Twin Control Moment Gyros for Small Satellites

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Summary

Future spacecraft will, of necessity, require rapid rotational manoeuvrability or agility. Such rapid retargeting manoeuvres are often subject to the physical limits of actuators. Control Moment Gyros (CMGs) are employed as primary actuators because of their high torque control capability. A typical configuration used is the 4 Single-Gimbal Control Moment (4SGCMG) pyramid mounting arrangement [48].

In the 1960s researchers studied an arrangement of CMGs called Twin Control Moment Gyros (TCMGs) [6, 11,20]. At the time these configurations were considered unsuitable for large satellite platforms, and the 4SGCMG pyramid configuration became the most common arrangement. The research in this report revisits their work, to address the suitability of TCMGs to future small satellite missions. Candidate configurations of both SGCMG and TCMG arrangements are considered as possible replacements for the 4SGCMG pyramid configuration. The candidate configurations are compared in terms of torque capability, power consumed and mass and volume occupied.

In this report the principles of Single-Control Moment Gyro and Twin-Control Moment Gyro operation are studied. The study of Euler’s equations and controller principles lead to the implementation of an attitude control system in SIMULINK, for which results are compared with previous work [46].

The results of a sizing/selection process show that the 4SGCMG pyramid configuration performs the best all round in terms of torque developed, power consumed and mass and volume occupied. Of the candidate configurations the 6SGCMG pyramid configuration stands out as a possible replacement for the baseline configuration. Of the TCMG configurations the 3TCMG pyramid configuration is the most promising. It is recommended that the 3TCMG pyramid configuration is investigated further. If the total mass of this system could be reduced it could be a contender as a possible replacement for the 4SGCMG pyramid configuration.
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Nomenclature

d ratio of total cluster number of CMGs
f some function
\( \vec{h} \) angular momentum
k number of optimization/sizing algorithm iterations, no units
n total number of configurations considered, no units
\( r_i \) inner rotor radius, m
\( r_o \) outer rotor radius, m
t constant relating torque capability of configuration i, to the torque capability of the baseline
\( \vec{u} \) control torque input vector
\( \vec{x} \) vector of decision variables
\( \vec{x}^* \) vector of the optimal decision variables
F the feasible region of the multobjective optimization problem
\( I_{rot} \) is the inertial of the rotor about the rotor spin axis, \( kgm^2 \)
I inertia, \( kgm^2 \)
M mass, kg
P power consumption, Watts
T torque capability, Nm
V occupied volume, \( m^3 \)

\( \chi \) efficiency, no units
\( \Omega \) rotor spin speed, \( rad/s \)
\( \ddot{\Omega} \) rotor acceleration, \( rad/s^2 \)
\( \delta \) gimbal angle, rad
\( \ddot{\delta} \) gimbal rate, rad/s
|\( \Phi_{le} \)| torque capability comparison, no units
\( \eta \) total number of rotors in a unit
\( \mu \) number of rotors in a unit
\( \eta \) total number of rotors in the configuration, no units
\( \dot{\omega} \) body angular rate, \( rad/s \)
\( \ddot{\omega} \) body angular acceleration, \( rad/s^2 \)
\( \gamma \) objective function weighting constants, no units
\( \ddot{\theta} \) body angular acceleration
\( \Gamma \) torque acting on the CMG rotor (Nm)
\( \tau \) torque acting on the satellite (Nm)

Subscripts
a actual
b baseline
B  Body Axis Coordinate

 Gimbal motor
i  identifier for configuration
LO  Local Orbit Coordinate
m  design margin
max  maximum value
min  minimum value
r  required
rmotor  rotor motor
rot  rotor
s  about the spin axis

**Superscripts**

DISC  the disc part of the simple rotor model
GIMBAL  in power equations—the contribution by the gimbal motor
RING  the ring part of the simple rotor model
ROT  used in power equations—the contribution by the rotor motor
STAT  stationary—when the rotor is spinning about its spin axis but not gimbaled
SWEPT  when the rotor is gimbaled
UNIT  a single unit in the cluster (consisting of 1 CMG for SGCMG configurations, and 2 CMGs for TCMG configurations)

**Definitions**

CMG  Control Moment Gyroscope
SGCMG  Single Gimbal Control Moment Gyroscope
TCMG  Twin Control Moment Gyroscope
1 Introduction

1.1 Previous Work

Some of the first spacecraft built in the early 1960s used on-board gyroscopes to sense 3-dimensional vehicle attitude. It was noted by White and Hansen that these devices imparted a small disturbance torque during operation, which could be harnessed and used for the control of the attitude of the satellites [41]. CMGs have been used for key spacecraft programs, including Skylab, the Hubble Space Telescope, and the International Space Station (ISS). Primarily, though, these devices have been used for medium to large satellites. They are a simple way of providing attitude control without the costly use of liquid fuel or other types of thrusters. Most of these CMGs are quite large, up to a few meters in diameter. However, quite recently, CMGs have been designed and flown by SSTL and NASA for small satellites [3, 13, 17].

A CMG contains a spinning rotor with large, constant angular momentum, whose angular momentum vector direction can be changed with respect to the spacecraft by gimballing the spinning rotor. The spinning rotor is mounted on a gimbal (or a set of gimbals), and torquing the gimbal results in a precessional, gyroscopic reaction torque orthogonal to both the rotor spin and gimbal axes. The CMG is a torque amplification device because small gimbal torque input produces large control torque output on the spacecraft [17].

Margulies and Aubrun (1978), and Oh, Vadali and Walker (1989), made contributions by capturing the exact nonlinear equations of satellite motion using CMGs, including single- and double-gimbal configurations, gimbal lock singularities, and CMG steering control laws [22, 39]. Further work describing different CMG singularity avoidance methods was presented in the early 1990s by Bedrossian, Paradiso, and Bergmann [2].

One of the key sources for familiarization with attitude control systems is presented by Wie in 1998, [46] in which he covers the mathematical modeling of single gimbal CMGs. He also introduces the existing steering laws, attitude control laws and equations required for modeling CMG based attitude control systems.
In reference [48] Wie develops a nonlinear feedback control logic for large-angle, rapid multitarget acquisition and pointing manoeuvres subject to various physical constraints, including actuator saturation, slew rate limit, and control bandwidth limit. In his work the proposed logic is applied to a more realistic problem of controlling an agile spacecraft using redundant SGCMGs. His results are replicated and discussed further in chapter 3 of this report.

1.2 Twin Control Moment Gyros (TCMGs)

In the 1960s researchers at NASA studied the effects of the performance of Twin CMG attitude control systems. In 1964 Havill and Ratcliff report that the TCMG attitude control system would be especially adaptable to space vehicles which require precise attitude control with high dynamic response and low values of cross-coupling torque [11]. Later that same year Havill, Ratcliff and Lopez demonstrated that an automatic TCMG system can stabilize a large space vehicle to within 1 sec of arc [20, 21].

In 1970 Campbell developed a nonlinear control law [6], the results of which were analytically and experimentally verified. The research outcomes led to the conclusion that TCMG systems appeared most applicable for space vehicle missions requiring many fast, accurate, large angle manoeuvres.

In 1971 Riper and Liden developed a unique arrangement of four single gimbal CMGs, where each pair of CMGs with collinear axes can be considered as a scissor pair. In this arrangement, referred to as a 4 SGCMG fine attitude control system (4-FACS), the control moment gyroscopes are arranged in two pairs, where the two gimbal axes in each pair are parallel or collinear, and where the gimbal axes of one pair are typically perpendicular to the gimbal axes of the other pair. The gimbal axes lie in the yz-plane of the CMG system coordinates, with angles of 45°, 135°, 225° and 315° from the y-axis and counterclockwise about the x-axis. This configuration permits the use of a relatively simple constant gain steering law. The angular relationships between the gimbals and between the gimbal pairs may be modified if it is desired to modify the relative angular momentum capacity of the configuration, without unduly complicating the steering law [30].

In operation, both the magnitude and direction of the net angular momentum vector of the control moment gyro system is controlled by controlling the relative magnitude and direction of
the resultant angular momentum of each pair of gyros, each pair being arranged as a scissored pair [30]. This research is particularly interesting as its results indicated potential flexibility in the positioning of the individual CMGs in TCMG pairs; this idea is discussed in further work in Chapter 5, as the principle of electronic synchronization. The invention provides a CMG configuration wherein only minor modifications of the steering law are required when converting from a four gyro operation to a three gyro operation.

Figure 1-1 Schematic Illustration of the CMG Configuration and Its Orientation Relative to a Set of Reference Axes, adapted from [8].

Despite interest in the 1960s and 70s TCMG systems were deemed not to be viable with the current technology and research halted. However the idea has become much more plausible when one considers the recent development of smaller CMGs. In their work from 1999-2004, Lappas et al have shown the potential of Control Moment Gyroscopes (CMG) as an alternative and more efficient actuator for microsatellites [15, 16, 17, 18]. Also considering the recent advancements in microprocessor technology, and control laws, alongside the developments of small scale CMGs, it is viable to investigate the potential of TCMG systems for the attitude control of small satellites.

More recently in 2005/2006 TCMG systems have been researched for multibody robotic systems by Peck [25, 26, 27]. His work on a satellite "MaintainanceBot" with high agility and low power requirements benefits dramatically from the dynamics and control of a multibody
robotic arm whose joints are driven by CMGs. The baseline concept includes a scissored pair of CMGs for each rotational joint. This work has progressed to the investigation of TCMG systems in prosthetic limbs [28]. Peck and his team at Cornell University have designed a novel, three degree-of-freedom prosthetic arm actuated with small-scale CMGs. Each of the three segments contains one CMG scissored pair, which allows precise control over joint torques and accelerations. This work demonstrates the potential of small-scale CMGs in TCMG systems and one can see the potential transition to small satellite applications.
1.3 Contributions of this Thesis

The research presented in this document compares various SGCMG and TCMG actuator configurations to answer the question of TCMG suitability to future spacecraft missions. To compare SGCMG and TCMG configurations, the torque, mass, volume and power consumed are calculated and compared. The results from the sizing process are considered, for two cases. The first case in which the torque developed by the candidate configurations is maximized for a configuration mass equivalent to the baseline configuration mass. The second case for which the mass of the candidate configuration is minimized, for torque equivalent to the torque developed by the baseline configuration.

In order to size a cluster the inertia of the CMG rotor is varied by changing (sizing) the rotor length. All other rotor dimensions are fixed and equal for all configurations. The effects of the rotor and gimbal motor masses and volumes are not considered at this stage.

The main objectives for this study are:

1. To understand the principles of Single Control Moment Gyro (SGCMG) operation
2. To understand the principles behind Twin Control Moment Gyro (TCMG) operation
3. To develop a Dynamic Attitude Control System (ACS) Model using SIMULINK
4. To develop a sizing/selection process in MATLAB to compare the applicability of SGCMG and TCMG configurations to small satellite missions

1.4 Outline of the Thesis

Figure 1-2 shows an overview of the modules which contribute to sections of the thesis.

Chapter 1 includes a review of previous work on SGCMGs, and TCMGs and optimization and sizing, it highlights the contributions of the thesis and an outline of the thesis.

In Chapter 2 a comparison is made between different types of CMGs. Twin control moment gyros are discussed in detail and an analysis of the principles of a TCMG platform is presented. Steering laws, attitude controllers and satellite dynamics are covered and the equations used to set up the Dynamic Attitude Control System model in SIMULINK are introduced and explained, and simulations generated by the developed model are presented. Block diagrams of the developed ACS are included in Appendix A.
Chapter 3 introduces the six candidate configurations that will be sized, and there follows a detailed angular momentum vector analysis of each configuration, resulting in the torque capabilities of each configuration, the values of which are needed in sizing and selection.

Chapter 4 introduces the sizing/selection process, and includes a detailed description of the method used and the program written is presented in Appendix B. Results for two cases are presented and analyzed and conclusions are drawn.

In Chapter 5 open problems and areas for future work are discussed.

Figure 1-2 Overview of the Sizing/Selection Program and the dynamic Attitude Control System model
2 CMG Fundamentals

CMGs are currently used on many space platforms, including the International Space Station. They are a simple way of providing attitude control without the costly use of liquid fuel or other types of thrusters. Most of these CMGs are quite large, up to a few meters in diameter.

A CMG consists of a spinning rotor which is gimbaled about an axis which is perpendicular to the rotor’s axis of rotation. This gimbal requires a torque, to induce the vector change in the satellite’s angular momentum. Due to Newton’s law, this torque applied to the CMG by the satellite is also applied by the CMG on the body (in accordance with the conservation of angular momentum).

CMGs obey the following equations:

\[ \tau = \frac{d}{dt} H \]  

\[ H = \omega_g \times h_{\text{rotor}} \]

It is important to note that when a CMG is gimbaled its angular momentum vector changes direction.

2.1 Single Gimbal Control Moment Gyro (SGCMG)

One type of CMG actuator is the Single Gimbal Control Moment Gyro (SGCMG), see Figure 2-1, which is a powerful torque generator for spacecraft attitude control. Because the overall cost and effectiveness of such agile spacecraft are greatly affected by the average retargeting time, the development of a low-cost attitude control system employing smaller and inexpensive CMGs, called mini-CMGs, is of practical importance for developing future agile scientific spacecraft [17].

The SGCMGs have the advantages of possessing relative mechanical simplicity and producing amplified torques (for low spacecraft angular velocities) on the spacecraft [1].
In a SGCMG configuration the following statements hold

1. gimbal position imparts a satellite body rate
2. gimbal velocity imparts a satellite body acceleration
3. gimbal acceleration imparts satellite body jerk

The rate of change of angular momentum between the CMG and the body is dependent on the gimbal rate. The developed control torque lies in a plane perpendicular to the gimbal shaft, thus producing two-axis coupling. The control torque is transmitted through the gimbal axis bearings to the satellite. The direction of the output torque changes in accordance with the gimbal axis motion. The CMG output torque is given in vector form as

\[ N = h \times \dot{\delta} \]  \hspace{1cm} 2-3

Where, \( h \) is the angular momentum and \( \dot{\delta} \) is the gimbal rate.

A system of several SGCMG or DGCMG units is needed to obtain the required torque. A system of such SGCMGs is known to be severely cross coupled and requires a complex control computer to decouple the control torques so that three-axis control can be attained. These cross coupling torques can be essentially eliminated by a twin gyro control system.
2.2 Variable Speed Control Moment Gyro (VSCMG)

A second actuator is the Variable Speed CMG (VSCMG). A VSCMG is essentially a gimballed Momentum Wheel (MW). It is comprised of a rotor and a gimbal, both driven by independent, direct-current (DC) motors [29].

2.3 Energy Storage and Attitude Control System (ESACS)

A method to reduce mass (and thus cost) is to combine key satellite functions. For example, a satellite's Energy Storage (ES) function, usually achieved via rechargeable batteries, can be combined with its pointing system (i.e. the ACS), forming an ESACS. Such an ESACS can entail using flywheel-based, three-axis stabilising, momentum exchange actuators such as reaction wheels (RWs), MWs, CMGs, or VSCMGs as energy storage devices. It operates by increasing and decreasing the wheel speed through the wheel motor. This wheel torque produces an equal-and-opposite torque on the spacecraft. Secondly, the wheel can be gimballed as is done in a CMG, producing an amplified output torque magnitude as compared its MW torque. This torque comes from redirecting existing high speed rotational wheel energy with a small input gimbal torque rather than creating high-energy torque from scratch [29].

2.4 Double Gimbal Control Moment Gyro (DGCMG)

A third actuator is the double gimbal CMG (DGCMG) which is a momentum wheel suspended inside two gimbals and therefore the momentum vector can be aimed along any direction on a sphere, see Figure 2-2. The gimbal steering logic of a DGCMG can more easily avoid singularities since it has an extra degree of freedom. However, a SGCMG is a lot simpler from a hardware point of view and it has significant cost, power, weight and reliability advantages over a DGCMG [14]. The SGCMG also has the advantage of producing an output torque across the gimbal bearings (directly into the supporting structure) considerably larger than the input torque initially required to drive the gimbal suspension, provided that the spacecraft has initially low inertial angular rates. By contrast, double gimbal CMGs must balance the same output torque with their outer gimbal motors before transmitting it to the supporting structure, and more problems arise from their greater mechanical complexity [14].
2.5 Twin Control Moment Gyro (TCMG)

A fourth actuator is the Twin CMG (TCMG), see Figure 2-3. A TCMG platform consists of two one-degree-of-freedom CMGs geared together so that their angular momentum vectors always remain in a position that is the mirror image of the other, as shown in Figure 2-5. An advantage of this type of controller is that it eliminates the gyroscopic cross coupling inherent in a single gyro system, thereby allowing large gimbal angle deflections so that most of the momentum stored in the gyros can be transferred to the vehicle. The elimination of cross coupling also permits the use of an independent control system about each axis, which drastically simplifies the control algorithms [6].
In a TCMG, two simple constraints are constructed resulting in angular momentum always falling along a straight line. With the two gimbal axes parallel, if the two CMGs are constrained to rotate with equal and opposite gimbal angles, their net angular momentum will always fall on a straight line. This straight line can be aligned with the axis of rotation of the satellite to achieve motion. Note that in the neutral position, the angular momentum vectors sum to zero, while in the active position their vector sum will always lie along the straight line [6,11].

The limit of the transfer of angular momentum occurs when the angular momentum vectors achieve a parallel orientation, the maximum possible amount of angular momentum to the satellite has been transferred (it is moving at its peak rate). This says that a torque in one direction indefinitely cannot be applied. Once the angular momentum vectors are parallel, no more torque can be applied. This means that if the arm is required to apply some constant torque, it will not be able to be achieved purely through CMG actuation. This also means that the dynamics of the satellite are bounded by the duration for which torque can be applied. The satellite cannot accelerate indefinitely.

This feature allows the use of large gimbal angles so that a major portion of the momentum stored in the gyros can be transferred to the vehicle without introducing cross-coupling torques. This is an advantage over the single gyro system which must be restricted in gimbal angle to minimize the cross-coupling torques. The advantage of eliminating cross-coupling torque is the requirement of the simplest control computer. A TCMG control system typically requires three TCMG platforms to provide torques about three orthogonal vehicle axes.

Figure 2-4 Momentum lines of the TCMG resulting in angular momentum [1]
The spin reference axis of each CMG can be arbitrarily oriented in the plane perpendicular to its momentum exchange axis, since for the twin-gyro configuration no net angular momentum exists about the spin reference axis. A further feature of this configuration is the ability to turn both gyros in a platform on or off during flight, with no disturbance torque. This achieved by caging both gyro spin axis i.e. holding them both in the zero gimbal angle.

In Figure 2-5 T is the torque acting on the gimbal shaft generated by each CMG in Nm. Where $\delta$ is the gimbal rate in rad/s and the angular momentum vector of each CMG is designated $h_{i1}$ and $h_{i2}$. The first subscript associates the angular momentum vector with a particular platform; the second subscript indicates the gimbal axes, axis 1 being the torquer driven shaft.
In Figure 2-6, $\delta_1$ is the gimbal rate, $\delta_2$ is the gimbal rate and for a TCMG $\delta_1 = \delta_2$.

2.6 TCMG Actuator Dynamics

Throughout this report the complete assemblage of the twin CMG will be referred to as being a TCMG platform. The platforms are assumed to be fastened rigidly to the satellite vehicle structure.

For all SGCMG and TCMG configurations it is assumed that:

1. The CMGs are rigidly fastened to the vehicle structure; that is the body-fixed coordinate system corresponds to the CMG fixed coordinate system.

2. The mass moments of inertia of each rotor and each of their gimbal structures are equal.

3. The magnitudes of the rotor's angular momentum vectors are equal and constant.
4. No mass unbalance exists about the gimbal axis; that is, the centre of mass of the CMG lies on the gimbal axis.

The three TCMG platforms shown in Figure 2-7 are designated as X, Y and Z, corresponding to the vehicle’s X-, Y- and Z-axes, respectively. These platforms are aligned so that the developed control torques act along the vehicle’s principle axes. From this point forward, this configuration will be called the 3TCMG axis aligned configuration. This is one of the candidate TCMG configurations which will be considered in detail in Chapters 3 and 4.

![Figure 2-7 Three-axis Twin Control Moment Gyro Control System (3TCMG Axis Aligned Configuration) [6]](image_url)

A vector schematic of the X-axis platform is shown in

Figure 2-8. The angular momentum vector of each CMG is designated $h_{1,1}$ and $h_{1,2}$
The angular momentum vectors, written in terms of the satellite body-fixed coordinates, are [6]

\[
\tilde{h}_{x_1} = h_{x_1} \cos \delta \hat{v} + h_{x_1} \sin \delta \hat{k} \tag{2-4}
\]

\[
\tilde{h}_{x_1} = h_{x_1} \begin{bmatrix} 0 \\ \cos \delta \\ \sin \delta \end{bmatrix}
\]

\[
\tilde{h}_{x_2} = -h_{x_2} \cos \delta \hat{v} + h_{x_2} \sin \delta \hat{k} \tag{2-5}
\]
\[ \vec{\tau} = \frac{d\vec{h}}{dt} \bigg|_{\text{ref}} - \vec{\omega}_h \times \vec{h} \]

Equation 2-6 yields the torques acting on the rotors. To obtain the torques acting on the satellite, one must take the negative of Equation 2-6, since the satellite vehicle control torques are actually the reaction torques of those described by Equation 2-6. Therefore

\[ \vec{\tau}_{\text{control}} = \frac{d\vec{h}}{dt} \bigg|_{\text{ref}} - \vec{\omega}_h \times \vec{h} \]

The first term in Equation 2-7 gives the torques produced by the rotor's precession mechanism, while the second term, \( \vec{\omega}_h \times \vec{h} \) is produced by the angular velocity, \( \vec{\omega}_h \) of the satellite. The angular velocity is defined as

\[ \vec{\omega}_h = [\omega_x \quad \omega_y \quad \omega_z]^T \]

The torques acting on the rotors are found by substituting Equations 2-4, 2-5, and 2-8 into 2-7.

