \sin \alpha \sin \beta + \nu \sin \gamma \cos \beta \right] M(\sigma, \nu) d\nu,
$$

(6.37)

$$
B(\sigma) \cos(\sigma \log n) = -\frac{\sin \theta_0}{2} \int_0^\infty B(\nu) \left[ \sigma \sin \alpha \cos \beta + \nu \sin \gamma \sin \beta \right] M(\sigma, \nu) d\nu,
$$

(6.38)

$$
C(\sigma) \cos(\sigma \log n) = \frac{(1-\nu^2)}{2} \int_0^\infty \left[ \sigma \cos \alpha \sin \alpha - \nu \sin \gamma \sin \alpha \right] C(\nu) \cos \theta_0
$$

$$
+ \left[ \nu \sin \gamma \cos \alpha + \sigma \cos \alpha \sin \gamma \right] \left[ \cos \theta_0 \right] \left[ L(\sigma, \nu) \right] \left[ d\nu \right],
$$

(6.39)

$$
D(\sigma) \cos(\sigma \log n) = -\frac{(1-\nu^2)}{2} \int_0^\infty \left[ \sigma \cos \alpha \sin \alpha - \nu \sin \gamma \sin \alpha \right] \sin \theta_0
$$

$$
+ \frac{\nu \sin \gamma \cos \alpha + \sigma \cos \alpha \sin \gamma}{2} \left[ L(\sigma, \nu) \right] \left[ d\nu \right],
$$

(6.40)

On expanding the hypergeometric series which appear in $L(\sigma, \nu)$ and $M(\sigma, \nu)$ it can be seen that $L(\sigma, \nu)$ and $M(\sigma, \nu)$ will
consist of a power series in terms of \((1-1/n^2)\), the expansions being valid for \(|1-1/n^2| < 1\). The perturbation method which will be used will consist of expanding out all the functions in which \(n\) appears in series whose terms involve \((1-1/n^2)\). For rapid convergence of these series \(n \sim 1\). It is also assumed that \(A, B, C,\) and \(D\) can also be expanded in an infinite series of term \((1-1/n^2)\). Thus

\[
L(\sigma, \nu) = \sum_{m=0}^{\infty} L^{(m)}(\sigma, \nu)(1 - \frac{1}{n^2})^m, \quad (6.41)
\]

\[
M(\sigma, \nu) = \sum_{n=0}^{\infty} L^{(n)}(\nu, \sigma)(1 - \frac{1}{n^2})^m = \sum_{n=0}^{\infty} M^{(n)}(\sigma, \nu)(1 - \frac{1}{n^2})^m, \quad (6.42)
\]

\[
A(\sigma) = \sum_{m=0}^{\infty} A^{(m)}(\sigma)(1 - \frac{1}{n^2})^m, \quad B(\sigma) = \sum_{m=0}^{\infty} B^{(m)}(\sigma)(1 - \frac{1}{n^2})^m, \quad (6.43)
\]

\[
C(\sigma) = \sum_{n=0}^{\infty} C^{(n)}(\sigma)(1 - \frac{1}{n^2})^m, \quad D(\sigma) = \sum_{n=0}^{\infty} D^{(n)}(\sigma)(1 - \frac{1}{n^2})^m.
\]

where all the quantities with a suffix \(m\) are independent of \(n\). The following result will also be needed.

\[
\cos(\sigma \log n) = \frac{n^{i\sigma} + n^{-i\sigma}}{2}, \quad \text{and since}
\]

\[
n^{i\sigma} = \frac{(i\sigma)}{2} \, _2F_1 \left(\frac{i\sigma}{2}, 1; 1; 1-1/n^2\right) \quad (6.44)
\]

then

\[
\cos(\sigma \log n) = \frac{1}{2} \left\{ _2F_1 \left(\frac{i\sigma}{2}, 1; 1; 1-1/n^2\right) + _2F_1 \left(\frac{-i\sigma}{2}, 1; 1; 1-1/n^2\right) \right\},
\]

\[
= \sum_{m=0}^{\infty} a_m (1-1/n^2)^m, \quad (6.45)
\]
where \( a_0 = 1, \ a_1 = 0 \).

From the above expansions the following is obtained

\[
A(\sigma) \cos(\sigma \log n) = \sum_{m=0}^{\infty} a_m (1 - \frac{1}{n^2})^m \sum_{m=0}^{\infty} A^{(m)}(\sigma) (1 - \frac{1}{n^2})^m
\]

\[
= \sum_{m=0}^{\infty} \left( \sum_{s=0}^{m} A^{(s)}(\sigma) a_{m-s} \right) (1 - \frac{1}{n^2})^m
\]

and similar results are obtained for \( A \) replaced by \( B, C, \) and \( D \); also

\[
(1 - \frac{1}{n^2}) A(\nu) M(\sigma, \nu) = \sum_{m=0}^{\infty} \left( \sum_{s=0}^{m} A^{(s)}(\nu) M^{(m-s)}(\sigma, \nu) \right) (1 - \frac{1}{n^2})^{m+1}
\]

Using the expressions (6.46) and (6.47) and there analogous expression for \( B, C, \) and \( D \), the equations (6.37) to (6.40) can be written in the following form

\[
\sum_{m=0}^{\infty} \left( \sum_{s=0}^{m} A^{(s)}(\sigma) a_{m-s} \right) (1 - \frac{1}{n^2})^m = \mathrm{ch}_\sigma \theta_0
\]

\[
+ \frac{1}{2} \sum_{m=0}^{\infty} \left\{ \sum_{s=0}^{m} A^{(s)}(\nu) M^{(m-s)}(\sigma, \nu) \chi_{\nu} \chi_{\nu} \chi_{\nu} \right\} (1 - \frac{1}{n^2})^{m+1}
\]

(6.48)

\[
\sum_{m=0}^{\infty} \left( \sum_{s=0}^{m} B^{(s)}(\sigma) a_{m-s} \right) (1 - \frac{1}{n^2})^m = -\mathrm{sh}_\sigma \theta_0
\]

\[
+ \frac{1}{2} \sum_{m=0}^{\infty} \left\{ \sum_{s=0}^{m} B^{(s)}(\nu) M^{(m-s)}(\sigma, \nu) \chi_{\nu} \chi_{\nu} \chi_{\nu} \right\} (1 - \frac{1}{n^2})^{m+1}
\]

(6.49)
\[
\sum_{m=0}^{\infty} \left( \sum_{s=0}^{m} C^{(s)} a_{m-s} \right) (1-1/n^2)^m = \frac{1}{2} \sum_{m=0}^{\infty} \left\{ \left[ \sigma \alpha_{\chi \psi} \chi_{\alpha} - \sigma \nu_{\psi} \chi_{\nu} \chi_{\sigma} \right] \chi \nu_{\theta} L^{(m)}(\sigma, \nu) \right\} d\nu \]
\]
\[
+ \int_{0}^{\infty} \sum_{m=0}^{\infty} \left( \sum_{s=0}^{m} D^{(s)} a_{m-s} \right) (1-1/n^2)^m \frac{1}{2} \sum_{m=0}^{\infty} \left\{ \left[ \sigma \alpha_{\chi \psi} \chi_{\alpha} - \sigma \nu_{\psi} \chi_{\nu} \chi_{\sigma} \right] \chi \nu_{\theta} L^{(m)}(\sigma, \nu) \right\} d\nu \left( 1-1/n^2 \right)^{m+1}.
\]

By equating like powers of \((1-1/n^2)\) on both sides of the equality sign in the expressions (6.48), (6.49), (6.50) and (6.51) an iteration scheme is obtained for finding all the coefficients \(A^{(m)}(\sigma), B^{(m)}(\sigma), C^{(m)}(\sigma)\) and \(D^{(m)}(\sigma)\). Thus the expansion for \(A, B, C,\) and \(D\) can be obtained to any desired order of \((1-1/n^2)\).

Assuming one has calculated \(A^{(m)}, B^{(m)}, C^{(m)},\) and \(D^{(m)}\) to the desired order of accuracy, say \(m = N\); this being theoretically possible although the work involved would be tedious for \(N\) large, then

\[
A(\sigma) = \sum_{m=0}^{N} A^{(m)}(\sigma) (1-1/n^2)^m + \mathcal{O}(1-1/n^2)^{N+1},
\]
\[
B(\sigma) = \sum_{m=0}^{N} B^{(m)}(\sigma) (1-1/n^2)^m + \mathcal{O}(1-1/n^2)^{N+1},
\]
\[
C(\sigma) = \sum_{m=0}^{N} C^{(m)}(\sigma) (1-1/n^2)^m + \mathcal{O}(1-1/n^2)^{N+1},
\]
\[
D(\sigma) = \sum_{m=0}^{N} D^{(m)}(\sigma) (1-1/n^2)^m + \mathcal{O}(1-1/n^2)^{N+1}.
\]
Now substituting the expressions (6.52) into (6.9) and (6.10) gives

\[
\begin{align*}
\psi_{r}(r, \theta) &= e^{2i \kappa \cos(\theta - \theta_0)} + \frac{2}{\pi} \sum_{m=0}^{N} \left(1 - \frac{1}{n^2}\right)^m \int_{0}^{\infty} \left[ \mathcal{C}(v) \sin \theta + \mathcal{D}(v) \cos \theta \right] K_{i \nu}(x r) \, dv \\
&\quad + O(1 - \frac{1}{n^2})^{N+1}, \\
\psi_{d}(r, \phi) &= \frac{2}{\pi} \sum_{m=0}^{N} \left(1 - \frac{1}{n^2}\right)^m \int_{0}^{\infty} \left[ \mathcal{A}(v) \sin \phi + \mathcal{B}(v) \cos \phi \right] K_{i \nu}(x r) \, dv \\
&\quad + O(1 - \frac{1}{n^2})^{N+1}.
\end{align*}
\]

(6.53)

(6.54)

In order that \( \psi_{r} \) and \( \psi_{d} \) given by (6.53) and (6.54) be solutions of the wave equation it is necessary to replace \( \kappa \) by \( -i \kappa \). In doing this (depending on the form of the integrands) it is possible that the resulting expression will be divergent. To avoid this happening one can evaluate the integrals explicitly and then replace \( \kappa \) by \( -i \kappa \). If it is not possible to evaluate the integrals explicitly, which will probably be the case, the contours of integration are changed in such a way that by replacing \( \kappa \) by \( -i \kappa \) the integrals remain convergent. Naturally any poles captured in changing the contours of integration must be accounted for. To illustrate what has just been stated the field inside and outside the wedge up to the order \( (1 - 1/n^2) \) will be determined.

In order to determine the field \( E_{z} \) to the order \( (1 - 1/n^2) \) it is necessary to calculate \( A^{(0)}(c) \) and \( A^{(1)}(c) \) and the corresponding terms for \( B, C, \) and \( D, \) explicitly. Expanding out the expression (6.48) to order \( (1 - 1/n^2) \) gives
\[ A^{(0)}(o) a_0 + (A^{(0)}(o) a_1 + A^{(1)}(o) a_0) (1 - \frac{1}{n^2}) + O(1 - \frac{1}{n^2})^2 \]

\[ = \angle \sigma \theta_0 + \frac{1}{2} \int_0^\infty A^{(0)}(\psi) \left( \sigma \chi \psi \sigma \varphi + \upsilon \psi \chi \sigma \varphi \right) d\psi \left(1 - \frac{1}{n^2}\right) + O(1 - \frac{1}{n^2})^2, \]  

which on equating like powers of \((1 - \frac{1}{n^2})\) gives

\[ A^{(0)}(o) = \angle \sigma \theta_0, \quad (6.56) \]

\[ A^{(1)}(o) = \frac{1}{2} \int_0^\infty \angle \sigma \theta_0 \left[ \sigma \chi \psi \sigma \varphi + \upsilon \psi \chi \sigma \varphi \right] d\psi \quad (6.57) \]

Similarly expanding out \((6.49)\) gives

\[ B^{(0)}(o) = -\angle \sigma \theta_0, \quad (6.58) \]

\[ B^{(1)}(o) = -\frac{1}{2} \int_0^\infty \angle \sigma \theta_0 \left[ \sigma \chi \psi \sigma \varphi + \upsilon \psi \chi \sigma \varphi \right] d\psi \quad (6.59) \]

Expanding out equation \((6.50)\) to order \((1 - \frac{1}{n^2})\) gives

\[ C^{(0)}(o) a_0 + (C^{(0)}(o) a_1 + C^{(1)}(o) a_0) (1 - \frac{1}{n^2}) + O(1 - \frac{1}{n^2})^2 \]

\[ = \frac{1}{2} \left\{ \int_0^\infty \left[ \sigma \chi \psi \sigma \varphi - \upsilon \sigma \chi \psi \sigma \varphi \right] \angle \sigma \theta_0 \right. \]

\[ + C^{(0)}(\psi) \left[ \upsilon \sigma \chi \psi \sigma \varphi + \sigma \chi \psi \sigma \varphi \right] L^{(1)}(\sigma, \psi) d\psi \left(1 - \frac{1}{n^2}\right) + O(1 - \frac{1}{n^2})^2, \]

\[ (6.60) \]

from which
\[ a(0) = 0, \quad (6.61) \]

\[
c^{(1)}(a) = \frac{1}{2} \int_0^\infty \frac{e^{\pm \omega}}{e^{\pm \omega} - e^{\pm \omega}} \, \text{d}v \quad (6.62) \]

