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VARIATIONAL METHODS AND PERIODIC SOLUTIONS OF N-BODY AND N-CENTRE PROBLEMS

A Thesis by John Southall
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy.

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©John Southall, 2006.
Dedicated to my wonderful wife Jia and in loving memory of Mum and Nan.
Abstract

In this thesis we study periodic solutions of several $N$-body and $N$-centre systems with different potentials from a variational viewpoint. The underlying focus is on understanding the structure of various action functionals, and the relationship between this and the system's periodic orbits and their properties.

In particular we:

1. investigate the integrable central force problems with potentials $V_\alpha(x) = -\frac{1}{|x|^{\alpha}}$ for $1 \leq \alpha \leq 2$. We show that for $1 < \alpha < 2$ there are only finitely many homotopy classes that do not contain a prime-period, but that this number diverges as $\alpha \to 1^+$ or $\alpha \to 2^-$. Given any non-null homotopy class of loops and any period $T > 0$ we list the finitely many distinct critical manifolds of collisionless orbits in that class in order of their action, $\mathcal{A}_T^R$, and label them with their Morse indices with respect to the action functional.

2. investigate the 2-body/1-centre problem with Lennard-Jones potential. We find the region of energy-momentum space that supports the existence of periodic or quasiperiodic motion. We also show there exists $m \in \mathbb{N}$ such that for all $q \geq m$ there are periodic orbits in $\Omega^p = \bigcup_{T>0} \Omega_T^p$ ($p \in \mathbb{Z} - \{0\}$) with $q$ 'radial oscillations' in one period.

3. Obtain results on which homotopy and homology classes of loops contain periodic solutions of the symmetric planar Newtonian 2-centre problem, see theorem 4.4.1. In particular we find that the integrability of the system places strong constraints on which homology classes of loops contain periodic solutions and obtain some interesting results on prime-period solutions. We also order by action 'P1 orbits' in all homotopy classes of loops that contain them and label them by their Morse indices with respect to the action functional.

4. investigate the $N$-body problem with identical particles interacting through a potential of Lennard-Jones (LJ) type. We consider any subset of loop space that satisfies a few basic conditions, one of which corresponds to the notion of ‘tiedness’ introduced by Gordon in [41]. We show that this system admits periodic solutions in every homotopy class of this subset of loop space. More precisely we show that every homotopy class contains at least two periodic solutions for sufficiently large periods. One of these solutions is a local minimum and the other is a mountain-pass critical point of the action functional. We also prove that given a homotopy class of one of these subsets there do not exist any periodic solutions in it for sufficiently small periods. The results have wide applicability. For example, one can consider the space of choreographies and prove existence results for choreographical solutions.

Our existence proof relies upon an assumption that global minimizers of standard strong force potentials on suitable spaces are nondegenerate up to some symmetries. We also find periodic solutions in some classes of loops that do not satisfy any tiedness condition. In particular we use a result in [27] to construct a periodic solution of the restricted spatial $(2N + 2)$-body problem.
• present a note on McCord, Montaldi, Roberts and Sbano's paper on relative periodic orbits in symmetric Lagrangian systems, see [54]. For the $N$-centre problem with a strong force potential that is bounded above we find those homotopy classes of relative loops on which the action functional is coercive. We describe the homotopy types of the homotopy classes of relative loops. Under the assumption that the action functional is an $S^1$-invariant Morse function we describe a set of homotopy classes that contain infinitely many periodic orbits. The work can be viewed as an expansion on an example presented at the end of [54] which had $N = 2$; in particular we correct and generalize some assertions made regarding coercivity of the action functional and the centralizers of the $g$-twisted action.

Some of the results are subject to numerically motivated assumptions detailed in the thesis.
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Chapter 1

Introduction/Overview

*Imagination is more important than knowledge.*
- Albert Einstein

*Think globally, act locally.*
- Activist bumper sticker

1.1 Basic Concepts and Principles

1.1.1 N-Body and N-Centre Problems

The N-body problem is the problem of finding, given a potential energy and the initial positions, masses and velocities of N bodies, their subsequent motions as determined by classical mechanics. The N-centre problem is an approximation of the N-body problem in which a test particle moves under the influence of a potential field generated by N fixed points. We will look at N-body and N-centre systems with a variety of different potentials.

1.1.1.1 Mathematical Perspective

The N-body problem has fascinated mathematicians past and present. Given N bodies and a potential energy $V$ we can write down the equations of motion of the system. In this thesis we shall consider potentials that can be written in the form

\[ V(x) = V(r_{1,1}, r_{1,2}, ..., r_{N-1,N}) \]

where $x = (x_1, ..., x_N)$ are the positions of the N bodies and $r_{i,j} = \|x_i - x_j\|$. The equations of motion are

\[ m_i \ddot{x}_i = -\sum_{j \neq i} \frac{\partial V(x)}{\partial r_{i,j}} \left( \frac{x_i - x_j}{r_{i,j}} \right), \quad i = 1, ..., N \]

(1.1)

where $m = (m_1, ..., m_N)$ are the masses of the N bodies. A solution to the N-body problem is a solution to (1.1). This notion should be contrasted with the notion of 'solving the N-body problem' which means finding an explicit expression for the general solution. Central force problems (which are equivalent to 2-body problems where the forces act in the direction of the vector pointing from one body to the other) and some other special cases of the N-centre problem can be solved.
Poincaré showed that the gravitational $N$-body problem effectively cannot be solved for $N > 2$. The $N$-body problem with other potentials suffers similar problems when $N$ is not small. We will concentrate on periodic solutions. A periodic orbit is a special type of solution for a dynamical system, namely one which repeats itself in time. If a solution $x(.)$ of (1.1) satisfies the boundary conditions

$$x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T),$$

then we call it a $T$-periodic solution. A $T$-periodic solution is said to be of prime-period if $T$ is the least such strictly positive number. We shall sometimes also consider relative periodic orbits (RPOs) for which we instead have the boundary condition:

$$x(T) = g \cdot x(0), \quad \dot{x}(T) = g \cdot \dot{x}(0)$$

for some fixed diffeomorphism $g$ of configuration space. They are periodic modulo the symmetry $g$ of configuration space. If $g$ has finite order then one can construct periodic solutions by gluing together relative periodic solutions, see [54]. The set of (relative) periodic orbits carries essential information concerning characteristic features of the dynamical system. (Relative) periodic orbits play a fundamental role as organizing centres for more complex dynamics (see for example [55]), including invariant tori and chaotic phenomena.

1.1.1.2 Physical Perspective

The gravitational $N$-body problem, which has potential

$$V = - \sum_{i<j} \frac{m_i m_j}{r_{ij}},$$

has attracted the most attention of researchers due to its relevance to celestial mechanics. However any force between the pairs of bodies may be considered. According to modern physics there are just four fundamental physical forces. These are gravity, electromagnetic, strong nuclear and weak nuclear. The $N$-body problem helps links these forces to the complicated behaviour that we observe in the universe. The $N$-body model is also useful when the forces modelled are not ‘fundamental’. For example the motion of $N$ atoms or molecules that are not chemically bonded to each other could be modelled as an $M$-body problem (where $M > N$) with electromagnetic interactions between all the electrons and nuclei. However a more tractable problem might be to model the atoms/molecules as point particles and model the system as an $N$-body problem with ‘molecular’ interactions. Such systems are studied in chapter 5. Although physical applications of periodic orbits are not investigated in this thesis we nevertheless comment that periodic orbits of $N$-body systems are of significance to celestial mechanics, space travel, atomic and molecular physics. For systems on a microscopic scale the periodic orbits are only those of a classical approximation but they are an important ingredient in semi-classical approximations to quantum mechanics, see [12]. For example in chapter 4 we study the Newtonian 2-centre problem which is a useful model (called the Born-Oppenheimer approximation) for the hydrogen molecular ion $H_2^+$. 

1.1.2 Configuration Spaces

The configuration space of the planar $N$-body problem is

$$\mathcal{M}^1 = \{ x \in (\mathbb{R}^2)^N : \sum_{i=1}^N m_i x_i = 0 \} \setminus K_e^1$$
where
\[ K^c = \{ x \in (\mathbb{R}^2)^N : \sum_{i=1}^{N} m_i x_i = 0, \quad \text{and} \quad x_i = x_j \quad \text{for some} \quad i \neq j \} \]
and \( \{m_i\} \) are the masses of the bodies. Note that we have fixed the centre of mass at the origin. We can do this without loss of generality because, by conservation of linear momentum, the centre of mass of the system of \( N \) bodies moves with constant velocity in a straight line, so we can make a Galilean coordinate transformation to a coordinate system where the centre of mass is stationary at the origin. The set \( K^c \) is the set of collision configurations. The configuration space of the planar \( N \)-centre problem is:
\[ \mathcal{M}^2 = \mathbb{R}^2 \setminus K^c_{c} \]
where \( K^2_{c} = \{ R_i \}_{i=1}^{N} \) is the set of centres.

Both \( \mathcal{M}^1 \) and \( \mathcal{M}^2 \) are \( K(\pi, 1) \)'s. This means that, except for the fundamental group, all their homotopy groups are trivial. The set of 'free homotopy classes' of loops in \( \mathcal{M}^i \) is the set of connected components of the space of loops in \( \mathcal{M}^i \). We are interested in these because periodic solutions are examples of loops in \( \mathcal{M}^i \). The set of free homotopy classes of loops in \( \mathcal{M}^i \) is in bijection with the set of conjugacy classes of \( \mathcal{M}^i \)'s fundamental group, \( \pi_1(\mathcal{M}^i) \). We have that:

- \( \pi_1(\mathcal{M}^1) \) is the \textbf{coloured braid group on} \( N \) \textbf{strands}. The coloured braid group on \( N \) strands is the subgroup of the usual braid group which corresponds to each braid returning to its starting point. The coloured braid group is generated by the \( ij \)-binaries: the loops where masses \( i \) and \( j \) make one turn around each other whilst the other masses stay still, far away.

- \( \pi_1(\mathcal{M}^2) \) is the \textbf{free group on} \( N \) \textbf{generators} i.e. the group is generated by \( N \) elements which have no nontrivial relations between each other. This group is generated by small loops that wind once around each of the centres. Unlike for the coloured braid group there are no nontrivial relations between the generators.

One of the advantages of the \( N \)-centre approximation to the \( N \)-body problem is that, in the planar case, the fundamental group of the configuration space is easier to manage than that of the \( N \)-body problem because of the lack of relations between generators.

### 1.1.3 Integrable Systems

In Hamiltonian mechanics, an integrable system refers to a Hamiltonian system that has \( n \) degrees of freedom, \( n \) constants of motion, and whose constants of motion are in involution: that is, the Poisson bracket between each pair of constants of motion vanishes. When a system is completely integrable, there is a special set of variables on the phase space of the system, known as action-angle coordinates, see [4]. The actions are the constants of motion, and all motion occurs on the surface of a torus, known as an invariant torus. The coordinates on the torus are the angle variables. Systems which are not completely integrable are in general chaotic. Although important in their own right, integrable systems also form islands of solvable problems in a sea of unsolvable problems. They can be used to investigate properties of 'nearby' non-integrable systems by perturbation theory (see again [4]). We comment that just because a system is integrable does not mean all the properties of that system follow trivially (see, for example, [28]). Indeed in chapters 2, 3 and 4 we investigate nontrivial aspects of three different integrable systems.
1.1.4 The Principle of Least Action

This thesis will take a variational approach to the study of \( N \)-body and \( N \)-centre systems. Mathematically the \( N \)-body problem is about understanding the solutions of a set of differential equations. The system of differential equations is a conservative Hamiltonian system with Hamiltonian function

\[
H(q, p) = \frac{1}{2} \sum_{i=1}^{N} \frac{\|p_i\|^2}{m_i} + V(q).
\]

Here \((q, p)\) are local coordinates on the cotangent bundle \( T^*M \) where \( M \) denotes configuration space. Many interesting classes of solutions can be found via techniques such as Kolmogorov-Arnold-Moser (KAM) theory when the \( N \)-body problem is studied from the Hamiltonian perspective, see for example [21]. However the \( N \)-body problem can also be approached using its characterization as a Lagrangian dynamical system. In this formulation the Lagrangian function on the tangent bundle \( TM \) is

\[
L(x, \dot{x}) = \frac{1}{2} \sum_{i=1}^{N} m_i \|\dot{x}_i\|^2 - V.
\]

The action is defined by

\[
\mathcal{A}_T[x] = \int_0^T L(x, \dot{x}) dt.
\]

The 'principle of least action' says that if \( \theta \) is an 'appropriate class of curves' \( x : [0, T] \rightarrow M \) then \( d\mathcal{A}_T[x] = 0 \) on \( \theta \) if and only if \( x \) is a solution of the equations of motion. From the point of view of the calculus of variations, a principle of stationary (rather than 'least') action is a more accurate formulation. Indeed many of the solutions that we find will correspond to critical points of the action functional that are saddles or local minima rather than global minima.