Using the chain rule,

\[ \frac{d\vec{h}}{dt} = \frac{d\vec{h}}{d\delta} \times \frac{d\delta}{dt} \]

Where

\[ \frac{d\delta}{dt} = \dot{\delta} \]

\[ \frac{d\vec{h}_{\delta}}{d\delta} = \vec{h}_{\delta} \begin{bmatrix} 0 \\ -\sin \delta \\ \cos \delta \end{bmatrix} \]
\[
\frac{dh_{x2}}{d\delta} = h_{x2} \begin{bmatrix} 0 \\ \sin \delta \\ \cos \delta \end{bmatrix}
\]

\[
\bar{\Gamma}_{x1} = h_{x1} \begin{bmatrix} (\omega_y \sin \delta - \omega_z \cos \delta) \\ -(\omega_z + \dot{\delta}) \sin \delta \\ (\omega_z + \dot{\delta}) \cos \delta \end{bmatrix}
\]

\[
\bar{\Gamma}_{x2} = h_{x2} \begin{bmatrix} (\omega_y \sin \delta + \omega_z \cos \delta) \\ -(\omega_z - \dot{\delta}) \sin \delta \\ -(\omega_z - \dot{\delta}) \cos \delta \end{bmatrix}
\]

A free body sketch of the X-axis platform is shown in Figure 2-9. \( T_m \) is the torque applied by the gimbal shaft torque motor, and \( D\dot{\delta} \) is a rate damping torque created by an eddy current damper, whose damping coefficient D has the units \( \frac{Nm}{rad/s} \).
Figure 2-9 Free Body Diagram of TCMG on the X-Axis Platform, adapted from [6]
The torque components of Equations 2-13 and 2-14 are shown acting on their respective rotors. The inverse of these rotor torques is then shown acting on the gimbal shaft's structure. The result of these rotor torques, as reflected on the number one gimbal shaft, is obtained by adding their respective components. The torques about the Y- and X-axis add directly but, because of the gear configuration, a positive torque about the X-axis of the number two gimbal shaft will be reflected onto the number one shaft as a negative, thus [6];

\[
T_{\text{nx}}^v = T_{x11}^v - T_{x22}^v = -h_{x1} (\omega_x \sin \delta - \omega_z \cos \delta) + h_{x2} (\omega_x \sin \delta + \omega_z \cos \delta) \quad 2-15
\]

\[
T_{\text{nx}}^v = T_{x1x}^v + T_{x2x}^v = h_{x1} (\omega_x + \delta) \sin \delta + h_{x2} (\omega_x - \delta) \sin \delta \quad 2-16
\]

\[
T_{\text{nx}}^v = T_{x1z}^v + T_{x2z}^v = -h_{x1} (\omega_x + \delta) \cos \delta + h_{x2} (\omega_x - \delta) \cos \delta \quad 2-17
\]

The second subscript indicates the component of the torque with respect to the first subscript's platform. The subscript \( v \) indicates the vehicle coordinate system. Since it was assumed that the magnitude of the rotor's angular momentum vectors were equal,

\[
|\omega_x| = |\omega_z| = |\omega_t| \quad 2-18
\]

Equations 2-15, 2-16 and 2-17 are reduced to

\[
T_{\text{nx}}^v = 2h_x \omega_x \cos \delta \quad 2-19
\]

\[
T_{\text{ny}}^v = 2h_x \omega_y \sin \delta \quad 2-20
\]

\[
T_{\text{nz}}^v = -2h_x \omega_z \cos \delta \quad 2-21
\]

The torques \( T_{\text{nx}}^v \) and \( T_{\text{ny}}^v \) are transmitted through the gimbal axis bearings to the vehicle, while \( T_{\text{nz}}^v \) acts upon the gimbal shaft [6].

2.7 Mission Maneouvre Requirements

For manoeuvring, the actuators can be sized by knowing the torque they must produce. This torque itself can be found by determining how quickly the spacecraft must rotate. This is often a matter of using sound engineering judgment. For example, it may not be reasonable for a spacecraft to take 30 minutes to rotate through 90 degrees, even though the mission requirements may not give a specific manoeuvring rate. In some cases, rotation rates may be
given explicitly. In either case, these rates can be used to estimate what angular accelerations are required, and these accelerations lead directly to torques, which in turn define the size of the actuators.

Requirements that must be considered include the following;

1. Mission related operation scenarios, large slew manoeuvres, and small slew manoeuvres
2. Effect of maximum slew angle to spacecraft structure/stability
3. Angular rate profile (bang-bang or bang-off-bang), torque profile
4. Gimbal rate/angular momentum selection/ trade-off
5. Hardware constraints (motor capability, subsystem/spacecraft mass, volume and power budgets.

Given a the requirements of a maximum slew rate of \( \omega_{\text{max}} \) and a time to reach the maximum slew rate, \( t_s \) the required body angular acceleration, \( \ddot{\vartheta} \) is calculated using;

\[
\ddot{\vartheta} = \frac{\omega_{\text{max}}}{t_s}
\]

 Knowing the principle Moment of Inertia (MOI) for the CMG wheel about its spin axis \( I_{ws} \), and the spin speed \( \Omega \), combined with the knowledge of the satellite principle axis inertia, \( I_{r,\text{max}} \), the torque required to perform the manoeuvre is given by;

\[
N_r = I_{ws} \Omega = I_{r,\text{max}} \ddot{\vartheta}
\]

The slew manoeuvre speed depends on the rotor motor’s torque and the rotor’s momentum capability.

### 2.7.1 Bang-Bang Manoeuvre

The bang-bang manoeuvre is depicted in Figure 2-10. The assumption is that the manoeuvre is a rest to rest manoeuvre which begins at \( \vartheta_0 = 0 \text{ with the satellite attitude equal to } 0 \text{ deg.} \) The satellite then undergoes an acceleration phase caused by an angular momentum exchange with the CMG actuator cluster. This is the initial bang phase which lasts for half of the manoeuvre. For the second half of the manoeuvre the satellite decelerates to reach the desired angle, \( \vartheta_f \) in
the allotted time, $t_r$. The torque required to achieve the manoeuvre is $T_r$ and the angular momentum of the satellite, $H_n$ is given as:

$$H_n = I_{r_{\text{max}} \omega}$$  \hspace{1cm} 2-24

Where $I_{r_{\text{max}}}$ is the satellite principle-axis inertia, and $\omega$ is the satellite angular velocity.

Knowing the total required angular momentum, $H_n$ and the torque capability, $|\vec{\omega}|_{\text{le}}$ of a configuration, the angular momentum required from each CMG, $h$ in a cluster can be calculated using:

$$h = \frac{H_n}{|\vec{\omega}|_{\text{le}}}$$  \hspace{1cm} 2-25

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2-10.png}
\caption{Bang-bang slew manoeuvre}
\end{figure}

2.7.2 Bang-off-Bang Manoeuvre

Typical thrust or torque manoeuvre utilize a “bang-off-bang” control sequence. That is, the spacecraft first commands an initial control pulse to build a manoeuvre rate, followed by a coast phase until it reaches the desired final attitude, at which point another control pulse commands the final desired angular rates. Generally, the “bang-off-bang” trajectory proves fuel-optimal; however, the penalty associated with the “bang-off-bang” manoeuvre is a small increase in the completion time.

When sizing a CMG for a proposed minisatellite mission a bang-off-bang type of manoeuvre is used [17]. In order to efficiently complete the manoeuvre in the desired time it the angular acceleration is increased in order to rapidly reach the maximum allowable slew rate. In this type
of manoeuvre the maximum angular speed is constrained, this avoids all kinds of spacecraft level complications that excessive slew rates can cause. The same satellite characteristics as described for the previous "bang-bang" manoeuvre are used, but the assumptions are extended to include a deadband phase.

The bang-off-bang manoeuvre is depicted in Figure 2-11. Where \( \theta_f \) is the desired angle which is reached in the allotted time \( t_f \), which includes a deadband period of \( t_{of} \) seconds. The torque required to achieve the manoeuvre is \( T_r \), \( \omega \) is the satellite angular velocity and \( \dot{\theta}_{\text{max}} \) is the maximum angular acceleration of the satellite.

2.8 Dynamics Attitude Control System (ACS) Model

Control is the process of orienting and moving the spacecraft in the desired direction depicted by the guidance. This includes attitude stabilization (maintaining the attitude in a desired state), the attitude manoeuvre control (changing the attitude from one orientation, or the old state, to another orientation, the new state), and moving the spacecraft to the desired trajectory. This process involves the use of control hardware (actuators such as CMGs), on-board or remote computers to generate commands and the relevant software [35].

Once the attitude of a spacecraft is found using the various sensors, its attitude dynamics must be modeled mathematically before it can be controlled. In this section, basic information by which spacecraft can be modeled will be presented. Equations relating the angular momentum of the spacecraft about the set of principal axis will be derived. The basic problem of attitude
dynamics is the determination of a spacecraft's motion about a given set of axes. In the case of a rigid body with no external or internal torques, this motion is derived from the angular momentum of the body. In its simplest form, the equations for rigid body momentum reduce to:

\[ h = I \omega \]  

2.9 Spacecraft Dynamics, Euler's Equations

The general equation of angular motion can be found after the components of angular momentum have been found. In general, the moment about the center of mass with respect to the body coordinate axis (B) is given by:

\[ I \ddot{\omega}_n + \ddot{\omega}_n \times I \dot{\omega}_n = \ddot{u} \]  

Where \( I \) is the satellite inertia tensor.

This equation reduces to Euler's rigid body moment equations:

\[ I_x \dot{\omega}_x = u_x - (I_z - I_y)\omega_y \omega_z \]  
\[ I_y \dot{\omega}_y = u_y - (I_z - I_x)\omega_z \omega_x \]  
\[ I_z \dot{\omega}_z = u_z - (I_y - I_x)\omega_x \omega_y \]  

Where

\[ I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \]  

\[ \ddot{\omega}_n = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} \]  

\[ \ddot{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \]  

2.10 Spacecraft Attitude Kinematics

The formulation of spacecraft attitude dynamics and control problems involves considerations of kinematics. In kinematics we are concerned with describing the orientation of a body that is in
rotational motion. The subject of rotational kinematics does not involve any forces associated with motion.

2.10.1 Direction Cosine Matrix

Consider a reference frame A with a right-hand set of three orthogonal unit vectors \( \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \) and a reference frame B with another right-hand set of three orthogonal unit vectors \( \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \). Basis vectors \( \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \) of B are expressed in terms of basis vectors \( \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \) of A as follows:

\[
\begin{bmatrix}
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\vec{a}_1 \\
\vec{a}_2 \\
\vec{a}_3
\end{bmatrix} = \begin{bmatrix}
\vec{a}_1 \\
\vec{a}_2 \\
\vec{a}_3
\end{bmatrix}
\]

Where \( \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \) is called the direction cosine matrix, which describes the orientation of B relative to A. The direction cosine matrix is also called the rotation matrix or coordinate transformation matrix to B from A. Such a coordinate transformation is symbolically represented as

\[ \bar{C}^{B/A} : B \leftarrow A \]

For brevity, we will use \( \bar{C} \) for \( \bar{C}^{B/A} \). Because each set of basis vectors of A and B consists of orthogonal unit vectors, the direction cosine matrix \( \bar{C} \) is an orthonormal matrix.

Three elementary rotations respectively about the first, second and third axes of the reference frame A are described by the following rotation matrices [46]:

\[
\bar{C}_i(\theta_i) =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_i & \sin \theta_i \\
0 & -\sin \theta_i & \cos \theta_i
\end{bmatrix}
\]

\[ \bar{C}_i(\theta_i) \]
\[ \tilde{C}_2(\theta_2) = \begin{bmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 1 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix} \quad 2-37 \]

\[ \tilde{C}_3(\theta_3) = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 2-38 \]

Where \( \tilde{C}_i(\theta_i) \) denotes the direction cosine matrix \( \tilde{C} \) of an elementary rotation about the \( i \)th axis of \( A \) with angle \( \theta_i \).

### 2.10.2 Euler Angles

One scheme for orientating a rigid body to a desired attitude is called a body-axis rotation; it involves successfully rotating three times about the axes of the rotated, body-fixed reference frame. The first rotation is about any axis. The second rotation is about either of the two axes not used for the first rotation. The third rotation is then about either of the two axes not used for the second rotation. There are 12 sets of Euler angles for such successive rotations about the axes fixed in the body.

Consider three successive body-axis rotations that describe the orientation of a reference frame \( B \) relative to a reference frame \( A \). A particular sequence chosen here is symbolically represented as

\[ \tilde{C}_3(\theta_3): B \leftarrow A' \]
\[ \tilde{C}_2(\theta_2): A' \leftarrow A \]
\[ \tilde{C}_1(\theta_1): A' \leftarrow A \]

Where each rotation is described as

\[
\begin{bmatrix}
\tilde{a}_1' \\
\tilde{a}_2' \\
\tilde{a}_3'
\end{bmatrix} = \begin{bmatrix}
\cos \theta_3 & \sin \theta_3 & 0 \\
-\sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix} = \tilde{C}_3(\theta_3) \begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix} \quad 2-40
\]
\[
\begin{bmatrix}
\tilde{a}_1^* \\
\tilde{a}_2^* \\
\tilde{a}_3^*
\end{bmatrix}
= \begin{bmatrix}
\cos \theta_2 & 0 & -\sin \theta_2 \\
0 & 1 & 0 \\
\sin \theta_2 & 0 & \cos \theta_2
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix}
= \tilde{C}_2(\theta_2)
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix}
\]
\[
\begin{bmatrix}
\tilde{b}_1 \\
\tilde{b}_2 \\
\tilde{b}_3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 & \sin \theta_1 \\
0 & -\sin \theta_1 & \cos \theta_1
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1^* \\
\tilde{a}_2^* \\
\tilde{a}_3^*
\end{bmatrix}
= \tilde{C}_1(\theta_1)
\begin{bmatrix}
\tilde{a}_1^* \\
\tilde{a}_2^* \\
\tilde{a}_3^*
\end{bmatrix}
\]

And \(A'\) and \(A''\) are two intermediate reference frames with basis vectors \(\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\}\) and \(\{\tilde{a}_1', \tilde{a}_2', \tilde{a}_3'\}\), respectively. The three angles \(\theta_1, \theta_2\) and \(\theta_3\) are called Euler angles.

By combining the preceding sequence of rotations we obtain

\[
\begin{bmatrix}
\tilde{b}_1 \\
\tilde{b}_2 \\
\tilde{b}_3
\end{bmatrix}
= \tilde{C}_1(\theta_1)
\begin{bmatrix}
\tilde{a}_1^* \\
\tilde{a}_2^* \\
\tilde{a}_3^*
\end{bmatrix}
= \tilde{C}_1(\theta_1)\tilde{C}_2(\theta_2)
\begin{bmatrix}
\tilde{a}_1^* \\
\tilde{a}_2^* \\
\tilde{a}_3^*
\end{bmatrix}
= \tilde{C}_1(\theta_1)\tilde{C}_2(\theta_2)\tilde{C}_3(\theta_3)
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{bmatrix}
\]

The rotation matrix from \(B\) to \(A\), or the direction cosine matrix of \(B\) relative to \(A\), is then defined as

\[
\tilde{C}^{B/A} = \tilde{C}_1(\theta_1)\tilde{C}_2(\theta_2)\tilde{C}_3(\theta_3)
\]

\[
= \begin{bmatrix}
-c_2c_3 & c_2s_3 & -s_2 \\
c_1s_2c_3 - c_3s_1 & c_1s_2s_3 + c_3c_1 & s_1c_2 \\
c_1s_2s_3 - c_3c_1 & c_1s_2c_3 + s_1s_3 & c_1c_2
\end{bmatrix}
\]

Where \(c_i = \cos \theta_i\) and \(s_i = \sin \theta_i\).

The preceding sequence of rotations to \(B\) from \(A\) is also symbolically denoted by

\[
\tilde{C}_1(\theta_1) \leftarrow \tilde{C}_2(\theta_2) \leftarrow \tilde{C}_3(\theta_3)
\]

In general there are 12 sets of Euler angles, each resulting in a different form of the rotation matrix \(\tilde{C}^{B/A}\). The rotation sequence of Euler angles, \(\theta_1, \theta_2\) and \(\theta_3\), becomes unimportant for infinitesimal rotations, whereas rotation sequence is important for finite rotations. In general,
Euler angles have an advantage over direction cosines in that three Euler angles determine a unique orientation, although there is no unique set of Euler angles for a given orientation [46].

2.10.3 Quaternions

We consider Euler's eigenaxis rotation about an arbitrary axis fixed both in a body-fixed reference frame B and in an inertial reference frame A. A unit vector \( \vec{e} \) along the Euler axis is defined as [44]

\[
\vec{e} = e_1 \vec{a}_1 + e_2 \vec{a}_2 + e_3 \vec{a}_3 = e_1 \vec{b}_1 + e_2 \vec{b}_2 + e_3 \vec{b}_3
\]

Where \( e_i \) are the direction cosines of the Euler axis relative to both A and B, and

\[
e_1^2 + e_2^2 + e_3^2 = 1.
\]

Then we define the four Euler parameters as follows:

\[
q_1 = e_1 \sin(\theta/2)
\]

\[
q_2 = e_2 \sin(\theta/2)
\]

\[
q_3 = e_3 \sin(\theta/2)
\]

\[
q_4 = \cos(\theta/2)
\]

Where \( \theta \) denotes the rotation angle about the Euler axis. Like the eigenaxis vector \( \vec{e} = (e_1, e_2, e_3) \), we define a vector \( \vec{q} = (q_1, q_2, q_3) \) such that

\[
\vec{q} = \vec{e} \sin(\theta/2)
\]

Euler parameters are not independent of each other, but are constrained by the relationship

\[
q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1.
\]

Euler parameters are also called quaternions.

Quaternions have no inherent geometric singularity as do Euler angles. Moreover, quaternions are well suited for onboard real-time computations because only products and no trigonometric relations exist in the quaternion kinematic differential equations. Thus, spacecraft orientation is now commonly described in terms of quaternions [46].

2.11 Rotational Manoeuvres and Attitude Control

2.11.1 Single-Axis Attitude Control Problem

For the purpose of illustrating the nonlinear feedback control logic proposed in [46], a single-axis attitude control problem of a rigid spacecraft described by
\[ J\ddot{\theta} = u \quad |u(t)| \leq U \quad 2-50 \]

Where \( J \) is the spacecraft moment of inertia, \( \theta \), the attitude angle, and \( u \) the control torque input with the saturation limit of \( \pm U \). The time-optimal feedback control logic for the commanded constant attitude angle of \( \theta_c \) is given by

\[ u = -U \text{sgn}|e + (1/2a)\dot{\theta}| \quad 2-51 \]

Where \( e = \theta - \theta_c \) is the attitude error and \( a = U/J \) is the maximum control acceleration. The signum function is defined as \( \text{sgn}(x) = 1 \) if \( x > 0 \) and \( \text{sgn}(x) = -1 \) if \( x < 0 \).

In practice, a direct implementation of such an ideal, time-optimal switching control logic results in a chattering problem. Consequently, there exists various ways of avoiding such a chattering problem inherent in the ideal, time-optimal switching control logic [48]. Consider a feedback control logic of the form

\[ w = -\text{sat}(K\text{sat}(e) + C\dot{\theta}) \quad 2-52 \]

Where \( K \) and \( C \) are, respectively, the attitude and attitude rate gains. The saturation function is defined as

\[ \text{sat}_L(e) = \begin{cases} 
L & \text{if } e \geq L \\
e & \text{if } |e| < L \\
-L & \text{if } e \leq -L 
\end{cases} \quad 2-53 \]

It can also be represented as

\[ \text{sat}_L(e) = \text{sgn}(e)\min(|e|, L) \quad 2-54 \]

Because of the presence of a limiter in the attitude-error feedback loop, the attitude rate becomes constrained as

\[ -|\dot{\theta}|_{\text{max}} \leq \dot{\theta} \leq |\dot{\theta}|_{\text{max}} \quad 2-55 \]

Where \( |\dot{\theta}|_{\text{max}} = LK/C \)

For most practical cases, a proper use of the feedback logic in Equation 2-52 will result in a typical bang-off-bang control.
For the nominal range of attitude-error signals that do not saturate the actuator, the controller gains, \( K = kJ \) and \( C = cJ \), can be determined such that
\[
k = \omega_n^2 \quad \text{and} \quad c = 2\zeta \omega_n
\]

Where \( \omega_n \) and \( \zeta \) are, respectively, the desired or specified linear control bandwidth and damping ratio. Furthermore, if the maximum slew rate is specified as \( |\dot{\theta}|_{\text{max}} \), then the limiter in the attitude-error feedback loop can be simply selected as
\[
L = \frac{C}{K} |\dot{\theta}|_{\text{max}} = \frac{c}{k} |\dot{\theta}|_{\text{max}}
\]

However, as the attitude-error signal gets larger, and also as the slew rate limit becomes larger for rapid manoeuvres, the overall response becomes sluggish with increased transient overshoot because of the actuator saturation. To achieve rapid transient settlings even for large commanded attitude angles, the slew rate limit needs to be adjusted as follows [48]
\[
|\dot{\theta}|_{\text{max}} = \min \left\{ \sqrt{2d|\dot{e}|}, |\omega|_{\text{max}} \right\}
\]

A smaller value than the nominal \( a \) is to be used to accommodate various uncertainties in the spacecraft inertia and actuator dynamics. Such a variable limited in the attitude-error feedback loop has the self-adjusting saturation limit,
\[
L = \frac{C}{K} |\dot{\theta}|_{\text{max}} = \frac{c}{k} |\dot{\theta}|_{\text{max}} = \frac{c}{k} \min \left\{ \sqrt{2d|\dot{e}|}, |\omega|_{\text{max}} \right\}
\]

And we obtain a nonlinear control logic of the form
\[
u = -sat\{Ksat(e) + C\dot{e}\} = -sat\{kJ \text{sgn}(e) \min \{ |e|, (c/k) \sqrt{2d|\dot{e}|}, (c/k) |\omega|_{\text{max}} \} + CJ\dot{\theta}\}
\]

An integral control is necessary to eliminate a steady-state pointing error due to any constant external disturbance, this lead to the modification of the control logic into the following proportional-integral-derivative (PID) saturation control logic.

39
\[ u = -\text{sat}\left( K \text{sat}\left( e + \frac{1}{T} \int e \right) + C \dot{\theta} \right) \]  

Where \( T \) is the time constant of integral control and \( L \) is given by Reference 48. In terms of the standard notation for PID controller gains, \( K_p, K_i \) and \( K_D \), we have
\[
K_p = K, \quad K_i = K / T, \quad \text{and} \quad K_D = C
\]
The PID controller gains can be determined as [48]
\[
K_p = J \left( \omega_n^2 + 2 \zeta \omega_n / T \right) \quad 2-62
\]
\[
K_i = J \left( \omega_n^2 / T \right) \quad 2-63
\]
\[
K_D = J \left( 2 \zeta \omega_n + 1 / T \right) \quad 2-64
\]
And the time constant \( T \) of integral control is often selected as \( T \approx 10 / (\zeta \omega_n) \) [46].