Similarly expanding out (6.51) gives

\[ d(0) = 0, \quad (6.63) \]

\[
d^{(1)}(a) = \frac{1}{2} \int_0^\infty \frac{e^{\pm \omega}}{e^{\pm \omega} - e^{\pm \omega}} \, \text{d}v \quad (6.64) \]

To evaluate the Cauchy principle value integrals (6.57), (6.59), (6.62) and (6.64) the following results are used, see appendix 6B,

\[
\int_0^\infty \frac{e^{\pm \omega}}{(e^{\pm \omega} - e^{\pm \omega})} \, \text{d}v = -\frac{\sinh \psi}{\sinh \psi} \cot \psi, \quad \sigma > 0, \ |\psi| < \pi, \quad (6.65) \]

\[
\int_0^\infty \frac{\sinh \psi}{(e^{\pm \omega} - e^{\pm \omega})} \, \text{d}v = -\frac{1}{\sinh \psi} \left( \cos \psi \cosh \psi - \frac{\sinh \psi}{\psi} \right), \quad \sigma > 0, \ |\psi| < \pi, \quad (6.66) \]

In order to apply the results (6.65), the integrals (6.57), (6.59), (6.62) and (6.64) are re-written as

\[
A^{(1)}(a) = \frac{1}{4} \int_0^\infty \frac{[\cosh(\psi + \theta) + \cosh(\psi - \theta)] + \cosh(\psi + \theta) + \cosh(\psi - \theta)]}{(e^{\pm \omega} - e^{\pm \omega})} \, \text{d}v \quad (6.66) \]
\[ B(1)(\sigma) = \frac{1}{4} \int_0^\infty \left[ \sigma \cos(\sigma(\alpha + \theta_0) - \sigma(\alpha - \theta_0)) + \sigma \sin(\sigma(\alpha + \theta_0) - \sigma(\alpha - \theta_0)) \right] d\sigma, \]
\[ \text{chim} - \text{chim} \tag{6.67} \]

\[ C(1)(\sigma) = \frac{1}{4} \int_0^\infty \left[ \sigma \sin(\sigma(\alpha + \theta_0) + \sigma(\alpha - \theta_0)) - \sigma \cos(\sigma(\alpha + \theta_0) + \sigma(\alpha - \theta_0)) \right] d\sigma, \]
\[ \text{chirn} - \text{chirn} \tag{6.68} \]

\[ D(1)(\sigma) = \frac{1}{4} \int_0^\infty \left[ \sigma \cos(\sigma(\alpha + \theta_0) - \sigma(\alpha - \theta_0)) - \sigma \sin(\sigma(\alpha + \theta_0) - \sigma(\alpha - \theta_0)) \right] d\sigma, \]
\[ \text{chirn} - \text{chirn} \tag{6.69} \]

Using the results given by (6.65) in the expressions above gives

\[ A(1)(\sigma) = \frac{1}{4 \sinh \sigma} \left\{ \sigma \cot(\alpha + \theta_0) \cot(\alpha - \theta_0) + \sigma \cot(\alpha - \theta_0) \cot(\alpha + \theta_0) \right\}, \]
\[ \tag{6.70} \]

\[ B(1)(\sigma) = \frac{1}{4 \sinh \sigma} \left\{ - \sigma \cosh(\pi + \theta_0) \csc(\alpha + \theta_0) + \sigma \sinh(\pi - \theta_0) \csc(\alpha - \theta_0) \right\} + \sinh \sigma \left( \csc(\alpha + \theta_0) - \csc(\alpha - \theta_0) \right), \]
\[ \tag{6.71} \]

\[ C(1)(\sigma) = \frac{1}{4 \sinh \sigma} \left\{ \sigma \cot(\alpha + \theta_0) \cot(\alpha - \theta_0) \right\}, \]
\[ \tag{6.72} \]

\[ D(1)(\sigma) = \frac{1}{4 \sinh \sigma} \left\{ - \sigma \cosh(\pi + \theta_0) \csc(\alpha + \theta_0) + \sigma \sinh(\pi - \theta_0) \csc(\alpha - \theta_0) \right\} - \sinh \sigma \left( \csc(\alpha + \theta_0) - \csc(\alpha - \theta_0) \right), \]
The expressions (6.70) to (6.73) are valid for $|\alpha \pm \theta_o| < \pi$, i.e. $|\theta_o| < \beta$.

Having found $A^{(0)}$ and $A^{(1)}$, and the corresponding terms for $B, C,$ and $D$, they are now substituted into the expressions (6.53) and (6.54) to obtain the field inside and outside the wedge. To avoid lengthy formulae the contribution from $A^{(0)}, B^{(0)}, C^{(0)},$ and $D^{(0)}$ will be evaluated and then the contribution from $A^{(1)}, B^{(1)}, C^{(1)},$ and $D^{(1)},$ which gives rise to the term of $O(1-1/n^2)$.

Substituting (6.61) and (6.63) into (6.53) gives

$$u_v(r, \theta) = e^{kr \cos(\theta - \theta_o)} + O(1-1/n^2). \quad (6.74)$$

Substituting $A^{(0)}(\sigma)$ and $B^{(0)}(\sigma)$, given by (6.56) and (6.58) into (6.54) gives

$$u_d(r, \sigma) = \frac{2}{\pi} \int_0^\infty \left\{ \mathrm{ch} \sigma \mathrm{ch} \sigma - \mathrm{sh} \sigma \mathrm{sh} \sigma \right\} K_{i\nu}(knr)dv + O(1-1/n^2),$$

$$= \frac{2}{\pi} \int_0^\infty \mathrm{ch} \sigma(\sigma - \theta_o)K_{i\nu}(knr)dv + O(1-1/n^2),$$

$$u_d(r, \sigma) = \frac{2}{\pi} \int_0^\infty \mathrm{ch} \sigma(\pi - \sigma + \theta)K_{i\nu}(knr)dv + O(1-1/n^2),$$

$$= e^{kr \cos(\theta - \theta_o)} + O(1-1/n^2), \quad (6.75)$$
where the expression (6.11) has been used. Substituting (6.72) and (6.73) into (6.53) gives

\[
u_\nu(x, \theta) = e^{\frac{kr \cos(\theta - \theta_0)}{2\pi}} \left( 1 - \frac{1}{n^2} \right) \int_0^{\infty} \frac{K_\nu(kr)}{k \sin \nu} d\nu
\]

\[
\left( \cot(\alpha + \theta_0) + \cot(\alpha - \theta_0) \right) v \cos(\theta - \theta_0) \frac{\sin(\alpha + \theta_0)}{\sin^2(\alpha + \theta_0)} \cos(\theta + \alpha)
\]

\[
= \frac{\sin(\alpha - \theta_0)}{\sin^2(\alpha - \theta_0)} \cos(\theta - \alpha)
\]

\[
\int_0^{\infty} d\nu + O(1 - n^{-2})^2.
\]

Noting that the integrand of the above expression is even in \( \nu \), \( K_\nu(x) = K_{-\nu}(x) \) the above expression can be re-written as

\[
u_\nu(x, \theta) = e^{\frac{kr \cos(\theta - \theta_0)}{2\pi}} \left( 1 - \frac{1}{n^2} \right) \int_0^{\infty} \frac{K_\nu(kr)}{k \sin \nu} d\nu
\]

\[
\frac{1}{\sin^2(\alpha + \theta_0)} \int_{-\infty}^{\infty} \frac{\sin(\alpha + \theta_0) \cos(\theta + \alpha)}{\sin \nu} K_\nu(kr) d\nu
\]

\[
= \frac{1}{\sin^2(\alpha - \theta_0)} \int_{-\infty}^{\infty} \frac{\sin(\alpha - \theta_0) \cos(\theta - \alpha)}{\sin \nu} K_\nu(kr) d\nu (1 - \frac{1}{n^2}) + O(1 - \frac{1}{n^2})^2.
\]

The second term for \( u_\phi(x, \phi) \) is obtained by substituting (6.70) and (6.71) into (6.54); the first term is given by (6.75). Thus
\[ u_d(x, y) = e^{i \text{arccos}(\theta_{\theta_0})} + \left( -\frac{1}{\eta^2} \right) \int_0^\infty K_i(xnr) \left\{ \nu_\theta \cot(\pi - \theta_0 + \phi) \text{shv}(\pi - \theta_0 - \phi) + \text{shv}(\alpha + \theta_0) \text{dv} \right\} \text{dv} + O(1 - \eta^2), \]

and noting that the integrands are even in \( \nu \) this expression can be written as

\[ u_d(x, y) = e^{i \text{arccos}(\theta_{\theta_0})} + \frac{1}{4\pi} \int_0^\infty \frac{\cot(\pi - \theta_0 + \phi) K_i(xnr) dv}{\text{shn}^2(\alpha + \theta_0)} \]

\[ + \frac{1}{\sin^2(\alpha + \theta_0)} \int_0^\infty \frac{\text{shv}(\pi - \theta_0 + \phi) K_i(xnr) dv}{\text{shn}^2(\alpha + \theta_0)} \]

\[ - \frac{1}{\sin^2(\pi - \theta_0)} \int_0^\infty \frac{\text{shv}(\alpha + \theta_0) \text{chv}(\phi - \beta) K_i(xnr) dv}{\text{shn}^2(\alpha + \theta_0)} \]

\[ - \frac{1}{\sin^2(\pi - \theta_0)} \int_0^\infty \frac{\text{shv}(\alpha - \theta_0) \text{chv}(\beta + \phi) K_i(xnr) dv}{\text{shn}^2(\alpha + \theta_0)} \left\{ 1 - \eta^2 \right\} \]

\[ + O(1 - \eta^2)^2. \]

(6.77)

The solutions for the diffusion problem, given by (6.76) and (6.77) are now converted into the solution of the diffraction
problem by replacing $k$ by $-ik$ and in order that the resulting expression be convergent $i\nu$ is replaced by $\nu$. Thus using the relationship

$$K_\nu(-iz) = \frac{\pi i}{2} e^{\frac{i\mu i}{2}} H_\nu^{(1)}(z),$$

(6.76) and (6.77) become

$$u_\nu(x, \Theta) = e^{-ikrcos(\Theta-\Theta_0)} + \frac{1}{\zeta}(1-\frac{1}{\nu^2}) \left\{ \begin{array}{l}
-\cot(\alpha+\Theta_0) + \cot(\alpha-\Theta_0) \int_{-\infty}^{\infty} v\cos v(\Theta-\Theta_0) e^{i\nu r} H_\nu^{(1)}(kr) dv \\
+ \frac{1}{\sin^2(\alpha+\Theta_0)} \int_{-\infty}^{\infty} \sin v(\alpha+\Theta_0) \cos v(\alpha+\Theta) e^{i\nu r} H_\nu^{(1)}(kr) dv \\
+ \frac{1}{\sin^2(\alpha-\Theta_0)} \int_{-\infty}^{\infty} \sin v(\alpha-\Theta_0) \cos v(\alpha-\Theta) e^{i\nu r} H_\nu^{(1)}(kr) dv \right\} + O(1-\frac{1}{\nu^2})$$

(6.78)

$$v_\nu(x, \phi) = e^{-ikr\cos(\Theta-\Theta_0)} + \frac{1}{\zeta}(1-\frac{1}{\nu^2}) \left\{ \begin{array}{l}
-\cot(\alpha+\Theta_0) \int_{-\infty}^{\infty} v\cos v(\Theta+\phi) e^{i\nu r} H_\nu^{(1)}(kr) dv \\
- \cot(\alpha-\Theta_0) \int_{-\infty}^{\infty} \sin v(\alpha+\Theta_0) \cos v(\alpha+\phi) e^{i\nu r} H_\nu^{(1)}(kr) dv \\
+ \frac{1}{\sin^2(\alpha+\Theta_0)} \int_{-\infty}^{\infty} \sin v(\alpha+\Theta_0) \cos v(\phi-\Theta) e^{i\nu r} H_\nu^{(1)}(kr) dv \right\}$$
Using the well known results

\[ J_\nu(z) \sim \frac{e^{-i\nu z/2}}{\sqrt{2\pi z}} \left[ 1 + O\left( \frac{1}{\nu} \right) \right], \]

(6.80)

\[ \tilde{J}_\nu(z) = \int \frac{1}{\pi v} \sin \pi v e^{i(v+\log(2/\nu)-\log z)} \]

\[ \tilde{J}_\nu(z) \sim \int \frac{1}{\pi v} \sin \pi v e^{i(v+\log(2/\nu)-\log z)} \]

\[ \left| e^{i\frac{\pi v}{2}} \tilde{J}_\nu^i(z) \right| \leq \frac{e^{-|\arg z|/2}}{\sqrt{2\pi |z|}}, \]

(6.81)

it is not difficult to show that

\[ \left| e^{i\frac{\pi v}{2}} \tilde{J}_\nu^i(z) \right| \leq \frac{e^{-|\arg z|/2}}{\sqrt{2\pi |z|}}, \]

(6.82)

\[ 1x1 \to \infty, \quad 0 < \arg z < \pi/2. \]

The result (6.82) shows that the integrals in the expressions (6.78) and (6.79) converge since \( \Im k > 0 \), provided the trigonometrical terms decay exponentially or are at least \( O(1) \) for \( \Im k > 0 \). This restricts the range of values which the angular variables can take. However, on evaluating the various integrals this restriction can be removed by analytic continuation.