1.1.5 Sobolev Spaces

The 'appropriate classes of curves' of subsection 1.1.4 are Sobolev spaces. Suppose \( \Omega \) is an open subset of \( \mathbb{R}^d \), \( m \in \mathbb{N} \cup \{0\} \) and \( p \in [1, \infty) \). The spaces \( W^{0,p}(\Omega) \) are defined to be \( L^p(\Omega) \) i.e. the Lebesgue \( p \)-integrable functions. The space \( W^{m,p}(\Omega) \) is defined to be the space of functions \( u \in L^p(\Omega) \) which have weak derivatives \( g = (g_1, \ldots, g_d) \) such that \( g_i \in W^{m-1,p}(\Omega) \). They are Banach spaces. Of particular interest are the spaces \( H^m(\Omega) = W^{m,2}(\Omega) \) which turn out to be Hilbert spaces. The space \( H^m \) can be thought of the space of functions all of whose partial derivatives up to order \( m \) are square integrable. We will be interested in functions that have values in \( M \cup K_c = \mathbb{R}^{2N} \) or \( \mathbb{R}^2 \) and have \( m = d = 1 \) (we take \( d = 1 \) since, for the \( N \)-body problem, \( x \) is a function only of time). Furthermore we impose periodic boundary conditions by replacing \( \Omega = \mathbb{R} \) with \( \mathbb{R}/\mathbb{T} \). We denote this space of functions by \( H^1(\mathbb{R}/\mathbb{T}, \mathcal{M} \cup K_c) \); it has a norm \( \|\cdot\|_1 \) coming from an inner product, see [9]. The norm \( \|\cdot\|_1 \) is given by:

\[
\|\cdot\|_1^2 = \int_0^T \|\dot{x}\|^2 dt + \int_0^T \|x\|^2 dt.
\]

We call \( \Lambda^1 = H^1(\mathbb{R}/\mathbb{T}, \mathcal{M}) \) loop space. Loop space is an open submanifold of the Hilbert space \( H^1(\mathbb{R}/\mathbb{T}, \mathcal{M} \cup K_c) \). An important point is that by the well-known Sobolev embedding \( H^1 \subset C^0 \), the functions belonging to \( H^1 \) are continuous. Further analytic details can be found in [41] and [43].
The connected components of $\Lambda^1$ are called (free) homotopy classes of loops. These correspond to the topological homotopy classes of configuration space given in subsection 1.1.2. Analogously, the spaces of relative periodic loops we consider are:

$$\Lambda^0(\mathcal{M}) = \{x \in H^1([0,T],\mathcal{M}) : x(T) = g \cdot x(0)\}.$$

### 1.1.6 Strong Force Potentials

**Definition 1.1.1.** Let

$$\sigma(x) = \begin{cases} \min_{i,j} \|x_i - x_j\| & \text{for the } N\text{-body case,} \\ \min_i \|x - R\| & \text{for the } N\text{-centre case.} \end{cases}$$

**Definition 1.1.2.** A potential $V : \mathcal{M} \to \mathbb{R}$ is said to be a strong force (SF) potential if for some $s \in \{\pm 1\}$ we have that there exist constants $c, R > 0$ such that:

$$V(x) < -\frac{sc}{\sigma(x)^2} \text{ for all } x \in \mathcal{M} \text{ such that } \sigma(x) < R.$$ 

It is readily shown (see lemma 2.3.7) that for strong force potentials $sA_T$ diverges to $+\infty$ on sequences of loops approaching a loop with collisions. The usual case considered is $s = 1$; this will be called 'standard' strong force when we need to distinguish it from the case $s = -1$. The typical example of a strong force potential with $s = 1$ is

$$V(x) = -\sum_{i < j} \frac{1}{\|x_i - x_j\|^\alpha}, \quad \alpha \geq 2. \quad (1.3)$$

with $\alpha \geq 2$. In chapter 5 we consider a strong force potential with $s = -1$; more precisely we consider the so called Lennard-Jones potential:

$$V_{LJ}(x) = \sum_{1 \leq i < j \leq N} \left[\frac{1}{\|x_i - x_j\|^\beta} - \frac{1}{\|x_i - x_j\|^\alpha}\right], \quad \beta > \alpha \geq 2. \quad (1.4)$$

This potential is repulsive at short distances, attractive at large distances and is used to model interactions between $N$ identical atoms or neutral molecules. In this thesis we will often refer to it as a 'molecular potential' and the corresponding $N$-body problem as the 'molecular $N$-body problem.' We call an action functional (standard) strong force if and only if it corresponds to a potential that is (standard) strong force.

### 1.1.7 The Palais-Smale Condition

Suppose $E$ is a real Banach space (i.e. a complete normed vector space). Let $C^1(E,\mathbb{R})$ denote the set of functionals that are Fréchet differentiable on $E$ with continuous derivatives. For $I \in C^1(E,\mathbb{R})$ we say that $I$ satisfies the Palais-Smale condition (PS) if any sequence $(x^{(n)}) \subset E$ for which $I(x^{(n)})$ is bounded and $dI(x^{(n)}) \to 0$ as $n \to \infty$ possesses a convergent subsequence. PS is a necessary condition for some theorems of the calculus of variations. More precisely, PS is a necessary condition for the 'deformation lemma' which is required for both minimization (used to find minimizers) and for mountain-pass theorems (which are used to find saddle critical points), see [64]. Defining
The deformation lemma allows us to deform the set $R_{c+\epsilon}$ to the set $R_{c-\epsilon}$ ($\epsilon > 0$ small) provided that $c$ is not a critical value of $I$. The condition is necessary because the calculus of variations studies function spaces (here loop space) that are infinite dimensional - therefore some extra notion of compactness beyond simple boundedness is required. Indeed note that if $I$ satisfies PS then each set $\{u \in E : I(u) = c, \ dI(u) = 0\}$, i.e. set of critical points having critical value $c$, is compact for any $c \in \mathbb{R}$.

1.1.8 The Direct Method

Typically the existence of critical points of the action functional for non-integrable systems is demonstrated by the so-called direct method. This involves finding a critical point by minimization. Suppose $V(x) \leq 0$ for all $x \in \mathcal{M}$ with $V(x) \rightarrow 0^-$ for $\sigma(x)$ large and increasing. Minimizing $\mathcal{A}_T$ over the full functional space is not rewarding because the minimum value (zero) is attained 'at infinity' with all bodies infinitely separated and moving infinitely slowly on small closed curves (see for example [17]); the Palais-Smale condition fails. We define:

$$\Delta : H^1(\mathbb{R}/\mathbb{T}, \mathcal{M}) \rightarrow \mathbb{R}; \quad x \mapsto \begin{cases} \text{sup}_t \text{max}_i \|x(t) - x_j(t)\| & \text{if } k = 1 \\ \text{sup}_t \text{max}_i \|x(t) - R_i\| & \text{if } k = 2. \end{cases}$$

Suppose we have a subset of loops $\theta \subset H^1(\mathbb{R}/\mathbb{T}, \mathcal{M})$. We say that the action functional is coercive on $\theta$ if for all sequences of loops $(x^{(n)}) \subset \theta$ such that $\Delta(x^{(n)}) \rightarrow \infty$ as $n \rightarrow \infty$ we have $\mathcal{A}_T(x^{(n)}) \rightarrow \infty$ as $n \rightarrow \infty$. The advantage of coercivity is that minimizers cannot be attained 'at infinity'.

A space of loops $\theta$ is 'tied' in the sense of Gordon (see [41], [59]) if for all sequences of loops $(x^{(n)}) \subset \theta$ such that $\Delta(x^{(n)}) \rightarrow \infty$ as $n \rightarrow \infty$ we have that $\frac{1}{2} \int_0^T \sum_{i=1}^N \|x_i^{(n)}(t)\|^2 dt \rightarrow \infty$. For a system with a potential that is bounded above the action functional is coercive on all tied classes. For the $N$-centre problem all non-null homotopy classes of loops (i.e. all the classes except the one containing the constant loops) are tied.

To be able to apply the action principle we require that critical points of $\mathcal{A}_T|\theta$ are critical points of $\mathcal{A}_T$. If $\theta$ is an open subset of $H^1(\mathbb{R}/\mathbb{T}, \mathcal{M})$, for example a homotopy class of loops then this holds immediately. Furthermore, homotopy classes of loops are often tied. Palais' principle of symmetric criticality (see [62]) is another useful tool in the $N$-body problem for forcing coercivity (see [36]). Palais' principle tells us that critical points of the action functional restricted to a symmetry class $\theta$ are also critical points in the full loop space. A famous example of a symmetry class on which the Newtonian action functional is coercive is the space of choreographies. Choreographies are discussed in subsection 5.5.1, they are motions in which the bodies chase each other along a fixed curve in configuration space with a constant phase-shift. Note that an $\mathcal{A}_T$-minimizer of a symmetry class $\theta$ is not necessarily a minimum in the full loop space; it may be a saddle critical point.

The theorem we use when applying the direct method is (see [64]):

**Theorem 1.1.1.** Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy Palais-Smale. If $I$ is bounded below then $a = \inf_E I$ is a critical value of $I$.

Below we discuss what happens when one tries to apply theorem 1.1.1 to systems with different types of potentials:
Standard Strong Force Potentials

Suppose that $V$ is standard strong force and that $\theta$ is a tied class of loops such that $V$ is bounded above on $\theta$. By coercivity the minimizer is not ‘at infinity’. Furthermore Poincaré noticed that the limit of $A_T$-minimizing sequences of loops for systems with (standard) strong force potentials are necessarily collisionless since $A_T$ diverges to $+\infty$ as we approach any element of $K_c$, see [63]. Indeed the Palais-Smale condition holds for tied classes of standard strong force $N$-body/centre systems, see [1]. Thus by the direct method strong force $N$-body and $N$-centre problems contain periodic orbits in all classes on which the action functional is coercive, see [41]. This fact is used in chapter 6.

Gravitational Potential

Suppose $V$ is the Newtonian potential and $\theta$ is a tied class. Although $A_T$ is bounded below we can (and often do) have that the limit of $A_T$-minimizing sequences on $\theta$ are in $K_c$ because the potential is not strong force; the action may be finite on collisions. This prevents Palais-Smale from holding and is the main obstacle to the direct method in the gravitational problem. In 2000 Chenciner and Montgomery found a new and topologically interesting solution to the equal mass Newtonian three-body problem in which the orbit lies in a fixed figure-eight in the plane, see [19]. The basic idea of their proof was to:

(i) Take $\{(x_1, \ldots, x_N) \in (\mathbb{R}^2)^N\}$ as our configuration space and $H^1(\mathbb{R}/T\mathbb{Z}, (\mathbb{R}^2)^N)$ as our loop space, so collision loops are included.

(ii) Restrict to a symmetry class of $H^1(\mathbb{R}/T\mathbb{Z}, (\mathbb{R}^2)^N)$ that induces coercivity of the Newtonian action functional; this class satisfies the Palais-Smale condition (see [1], [19]).

(iii) Action minimize on the symmetry class. To prove the minimizer is without collisions they find a lower bound on the action for the set of all loops with collisions and then construct a ‘test loop’ with lower action than this bound.

This method has recently been replaced with a more systematic approach of Ferrario and Terracini in which a ‘factory’ of orbits is created, see [36] and [69]. The idea combines symmetry constraints satisfying a ‘rotating circle property’ with Marchal’s theorem. Marchal’s theorem says that any two configurations of $N$ bodies are joined by a path that is the minimizer of the Newtonian action functional that is collision-free for all times different from the initial and final ones. This is a useful mechanism for avoiding collisions in cases where you can construct the periodic orbit by gluing together relative periodic orbits.

Lennard-Jones Potential

The direct method can be used in problems where the action functional is bounded below. However it fails for say the Lennard-Jones action functional because this functional is not bounded below on loop space. More precisely, the action diverges to $-\infty$ as we approach an element of $K_c$. Therefore a different approach is required; this is what chapter 5 is concerned with. As part of our argument we construct a proof that PS holds for the Lennard-Jones action functional defined on tied classes of the $N$-body problem, see proposition 5.4.3.
1.1.9 Minimax Methods

Minimax methods, see [64], are methods that characterize a critical value $c$ of a functional $I$ as a minimax over a suitable class of sets $S$:

$$c = \inf_{A \in S} \sup_{x \in A} I(x).$$

The mountain-pass theorem, of which there are several different forms, is a powerful minimax result that enables us to infer the existence of saddle critical points, i.e. critical points that are neither local minima or local maxima. We shall use a version of it in chapter 5 to prove the existence of a mountain-pass solution of the molecular $N$-body problem. Later in that chapter we will also use a result by Coti-Zelati that he proved by a minimax method for the (generalized) molecular 1-centre problem in [27].

1.1.10 Variational Calculus in the Large

Variational calculus in the large is the branch of mathematics involving the use of topological concepts and methods in qualitative studies of variational problems. In particular it can be used for estimating the number of critical points and finding qualitative properties of them. Unlike for, say, the direct method all critical points are considered to be of interest, not just minima. Use is made of the global, topological properties of the functional space of the variational problem. The general method of study in variational calculus in the large may be described as follows. For a given functional $I$, considered as a function on an infinite-dimensional functional space $\mathcal{F}$, one studies the changes of topological properties of the sets

$$\theta^c = \{x \in \theta : I(x) < c\}$$

as the value of $c$ is changed. The aim is to show that the topology of these sets changes only when $c$ passes through a critical value and to describe the connection between the changes accompanying such transitions and the properties of the respective critical points. In [54] topological results were announced that allow one to compute the topology of a homotopy class of relative loops of a symmetric Lagrangian system. More precisely in the case that configuration space is a $K(n,1)$ they reduced the problem of finding the topology of a homotopy class of relative loops to the computation of a centralizer of a particular action, called a ‘$g$-twisted’ action. The aim was to use this known topology of homotopy classes of loops with variational calculus in the large to infer information about the critical points of the action functional, and hence the periodic solutions, in those classes.