If the attitude reference input is tracked to be a smooth function, instead of a multistep input, we employ a PID saturation control logic of the following form
\[
\begin{align*}
\dot{\omega} &= -\text{sat}\left( kJ \text{sat}\left( e + \frac{1}{T} \int e \right) + C \dot{\theta} \right) \\
\end{align*}
\]
For a PID-type saturation control logic, the so-called integrator anti windup or integrator synchronisation is necessary to avoid the phenomenon known as integrator windup, inherent in all PID-type controllers with actuator saturation. Such integrator windup results in substantial transient overshoot and control effort [48].

**2.11.2 Three-Axis Attitude Control Problem**

In the following section a three-axis quaternion feedback control problem is described, as developed in Wie [47]. We first consider the rotational equations of motion of a rigid spacecraft described by:
\[
\bar{J} \ddot{\omega} + \bar{\omega} \times \bar{J} \bar{\omega} = \bar{u}
\]
Where \( J \) is the inertia matrix, \( \bar{\omega} = (\omega_1, \omega_2, \omega_3) \) the angular velocity vector and \( \bar{u} = (u_1, u_2, u_3) \) the control torque input vector. The cross product of two vectors is represented in matrix notation as
Here $h = J\omega$ is the angular momentum vector. It is assumed that the angular velocity vector components $\omega_i$ along the body-fixed control axes are measured by rate gyros. Let a unit vector along the Euler axis be denoted by $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_i$ are the direction cosines of the Euler axis relative to either an inertial reference frame or the body-fixed control axis. The four elements of quaternions are the defined as

\begin{align*}
q_1 &= \lambda_1 \sin(\theta/2) \\
q_2 &= \lambda_2 \sin(\theta/2) \\
q_3 &= \lambda_3 \sin(\theta/2) \\
q_4 &= \cos(\theta/2)
\end{align*}

Where $\theta$ denotes the rotation angle about the Euler axis, and we have $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$.

The quaternion kinematic differential equations are given by

\begin{align*}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4
\end{bmatrix} &= \frac{1}{2} \begin{bmatrix}
0 & \omega_3 & -\omega_2 & \omega_1 \\
-\omega_3 & 0 & \omega_1 & \omega_2 \\
\omega_2 & -\omega_1 & 0 & \omega_3 \\
-\omega_1 & -\omega_2 & -\omega_3 & 0
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
\end{align*}

Like the Euler-axis vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, we define a quaternion vector $\bar{q} = \lambda \sin(\theta/2)$. The vector part of the quaternions, denoted by $\bar{q} = (q_1, q_2, q_3)$, is simply called the quaternion vector. Then Equation 2-72 can be rewritten as

\begin{align*}
\bar{q} &= -\frac{1}{2} \bar{\omega} \times \bar{q} + \frac{1}{2} q_4 \bar{\omega} \\
\bar{q}_4 &= -\frac{1}{2} \bar{\omega}^T \bar{q}
\end{align*}

Where

\begin{align*}
\bar{\omega} \times \bar{q} &= \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix}
\end{align*}
A linear state-feedback controller of the following form can be considered for real-time implementation:

\[ u = -Kq - C\omega \quad 2-76 \]

Where \( K \) and \( C \) are controller gain matrices to be properly determined. If the commanded attitude quaternion vector is given as \( \tilde{q}_c = (q_{1c}, q_{2c}, q_{3c}, q_{4c}) \), then the control logic of Equation 2-76 can be simply modified into the following form:

\[ u = -K\tilde{e} - C\omega \quad 2-77 \]

Where \( \tilde{e} = (e_1, e_2, e_3) \) is called the attitude-error vector. The commanded attitude quaternions \( (q_{1c}, q_{2c}, q_{3c}, q_{4c}) \) and the current attitude quaternions \( (q_1, q_2, q_3, q_4) \) are related to the attitude-error quaternions \( (e_1, e_2, e_3, e_4) \) as follows:

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4
\end{bmatrix} =
\begin{bmatrix}
q_{4c} & q_{3c} & -q_{2c} & -q_{1c} \\
-q_{3c} & q_{4c} & q_{1c} & -q_{2c} \\
q_{2c} & -q_{1c} & q_{4c} & -q_{3c} \\
-q_{1c} & q_{2c} & -q_{3c} & q_{4c}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
\quad 2-78
\]

It was shown in References [42] and [43] that the closed-loop nonlinear system of a rigid spacecraft with the linear state-feedback controller of the form of Equation 2-76 or Equation 2-77 is globally asymptotically stable for the following gain selections:

1) controller 1: \( \bar{K} = 2k\bar{I}, \quad \bar{C} = diag(c_1, c_2, c_3) \)

2) controller 2: \( \bar{K} = \left( \frac{2k}{q_4} \right) \bar{I}, \quad \bar{C} = diag(c_1, c_2, c_3) \)

3) controller 3: \( \bar{K} = 2k \text{sgn}(q_4)\bar{I}, \quad \bar{C} = diag(c_1, c_2, c_3) \)

4) controller 4: \( \bar{K} = \left[ \alpha \bar{I} + \beta \bar{I}^2 \right]^{-1}, \quad C > 0 \)

where \( k \) and \( c_i \) are positive scalar constants, \( \bar{I} \) is a 3x3 identity matrix, \( \text{sgn}() \) denotes the signum function, and \( \alpha \) and \( \beta \) are nonnegative scalars. Controller 1 is a special case of controller 4 and \( \beta \) can be selected as zero when \( \alpha \neq 0 \). Controllers 2 and 3 approach the origin, either \((0,0,0,1)\) or \((0,0,0,-1)\), by taking a shorter angular path.
The gyroscopic term of Euler's rotational equation of motion is not significant for most practical rotational manoeuvres. However, in some cases, it may be desirable to directly counteract the term by control torque as [48]

\[ \ddot{\theta} = -K \dot{\theta} - C \dot{\omega} + \tilde{\omega} \times \tilde{J} \omega \]

It was shown in Ref [43] that the closed-loop system with the controller 18 is globally asymptotically stable if the matrix \( \tilde{K} \) is positive definite. A natural selection of \( \tilde{K} \) and \( \tilde{C} \) for guaranteeing such a condition is \( \tilde{K} = 2k \tilde{J} \) and \( \tilde{C} = c \tilde{J} \), where \( k \) and \( c \) are positive scalar constants to be properly selected. The positive scalar constants \( k \) and \( c \) are often chosen as \( k \approx \omega_n^2 \) and \( c \approx 2\zeta \omega_n \), where \( \omega_n \) and \( \zeta \) are, respectively the desired or specified linear control bandwidth and damping ratio of the three-axis attitude control system.

Furthermore, a rigid spacecraft with the controller

\[ \ddot{\theta} = -\tilde{J} (2k \tilde{\theta} + c \tilde{\omega}) + \tilde{\omega} \times \tilde{J} \tilde{\omega} \]

Performs a rest-to-rest reorientation manoeuvre about an eigenaxis along the commanded quaternion vector \( \tilde{q}_e \) [48]. An integral control can also be added to the quaternion-error feedback control logic 2-84, as follows [48]:

\[ \ddot{\theta} = -\tilde{J} \left( 2k \tilde{\theta} + \frac{2k}{T} \int \tilde{\theta} + c \tilde{\omega} \right) + \tilde{\omega} \times \tilde{J} \tilde{\omega} \]

Where \( T \) is the time constant of the quaternion-error integral control. Because the gyroscopic decoupling term, \( \tilde{\omega} \times \tilde{J} \tilde{\omega} \) is not needed for most practical rotational manoeuvres, the term is not considered [48].

Consider the rotational equations of motion of a rigid spacecraft described by [46, 48]

\[ \ddot{q} = \tilde{f}(\tilde{q}, \tilde{\omega}) = -\frac{1}{2} \tilde{\omega} \times \tilde{q} \pm \frac{1}{2} \sqrt{1 - \|\tilde{q}\|^2} \tilde{\omega} \]

\[ \ddot{\omega} = \tilde{g}(\tilde{\omega}, \tilde{\omega}) = \tilde{J}^{-1} (-\omega \times \tilde{J} \tilde{\omega} + \tilde{u}) \]

Where \( \tilde{J} \) is the inertia matrix, \( \tilde{q} = (q_1, q_2, q_3) \) is the quaternion vector, \( \tilde{\omega} = (\omega_1, \omega_2, \omega_3) \) is the angular velocity vector, \( \tilde{u} = (u_1, u_2, u_3) \) is the control input vector, and \( \|\tilde{q}\|^2 = \tilde{q}^T \tilde{q} = q_1^2 + q_2^2 + q_3^2 \)

The state vector of the system, denoted by \( \tilde{x} \), is then defined as
A dynamic system described by a set of differential equations of the form of Equation 2-86 and Equation 2-87 is called a cascaded system because \( \dot{q} \) does not appear in 2-87. Saturation functions employed for the cascade saturation controller can be defined as follows [48].

A saturation function of an n-dimensional vector \( \bar{x} = (x_1, \ldots, x_n) \) is defined as

\[
\text{sat}_L(x) = \begin{bmatrix}
sat_{L_1}(x_1) \\
sat_{L_2}(x_2) \\
\vdots \\
sat_{L_n}(x_n)
\end{bmatrix}
\]

Similarly, a signum function of an n-dimensional vector \( \bar{x} \) is defined as

\[
\text{sgn}(x) = \begin{bmatrix}
\text{sgn}(x_1) \\
\text{sgn}(x_2) \\
\vdots \\
\text{sgn}(x_n)
\end{bmatrix}
\]

A normalized saturation function of an n-dimensional vector \( \bar{x} \) is defined as [48]

\[
\text{sat}_U(\bar{x}) = \begin{cases} 
\bar{x} & \text{if } \mu(\bar{x}) < U \\
\bar{x}[U/\mu(\bar{x})] & \text{if } \mu(\bar{x}) \geq U
\end{cases}
\]

Where \( \mu(\bar{x}) \) is a positive scalar function of \( \bar{x} \) that characterizes the largeness of the vector \( \bar{x} \). Because the largeness of a vector \( \bar{x} \) is often characterized by its norms, we may choose \( \mu(\bar{x}) = \|\bar{x}\|_2 = \sqrt{\bar{x}^T \bar{x}} \) or \( \mu(\bar{x}) = \|\bar{x}\|_{\infty} = \max\{x_1, \ldots, x_n\} \). The normalized saturation of a vector \( \bar{x} \) has the same direction of the vector \( \bar{x} \) itself before saturation, the this means that the it maintains the direction of the vector [44].

The simplest form of a two-layer cascade-saturation control logic for a rigid spacecraft can be expressed as [48]:

\[
\bar{u} = -\text{sat}_U \left( \bar{P} \bar{\omega} + \bar{Q} \text{sat}(\bar{q}) \right)
\]

Where \( \bar{P} \) and \( \bar{Q} \) are the controller gain matrices.
We now consider a rigid spacecraft that is required to manoeuvre about an inertially fixed axis as fast as possible, but not exceeding the specified maximum slew rate about that axis. The following saturation control logic provides such a rest-to-rest eigeneaxis rotation under slew rate constraint [48]

\[ \ddot{\vec{q}} = -K_{sat} + \frac{1}{T} \int \dot{\vec{q}} - \vec{C} \dot{\vec{q}} \]

\[ = -J \left\{ 2k_{sat} \left( \vec{e} + \frac{1}{T} \int \dot{\vec{e}} \right) - c \dot{\vec{q}} \right\} \]

Where \( \vec{e} = (e_1, e_2, e_3) \) is the quaternion-error vector, \( \vec{K} = 2k \vec{J} \) and \( \vec{C} = c \vec{J} \), and the saturation limits \( L_i \) are determined as

\[ L_i = (c/2k) |\omega_i|_{\text{max}} \]

Where \( |\omega_i|_{\text{max}} \) is the specified maximum angular rate about each axis. It is assumed that the control torque input for each axis is constrained as

\[ -U \leq u_i(t) \leq +U \quad i = 1, 2, 3 \]

Where \( U \) is the saturation limit of each control input. Then, a control logic that accommodates possible control torque input saturation but that still provides an eigeneaxis rotation under slew rate constrained can be expressed as

\[ \vec{\omega} = -J \left\{ 2k_{sat} \left( \vec{e} + \frac{1}{T} \int \dot{\vec{e}} \right) - c \dot{\vec{q}} \right\} \]

\[ \ddot{\vec{q}} = sat(\vec{e}) \]

\[ \begin{cases} \vec{\omega} = \vec{e} & \text{if } \|\vec{\omega}\| < U \\ \|\vec{\omega}\| = \frac{U}{\|\vec{e}\|} \end{cases} \]

Where \( \|\vec{e}\| = \max\{\|\vec{r}_1\|, \|\vec{r}_2\|, \|\vec{r}_3\|\} \).

The slew rate limit is

\[ L_i = (c/2k) \min\left\{ \sqrt{4a_i \|e_i\| \|\omega_i\|_{\text{max}}} \right\} \]

Where \( a_i = U / J \) is the maximum control acceleration about the \( i \)th control axis and where \( |\omega_i|_{\text{max}} \) is the specified maximum angular rate about each axis.
2.12 Steering Logic for Single-Gimbal Control Moment Gyros

2.12.1 Pseudoinverse Steering Logic

For a CMG configuration the total angular momentum vector \( \vec{H} = [H_x, H_y, H_z]^T \) is expressed in spacecraft reference frame as

\[
\vec{H} = \sum_{i=1}^{n} \vec{h}_i(\delta_i) \tag{2-99}
\]

Where \( \vec{h}_i \) is the angular momentum vector of the \( i \)th CMG expressed in spacecraft reference frame, \( \delta_i \) are the gimbal angles, \( n \) is the number of CMGs and constant unit momentum magnitudes are assumed without loss of generality [44].

One way to determine the gimbal angle trajectories that generate the commanded \( \vec{H} \) trajectory is to solve a constrained optimization problem by minimizing a suitable performance index \( J(\delta) \) subject to nonlinear constraints. However this approach is not suitable for real-time implementation [44]. A more suitable approach to solving the inverse kinematic problem is to utilize the differential relationship between gimbal angles and the CMG momentum vector. For such local inversion or tangent methods, the time derivative of the CMG angular momentum vector Equation 2-99, can be obtained as

\[
\ddot{\vec{H}} = \sum_{i=1}^{n} \ddot{\vec{h}}_i = \sum_{i=1}^{n} \ddot{\vec{a}}_i(\delta_i) \dot{\delta}_i = \dot{A} \ddot{\delta} \tag{2-100}
\]

Where \( \ddot{\delta} = (\ddot{\delta}_1, \ddot{\delta}_2, \ddot{\delta}_3, \ddot{\delta}_4) \) is the gimbal angle vector, \( \ddot{\vec{a}}_i \) is the \( i \)th column of \( \dot{A} \), where \( \dot{A} \) is called the Jacobian matrix [48]. For the commanded control torque input \( \ddot{\vec{u}} \), the CMG momentum rate command \( \ddot{\vec{H}} \) is chosen as;

\[
\ddot{\vec{H}} = -\ddot{\vec{u}} - \vec{\omega} \times \vec{h} \tag{2-101}
\]

And the gimbal rate command \( \ddot{\delta} \) is then obtained as

\[
\ddot{\delta} = \dot{A}^T \ddot{h} = \dot{A}^T \left( \dddot{A} \dot{A}^T \right)^T \ddot{h} \tag{2-102}
\]

Which is often referred to as the pseudoinverse steering logic [46]. Most CMG steering laws determine the gimbal rate commands with some variant of pseudoinverse.
2.12.2 Singular States

If \( \text{rank}(\tilde{A}) < 3 \) for certain sets of gimbal angles, or, equivalently, \( \text{rank}(\tilde{A}\tilde{A}^T) < 3 \), the pseudoinverse does not exist and the pseudoinverse steering logic encounters singular states. This singular situation occurs when all individual CMG torque output vectors \( \tilde{a}_i \) are perpendicular to the commanded torque direction. Equivalently, the singular situation occurs when all individual CMG momentum vectors have extremal projections onto the commanded torque vector direction [46].

In general the singularity condition

\[
\text{det}(\tilde{A}\tilde{A}^T) = 0 
\]

defines a set of surfaces in \( \tilde{\delta} - \text{space} \), or, equivalently in \( \tilde{H} - \text{space} \).

The pseudoinverse steering logic tends to leave inefficiently positioned CMGs alone, causing the gimbal angles to eventually hang-up in singular antiparallel arrangements. That is, it tends to steer gimbals towards singular states [46].

2.12.3 Singularity-Avoidance Steering Logic

Equation 2-102 can be considered as a particular solution to Equation 2-101. The corresponding homogeneous solution is then obtained through null motion such that

\[
\tilde{A}\tilde{n} = 0
\]

Where \( \tilde{n} \) denotes the null vector spanning the null space of \( \tilde{A} \). The general solution to Equation 2-100 is then given by

\[
\tilde{\delta} = \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{H} + \gamma \tilde{n}
\]

Where \( \gamma \) represents the amount of null motion to be properly added [46]. The amount of null motion may be chosen as

\[
\gamma = \begin{cases} 
  m^e & \text{for } m \geq 1 \\
  m^n & \text{for } m < 1 
\end{cases}
\]

Where \( m = \sqrt{\text{det}(\tilde{A}\tilde{A}^T)} \) is the singularity measure, also called the CMG gain

\( \tilde{n} = (C_1, C_2, C_3, C_4) \) is the Jacobian null vector

\( C_i = (-1)^{i+1} M_i \) is the order 3 Jacobian cofactor
\( M_i = \det(\tilde{A}_i) \) is the order 3 Jacobian minor
\( \tilde{A}_i = \tilde{A} \) with the \( i \)th column removed

This choice of scaling factor \( \gamma \) arises from the representation of \( m \) as a measure of distance from singularity, as well as the fact that \[44\]

\[
\det(\tilde{A}\tilde{A}^T) = \sum^\infty_{i=1} M_i = \bar{n}^T\bar{n} \quad 2-107
\]

This nondirectional null-motion approach introduces substantial null motion even when the system is far from being singular and tries to prevent the gimbal angles from settling into locally optimal configurations, which may eventually result in a singularity. Although the null vector can be obtained through a variety of ways e.g. using singular value decomposition, a projection operator, or the generalized cross or wedge product, it is often expressed as

\[
\bar{n} = [\tilde{I} - \tilde{A}^T\tilde{A}]\gamma \quad 2-108
\]

Where \( \tilde{A}^* = \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1} \), \( \tilde{I} \) is an identity matrix, and \( [\tilde{I} - \tilde{A}^T\tilde{A}] \) is a projection matrix and \( \bar{n} \) is an arbitrary \( n \)-dimensional nonzero vector \[46\].

A variety of analytic and heuristic approaches have been developed in the past to determine a proper null motion for singularity avoidance, i.e. to properly select the scalar \( \gamma \) and the \( n \)-dimensional vector \( \bar{n} \).

### 2.12.4 Singularity Robust Steering Law

A heuristic modification of the pseudoinverse-based steering logic is to employ a singularity robust inverse algorithm of the form

\[
\tilde{A}^* = \tilde{A}^T(\tilde{A}\tilde{A}^T + \lambda I)^{-1} \quad 2-109
\]

Where \( \tilde{I} \) is an identity matrix and \( \lambda \) is a positive scale factor that may be automatically adjusted as

\[
\lambda = \begin{cases} \\
\lambda_0 (1 - m/m_0)^2 & \text{for } m < m_0 \\
\frac{m}{m_0} & \text{for } m \geq m_0 
\end{cases} \quad 2-110
\]

Where \( m = \sqrt{\det(\tilde{A}\tilde{A}^T)} \) and \( \lambda_0 \) and \( m_0 \) are to be properly selected. However, a small positive constant of the order 0.01 may be simply selected for \( \lambda \) \[46\].
In Reference 48 Wie presents a CMG steering logic based on the generalized singularity robust inverse steering logic, where,

\[ \tilde{\delta} = \tilde{A}^s \tilde{u} \]

\[ \tilde{A}^s = \tilde{A}^r (\tilde{A} \tilde{A}^r + \lambda \tilde{E})^{-1} \]

And

\[ E = \begin{bmatrix} 1 & \varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 1 & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_1 & 1 \end{bmatrix} \]

The scalar \( \lambda \), and the off-diagonal elements \( \varepsilon \) are to be selected such that \( \tilde{A}^s \tilde{u} \neq 0 \) for any nonzero constant \( \tilde{u} \).

In Wie [48] \( \varepsilon_1 \), is continuously modulated as \( \varepsilon_1 = \varepsilon_0 \sin(\omega t + \phi_1) \) where the amplitude \( \varepsilon_0 = 0.01 \), the modulation frequency \( \omega = 0.5\pi \) and the phases \( \phi_1 \) are selected as \( 0, \pi/2, \pi \). The logic, presented by Wie, approaches and rapidly transits unavoidable singularities whenever needed. The logic effectively generates deterministic dither signals when the system becomes near singular. Any internal singularities can be escaped for any nonzero constant torque commands using the steering logic [48].

2.13 Attitude Control and CMG Steering Logic Simulation Results

An Attitude Control System (ACS) model was developed in SIMULINK for the dual purposes of familiarization with the equations and as a tool to contribute dynamic data to the question of TCMG suitability to future spacecraft missions, and further to assist in the selection of a suitable TCMG actuator. The developed ACS is presented in Appendix A.

In order to validate the author’s developed ACS SIMULINK model, results generated in work by Wie [48] were replicated. His selected satellite, actuator and controller parameters were input to the author’s model with the aim of matching the results. To check the error between the author’s SIMULINK model and Wie’s dual M-File model the M-Files originally developed by Wie were obtained by the author. It consists of two M-Files which together model satellite and actuator dynamics and generates his results presented in [48].
In these simulations an agile satellite with a typical 4SGCMG pyramid mounting is considered. The nonlinear control algorithm developed by Wie [48] is applied to the three-axis control problem, to control an agile spacecraft, to replicate the results of [48]. A generalized singularity robust steering law is implemented. In Appendix A a functional block diagram of a general attitude control system is shown, it is noted that sensors are not modeled for this simulation case. The simulation parameters were selected to match Wie’s results for verification of the developed ACS model.