By substituting the expression (6.81) into (6.78) and
(6.79) and using the fact that the integrands in the latter two expressions are even, one obtains,

\[ u_\nu(r, \theta) = e^{-ikr \cos(\theta - \theta_0)} \left[ \frac{1}{4} \left( 1 - \frac{i}{n^2} \right) \right] \left\{ \begin{array}{l}
\frac{i}{\sin^2(\alpha + \theta)} \int_{-\infty}^{\infty} \sin\nu(\alpha + \theta_0) \cos\nu(\alpha + \theta) e^{-i\nu r} J_\nu(kr) d\nu \\
\frac{i}{\sin^2(\alpha + \theta)} \int_{-\infty}^{\infty} \sin\nu(\alpha + \theta_0) \cos\nu(\alpha + \theta) e^{-i\nu r} J_\nu(kr) d\nu
\end{array} \right\} + O \left( \frac{1}{n^2} \right)^2 
\]

(6.83)

\[ u_d(r, \theta) = e^{-ikr \cos(\theta - \theta_0)} \left[ \frac{1}{4} \left( 1 - \frac{i}{n^2} \right) \right] \left\{ \begin{array}{l}
\frac{i}{\sin^2(\alpha + \theta)} \int_{-\infty}^{\infty} \sin\nu(\alpha + \theta_0) \cos\nu(\alpha + \theta) e^{-i\nu r} J_\nu(kr) d\nu \\
\frac{i}{\sin^2(\alpha + \theta)} \int_{-\infty}^{\infty} \sin\nu(\alpha + \theta_0) \cos\nu(\alpha + \theta) e^{-i\nu r} J_\nu(kr) d\nu
\end{array} \right\} + O \left( \frac{1}{n^2} \right)^2 
\]

(6.84)
In the expressions (6.73) and (6.83) the first term corresponds to the incident plane wave outside the dielectric wedge. In the expressions (6.79) and (6.84) the first term corresponds to the transmitted wave inside the dielectric wedge. Since it has been assumed \( n \approx 1 \) the transmitted wave corresponds to the Rayleigh-Gans-Born approximation where the propagation constant for the plane wave is appropriate for the medium in which it is propagating.

By deforming the contours of integration in the expressions (6.83) and (6.84) so that they take up paths in the right hand \( \nu \)-plane, enclosing the real axis the integrals can be evaluated in terms of the residues at the poles \( \nu = n\pi \) on the real axis, \( (n = 0, 1, \ldots) \), see figure 61. The contribution from the infinite semi-circle is zero because from (6.80)

\[
\left| \frac{\varepsilon_{\nu}}{\sin \nu \pi} \right| \lessgtr \frac{\varepsilon_{\nu}}{\sqrt{2\pi |\nu|}},
\]

where

\[
z = |z| e^{ib}, \quad \nu = |\nu| e^{ia}, \quad -\frac{\pi}{2} < a < \frac{\pi}{2}.
\]
All the poles for Re $\nu > 0$, except at the origin which is a simple pole, are double poles. Thus

$$u_\nu(r,\theta) = e^{-ikr \cos(\theta-\theta_0)}$$

$$= -\frac{1}{4\pi} \left[ \frac{1}{n^2} \right] \left[ \cot(\alpha+\theta_0) + \cot(\alpha-\theta_0) \right] \left\{ 2J_0(kr) \right\} + 2 \sum_{s=1}^{\infty} \left\{ \frac{\partial}{\partial \nu} \left( \nu \cos \nu(\theta-\theta_0) e^{-\frac{\nu x_0}{2} J_0(kr)} \right) \right\}_{\nu=s}$$

Figure 61
\[
\frac{1}{\sin^2(\alpha + \theta_0)} \left\{ J_0(kr) (\alpha + \theta_0) + 2 \sum_{s=1}^{\infty} \left[ \frac{2}{\partial \nu} \left( \sin\nu(\alpha+\theta_0) \cos\nu(\alpha+\theta) e^{-\frac{i\nu\pi}{2} J_0(kr)} \right) \right] \right\}_{\nu=\delta} \\
- \frac{1}{\sin^2(\alpha - \theta_0)} \left\{ J_0(kr) (\alpha - \theta_0) + 2 \sum_{s=1}^{\infty} \left[ \frac{2}{\partial \nu} \left( \sin\nu(\alpha-\theta_0) \cos\nu(\alpha-\theta) e^{-\frac{i\nu\pi}{2} J_0(kr)} \right) \right] \right\}_{\nu=\delta}
\]
\[+ O \left( \frac{1}{n^2} \right)^2. \tag{6.85} \]

\[
u_d(x, \beta) \quad e^{-i k n r \cos(\theta_0 - \beta)} - \frac{1}{4 \pi} \left( 1 - \frac{1}{n^2} \right) \left[ \cot(\alpha + \theta_0) \left\{ J_0(knr) \right\}_{\nu=\delta} + 2 \sum_{s=1}^{\infty} \left[ \frac{2}{\partial \nu} \left( \nu \cos\nu(\beta - \theta_0 + \phi) e^{-\frac{i\nu\pi}{2} J_0(knr)} \right) \right] \right]_{\nu=\delta}
\]
\[+ \cot(\alpha - \theta_0) \left\{ J_0(knr) + 2 \sum_{s=1}^{\infty} \left[ \frac{2}{\partial \nu} \left( \nu \cos\nu(\pi + \theta_0 + \phi) e^{-\frac{i\nu\pi}{2} J_0(knr)} \right) \right] \right\}_{\nu=\delta}
\]
\[- \frac{1}{\sin^2(\alpha + \theta_0)} \left\{ J_0(knr) (\alpha + \theta_0) + 2 \sum_{s=1}^{\infty} \left[ \frac{2}{\partial \nu} \left( \sin\nu(\alpha + \theta_0) \cos\nu(\beta - \phi) e^{-\frac{i\nu\pi}{2} J_0(knr)} \right) \right] \right\}_{\nu=\delta} \\
- \frac{1}{\sin^2(\alpha - \theta_0)} \left\{ J_0(knr) (\alpha - \theta_0) + 2 \sum_{s=1}^{\infty} \left[ \frac{2}{\partial \nu} \left( \sin\nu(\alpha - \theta_0) \cos\nu(\beta + \phi) e^{-\frac{i\nu\pi}{2} J_0(knr)} \right) \right] \right\}_{\nu=\delta}
\]
\[+ O \left( \frac{1}{n^2} \right)^2. \tag{6.86} \]

For the field at the origin putting \( r = 0 \) in the expressions (6.85) and (6.86) gives
\[ u_v(\theta, 0) = u_d(\phi, 0) = 1 - \frac{1}{2\pi} \left( 1 - \frac{1}{n^2} \right) \left\{ \cot(\alpha + \theta) + \cot(\alpha - \theta) \right\} \]

\[ \frac{\alpha + \theta}{\sin^2(\alpha + \theta)} - \frac{\alpha - \theta}{\sin^2(\alpha - \theta)} \right\} + O(1 - 1/n^2)^2. \]  

(6.87)

The result (6.87) is in agreement for \( \alpha = \pi/4 \), \( \theta_0 = \frac{\pi}{4} - \theta_0 \) with the expression obtained in chapter 5 (5.48), for the particular case of a right angle wedge.

The expressions (6.85) and (6.86) are only suitable if \( kr \) is not large. An expression will now be obtained for the diffracted far field. If \( kr \) is large then substituting the well known asymptotic representation

\[ H_0^{(1)}(kr) \sim \frac{2}{\sqrt{\pi kr}} e^{i(kr - \frac{\pi}{2} - \pi/4)} [1 + O(\frac{1}{kr})], \]

into the expressions (6.78) and (6.79) gives (ignoring the effects of captured poles and geometrical optics terms),

**Diffracted far field in** \( S_v = D_v =

\[ \frac{1}{\Delta} \left\{ \cot(\alpha + \theta) + \cot(\alpha - \theta) \right\} \left[ \nu \cos \nabla(\theta - \theta_0) \right] \]

\[ + \frac{1}{\sin^2(\alpha + \theta_0)} \left[ \sin \nu(\alpha + \theta_0) \cos \nabla(\theta + \alpha) \right] \]

\[ + \frac{1}{\sin^2(\alpha - \theta_0)} \left[ \sin \nu(\alpha - \theta_0) \cos \nabla(\theta - \alpha) \right] \]

\[ + O \left( \frac{1}{\nu} \right) + O \left( \frac{1}{kr} \nu \right), \]

**Diffracted field in** \( S_d = D_d =

\[
\begin{align*}
\mathcal{D}_V &= \frac{1}{(1-n^2)} \left\{ \sum_{k=-n}^{n} i^{kr} \left[ \cot(\alpha-\theta_0) \int_{-\infty}^{\infty} \frac{\cos\psi(\pi-\theta_0-\phi)}{\sin n\psi} \, d\psi \\ + \frac{1}{\sin^2(\alpha-\theta)} \int_{-\infty}^{\infty} \frac{\sin\psi(\alpha+\theta_0) \cos\psi(\phi-\alpha)}{\sin n\psi} \, d\psi \\ + \frac{1}{\sin^2(\alpha-\theta)} \int_{-\infty}^{\infty} \frac{\sin\psi(\alpha-\theta_0) \cos\psi(\phi+\alpha)}{\sin n\psi} \, d\psi \right] \right\} + O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{k^2n^2}\right) \\
\mathcal{D}_d &= \frac{1}{(1-n^2)} \left\{ \sum_{k=-n}^{n} i^{kr} \left[ \frac{\sin 2\alpha}{(\cos 2\alpha - \cos 2\theta_0)(1 - \cos(\theta_0-\phi))} \\ + \frac{1}{2\sin(\alpha+\theta_0)\left[\cos(\alpha+\theta_0)+\cos(\alpha-\theta_0)\right]} + \frac{1}{2\sin(\alpha-\theta_0)\left[\cos(\alpha-\theta_0)+\cos(\phi+\beta)\right]} \right] \right\} + O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{k^2n^2}\right)
\end{align*}
\]

The integrals appearing in the curly brackets of the above expressions have been evaluated in appendix 60, and thus

\[
\mathcal{D}_V = \frac{(1-n^2)}{2\sqrt{2\pi}kr^2} \left\{ \frac{\sin 2\alpha}{(\cos 2\alpha - \cos 2\theta_0)(1 - \cos(\theta_0-\phi))} \\ + \frac{1}{2\sin(\alpha+\theta_0)\left[\cos(\alpha+\theta_0)+\cos(\alpha-\theta_0)\right]} + \frac{1}{2\sin(\alpha-\theta_0)\left[\cos(\alpha-\theta_0)+\cos(\phi+\beta)\right]} \right\}
\]

\[
\mathcal{D}_d = \frac{(1-n^2)}{2\sqrt{2\pi}kr^2} \left\{ \frac{\sin 2\alpha}{(\cos 2\alpha - \cos 2\theta_0)(1 - \cos(\theta_0-\phi))} \\ + \frac{1}{2\sin(\alpha+\theta_0)\left[\cos(\alpha+\theta_0)+\cos(\phi+\beta)\right]} + \frac{1}{2\sin(\alpha-\theta_0)\left[\cos(\alpha-\theta_0)+\cos(\phi+\beta)\right]} \right\}
\]

(6.88)
\[ + O \left( \frac{1}{\mu^2} \right)^2 + \left( \frac{1-\frac{1}{\mu^2}}{(kr)^{3/2}} \right) \]  

(6.89)

For the particular case of a right angle dielectric wedge 
\( \alpha = \pi/4 \), making the change of variables \( \theta_\circ = \pi/4 - \theta_\circ \), 
\( \theta = \pi/4 - \theta \), so that the system of angular measurements is 
the same as chapter 5 gives

\[
D_v = \frac{(1-1/\mu^2) e^{i(kr\pi/4)}}{2\sqrt{2\pi kr}} \left\{ \frac{-1}{\sin 2\theta_\circ (1-\cos(\theta-\theta_\circ))} \right. \\
+ \frac{1}{2 \cos \theta_\circ [\sin \theta_\circ - \sin \theta]} + \frac{1}{2 \sin \theta_\circ [\cos \theta_\circ - \cos \theta]} \left\} + O \left( \frac{1}{\mu^2} \right)^2 \\
+ O \left( \frac{1}{\mu^2} \right)^2,
\]

which agrees with the result obtained in chapter 5. Similarly 
using the fact that \( \alpha + \beta = \pi \), \( \theta + \beta = \pi \), agreement is also found
for \( D_d \) for \( n \approx 1 \).
In this chapter an expression will be obtained for the electromagnetic field near to the tip of an arbitrary angle dielectric wedge of arbitrary refractive index 'n'. The primary source will be a line source of either E or H polarisation. Meixner [17] showed in a quantitative manner how the electric and magnetic field behave as a function of the radial distance from a wedge tip. He obtained his results directly from Maxwell's equations by assuming the fields near the wedge tip could be expanded in the form 

$$A + B r^\alpha + C r^{\alpha+1}$$

His analysis enabled him to determine \( \alpha \) but not, \( A, B, C, \) etc. which are functions of the angular variables and the refractive index, he also does not consider an incident field.