1.1.10.1 Category Theory

Lusternik-Schnirelmann (LS) category theory, see [13], is an example of variational calculus in the large. The LS category $Cat_X(\theta)$ of a topological space $\theta \subset X$ with respect to $X$ is the topological invariant defined as the smallest cardinality of a covering of $\theta$ by contractible open subsets of $X$. For example, if $X$ is an $n$-sphere then $Cat_X(X) = 2$. We quote the following theorem from [54]:

**Proposition 1.1.1.** Suppose that $A_T : \theta \to \mathbb{R}$ is a smooth functional bounded below and that PS holds on $\theta^c$. Then the number of critical points of $A_T$ on $\theta$ is greater than or equal to $Cat_\theta(\theta^c)$. 
Note that proposition 1.1.1:

- does not apply to non-tied classes in general because PS fails due to a lack of coercivity. We say 'in general' because one could construct potentials such that \( V \to -\infty \) as \( \Delta \to \infty \) which would give coercivity without necessarily restricting to a tied class. Such potentials shall not be considered in this thesis - all the potentials we consider tend to zero at large distances.

- does not apply to the Newtonian N-body/centre problem; PS fails due to the existence of finite action collisions on the boundaries of classes.

- does not apply to the molecular N-body/centre problem; the LJ action functional is not bounded below.

- does apply to tied classes and an action functional corresponding to a potential that is strong force and bounded above.

1.1.10.2 Morse Theory

Morse theory is another example of variational calculus in the large. In this thesis we shall consider functionals \( I : \theta \to \mathbb{R} \) that are \('G-invariant', that is for some Lie group \( G \) acting smoothly on \( \theta \) we have that

\[
I(g \cdot x) = I(x) \quad \text{for all } g \in G.
\]

For example, if a system has a time-independent potential then there is an \( S^1 \) symmetry of the action functional given by phase shifts of the loops: given any \( s \in \mathbb{R}/T\mathbb{Z} \) and \( x_0 \in H^1(\mathbb{R}/T\mathbb{Z}, \mathcal{M}) \) we have that \( \mathcal{A}_T(x_0) = \mathcal{A}_T(x_s) \) where \( x_s(t) = x(t + s) \) for all \( t \in \mathbb{R}/T\mathbb{Z} \). A \( G \)-invariant Morse functional \( I \) is a \( G \)-invariant functional such that at all critical points \( x \) of \( I \) we have:

\[
\ker d^2I(x) = T_x(G \cdot x),
\]

where \( G \cdot x = \{ g \cdot x : g \in G \} \). If \( x \) is a critical point of the \( G \)-invariant action functional \( \mathcal{A}_T \) then \( G \cdot x \) is called a 'critical orbit' of \( x \). The Morse index of a critical point with respect to the functional is defined to be the maximal dimension of a subspace on which the Hessian of that functional is negative definite. We can define the Morse index of \( G \cdot x \) to be the Morse index of \( x \).

Morse theory explains how the topology of a space is linked to the critical points of a \( G \)-invariant Morse function on that space and its Morse indices. In particular the change in topology at the critical value \( c \) is determined by the values of the Morse indices of the critical points at that value, see for example Milnor's book, [56]. The Morse index can be defined for a critical point of any functional; however to be able to apply Morse theory it is necessary that the functional is Morse. For an action functional that is a \( G \)-invariant Morse function, we can use the 'Morse inequalities' as an alternative to category in proposition 1.1.1. The Morse inequalities are:

\[
\sum_{i \in \mathbb{N} \cup \{0\}} n_i t^i P(t) - Q(t) = (1 + t) \sum_{i \in \mathbb{N} \cup \{0\}} q_i t^i \text{ for all } t \in \mathbb{R} \text{ with } q_i \geq 0 \text{ for all } i,
\]

where \( P(t) \) and \( Q(t) \) are the Poincaré polynomials of \( G \) and the topological space respectively. (The coefficients in Poincaré polynomials are defined by Betti numbers.) The quantities \( n_i \) are the number of critical manifolds of Morse index \( i \) for each \( i \). The inequalities (1.6) were used in [54] and are used in chapter 6. The bounds given by the Morse inequalities are usually stronger than that given by the category. However the drawback is that to obtain the bounds we require the assumption (1.5), unlike for category.
1.2 The Systems We Consider

The systems we shall study are:

- Central force problems with potentials $-\frac{1}{|x|^\alpha}$, for $1 \leq \alpha \leq 2$ (Chapter 2).
- The molecular 2-body problem with Lennard-Jones potential (Chapter 3).
- The symmetric planar Newtonian two centre problem (Chapter 4).
- The molecular $N$-body problem for $N > 2$ (Chapter 5: joint work with Luca Sbano).
- Symmetric strong force $N$-centre problems (Chapter 6: a note on [54]).

The first three of these systems are integrable. The last two systems are not integrable but this does not mean to say we cannot infer the existence (and sometimes non-existence) of periodic solutions using variational methods. Each chapter of this thesis is self contained with its own abstract, introduction (which includes literature review and motivation), main body and conclusion. Links between chapters and underlying themes are pointed out in the introductions, conclusions and remarks. Rather than repeating the chapter abstracts or introductions in the next section we shall list the important questions that we ask of the systems in general. This will clarify the underlying themes of the thesis. We give some motivation for each question and point to parts of the thesis where the questions have been addressed.

1.3 The Questions We Ask

1.3.1 Which Homotopy Classes of Loops contain Periodic Orbits?

Integrable Systems (Chapters 2, 3 and 4).

We look at the ratio of angular frequencies in the separating coordinate system; the action-angle coordinates. Rational values of this ratio correspond to periodic orbits. We study the relationship between this ratio and the homotopy class. In chapter 4 we find that the integrability of the Newtonian two centre problem puts strong constraints on the set of homotopy classes containing periodic orbits, see theorem 4.4.1. Indeed, in a sense that is made precise in the chapter (see subsection 4.5.2), we find that there are ‘very few’ homology classes of the symmetric planar Newtonian two centre problem that contain periodic orbits. This can be contrasted with planar two centre problems with (standard) strong force potentials for which the direct method yields periodic solutions in all non-null homotopy classes of loops.

Nonintegrable Systems (Chapters 5 and 6).

As mentioned above the main difficulty with molecular systems is that the action functional is not bounded below on loop space. Chapter 5 can be thought of as the study of the Lennard-Jones action functional and how its structure changes with period. We find nonexistence results for small periods (see proposition 5.3.2) and, subject to an assumption, existence results for large periods (see theorem 5.4.2). The idea of the existence proof is that for large periods the LJ action functional can be perturbed from a standard strong force action functional. Indeed our existence (and nonexistence) results only apply to tied classes. In chapter 6 we consider RPOs of symmetric standard
strong force $N$-centre problems with potentials bounded above. We find all homotopy classes of relative loops on which the action functional is coercive. By the direct method each of these classes contains a solution of the equations of motion.

1.3.2 Which Homotopy Classes of Loops contain Prime-Period Periodic Orbits?

This is a more specific question than the previous one. All periodic orbits are multiples of prime-period orbits. If we find a periodic orbit in a particular homotopy class of loops it is interesting to know whether or not one could have obtained that orbit simply by 'repeating' a lower period orbit an integer (> 1) number of times. If not then it is of prime-period and is 'new' in a sense. We describe which homotopy classes contain prime-period periodic orbits for the integrable systems in chapters 2 and 4. For central force problems with potential $-\frac{1}{|x|^k}$ we find that the number of homotopy classes not containing prime-period orbits is finite but nonzero for $1 < \alpha < 2$ (see proposition 2.3.3) but diverges as $\alpha \to 2^-$ or $\alpha \to 1^+$ (see lemmas 2.3.4 and 2.3.5). The two centre problem of chapter 2 provides a more complicated example. All periodic orbits can be classified into one of the types $P_1$, $P_2$ or $P_3$ (see definition 4.3.1) depending on some qualitative features of the orbit. In theorem 4.4.1 we describe all those homology classes of the planar Newtonian two centre problem that can contain prime-periodic periodic orbits. It is found that there are no homotopy classes that contain both a prime-period and a non-prime period orbit of type $P_i$ if $i \in \{2,3\}$ but that, apart from two classes, all non-null homotopy classes containing $P_1$ orbits contain both prime-period and non-prime-period orbits of type $P_1$.

1.3.3 How many Periodic Orbits are in each Homotopy Class of Loops?

Any system that possesses continuous symmetries will contain an (uncountably) infinite number of $T$-periodic orbits if it contains at least one. Therefore when we count periodic orbits we shall do so up to such symmetries; in other words we only count the number of different critical orbits of $A_T$. For integrable systems with two degrees of freedom this can be done by examining the ratio of angular frequencies, say $\Delta = \frac{p}{q}$. Fixing the homotopy class puts a constraint on $p$ and $q$. The strategy is to count how many pairs satisfy this constraint and also lie in the allowed range of $\Delta$. This is explored in chapters 2, 3 and 4.

For nonintegrable systems the situation is more complicated. In chapter 5 we find, see theorem 5.4.2, two distinct manifolds of periodic orbits in every class of loops satisfying certain conditions (see definition 5.2.6) of the molecular $N$-body problem. At the end of [54], under the assumption that the action functional of the autonomous system is an $S^1$-invariant Morse function, equivariant Morse inequalities were used to show the existence of infinitely many periodic orbits in some classes of relative loops of the planar strong force 2-centre problem. The example given there contains a mistake. In chapter 6 we correct the mistake and generalize the result to the $N$-centre problem.

1.3.4 What is the 'Action Spectrum'?

In [57] Richard Montgomery defined an action spectrum as 'the value of the actions of orbits plotted against some way of indexing those orbits'. In this thesis we will sometimes plot the actions of periodic orbits in particular classes of loops against parameters of the systems. We will sometimes also label the periodic orbits with their Morse indices with respect to the action functional. More precisely, we ask:
What are the actions of the $T$-periodic orbits relative to other $T$-periodic orbits in the same homotopy class?

This will be investigated numerically but we will hint at possible analytical approaches. We find it for $-\frac{1}{|x|^{\alpha}}$ ($1 \leq \alpha \leq 2$) potentials and for P1 orbits of the planar symmetric Newtonian 2-centre problem. Answering this question helps us answer the following question:

What are the Morse indices of the periodic orbits with respect to the action functional?

We can sometimes track periodic orbits as parameters of the problem (e.g. $\alpha$ or $T$) are varied. By examining plots of the action spectrum and calculating Morse index exchanges at bifurcations we are sometimes able to infer the Morse indices of periodic orbits. In chapter 2 we find the Morse indices of collisionless periodic orbits of central force systems with potentials $-\frac{1}{|x|^\alpha}$ with $1 \leq \alpha \leq 2$ by tracking the Morse index from the cases $\alpha \geq 2$ which are strong force. In particular we find the Morse indices of Kepler elliptical orbits with respect to the action functional. We expected this result to already exist in the literature but cannot find it; perhaps it is new. In chapter 4 (subsection 4.6.3) we track the Morse indices of Kepler orbits to certain types of orbits (called P1) of the symmetric planar Newtonian two centre problem. In chapter 5 we use a perturbation argument to track a minimizer of the standard SF action functional to a local minimizer of the LJ action functional for large periods; this has Morse index 0. To perform the perturbation we make an assumption of the form (1.5). We then use a mountain-pass argument to construct a mountain-pass critical point with Morse index $\geq 1$. In chapter 6 we find there are critical manifolds with Morse index $m$ for every even natural number $m$ in some classes of relative loops of symmetric $N$-centre systems.

1.3.5 What values of the period and constants of motion support the existence of periodic solutions?

The equations of motion (1.1) of the $N$-body problem with potential of the form (1.3) have a natural scaling symmetry given by:

$$x(t) \mapsto \lambda x(\lambda^{-(\alpha+2)}t), \quad \lambda > 0.$$  

This symmetry implies that if there exists a periodic orbit in a homotopy class for a value of the period then there exists periodic orbits in that class for all periods. However for the $N$-centre problem and the $N$-body problem with nonhomogeneous potentials (such as the Lennard-Jones potential), no such symmetry exists and the existence of periodic orbits of period $T$ in a particular class may depend on the value of $T$. This is explored for the 2-centre problem, see subsection 4.6.2.2, and for the molecular $N$-body problem in chapter 5. Chapter 5 is concerned with understanding the structure of the LJ action functional and how this changes with period.

The set of all possible pairs of values of the two constants of motion for quasi-periodic motion is known for systems with potentials $-\frac{1}{|x|^\alpha}$ (see [38]) and the 2-centre problem (see [72]). In chapter 3 we find this set for the molecular 2-body problem with potential (1.4) with $\alpha = 6$, $\beta = 12$ (see theorem 3.3.2).
1.3.6 What do the Periodic Orbits look like? What Symmetries do they have?

The study of periodic orbits is aesthetically appealing in that the solutions often trace out interesting curves in configuration space. Moreover the shape of the orbits may be of physical significance - for example, in planning the trajectory of a satellite. We find remarkable curves in each of the systems that we study. Some of the solutions in particular homotopy classes necessarily possess symmetries. For example, in chapter 4 we find that some periodic orbits necessarily possess a symmetry which we denote by (S). In general periodic solutions do not necessarily possess any symmetry. However we can often impose symmetries and employ Palais’ principle of symmetric criticality. This is used in chapter 5, for example in finding choreographical motions.