The spacecraft simulated by Wie has the following nominal inertia:

\[
J = \text{diag}[21400, 20100, 5000] \text{kgm}^2
\]

The pyramid skew angle is selected as \( \beta = 53.13 \text{deg} \) and the constant angular momentum magnitude for each CMG as 1000Nms. In his simulations Wie assumes that the gimbal rate command limit of each CMG is \( \delta = 2 \text{rad/s} \). He also makes the assumptions that the attitude control bandwidth needs to be lower than 5 rad/s and the maximum slew rate less than \( |\omega|_{\text{max}} = 10 \text{deg/s} \). The transverse axes of this near symmetrical spacecraft are the roll and pitch axes. The symmetry axis with the smallest momentum of inertia is the yaw axis, which is pointing towards a target. The commanded quaternion vector for a rest to rest, 47 deg roll-axis reorientation manoeuvre is given as \((q_{47}, q_{34}, q_{32}) = (0, 4, 0, 0)\). The time-optimal reorientation for this particular manoeuvre should ideally be completed in 8s. The initial gimbal angles considered are \( \delta = (0, 0, 0) \).

When \( \omega_s = 3 \text{rad/s}, \zeta = 0.9 \) and \( T = 10 \) the controller gains, \( k \) and \( c \) are calculated as \( k = 9.54 \) and \( c = 5.5 \). The control acceleration \( \alpha \) is calculated as 40% of the actual maximum acceleration to accommodate the actuator dynamics and the nonlinear nature of quaternion kinematics. For the normalized Jacobian matrix, \( A \), the scale factor, and \( E \) are chosen as:

\[
\lambda = 0.01 \exp[-10 \det(ATA^T)]
\]

\[
E = \begin{bmatrix} 1 & e_1 & e_2 \\ e_1 & 1 & e_3 \\ e_2 & e_3 & 1 \end{bmatrix} > 0
\]
Where \( e_i = 0.01 \sin(0.5 \pi t + \phi_i) \) with \( \phi_i = 0 \), \( \phi_2 = \pi/2 \) and \( \phi_3 = \pi \).

As can be seen from Figure 2-12 the roll-axis reorientation is successfully completed within 12 seconds in the presence of the CMG singularity encounters, momentum saturation, and gimbal rate limits. The cross axis pitch/yaw pointing error during the singularity transit is relatively small compared to the actual roll manoeuvre as shown in Figure 2-13 and also seen in the results of [48]. Figure 2-18 shows the logic approaching and rapidly transiting the internal elliptic singularity \( \delta = (90,0,-90,0) \)deg. Figure 2-19 further indicate that CMG system successfully passed through the internal elliptic singularity (i.e. the points where the singularity measure becomes zero, \( \det(\mathbf{A}^T) = 0 \).

Figure 2-12 Plot of a 47 degree roll maneouvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figure 2-13 Plots of pitch and yaw for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figure 2-14 Author's Reproductions of quaternions over plot on Wie's data for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figure 2-15 Plot showing roll axis angular rate for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]

Figure 2-16 Plot showing pitch axis angular rate for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figure 2-17 Plot showing yaw axis angular rate for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48].
Figure 2-18 Plots to show the author’s reproductions of CMG gimbal angles over plot on Wie’s data, for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figure 2-19 Plots to show the author's CMG momentum and singularity measures, for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]

2.14 Error Plots and Discussion

Comparing the data produced by the author's SIMULINK model and the original data produced by Wic in [48] it was seen that the two sets of data are close but not identical. In this section the differences, or errors between the data sets calculated by both models will be analysed, and an explanation for these differences will be found.

If the value of quantities calculated by Wic in Reference [48] are considered to be the correct values and are called the true values, and the authors ACS simulated values of the same quantity are considered to be the approximations \( \tilde{a} \), the difference is called the error, \( \varepsilon \) of \( \tilde{a} \) [12];

\[
\varepsilon = a - \tilde{a}
\]

Figures 2-20 and 2-21 show the errors about each of the satellite’s principle axis for the 47 degree manoeuvre. The error in the roll axis is shown in Figure 2-20 as having a maximum of \(-0.87^\circ\) degrees. Figure 2-21 shows that the pitch and yaw axes reach maximum errors of \(-0.9^\circ\) and \(-0.25^\circ\) respectively. Figures 2-22, and 2-23 show errors in the angular rates, which are all less than \(\pm 0.5^\circ\).
Figure 2-20 Plot showing the roll axis errors for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]

Figure 2-21 Plot showing the pitch and yaw axes errors for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figure 2-22 Plot showing the error in angular rate about the roll axis for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]

Figure 2-23 Plot showing the error in angular rates about the pitch and yaw axis for a 47 degree roll manoeuvre using nonlinear control logic, and generalized singularity robust steering logic, under rate and control saturation limits, developed in [48]
Figures 2-12 to 2-19 show the overall shape of the plots of both sets of data, but it can be seen that the data produced by the author’s SIMULINK model and the original data produced by Wie in [48] are close but not identical. Errors propagate into the computation and affect the accuracy. Such errors or differences between values may result from a combination of effects including for example round-off errors, and truncating errors. These differences depend on the computational method used. Wie has used MATLAB M-Files and the author has used SIMULINK. Both data sets are formed using ode45 solver which is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair.

Wie’s model calculates 1957 values during the 15 second simulation time, i.e a value approximately every 0.00766 seconds, it is more sensitive to changes in commanded inputs than the author’s model which calculates 444 values during the same simulation time, i.e. a value approximately every 0.338 seconds. This difference in number of steps can explain the difference in values of the two models. More steps means that Wie’s model will be more sensitive to changes, i.e. as the gradient of the plots changes from positive to negative and vice versa.

The relative errors seen at the start of the manoeuvre, can be explained by the difference in step intervals of the two models. Wie’s model, having a smaller step size between calculated values is able to respond quicker to the changes in the commanded input. The large relative errors which occur at \( t > 8 \) seconds, as the manoeuvre is completing can also be explained by the difference sensitivity of each model, as the commanded signals settle to their final values, from high rate of change.
3 Configuration Geometric Analysis

3.1 CMG Designs- Configurations

Research into CMG hardware and software systems started in the mid 1960s. Evaluation of various types and configurations of CMGs were made in terms of weight and power consumption. A large number of CMG configurations have been proposed and analyzed in varying detail over the past several years [14]. However, little published material is available comparing the various systems and assessing their utility for the types of missions where CMGs are practical. There are a large number of CMG configurations which may be formulated from the basic building block of Single Gimbal CMG (SGCMG). Each of these configurations has a unique set of control laws which are formulated to provide constant gain and minimum cross coupling. In order to apply the sizing/selection process, we identify a few key configuration designs, listed below. A full description of each configuration is contained in the following sections of this chapter.

1. The 4SGCMG pyramid configuration is a pyramid of 4 Single Gimbal CMGs, as shown in Figure 3-1.

![Figure 3-1 4SGCMG pyramid](image)

2. The 4TCMG pyramid configuration is a pyramid of 4 Twin CMGs, as shown in Figure 3-2.
3. The 3SGCMG axis aligned configuration is a cluster of 3 Single Gimbal CMGs, each aligned with an axis of the satellite, as shown in Figure 3-3.

4. The 3TCMG axis aligned configuration is a cluster of 3 Twin CMGs, each aligned with an axis of the satellite, similarly to the 3SGA configuration shown in Figure 3-3, with TCMGs replacing the SGCMGs.

5. The 3TCMG pyramid configuration is a pyramid of 3 Twin CMGs, as shown in Figure 3-4.
6. The 6SGCMG pyramid configuration is a pyramid of 6 Single Gimbal CMGs, as shown in Figure 3-5.

3.2 4SGCMG Pyramid Configuration

Most previous research has dealt with a pyramid type system, see Figure 3-1, which comprises four single gimbal CMG units; four are the minimum for the system to have one degree of redundancy. This section provides a mathematical description of the geometry of the 4SGCMG configuration.

The total angular momentum is the sum of all $\delta_i$ multiplied by the units' angular momentum value which is denoted by $h$. The angular momentum is arbitrarily taken as a unit vector. In this work, $H$ denotes the total angular momentum:
To find the maximum angular momentum achievable about each axis, appropriate values of gimbal angles are substituted into Equation 3-1. The gimbal angles and the resulting values of angular momentum are displayed in Table 3-1.

<table>
<thead>
<tr>
<th>$\delta_1$(deg)</th>
<th>$\delta_2$(deg)</th>
<th>$\delta_3$(deg)</th>
<th>$\delta_4$(deg)</th>
<th>$H_x$(Nms)</th>
<th>$H_y$(Nms)</th>
<th>$H_z$(Nms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-90</td>
<td>180</td>
<td>90</td>
<td>0</td>
<td>2 + 2$c$/$\beta$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-90</td>
<td>180</td>
<td>90</td>
<td>0</td>
<td>2 + 2$c$/$\beta$</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>4$s$/$\beta$</td>
</tr>
</tbody>
</table>

Table 3-1 Maximum angular momentum values in X- Y- and Z- axis, 4SGCMG pyramid configuration

Where $c$/$\beta$ = cos $\beta$, $s$/$\beta$ = sin $\beta$ and $\beta$ is the control moment gyro pyramid angle measured in radians. With $\beta$ = 54.7 deg the maximum possible $h$ along each vehicle axis is:

$$h(3.156, 3.156, 3.265)$$

From Equation 3-2 the minimum maximum value of angular momentum is 3.156 about the X-axis hence value of the unit vector representing the peak cluster torque is:

$$|\hat{\tau}| = 2 + 2c\beta$$

3.3 4TCMG Pyramid Configuration

One of the candidate configurations considered in past research [6, 11, 21] was a twin type system made of two single gimbal CMGs driven in opposite directions. The first of several Twin CMG (TCMG) configurations studied in this work is shown below in Figure 3-6. Study of this configuration will allow a direct comparison between the classical 4SGP configuration which is commonly used onboard satellites and a geometrically similar twin CMG configuration.
Where, P1, P2, P3 and P4 are platforms consisting of a TCMG pair. A pair of CMGs with parallel axes will be termed a platform in this document. The following section provides a mathematical description of the geometry of the 4TP configuration. Analysis is started by considering the angular momentum components of each SGCMG on a platform in turn, and combining these components;

For TCMG configuration analysis it is imposed that the angular momentum of each CMG in a platform are equal. It is further imposed that the rotors of each CMG in each platform all have the same angular momentum. It is also imposed that the magnitudes of the gimbal angles in a platform are equal.

Platform 1 analysis

\[
\begin{align*}
\vec{\mathbf{h}}_{1a} &= \begin{bmatrix} -c\beta s\delta_1 \\ c\delta_1 \\ s\beta s\delta_1 \end{bmatrix} \\
\vec{\mathbf{h}}_{1b} &= \begin{bmatrix} -c\beta s\delta_1 \\ -c\delta_1 \\ s\beta s\delta_1 \end{bmatrix} \\
\vec{\mathbf{h}}_{p1} &= \begin{bmatrix} -2c\beta s\delta_1 \\ 0 \\ 2s\beta s\delta_1 \end{bmatrix}
\end{align*}
\]

Platform 2 analysis

\[
\begin{align*}
\vec{\mathbf{h}}_{2a} &= \begin{bmatrix} -c\delta_2 \\ -c\beta s\delta_2 \\ s\beta s\delta_2 \end{bmatrix} \\
\vec{\mathbf{h}}_{2b} &= \begin{bmatrix} c\delta_2 \\ -c\beta s\delta_2 \\ s\beta s\delta_2 \end{bmatrix} \\
\vec{\mathbf{h}}_{p2} &= \begin{bmatrix} 0 \\ -2c\beta s\delta_2 \\ 2s\beta s\delta_2 \end{bmatrix}
\end{align*}
\]

Platform 3 analysis

\[
\begin{align*}
\vec{\mathbf{h}}_{3a} &= \begin{bmatrix} c\beta s\delta_3 \\ -c\delta_3 \\ s\beta s\delta_3 \end{bmatrix} \\
\vec{\mathbf{h}}_{3b} &= \begin{bmatrix} c\beta s\delta_3 \\ c\delta_3 \\ s\beta s\delta_3 \end{bmatrix} \\
\vec{\mathbf{h}}_{p3} &= \begin{bmatrix} 2c\beta s\delta_3 \\ 0 \\ 2s\beta s\delta_3 \end{bmatrix}
\end{align*}
\]

Platform 4 analysis
Combining the angular momentum components from each platform gives the total angular momentum of the cluster. The maximum possible value of angular momentum about each of the satellite vehicle axis is obtained by substituting values of \( \delta_1, \delta_2, \delta_3, \delta_4 \) into

\[
\begin{bmatrix}
    c\delta_4 \\
    c\beta s\delta_4 \\
    s\beta s\delta_4
\end{bmatrix}, \quad
\begin{bmatrix}
    -c\delta_4 \\
    c\beta s\delta_4 \\
    s\beta s\delta_4
\end{bmatrix}, \quad
\begin{bmatrix}
    0 \\
    2c\beta s\delta_4 \\
    2s\beta s\delta_4
\end{bmatrix}
\]

Maximizing the angular momentum along each vehicle axis;

\[
H_{xy} = \sum_{i=1}^{4} h_i(\delta_i) = h
\begin{bmatrix}
    -2c\beta s\delta_1 & 0 & 2c\beta s\delta_3 & 0 \\
    0 & -2c\beta s\delta_2 & 0 & 2c\beta s\delta_4 \\
    2s\beta s\delta_1 & 2s\beta s\delta_2 & 2s\beta s\delta_3 & 2s\beta s\delta_4
\end{bmatrix}
\]

Table 3-2 Maximum angular momentum values in X- Y- and Z- axis, 4TCMG pyramid configuration

<table>
<thead>
<tr>
<th>( \delta_1(\text{deg}) )</th>
<th>( \delta_2(\text{deg}) )</th>
<th>( \delta_3(\text{deg}) )</th>
<th>( \delta_4(\text{deg}) )</th>
<th>( H_x(\text{Nms}) )</th>
<th>( H_y(\text{Nms}) )</th>
<th>( H_z(\text{Nms}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-90</td>
<td>0</td>
<td>90</td>
<td>0</td>
<td>4c\beta</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-90</td>
<td>0</td>
<td>90</td>
<td>0</td>
<td>4c\beta</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>8s\beta</td>
</tr>
</tbody>
</table>

Where \( c\beta = \cos\beta, \ s\beta = \sin\beta \) and \( \beta \) is the control moment gyro pyramid angle measured in radians

With \( \beta = 54.7^\circ \) the minimum maximum value of \( H \) is

\[
H(2.311 \ 2.311 \ 6.53)
\]

The peak torque capability, \( |v| \), is

\[
|v| = 2.311 = 4c\beta
\]

3.4 3SGCMG Axis Aligned Configuration

Although four SGCMG units are required to meet redundancy requirements, the 3SGA configuration is included for completeness, enabling another direct comparison between like-for-like SGCMG systems and TCMG systems. The 3SGA configuration comprises of 3SGCMGs each with its gimbal axis aligned with one of the satellite vehicle axes, as shown in Figure 3-7.
CMG #1 is chosen by the author to be aligned with the gimbal axis in the direction of the vehicle's X-Axis as is shown in Figure 3-8.

Resolving the angular momentum generated by CMG #1, $h_1$ into its components in the vehicle frame we obtain:

$$\bar{h}_i = \begin{bmatrix} 0 \\ s\delta_i \\ -c\delta_i \end{bmatrix}$$

3-11

The same process is applied to CMG #2 which is chosen to have the gimbal axis aligned with the Y-Axis of the vehicle. Figure 3-9 shows the vehicle axis rotated clockwise about
the Z-Axis through 90 degrees and the angular momentum generated by CMG #2 in the vehicle axis is seen.

\[
\begin{bmatrix}
-c\delta_2 \\
0 \\
 s\delta_2 
\end{bmatrix}
\]

3-12

Once again the perspective of the vehicle axis is changed to show the angular momentum generated by CMG #3 which is chosen to have gimbal axis aligned with the Z-Axis of the vehicle.

\[
\begin{bmatrix}
-c\delta_2 \\
0 \\
 s\delta_2 
\end{bmatrix}
\]

3-12
Summing Equations 3-11, 3-12, and 3-13 the resultant angular momentum developed along each of the axis is obtained as in Equation 3-14.

$$H = \sum_{i=1}^{3} h_i(\delta_i) = \begin{bmatrix} 0 & -c\delta_2 & s\delta_3 \\ s\delta_1 & 0 & -c\delta_3 \\ -c\delta_1 & s\delta_3 & 0 \end{bmatrix}$$

3-14

Maximizing the angular momentum along each vehicle axis;

<table>
<thead>
<tr>
<th>$\delta_1$ (deg)</th>
<th>$\delta_2$ (deg)</th>
<th>$\delta_3$ (deg)</th>
<th>$H_x$ (Nms)</th>
<th>$H_y$ (Nms)</th>
<th>$H_z$ (Nms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>180</td>
<td>90</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>0</td>
<td>180</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>180</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3-3 Maximum angular momentum values in X- Y- and Z- axis, 3SGCMG axis aligned configuration

Where $c\beta = \cos \beta$, $s\beta = \sin \beta$ and $\beta$ is the control moment gyro pyramid angle measured in radians. With $\beta = 54.7^\circ$ the maximum possible angular momentum along each vehicle axis is;

$$h(2 \ 2 \ 2)$$

3-15

Hence the peak cluster torque capability is

$$|v|_{le} = 2$$

3-16

3.5 3TCMG Pyramid Configuration

The axis aligned configuration provides no redundancy / failure survivability. The natural progression is to analyze a 3TCMG pyramid configuration. In this section the angular momentum generated by each TCMG platform in a 3TCMG configuration with a skew angle of $\beta = 54.7^\circ$ is analyzed. The 3TCMG pyramid is a tetrahedron which is composed of four triangular faces, three of which meet at each vertex.
Figure 3-11 3TCMG pyramid

For this analysis it is imposed that the angular momentum of each CMG in a platform are equal i.e.

\[ h_{1a} = h_{1b} = h_1 \]  
Assumption 1

\[ h_{2a} = h_{2b} = h_2 \]  
Assumption 2

\[ h_{3a} = h_{3b} = h_3 \]  
Assumption 3

It is further imposed that the rotors of each CMG in each platform all have the same angular momentum;

\[ h_1 = h_2 = h_3 = h \]  
Assumption 4

It is also imposed that the magnitudes of the gimbal angles in a platform are equal i.e

\[ \delta_{1a} = \delta_{1b} = \delta_1 \]  
Assumption 5

\[ \delta_{2a} = \delta_{2b} = \delta_2 \]  
Assumption 6

\[ \delta_{3a} = \delta_{3b} = \delta_3 \]  
Assumption 7

First considering the two CMGs which make up TCMG platform 1 (labeled P1 on Figure 3-11), the angular momentum generated by each CMG can be resolved into components in the vehicle axis frame.
Resolving the angular momentum generated by this platform into its components in the vehicle frame we obtain Equations 3-17, 3-18, and 3-19.

\[
\begin{align*}
\vec{\tau}_{1a} &= h \begin{bmatrix} -c\beta s\delta_1 \\ -c\delta_1 \\ s\beta s\delta_1 \end{bmatrix} \\
\vec{\tau}_{1b} &= h \begin{bmatrix} -c\beta s\delta_1 \\ c\delta_1 \\ s\beta s\delta_1 \end{bmatrix} \\
\vec{\tau}_{11} &= h \begin{bmatrix} -2c\beta s\delta_1 \\ 0 \\ 2s\beta s\delta_1 \end{bmatrix}
\end{align*}
\]

Platform 2, shown in Figure 3-14 can be treated in a similar manner. First by resolving the components of \(\tau_{2a}\) and \(\tau_{2b}\) along the base of the pyramid as shown in Figure 3-15.
Figure 3-13 3TCMG pyramid – platform 2 (3D side view)

Figure 3-14 3TCMG pyramid – platform 2 (side view) angular momentum components
As shown in Figure 3-15 the component vector $h_2s\delta_2s\beta$ lies purely along the vehicle $Z$-axis. However $h_2s\delta_2c\beta$ and both of the $h_2c\delta_2$ component vectors have components in the $X$- and $Y$-axis. These vectors are resolved along the vehicle axis, as shown in Figure 3-16.

Combining the angular momentum vector components of Equation 3-20 and 3-21, gives Equation 3-22.
Next considering the two CMGs which make up TCMG platform 3 (labeled P3 on Figure 3-11), the angular momentum generated by each CMG can be resolved into components in the vehicle axis frame.

\[
\begin{align*}
\bar{\bar{h}}_{2a} &= h_2 \begin{bmatrix} c\beta s \delta_2 c 60 + c\delta_2 s 60 \\ -c\beta s \delta_2 s 60 + c\delta_2 c 60 \\ s\beta s \delta_2 \end{bmatrix} & 3-20 \\
\bar{\bar{h}}_{2b} &= h_2 \begin{bmatrix} c\beta s \delta_2 c 60 - c\delta_2 s 60 \\ -c\beta s \delta_2 s 60 - c\delta_2 c 60 \\ s\beta s \delta_2 \end{bmatrix} & 3-21 \\
\bar{\bar{h}}_{p3} &= h_2 \begin{bmatrix} 2c\beta s \delta_2 c 60 \\ -2c\beta s \delta_2 s 60 \\ 2s\beta s \delta_2 \end{bmatrix} & 3-22
\end{align*}
\]

Figure 3-17 3TCMG pyramid – platform 3 (3D side view)
Figure 3-18 3TCMG pyramid – platform 3, angular momentum vector components of each CMG in platform 3 resolved in the vehicle X- and Y- and Z-axis.

Figure 3-19 3TCMG pyramid – platform 3, angular momentum vector components of each CMG in platform 3 resolved in the vehicle X- and Y- axis.

Resolving the angular momentum generated by this platform into its components in the vehicle frame we obtain Equations 3-23, 3-24, leading to 3-25.

\[
\hat{h}_{2n} = \begin{bmatrix}
    h_3 c_3 b_3 s_3 c_6 0 - c_3 s_3 s_6 0 \\
    h_3 c_3 b_3 s_3 s_6 0 + c_3 c_3 c_6 0 \\
    s_3 b_3 s_3
\end{bmatrix}
\]  

\[3-23\]
Calculating the angular momentum contributed by each CMG about each vehicle axis, and combining those from each platform (i.e. \( h_{p1}, h_{p2}, \) and \( h_{p3} \)) to obtain the total angular momentum in each vehicle axis, one obtains:

\[
\vec{h}_{p3} = \begin{bmatrix} 2h_3c\beta s_3\delta_3c60 \\ 2h_3c\beta s_3\delta_3s60 \\ 2s\beta s_3 \end{bmatrix} \] \quad 3-25

From Equation 3-26 the gimbal angles that contribute the maximum possible amount of the individual CMG angular momentum in each vehicle axis can be derived. Table 3-4 summarizes these angles and the maximum possible values of angular momentum in each vehicle axis.