**Formulation of the boundary value problem**

The geometry of the wedge system is the same as that used in chapter 6, except in the present problem a line source of either \( E_z \) or \( H_z \) polarisation is assumed to be of the form

$$u_1 = H_0^{(1)}(kr), \quad R^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0), \quad -\beta < \theta < \beta.$$
The conditions which the field in $S_v$ and $S_d$ must satisfy are the same as given in chapter 6.

It is not difficult to show that the appropriate general solutions for $u_v$ and $u_d$ are given by

$$u_v(r, \theta) = i \int_{-\infty}^{\infty} \left\{ \cos \nu (\pi - 10 - \theta_0) \right\} H_0^{(1)}(k \rho) + C'(\nu) \cos \nu \theta + D'(\nu) \sin \nu \theta \right\} J_v(kr) \, d\nu,$$

$$u_d(r, \theta) = i \int_{-\infty}^{\infty} \left\{ A(\nu) \cos \nu \phi + B(\nu) \sin \nu \phi \right\} J_v(k \rho \rho) \, d\nu,$$

where the formula, see appendix 7A,

$$H_0^{(1)}(k \rho) = \frac{i}{2} \int_{-\infty}^{\infty} \frac{\cos \nu (\pi - 10 - \theta_0)}{\sin \nu} J_v(k \rho) H_0^{(1)}(k \rho) \, d\nu,$$

$$0 \leq |\theta - \theta_0| \leq 2\pi, \quad r < r_0,$$

has been used. In the present problem we are only interested in the region very close to the wedge tip, that is $r \approx 0$; and thus for a fixed value of $\nu (\geq 1)$ the Bessel functions $J_v(k \rho \rho), J_v(k \rho)$ can be replaced by their asymptotic approximation for small argument, i.e.

$$J_v(x) = \frac{x^v}{2^v \Gamma(1+v)} + O(x^{2+v}) \quad \text{as} \; x \rightarrow 0. \quad (7.3)$$

Substituting (7.3) into the expressions (7.1) and (7.2) for $r \approx 0$, and introducing new arbitrary constants $A(\nu)$, $B(\nu)$, $C(\nu)$, and $D(\nu)$ for computational convenience gives
\[ u_{\psi}(x, \theta) = i \int_{-\infty}^{\infty} \left\{ \cos \nu(\pi - \theta - \theta_0) H_{\nu}^{(0)}(kr) + C(\nu) \cos \nu \theta + D(\nu) \sin \nu \theta \right\} (kr)^{i} dv \\
+ O[(kr)^3], \quad (7.4) \]

\[ u_{\phi}(x, \phi) = i \int_{-\infty}^{\infty} \left\{ A(\nu) \cos \nu \phi + B(\nu) \sin \nu \phi \right\} n^{\nu}(kr)^{i} d\nu + O[(kr)^3]. \quad (7.5) \]

Substituting the expressions (7.4), (7.5) and

\[ \frac{\partial u_{\psi}}{\partial \theta} \bigg|_{\theta > \theta_0} = \int_{-\infty}^{\infty} \left\{ \sin \nu(\pi - \theta + \theta_0) H_{\nu}^{(0)}(kr) - C(\nu) \sin \nu \theta + D(\nu) \cos \nu \theta \right\} (kr)^{i} dv, \]

\[ \frac{\partial u_{\psi}}{\partial \theta} \bigg|_{\theta < \theta_0} = \int_{-\infty}^{\infty} 2^{\nu} \Gamma(\nu + 1) \sin \nu \theta d\nu \]

\[ \frac{\partial u_{\phi}}{\partial \phi} = \int_{-\infty}^{\infty} \left\{ -A(\nu) \sin \nu \phi + B(\nu) \cos \nu \phi \right\} n^{\nu}(kr)^{i} d\nu, \]

into the boundary conditions (6.6) of chapter 6, gives

\[ n^{\nu} A \cos \nu \alpha - n^{\nu} B \sin \nu \alpha - C \cos \nu \beta - D \sin \nu \beta = \frac{\cos \nu(\pi - \beta + \theta_0) H_{\nu}^{(0)}(kr)}{2^{\nu} \Gamma(\nu + 1) \sin \nu \theta}, \]

\[ n^{\nu} A \cos \nu \alpha + n^{\nu} B \sin \nu \alpha - C \cos \nu \beta + D \sin \nu \beta = \frac{\cos \nu(\pi - \beta - \theta_0) H_{\nu}^{(0)}(kr)}{2^{\nu} \Gamma(\nu + 1) \sin \nu \theta}, \]

\[ \gamma n^{\nu} A \sin \nu \alpha + \gamma n^{\nu} B \cos \nu \alpha + C \sin \nu \beta - D \cos \nu \beta = \frac{\cos \nu(\pi - \beta + \theta_0) H_{\nu}^{(0)}(kr)}{2^{\nu} \Gamma(\nu + 1) \sin \nu \theta}. \]
By adding and subtracting appropriately the above equations can be reduced to the simpler equations

\[ A^\nu \cos \alpha - C \cos \beta = \frac{\cos \nu \cos \theta_e H_0^{(0)}(kr_0)}{2^\nu \Gamma(\nu+1) \sin \nu \pi}, \]

\[ A^\nu \sin \alpha + C \sin \beta = \frac{\sin \nu \cos \theta_e H_0^{(0)}(kr_0)}{2^\nu \Gamma(\nu+1) \sin \nu \pi}, \]

\[ n^\nu B \sin \alpha + D \sin \beta = \frac{\sin \nu \sin \theta_e H_0^{(0)}(kr_0)}{2^\nu \Gamma(\nu+1) \sin \nu \pi}, \]

\[ -n^\nu B \cos \alpha + D \cos \beta = \frac{-\cos \nu \sin \theta_e H_0^{(0)}(kr_0)}{2^\nu \Gamma(\nu+1) \sin \nu \pi}. \]

These equations are easily solved to give

\[ A(\nu) = \frac{H_0^{(0)}(kr_0) \cos \theta_e}{(2\nu)^\nu \Gamma(\nu+1) \Delta_{\alpha \beta}^{(0)}}, \quad (7.6) \]

\[ B(\nu) = \frac{H_0^{(0)}(kr_0) \sin \theta_e}{(2\nu)^\nu \Gamma(\nu+1) \Delta_{\alpha \beta}^{(0)}}, \quad (7.7) \]

\[ C(\nu) = \frac{(1-\gamma) H_0^{(0)}(kr_0) \sin 2\nu \cos \theta_e}{2^{\nu+1} \Gamma(\nu+1) \sin \nu \pi \Delta_{\alpha \beta}^{(0)}}, \quad (7.8) \]

\[ D(\nu) = \frac{(1-\gamma) H_0^{(0)}(kr_0) \sin 2\nu \sin \theta_e}{2^{\nu+1} \Gamma(\nu+1) \sin \nu \pi \Delta_{\alpha \beta}^{(0)}}. \quad (7.9) \]
where

\[
\Delta_{\alpha\beta}^{(\nu)} = \cos \alpha \sin \beta + \tau \cos \beta \sin \alpha, \quad (7.10)
\]

\[
\Delta_{\beta\alpha}^{(\nu)} = \cos \beta \sin \alpha + \tau \cos \alpha \sin \beta. \quad (7.11)
\]

Substituting the expressions (7.6) to (7.9) into (7.4) and (7.5) gives

\[
u(x, \theta) = i \int_{-\infty}^{\infty} \left\{ \frac{\cos \nu (\pi - \theta - \phi)}{2 \sin \nu \pi} \Delta_{\alpha\beta}^{(\nu)} + \frac{(1 - \tau) \sin 2 \nu \alpha \cos \theta \cos \phi}{2 \sin \nu \pi} \Delta_{\alpha\beta}^{(\nu)} \right\} H_0^{(1)}(kr) \left( \frac{kr}{\pi} \right)^\nu + O(kr),
\]

\[
u_{\alpha}(x, \theta) = \int_{-\infty}^{\infty} \left\{ \frac{\cos \theta \cos \phi}{\Delta_{\alpha\beta}^{(\nu)}} + \frac{\sin \theta \sin \phi}{\Delta_{\beta\alpha}^{(\nu)}} \right\} H_0^{(1)}(kr) \left( \frac{kr}{\pi} \right)^\nu + O(kr).
\]

It is now shown that the above expressions are convergent for

\[1 \arg \nu \leq \frac{\pi}{2}, \ 1 \nu \rightarrow \infty.\]

From appendix 7A

\[
\frac{H_0^{(1)}(kr)}{\Gamma(\nu + 1)} \left( \frac{kr}{2} \right)^\nu \sim \frac{1}{1 \nu \pi} \left( \frac{r}{\rho_0} \right)^\nu,
\]

\[1 \nu \rightarrow \infty, \ r < \rho_0, \ \arg \nu < \pi.\]

Now

\[
|\frac{(1 - \tau) \sin 2 \nu \alpha \cos \theta \cos \phi}{2 \sin \nu \pi \Delta_{\alpha\beta}^{(\nu)}} | \leq \text{constant} e^{-\nu \gamma(1 - \nu \theta + \pi - \nu \phi - \nu \delta)},
\]

\[\nu = x + iy, \ 1 \nu \rightarrow \infty,\]
and since $\pi > 1\alpha + 1\theta_1$, then

$$\left| \frac{(1-\tau) \sin 2\nu x \cos \theta_0 \cos \theta}{2 \sin \nu \pi \Delta_{\alpha \theta}} \right| \leq \text{constant} e^{-\Im(1\beta - 1\theta_1)}$$

and thus this expression decays exponentially since $|\beta| > 1\theta_1$. Similarly

$$\left| \frac{(1-\tau) \sin 2\nu \pi \sin \theta_0 \sin \theta}{2 \sin \nu \pi \Delta_{\alpha \theta}} \right| \leq \text{constant} e^{-\Im(1\beta - 1\theta_1)}$$

Now,

$$\left| \frac{\cos \theta_0 \cos \phi}{\Delta_{\alpha \theta}} \right| \leq \text{constant} e^{-\Im(1\xi - 1\phi + 1\beta_1 - 1\theta_1)}$$

and since $1\alpha > 1\beta_1$, then

$$\left| \frac{\cos \theta_0 \cos \phi}{\Delta_{\alpha \theta}} \right| \leq \text{constant} e^{-\Im(1\beta - 1\theta_1)}$$

Similarly,

$$\left| \frac{\sin \theta_0 \sin \phi}{\Delta_{\alpha \theta}} \right| \leq \text{constant} e^{-\Im(1\beta - 1\theta_1)}$$

From the above approximations clearly the representations (7.12) and (7.13) are convergent, and convergence is enhanced for $(r/r_0)$ small.

A power series representation of the field near the wedge tip can be obtained by closing the contours of integration in the expressions (7.12) and (7.13) by an infinite semi-circle in the right-hand $\nu$-plane. Splitting off the residue contribution of the expressions (7.12) and (7.13), at
\( \nu = 0 \) gives

\[
u_v(r, \theta) = \frac{\pi H_v(kr)}{(\nu + \delta)} + \frac{1}{\nu + \delta} \left\{ \frac{\cos \nu(\pi - \theta - 0.4) + (1 - \tau) \sin 2\nu \cos \theta \cos \phi}{(1 - \tau) \sin \nu - (\nu - 1) \sin \nu(\pi - 2\alpha)} \right\} \frac{H_v(kr)}{(\nu + 1)} \frac{(kr)^\nu}{2} \sin^\nu \theta \]

\[
+ \frac{(1 - \tau) \sin 2\nu \cos \theta \cos \phi}{(1 - \tau) \sin \nu - (\nu - 1) \sin \nu(\pi - 2\alpha)} \right\} \frac{H_v(kr)}{(\nu + 1)} \frac{(kr)^\nu}{2} \sin^\nu \theta \]

\[0 < \delta < \frac{1}{2}, \quad (7.14)\]

\[
u_a(r, \theta) = \frac{\pi H_\nu(kr)}{(\nu + \delta)} + \frac{1}{\nu + \delta} \left\{ \frac{2 \cos \theta \cos \phi}{(1 - \tau) \sin \nu(\pi - 2\alpha)} \right\} \frac{H_\nu(kr)}{(\nu + 1)} \frac{(kr)^\nu}{2} \sin^\nu \theta \]

\[
+ \frac{2 \sin \nu \theta \sin \nu \phi}{(1 - \tau) \sin \nu(\pi - 2\alpha)} \right\} \frac{H_\nu(kr)}{(\nu + 1)} \frac{(kr)^\nu}{2} \sin^\nu \theta \]

\[0 < \delta < \frac{1}{2}, \quad (7.15)\]

where the alternative representations

\[A^{(\nu)}_{\alpha \beta} = \frac{(1 + \tau)}{2} \sin \nu \pi - \frac{(\nu - 1)}{2} \sin \nu(\pi - 2\alpha). \quad (7.16)\]

\[A^{(\nu)}_{\beta \alpha} = \frac{(1 + \tau)}{2} \sin \nu \pi - \frac{(\nu - 1)}{2} \sin \nu(\pi - 2\alpha). \quad (7.17)\]

have been used. The condition \( 0 < \delta < \frac{1}{2} \) ensures, as will be seen later, that in shifting the contour of integration to the right no poles which lie on the real axis are captured.
From the expressions (7.14) and (7.15) it can be seen that the only poles that occur in the right-hand $\nu$-plane are given by

\[
\Delta_{\alpha\beta}^{(\nu)} = 0 = \sin \nu \pi = \frac{\nu-1}{\nu+1} \sin \nu(\pi-2\alpha), \quad (7.18)
\]

\[
\Delta_{\rho\alpha}^{(\nu)} = 0 = \sin \nu \pi = \frac{1-\rho}{\nu+1} \sin \nu(\pi-2\alpha). \quad (7.19)
\]

In the expression (7.14) the part of the integrand in the curly bracket vanishes for $\nu = n\pi$, ($n = 1, 2, \ldots$) and thus no poles exist for $\sin \nu \pi = 0$.