1.3.7 Do Nonplanar Solutions Exist?

So far we have we have only discussed the $N$-body and $N$-centre problems restricted to the plane. An interesting question is whether or not we can show the existence of nonplanar ('spatial') solutions. The trouble with no longer restricting to the plane is that the configuration spaces lose their topology: they become simply connected. In [20] Chenciner and Venturelli proved the existence of “hip-hop” solutions of the Newtonian 4-body problem. Their idea has since been generalized to more classes (see [67]) and to $2N$-bodies (see [70]). The idea of the proof of hip-hop solutions is to:

(i) restrict to a tied symmetry class. Moreover, the symmetry class is chosen such that restriction of the symmetry class to planar loops corresponds precisely to choreographical motions in the plane. For potentials of the form (1.3), with $\alpha > 0$, the action minimizer of planar choreographical motions is known to be a Lagrange solution (a relative equilibrium), see [7].

(ii) prove that the minimizer on the whole symmetry class cannot be planar. This is done by constructing a small deformation of the Lagrange solution that results in a loop that is in the symmetry class but nonplanar and has lower action than the Lagrange solution.

(iii) use the direct method. The symmetry class induces coercivity. Collisions are excluded by using Marchal’s theorem in a similar way to [36].

In subsection 5.5.3 we show the existence of nonplanar solutions of the molecular 4-body problem. We do this by perturbing from the standard strong force problem with potential (1.3) with $\alpha \geq 2$, at large periods. We generalize Salomone and Xia’s argument (see [67]) for the Newtonian potential to this potential. We then use a perturbation argument to ‘track’ this solution to a solution of the molecular problem for large periods, noting that for sufficiently large periods the solution is still nonplanar.

Also, using a result from Coti Zelatti’s paper [27] (which deals with a generalized molecular 1-centre problem), we prove in section 5.6 the existence of periodic solutions of the restricted spatial molecular $(2N+2)$-body problem.
Chapter 2

A Family Of Central Force Problems

*It is through science that we prove but through intuition that we discover.*
- Jules H. Poincaré

Abstract

We investigate the integrable central force problem with potentials $V_\alpha(x) = -\frac{1}{\|x\|^\alpha}$ for $1 \leq \alpha \leq 2$. In particular we:

- Prove that for $1 < \alpha < 2$ there are only finitely many homotopy classes that do not contain a prime-period periodic orbit, but that this number diverges as $\alpha \to 1^+$ or $\alpha \to 2^-$. 
- Use numerics to describe what we call the 'action spectrum' of each homotopy class. Given any non-null homotopy class and any period $T > 0$ we list all collisionless orbits ordered according to their action, $A_T^T$.
- Label the periodic orbits by their Morse indices.

2.1 Introduction

Central force problems are extremely important. They are one of the most fundamental models of both celestial and atomic motion. We will study the periodic solutions of a class of central force problems from a variational viewpoint using the calculus of variations. The spaces we shall consider are homotopy classes of loop space and the function we shall consider is the action functional, denoted by $A_T^T$. The action functional of particular interest because, by the principle of least action, critical points of it on a homotopy class correspond precisely to periodic orbits in that class. Crucially central force problems are *integrable*. If one is interested in the action functional then it is fruitful to study integrable systems from a variational perspective because, by examining the ratios of angular frequencies of the separating coordinate system, we can find *all* periodic solutions in a given homotopy class and, by implication, all critical points of the action functional on that homotopy class. We can track all periodic solutions as parameters of the system are varied and plot their actions. Our study of the structure of the action functional and Morse indices of the action functional on the homotopy classes of loops is also motivated by Montaldi, McCord, Roberts and Sbano's paper [54] in which a method is found for computing the topology of the homotopy classes
of the loop space of a symmetric Lagrangian system. The aim of this chapter is to investigate the periodic orbits of these systems and the structure of the action functional. This chapter also serves as an introduction to chapter 4 where the Newtonian 2-centre problem is analyzed in an analogous fashion.

We begin this chapter by showing how central force problems can be integrated in polar coordinates \((r, \theta)\). Action-angle variables are defined and we calculate the ratio of angular frequencies \(\Delta_\alpha = \frac{\omega_\theta}{\omega_r}\) as a function of the two constants of motion. Plots of \(\Delta_\alpha(c, h)\) for \(1 \leq \alpha \leq 2\) are given and properties of \(\Delta_\alpha\) are discussed. We calculate the quantities \(\sup(\Delta_\alpha)\) and \(\inf(\Delta_\alpha)\) in terms of \(\alpha\). This result allows us to give a prescription for finding all those orbits that exist for a given \(\alpha\). For any given orbit we find the set of \(\alpha\) for which this orbit exists. We then describe how the list of homotopy classes not containing any prime-period orbits changes as a function of \(\alpha\). We prove that the size of the list is finite for all \(1 < \alpha < 2\) by calculating an upper bound in terms of \(\alpha\) for the length of the list. However we show that the list grows to infinite size as \(\alpha \to 1^+\) or \(\alpha \to 2^-\). Then what we call the 'action spectrum' as introduced by Montgomery, see [57], is discussed: for a given \(\alpha\) we list all the possible orbits in each connected component of loop space in order of increasing action coefficient. The 'birth' and 'death' of orbits as \(\alpha\) is varied from 1 to 2 is explained; these occur as bifurcations off 'collision orbits' and 'circular orbits.' These births and deaths can be visualized in a plot of the action spectrum. We compute the Morse indices of the orbits with respect to the action functional using bifurcation theory. As an interesting corollary the Morse index with respect to the action functional of a Kepler orbit that winds \(p\) times around the centre in one period is calculated.

2.2 Setup

2.2.1 Integration

Central Force Problems (CFPs) are necessarily planar. We are therefore dealing with systems with two degrees of freedom. There are always at least two first integrals (conserved quantities) - these being energy and angular momentum. This implies that such systems are always Liouville integrable. Plane-polar coordinates \((r, \theta)\) are the most natural choice of coordinates due to rotational symmetry of the system.

Suppose that the potential energy of the system is \(V(r)\). We have

\[
H = \frac{1}{2} r^2 + \frac{1}{2} r^2 \theta^2 + V(r).
\]  

(2.1)

If the distance metric is \(ds\) then using plane-polar coordinates \((r, \theta)\) we have \(ds^2 = a_r^2dr^2 + a_\theta^2d\theta^2\) where \(a_r = 1\) and \(a_\theta = r\). Therefore \(p_r = a_r^2 \dot{r} = \dot{r}\) and \(p_\theta = a_\theta^2 \dot{\theta} = r^2 \dot{\theta}\) i.e. \(\dot{\theta} = \frac{p_\theta}{r^2}\). Substituting these into (2.1) gives:

\[
H = \frac{1}{2} p_r^2 + \frac{p_\theta^2}{2r^2} + V(r).
\]

Now we can write \(p_r = \frac{\partial S}{\partial r}\) and \(p_\theta = \frac{\partial S}{\partial \theta}\), where \(S\) is Hamilton's principal function, to get:

\[
2hr^2 - 2r^2 V(r) - r^2 \left(\frac{\partial S}{\partial r}\right)^2 = \left(\frac{\partial S}{\partial \theta}\right)^2.
\]
To complete the separation of variables we set $LHS = RHS = c^2$, where for convenience we pick $c$ to be positive (corresponding to anticlockwise motion). From $RHS = c^2$ we then obtain that:

$$p_\theta = c \quad \text{i.e.} \quad r^2 \dot{\theta} = c.$$

From $LHS = c^2$ we obtain that:

$$\dot{r} = p_r = \frac{\partial S}{\partial r} = \pm \frac{1}{r} \sqrt{2nr^2 - 2r^2V(r) - c^2}.$$

**Definition 2.2.1.** We will denote

$$P(r) \equiv 2nr^2 - 2r^2V(r) - c^2.$$

Equation (2.2) then becomes

$$\dot{r} = \pm \frac{1}{r} \sqrt{P(r)}.$$

### 2.2.2 Flows of $C$ and $H$

The flow of $H$ gives phase-shifts. To compute the flow of $C$ we treat $C$ as if it were the Hamiltonian and write down Hamilton's equations of motion:

$$\dot{r}' = \frac{\partial C}{\partial p_r} = 0, \quad \dot{\theta}' = \frac{\partial C}{\partial p_\theta} = 1,$$

$$p_r' = -\frac{\partial C}{\partial r} = 0, \quad p_\theta' = -\frac{\partial C}{\partial \theta} = 0.$$  

Here $s'$ gives a parametrization of the flow. We see then, that under the flow of $C$, $r$, $p_r$ and $p_\theta$ stay the same, and $\theta(s) = s + s_0$ where $s_0$ is a constant. The flow of $C$ is therefore rotations about the centre. We can see that the flows of $H$ and $C$ become dependent for circular orbits. The flows of $C$ and $H$ here do not allow us to move from one orbit to another of the opposite orientation.

### 2.2.3 Action-Angle variables

Since we can separate in $(r, \theta)$ it is natural to define:

$$I_r = \frac{1}{2\pi} \int_{\gamma_r} p_r dr \quad \text{and} \quad I_\theta = \frac{1}{2\pi} \int_{\gamma_\theta} p_\theta d\theta,$$

where $\gamma_r$ and $\gamma_\theta$ are the projections of the orbits in phase-space to the $r - p_r$ and $\theta - p_\theta$ planes respectively. For bounded collisionless orbits, $r$ will oscillate between a minimum value, say $r_{\min}$, and a maximum value, say $r_{\max}$. The variable $\theta$ will vary from 0 to $2\pi$. Therefore:

$$I_r = \frac{2}{2\pi} \int_{r_{\min}}^{r_{\max}} p_r dr = \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} \dot{r} dr$$

and

$$I_\theta = \frac{1}{2\pi} \int_0^{2\pi} p_\theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} c d\theta = c,$$

where $c$ is the angular momentum. We let $\omega_r$ and $\omega_\theta$ denote the corresponding angular frequencies.
2.2.4 Ratio of frequencies

Definition 2.2.2. Let

\[ M = \frac{\partial (I_r, I_\theta)}{\partial (h, c)} = \left( \begin{array}{cc} \frac{\partial I_r}{\partial h} & \frac{\partial I_r}{\partial c} \\ \frac{\partial I_\theta}{\partial h} & \frac{\partial I_\theta}{\partial c} \end{array} \right). \]

Lemma 2.2.1. We have the identity

\[ (1 0) = (\omega_r \quad \omega_\theta) M. \]

Proof. This follows from the chain rule using the fact that \( \omega_j = \frac{\partial I_j}{\partial I_j} \) where \( j \) is \( r \) or \( \theta \). \( \square \)

Remark 2.2.1. We can write lemma 2.2.1 as

\[ \left( \begin{array}{c} \omega_r \\ \omega_\theta \end{array} \right) = (M^t)^{-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right). \]

Note that because \( I_\theta \equiv c \) we have

\[ M = \left( \begin{array}{cc} J_h & J_c \\ 0 & 1 \end{array} \right), \]

where

\[ J_h = \frac{\partial I_r}{\partial h} \quad \text{and} \quad J_c = \frac{\partial I_r}{\partial c}. \]

(2.3)

It follows that

\[ (M^{-1})^t = \frac{1}{J_h} \left( \begin{array}{cc} 1 & 0 \\ -J_c & J_h \end{array} \right), \]

from which we find \( \omega_r = \frac{J_h}{J_h} \) and \( \omega_\theta = -\frac{J_c}{J_c} \).

Definition 2.2.3. Let \( T_0 \) be the time taken for \( \theta \) to travel through \( 2\pi \) radians. Let \( T_r \) be the time taken for a single \( r \)-oscillation. Let \( \Delta = \frac{\omega_\theta}{\omega_r} = \frac{J_c}{J_h} \).

Lemma 2.2.2. We have that

\[ \Delta = -J_c, \quad T_r = 2\pi J_h \quad \text{and} \quad T_\theta = -2\pi J_h/J_c, \]

(2.4)

where \( J_h \) and \( J_c \) are as defined in (2.3).

To get computable expressions for \( J_h \) and \( J_c \) we need a lemma that allows us to interchange integration and partial differentiation.

Lemma 2.2.3. If \( I(k) = \int_{F_1(h)}^{F_2(h)} f(k,r)dr \), then:

\[ \frac{\partial I(k)}{\partial k} = F_2'(k)f(k,F_2(k)) - F_1'(k)f(k,F_1(k)) + \int_{F_1(k)}^{F_2(k)} \frac{\partial f(k,r)}{\partial k} dr. \]

Lemma 2.2.4. A formula for \( \Delta(c,h) \) is given by

\[ \Delta(c,h) = \frac{\omega_\theta}{\omega_r} = -J_c = \frac{1}{\pi} \int_{\tau_{\min}}^{\tau_{\max}} \frac{c}{\tau \sqrt{P(r)}} dr. \]

(2.5)
Proof. Applying lemma (2.2.3) to $I = I_r$, with $k \in \{c, h\}$, noting that $r$ vanishes for $r \in \{r_{\min}, r_{\max}\}$ and using definition 2.2.1, we obtain:

$$J_k = \frac{\partial I_r}{\partial h} = \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} \frac{1}{r} \frac{\partial}{\partial k} \sqrt{P(r)} dr.$$ 

More concretely:

$$J_h = \frac{\partial I_r}{\partial h} = \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} \frac{r}{\sqrt{P(r)}} dr$$ and

$$J_c = \frac{\partial I_r}{\partial c} = -\frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} \frac{c}{r \sqrt{P(r)}} dr.$$ 

Then using (2.4), we get the result. □

**Remark 2.2.2.** $\Delta$ represents the fraction of one-revolution that one goes through between successive meetings of the orbit with the apocentre (which is the circle $r = r_{\max}$) or pericentre (the circle $r = r_{\min}$). The quantity $\pi \Delta$ is called the apsidal angle.