<table>
<thead>
<tr>
<th>( \delta_1(\text{deg}) )</th>
<th>( \delta_2(\text{deg}) )</th>
<th>( \delta_3(\text{deg}) )</th>
<th>( H_x(\text{Nms}) )</th>
<th>( H_y(\text{Nms}) )</th>
<th>( H_z(\text{Nms}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-90</td>
<td>90</td>
<td>90</td>
<td>2.31</td>
<td>0</td>
<td>1.63</td>
</tr>
<tr>
<td>0</td>
<td>-90</td>
<td>90</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>90</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>4.90</td>
</tr>
</tbody>
</table>

Table 3-4 Maximum angular momentum values in X-, Y- and Z-axis, 3TCMG pyramid configuration

Where \( c\beta = \cos \beta \), \( s\beta = \sin \beta \) and \( \beta \) is the control moment gyro pyramid angle measured in radians. With \( \beta = 54.7^\circ \) the maximum possible angular momentum along each vehicle axis is:

\[
h(2.31 \ 2 \ 4.90) \] \quad 3-27

Hence the cluster peak torque capability is

\[
\|\vec{v}\|_{vc} = 2 = 3.464c\beta \] \quad 3-28
3.6 6SGCMG Pyramid Configuration

Choosing an xyz Cartesian coordinate system in space, the components of the angular momentum vectors are found with respect to this coordinate system. The 6SGCMG pyramid system, shown in Figure 3-20 and Figure 3-21, is a system with six CMG units arranged on the sides of a 6-sided pyramid. The centre of this pyramid is aligned with the origin of the Cartesian coordinate system.

![Figure 3-20 6SGCMG pyramid configuration](image-url)
Figure 3-21 6SGCMG pyramid configuration, view of the base in the XY plane of the vehicle axis

For the 6SGCMG pyramid cluster the total CMG angular momentum vector $\mathbf{H}_{6SGCMG}$ is expressed in spacecraft reference frame as

$$\mathbf{H}_{6SGCMG} = \sum_{i=1}^{6} \mathbf{h}_i(\delta_i)$$

Where $\mathbf{h}_i$ is the angular momentum vector of the $i$th CMG expressed in spacecraft reference frame. The magnitude of the angular momentum, is arbitrarily taken as a unit.

In the following sections the general solution to Equation 3-33 is presented.

3.6.1 6SGCMG pyramid, analysis of CMG #1

Figure 3-22 shows the plane in which CMG#1 lies.
The angular momentum $hc\delta_1$ can be replaced by two components, which are equivalent in action, as shown in Figure 3-22. One component $hc\delta_1$ is in the direction of the Y-axis and the other component, $hs\delta_1$, along the height of the triangular plane. The latter component can be resolved along the X- and Z-axis, as $hs\delta_1c\beta$ and $hs\delta_1s\beta$ respectively. The vector components of angular momentum of CMG#1 can be described by a matrix;

$$\vec{h}_{CMG#1} = h \begin{bmatrix} -s\delta_1c\beta \\ c\delta_1 \\ s\delta_1s\beta \end{bmatrix}$$ 3-30

3.6.2 6SGCMG pyramid, analysis of CMG #2

This section resolves the components related to CMG#2, which is shown in the base plane (XY plane) of the vehicle axis in Figure 3-23.
Figure 3-23 6SGCMG pyramid, showing the vector analysis of CMG #2, viewed from the XY plane of the vehicle axis

The vector $hc\delta_2$ in the XY plane of the pyramid can be resolved along the X- and Y-axes, as the components $hc\delta_2 \sin \theta_60$ and $hc\delta_2 \cos \theta_60$ respectively, described by Equation 3-31.

$$
\vec{h}_{CMG#2} = h \begin{bmatrix}
-c\delta_2 \sin \theta_60 \\
c\delta_2 \cos \theta_60 \\
0
\end{bmatrix}
$$

3-31

Component $h\delta_2$, which lies along the height of the triangle, as shown in Figure 3-24, can be resolved into components along the X-, Y- and Z-axes. The components can be added to Equation 3-31 to give Equation 3-32.
Figure 3-24 6SGCMG pyramid, showing the vector analysis of CMG #2, 3D side view in the vehicle axis

\[
\bar{\mathbf{h}}_{\text{CMG#2}} = \begin{bmatrix}
-c\delta_2s60 \\
c\delta_2c60 + s\delta_2c\beta \\
s\delta_2s\beta
\end{bmatrix}
\]

3.6.3 6SGCMG pyramid, analysis of platform CMG #3

This section resolves the components related to CMG#3, which is shown in the base plane (XY plane) of the vehicle axis in Figure 3-25.
The vector $hc\delta_3$ in the XY plane of the pyramid can be resolved along the X- and Y-axes, as the components $hc\delta_3s60$ and $-hc\delta_3c60$ respectively, described by Equation 3-33.

$$\tilde{h}_{CMG#3} = h \begin{bmatrix} -c\delta_3s60 \\ -c\delta_3c60 \\ 0 \end{bmatrix}$$  \hspace{1cm} 3-33$$

Component $hs\delta$, which lies along the height of the triangle, as shown in Figure 3-26, can be resolved into components along the X-, Y- and Z-axes. The components can be added to Equation 3-33 to give Equation 3-34.
Figure 3-26 6SGCMG pyramid, showing the vector analysis of CMG #3, 3D side view in the vehicle axis

\[ \tilde{h}_{CMG3} = \begin{bmatrix} -c\delta_3 s60 + s\delta_3 c\beta s60 \\ -c\delta_3 c60 - s\delta_3 c\beta s60 \\ s\delta_3 s\beta \end{bmatrix} \] 

3.6.4 6SGCMG pyramid, analysis of CMG #4

CMG #4 is positioned geometrically opposite CMG#1, the angular momentum components are the “mirror image”, as shown in Figure 3-27.
The angular momentum components of CMG#4 are shown with respect to the components of CMG#1, in Figure 3-28, and in matrix form in Equation 3-35.
3.6.5 6SGCMG pyramid, analysis of CMG #5

CMG #5 is positioned geometrically opposite CMG#2, hence it can be said that the angular momentum components are the "mirror image", as shown in Figure 3-29.

\[
h_{CMG#5} = h \begin{bmatrix} s\delta_5 c\beta \\ -c\delta_5 \\ s\delta_5 s\beta \end{bmatrix}
\]

Figure 3-29 6SGCMG pyramid, showing the vector analysis of CMG #5 viewed from the XY plane of the vehicle axis

The vector \(hc\delta_5\) in the XY plane of the pyramid can be resolved along the X- and Y-axes, as shown in Figure 3-30 and as described by Equation 3-36.
Figure 3-30 6SGCMG pyramid, showing the relative components of CMG #2 and CMG #5 in the XY plane of the vehicle axis

\[
\vec{h}_{\text{CMG#5}} = \begin{bmatrix}
  c\delta_x s60 \\
  -c\delta_x c60 \\
  0
\end{bmatrix}
\]

3-36

Component \( h\delta_x \), which lies along the height of the triangle, as shown in Figure 3-31 and, can be resolved into components along the X-, Y- and Z-axes, as shown in Figure 3-32. The components can be summed to those in Equation 3-36 to give Equation 3-37.
Figure 3-31 6SGCMG pyramid, showing the vector analysis of CMG #5, 3D side view in the vehicle axis

Figure 3-32 6SGCMG pyramid, showing the vector analysis of CMG #5, in the base plane (the XY plane)

\[
\vec{h}_{CMG5} = \begin{bmatrix}
c\delta_2 \sin 60 + s\delta_2 \cos 60 \\
-c\delta_2 \cos 60 + s\delta_2 \sin 60 \\
s\delta_2 \sin \beta
\end{bmatrix}
\] 3-37
3.6.6 6SGCMG pyramid, analysis of CMG #6

CMG #6 is positioned geometrically opposite CMG#3, hence it can be said that the angular momentum components are the “mirror image”, as shown in Figure 3-33.

![Diagram showing CMG #3 and CMG #6](image)

Figure 3-33 6SGCMG pyramid, showing the angular momentum vector of CMG #6 relative to angular momentum vector of CMG #3 viewed from the XY plane of the vehicle axis

The vector components $hc\delta_3$ and $hs\delta_6$ are shown in Figure 3-34 and Figure 3-35 respectively. The resolution of these components along the X- Y- and Z- axes are also shown in these figures, and given in their matrix form in Equation 3-38.
Figure 3-34 6SGCMG pyramid, showing the relative components of CMG #3 and CMG#6 in the XY plane of the vehicle axis.

Figure 3-35 6SGCMG pyramid, showing the relative components of CMG #3 and CMG#6 in the XYZ plane of the vehicle axis (3D view).
The total angular momentum vector for the 6SGCMG pyramid is expressed in reference frame as:

\[ \mathbf{\bar{H}}_{6S} = \sum_{i=1}^{6} \mathbf{h}_i(\delta_i) \]  

3-39

Summing the angular momentum vector components of each CMG in the configuration, which are given in Equations 3-30, 3-32, 3-34, 3-35, 3-37, and 3-38, to give Equation 3-40. The maximum angular momentum about each axis can be found by selecting appropriate gimbal angle values, \( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \) which maximize each row of the total angular momentum matrix in Equation 3-40. These gimbal angles and the resulting angular momentum about each axis is tabulated in Table 3-5.

\[
\begin{bmatrix}
-\sin\beta \cos\alpha & -\cos\beta \cos\alpha & \sin\alpha \\
-\sin\beta \sin\alpha & -\cos\beta \sin\alpha & \cos\alpha \\
-\cos\beta & \sin\beta & 0 \\
\end{bmatrix} \]

3-40

<table>
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<th>( \delta_1 ) (deg)</th>
<th>( \delta_2 ) (deg)</th>
<th>( \delta_3 ) (deg)</th>
<th>( \delta_4 ) (deg)</th>
<th>( \delta_5 ) (deg)</th>
<th>( \delta_6 ) (deg)</th>
<th>( H_x ) (Nms)</th>
<th>( H_y ) (Nms)</th>
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<td>4.00</td>
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<tr>
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<td>90</td>
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<td>90</td>
<td>90</td>
<td>0.87</td>
<td>4.90</td>
<td></td>
</tr>
</tbody>
</table>

Table 3-5 Maximum angular momentum values in X- Y- and Z- axis, 6SGCMG pyramid configuration

Where \( c\beta = \cos\beta \), \( s\beta = \sin\beta \) and \( \beta \) is the control moment gyro pyramid angle measured in radians. With \( \beta = 54.7^\circ \) the maximum possible angular momentum along each vehicle axis is;

\[ h(4.04 \ 4.00 \ 4.90) \]  

3-41

Hence the cluster peak torque capability is

\[ |\mathbf{\bar{v}}|_e = 2 + 2\sqrt{3}c\beta \]  

3-42
4 Sizing/Selection Process for SG- and T- CMG Configurations

In this chapter a sizing/selection process for SG- and T-CMG configurations will be described. The process has been implemented in MATLAB, and in this chapter results will be presented, analyzed and a recommendation for the most viable configuration for a particular mission scenario will be made.

4.1 Design Margins

The engineering objectives required from the sizing/selection process are described mathematically using a set of four design margins, presented below

\[ T_m = T_a - T_r \]  
\[ M_m = M_r - M_a \]  
\[ V_m = V_r^{SWEPT} - V_a^{SWEPT} \]  
\[ P_m = P_r - P_a \]

Where \( T_m, M_m, P_m, V_m \) are the torque, mass, power and volume design margins for each configuration.

\( T_a, M_a, P_a, V_a \) are the actual values of torque, mass, power and volume for each configuration.

\( T_r, M_r, P_r, V_r \) are the required values of torque, mass, power and volume for all configurations.

The decision variables \( (\ell_{rot}, \rho_{rot}, r_q, r_r) \) are the dimensions of the rotor of a CMG, which are shown in Figure 4-1 and Figure 4-2.
4.2 Design Margins Described in Terms of Decision Variables

In order to describe the design margins in terms of the decision variables four cluster values need to be calculated:

1. The actual total cluster torque developed by a candidate configuration, $T_a$.
2. The actual total cluster mass of a candidate configuration, $M_a$.
3. The actual total cluster swept volume of a candidate configuration, $V_a$.
4. The actual total cluster power consumed by a candidate configuration, $P_a$.

In the following sections of this chapter, these values will be derived in terms of the decision variables and then they will be substituted into the design margin equations. The four design margins described
terms of the decision variables can be used to form a set of four sizing/selection functions which will be calculated using a program written in MATLAB.

### 4.2.1 Total Cluster Torque Developed

The cluster total torque, $T_a$, developed by a configuration is calculated using [18];

$$ T_a = \chi I_{rot} \Omega \delta |\omega| $$  \hspace{1cm} \text{(4.5)}

A CMG rotor's spin axis moment of inertia, as shown by Lappas [16] dominates the CMG physical size. When deriving the equations for the inertia of the rotor, a simple model of a pierced-disc rotor is considered. Figure 4-2 shows the rotor as a ring of length, $\ell_{rot}/2$ on top of a disc of length, $\ell_{rot}/2$. The radius of the disc section is $r_o$, which is the same as the outer radius of the ring section, the inner radius of the ring is $r_i$. It is known that the inertia of the rotor about its spin axis is dependent on the dimensions of the rotor which are in turn are the decision variables, $(\ell_{rot}, \rho_{rot}, r_o, r_i)$.

The Moment of Inertia of the disc section of the rotor about its spin axis is derived as:

$$ I_{rot}^{\text{Disc}} = 0.5 \pi \rho_{rot} \left( \frac{\ell_{rot}}{2} \right) r_o^4 $$  \hspace{1cm} \text{(4.6)}

And the Moment of Inertia of the ring section of the rotor about its spin axis is:

$$ I_{rot}^{\text{Ring}} = 0.5 \pi \rho_{rot} \left( \frac{\ell_{rot}}{2} \right) \left( r_o^4 - r_i^4 \right) $$  \hspace{1cm} \text{(4.7)}

Summing the inertias of the ring and disc sections of the rotor about its spin axis, the equation for the inertia of the rotor about its spin axis can be obtained;

$$ I_{rot} = 0.5 \pi \rho_{rot} \left( \frac{\ell_{rot}}{2} \right) \left( 2r_o^4 - r_i^4 \right) $$  \hspace{1cm} \text{(4.8)}

Therefore, the total cluster torque available from a configuration in terms of the decision variables can be found;

$$ T_a = \chi \Omega \delta |\omega| 0.5 \pi \rho_{rot} \left( \frac{\ell_{rot}}{2} \right) \left( 2r_o^4 - r_i^4 \right) $$  \hspace{1cm} \text{(4.9)}
The torque capability term, $|\psi|_{tc}$, is a specific constant for a particular candidate configuration. In the first instance it is chosen to fix all rotor dimensions (i.e. decision variables, $(\rho_{rot}, r_o, r_i)$) with the exception of the rotor length, $\ell_{rot}$, which will be sized for each configuration via the sizing process. Of interest to the designer is the variation in sized rotor length for each configuration as the gimbal rate is varied over the range $\dot{\delta}_{\text{min}} \leq \dot{\delta}_{\text{dis}} \leq \dot{\delta}_{\text{max}}$ for k discrete steps. The efficiency of torque development, $\chi$, is assumed to be the same for each configuration, and is chosen as 0.8%, based on the heuristic value selected in Lappas [19], which was attained from simulations and practical experience. The rotor spin speed is also chosen to be a fixed value which is the same for all configuration for all gimbal rates.

As the decision variables $(\rho_{rot}, r_o, r_i)$ have been fixed, a constant $C_i$ can be defined as

$$C_i = \frac{1}{4} \pi \Omega \rho_{rot} \left(2r_o^4 - r_i^4\right)$$

4-10

The equation for total cluster torque available can be reduced by substituting in the constant $C_i$.

$$T_a = C_i |\psi|_{tc} \ell_{rot} \delta$$

4-11

This section has yielded an equation for the actual total cluster torque available from a candidate configuration.

### 4.2.2 Total Cluster Mass

The cluster total mass of the candidate configuration, $M_a$, can be written as

$$M_a = \eta M_{rot}$$

4-12

Where;

$\eta$ is the total number of rotors in the configuration

$M_{rot}$ is the mass of one rotor

In the following sections only the rotor mass is considered, and it is assumed that all the other components of the CMGs are identical across the configurations.
The mass of one rotor in the configuration is dependent on the dimensions of the rotor, i.e. the values of the decision variables, \( (\ell_{\text{rot}}, \rho_{\text{rot}}, r_o, r_i) \). In forming an equation for the mass of the rotor, the rotor is divided into two separate parts, the ring and the disc parts; The mass of the disc section of the rotor is calculated using:

\[
M_{\text{rot}}^{\text{disc}} = \pi \rho_{\text{rot}} \left( \frac{\ell_{\text{rot}}}{2} \right) (r_o^2)
\]

And the mass of the ring is calculated using:

\[
M_{\text{rot}}^{\text{ring}} = \pi \rho_{\text{rot}} \left( \frac{\ell_{\text{rot}}}{2} \right) (r_o^2 - r_i^2)
\]

Summing the mass of the rotor ring and disc sections the equation for the total mass of one rotor is obtained;

\[
M_{\text{rot}} = 0.5 \pi \rho_{\text{rot}} \left( \frac{\ell_{\text{rot}}}{2} \right) (2r_o^2 - r_i^2)
\]

Therefore the total cluster rotor mass is;

\[
M_C = \eta 0.5 \pi \rho_{\text{rot}} \left( \frac{\ell_{\text{rot}}}{2} \right) (2r_o^2 - r_i^2)
\]

Rearranging terms to obtain;

\[
M_C = \eta \frac{\pi}{4} \rho_{\text{rot}} \ell_{\text{rot}} (2r_o^2 - r_i^2)
\]

A second constant can be defined as

\[
C_2 = \frac{1}{4} \pi \rho_{\text{rot}} (2r_o^2 - r_i^2)
\]

Which reduces the equation for total cluster mass to;

\[
M_C = \eta C_2 \ell_{\text{rot}}
\]

The work in this section has yielded an equation for the actual total cluster mass for a candidate configuration.

4.2.3 Total Cluster Volume

The total cluster rotor volume is derived by dealing with the rotor in two parts, the ring and disc parts, as was done for the calculations of rotor inertia and mass. Two volumes are considered;

1. The volume of a stationary rotor, \( V_{\text{rot}} \).
2. The volume which is swept by a gimbaled rotor, $V_{rot}^{SWEPT}$

First the volume occupied by a stationary rotor is calculated. The volume occupied by the disc section of the rotor is

$$V_{rot}^{DISC} = \frac{\pi r_o^2 \ell_{rot}}{2}$$  

And the volume occupied by the ring section of the rotor is

$$V_{rot}^{RING} = \pi \left( r_o^2 - r_i^2 \right) \ell_{rot}/2$$  

The total volume of a stationary rotor is the sum of the ring and disc volumes;

$$V_{rot} = \pi \left( 2r_o^2 - r_i^2 \right) \ell_{rot}/2$$  

Therefore the total cluster rotor volume is

$$V_{a}^{STAT} = n V_{rot}$$  

Substituting $V_{rot}$ into the equation for $V_{a}^{STAT}$

$$V_{a}^{STAT} = \eta \pi \left( 2r_o^2 - r_i^2 \right) \ell_{rot}/2$$  

Now consider the swept volume of the rotor

$$V_{rot}^{SWEPT} = \frac{4}{3} \pi r_o^3$$  

The cluster total swept rotor volume is

$$V_{a}^{SWEPT} = \eta V_{rot}^{SWEPT}$$  

Substituting $V_{rot}^{SWEPT}$ into the equation for $V_{a}^{SWEPT}$

$$V_{a}^{SWEPT} = \eta \frac{4}{3} \pi r_o^3$$  

The work in this section has yielded an equation for the actual total volume swept by the rotors in a candidate configuration.

4.2.4 Cluster Total Power Consumed

The actual cluster total power consumed by a configuration is the sum of two parts, the power consumed by the rotor motors, and the power consumed by the gimbal motors;

$$P_a = P_{rot} + P_G$$  

In the following subsections each term is considered separately
4.2.5 Total Cluster Rotor Power

The total cluster rotor power consumed is;

\[ P_{\text{ROT}} = \eta P_{\text{UNIT}} \]  

Where;

\[ P_{\text{UNIT}} \] is the power consumed by the rotors in a single unit of the cluster

The power consumed by one rotor in a single unit of the cluster is calculated using;

\[ P_{\text{ROT}} = \mu M_{\text{rot}} \Omega \hat{\Omega} \]  

Where

\[ \mu \] is the number of rotors in a unit.