The equations (7.18) and (7.19) can be put in the general
form

\[ \sin x = R \sin ax, \]  

(7.20)

where \( a = 1 - \frac{2\alpha}{\pi} \), \( x = \omega t \), and for (7.18) \( R = \frac{\pi - 1}{\pi + 1} \); and for (7.19) \( R = \frac{\pi}{\pi + 1} \). The equation (7.20) is a transcendental equation and is best solved graphically. To solve it graphically two separate situations must be considered.

Case 1 \( a \) and \( R \) both of the same sign.

In this case \( y = R \sin ax \) is positive for \( 0 < x < \frac{\pi}{|a|} \). Thus imposing the curve \( y = \sin x \) on the curve \( y = R \sin ax \) gives the root of (7.20) by reading off \( x_1 \) at the intersection point.

![Figure 64](image)

Clearly \( \frac{\pi}{2} < x_1 \) or \( \frac{\pi}{2} < \text{Re } \nu_1 < 1 \) is the range for the first root of the equation (7.20) for \( \text{sgn } R = \text{sgn } a \).

Case 2 \( a \) and \( R \) of opposite sign.

In this case the curve \( y = R \sin ax \) is negative for \( 0 < x < \frac{\pi}{|a|} \). Thus imposing the curve \( y = \sin x \) on the curve \( y = R \sin ax \) gives the root of (7.20) by reading off \( x_1 \) at the
point of intersection of these curves.

Clearly \( \pi < x_1 < 2\pi \) or \( 1 < \text{Re} \nu_1 < 2 \) is the range for the first root of the equation (7.20) for \( \text{sgn} \, R = -\text{sgn} \, a \). If case 1 applies to the equation (7.18) then equation (7.19) must be case 2, and vice versa. Thus in the range \( \frac{1}{2} < \text{Re} \nu < 2 \) the first roots of the equations (7.18) and (7.19) exist; also in the situation of case 1 the second root also exists in this range of \( \nu \). Roots greater than \( \nu = 2 \) cannot be included because the approximation of the Bessel function (7.3) is only valid up to order \((kr)^2\). The equations (7.18) and (7.19), the roots of which give the order of singularity of the gradient of the field at the wedge tip, agree with those obtained by Meixner [17] for both E and H polarisation. Meixner [17] gives a nomogram, see figure 66 and figure 67 reproduced here, for the roots of the equations (7.18) and (7.19) for the ranges of interest. By using this nomogram and the convergent iteration

\[ y = R \sin \alpha \]

\[ y = \sin \nu \]

**Figure 65**
Figure 66

NOMOGRAM of $\frac{\sin \frac{\pi \alpha}{2\epsilon}}{\sin \psi (\pi-2\alpha)} = \frac{1-\psi}{1+\psi} \text{ or } \frac{x-1}{x+1}$, for $\frac{1}{2} < u < \frac{3}{2}$
NOMOGRAM OF \( \frac{\sin \alpha}{\sin \alpha(\pi - 2 \alpha)} = \frac{1 - \varphi}{1 + \varphi} \quad \frac{\varphi - 1}{\varphi + 1} \) for \( \frac{3}{2} < \varphi < 2 \)

Figure 67
\[
\sin \pi \nu_{r+1} = R \sin \nu_r (\pi - 2\alpha),
\]

where \( R = \frac{1 - \tau}{1 + \tau} \) or \( \frac{\tau - 1}{1 + \tau} \), the value of the root for given \( R \) and \( \alpha \) can be found to any desired degree of accuracy.

Thus closing the contours of integration in the expressions (7.14) and (7.15) by the infinite semi-circle \( C \), see figure 63, and using Cauchy's residue theorem gives

\[
u_r, \theta = \frac{\pi H_1^{(1)}(kr_0)}{(r_0 + \beta)}
\]

\[
+ 2(1 - \tau) \sum_{n=1}^{i} \frac{\sin 2\nu_\alpha \cos \nu_\theta \cos \nu_\theta H_0^0(kr_0)}{\left[ (\tau + i) \cos \nu_\pi - (\tau - i)(1 - 2\alpha \cos \nu_{n}(\pi - 2\alpha)) \right] \Gamma (\nu_{n+1}) \sin \nu_\pi} \left( \frac{kr}{2} \right)^{\nu_\pi}
\]

\[
+ 2 \left( \tau - 1 \right) \sum_{n=1}^{j} \frac{\sin 2\nu_\alpha \sin \nu_\theta \sin \nu_\theta H_0^0(kr_0)}{\left[ (\tau + i) \cos \nu_\pi - (\tau - i)(1 - 2\alpha \cos \nu_{n}(\pi - 2\alpha)) \right] \Gamma (\nu_{n+1}) \sin \nu_\pi} \left( \frac{kr}{2} \right)^{\nu_\pi}
\]

\[
+ O[(kr)^2], \quad (7.21)
\]

\[
u_\alpha, \theta = \frac{\pi H_1^{(1)}(kr_0)}{(r_0 + \beta)}
\]

\[
+ 4 \sum_{n=1}^{i} \frac{\cos \nu_\theta \cos \nu_\theta \sin \nu_\theta H_0^0(kr_0)}{\left[ (\tau + i) \cos \nu_\pi - (\tau - i)(1 - 2\alpha \cos \nu_{n}(\pi - 2\alpha)) \right] \Gamma (\nu_{n+1})} \left( \frac{kr}{2} \right)^{\nu_\pi}
\]

\[
+ 4 \sum_{n=1}^{j} \frac{\sin \nu_\theta \sin \nu_\theta \sin \nu_\theta H_0^0(kr_0)}{\left[ (\tau + i) \cos \nu_\pi - (\tau - i)(1 - 2\alpha \cos \nu_{n}(\pi - 2\alpha)) \right] \Gamma (\nu_{n+1})} \left( \frac{kr}{2} \right)^{\nu_\pi}
\]

\[
+ O[(kr)^2], \quad (7.22)
\]
where \( v_i \) are the roots of (7.18) such that \( 0 < v_i < 2 \) and
\( v_j \) are the roots of (7.19) such that \( 0 < v_j < 2 \). It looks
as though (7.21) becomes infinite for \( v_n = 1 \) but in fact this
corresponds to the situation where \( \alpha = 0 \) and we have assumed
from the start that \( |\alpha| > 0, |\beta| > 0 \). \( \alpha = 0 \) corresponds to
a wedge of no thickness and consequently no material, and
this is why it is difficult to formulate the dielectric half-
plane problem.
Appendix 5A

Asymptotic evaluation of the integral

\[ I = \int_{-\infty}^{\infty} \frac{f(v) e^{i(\nu x + \sqrt{k^2 - \nu^2} |y|)}}{\sqrt{k^2 - \nu^2} (\nu - \nu_p)} \, dv, \quad |y| < \infty, \ |x| < \infty. \quad (5A.1) \]

The pole \( \nu_p \) can approach the saddle point, \( = \nu_S \) and the normal saddle point method breaks down. The method used to evaluate \((5A.1)\) asymptotically is based on Jones [12]. The assumptions that will be made are:

\[ k = |k| e^{i\xi} \ (\xi > 0), \quad \text{Im} \sqrt{k^2 - \nu^2} > 0, \]

\[ f(\nu_S) = O(1), \text{ where } \nu_S = \frac{kx}{r}, \quad r^2 = x^2 + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta. \]

Let \( g(v) = \nu x + \sqrt{k^2 - \nu^2} |y| \),

then \[ g'(v) = x - \frac{\nu |y|}{\sqrt{k^2 - \nu^2}}, \quad (5A.2) \]

\[ g''(v) = -\frac{|y| k^2}{(k^2 - \nu^2)^{3/2}}. \quad (5A.3) \]

The saddle point occurs at the point \( g'(\nu_S) = 0; \) and thus from \((5A.2)\) \( \nu_S = \frac{kx}{r}, \) where \( \sqrt{k^2 - \nu_S^2} = \frac{k |y|}{r} \) since \( \text{Im} \sqrt{k^2 - \nu^2} > 0. \)

The saddle point is also a simple saddle point because

\[ g''(\nu_S) = -\frac{3}{k |y|^2} \neq 0. \]
For clarity's sake we will assume that \( \text{Im} \nu_p > 0 \), thus the situation is as shown in figure 68.

\[
g(\nu) = kT - \frac{x^3}{2k|\gamma|^2} (\nu - \nu_p)^2 + O(\nu - \nu_p)^3, \quad (5A.4)
\]

and the real path of integration is deformed such that it runs through the saddle point, and the phase of the second term of the expansion \((5A.4)\) is constant. Effectively one shifts the real contour of integration so that it runs through the point \( \nu_s \) and then this contour is rotated about \( \nu_p \) until \( \text{Im}[((\nu - \nu_s)e^{-\nu/2}] = 0 \). In the process of deforming the contour of integration it is quite possible that the pole \( \nu_p \) can be captured or can be very close to the saddle point. On the other hand it might not be captured at all but may still be close to the saddle point. Thus two possible situations must be considered and these are defined analytically by the inequalities.
Case (1) \( \text{Im}(\nu_p - \nu_s) e^{-\frac{i\xi}{2}} > 0 \), no pole captured. \hspace{1cm} (5A.5)

Case (2) \( \text{Im}(\nu_p - \nu_s) e^{-\frac{i\xi}{2}} < 0 \), pole captured. \hspace{1cm} (5A.6)

Case (1) \( \text{Im}(\nu_p - \nu_s) e^{-i\xi/2} > 0 \).

After distorting the path of integration and substituting the Taylor series for \( g(\nu) \), (5A.1) becomes

\[
I \sim e^{ikr} \int_{-\infty}^{\infty} \frac{2|k|y^2}{\sqrt{k^2 - \nu^2}} \cdot \frac{2|k|y^2}{\sqrt{k^2 - \nu^2}} \left[ (\nu - \nu_s) e^{-i\xi_1} \right]^2 d\nu,
\]

and since the major contribution comes from the region \( \nu_s \),

\[
\frac{f(\nu)}{\sqrt{k^2 - \nu^2}} \sim \frac{f(\nu_s)}{\sqrt{k^2 - \nu_s^2}},
\]

so that

\[
I \sim e^{ikr} f(\nu_s) \int_{-\infty}^{\infty} \frac{2|k|y^2}{\sqrt{k^2 - \nu^2}} \left[ (\nu - \nu_s) e^{-i\xi_1} \right]^2 d\nu.
\]
Making the change of variable

\[ u = \sqrt{\frac{r^2}{2 \lambda k^2}} (v - v_s) e^{-i \epsilon/2} = \sqrt{\frac{r}{2k}} \frac{(v - v_s)}{\sin \theta} , \]

then

\[ I \sim \frac{r f(k \cos \theta) e^{ikr}}{k \lambda^2} \int_{-\infty}^{\infty} \frac{e^{-iu^2}}{u-Q} du , \]

where

\[ Q = \sqrt{\frac{r^2}{2 \gamma^2 k^2}} \frac{-i \epsilon}{2} (v_p - v_s) = \sqrt{\frac{r}{2k}} \frac{(v_p - v_s)}{\sin \theta} , \]

and from (5A.5), \( \text{Im} Q > 0 \).