### 2.2.5 Homotopy Classes

We define loop space by:

$$\Omega_T = \{ x \in H^1(\mathbb{R}/T \mathbb{Z}, \mathbb{R}^2) : x(t) \neq 0 \text{ for every } t \in \mathbb{R}/T \mathbb{Z} \}.$$ 

Then $\Omega_T$ is an open subset of $H^1(\mathbb{R}/T \mathbb{Z}, \mathbb{R}^2)$. We have that

$$\mathcal{D} \Omega_T = \{ x \in H^1(\mathbb{R}/T \mathbb{Z}, \mathbb{R}^2) : \exists \bar{t} \in \mathbb{R}/T \mathbb{Z} \text{ such that } x(\bar{t}) = 0 \};$$ these are called 'collision loops'. On $H^1(\mathbb{R}/T \mathbb{Z}, \mathbb{R}^2)$ we take the norm:

$$\|x\|_1^2 = \int_0^T \|x(t)\|^2 dt + \int_0^T \|\dot{x}(t)\|^2 dt.$$ 

We denote the connected components of $\Omega_T$, called homotopy classes, by $\Omega_T^p$, where $\Omega_T^p$ consists of all those loops in $\Omega_T$ that wind clockwise $p$ times around the origin in period $T$. Note that $\Omega_T = \bigcup_{p \in \mathbb{Z}} \Omega_T^p$.

### 2.2.6 Prime-Period Orbits

An orbit is periodic if and only if $\Delta \in \mathbb{Q}$. A periodic orbit is said to be of prime-period $T$ if $T > 0$ is its least period. We will refer to a periodic orbit with $p$ $\theta$-oscillations and $q$ $r$-oscillations as an orbit of type $\frac{p}{q}$. The orbit is of prime-period if and only if $\gcd(p, q) = 1$. Note that the periodic orbit has $\Delta_\alpha = \frac{\pi}{2}$. A $T$-periodic orbit of type $\frac{p}{q}$ is in the homotopy class $\Omega_T^p$ if $c > 0$ and the homotopy class $\Omega_T^{-p}$ if $c < 0$ and $\Omega_T^p$ if $c > 0$. All non-prime-period orbits of type $\frac{p}{q}$ can be obtained by repeating a prime-period orbit (of smaller period) $\gcd(p, q)$ times.

### 2.2.7 The Principle of Least Action

**Definition 2.2.4.** Let

$$\mathcal{A}_T : \Omega_T^p \to \mathbb{R}; \quad x \mapsto \int_0^T L_\alpha(t) dt,$$

where $L(t) = \frac{1}{2} \dot{x}(t)^2 - V(x)$ is the Lagrangian.

The principle of least action states that $T$-periodic orbits of the system with potential $V(x)$ correspond to critical points of $\mathcal{A}_T$.
2.3 Systems with Potential $-\frac{1}{\|x\|^\alpha}$, $1 \leq \alpha \leq 2$

Definition 2.3.1. We define $V_\alpha(x) = -\frac{1}{\|x\|^\alpha}$ and denote the corresponding Lagrangian, action functional and ratio of angular frequencies by $L_\alpha$, $A_T^\alpha$ and $\Delta_\alpha$ respectively.

For the remainder of this chapter we will only consider central force systems with potential $V_\alpha(x)$ for $1 \leq \alpha \leq 2$. However in chapter 3 we shall consider a central force problem with a different potential and use the results from section 2.2.

2.3.1 The function $\Delta_\alpha(c, h)$

We use $\Delta_\alpha$ to denote the function $\Delta$ for the potential $-\frac{1}{\|x\|^\alpha}$. To begin with we briefly discuss the cases $\alpha = 1$ and $\alpha = 2$:

- The case $\alpha = 1$ is the famous Kepler problem. All periodic solutions trace out ellipses in configuration space. Since $|\Delta_1|$ is the fraction of a revolution that one goes through between successive meetings of the orbit with the apocentre, $|\Delta_1|$ is identically equal to 1 everywhere that it is defined. Prime-period periodic orbits of period $T$ are only found in the homotopy classes $\Omega_p^{\pm 1}$.

- The case $\alpha = 2$ is the lowest value of $\alpha$ for which the system is strong force (see definition 1.1.2). We will see that, for $\alpha \geq 2$, all orbits are circular and so again prime-period periodic orbits of period $T$ can only found in the homotopy classes $\Omega_T^{\pm 1}$. The function $\Delta_\alpha$ is not defined for $\alpha \geq 2$.

However we will find that for $1 < \alpha < 2$ there are other homotopy classes that contain prime-period orbits.

We would like to know what the function $\Delta_\alpha(c, h)$ looks like for $1 < \alpha < 2$. Recalling (2.2) we see that we need to examine the properties of the polynomial $P(r)$. Differentiating $P(r)$ we get

$$P'(r) = 4hr + 2(2 - \alpha)r^{1-\alpha}.$$
Lemma 2.3.1. If $h > 0$ then there are no bounded orbits of the system with potential $-\frac{1}{\|x\|^\alpha}$ for $1 \leq \alpha < 2$.

Proof. If $h > 0$ then $P'(r) > 0$ for all $r > 0$. Then if $\dot{r} = 0$ when $r = r_0$ we have that $\dot{r}$ always has the same sign for $r > r_0$. \hfill \Box

We are only interested in closed orbits so we restrict to

$$h < 0.$$  \hfill (2.6)

We then have that $P'(r) = 0$ only at $r^* = (\frac{\alpha-2}{2h})^\frac{1}{\alpha} > 0$. Note that $P(0) = -c^2$ and that $P \to -\infty$ as $r \to \infty$. We therefore have three possible cases:

(i) $P(r)$ doesn’t cross the $r$-axis for $r > 0$. In this case there are no orbits, as $\dot{r} \notin R$.

(ii) The critical case: The $r$-axis is tangential to the curve and $P(r^*) = 0$. The condition $P(r^*) = 0$ is readily seen to be equivalent to

$$c^2 = \alpha(\frac{\alpha-2}{2h})^{\frac{2-\alpha}{\alpha}}.$$  \hfill (2.7)

In this case we get circular orbits with radius $r^*$.

(iii) $P(r)$ cuts the $r$-axis at $r_{\min}$ and $r_{\max}$ where $r_{\min}$ and $r_{\max}$ are the only two real and positive roots of $P$. Motion oscillates between these two values of $r$ as can be seen from the phase-portrait for $r$.

We use (2.6) and (2.7) to define a region of $(c, h)$-space:

Definition 2.3.2. Let

$$R_{\alpha} = \{(c, h) \in \mathbb{R}^2 : h < 0, c^2 \leq \alpha(\frac{\alpha-2}{2h})^{\frac{2-\alpha}{\alpha}}, c \neq 0\}.$$  

The point is that, for $1 \leq \alpha < 2$, $R_{\alpha}$ is precisely the region of energy-momentum space that supports (quasi-)periodic motion.

Definition 2.3.3. Let

$$R_{\alpha}^+ = \{(c, h) \in \mathbb{R}^2 : h < 0, c > 0, c^2 \leq \alpha(\frac{\alpha-2}{2h})^{\frac{2-\alpha}{\alpha}}\}.$$  

Definition 2.3.3 is sometimes convenient because $\Delta_{\alpha}(-c, h) = -\Delta_{\alpha}(c, h)$ and so $|\Delta_{\alpha}(-c, h)| = |\Delta_{\alpha}(c, h)|$. The map $c \mapsto -c$ corresponds to a change of orientation.

Remark 2.3.1.

- In the limit $\alpha \to 1^+$ we obtain

$$R_1^+ = \{(c, h) : c > 0, h < 0, c^2 \leq \frac{-1}{2h}\}.$$  

We have $\Delta_1(c, h) = 1$ on $R_1^+$.  

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In the limit $\alpha \to 2^-$, $c^2 \to 2$, so $c \to \pm \sqrt{2}$. Then $r = \sqrt{2|h|}$. So for periodic solutions $h = 0$.

Thus $R_0^+ = \lim_{\alpha \to 2^-} R_0^+ = \{(\sqrt{2}, 0)\}$. The function $\Delta_2(c, h)$ is not defined on $R_0^+$. The only possible motions are circular.

Definition 2.3.4. Let $D_\alpha = \{\Delta_\alpha(c, h) : (c, h) \in R_\alpha\} = \{\Delta_\alpha(c, h) : (c, h) \in R_0^+\}$.

We are interested in the the values of $\sup(D_\alpha)$ and $\inf(D_\alpha)$. To help find these we first look at numerically obtained contour and surface plots of $\Delta_\alpha(c, h)$ for various values of $\alpha$. Figures 2.3.1 and 2.3.2 show such plots for $\alpha = 1.2$.

2.3.2 Scaling Symmetries

Periodic solutions of the central force problem with potential $V_\alpha(x) = \frac{-1}{|x|^{2\alpha}}$ admit a scaling symmetry:

$$\Omega_T^p \to \Omega_{T_\lambda}^{p \lambda^{-\alpha}}; \quad x(t) \mapsto \lambda x(\lambda^{-\alpha} t) \quad \text{for all } p \in \mathbb{Z}, \quad T > 0, \lambda > 0. \quad (2.8)$$

We note how some quantities map under the scaling (2.8):

$$T \mapsto \lambda^{-\alpha} T, \quad h \mapsto \lambda^{-\alpha} h, \quad c \mapsto \lambda^{\frac{2-\alpha}{2}} c, \quad A_T^\alpha \mapsto A_{T_\lambda}^{\alpha \lambda^{-\alpha}} = \lambda^{\frac{2-\alpha}{2}} A_T^\alpha, \quad \Delta_\alpha \mapsto \Delta_\alpha. \quad (2.9)$$

The scaling (2.8) naturally preserves $\Delta_\alpha$.

We ask what this scaling corresponds to on a contour plot of $\Delta_\alpha(c, h)$. Since $\Delta_\alpha(c, h)$ is preserved one must ‘move along the contours’. The scaling is transitive on contours because given any point
Figure 2.3:

Surface plot of $\Delta_{1,2}(c, h)$
Contour plot of $\Delta(c, h)$ for $c > 0$ and $\alpha = 1.2$

Figure 2.4:
(c_0, h_0) on a contour all other points on that contour can be reached via a scaling. The general equation of a contour is:

\[ h = K c^{\frac{\alpha - 2}{\alpha}}, \]  

where \( K \) is a constant. Later we find numerical evidence that, for \( 1 < \alpha < 2 \), if you fix \( c > 0 \), then \( \Delta_\alpha(c, h) \) is a strictly monotonically increasing function of \( h \). It follows that the contour of \( \Delta_\alpha \) on \( R \) is connected and there are no discrete symmetries that preserve \( \Delta_\alpha \) and change \( (c, h) \). The only symmetry that preserves \( \Delta_\alpha \) and changes \( c \) and \( h \) nontrivially are the scalings of equation (2.8).

**Remark 2.3.2.** There are other symmetries that preserve \( h \) and \( c \) and consequently also preserve \( \Delta_\alpha \):

- **continuous symmetries**: time translation, corresponding to the flow generated by \( h \), and rotations, corresponding to the flow generated by \( c \).
- **discrete symmetries**: e.g. reflections, e.g. mapping a closed periodic orbit to a 'multiple' of itself.

### 2.3.3 Action of a Prime-Period Periodic Orbit.

For a prime-periodic orbit we have from definition 2.2.4 that

\[ A^p_\alpha = \int_0^T L_\alpha(t) dt = \int_0^T \left( \sum_i p_i q_i - h \right) dt = \left( 2\pi \sum_i N(q_i) I_{q_i} \right) - hT, \]

where \( N(q_i) \) is the number of \( q_i \)-oscillations. For orbits in a central force potential with \( \Delta_\alpha = \frac{p}{q} \) we have that \( N(r) = q, N(\theta) = p \) and \( T = pT_\theta \). Therefore:

\[ A^p_\alpha = 2\pi q I_r + 2\pi p I_\theta - hqT_\theta = 2q \int_{r_{\min}}^{r_{\max}} \frac{\sqrt{P(r)}}{r} dr + 2\pi pc - 2\pi qJ_h. \]

Finally, using our expression for \( J_h \) we get that:

\[ A^p_\alpha = 2q \int_{r_{\min}}^{r_{\max}} \frac{\sqrt{P(r)}}{r} dr + 2\pi pc - 2q \int_{r_{\min}}^{r_{\max}} \frac{r}{\sqrt{P(r)}} dr. \]  

(2.11)

Therefore if we are given a pair of values \((c, h)\) of angular momentum and energy that gives rise to a rational value of \( \Delta_\alpha \) equal to \( \frac{p}{q} \), where \( p \) and \( q \) are coprime, then we can calculate the action using (2.11).

### 2.3.4 Action Coefficients

From our knowledge of how \( A^p_\alpha \) and \( T \) transform under the scaling (2.8) we deduce that \( A^p_\alpha \) is proportional to \( T^{\frac{\alpha - 2}{\alpha}} \) along contours.

**Definition 2.3.5.** We call the coefficient of \( T^{\frac{\alpha - 2}{\alpha}} \) the action coefficient. We will denote the action coefficient by \( \Psi \).