As derived above the mass of a rotor is;

\[ M_{\text{rot}} = 0.5 \pi \rho_{\text{rot}} \left[ \left( \frac{e_{\text{rot}}}{2} \right)^2 - r_i^2 \right] \]

Substituting into the equation for total rotor power for a unit \( P_{\text{UNIT}} \)

\[ P_{\text{UNIT}} = \mu 0.5 \pi \rho_{\text{rot}} \left[ \left( \frac{e_{\text{rot}}}{2} \right)^2 - r_i^2 \right] \Omega \hat{\Omega} \]  

The total cluster rotor power, \( P_{\text{ROT}} \), consumed is

\[ P_{\text{ROT}} = \eta 0.5 \pi \rho_{\text{rot}} \left[ \left( \frac{e_{\text{rot}}}{2} \right)^2 - r_i^2 \right] \Omega \hat{\Omega} \]  

Knowing that \( C_s = \frac{1}{4} \pi \rho_{\text{rot}} \left( 2r_o^2 - r_i^2 \right) \), the equation becomes

\[ P_{\text{ROT}} = C_s \rho_{\text{rot}} \Omega \hat{\Omega} \]

4.2.6 Total Cluster Gimbal Power

Next consider the total cluster power consumed by the gimbal motors;

\[ P_G = \frac{\eta P_{\text{UNIT}}}{\mu} \]  

The total power consumed by the gimbal motor in a single unit in a cluster is;

\[ P_{\text{UNIT}} = \mu T_{\text{rot}} \delta \]  

Where;
\( T_{\text{rot}} \) is the torque developed by a single rotor in the unit.

\[
T_{\text{rot}} = \left[ \frac{\pi}{4} \chi \Omega_{\text{rot}} \left( 2r^3 - r^4 \right) \right] \ell_{\text{rot}} \delta
\]

Reduced to;

\[
T_{\text{rot}} = C_1 \ell_{\text{rot}} \delta
\]

Therefore the total power consumed by the gimbal motor in a single unit in a cluster can be written in terms of the decision variables as;

\[
P_{G}^{\text{UNIT}} = \mu C_1 \ell_{\text{rot}} \delta^2
\]

When calculating the total cluster power consumed by the gimbal motors we multiply \( P_{G}^{\text{UNIT}} \) by the number of gimbals in the cluster;

The number of gimbals in the cluster is defined as \( \frac{\eta}{\mu} \);

\[
P_{G}^{\text{GIMBAL}} = \frac{\eta}{\mu} \pi \chi \Omega_{\text{rot}} \ell_{\text{rot}} \left( 2r^3 - r^4 \right) \delta^2
\]

Which reduces to;

\[
P_{G}^{\text{GIMBAL}} = C_1 \eta \ell_{\text{rot}} \delta^2
\]

Summing the power consumed by the rotor motors and the gimbal motors in a unit, gives the total power consumed by a unit in the cluster;

\[
P_{a}^{\text{UNIT}} = P_{\text{ROT}}^{\text{UNIT}} + P_{G}^{\text{UNIT}}
\]

Substituting in derived equations;

\[
P_{a}^{\text{UNIT}} = \mu \ell_{\text{rot}} \left( C_1 \delta^2 + C_2 \Omega \delta \right)
\]

Summing the power consumed by all the rotor motors, and all the gimbal motors in a cluster gives the total cluster power consumed;

\[
P_{a} = P_{a}^{\text{ROT}} + P_{a}^{\text{GIMBAL}}
\]

Substituting in equations for total cluster rotor power and total cluster gimbal power consumed;

With constants \( C_1, C_2 \);

\[
P_{a} = \eta \ell_{\text{rot}} \left( C_1 \delta^2 + C_2 \Omega \delta \right)
\]
The work in Sections 4.2.4, 4.2.5, and 4.2.6 have resulted in an equation to describe the actual total power consumed by a candidate configuration.

Now the equations which describe total cluster torque developed, mass, volume and power consumed in terms of the decision variables have been derived.

4.3 Forming the Sizing Functions

In Section 4.2 equations describing the actual values of total torque developed, total mass, total volume occupied, and total power consumed have been derived. These equations can be used to form a set of sizing functions.

In order to make a comparison of the alternative CMG configurations, the 4 SGCMG pyramid has been selected as a baseline against which all others will be compared. For this aim, a geometric approach has been taken. A typical configuration of CMGs is the pyramid mounting arrangement of four single gimbal CMGs with a skew angle, $\beta$, of 54.7 deg, shown in Figure 3-1. This configuration os selected as the baseline configuration as most previous research works have dealt with this pyramid type system [14, 15, 46].

The research considers a number of alternative configurations of CMGs, some are SGCMG configurations and others are TCMG configurations. Each configuration is labeled from $i = 2...n$, with the first configuration, $i = 1$ being the baseline, see Table 4-1.

| I | CONFIGURATION                      | TORQUE CAPABILITY, $|\tau|_{bc}$ |
|---|------------------------------------|----------------------------------|
| 1 | 4 SGCMG pyramid (baseline)         | 3.16                             |
| 2 | 3 SGCMG axis aligned               | 2                                |
| 3 | 3 TCMG axis aligned                | 2                                |
| 4 | 3 TCMG pyramid                     | 2                                |
| 5 | 4 TCMG pyramid                     | 2.31                             |
| 6 | 6 SGCMG pyramid                    | 4                                |

Table 4-1 Torque capability for each configuration to be compared
We have four design margins which we can describe in terms of sizing/selection functions. For this study we wish to compare CMG configurations by finding the difference between the design margins of the candidate configurations, (i.e. \( i = 2 \ldots n \)), and the baseline configuration design margins.

The first selection function pertains to the torque design margin. We begin by setting the required candidate configuration value for torque developed to be equal to the value of the baseline actual torque developed;

\[
T_{ob} = T_{ni} = T_{nh} = T_r
\]

In words, this states that the torque available from the baseline is required from the alternative configuration, and obviously, but included for clarity, this value is also required from the baseline itself, and will be called \( T_r \), torque required. The total cluster mass, volume, and power limits / budgets can also be set to be equal to the baseline actual values.

\[
M_r = M_{ob}
\]

\[
V_r = V_{ab}
\]

\[
P_r = P_{ab}
\]

We consider two of the sizing/selection functions;

\[
f_1(\bar{x}) = T_{na}
\]

Where; \( T_{na} = T_{ni} - T_r \)

\[
f_2(\bar{x}) = M_{na}
\]

Where; \( M_{na} = M_r - M_{na} \)

### 4.3.1 Fixed Baseline Sizing/Selection Process

Consider a particular CMG configuration produced by a manufacturer, which will be the baseline configuration. The manufacturer wants to compare their configuration to alternative designs, (i.e. the candidate configurations), in terms of the torque that each configuration can develop, the mass and volume that it takes up on the satellite, and the power it consumes for particular mission scenarios. The fixed baseline sizing/selection process seeks to answer two main questions for the manufacturer:

1. Can more torque be developed for the same total mass of configuration?
2. Can the same torque be developed for less configuration mass?

The candidate configurations are sized by changing the rotor length. Figure 4-3 is a flow chart for the actuator configuration sizing process.
Basic Optimal Sizing Program, Richie [27]  

Author's Fixed and Moving Baseline Sizing/Selection Program  

Figure 4-3 Comparison of Richie’s Basic Optimal Sizing Program (subfigure a), with the author’s implementation of a Fixed Baseline Sizing/Selection Program (dash-dot lines), and developed Moving Baseline Sizing/Selection Program (dotted lines), (subfigure b)
4.4 Sizing/Selection Process Case Studies

There are two prominent strategies that will be considered in this research to complete a design sizing comparison of this nature;

1. Case 1, sizing for maximum torque at a total cluster mass equivalent to the baseline, or
2. Case 2, sizing for minimum mass at a total cluster torque developed value equivalent to the baseline system.

For each case the chosen parameters are presented in Table 4-2

The fixed simulation parameters shown in Table 4-2 are chosen based on previous research experience in the field. A BILSAT CMG rotor provided by Surrey Space Centre was measured and these dimensions were used for the fixed rotor values in simulation. The rotor spin speed was selected according to experience with reaction/momentum wheels and CMGs in previous sizing work for minisatellites [19].

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</tr>
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<td>$\ell_{rot}$, m</td>
<td>0.012</td>
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</tbody>
</table>

Table 4-2 Simulation Parameters

Where

$\chi$ is the efficiency

$\Omega$ is the rotor spin speed.

$\Omega^*$ is the rotor acceleration.

$|\mathbf{v}_{zr}|$ is the torque capability.
For the sizing of each configuration the gimbal rate is varied over the chosen range \( \delta_{\text{min}} \leq \delta \leq \delta_{\text{max}} \), where \( \delta_{\text{min}} = 0.5 \text{rad/s} \) and \( \delta_{\text{max}} = 10 \text{rad/s} \).

### 4.5 Case 1 Maximizing Torque, Equating Mass

The sizing/selection function is required to size candidate configuration to maximize the torque developed by the candidate configuration for an equivalent mass of baseline configuration, (i.e. \( M_{\text{al}} = M_{b} \)) implying the following constraints:

The constraint of equal mass allows a fair comparison of the torque developed by each configuration. In order to develop the largest value of torque the rotor length is sized. The parameters for case 1 are presented in Table 4-2. Each candidate configuration is sized at discrete gimbal rates over the range \( \delta_{\text{min}} \leq \delta \leq \delta_{\text{max}} \), where \( \delta_{\text{min}} = 0.5 \text{rad/s} \) and \( \delta_{\text{max}} = 10 \text{rad/s} \). The BILSAT-1 satellite was chosen for the case study.

The sizing/selection function for the torque design margin was derived in Section 4.3 as;

\[
f_i(\bar{x}) = T_{ai} - T_{b}
\]

It has been defined that \( \bar{x} = [\ell_{\text{rot}}, \rho_{\text{rot}}, r_{\text{a}}, r_{\text{i}}]^T \) where \( \rho_{\text{rot}} = \rho_{\text{rot}}^b, r_{\text{a}} = r_{\text{a}}^b, r_{\text{i}} = r_{\text{i}}^b \) and \( \chi_i = \chi_b, \Omega_i = \Omega_b \).

The sizing/selection function is now studied in more detail, so that the effects of the variables on the value of the sizing/selection function can be seen. The equations for the torque margins

\[
T_{ai} - T_{b}
\]

for the configuration, i, and the baseline configuration are substituted into the sizing function \( f_i(\bar{x}) \);

\[
f_i(\bar{x}) = T_{ai} - T_{b}
\]

The equation for the total cluster torque available from a configuration, i, in terms of the decision variables has already been defined in Section 4.2.1, as Equation 4-9;

\[
T_{ai} = \chi_i \Omega_i \delta [\bar{v}]_{\text{sci}} 0.5 \pi \rho_{\text{rot}} \left( \frac{\ell_{\text{rot}}}{2} \right) (2r_{a}^4 - r_{i}^4)
\]

Where \( C_i = \frac{\pi}{4} \rho_{\text{rot}} \Omega_i (2r_{a}^4 - r_{i}^4) \) simplifying to Equation 4-11;

\[
T_{ai} = C_i [\bar{v}]_{\text{sci}} \ell_{\text{rot}} \delta
\]
And for the baseline configuration the total torque developed by can be defined as
\[ T_{ab} = \chi_h \Omega_b \delta \left| \vec{v}_{\text{rel}} \right| 0.5 \pi \rho_{\text{rot}} \left( \left( \frac{\ell_{\text{rot}}}{2} \right) (2r_{1a}^4 - r_{1b}^4) \right) \]
4-52

Which can be simplified to
\[ T_{ab} = C_1 \left| \vec{v}_{\text{rel}} \right| \ell_{\text{rot}} \delta \]
4-53

Substituting Equations 4-9, 4-11, 4-55, and 4-56 into Equation 4-54
\[ f_i(\vec{x}) = T_{ai} - T_{ab} = \frac{1}{4} \chi \Omega \delta \pi \rho_{\text{rot}} \left( 2r_{a1}^4 - r_{a1}^4 \right) (\ell_{\text{rot}} \left| \vec{v}_{\text{rel}} \right| - \ell_{\text{rot}} \left| \vec{v}_{\text{rel}} \right|) \]
4-54

Which can be simplified to
\[ f_i(\vec{x}) = C_1 \delta (\ell_{\text{rot}} \left| \vec{v}_{\text{rel}} \right| - \ell_{\text{rot}} \left| \vec{v}_{\text{rel}} \right|) \]
4-55

To maximize the torque available from a candidate configuration, such that it is larger or equal to the value of torque developed from the baseline configuration, Equation 4-58 must be maximized. The value of torque capability, \( \left| \vec{v}_{\text{rel}} \right| \), is individual to each candidate configuration, and dependent on configuration geometry. The length of each rotor in the configuration, \( \ell_{\text{rot}} \), is sized by the sizing process and changes at each discrete \( \delta \), gimbal rate.

With the constraints;
\[ M_{ai} = M_{ab} \]
\[ \chi_i = \chi_b, \Omega_i = \Omega_b, r_{ai} = r_{ib}, r_{ai} = r_{ib}, \rho_{\text{rot}} = \rho_{\text{rot}} \]
\[ \ell_{\text{rot}} \geq 0 \]

With \( C_1 = \frac{1}{4} \chi \Omega \pi \rho_{\text{rot}} \left( 2r_{a1}^4 - r_{a1}^4 \right) \)

4.5.1 Rotor Length Results
In this case the torque developed by each candidate configuration is maximized by changing the rotor length, \( \ell_{\text{rot}} \). In order to do this the largest allowable value for the rotor length will be selected by the sizing program. However, it is constrained that the total cluster mass of the candidate configuration, \( i \), is to be equal to the total cluster mass of the baseline configuration; i.e. \( M_{ai} = M_{ai} \). This fixes the rotor length that can be selected by the sizing algorithm. A
mathematical equation to represent this value for the rotor length is presented in the following lines;

From Equation 4-17 derived in Section 4.2.2 the total cluster mass of the candidate configuration, \(i\), is

\[ M_{ai} = \eta_i \frac{\pi}{4} \rho_{rot} \ell_{rot} \left(2r_o^2 - r_i^2 \right) \]

The total cluster mass of the baseline configuration can be written as

\[ M_b = \eta_b \frac{\pi}{4} \rho_{rot} \ell_{rots} \left(2r_o^2 - r_i^2 \right) \]

Setting the fixed values in the equation equal to the constant, \(\frac{\pi}{4} \rho_{rot} \left(2r_o^2 - r_i^2 \right) = C_2\) and substituting into \(M_{ai} = M_b\), gives

\[ \eta_i C_2 \ell_{rot} = \eta_b C_2 \ell_{rots} \]

Rearranging

\[ \ell_{rot} = \frac{\eta_b C_2 \ell_{rots}}{\eta_i C_2} = \frac{\eta_b \ell_{rots}}{\eta_i} \]

This last equation shows that the maximum allowable rotor length is dependent on the number of CMGs in the configuration. The effect of equating the mass of the candidate configuration to the baseline configuration fixes the value of rotor length in the sizing process, independently of the gimbal rate. The calculated values of rotor length from the sizing process are presented in Table 4-3.

The number of CMGs in a candidate configuration can be represented as a percentage of the number of CMGs in the baseline configuration.

\[ \eta_i = d_i \eta_b \]

Where \(d_i\) is a constant for each candidate configuration. Substituting \(d_i\) into the equation for the rotor length of a configuration gives

\[ \ell_{rots} = \frac{\ell_{rots}}{d_i} \]
The configurations have been listed in the table in decreasing order of rotor length. As expected from the above equations the value of rotor length was fixed for each candidate configuration, and did not vary with gimbal rate.

| I  | CONFIGURATION                      | $|\vec{\tau}|_{lc}$ | $\eta$ | $d_i = \frac{\eta_i}{\eta_b}$ | ROTOR LENGTH (CM) |
|----|-----------------------------------|---------------------|--------|-------------------------------|-------------------|
| 1  | 4SGCMG Pyramid (baseline)         | 3.16                | 4      | 1                             | 1.2               |
| 2  | 3SGCMG Axis Aligned               | 2                   | 3      | 0.75                          | 1.6               |
| 6  | 6SGCMG Pyramid                    | 4                   | 6      | 1.5                           | 0.8               |
| 3  | 3TCMG Axis Aligned                | 2                   | 6      | 1.5                           | 0.8               |
| 4  | 3TCMG Pyramid                     | 2                   | 6      | 1.5                           | 0.8               |
| 5  | 4TCMG Pyramid                     | 2.31                | 8      | 2                             | 0.6               |

Table 4-3 Case 1 Results for sized rotor length for the case in which the sizing/selection function is required to size the candidate configuration to maximize the torque developed by the candidate configuration for an equivalent mass of baseline configuration.

Where

$|\vec{\tau}|_{lc}$ is the torque capability

$\eta$ is the total number of rotors in the configuration.

In Table 4-3 it can be seen that the rotor length is fixed for each configuration, and it is independent of the torque capability. The value of the rotor length is determined by the number of CMGs in a configuration $\eta_i$, and its value decreases with increase in the ratio of the number of candidate to baseline rotors, (i.e the value of $d_i = \frac{\eta_i}{\eta_b}$).

4.5.2 Total Cluster Mass Results

For this case the requirement of the sizing/selection function is to size candidate configurations so that the torque developed by the candidate configuration is maximized for an equivalent
configuration mass to the baseline configuration mass. This requirement leads to the expectation that the values of candidate configuration total cluster masses will be equal to the baseline values, these values will be fixed and not vary with gimbal rate $M_{\omega} = M_b$. The total mass of the baseline configuration is calculated by the sizing program to be 2.7890 kg. However, the results for total masses show that the mass is not equated exactly but the maximum difference is relatively very small (0.0139 g for the 4TCMG pyramid). The mass differences (i.e. $M_{\omega} - M_b$) are presented in Table 4-4. The torque capability of a particular candidate configuration, i, can be related to the torque capability of the baseline configuration through a constant, $t_i$ (see Table 4-4).

<table>
<thead>
<tr>
<th>I</th>
<th>CONFIGURATION</th>
<th>$\eta$</th>
<th>$d_i = \frac{\eta_i}{\eta_b}$</th>
<th>MASS DIFFERENCE (G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4SGCMG Pyramid (baseline)</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3SGCMG Axis Aligned</td>
<td>3</td>
<td>0.75</td>
<td>0.0052</td>
</tr>
<tr>
<td>6</td>
<td>6SGCMG Pyramid</td>
<td>6</td>
<td>1.5</td>
<td>0.0104</td>
</tr>
<tr>
<td>3</td>
<td>3TCMG Axis Aligned</td>
<td>6</td>
<td>1.5</td>
<td>0.0104</td>
</tr>
<tr>
<td>4</td>
<td>3TCMG Pyramid</td>
<td>6</td>
<td>1.5</td>
<td>0.0104</td>
</tr>
<tr>
<td>5</td>
<td>4TCMG Pyramid</td>
<td>8</td>
<td>2</td>
<td>0.0139</td>
</tr>
</tbody>
</table>

Table 4-4 Table to show the mass error results, and the $t_i$ and $d_i$ for each configuration for the case in which the sizing/selection function is required to size candidate configuration to maximize the torque developed by the candidate configuration for an equivalent mass of baseline configuration.

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Figure 4-4 Plot demonstrating the effect of the increase in number of CMGs in the candidate configuration on the increase in the mass difference to the baseline configuration

Figure 4-4 shows the linear trend in the increase of the mass difference of a candidate configuration to the baseline configuration, with increase in the number of CMGs in the configuration. This linear trend indicates that the differences are possibly due to truncation or rounding errors when the values of rotor length are progressed through the sizing program. However the relative error is small, when compared with the value of the total mass of the configuration.

4.5.3 Total Torque Generated Results

For Case 1 the requirement of the sizing/selection function is to size candidate configurations to maximize the torque developed by the candidate configuration, were the candidate configuration mass is equivalent to the baseline configuration mass. In this section the equation for total cluster torque developed will be examined to determine which parameters affect its value.
In Section 4.2.1, Equation 4-11 the torque available from the candidate configuration, is derived as;

\[ T_{\text{av}} = C_i \delta \| \vec{\nu} \|_{\text{av}} \ell_{\text{ref}} \]

The baseline available torque is

\[ T_{\text{bas}} = C_i \delta \| \vec{\nu} \|_{\text{bas}} \ell_{\text{ref}} \]

These equations show that as the gimbal rate, \( \delta \) is increased, the torque available from the candidate configuration, and the baseline configuration will increase. Remembering that \( \| \vec{\nu} \|_{\text{av}} \) is a constant for each configuration and its value is geometrically derived it is therefore independent of \( \delta \).

The torque capability of a candidate configuration \( \| \vec{\nu} \|_{\text{av}} \), can be defined in terms of the torque capability of the baseline configuration, \( \| \vec{\nu} \|_{\text{bas}} \) through a constant \( t_i \).

\[ \| \vec{\nu} \|_{\text{av}} = t_i \| \vec{\nu} \|_{\text{bas}} \quad 4-61 \]

Substituting Equation 4-59, and Equation 4-60, (which describes \( \ell_{\text{ref}} \) in terms of \( \ell_{\text{ref}} \)) in Equation 4-11, the total cluster torque available from a candidate configuration can be written in terms of the baseline parameters;

\[ T_{\text{av}} = C_i \frac{f_i}{d_i} \| \vec{\nu} \|_{\text{bas}} \ell_{\text{ref}} \delta \quad 4-62 \]

Gathering constants and introducing a new constant;

\[ C_i = C_i \| \vec{\nu} \|_{\text{bas}} \ell_{\text{ref}} \quad 4-63 \]

Gives

\[ T_{\text{av}} = C_i \frac{f_i}{d_i} \delta \quad 4-64 \]

The torque available from a particular candidate configuration is fixed for a particular gimbal rate, and increases proportionally with gimbal rate increase, as the constants \( C_i, t_i, d_i \) are fixed values for each candidate configuration. Candidate configurations with higher values of \( \frac{f_i}{d_i} \) have higher values of total cluster torque available. In Table 4-5 the calculated values of \( \frac{f_i}{d_i} \) for the
candidate configurations are presented. Figure 4-5 shows the total cluster torque generated by candidate configurations, which have a total cluster mass equal to the baseline configuration mass.

| I | CONFIGURATION                  | $|v|_{in}$ | $\frac{t_l}{d_i}$ |
|---|--------------------------------|----------|--------------------|
| 1 | 4SGCMG Pyramid (baseline)      | 3.16     | 1                  |
| 2 | 3SGCMG Axis Aligned            | 2        | 0.8439             |
| 3 | 6SGCMG Pyramid                 | 4        | 0.8439             |
| 4 | 3TCMG Axis Aligned             | 2        | 0.4219             |
| 5 | 4TCMG Pyramid                  | 2.31     | 0.3655             |

Table 4-5 Table to show calculated values of $|v|_{in}$ and $\frac{t_l}{d_i}$ for candidate configurations, for the case in which the sizing/selection function is required to size candidate configuration to maximize the torque developed by the candidate configuration for an equivalent mass of baseline configuration.

Table 4-5 shows that the 3TCMG axis aligned and the 3TCMG pyramid configurations have the same value of $\frac{t_l}{d_i}$, this indicates that the values of torque developed for these candidate configurations should be equal. The plots for the 3TCMG axis aligned and the 3TCMG pyramid configurations appear over plotted, and when calculating the difference between the plots i.e. $(T_{a(3TCMG, axis aligned)} - T_{a(3TCMG, pyramid)})$ it is found that the difference is zero, i.e. the plots are exactly the same for each gimbal rate. Table 4-5 shows a further two configurations, the 6SGCMG pyramid and 3SGCMG axis aligned configurations have the same value of $\frac{t_l}{d_i}$. The plots for these configurations also appear to be the same, but there is actually a small difference between the two plots.
i.e. \( T_{3TCMG, \text{pyramid}} - T_{3TCMG, \text{axis, aligned}} \), this difference is shown in Figure 4-6. The difference follows a linear trend, with the equation \( y = 0.003x - 0.00002 \), were \( y \) is the difference, and \( x \) is the gimbal rate.