Now

\[ \int_{-\infty}^{\infty} \frac{e^{-iu^2}}{(u-Q)} du = i \int_{-\infty}^{\infty} e^{-iu^2} du \int_{0}^{\infty} e^{-i (u-Q) \psi} d\psi , \text{ Im} Q > 0 , \]

\[ = i \int_{-\infty}^{\infty} e^{-i (u+\gamma/2)^2} du \int_{0}^{\infty} e^{i (\gamma^2/4 + \gamma \psi)} d\psi , \]

\[ = \sqrt{\pi} e^{i \pi/4} \int_{0}^{\infty} e^{i (\gamma^2/4 + \gamma \psi)} d\psi , \]

\[ = \sqrt{\pi} e^{i \pi/4} e^{-i \gamma^2} \int_{0}^{\infty} e^{i (\gamma/2 + \gamma \psi)^2} d\psi , \]

and letting \( \gamma/2 + \psi = t \),

\[ \int_{-\infty}^{\infty} \frac{e^{-iu^2}}{(u-Q)} du = 2\sqrt{\pi} e^{i (\pi/4 - \gamma^2)} \int_{Q}^{\infty} e^{it^2} dt . \]

If we let \( F(Q) = e^{-i \gamma^2} \int_{Q}^{\infty} e^{it^2} dt , \)

...
and note that

\[ \frac{r}{k^{1/2}} = \frac{Q}{(y_p - y_s)^2} \sqrt{\frac{2}{kr}}, \]

then

\[ I \sim 2 \sqrt{\frac{2\pi}{k^2}} \frac{f(k \cos \theta)}{(y_p - y_s)^2} \exp \left( i(kr + \pi/4)f(Q) \right), \]

(5A.7)

\[ \text{Im}(y_p - y_s)^2 e^{-i\varepsilon/2} > 0. \]

Case (2) \quad \text{Im}(y_p - y_s)^2 e^{-i\varepsilon/2} < 0.

The analysis for this case is similar to the last except the contribution from the captured pole \( y_p \) is included. Thus

\[ I \sim \frac{2\pi i}{\sqrt{k^2 - y_p^2}} \int \frac{e^{i(y_p x + \sqrt{k^2 - y_p^2} y)}}{klyy | y|} \left[ \int e^{-iu^2} \right] \int_{-\infty}^{\infty} \frac{e^{-iu^2}}{u - Q}, \]

\[ \text{Im}Q < 0. \]

Now for \( \text{Im}Q < 0. \)

\[ \int_{-\infty}^{\infty} \frac{e^{-iu^2}}{u - Q} \, du = -i \int_{-\infty}^{\infty} e^{-iu^2} \, du = \int_{0}^{\infty} e^{i\xi(u-Q)} \, d\xi, \]

\[ = -ie^{-iQ^2} \int_{-\infty}^{\infty} e^{-i(u-\xi/2)^2} \, du \int_{0}^{\infty} e^{i(\xi/2-Q)^2} \, d\xi, \]

\[ = -2/\pi e^{i(\pi/4-Q^2)} \int_{-\infty}^{\infty} e^{it^2} \, dt = -2/\pi e^{i\pi/4} T(-Q). \]

Thus
For the case when \( \text{Im}v_s \leq 0 \) then the saddle point occurs below the line \( \text{Im}v = 0 \) and the appropriate expressions for \( I \) are:

**case (1) \( \text{Im}(v_p - v_s)e^{-i\xi/2} > 0 \).**

\[
I \sim 2\sqrt{\frac{2\pi}{kr}} f(k\cos \theta) \left[ \frac{-Q}{\nu_p - \nu_s} \right] e^{i(kr+\pi/4)} F(-Q)
\]

\[+ \frac{2\pi i f(v_p)}{\sqrt{k^2 - \nu_p^2}} e^{i(v_p x + \sqrt{k^2 - \nu_P^2} \gamma_1)} \left\{ (v_p - \nu_s) \left[ 2Q F(\alpha) \right] e^{i(kr+\pi/4)} \right\} \]

for \( \text{Im}(v_p - v_s)e^{-i\xi/2} < 0 \) .

(5A.8)

Thus generally

**case (1) \( \text{Im}(v_p - v_s)e^{-i\xi/2} > 0 \).**

\[
I \sim -2\pi i f(v_p) e^{i(v_p x + \sqrt{k^2 - \nu_P^2} \gamma_1)} \frac{1}{\sqrt{k^2 - \nu_P^2}} \left\{ (v_p - \nu_s) \left[ 2Q F(\alpha) \right] e^{i(kr+\pi/4)} \right\}
\]

(5A.9)

**case (2)**

\[
I \sim \sqrt{\frac{2\pi}{kr}} f(v_s) \left[ \frac{-2Q}{\nu_p - \nu_s} \right] e^{i(kr+\pi/4)} F(-Q) .
\]

(5A.10)

Thus generally

**case (1) \( \text{Im}(v_p - v_s)e^{-i\xi/2} > 0 \).**

\[
I \sim \frac{2\pi i f(v_p) e^{i(v_p x + \sqrt{k^2 - \nu_P^2} \gamma_1)}}{\sqrt{k^2 - \nu_P^2}} \left\{ (v_p - \nu_s) \left[ 2Q F(\alpha) \right] e^{i(kr+\pi/4)} \right\}
\]

(5A.11)
case (2) \( \text{Im}(\nu_p - \nu_s) e^{-i\varepsilon/2} < 0 \).

\[
I \sim 2\pi i \int (\nu_p) \, e^{i(\nu_p x + \sqrt{k^2 - \nu_p^2} y)} \frac{H(\text{Im}\nu_p) H(\text{Im}\nu_s) + \sqrt{2\pi} f(\nu_s) e^{-2iq} e^{i(kr - \pi/4)}}{\sqrt{k^2 - \nu_p^2}} \left( \nu_p - \nu_s \right) e^{i(kr - \pi/4)}. \]  

(5A.12)

where \( \nu_s = \frac{kx}{r} \) and \( H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \).

Properties of \( F(Q) \)

\[
F(Q) + F(-Q) = \sqrt{\pi} e^{-iQ^2 + i\pi/4},
\]

\[
F(Q) \sim \frac{1}{2Q} + O(Q^{-2}), \quad |Q| \to \infty, \quad (5A.13)
\]

\[
F(Q) \sim \sqrt{\pi} e^{i\pi/4} + O(Q^2), \quad |Q| \to 0. \quad (5A.14)
\]

If the pole \( \nu_p \) is well removed from the saddle point \( \nu_s \) then using (5A.13) the expressions and (5A.12) reduce to

case (1) \( \text{Im}(\nu_p - \nu_s) e^{-i\varepsilon/2} > 0 \),

\[
I \sim -2\pi i \int (\nu_p) \, e^{i(\nu_p x + \sqrt{k^2 - \nu_p^2} y)} \frac{H(-\text{Im}\nu_p) H(\text{Im}\nu_s) + \sqrt{2\pi} f(\nu_s) e^{-2iq} e^{i(kr - \pi/4)}}{\sqrt{k^2 - \nu_p^2}} \left( \nu_p - \nu_s \right) e^{i(kr - \pi/4)}. \]

(5A.15)

case (2) \( \text{Im}(\nu_p - \nu_s) e^{-i\varepsilon/2} < 0 \),

\[
I \sim 2\pi i \int (\nu_p) \, e^{i(\nu_p x + \sqrt{k^2 - \nu_p^2} y)} \frac{H(\text{Im}\nu_s) H(\text{Im}\nu_p) + \sqrt{2\pi} f(\nu_s) e^{-2iq} e^{i(kr - \pi/4)}}{\sqrt{k^2 - \nu_p^2}} \left( \nu_p - \nu_s \right) e^{i(kr - \pi/4)}. \]  

(5A.16)
where

\[ \nu_s = \frac{kr}{r} = k \cos \theta, \quad f_1(\nu) = \frac{f(\nu)}{\sqrt{k^2 - v^2} (v - v_0)} \]

The term \( H(\pm i \nu_p) \) is included because if \( \text{Im}(\nu_p) > 0 \) and \( \text{Im}(\nu_s) < 0 \), then there is no possibility of capture.
Appendix 5B

Inequalities

2nd quadrant \( \frac{\pi}{2} \leq \theta \leq \pi, \quad 0 < \theta_0 < \frac{\pi}{2} \)

Case (1) \[- (\cos \theta + \cos \theta_0) > 0 \] \hspace{1cm} (5B.1)

or \[- 2 \cos \left( \frac{\theta + \theta_0}{2} \right) \cos \left( \frac{\theta - \theta_0}{2} \right) > 0 . \]

Case (2) \[- (\cos \theta + \cos \theta_0) < 0 \] \hspace{1cm} (5B.2)

or \[- 2 \cos \left( \frac{\theta + \theta_0}{2} \right) \cos \left( \frac{\theta - \theta_0}{2} \right) < 0 . \]

For the range of \( \theta \) and \( \theta_0 \)

\[ \frac{\pi}{2} < \theta + \theta_0 < \frac{3\pi}{2}, \quad 0 < \theta - \theta_0 < \pi . \]

Hence

![Figure 70](image-url)
Clearly \( \cos \left(\frac{\theta - \theta_0}{2}\right) > 0 \), and

\[
\cos \left(\frac{\theta + \theta_0}{2}\right) > 0 \text{ for } \frac{\pi}{2} < \theta + \theta_0 < \pi,
\]

\[
\cos \left(\frac{\theta - \theta_0}{2}\right) < 0 \text{ for } \pi < \theta + \theta_0 < \frac{3\pi}{2}.
\]

Clearly (5B.1) is satisfied iff \( \theta + \theta_0 > \pi \).

Clearly (5B.2) is satisfied iff \( \theta + \theta_0 < \pi \).

3rd and 4th quadrant \( 0 < \theta_0 < \frac{\pi}{2} \), \( \pi < \theta < 2\pi \)

**Case (1)** \( (\cos \theta_0 - \cos \theta) > 0 \) \hspace{1cm} (5B.3)

or \( 2\sin \left(\frac{\theta + \theta_0}{2}\right) \sin \left(\frac{\theta - \theta_0}{2}\right) > 0 \).

**Case (2)** \( (\cos \theta_0 - \cos \theta) < 0 \) \hspace{1cm} (5B.4)

or \( 2\sin \left(\frac{\theta + \theta_0}{2}\right) \sin \left(\frac{\theta - \theta_0}{2}\right) < 0 \).

For the range of \( \theta \) and \( \theta_0 \), then \( \pi < \theta + \theta_0 < \frac{5\pi}{2} \), \( \frac{\pi}{2} < \theta - \theta_0 < 2\pi \).

Hence
Clearly \( \sin \frac{\theta - \theta_o}{2} > 0 \), and

\[
\sin \frac{\theta + \theta_o}{2} > 0 \text{ for } \pi < \theta + \theta_o < 2\pi,
\]

\[
\sin \frac{\theta + \theta_o}{2} < 0 \text{ for } 2\pi < \theta + \theta_o < \frac{5\pi}{2},
\]

hence (5B.3) is satisfied iff \( \theta + \theta_o < 2\pi \).

Hence (5B.4) is satisfied iff \( \theta + \theta_o > 2\pi \).
Appendix 6A

Some properties of $K_{i\nu}(x)$.

The fact that $K_{i\nu}(0) = \pi \delta(\nu)$ can be shown quite simply from the integral definition of $K_{i\nu}(x)$, Lebedev [22]

$$K_{i\nu}(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x \cosh -i\nu t} dt.$$  \hspace{1cm} (6A.1)

Letting $x \to 0^+$ give immediately the Fourier representation of the delta function, i.e.

$$K_{i\nu}(0) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-i\nu t} dt = \pi \delta(\nu). \hspace{1cm} (6A.2)$$

The above derivation is not rigorous, however it is well known that the delta function is characterised by the following properties

$$\delta(\nu) = 0 \text{ if } \nu \neq 0, \hspace{1cm} (6A.3)$$

$$\int_{-\infty}^{\infty} \delta(\nu) d\nu = 1. \hspace{1cm} (6A.4)$$

It will now be shown that the generalised function $\lim_{x \to 0^+} \frac{K_{i\nu}(x)}{x}$ meets the requirements imposed by (6A.3) and (6A.4). $K_{i\nu}(x)$ is defined by Lebedev [22],

$$K_{i\nu}(x) = \frac{\pi}{2 \sinh \nu x} [I_{-i\nu}(x) - I_{i\nu}(x)]. \hspace{1cm} (6A.5)$$

To find the correct behaviour as $x \to 0^+$ the change of variable $x = 2e^{-t}$ is made in (6A.5) giving
\[ K_{1 \nu}(2e^{-t}) = \frac{\pi}{2i\sin \nu \pi} \left[ I_{-1 \nu}(2e^{-t}) - I_{1 \nu}(2e^{-t}) \right], \quad t \to \infty. \]

Substituting the series representation for \( I_{\pm \nu} \) gives

\[
K_{1 \nu}(2e^{-t}) = \frac{\pi}{2i\sinh \nu \pi} \left[ \frac{e^{ivt}}{\Gamma(1 - i\nu)} - \frac{e^{-ivt}}{\Gamma(1 + i\nu)} \right]
\]

\[ + O(e^{-4t} + B_0 e^{-2t}) \tag{6A.6} \]

and for \( \nu \neq 0 \) as \( t \to \infty \) the expression (6B.6) is zero almost everywhere; thus the condition (6A.3) has been satisfied. The condition (6A.4) is also satisfied since

\[ \lim_{x \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} K_{1 \nu}(x) dv = \lim_{x \to 0} \frac{2}{\pi} \int_{0}^{\infty} K_{1 \nu}(x) dv, \quad \text{since} \quad K_{1 \nu}(x) = K_{-1 \nu}(x), \]

and using the result \( \int_{0}^{\infty} K_{1 \nu}(x) dv = \frac{\pi}{2} e^{-x} \),

then

\[ \lim_{x \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} K_{1 \nu}(x) dv = \lim_{x \to 0} e^{-x} = 1. \]

In dealing with Bessel functions of imaginary order one is in effect dealing with generalised functions of the form \( x^{i\nu} \), where \( x \) and \( \nu \) are real. In the work which follows the following result will be used

\[ f_{x}(\nu) = \lim_{x \to 0^+} \frac{(x^{i\nu})}{i\nu x} = \pm \delta(\nu), \tag{6A.7} \]

which will now be proved. Clearly for (6A.7) to be true then
(i) \( f_x(v) = 0, \ v \neq 0 \),

(ii) \( \int_{-\infty}^{\infty} f_x(v)dv = \pm 1 \).