Note that every strictly positive value of \( T \) between 0 and \( \infty \) is possible via an appropriate choice of \( \lambda \) in (2.9).

The action coefficient of a periodic orbit of prime-period can be calculated numerically. Given values \( c, h \) such that \( \Delta_\alpha(c, h) = \frac{p}{q} \) where \( \gcd(p, q) = 1 \), we can calculate \( A^p_\alpha \) and \( T \) using formulae (2.11) and \( T = qT_\theta = 2\pi qJ_h \). The action coefficient is then \( \Psi = A^p_\alpha T^{\frac{\alpha - 2}{\alpha}} \).
2.3.4.1 Action coefficient of multiples of prime period orbits

'Repeating' a prime-period periodic orbit does not change $\Delta$. However, $T$, $A^2_T$ and $\Psi$ are affected. For suppose we repeat an orbit $n$ times. Then $A^2_T \mapsto nA^2_T$ and $T \mapsto nT$. Therefore $\Psi = A^2_T \frac{2\pi}{nT} \mapsto n\frac{2\pi}{T} \Psi = n^2 \frac{2\pi}{T} \Psi$. Therefore the action coefficient, $\Psi$, gets multiplied by $n^2 \frac{2\pi}{T}$.

2.3.5 Existence of Periodic Orbits

2.3.5.1 Finding sup$(D_\alpha)$ and inf$(D_\alpha)$.

**Assumption (I)**

For $1 < \alpha < 2$ and fixed $c > 0$, $\Delta_\alpha(c, h)$ is strictly monotonically increasing as a function of $h$.

Due to the shape of the contours, assumption (I) is equivalent to assuming that if you fix $h$ then $\Delta_\alpha$ strictly monotonically decreases as $c > 0$ increases.

**Remark 2.3.3.** The claim has been verified by producing contour plots for several different values of $\alpha$ but has not yet been proved. It is possible that one may be able to prove it using some work by Chow and Sanders on elliptic integrals, see [22].

The proofs of the following two lemmas, lemmas 2.3.2 and 2.3.3, are based upon the proof of Bertrand's theorem found in [38].

**Lemma 2.3.2.** Subject to assumption (I) we have:

$$\text{inf}(D_\alpha) = \frac{1}{\sqrt{2 - \alpha}}.$$

Proof. From the contour plots, the general equation of a contour and Assumption (I), we see that $\Delta_\alpha$ takes its minimum value in the limit as you approach the curve $c^2 = \alpha \frac{(2 - \alpha)^2}{2\pi}$. On the curve itself we have circular motion for which $\Delta_\alpha$ is not defined. The quantity $\text{inf}(D_\alpha)$ will be the value of $\Delta_\alpha$ for an orbit close to a circle.

The equation of radial motion in plane-polar coordinates is:

$$\ddot{r} - r \dot{\theta}^2 = -V'(r).$$

Using $\dot{\theta} = \frac{c^2}{r^3}$ we have that

$$\ddot{r} - \frac{c^2}{r^3} = -V'(r). \quad (2.12)$$

For a circular orbit, $r = r_0$ (constant) and $\dot{r} = 0$. Therefore (2.12) becomes:

$$\frac{c^2}{r_0^3} = -V'(r_0). \quad (2.13)$$

If the circular orbit is perturbed, then writing $\varepsilon = r - r_0$, the perturbation, we obtain from (2.12):

$$\ddot{\varepsilon} - c^2(\varepsilon + r_0)^{-3} = -V'(\varepsilon + r_0).$$

We expand this as a series in the small quantity $\varepsilon$:

$$\ddot{\varepsilon} - \frac{c^2}{r_0^3} (1 - 3 \frac{\varepsilon}{r_0} + ...) = -V'(r_0) - V''(r_0) \varepsilon + ...$$

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To finish, using equation (2.13), we get:

\[ \varepsilon + (3V'(r_0) \frac{1}{r_0} + V''(r_0) + \ldots) \varepsilon = 0. \]

Assuming \( 3V'(r_0) \frac{1}{r_0} + V''(r_0) > 0 \) this approximates simple harmonic motion. The squared frequency of the oscillations of the perturbations about \( r_0 \) is:

\[ \omega_r^2 = 3V'(r_0) \frac{1}{r_0} + V''(r_0). \]

From (2.13):

\[ \frac{\omega_0}{\omega_r} = \frac{\sqrt{V'(r_0)}}{r_0 V''(r_0) + 3V'(r_0)}. \]

Therefore:

\[ \Delta_\alpha = \frac{\omega_0}{\omega_r} = \frac{\sqrt{V'(r_0)}}{r_0 V''(r_0) + 3V'(r_0)}. \] (2.14)

Here we have potential \( V_\alpha(r) = -\frac{1}{r^\alpha} \). Therefore \( V''_\alpha(r_0) = -\frac{\alpha(\alpha+1)}{r_0^{\alpha+2}} \) and \( V''_\alpha(r_0) = -\frac{\alpha(\alpha+1)}{r_0^{\alpha+2}} \). Substituting these into (2.14) and using assumption (I) gives the desired result. \( \square \)

**Lemma 2.3.3.** Subject to assumption (I) we have:

\[ \text{sup}(D_\alpha) = \frac{1}{2 - \alpha}. \]

**Proof.** To find \( \text{sup}(D_\alpha) \) we need to find the value of \( \Delta_\alpha \) in the limit as \( h \to 0^- \). From (2.5) we have:

\[ \pi \Delta_\alpha = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\sqrt{c}}{r^2 \sqrt{2\pi^2(h - V(r)) - c^2}} dr. \] (2.15)

We now make a change of variables given by \( u = \frac{\varepsilon}{r}, dr = -\frac{1}{c^2} du \). The equation (2.15) becomes:

\[ \pi \Delta_\alpha = \int_{u_{\text{min}}}^{u_{\text{max}}} \frac{du}{\sqrt{2(h - V_{\text{eff}}(u))}}. \] (2.16)

where \( V_{\text{eff}}(u) = V(u) + \frac{u^2}{2} \). Next setting \( y = \frac{u}{u_{\text{max}}} \), \( du = u_{\text{max}} dy \) we obtain:

\[ \pi \Delta_\alpha = \int_{y_{\text{min}}}^{1} \frac{dy}{\sqrt{\frac{2}{u_{\text{max}}^2}(h - V_{\text{eff}}(u_{\text{max}}y))}}. \] (2.17)

Defining \( W(y) = \frac{1}{2} y^2 + \frac{1}{u_{\text{max}}^2} V\left(\frac{y}{u_{\text{max}}}\right) \) we note that for \( r = r_{\text{min}} \) we have that \( r^* = 0 \) and so \( h = \frac{c^2}{2u_{\text{min}}^2} + V(r_{\text{min}}) = \frac{1}{2} u_{\text{max}}^2 + V\left(\frac{c}{u_{\text{max}}}\right) = u_{\text{max}}^2 W(1) \). So we get that \( W(1) = \frac{h}{u_{\text{max}}^2} \). Therefore \( W(1) - W(y) = \frac{h}{u_{\text{max}}^2} - \frac{1}{2} y^2 - \frac{1}{u_{\text{max}}^2} V\left(\frac{y}{u_{\text{max}}}\right) \). Therefore

\[ \frac{2}{u_{\text{max}}^2} (h - V_{\text{eff}}(u_{\text{max}}y)) = W(1) - W(y). \] (2.18)

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Using (2.18), (2.17) can be written

\[ \pi \Delta_\alpha = \int_{y_{\text{min}}}^{1} \frac{dy}{\sqrt{2W(y)}}. \]

Note that as \( h \to 0^- \), \( W(1) = \frac{h}{u_{\text{max}}} \to 0 \), so we get:

\[ \pi \Delta_\alpha = \int_{y_{\text{min}}}^{1} \frac{dy}{\sqrt{-2W(y)}}. \tag{2.19} \]

Now \(-2W(y) = -2 \left( \frac{y^2}{2r_{\text{min}}} - \frac{1}{2} \left( \frac{c}{r_{\text{min}}} \right)^{-\alpha} \right) = y^\alpha \left( 2r_{\text{min}}^{-\alpha} \left( \frac{c}{r_{\text{min}}} \right)^{-2} \right) - y^2. \)

We have \( h = 0 \) and that at \( r = r_{\text{min}}, r = 0 \). So from the energy equation we get \( \frac{c^2}{2r_{\text{min}}} + V(r_{\text{min}}) = 0 \)

which implies that \( \left( \frac{c}{r_{\text{min}}} \right)^2 = -2V(r_{\text{min}}) = 2r_{\text{min}}^{-\alpha}. \)

Therefore \(-2W(y) = y^\alpha - y^2. \) Substituting this into (2.19) gives

\[ \pi \Delta_\alpha = \int_{0}^{1} \frac{dy}{\sqrt{y^\alpha - y^2}} = \frac{\pi}{2 - \alpha}, \]

and so, subject to assumption (1), we get:

\[ \sup(D_\alpha) = \Delta = \frac{1}{2 - \alpha}. \]

Remark 2.3.4. The plots and the form of the equations of contours given in (2.10) imply that the above limit value is the same as for \( c \to 0^+ \) with \( h \) held constant.

Lemmas 2.3.2 and 2.3.3 together imply:

Proposition 2.3.1. Subject to assumption (1),

\[ \frac{1}{\sqrt{2 - \alpha}} < \Delta_\alpha(c, h) < \frac{1}{2 - \alpha} \text{ for all } (c, h) \in \mathbb{R}^+. \]

Remark 2.3.5. Bertrand's theorem states that in problems where the kinetic energy is the sum of the squares of the velocities and the potential is spherically symmetrical, the only potentials for which all orbits are bounded and closed are those for which \( V(r) = \frac{k}{r} \) where \( k > 0 \) and \( V(r) = kr^2, \ k > 0. \) Note that for all bounded orbits to be closed we require that \( \Delta_\alpha \) have constant value on \( \mathbb{R}^+. \) For otherwise the continuity of \( \Delta_\alpha(c, h) \) for \( c > 0 \) implies there exists values of \( (c, h) \) for which \( \Delta_\alpha(c, h) \) is not rational and therefore corresponds to non-closed orbits. Indeed \( \Delta_1 \equiv 1 \) on \( \mathbb{R}^+. \) The proof of Bertrand's theorem does not recognise the values \( \frac{1}{2-\alpha} \) and \( \sqrt{\frac{1}{2-\alpha}} \) as the supremum and infimum values of \( \Delta_\alpha \) on \( \mathbb{R}^+. \)

Remark 2.3.6. It has been proved, see [51], that planar systems that only have either closed or unbounded motion are necessarily superintegrable; that is they have more independent constants of motion than degrees of freedom.
2.3.6 Applications of $\frac{1}{\sqrt{2-\alpha}} < \Delta < \frac{1}{\sqrt{2-\alpha}}$

**Proposition 2.3.2.** Subject to assumption (I), for any $T \in (0, \infty)$ there exist orbits of type $\frac{p}{q}$ if and only if $\frac{2p-q}{p} < \alpha < \frac{2p^2+q^2}{p^2}$.

*Proof.* For $\frac{p}{q}$ type orbits we are interested in the contour defined by $\Delta = \frac{p}{q}$. Notice that, if this contour exists, then the scaling symmetries imply that $T$ can take every value in $(0, \infty)$ along the contour. More precisely, given there is an orbit of period $T_0$ any other $T$ can be obtained via an appropriate choice of $\lambda$ since $T = \lambda^{\frac{q}{p}} T_0$. From the inequality $\frac{1}{\sqrt{2-\alpha}} < \Delta < \frac{1}{\sqrt{2-\alpha}}$ we see that this contour exists if and only if $\frac{1}{\sqrt{2-\alpha}} < \frac{q}{p} < \frac{1}{\sqrt{2-\alpha}}$. The left hand inequality is equivalent to $\frac{1}{\sqrt{2-\alpha}} < \frac{q^2}{p^2}$ and the right hand equality is equivalent to $2 - \alpha < \frac{q}{p} \Rightarrow \alpha > \frac{2p-q}{p}$. □

All homotopy classes $\Omega^p_T$ for $p \neq 0$ contain periodic orbits. To see this note that we can take a circular orbit with the required winding number. However, we pose the questions:

(i) Given a potential $-\frac{1}{\sqrt{2}}$ are there orbits of prime-period in every non-null homotopy class?

(ii) If the answer to (i) is no, which of the homotopy classes do not have an orbit of prime-period?

To answer this we observe that, for $p > 1$, there is an orbit of prime-period in the homotopy class $\Omega^p_T$ if and only if there exists $q$, coprime to $p$, such that $\frac{1}{\sqrt{2-\alpha}} < \frac{q}{p} < \frac{1}{\sqrt{2-\alpha}}$. The case $p = 1$ is special: since $\frac{1}{\sqrt{2}} > 1$ for $1 < \alpha < 2$, there is no natural number $q$ such that $\frac{1}{\sqrt{2-\alpha}} < \frac{1}{q} < \frac{1}{\sqrt{2-\alpha}}$. But there is an orbit in the homotopy class $\Omega^1_T$ - the circular orbit that winds around the origin once in one period.