In the calculation of \( T_{ai} \), the pairing of 3TCMG axis aligned and 3TCMG pyramid have exactly the same values of \( t_i \) and \( d_i \), leading to the same value of \( \frac{t_i}{d_i} \). However the pairing of 6SGCMG pyramid and 3SGCMG axis aligned configurations have different values of \( t_i \) and \( d_i \), which although they work out to the give the same value \( \frac{t_i}{d_i} \) could give rise to the small differences seen in Figure 4-6, due to truncation and rounding in the program when values are passed.

Figure 4-5 Total cluster torque generated by candidate configurations with total mass equal to baseline configuration mass. Plots of 3SGCMG axis aligned and 6SGCMG pyramid are overplot.
Figure 4-6 Plot showing the difference in torque generated by the 3SGCMG axis aligned and 6SGCMG pyramid configurations (i.e. $T_{3SGCMG \text{ aligned}} - T_{6SGCMG \text{ aligned}}$), which appear overplotted in Figure 4-5.

From the results of torque generated it is concluded that for equivalent mass the candidate configurations generate less torque than the baseline configuration, as shown in Figure 4-5.

4.5.4 Total Configuration Volume Occupied Results

The sizing process is not required either equate or maximize/ minimize the total cluster volume occupied by a candidate configuration, only to calculate its value based on Equation 4-27, developed in Section 4.2.3.

$$V_{\text{swept}} = \eta \frac{3}{4} \pi r_o^3$$

This equation shows that the volume is dependent only on the number of rotors in a configuration. Table 4-6 shows the results of total swept volume for each configuration. These results show the increase in volume follows an increase in number of rotors in a configuration.
### Table 4-6 Total swept volume of configurations of equal torque, and minimum mass

<table>
<thead>
<tr>
<th>CONFIGURATION</th>
<th>η</th>
<th>VOLUME m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 4SGCMG Pyramid</td>
<td>4</td>
<td>0.0122</td>
</tr>
<tr>
<td>(baseline)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 3SGCMG Axis Aligned</td>
<td>3</td>
<td>0.0091</td>
</tr>
<tr>
<td>3 3TCMG Axis Aligned</td>
<td>6</td>
<td>0.0183</td>
</tr>
<tr>
<td>4 3TCMG Pyramid</td>
<td>6</td>
<td>0.0183</td>
</tr>
<tr>
<td>5 4TCMG Pyramid</td>
<td>8</td>
<td>0.0244</td>
</tr>
<tr>
<td>6 6SGCMG Pyramid</td>
<td>6</td>
<td>0.0183</td>
</tr>
</tbody>
</table>

It can be seen that the 3SGCMG axis aligned configuration occupies the least volume, and the 4TCMG pyramid occupies the most.

### 4.5.5 Total Configuration Power Consumed Results

In this section the equation for the total power consumed by a candidate configuration will be examined to determine which parameters affect its value. In Section 4.2.6, Equation 4-44 was developed to give the total cluster power consumed for a candidate configuration;

\[ P_a = \eta_c \dot{\theta} \left( C_1 \delta^2 + C_2 \Omega \dot{\theta} \right) \]

The mass of candidate configuration is equated to the mass of the baseline configuration, so

\[ \ell_{rel_i} = \frac{\ell_{rel_b}}{d_i} \]

Where \( \frac{\eta_i}{\eta_b} = d_i \)

Substituting into the equation for total cluster power consumed, to obtain an equation for the total power consumed by a candidate configuration, in terms of the baseline parameters;

\[ P_a = d_i \eta_b \frac{\ell_{rel_b}}{d_i} \left( C_1 \delta^2 + C_2 \Omega \dot{\theta} \right) \]

\[ P_a = \eta_b \ell_{rel_b} \left( C_1 \delta^2 + C_2 \Omega \dot{\theta} \right) \]

4-65

4-66
Equation 4-70 shows that the power consumed by each candidate configuration is equal to the total cluster power consumed by the baseline configuration.

![Graph showing power consumed by gimbal motors](image)

**Figure 4-7** Plot showing the total power consumed by the gimbal motors of the candidate configuration for the case in which the sizing/selection function is required to size the candidate configuration to maximize torque for an equivalent configuration mass to the baseline configuration mass.

Figure 4-7 shows the total power consumed by the gimbal motors of the candidate configurations for the case in which the candidate configurations have a total cluster mass equal to the baseline configuration mass, and the torque generated from each configuration is maximized. The shape of Figure 4-7 is due to the quadratic form of Equation 4-66. This plot indicates that the power consumed by each configuration is the same, but Figure 4-8 shows that there is a small difference between the values of power consumed for the candidate configurations and the baseline configuration.

The plots of difference for the 3TCMG axis aligned, 3TCMG pyramid and the 6SGCMG pyramid are equal. These plots are equal as the values of rotor length, and the number CMGs in each configuration are the same.
Figure 4-8 Plot showing the difference between the power consumed by candidate configurations and the baseline configuration, for configurations of equal mass, and for maximum torque generation. The plots for 3TCMG axis aligned, 3TCMG pyramid and 6SGCMG pyramid are over plotted, as power consumed by these configurations are equal.

The power can be broken down into total cluster gimbal power and total cluster rotor power for analysis. The gimbal motors draw the majority of the total power consumed, with the rotor motor power consuming relatively very little. From Equation 4-33 it is seen that the rotor motor power does not vary with gimbal rate variation. The power consumed by the rotor motors is $P_{\text{rot}} \approx 0.0033$ Watts, this value is exact for the 4SGCMG pyramid configuration, however there is a small difference for the candidate configurations. The difference for the 3SGCMG axis aligned is $+0.6203\times10^{-8}$ Watts, for the 4TCMG pyramid configuration it is $+0.1654\times10^{-7}$ Watts, and for the remaining configurations (3TCMG axis aligned, 3TCMG pyramid and 6SGCMG pyramid configurations) the difference is $+0.1241\times10^{-7}$ Watts.

The differences in values for the power consumed can be explained by the equations implemented in the sizing/selection program written in MATLAB which generates the results.
The expectation that the power consumed by the candidate configurations will be equal to the power consumed by the baseline configuration is based on Equation 4-66. However, the power consumed is not directly calculated using Equation 4-66 in the program. In the program the power is calculated following stages represented by Equations 4-31 to Equation 4-43. During the calculation of each of these stages errors can be built up due to rounding within the program.

4.6 Case 2 Minimizing Mass, Equating Torque

The sizing/selection function is required to size candidate configurations to minimize the mass of the candidate configuration, were the torque developed by the candidate configuration is equivalent to the torque developed by the baseline configuration, i.e. \( T_{ai} = T_{ab} \).

The constraints of equal torque allows a fair comparison of the torque developed by each configuration. In order to develop the largest value of torque the rotor length is sized. The parameters for case 1 are presented in Table 4-2. Each candidate configuration is sized at discrete gimbal rates over the range \( \delta_{min} \leq \delta_{ij} \leq \delta_{max} \), where \( \delta_{min} = 0.5 \text{rad/s} \) and \( \delta_{max} = 10 \text{rad/s} \). The BILSAT-1 satellite was chosen for the case study.

The sizing/selection function for the mass design margin was derived in Section 4.3 as, Equation 4-50;

\[
f_i(\bar{x}) = M_{ui}
\]

It has been defined that \( \bar{x} = [\ell_{rot}, r_{rot}, r_a, r_i] \) where \( \rho_{rot} = \rho_{rot b}, r_{rot} = r_{rot b}, r_i = r_i \) and \( \chi_i = \chi_b \), \( \Omega_i = \Omega_b \).

The sizing/selection function is now studied in more detail, so that the effects of the variables on the value of the objective function can be seen. The equations for the mass margins \( M_m = M_r - M_a \) for the configuration, i, and the baseline configuration are substituted into the sizing function \( f_i(\bar{x}) \).

The equation for the total cluster mass available from a configuration, \( i \) in terms of the decision variables has already been defined as Equation 4-17;

\[
M_a = \eta \frac{\pi}{4} \rho_{rot} \ell_{rot} \left( 2r_a^2 - r_i^2 \right)
\]

And the equation for total cluster mass reduces to Equation 4-19

\[
M_a = \eta C2 \ell_{rot}
\]
For the baseline configuration the total mass of the configuration can be defined as

\[ M_{ab} = \eta_b \frac{\pi}{4} \rho \, r_i \, (2r_i^2 - r_o^2) \]

Which can be simplified to

\[ M_{ab} = \eta_b C_2 \, l_{rotb} \]

Substituting Equations 4-19 and 4-68 into Equation 4-50

\[ f_2(x) = M_{wi} = \eta_w C_2 \, l_{roti} - \eta_i C_2 \, l_{roti} \]

Which can be simplified to

\[ f_2(x) = M_{wi} = C_2 (l_{rotb} - l_{roti}) \]

To minimize the mass of each candidate configuration, such that it is less than or equal to the value mass of the baseline configuration, Equation 4-70 must be maximized. The value of number of CMGs per configuration, \( \eta \) is individual to each candidate configuration, and dependent on configuration geometry. The length of each rotor in the configuration, \( l_{rot} \) is sized by the sizing process and changes at each discrete \( \delta_s \), gimbal rate.

With the constraints;

\[ T_{ai} = T_{ib} \]

\[ x_i = x_b, \, \Omega_i = \Omega_b, \, r_{ai} = r_{ib}, \, r_{oi} = r_{ob}, \, \rho_{roti} = \rho_{rotb} \]

To minimize the mass of each candidate configuration, such that it is less than or equal to the value mass of the baseline configuration, Equation 4-70 must be maximized. The value of number of CMGs per configuration, \( \eta \) is individual to each candidate configuration, and dependent on configuration geometry. The length of each rotor in the configuration, \( l_{rot} \) is sized by the sizing process and changes at each discrete \( \delta_s \), gimbal rate.

With the constraints;

\[ T_{ai} = T_{ib} \]

\[ x_i = x_b, \, \Omega_i = \Omega_b, \, r_{ai} = r_{ib}, \, r_{oi} = r_{ob}, \, \rho_{roti} = \rho_{rotb} \]

\[ l_{rot} \geq 0 \]

### 4.6.1 Rotor Length Results

In this case the mass of each candidate configuration is minimized by changing the rotor length, \( l_{rot} \). In order to do this the smallest allowable value for the rotor length will be selected by the sizing program.

However, it is constrained that the total cluster torque of the candidate configuration, \( i \), is to be equal to the total cluster torque of the baseline configuration; i.e. \( T_{ai} = T_{ib} \). This fixes the rotor length that can be selected by the sizing algorithm. A mathematical equation to represent this value for the rotor length is presented in the following lines;

From Equation 4.11, derived in Section 4.2.1 the total cluster torque of the candidate configuration, \( i \), is;
The total cluster torque of the baseline configuration can be written as

\[ T_{ab} = C_1 |v| \ell_{rot} \hat{\delta} \]

Where \( C_1 = \frac{\pi}{4} \rho_{rot} \Omega X (2r_o^4 - r_i^4) \)

And the total cluster torque of the baseline configuration can be written as

\[ T_{ab} = C_1 |v| \ell_{rot} \hat{\delta} \]

4-71

Substituting into \( T_{ab} = T_b \), gives

\[ C_1 |v| \ell_{rot} \hat{\delta} = C_1 |v| \ell_{rot} \hat{\delta} \]

Rearranging

\[ \ell_{rot} = \frac{|v| \ell_{rot}}{|v| \ell_{rot}} \]

4-72

This last equation shows that the minimum allowable rotor length is dependent on the torque capability of the candidate configuration, where a higher torque capability yields a smaller rotor length. The effect of equating the torques, fixes the value of rotor length from the sizing process, independently of the gimbal rate. The calculated values of rotor length from the sizing process are presented in Table 4-7.

The torque capability of a candidate configuration can be represented as a percentage of the torque capability of the baseline configuration, where \( t_i \) is a constant for each candidate configuration.

\[ t_i = \frac{|v|_{r_i}}{|v|_{b_i}} \]

The configurations have been listed in the table in decreasing order of rotor length. As expected from the above equations the value of rotor length was fixed for each candidate configuration, and did not vary with gimbal rate.
Table 4-7 Sized rotor lengths for the case in which the mass is minimized and the torque developed by the candidate configuration is equated to the torque developed by the baseline configuration

In Table 4-7 it can be seen that the rotor length is fixed for each configuration, and it is independent of the number of CMGs in a configuration. The value of the rotor length is determined by the torque capability of a candidate configuration \( \eta \), and its value decreases with increase in the ratio of the candidate torque capability to the baseline torque capability, (i.e the value of \( \eta = \frac{[\psi]_{c}}{[\psi]_{b}} \)). This result is the direct opposite to the results of Section 4.5.1, were the torque developed by the candidate configuration is maximized and for an equivalent configuration mass.

4.6.2 Total Torque Generated Results

For this case the sizing/selection function is required to size candidate configurations to minimize the mass of the candidate configuration, were the torque developed by the candidate configuration is equivalent to the torque developed by the baseline configuration,(i.e. \( T_{c} = T_{b} \)). This requirement leads to the expectation that the values of candidate configuration torque developed will be equal to the value of the baseline value torque developed.
Figure 4-9 Plot showing generated torque of the baseline configuration (4SGCMG pyramid configuration), over gimbal rate range 0-10 rad/s, with the simulation parameters presented in Table 4-2.

The results for total torque developed showed that the torque developed by the candidate configurations is not equal to the torque developed by the baseline configuration, these differences are plotted in Figure 4-10.
Figure 4-10 Torque errors between the baseline configuration and the candidate configurations, for the case in which the sizing/selection function is required to equate the torque developed by the candidate configuration to the torque developed by the baseline configuration.

In Figure 4-10 it is seen that the plots for 3SGCMG axis aligned, 3TCMG axis aligned and the 3TCMG pyramid configurations are over plotted. The calculated difference between these three plots is zero, and this is due all three configurations having the same torque capability, as shown in Table 4-7.

The plots in Figure 4-10 show that the errors are building up as the gimbal rate is increased, this indicates that the errors are rounding or truncation errors in the value of $\ell_{rot}$, which start small but are accumulated as the program is run. However the relative error is small, when compared with the value of torque developed.

4.6.3 Mass Results and Discussion

For this case the sizing/selection function is required to size candidate configurations to minimize the mass of the candidate configuration. In this section the equation for total cluster mass will be examined to determine which parameters affect its value.
In Section 4.2.2, Equation 4.19 for the total cluster mass of a candidate configuration was derived to be

\[ M_{cl} = \eta_i C_r^2 \ell_{rot} \]

This equation shows that the total configuration mass does not vary with increase in gimbal rate, and is a fixed value for a particular values of rotor length. In this case the sizing program is required to equate the torque developed by the candidate configuration to the torque developed by the baseline configuration. This requirement has been shown to fix the rotor length of a candidate configuration according to the following equation

\[ \ell_{rot} = \frac{|\mathbf{v}|_{hub}}{|\mathbf{v}|_{rot}} \ell_{rots} \]

This Equation in turn fixes the value of the mass of the candidate configuration in Equation 4.19.

| CONFIGURATION                  | \( \eta \) | \( t_i = \frac{|\mathbf{v}|_{ic}}{|\mathbf{v}|_{is}} \) | ROTOR LENGTH (CM) | MASS (KG) |
|-------------------------------|------------|-----------------------------------------------|-------------------|-----------|
| 1 4SGCMG Pyramid (baseline)   | 4          | 1                                             | 1.2               | 2.789     |
| 2 3SGCMG Axis Aligned         | 3          | 0.6329                                        | 1.9               | 3.300     |
| 3 3TCMG Axis Aligned          | 6          | 0.6329                                        | 1.9               | 6.601     |
| 4 3TCMG Pyramid               | 6          | 0.6329                                        | 1.9               | 6.601     |
| 5 4TCMG Pyramid               | 6          | 0.7310                                        | 1.6               | 7.615     |
| 6 6SGCMG Pyramid              | 6          | 1.2658                                        | 0.9               | 3.299     |

Table 4-8 Total configuration mass results for the case in which the mass is minimized and the torque developed by the candidate configuration is equated to the torque developed by the baseline configuration.

Table 4-8 shows that the masses for the 3TCMG axis aligned and the 3TCMG pyramid configurations are equal, this is explained by both configurations having the same number of
CMGs, the same torque capability and therefore the same values of $\eta$, $t$, and $\ell_{n}$. Table 4-8 shows that the mass increases with increase in number of rotors and increase in rotor length.

### 4.6.4 Total Power Consumed Results

The sizing process is not required either equate or maximize/minimize the total cluster power consumed by a candidate configuration, only to calculate its value based on Equation 4-44, developed in Section 4.2.

$$P_a = \eta \ell_{n} \left( C_1 \dot{\delta}^2 + C_2 \Omega \dot{\Omega} \right)$$

For the case in which the total torque developed by a candidate configuration is equated to the total torque developed by the baseline configuration, the rotor length is a fixed value.

$$\ell_{n} = \frac{\ell_{rot}}{t}$$

Substituting this into the equation for total cluster power consumed

$$P_a = d \eta \ell_{rot} \left( C_1 \dot{\delta}^2 + C_2 \Omega \dot{\Omega} \right)$$

This equation shows that the total power consumed by a candidate configuration is dependant on both the rotor length and the number of rotors in a configuration. The equation is seen to be quadratic in form; this indicates the expected shape of the plot. Figure 4-11 shows the plots of the total power consumed by each configuration.
Figure 4-11 Plot showing the total power consumed by the configurations, for the case in which the configuration mass is minimized and the torque developed by candidate configurations is equated to the torque developed by the baseline configuration.

The 3TCMG axis aligned and the 3TCMG pyramid configurations have exactly the same values of $t_i$ and $d_i$, in Equation 4-73, this explains why their plots are exactly the same. The plots for the 3SGCMG axis aligned and the 6SGCMG pyramid configurations are also the same, even though their values of $t_i$ and $d_i$ are not equal, however their ratios of $\frac{d_i}{t_i}$ in Equation 4-73 are exactly the same.

The power consumed by the rotor motors of the 4SGCMG pyramid configuration is $P^\text{rot}_n \approx 0.0033\text{ Watts}$. The 3SGCMG axis aligned configuration, and the 6SGCMG pyramid configuration rotor motors consume about the same power at $0.0039\text{ Watts}$. The 3TCMG pyramid and 3TCMG axis aligned configurations consume more power at $0.0079\text{ Watts}$, and the highest power is consumed by the 4TCMG pyramid configuration at $0.009\text{ Watts}$. 

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4.6.5 Total Volume Occupied Results

The sizing process is not required either equate or maximize/ minimize the total cluster volume occupied by a candidate configuration, only to calculate its value based on Equation 4-27, developed in Section 4.2.3

\[ V_{\text{swept}} = \eta \frac{3}{4} \pi \rho^3 \]

This Equation shows that the volume is dependant only on the number of rotors in a configuration. The results for this case are the same as the first case considered in Section 4.54, and Table 4-6 shows the results of total swept volume for each configuration.

4.7 Sizing/Selection Process Conclusions

In Chapter 2 the principles of Single-Control Moment Gyro and Twin-Control Moment Gyro operation where studied. The study of Euler’s equations and controller principles lead to the implementation of an attitude control system in SIMULNIK, for which results were compared with previous work [46] and differences were highlighted and discussed.

In Chapter 3 the geometry of each configuration was analysed and the torque capability of each was calculated. From this work it can be seen that the geometry of configurations is important when considering a replacement actuator to the 4SGCMG pyramid configuration. The 3SGCMG axis aligned, and 3TCMG axis aligned configurations have the disadvantage of being aligned with the axes of the satellite, it suffers from low failure survivability. If one of these SGCMGs or TCMGs fails, control is lost about one of the satellite principle axes, this situation is unacceptable to small satellite missions. For this reason both these configurations are considered to be unsuitable as replacements for the 4SGCMG pyramid configuration.

The results from the sizing process are now considered, starting with the first case in which the torque developed by the candidate configurations is maximized for a configuration mass equivalent to the baseline configuration mass, these results are presented in Section 4.5

These results show that the rotor length is fixed for each configuration, and is independent of torque capability. The value of the rotor length is determined by the number of rotors in a
configuration. The 6SGCMG pyramid configuration has the smallest rotor length, and the 3SGCMG axis aligned configuration has the largest rotor length. For equal configuration mass, the torque developed by the candidate configurations has been shown to be less than the torque developed by the baseline configuration. However, at low gimbal rates this difference is smaller than at high gimbal rates. Of the candidate configurations the 6SGCMG pyramid develops the greatest torque across gimbal rate.

It was expected that the power consumed by the candidate configurations would be equal to the power consumed by the baseline configuration is based on Equation 4-70. However, the power consumed is not directly calculated using Equation 4-70 in the program. This lead to errors in the power results, however these errors were relatively small, and will be considered to be equal for this analysis.

It is concluded that for the case in which the torque developed by the candidate configurations is maximized for a configuration mass equivalent to the baseline configuration mass, most of the candidate configurations are not suitable for replacing the 4SGCMG pyramid configuration. However, the 6SGCMG pyramid configuration performs the best out of the candidate configurations.

The results for the case in which the mass of the candidate configuration is minimized, for torque equivalent to the torque developed by the baseline configuration, are presented in Section 4.6. These results show that the rotor length is fixed for each configuration and is independent of the number of rotors in a configuration. The value of the rotor length is determined by the torque capability of a candidate configuration. This result is the direct opposite to the results of Section 4.5.1, for the case in which the torque developed by the candidate configurations is maximized for a configuration mass equivalent to the baseline configuration mass.

The results for total configuration cluster mass show that the value of mass increases with both an increase in number of rotors and an increase in rotor length. The baseline configuration (i.e. 4SGCMG pyramid) shows the least total cluster mass, when compared to candidate
configurations. The candidate configuration with the least configuration mass is found to be the 6SGCMG pyramid configuration.

The results have shown that the total power consumed by a candidate configuration is dependent on both the rotor length and the number of rotors in a configuration. The configuration consuming the least power is the baseline configuration (i.e. 4SGCMG pyramid configuration). Of the remaining candidate configurations the 6SGCMG pyramid and the 3SGCMG pyramid configurations consume the least power.

From the results of the total volume occupied by a configuration it was seen that the 3SGCMG axis aligned configuration occupies the least volume. However this configuration has been considered to be unsuitable as a replacement configuration due to its geometry. The configuration with the second smallest volume is the baseline configuration.