(i) is clearly satisfied by letting \( x \to 0 \). The proof of (ii) follows thus

\[
\lim_{x \to 0^+} \int_{-\infty}^{\infty} \frac{f(v)}{i\nu x} dv = \lim_{x \to 0^+} \int_{-\infty}^{\infty} \frac{e^{iv\log x}}{iv} dv,
\]

\[
= \lim_{x \to 0^+} \frac{1}{i\pi} \int_{-\infty}^{\infty} \left( \frac{\cos(v\log x)i\sin(v\log x)}{v} \right) dv.
\]

The first integral of the above expression is odd in \( v \) and hence

\[
\int_{-\infty}^{\infty} f_x(v)dv = \lim_{x \to 0^+} \int_{-\infty}^{\infty} \frac{\sin(v\log x)}{\nu x} dv.
\]

Making the change of variable \( \log x = -y \), (so that \( y \to \infty \) as \( x \to 0^+ \)) gives

\[
\int_{-\infty}^{\infty} f_x(v)dv = \pm \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{\sin vy}{\nu y} dv = \pm 1,
\]

by Dirichlet's integral.

A proof of the expression

\[
\int_0^{\infty} K_\nu(xu) K_\sigma(x) dx = \frac{\pi \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

\[
+ \left(1 - \frac{\nu + \sigma}{\nu - \sigma}\right) \frac{\pi \nu \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

\[
+ \left(1 - \frac{\nu + \sigma}{\nu - \sigma}\right) \frac{\pi \nu \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

by Dirichlet's integral.

A proof of the expression

\[
\int_0^{\infty} K_\nu(xu) K_\sigma(x) dx = \frac{\pi \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

\[
+ \left(1 - \frac{\nu + \sigma}{\nu - \sigma}\right) \frac{\pi \nu \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

by Dirichlet's integral.

A proof of the expression

\[
\int_0^{\infty} K_\nu(xu) K_\sigma(x) dx = \frac{\pi \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

\[
+ \left(1 - \frac{\nu + \sigma}{\nu - \sigma}\right) \frac{\pi \nu \sin(\sigma \log x)}{2 \sigma sh \sigma \pi}
\]

by Dirichlet's integral.
in more detail than given by Smith [16] will now be given. From the differential equations for \( K_{lv}(nx) \) and \( K_{ls}(x) \)

\[
- \left( \frac{d}{dx} - x^2 \right) K_{lv}(nx) = \frac{d}{dx} \left[ x \frac{d}{dx} K_{lv}(nx) \right] , \quad (6A.9)
\]

\[
- \left( \frac{d}{dx} - x \right) K_{ls}(x) = \frac{d}{dx} \left[ x \frac{d}{dx} K_{ls}(x) \right] . \quad (6A.10)
\]

Multiplying (6A.9) and (6A.10) by \( K_{ls}(x) \) and \( K_{lv}(nx) \) respectively, and integrating over \( x \) gives

\[
\lim_{\varepsilon \to 0} \left( \frac{\sigma^2 - \nu^2}{\varepsilon} \right) \int_0^\infty K_{lv}(nx)K_{ls}(x) \frac{dx}{x}
\]

\[
= (1-n^2) \lim_{\varepsilon \to 0} \left[ \int_0^\infty x K_{lv}(nx)K_{ls}(x)dx \right]
\]

\[+ \lim_{\varepsilon \to 0} \left[ K_{lv}(nx) \frac{\partial}{\partial x} K_{ls}(x) - K_{ls}(x) \frac{\partial}{\partial x} K_{lv}(nx) \right]_{x=\varepsilon} .
\]

Hence

\[
\int_0^\infty K_{lv}(nx)K_{ls}(x) \frac{dx}{x} = \frac{(1-n^2)}{(\sigma^2 - \nu^2)} \int_0^\infty x K_{lv}(nx)K_{ls}(x)dx
\]

\[+ \lim_{x \to 0} \frac{x}{(\sigma^2 - \nu^2)} \left[ K_{lv}(nx) \frac{\partial}{\partial x} K_{ls}(x) - K_{ls}(x) \frac{\partial}{\partial x} K_{lv}(nx) \right] .
\]

The first integral on the right hand side of the equality sign is given immediately from Erdelyi [24] Vol.2 page 145, formula No.49, and using the fact that

\[
\left[ \Gamma \left( \frac{i(\sigma - \nu)}{2} \right) \right]^{-1} \left[ \Gamma \left( \frac{i(\sigma + \nu)}{2} \right) \right]^{-1} \left[ \Gamma \left( \frac{i(\sigma - \nu)}{2} \right) \right]^{-1} = \frac{8n^2}{(\sigma^2 - \nu^2)(\sinh \pi \sigma - \sinh \pi \nu)} ,
\]
then

\[
\int_0^\infty K_{i\nu}(nx) K_{i\sigma}(x) \, \frac{dx}{x} = \left(1 - \frac{1}{n^2}\right) \pi x^{-i\sigma} \frac{\Gamma(1 + i\sigma)}{\Gamma^2(1 + i\sigma)} \frac{\Gamma(1 - i\nu)}{\Gamma^2(1 - i\nu)} \frac{\Gamma(1 - i\nu)}{\Gamma^2(1 - i\nu)} \frac{\Gamma(1 + i\sigma)}{\Gamma^2(1 + i\sigma)}
\]

(6A.11)

To evaluate the limit term of the above expression the following expressions are required which are obtained from (6A.5) and the series representation for \( I_{i\nu} \).

\[
K_{i\sigma}(x) = \frac{-\pi i}{2 \Gamma(1 - i\sigma)} \left[ \frac{(x/2)^{i\sigma}}{\Gamma(1 - i\sigma)} - \frac{(x/2)^{-i\sigma}}{\Gamma(1 + i\sigma)} \right] + O(x^{2i\sigma}),
\]

(6A.12)

\[
\frac{\partial K_{i\sigma}(x)}{\partial x} = \frac{-\pi \sigma}{2 \Gamma(1 - i\sigma)} \left[ \frac{x^{i\sigma - i\sigma - 1}}{\Gamma(1 - i\sigma)} + \frac{x^{-i\sigma - 1}}{\Gamma(1 + i\sigma)} \right] + O(x^{2i\sigma}),
\]

(6A.13)

\[
K_{i\nu}(nx) = \frac{-\pi i}{2 \Gamma(1 - i\nu)} \left[ \frac{2 i\nu x^{i\nu - 1}}{\Gamma(1 - i\nu)} - \frac{n\nu x^{i\nu}}{\Gamma(1 + i\nu)} \right] + O(x^{2i\nu}),
\]

(6A.14)

\[
\frac{\partial K_{i\nu}(nx)}{\partial x} = \frac{-\pi \nu}{2 \Gamma(1 - i\nu)} \left[ \frac{n\nu x^{i\nu - 1}}{\Gamma(1 - i\nu)} + \frac{x^{i\nu} \nu^{i\nu}}{\Gamma(1 + i\nu)} \right] + O(x^{2i\nu}),
\]

(6A.15)

\[ x \text{ small.} \]

Substituting the expressions (6A.12) to (6A.15) into the last term of the expression (6A.11) gives

\[
\chi = \lim_{x \to 0} \frac{x}{(\sigma^2 - \nu^2)} \left[ K_{i\nu}(nx) \frac{\partial}{\partial x} K_{i\sigma}(x) - K_{i\sigma}(x) \frac{\partial}{\partial x} K_{i\nu}(nx) \right]
\]
\[
\chi = -\frac{i\pi^2}{4\sin\Omega\sin\Omega}\left\{ \frac{-i\pi}{\eta^{\nu}\Gamma(1-i\sigma)\Gamma(1-i\nu)} \delta(\sigma_{j\nu}) + \frac{i\pi}{\eta^{\nu}\Gamma(1+i\sigma)\Gamma(1+i\nu)} \delta(\sigma_{j\nu}) \right. \\
\left. + \frac{\eta^{\nu}i\pi}{\Gamma(1+i\sigma)\Gamma(1-i\nu)} \delta(\sigma_{j\nu}) + \frac{\eta^{\nu}i\pi}{\Gamma(1+i\sigma)\Gamma(1+i\nu)} \delta(\sigma_{j\nu}) \right\}
\]

Using the result (6A.7) reduces the above expression in the limit to

\[
\chi = -\frac{i\pi^2}{4\sin\Omega\sin\Omega}\left\{ \frac{i\pi}{\eta^{\nu}\sin\Omega\sin\Omega} \delta(\sigma_{j\nu}) + \frac{i\pi}{\eta^{\nu}\sin\Omega\sin\Omega} \delta(\sigma_{j\nu}) \right. \\
\left. + \frac{\eta^{\nu}i\pi}{\Gamma(1+i\sigma)\Gamma(1+i\nu)} \delta(\sigma_{j\nu}) + \frac{\eta^{\nu}i\pi}{\Gamma(1+i\sigma)\Gamma(1+i\nu)} \delta(\sigma_{j\nu}) \right\}
\]

\[
= -\frac{\pi^3}{4\sin\Omega\sin\Omega}\left\{ \delta(\sigma_{j\nu}) (\eta^{\nu}\sin\Omega) + \delta(\nu_{j\sigma}) (\eta^{\nu}\sin\Omega) \right\}
\]
Using
\[ \frac{\text{ishom}}{\pi a} = \frac{1}{\Gamma(1-i\sigma)\Gamma(1+i\sigma)} \]

one finally obtains
\[ \chi = \frac{\pi^2}{4\sinh\omega} \left\{ \delta(-\nu, \omega)(e^{i\log n} - e^{-i\log n}) + \delta(\nu, \omega)(e^{i\log n} - e^{-i\log n}) \right\}, \]
\[ = \frac{\pi^2}{2\cosh\omega} \left[ \delta(\omega, \nu) + \delta(\omega, -\nu) \right] \cos(\omega \log n), \]

which proves the desired expression (6A.8).

It might be objected that one should be more careful about taking the derivative of a generalised function, i.e. \( \frac{\partial}{\partial x} x^{i\nu} \), but in fact \( \chi \) can be put in the form
\[ \chi = \lim_{x \to 0} \frac{x}{2(\omega^2 - \nu^2)} \left[ n K_{i\sigma}(x)(K_{i\nu+1}(nx) + K_{i\nu-1}(nx)) \right. \]
\[ - K_{i\nu}(nx)(K_{i\nu+1}(x) + K_{i\nu-1}(x)) \left. \right], \]

where the recurrence relationship
\[ K_{\sigma-1} + K_{\sigma+1} = -2K_{\sigma}, \]

has been used. If the limiting process is carried through one arrives at the same end result.
Consider the integral

$$I = \int \frac{e^{z\psi}dz}{\cosh z - \cosh \sigma}, \quad \sigma > 0, \quad |\psi| < \pi,$$

where the contour of integration is as shown in figure 72.