**Definition 2.3.6.** Let:

$$\Pi(\alpha) = \{ p \in \mathbb{Z} - \{0\} : \text{there does not exist a prime-period periodic orbit of the system with potential } -\frac{1}{\sqrt{x^2}} \text{ in the homotopy class } \Omega^p_T \}.$$  

**Lemma 2.3.4.** Suppose $N \geq 2$ is an integer. Then, subject to assumption (I), for $2 - \frac{1}{\sqrt{2}} < \alpha < 2$ we have that $\{2, 3, ..., N\} \subseteq \Pi(\alpha)$.

*Proof.* First note that:

$$2 - \frac{1}{N^2} < \alpha \iff \frac{1}{\sqrt{2-\alpha}} > N. \quad (2.20)$$

If $p \neq 1$ is such that $p \notin \Pi(\alpha)$ then there exists $q \in \mathbb{N}$ such that:

$$\frac{p}{q} > \frac{1}{\sqrt{2-\alpha}}.$$ 

Therefore by (2.20) we have

$$\frac{p}{q} > N,$$

so

$$p \geq \frac{p}{q} > N.$$

Therefore if $p \in \{2, ..., N\}$ we have $p \in \Pi(\alpha)$. □
Notice that as $N \to \infty$, $\alpha \to 2$ and $|\Pi(\alpha)| \to \infty$. In the limit we get the strong-force case in which the only homotopy classes containing prime-period orbits are $\Omega_T^{x+1}$. Indeed the only collisionless closed periodic orbits in the strong-force case are circular orbits.

**Lemma 2.3.5.** If $1 < \alpha \leq \frac{3}{2}$ then, subject to assumption $(I)$, $\{2, 3, \ldots, \lfloor \frac{1}{\alpha - 1} \rfloor \} \subset \Pi(\alpha)$.

**Proof.** Suppose we fix $p \neq 1$. Then, since $\frac{p}{q} > \frac{1}{\sqrt{2}-\alpha} > 1$ for $1 < \alpha < 2$, we have:

$$q \in \{1, 2, \ldots, p - 1\}.$$  \hspace{1cm} (2.21)

Note that since $\alpha \leq \frac{3}{2}$ we have that $\frac{1}{\alpha - 1} \geq 2$. If $p < \frac{1}{\alpha - 1}$ then $\frac{p}{p - 1} > \frac{1}{2 - \alpha}$. So, using (2.21), we have:

$$\frac{p}{q} > \frac{p}{p - 1} > \frac{1}{2 - \alpha}.$$  

Therefore $p \in \Pi(\alpha)$. \hfill \Box

Note that if $\alpha \to 1$ then $\lfloor \frac{1}{\alpha - 1} \rfloor \to \infty$. In the limit $\alpha = 1$ we get the Kepler problem in which all prime-period collisionless periodic orbits of period $T$ lie in the homotopy class $\Omega_T^1$ (they are ellipses).

**Proposition 2.3.3.** Subject to assumption $(I)$, for all $1 < \alpha < 2$ we have that:

(i) $\Pi(\alpha) \neq \emptyset$,

(ii) $|\Pi(\alpha)|$ is finite. More precisely, if $p \geq p_0 = \left(\frac{1}{\alpha - 2 + \sqrt{2 - \alpha}} + 1\right)^2$ then $p \notin \Pi(\alpha)$.

**Proof.** (i) If $1 < \alpha \leq \frac{3}{2}$ then $2 \in \Pi(\alpha)$ by lemma 2.3.5. If $\frac{3}{2} < \alpha < 2$ then $2 \in \Pi(\alpha)$ by lemma 2.3.4. So suppose that $\frac{3}{2} < \alpha \leq \frac{3}{2}$. We claim that in this case $p = \frac{4}{\alpha} \in \Pi(\alpha)$. To see this note that for all $\frac{3}{2} < \alpha \leq \frac{3}{2}$ we have $\Delta_\alpha \in (\sqrt{2}, 4) = I$. If $q = 1$ we get $\Delta_\alpha = \frac{2}{\alpha} = \frac{4}{\alpha} \notin I$. We can’t take $q = 2$ as then it is not coprime to $p$. If $q \geq 3$ then $\frac{p}{q} \leq \frac{4}{3} < \sqrt{2}$ so $\frac{p}{q} \notin I$.

(ii) We do not distinguish between $x$ and $\lfloor x \rfloor$, since only orders of magnitude matter. Consider fixing $p$. To begin with we are interested in the number of $q$ such that $q, p$ coprime and $\frac{a}{p} \leq q \leq \frac{b}{p}$. Suppose that $p = P_1^{\gamma_1}P_2^{\gamma_2}\ldots P_N^{\gamma_N}$, where $P_i$ is prime for all $1 \leq i \leq N$. We have that for all $1 \leq i \leq N$, the number of $pa \leq q \leq pb$ with $q$ not divisible by $P_i$ is $\approx p(b - a) - \frac{b - a - \gamma_i}{P_1} = (1 - \frac{1}{P_i})p(b - a)$. Therefore the number of $q, p$ coprime such that $\frac{a}{p} \in [a, b]$ is $\approx p(b - a)\prod_{d \mid p}(1 - \frac{1}{d})$ where $d$ runs over the prime-divisors of $p$. Letting $d$ run over all primes $\leq p$,

$$\prod_{d \mid p}(1 - \frac{1}{d}) \geq \prod_{d \leq p}(1 - \frac{1}{d}).$$

Now, since $p$ admits at most one prime divisor $> \sqrt{p}$,

$$\prod_{d \leq \sqrt{p}} \left(1 - \frac{1}{d}\right) \geq \prod_{d \leq \sqrt{p}} \left(1 - \frac{1}{\sqrt{d}}\right) \prod_{d \leq \sqrt{p}} \left(1 - \frac{1}{d}\right).$$

We note that

$$\prod_{d \leq \sqrt{p}} \left(1 - \frac{1}{d}\right) \geq \prod_{k \leq \sqrt{p}} \left(1 - \frac{1}{k}\right) = \frac{1}{\sqrt{p}},$$

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where $k$ now runs over all natural numbers $\leq \sqrt{p}$, and greater than 1. Thus we have that $c_p \geq (\sqrt{p} - 1)(b - a)$ which is certainly $\geq 1$ for sufficiently large $p$. The argument gives an effective $p_0$ such that $c_p \geq 1 \forall p \geq p_0$ : it is enough to take

$$p_0 = \left( \frac{1}{b - a} + 1 \right)^2.$$

Notice the equivalence $a \leq \frac{b}{p} \leq b \Leftrightarrow \frac{1}{b} \leq \frac{p}{q} \leq \frac{1}{a}$. Finally we choose $b = \sqrt{2 - \alpha}$ and $a = 2 - \alpha$. Then we get that $p_0 = \left( \frac{1}{\alpha - 2 + \sqrt{2 - \alpha}} + 1 \right)^2$, from which the result follows.

Note that $(\alpha - 2 + \sqrt{2 - \alpha} + 1)^2 \to \infty$ as $\alpha \to 1$ or $\alpha \to 2$ but that $(\alpha - 2 + \sqrt{2 - \alpha} + 1)^2$ is finite for all $1 < \alpha < 2$. Therefore given any $\alpha \in (1, 2)$ we only need check a finite number of possibilities to see if they are in $\Pi(\alpha)$.

**Example 2.3.1.** Take $\alpha = 1.3$. Then the above tells us that we only need check values of $p$ up to $[69.1791] = 69$. Checking these values one by one (by hand or using a computer) we find $\Pi(\alpha) = \{2, 3, 8, 10, 12, 20\}$.

**Example 2.3.2.** We can proceed in the same way as example 2.3.1 for other values of $\alpha$:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\Pi(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>${2, 3, 4, \ldots, 10, 20, \ldots, 70}$</td>
</tr>
<tr>
<td>1.2</td>
<td>${2, 3, 4, 5, 10, 12, 14, 18, 24, 30}$</td>
</tr>
<tr>
<td>1.3</td>
<td>${2, 3, 8, 10, 12, 20}$</td>
</tr>
<tr>
<td>1.4</td>
<td>${2, 5, 6, 9, 12}$</td>
</tr>
<tr>
<td>1.5</td>
<td>${2, 4, 6}$</td>
</tr>
<tr>
<td>1.6</td>
<td>${3, 4, 6, 10, 14}$</td>
</tr>
<tr>
<td>1.7</td>
<td>${4, 6, 10}$</td>
</tr>
<tr>
<td>1.8</td>
<td>${2, 6}$</td>
</tr>
<tr>
<td>1.9</td>
<td>${2, 3, 12}$</td>
</tr>
<tr>
<td>1.98</td>
<td>${2, 3, 4, 5, 6, 7}$</td>
</tr>
<tr>
<td>1.99</td>
<td>${2, 3, 4, 5, 6, 7, 8, 9}$</td>
</tr>
</tbody>
</table>

Note how $|\Pi(\alpha)|$ increases as $\alpha \to 1^+$ or $\alpha \to 2^-$.

**2.3.7 The Birth and Death of Periodic Orbits as $\alpha$ is varied.**

Suppose we fix $p$ and $q$ and $T$ and let $\alpha \to \frac{2p-q}{p}^+$. Then $\frac{1}{2 - \alpha} \to \frac{p}{q}^+$. The $c$- and $h$-axes therefore move closer to the contour $\Delta_\alpha = \frac{p}{q}$. We therefore have two possibilities for our orbit: either we tend towards an ejection-collision orbit ($c = 0$) or else we tend towards an unbounded orbit ($h = 0$). Since we have fixed $T$ at a finite value we see that the latter case is not possible because as $h \to 0$, $T \to \infty$. Thus we tend towards a ejection-collision orbit with $q \tau$-oscillations.
Remark 2.3.7. We chose to fix $T$ when varying $\alpha$. This meant that the c-axis (rather than the h-axis) approached the contour $\Delta = \frac{2\pi}{\alpha}$ (as $\alpha \to \frac{2\pi^2 - q^2}{p^2}$) and so we tended towards collision-ejection orbits rather than unbounded orbits. However we could have fixed $c$. The action coefficients would be the same but we would tend towards an unbounded orbit instead.

Next we examine what happens if we fix $p, q$ and $T$ and then let $\alpha \to \frac{2\pi^2 - q^2}{p^2}$. Then $\frac{1}{\sqrt{2-\alpha}} \to \frac{2\pi}{q}$. The curve $c^2 = \frac{(\alpha - 2\pi^2\alpha^2)}{2\pi^2 \alpha}$ gets closer to the contour $\Delta = \frac{2\pi}{q}$. We therefore tend towards a circular orbit that winds around the origin $p$ times in time $T$.

2.3.8 Bifurcations

We investigate the 'birth' and 'death' of periodic orbits of type $\frac{p}{q}$ as $\alpha$ is increased from 1 to 2. At $\alpha = \frac{2\pi^2 - q^2}{p^2}$ an orbit of type $\frac{p}{q}$ bifurcates off the collision-ejection orbit (which exists for all $\alpha$) with $q$ r-oscillations; the $\frac{p}{q}$ orbit is 'born'. However this orbit 'dies' when you reach $\alpha = \frac{2\pi^2 - q^2}{p^2}$ when it 'merges' with the circular orbit that winds around the origin $p$ times in one period. In other words, as $\alpha$ decreases there is a bifurcation at

$$\alpha = \frac{2\pi^2 - q^2}{p^2},$$

at which point an orbit of type $\frac{p}{q}$ branches off the circular orbit (which exists $\forall \alpha$) that winds around the origin $p$ times in one period.

In the next section we shall investigate what these bifurcations look like in a plot of action coefficients against $\alpha$.

2.3.9 Action coefficients of $\frac{p}{q}$ orbits

To be able to calculate the action coefficient of a $\frac{p}{q}$ type orbit we employ equation (2.11). However there are two other kinds of prime-period orbits whose action coefficients cannot be calculated using equation (2.11). These are the circular orbit that winds around the origin once and the collision-ejection orbit. Once we know the action coefficients of these two orbits too, it becomes possible to calculate the action coefficient of any periodic orbit by using the fact that if you repeat an orbit $n$ times, then its action coefficient gets multiplied by $n^2\frac{2\pi}{\alpha}$.

2.3.10 Action coefficient of the circular orbit

Lemma 2.3.6. The action coefficient of the circular orbit in the homotopy class $\Omega_1$ is:

$$2\pi^2 \left(\frac{\alpha}{4\pi^2}\right)^\frac{2}{\alpha + 2} + \left(\frac{4\pi^2}{\alpha}\right)^\frac{2}{\alpha + 2}.$$  

Proof. We have that

$$A_T^\alpha = \int_0^T L_\alpha(t) dt = \int_0^{2\pi} L_\alpha(\theta) d\theta = \int_0^{2\pi} L_\alpha \frac{\|x(t)\|^2}{c} d\theta,$$

where

$$L_\alpha = \frac{1}{2} \left(\frac{d\|x(t)\|}{dt}\right)^2 + \frac{c^2}{2\|x(t)\|^2} + \frac{1}{\|x(t)\|^\alpha}.$$
However, for a circular orbit, \( \frac{d}{dt} \|x(t)\| = 0 \) which implies that \( \|x(t)\| \) is constant, say \( \|x(t)\| = r_c \) for all \( t \). Since \( c = r_c^2 \theta(t) \) we also get that \( \theta(t) \) is constant. Therefore \( \theta(t) = \frac{2\pi}{T} \) and so we get \( c = \frac{2\pi r_c^2}{T^2} \). Plugging this all in we get:

\[
\mathcal{A}_T^\alpha = \int_0^{2\pi} \left( \left( \frac{2\pi r_c^2}{T} \right)^2 \frac{1}{2r_c^2} + \frac{1}{r_c^\alpha} \right) \frac{T}{2\pi} \, d\theta,
\]

and after a small amount of manipulation we obtain

\[
\mathcal{A}_T^\alpha = \frac{2\pi^2 r_c^2}{T} + \frac{T}{r_c^\alpha}.
\]

Next, we need to find \( r_c \) in terms of \( T \). To do this note the effective potential of the system is \( V_{\text{eff}}(\|x\|) = \frac{1}{2r_c^2} - \frac{|\|x\||}{r_c^\alpha} \) and that we must have that \( V_{\text{eff}}'(r_c) = 0 \). Using this it is easy to show that \( r_c = \left( \frac{2\pi^2}{4\pi^2} \right)^{2/\alpha} \). Substituting this into (2.23) we obtain:

\[
\mathcal{A}_T^\alpha = \left( \frac{2\pi^2}{(4\pi^2)^{2/\alpha}} \right)^{2/\alpha} + \left( \frac{4\pi^2}{\alpha} \right)^{3/\alpha} \right) T^{2-\alpha/3}. \]

\[\square\]

**2.3.11 Action coefficient of the ejection-collision orbit.**

**Lemma 2.3.7.** If \( V \) satisfies the conditions of definition 1.1.2 then the action functional diverges (to \( +\infty \)) on any sequence of loops approaching a loop with collisions.