It is concluded that for the case in which the mass of the candidate configuration is minimized, for torque equivalent to the torque developed by the baseline configuration, most of the candidate configurations are not suitable for replacing the 4SGCMG pyramid configuration. However, with the least configuration mass, and consuming the least power the 6SGCMG pyramid configuration performs the best out of the candidate configurations.

Considering the conclusions drawn from all the results presented in this Chapter, it can be seen that the 4SGCMG pyramid configuration performs the best all round in terms of torque developed, power consumed and mass and volume occupied. Of the candidate configurations the 6SGCMG pyramid configuration stands out as a possible replacement for the baseline configuration.

Of the TCMG configurations the 3TCMG axis aligned configuration is considered unsuitable due to its geometry. The 4TCMG pyramid configuration is heavy and for its mass it develops a relatively low torque, while consuming the most power. However, the 3TCMG pyramid configuration is the most promising. Although it is a relatively heavy configuration, about twice the mass of the 6SGCMG pyramid, it consumes the same power. It is recommended that the
3TCMG pyramid configuration is investigated further if the total mass of this system could be reduced it could be a contender as a possible replacement for the 4SGCMG pyramid configuration.
5 Future Work and Open Problems

5.1 Extensions of the Sizing/Selection Program

The inclusion of rotor motor and gimbal motor masses in the sizing/selection process would allow a more realistic comparison of the configurations. Including the gimbal shaft and gimbal casing structure volume would add to the calculation of the total cluster volume of the configuration. These suggested additions would make the results more realistic as the TCMG is "sold" on the fact that it may take less volume/mass than the SGCMG system.

The work could be enhanced by the inclusion of a function in the algorithm to calculate the forces if units are placed not at the centre of mass of the satellite but are situated anywhere within the satellite volume. Extending this idea a routine that finds the optimal positioning of the units could be investigated.

A further enhancement would be to provide an automated satellite command and control system dynamic performance optimization concept. This concept would assist a second objective to determine the effects of the performance of a small satellite with different candidate configurations. The optimisation concept would be automated and based on dynamic mathematical modelling of the satellite, its system and its system components. In the case where the CMGs are dispersed within the satellite volume and are of differing design, it is suggested that the design/implementation flexibility provided by this general concept would be extremely useful in achieving a cost effective system design and implementation.

An outcome of the general concept would be to provide a facility to perform flight clearance calculations defining the flight limitations. These could include in-flight variations in mass/inertia due to for example thruster fuel usage or probe deployment. Whilst the concept would also assist in the event of an 'eleventh hour' re-positioning of the system components to alleviate unplanned structural coupling effects following satellite ground based verification testing.
5.2 Moving Baseline Description

The approach to sizing taken in this report is to size candidate configurations to match/better the performance of a particular fixed baseline configuration. A further extension of the sizing/selection process, would be to also size the baseline configuration to specific mission requirements. Each candidate configuration would then in turn be compared to the sized baseline configuration. This process would allow the designer to select a value of required torque, or limits on mass, volume and power consumed for all gimbal rates.

5.3 Failure Survivability of TCMG systems

A further open problem is the evaluation and derivation of a control law that would enable a failure survivability twin CMG control system to perform large angle manoeuvres at high rates while maintaining the attitude acquisition accuracy expected from a CMG system. The results would then require evaluation through experimental simulation of the control system.

5.4 Electrically Synchronized TCMGs

A feasible solution to the failure survivability problem is to replace the mechanical coupling with electronic coupling by incorporating constraints into the CMG command system. Any pair of CMGs for which pairs of gimbal axes are collinear may be considered as potentially a scissor pair, and provide all the advantages as described for the mechanically geared TCMG platform [28]. The cross coupling can be eliminated from the control of the CMGs by a steering law which considers the CMGs always in pairs.

With careful positioning of each CMG and provided there is sufficient computational capacity on board the vehicle, the cross axis effects that would arise during a single CMG failure can be used to advantage to achieve fail operational characteristics. In this fail operational configuration a set of individual CMG gimbal rate loops are required. This would require a failure detection, identification, rejection and reconfiguration system associated with the CMG command system. The feature that enhances the reliability is its ability to operate in a back up mode with one CMG failed and still maintain sufficient three-axis vehicle control performance.
References


[19] Lappas, V. and Wie, B., CMG Sizing for Minisatellites, Technical Note 2, Surrey Space Centre, University of Surrey, Guildford, United Kingdom, for the European Space Agency, 2004


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Appendix A Attitude Control System (ACS) Block Diagrams

Figure 6-0-1 Attitude Control System (ACS) Simulink Implementation
Figure 6-0-2 PID Block Detail

Figure 6-0-3 GSR Block Detail
Figure 6-0-4 Saturation and CMG Dynamics Block Detail
Figure 6-0-6 Antiwind Up Logic Block Detail
Appendix B MATLAB M File Code For Sizing Program

M-file code for OPTSIZE_2712_calc.m

function J=OPTSIZE_2712_calc(x)

global methodDISP optDISP sizeDISP satDISP %FOR PLOTTING USING AUTHOR'S FUNCTION alicesGraphs.m

global log
global Tb Mb Pb Vb
global q_max beta ta Hr qdot I_max accel_rot

global actual values
global Ta Ma Pa Va %actuals a
global Irot Vswept Mcmg Tcmsg vtc %values a
global length_rot rot_o rot_i density_rot %rotor dimensions a

global required values
global Tr Fr Vr Mr

global input options
global size var optimize method satcase

%iterations

global deltal hr ho

%for power plots - interesting curve investigation

global P_GM_tot P_ROT

global P_unit P_unitRot P_unitGM %for unit power plots - check equations


global Tcmsg Tcmsga


global Ncmsg

%VARIABLE LOOP-USER INPUT
deltad=var;

%1: MISSION REQUIREMENTS: Define manœuvre

if satcase==1
    I_max=10; %max principle MOI of satellite kgm2 - Vehicle Z-Axis
    q_max=4*(pi/180); %Slew Rate_max about max I_V deg/s to rad/s - (otherwise known as omega max)
    ta=3; %Time to reach max slew rate sec
    qdot=q_max/ta; %Required acceleration rad/sec
elseif satcase==2
    I_max=70; %max principle MOI of satellite kgm2 - Vehicle Z-Axis
    q_max=4*(pi/180); %Slew Rate_max about max I_V deg/s to rad/s - (otherwise known as omega max)
    ta=3; %Time to reach max slew rate sec
    qdot=q_max/ta; %Required acceleration rad/sec
end

Tr=I_max*qdot;
Hr=I_max*q_max;

%Calc torque required for manœuvre

% theta=48*(pi/180);
% toff=0;
% tf=12;
% Tr=(4*I_max*theta)/(tf^2-toff^2);

% 2: CONFIGURATIONS SET UP

%TEST CONFIG ROTOR PARAMETERS

beta=54.7*pi/180;

rot_o=0.09; %m
rot_i=0.05; %m
accel_rot=0.003;%rad/s/s
density_rot=2700;%kg/m^3
omega_R=5000*pi/30; %rotor speed ras/sec
length_disc=x(1); %m
length_ring=length_disc; %equate disc and ring lengths for simple model of rotor

%Baseline Configuration

if i==1
    %4SGCMG pyramid
    vte=2+2*cos(beta);
    Ncmg=4;
    CMG_RATIO=1;
    if method==1 length_disc=0.012/2;
%3: CALCULATIONS

% INERTIA OF ONE ROTOR
I_{ring} = 0.5 \cdot \pi \cdot \text{density}_{rot} \cdot \text{length}_{ring} \cdot (\text{rot}_{o}^4 - \text{rot}_{i}^4); \ %RING
I_{disc} = 0.5 \cdot \pi \cdot \text{density}_{rot} \cdot \text{length}_{disc} \cdot (\text{rot}_{o}^4); \ %DISC
I_{rot} = I_{ring} + I_{disc}; \ %ROTOR = RING + DISC

% MASS OF ONE ROTOR
M_{ring} = \pi \cdot \text{density}_{rot} \cdot \text{length}_{ring} \cdot (\text{rot}_{o}^2 - \text{rot}_{i}^2); \ %RING
M_{disc} = \pi \cdot \text{density}_{rot} \cdot \text{length}_{disc} \cdot (\text{rot}_{o}^2); \ %DISC
M_{rot} = M_{ring} + M_{disc}; \ %ROTOR = RING + DISC
%TOTAL CONFIG. MASS
Ma=Mrot*Ncmg;

%VOLUME NEEDED FOR ONE ROTOR WHEN GIMBALED- CALLED THE SWEPT VOLUME
Vswept_rot=2*rot_o*pi*(rot_o)^2; % a cylinder with length 2ro and end surf area of (pi)ro^2

%TOTAL CONFIG. SWEPT VOLUME
Vswept=Vswept_rot*Ncmg;
Va=Vswept;

%POWER CONSUMED BY CONFIG.

Tcmga=0.8*Irot*omega_R*deltad;

ho=Hr/vtc; %angular mom required per CMG
Tcmga=0.8*ho*deltad; %?? check this out alice!
P_GM=CMG_RATIO*Tcmga1*deltad; %Power required from gimbal motor
P_GM_tot=P_GM*Ncmg/CMG_RATIO;

%Rotor Motor
%Torque required of one rotor motor = mass of rotor * acceleration required
T_lrot=frot*accel_rot; % torque to accelerate the baseline rotor
P_Lrot=T_lrot*omega_R;
%Total power to accelerate all rotors (depends on number of rotors in config.)
P_ROT=P_Lrot*Ncmg;

%TOTAL POWER = power required by gimbal motor and rotor motors
P=P_GM_tot+P_ROT;

P_unitRot=P_Lrot*CMG_RATIO; %SEND TO PLOT - multiply by cmg ratio for plotting a TCMG platform total
P_unitGM=P_GM;
P_unit=P_unitRot+P_unitGM;

%TORQUE AVAILABLE
chi=0.8;
Ta=chi*Irot*omega_R*deltad*vtc;
%added to remove the small errors. 04 March 2008
% if abs(Tr-Ta)<0.001
% Ta=Tr;
% else
% end

% \( \text{ho} = \frac{T_a}{(0.8 \cdot \text{deltad})} \);
% \( \text{ho} = \text{lcmg} \cdot \text{omega}_R \);
% \text{ho} = \text{vtc};

%4: MARGINS
%sets baseline values
if i==1 Mb=Ma;
  Tb=Ta;
  Vb=Va;
  Irotb=Irot;
  Pb=Pa;
else
end

Mr=Mb+10e-12;%Set Mr=Mb - if we do this we get a divide by zero
Vr=Vb+10e-12; %??? value

%TORQUE MARGIN
%margin for test config
Tm=Ta-Tr;
%margin for baseline config
Tmb=Tb-Tr;

%MASS MARGIN
%margin for test config
Mm=Mr-Ma;
%margin for baseline config
Mmb=Mr-Mb;

%VOLUME MARGIN
%margin for test config
Vm=Vr-Va;
%margin for baseline config
Vmb=Vr-Vb;

%POWER MARGIN
%margin for test config
Pm=Pr-Pa;
%margin for baseline config
\[ P_{mb} = P_r - P_b; \]

\[
\text{%5: OUTPUT if optimize == 1} \quad \text{%optimal mass (as a percentage of baseline value) at performance equivalent}
\]
\[
J = \left( \frac{-T_m + T_{mb}}{T_{mb}} \right); \quad \text{%optimal mass values}
\]
\[
\text{elseif optimize == 2}
\]
\[
J = \left( \frac{-M_m + M_{mb}}{M_{mb}} \right); \quad \text{%optimal torque values}
\]
\[
\text{else}
\]
\[
\text{end}
\]

M-file code for OPTSIZE_2712_Run.m

% IMPORTANT NOTES FOR USE:
% Letters used: \%i=\#of configurations; count=\#counts the var steps;
% Displays for graph ID are: methodDisp;optDisp;sizeDisp; satDisp

% IMPORTANT ASSUMPTION NOTE ABOUT ROTOR LENGTH
%assumption that length of ring + length of disc = length of rotor
% x(1)=length of disc
% BUT NOTE THAT: length of ring= length of rotor i.e. x(1)*2=length of
% rotor

global methodDisp optDisp sizeDisp satDisp %FOR PLOTTING FUNCTION
alicesGraphs.m
global Tlog dimlog varlog varmax graphNo

global Tb Mb Pb Vb %WHY DOES RUN NEED TO SEE THESE?? MAKES NO SENSE!!
???

%global missions requirements
global q_max beta ta Hr qdot I_max accel_rot
%global actual values
global T, M, P, V, %actuals a

global Irot, Vrot, Mrot, Trot %values a

global length_rot, rot_o, rot_i, density_rot % rotor dimensions a length ring = length disc

% global required values
global Tr, Pr, Vr, Mr % requireds
% global input options
global size var optimize method satcase
% iterations
global deltat i hr ho

% for power plots - interesting curve investigation
global P_GM_tot, P_ROT
global P_unit, P_unitRot, P_unitGM % for unit power plots - check equations

global Tcmsg, Tcmga1 % checking power

global Ncmsg

% required???
omega_GM = 5000 * pi / 30; % rad/sec

% elseif optChoice = 2
% x0 = [startguess1, startguess2]; % Make a starting guess at the solution x(1) = outer radius, x(2) = inner radius
% lb = [lowerbound1, lowerbound2]; % lower bound for length_rot
% ub = [upperbound1, upperbound2];
% end

%%%%%%%%%%%%%%%%%%%%% USER REQUIRED INPUTS

satcase = input('Choose case:
1... Small Satellite 10kgm^2,4deg/s,48deg,ta=3s
2... SSTL MicroSat-70 70kgm^2,4deg/s,48deg,ta=3s
');
method = input('Choose either method of optimization:
1... Fixed Baseline Method
2... Moving Baseline Method
');
optimize = input('Choose to optimize:
1... Torque (equate mass)
2... Mass (equate Torque)
');
rotsize = input('Choose rotor dimension to size:
1... Rotor Length
');
%graphtype = input('Choose graph type:
1... Colour
2... Black and White
');

% BOUNDS FOR INPUTS
% rotsize : ROTOR LENGTH BOUNDS - SIZING X
upperbound1=0.03;%for rotor length 0.04
lowerbound1=0.0005;%for rotor length 0.005
startguess1=1;

%1 GIMBAL RATE LIMITS
vamiin=0.5; %rad/sec- deltad min
varmax=10; %rad/sec- deltad max

%Identification variables - initialised
count=1; %variable for array cell identification - logging calculated values at each deltad value
k=1; %another logging variable for array ID for less points than deltad values
%var=1;
for var=linspace(varmin,varmax,50)
%for i=l;6 %CONFIGURATION SET - CYCLES THROUGH ALL BASELINE AND TEST CONFIGS
%for i=l;6
for i=l;6
    x0 = [startguess1]; % Make a starting guess at the solution
    lb = [lowerbound1]; % lower bound for length_rot
    ub = [upperbound1]; % upper bound
    options = optimset(’LargeScale’,’off’,’Display’,’iter’);
    [x,fval] = fmincon(@OPTSIZE_2712_calc,x0,[],[],[],[],lb,ub,@constraintsOPTSIZEa,options); % changed from VL to VL alice new idea or FB to MF in 1 program
    [c, ceq] = constraintsOPTSIZEa(x);
%END OPTIMIZATION
    dim=x(1)*2; %LOGGING TOTAL LENGTH OF ROTOR (ring + disc which are equal)

%LOGGING VALUES
if method==1
    if i==1 dimlog(count,i)=length_rot;
    else
        dimlog(count,i)=dim;
    end
elseif method==2
    dimlog(count,i)=dim;
end
varlog(count,:) = var;
% Tlog(count,i) = Ta;% LOG TORQUE
% Telog(count,i) = Ta-Tb;
Plog(count,i) = Pa;% LOG POWER
%protlog(count,i) = P ROT;% LOG POWER ROTORS
%pglog(count,i) = P_GM_tot;% LOG POWER GIMBAL

%P_unit_Log(count,i) = P unit;
%P unitRot_Log(count,i) = P unitRot;
%P unitGM_Log(count,i) = P unitGM;

%Vlog(count,i) = Va;% LOG VOLUME
%Mlog(count,i) = Ma;% LOG MASS
%Mlog(count,i) = Ma-Mb;
%vtcLog(count,i) = Vtc;% Log for plotting mass for MB
%NcmgLog(count,i) = Ncgm;% Log for plotting mass for MB
%Irotlog(count,i) = Irot;% LOG SINGLE ROTOR INERTIA

% checking calc of torque required and available
% from one rotor
% TcmlgaLog(count,i) = Tcmlga;
% Tcmlgalog(count,i) = Tcmlgal;

% if graphtype == 2
% BlackWhite
% else
% end

% if (count == 1 | count == 100 | count == 200 | count == 300 | count == 400 | count == 500) %
% varlog2(k,:) = var;
% dimlog2(k,i) = dim;
% Nlog2(k,i) = Na;
% Plog2(k,i) = Pa;
% Vlog2(k,i) = Va;
% k = k + 1;
% else k = k;
% end
end
count = count + 1;
end
% datelog=datestr(now,'ddmmyy');
% timelog=datestr(now,'HHMM');

%This section is included to title the graphs
if satcase==1
  satDISP='I_{max}=10kgm'^2,\omega=4deg/s,\theta_f=48deg,t_a=3s,\beta=54.7deg';
elseif satcase==2
  satDISP='MicroSat-70: I_{max}=70kgm'^2,\omega=4deg/s,\theta_f=48deg,t_a=3s,\beta=54.7deg';
end
if method==1
  methodDISP='Fixed Baseline';
else
  methodDISP='Moving Baseline';
end
if optimize==1
  optDISP='Optimize Torque Equate Mass';
else
  optDISP='Optimize Mass, Equate Torque';
end
if rotsize==1
  sizeDISP='Size Rotor Length';
else
end

% PLOTTING FUNCTION CALLED HERE
% figure
% axes('FontSize',18)
% surf(vtcLog(:,1:6),NcmgLog(:,1:6),Tlog(:,1:6))

% figure
% axes('FontSize',18)

% plot3(varlog,vtcLog,Mlog)
% xlabel('Gimbal Rate, (rad/s)', 'FontSize', 16);
% ylabel('Torque (Nm)', 'FontSize', 16);
% ylabel('Number of Rotors in Cluster \{\text{total}\}', 'FontSize', 16);
% ylabel('Torque Capability |v_{\text{te}}|', 'FontSize', 16);
% ylabel('Rotor Length (m)', 'FontSize', 16);
% ylabel('Total Cluster Mass, kg', 'FontSize', 16);
% legend('4SGCMG Pyramid','3SGCMG Axis Aligned','3TCMG Axis Aligned','3TCMG Pyramid','4TCMG Pyramid','6SGCMG Pyramid');
% grid on
function [c, ceq] = constraintsTCMGa(x)
% Nonlinear inequality constraints
global Mr Ma Mb Tb Ta rot o optChoice length_rot

% First inequality <=0,
% Nonlinear equality constraints

% Note: this constraint seems to cause problems
% the first time that the program is run!
equivalent performance - this is what I was doing before
if strategies==1
  if optChoice==1
    c=[(x(1)*2)-(rot_o/4)]; %x(1)=rotor length
c=[];
  end
if optimize==2 %optimal mass equate torque
  if method==1 ceq=[Ta-Tb]; %fixed baseline so equate torque to baseline Tb
    else ceq=[Ta-Tr]; %moving baseline so equate torque to value Tr
  end
else
  end
end
if optimize==1 %optimal torque equate mass
  if method==1 ceq=[Ma-Mb]; %fixed baseline so equate mass to baseline
    else ceq=[Ma-Mr]; %moving baseline so equate mass to value Mr
  end
else
  end
  elseif optChoice==2 c=[(length_rot*2)-(x(1)/4)]; %x(1)=outer radius
  ceq=[Ta-Tr];
  %ceq=[Ta-Tb 0.9*x(1)-x(2)];%x(1)=outer radius x(2)=inner radius
end
elseif strategies==2
  if optChoice==1 c=[(x(1)*2)-(rot_o/4)]; %x(1)=rotor length
  ceq=[Ma-Mb];
  elseif optChoice==2 c=[(length_rot*2)-(x(1)/4)]; %x(1)=outer radius
  ceq=[Ma-Mb];
  %ceq=[Ma-Mb 0.9*x(1)-x(2)]; %x(1)=outer radius x(2)=inner radius
  end
elseif strategies==3
  if optChoice==1 c=[(x(1)*2)-(rot_o/4)]; %x(1)=rotor length
  ceq=[Ta-Tr];
  elseif optChoice==2 c=[(length_rot*2)-(x(1)/4)]; %x(1)=outer radius
  ceq=[Ta-Tr];
  %ceq=[Ta-Tb 0.9*x(1)-x(2)];%x(1)=outer radius x(2)=inner radius
function alicesGraphs(log,y)
% INPUT log - variable to plot
%global methodDISP optDISP sizeDISP satDISP
%global varlog varmax graphNo

minLogColumns=min(log);
minALLvalues=min(minLogColumns);

%logs date and time of graph generated
datelog=datestr(now,'ddmm');
timelog=datestr(now,'HHMMSS');
%combines date and time into a unique number for identification
comb=[datelog timelog];

figure
axes('FontSize',18)
for i=1:6
if i==1 plot(varlog,log(:,i),'k','LineWidth',2);
elseif i==2 plot(varlog,log(:,i),'r','LineWidth',2);
elseif i==3 plot(varlog,log(:,i),'b','LineWidth',2);
elseif i==4 plot(varlog,log(:,i),'g','LineWidth',2);
elseif i==5 plot(varlog,log(:,i),'c','LineWidth',2);
elseif i==6 plot(varlog,log(:,i),'m','LineWidth',2);
end
hold on
ylabel(y,'FontSize',16);
xlabel('Gimbal Rate rad/s','FontSize',16);
legend('4SGCMG Pyramid','3SGCMG Axis Aligned','3TCMG Axis Aligned','3TCMG Pyramid','4TCMG Pyramid','6SGCMG Pyramid');
grid on

end
end

M-file code for alicesGraphs.m

%#########################################################################
%## NAME OF FILE: aIicesGraphs.m
%## AUTHOR: Alice Daihyshire
%## DATE: 27 DEC 2007
%#########################################################################

%IMPORTANT NOTES FOR USE
%INPUTS: LOG, AND Y WHICH ARE GENERATED BY: OPTSIZE_2712_Run.m
%#########################################################################
title(title, {methodDISP; optDISP; sizeDISP; satDISP}) \% indicates the cases shown
text(varmax, minALL.values, comb) \% identifies the date and time when the graph
was generated
end