The only poles of the integrand occur at $\cosh \pi = \cosh \sigma$, i.e. $z = \pm(\sigma \pm 2in)$, $n = 0, 1, 2, \ldots$. Since no poles are enclosed by C, Cauchy's theorem gives

$$\int_{C} \frac{e^{z\psi}dz}{\cosh z - \cosh \sigma} = 0$$

$$= \int_{-R}^{R} \frac{e^{x\psi}dx}{\cosh x - \cosh \sigma} + \int_{-R}^{R} \frac{e^{(x+2i)\psi}dx}{\cosh (2i + x) - \cosh \sigma}$$

$$+ \left\{ \int_{\Sigma_1} + \int_{\Sigma_2} + \int_{\Sigma_3} + \int_{\Sigma_4} + \int_{E_1} + \int_{E_2} \right\} \frac{e^{z\psi}dz}{\cosh z - \cosh \sigma} = 0.$$  \hspace{1cm} (6B.1)
Now
\[ \int_{\Sigma_2} dz = -\pi \lim_{z \to \sigma} \frac{(z-\sigma)e^{z\psi}}{\text{ch} x - \text{ch} \sigma} = -\frac{ie^{\sigma\psi}}{\text{sh} \sigma}, \]
\[ \int_{\Sigma_1} dz = -\pi \lim_{z \to \sigma} \frac{(z+\sigma)e^{z\psi}}{\text{ch} x - \text{ch} \sigma} = \frac{ie^{\sigma\psi}}{\text{sh} \sigma}, \]
\[ \int_{\Sigma_3} dz = -\pi \lim_{z \to \sigma+2i} \frac{(z-\sigma-2i)e^{z\psi}}{\text{ch} x - \text{ch} \sigma} = -\frac{ie^{\sigma+2i\psi}}{\text{sh} \sigma}, \]
\[ \int_{\Sigma_4} dz = -\pi \lim_{z \to \sigma+2i} \frac{(z+\sigma+2i)e^{z\psi}}{\text{ch} x - \text{ch} \sigma} = \frac{ie^{\sigma+2i\psi}}{\text{sh} \sigma}, \]
\[ \int_{\Sigma_1} dz = \int_{\Sigma_2} dz = O(e^{R(1+\psi-\pi)}) \text{ as } R \to \infty. \]

Using the above results in (6B.1) gives as \( R \to \infty \)
\[ (1-e^{2i\psi}) \int_{-\infty}^{\infty} \frac{e^{x\psi}dx}{\text{ch} x - \text{ch} \sigma} = \frac{2i\text{sh} \sigma}{\text{sh} \sigma} (1+e^{i2\psi}), \quad (6B.2) \]

and hence
\[ \int_{0}^{\infty} \frac{\text{ch} x\psi dx}{\text{ch} x - \text{ch} \sigma} = -\frac{\text{sh} \sigma}{\text{sh} \sigma} \cot \psi. \quad (6B.3) \]

Now considering the integral
\[ \int_{C} \frac{ze^{z\psi}}{\text{ch} x - \text{ch} \sigma} dz, \quad \sigma > 0, \quad |\psi| < \pi, \]

where \( C \) is the same contour as before. Then
\[ \int \frac{ze^{z\psi}}{\sin z - \sin \sigma} \, dz = 0, \quad \text{or} \quad \int_{-R}^{R} \frac{x e^{x\psi}}{\sin x - \sin \sigma} \, dx + \int_{R}^{\infty} \frac{e^{(x+2i)\psi}(x+2i)}{\sin x - \sin \sigma} \, dx \]

\[+ \left\{ \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} \right\} \frac{ze^{z\psi}}{\sin z - \sin \sigma} = 0. \quad (6B.4) \]

Now

\[ \int (\ )dz = -\pi i \lim_{z \to \sigma} \frac{(z-\sigma)ze^{z\psi}}{\sin z - \sin \sigma} = -\frac{i e^{\sigma\psi}}{\sin \sigma}; \]

\[ \int (\ )dz = -\pi i \lim_{z \to \sigma+2i} \frac{(z+2i-\sigma)ze^{z\psi}}{\sin z - \sin \sigma} = -\frac{i e^{(\sigma+2i)\psi}}{\sin \sigma}; \]

\[ \int (\ )dz = -\pi i \lim_{z \to \sigma+2i} \frac{(z+\sigma-2i)ze^{z\psi}}{\sin z - \sin \sigma} = -\frac{i e^{(\sigma-2i)\psi}}{\sin \sigma}; \]

\[ \int (\ )dz = \int (\ )dz = 0(\Re e^{R(|\psi| - \kappa)}). \]

Using the above results in (6B.4) gives as \( R \to \infty \)

\[
(1 - e^{2i\psi}) \int_{-\infty}^{\infty} \frac{xe^{x\psi}}{\sin x - \sin \sigma} \, dx - 2ie^{2i\psi} \int_{-\infty}^{\infty} \frac{e^{x\psi}}{\sin x - \sin \sigma} \, dx = \frac{i \sigma}{\sin \sigma} (1 + e^{2i\psi})(e^{\sigma\psi} + e^{-\sigma\psi}) - \frac{2e^{2i\psi}}{\sin \sigma} (e^{\sigma\psi} - e^{-\sigma\psi}).
\]

Substituting the result (6B.2) for the second integral of
the above expression gives

\[(1 - e^{2i\psi}) \int_0^\infty \frac{xe^{ix}}{\cosh x - \cosh\sigma} \, dx + 4ie^{2i\psi} \frac{\sh\psi}{\sh\sigma} \cot \psi\]

\[= \frac{2i\sigma}{\sh\sigma} (1 + e^{2i\psi}) \sh\psi = 4e^{2i\psi} \frac{\sh\psi}{\sh\sigma},\]

and hence

\[\int_0^\infty \frac{x\sh x \psi \, dx}{\cosh x - \cosh\sigma} = -\frac{1}{\sh\sigma} (\psi \cot \psi \sh\psi - \frac{\sh\psi}{\sin^2\psi}).\]

This last result could also have been obtained by differentiating the expression (6B.3) with respect to \(\psi\).
Consider the integral

\[ J = \int_{-\infty}^{\infty} \frac{\nu \cos \nu a}{\sin \nu \pi} \, dv. \]

Making the change of variable \( t = e^{-i\nu \pi} \) gives

\[ J = -\frac{i}{\pi^2} \int_{0}^{\infty} \frac{\log t}{\left(1-t^2\right)} \left(t^\frac{a}{\pi} + t^\frac{a}{\pi}\right) \, dt, \]

and since

\[ \int_{0}^{\infty} t^p \log t \, dt = -\frac{\pi^2}{4 \cos^2 \frac{p\pi}{2}}, \quad \text{Re} \, p < \pi, \]

Gradshteyn and Ryzhik [21] page 573 sec. 4.252, formula No.3, then

\[ J = \frac{i}{2 \cos^2 \frac{a}{2}} = \frac{i}{1 + \cos a}, \quad |a| < \pi. \]

Consider the integral

\[ K = \int_{-\infty}^{\infty} \frac{\sin \nu a \cos \nu b}{\sin \nu \pi} \, dv \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\sin \nu (a+b) + \sin \nu (a-b)}{\sin \nu \pi}\right) \, dv. \]

Making the change of variable \( t = e^{-i\nu \pi} \) gives

\[ K = \frac{i}{2\pi} \int_{0}^{\infty} \left(t^\frac{a+b}{\pi} t^\frac{a+b}{\pi} - t^\frac{a-b}{\pi} - t^\frac{a-b}{\pi}\right) \left(1-t^2\right) \, dt, \]

and since

\[ \int_{0}^{\infty} \frac{t^p}{(1-t^2)} \, dt = -\frac{\pi}{2} \tan \frac{p\pi}{2}, \quad \text{Re} \, p < 1, \]
Erdelyi [24] Vol. 1, page 309 formula No. 8, then

\[ K = \frac{i}{2} \left( \tan \left( \frac{a+b}{2} \right) + \tan \left( \frac{a-b}{2} \right) \right) = \frac{i \sin a}{\cos a + \cos b}, \]

\[ |a \pm b| < \pi. \]
Appendix 7A

The formula

\[ H_0^{(1)}(kr) = i \int_{-\infty}^{\infty} \frac{\cos \frac{\nu(\pi - \psi)}{\nu}}{\sin \frac{\nu}{\nu}} J_\nu(kr) \Im H_\nu^{(1)}(kr_0) d\nu , \]

where \( R^2 = r^2 + r_0^2 - 2rr_0 \cos \psi \), \( 0 < |\psi| < 2\pi \), \( r < r_0 \), will be derived. The dash through the integral sign means that the contour of integration is indented at \( \nu = 0 \).

Consider the integral

\[ \left[ i \int_{-\infty}^{\infty} + i \int_{C} \right] \frac{\cos \frac{\nu(\pi - \psi)}{\nu}}{\sin \frac{\nu}{\nu}} J_\nu(kr) \Re H_\nu^{(1)}(kr_0) d\nu , \]

where the contour of integration is the closed curve shown in figure 73.

Figure 73
An application of Cauchy's residue theorem, as the semi-circle radius tends to infinity gives

\[
\begin{align*}
&\quad i \int_{-\infty}^{\infty} \frac{\cos \psi(\pi-1\psi)}{\sin \psi} J_\nu(kr) H_\nu^{(1)}(kr_o) d\psi \\
&= J_0(kr) H_0^{(1)}(kr_o) + 2 \sum_{n=1}^{\infty} J_n(kr) H_n^{(1)}(kr_o) \cos n\psi \\
&= i \int_{0}^{\infty} \frac{\cos \psi(\pi-1\psi)}{\sin \psi} J_\nu(kr) H_\nu^{(1)}(kr_o) d\psi \\
&= H_0^{(1)}(kr) - i \int_{0}^{\infty} \frac{\cos \psi(\pi-1\psi)}{\sin \psi} J_\nu(kr) H_\nu^{(1)}(kr_o) d\psi,
\end{align*}
\]

where the well known addition formula

\[
H_0^{(1)}(kr) = J_0(kr) H_0^{(1)}(kr_o) + 2 \sum_{n=1}^{\infty} J_n(kr) H_n^{(1)}(kr_o) \cos n\psi
\]

Lebedev [22], has been used. Thus provided the contribution on the infinite semi-circle C vanishes, then (7A.1) is established. From the series representation of \( J_\nu(kr) \)

\[
J_\nu(kr) = \left( \frac{kr}{2} \right)^\nu \left[ \frac{1}{\Gamma(\nu+1)} - \frac{(kr/2)^2}{\Gamma(\nu+2)} + \cdots \right],
\]

\( \text{larg } \nu > \pi, \)

allowing \( \nu \to \infty \) and using the expression

\[
\Gamma(\nu+1) \sim \sqrt{2\pi\nu} e^\nu \log \nu - \nu, \quad \text{larg } \nu < \pi,
\]

\( \text{larg } \nu \to \infty \)

gives
\[
J_v(kr) \sim \left(\frac{kr}{2}\right)^v \frac{1}{\sqrt{2\pi v}} e^{v\log_v - v} \left[ 1 + O\left(\frac{1}{v}\right) \right] .
\]

\[
\text{as } v! \to \infty
\]
Similarly
\[
J_v(kr) = \left(\frac{kr}{2}\right)^v \left[ \frac{1}{\Gamma(1-v)} - \frac{(kr/2)^2}{\Gamma(2-v)} \ldots \right]
\]

which on using the relation \(\frac{1}{\Gamma(1-v)} = \frac{\Gamma(v+1)}{\sqrt{2\pi v} e^{v\log_v - v}}\), and the asymptotic approximation for \(\Gamma(v+1)\) gives
\[
J_v(kr) \sim \left(\frac{kr}{2}\right)^v \sin \frac{\pi}{v} \sqrt{\frac{2\pi}{v}} e^{v\log_v - v} \left[ 1 + O\left(\frac{1}{v}\right) \right] , \text{ as } v! < \pi .
\]

\[
\text{Now}
\]
\[
J_v(kr)H^{(1)}_v(kr) = J_v(kr) \frac{[J_v(kr_0) - e^{-\pi i} J_v(kr_0)]}{\sin \frac{\pi}{v}} ,
\]

and thus substituting into the right hand side the asymptotic forms for the Bessel functions gives
\[
J_v(kr)H^{(1)}_v(kr_0) \sim \frac{1}{\sqrt{2\pi v}} \left(\frac{r}{r_0}\right)^v \frac{e^{-\pi i} (kr/2)^{2v}}{2\pi v \sin \frac{\pi}{v} e^{2v\log_v - 2v}}
\]

\[
\sim \frac{1}{\sqrt{2\pi v}} \left(\frac{r}{r_0}\right)^v, \text{ as } v! < \pi ,
\]

thus
\[
|J_v(kr)H^{(1)}_v(kr_0)| \leq \frac{1}{\sqrt{2\pi v}} \left(\frac{r}{r_0}\right)^v \cos \psi
\]

The remaining term of the integran is
\[ \left| \frac{\cos \psi (\pi - 1 \psi)}{\sin \psi} \right| \leq e^{-\rho (\pi - 1 \psi)} |\sin \psi| \quad |\psi| \to \infty \]

and this expression exponentially decays for \(0 < |\psi| < 2\pi\).

Thus

\[ \left| i \int \frac{\cos \psi (\pi - 1 \psi)}{\sin \psi} J_0(kr) H_0^0(kr) \, d\psi \right| \leq \frac{1}{\pi} \int_0^{\pi/2} e^{-\rho (\pi - 1 \psi)} |\sin \psi| - \rho \cos \psi \log \left( \frac{r_0}{r} \right) \, d\psi, \]

\[ = \frac{2}{\pi} \int_0^{\pi/2} \frac{e^{-\rho (\pi - 1 \psi)} |\sin \psi| - \rho \cos \psi \log \left( \frac{r_0}{r} \right)}{e^{-\rho (\pi - 1 \psi)} - e^{-\rho \log \left( \frac{r_0}{r} \right)}} \, d\psi, \]

\[ \to 0 \quad \text{as} \quad \rho \to \infty, \text{since} \quad 0 < |\psi| < 2\pi, \quad r_0/r >> 1, \text{where} \]

the inequalities

\[ \sin \psi \geq \frac{2\psi}{\pi}, \quad \cos \psi \geq 1 - \frac{2\psi}{\pi}, \quad \text{for} \quad 0 \leq \psi \leq \pi/2 \]

have been used.

Thus the proof of (7A.1) is complete.
References