**Proof.** We take \( \sigma(x) \) as in definition 1.1.1. We have that if \( \sigma < \min\{1, R\} \) then:

\[
\frac{1}{2} \dot{\sigma}^2 - V(x) \geq \frac{1}{2} \dot{\sigma}^2 + \frac{1}{\delta(\sigma)^\alpha} \geq \frac{1}{2} \dot{\sigma}^2 + \frac{1}{2\sigma^\alpha} \geq \frac{\dot{\sigma}^2}{\sigma^\alpha} = 2|\frac{d}{dt} \ln(\sigma)|
\]

so \( \mathcal{A}_T^\alpha \geq \text{constant} \cdot \int d(\ln(\sigma)) \) which tends to infinity as \( \sigma \to 0 \). \[\square\]

For such potentials we can immediately exclude the possibility of minimizers of the action functional from having collisions. Potentials of the form \( -\frac{1}{|x|^\alpha} \) satisfy the criteria of definition 1.1.2 if and only if \( \alpha \geq 2 \). If \( \alpha < 2 \) it is possible for collision loops to have finite action, as we shall see below.

We comment here that ejection-collision orbits are not in \( \Omega_T^\alpha \); they are in \( \partial \Omega_T^\alpha \).

For an ejection-collision orbit we have \( \dot{c} = 0 \) and so the motion of the particle is along a straight line. Suppose the maximum distance of the particle from the origin is \( d \). Conservation of energy yields

\[
\frac{u^2}{2} = \|x\|^{-\alpha} - d^{-\alpha}.
\]

Now, since \( \mathcal{A}_T^\alpha \) is proportional to \( T^{\frac{2-\alpha}{3}} \), to find the action coefficient it is sufficient to calculate \( \mathcal{A}_T^\alpha \) for a single value of \( T \). The action coefficient will then be \( \Psi = \mathcal{A}_T^\alpha T^{\frac{2-\alpha}{3}} \). A single value of \( T \) corresponds to a single value of \( d \). For convenience we set \( d = 1 \). This gives, from (2.24), that
\[ v = \sqrt{2} \sqrt{x^{-\alpha} - 1}. \]

We now compute \( T \) and \( A^\beta \) for \( d = 1 \). For \( T \) we have that:

\[ \frac{T}{2} = \int dt = \int_0^1 \frac{1}{v} d||x|| \quad \text{i.e.} \quad T = \sqrt{2} \int_0^1 \frac{1}{\sqrt{||x||^{-\alpha} - 1}} d||x||. \]

For \( A^\beta \) we have that:

\[ \frac{A^\beta}{2} = \int_0^\frac{T}{2} \left( \frac{v^2}{2} + \frac{1}{||x||^\alpha} \right) d||x|| \quad \text{i.e.} \quad A^\beta = \sqrt{2} \int_0^1 \left( \frac{||x||^{-\alpha}}{\sqrt{||x||^{-\alpha} - 1}} + \frac{||x||^{-\alpha}}{\sqrt{||x||^{-\alpha} - 1}} \right) d||x||. \]

Thus we have that:

\[ \Psi = \left( \sqrt{2} \int_0^1 \left( \frac{||x||^{-\alpha}}{\sqrt{||x||^{-\alpha} - 1}} + \frac{||x||^{-\alpha}}{\sqrt{||x||^{-\alpha} - 1}} \right) d||x|| \right) \left( \sqrt{2} \int_0^1 \frac{1}{\sqrt{||x||^{-\alpha} - 1}} d||x|| \right)^{\frac{3\alpha^2}{3^3}}. \quad (2.25) \]

Given \( 1 < \alpha < 2 \) we can calculate \( \Psi \) for a collision-ejection orbit by using equation (2.25) above and doing the integrations numerically.

In figure 2.5 a plot of action coefficients of the circular orbit and the collision-ejection orbit for \( 1 < \alpha < 2 \) is given. Notice that the action coefficient of the ejection-collision orbit is the same as the action coefficient for the circle only at \( \alpha = 1 \); this corresponds to the Kepler problem in which all periodic orbits have the same action coefficient.

2.3.12 The Action Spectrum

Richard Montgomery in [57] defined an action spectrum to be the value of the action of orbits plotted against some way of indexing those orbits. We look at the action spectrum of our orbits within each homotopy class. This fixes \( p \); we then index all the non-circular orbits by \( q \). Alternatively, we can index the orbits by their Morse indices as we will see in the next section.

The key to obtaining the action spectrum is understanding the behaviour of action coefficients at bifurcations. Given \( \alpha \) we are now in a position to calculate all the action coefficients of all the different types of orbit in a particular homotopy class. We start with some numerical investigations.

Let us look at the \( \frac{3}{2} \) type orbit. This exists if and only if \( \frac{3}{2} < \alpha < \frac{14}{9} \). For this domain we have plotted the action coefficient of the \( \frac{3}{2} \) orbit, the circular orbit that winds around the origin three times in one period, and the ejection-collision orbit with \( 2 r \)-oscillations, see figure 2.6.

We observe that:

- the action coefficients of the \( \frac{3}{2} \) orbit and the ejection-collision orbit with \( 2 r \)-oscillations are the same in the limit as \( \alpha \to \frac{3}{2} \), and
- the action coefficients of the \( \frac{3}{2} \) orbit and the circular orbit that winds around the origin three times in one period are the same in the limit as \( \alpha \to \frac{14}{9} \).
- the \( \frac{3}{2} \) orbit always has a lower action coefficient than both ‘3-circles’ and ‘2-collisions’.

Remark 2.3.8. We anticipate that the above results can be proved more generally by using splitting lemmas and Taylor coefficients calculations.
Figure 2.5: Action Coefficients of Circular orbit with 1 winding and Collision Orbit with 1 collision.
Figure 2.6: Action Coefficients of 3-circles, 2-collisions and ‘3/2’-orbit. Bifurcations occur at A and B.

Figure 2.7: Subharmonic bifurcations off ‘p-circles’ orbit. Morse indices in green.
In figure 2.7 the black curve is the action coefficient of the 'p-circles' orbit. The red curves are the plots of action coefficients for the '\( \frac{p}{q} \)' orbits. The numbers in green are the equivariant Morse indices - these will be explained later.

**Definition 2.3.7.** Let \( \Psi_{\alpha}(p, q) \) be the action coefficient of a \( \frac{p}{q} \) orbit in the system with potential \( -\|x\|^\alpha, \ 1 \leq \alpha \leq 2 \).

Motivated by figures 2.6 and 2.7 we make the following observation: if there exists a \( \frac{p}{q} \) orbit and a \( \frac{p}{q'} \) orbit of the system with potential \( -\|x\|^\alpha \) in the homotopy class \( \Omega_T^p \) with \( Q > q \) then \( \Psi_{\alpha}(p, Q) > \Psi_{\alpha}(p, q) \).

This leads to the following conjecture:

**Conjecture 2.3.1.** Suppose we fix \( T > 0, \ 1 < \alpha < 2 \) and \( p \in \mathbb{Z} - \{0\} \). Then the orbits in the homotopy class \( \Omega_T^p \) in order of increasing action are:

\[
\frac{p}{q_{\text{min}}}, \ \ldots \ \frac{p}{q_{\text{max}}} \ \text{and} \ \text{`p-circles'}
\]

where \( q_{\text{min}} = \lfloor p(2 - \alpha) \rfloor \) and \( q_{\text{max}} = \lfloor q(2 - \alpha) \rfloor \).

**Remark 2.3.9.** There are \( \lfloor q(2 - \alpha) \rfloor - \lfloor p(2 - \alpha) \rfloor + 2 \) distinct critical orbits of \( \mathcal{A}_T^\alpha \) on \( \Omega_T^p \).

### 2.3.13 Morse Indices

**Definition 2.3.8.** Suppose that \( x \) is a periodic orbit. Let the Morse index of \( x \) be the maximal positive integer \( m \) such that the Hessian of \( \mathcal{A}_T^\alpha \) at \( x \) is negative definite on an \( m \)-dimensional subspace.

**Remark 2.3.10.** If you wanted to apply Morse theory and in particular the equivariant Morse inequalities you require that there exists a symmetry group \( G \) such that given any critical point \( x \) you have:

\[
\mathcal{A}_T^\alpha(x \cdot y) = \mathcal{A}_T^\alpha(x) \quad \text{for all} \quad y \in G \quad \text{and} \quad T_y(G \cdot x) = \ker d^2 \mathcal{A}_T^\alpha(y) \quad \text{for all} \quad y \in G \cdot x. \quad (2.26)
\]

It is difficult to prove that condition (2.26) holds in general.

#### 2.3.13.1 Explicit calculation of the kernel of \( d^2 \mathcal{A}_T^\alpha \) on the 'p-circles' orbits

**Proposition 2.3.4.** If \( x_p^\alpha \) is the circular periodic orbit in the homotopy class \( \Omega_T^p \) then

\[
\dim \ker d^2 \mathcal{A}_T^\alpha(x_p^\alpha) = \begin{cases} 3 & \text{if} \quad \alpha = 2 - \left( \frac{j}{p} \right)^2, \quad j \in \{1, \ldots, p\} \\ 1 & \text{otherwise.} \end{cases}
\]

**Remark 2.3.11.** One of the degenerate directions corresponds to phase-shiftings of the circular orbit (or, equivalently here, rotations).
Proof. Take 
\( x, u, v : \mathbb{R}/T \mathbb{Z} \to \mathbb{R}^2 \);

Let \( \langle ., . \rangle \) to be the scalar product on real vectors. We compute

\[
    dA_T^x(x)(u) = \int_0^T dt \left( \langle \dot{u}, \dot{x} \rangle - \alpha \frac{\langle x, u \rangle}{|x|^{\alpha+2}} \right)
\]

and

\[
    d^2A_T^x(x)(u, v) = \int_0^T dt \left( \langle \dot{u}, \dot{v} \rangle - \alpha \frac{\langle v, u \rangle}{|x|^{\alpha+2}} + \alpha(\alpha + 2) \frac{\langle x, u \rangle \langle x, v \rangle}{|x|^{\alpha+4}} \right).
\]

For \( x_p^\alpha \) a circular solution of radius \( p \) we have:

\[
    d^2A_T^x(x_p^\alpha)(u, v) = \int_0^T dt \left( \langle \dot{u}, \dot{v} \rangle - \alpha \frac{\langle v, u \rangle}{p^{\alpha+2}} + \alpha(\alpha + 2) \frac{\langle x_p^\alpha, u \rangle \langle x_p^\alpha, v \rangle}{p^{\alpha+4}} \right). \tag{2.27}
\]

Now for circular solutions that wind around the origin \( p \) times in period \( T \) we have:

\[
    \rho \omega^2 = V'(\rho) = \frac{\alpha}{\rho^{\alpha+1}}, \quad \omega = \frac{2\pi p}{T}. \tag{2.28}
\]

This gives:

\[
    \rho = \left( \frac{\alpha}{\omega^2} \right)^{\frac{1}{\alpha+2}}. \tag{2.28}
\]

Substituting equation (2.28) into equation (2.27) we get that:

\[
    d^2A_T^x(x_p^\alpha)(u, v) = J_1 + J_2,
\]

where

\[
    J_1 = \int_0^T dt \left( \langle \dot{u}, \dot{v} \rangle - \omega^2 \langle u, v \rangle \right), \quad J_2 = \int_0^T dt \left( \alpha(\alpha + 2) \frac{\omega^2}{\alpha} \langle x_p^\alpha, u \rangle \langle x_p^\alpha, v \rangle \right). \tag{2.29}
\]

Now \( \langle \dot{u}, \dot{v} \rangle = \frac{d}{dt} \langle \hat{u}, \hat{v} \rangle - \langle \hat{u}, \hat{v} \rangle \), so we can write the \( J_1 \) from equation (2.29) as:

\[
    J_1 = \langle \langle \dot{u}, \dot{v} \rangle \rangle - \int_0^T \langle \langle \hat{u}, \hat{v} \rangle + \omega^2 \langle u, v \rangle \rangle \ dt.
\]

From the periodicity of \( u \) this is

\[
    J_1 = - \int_0^T \langle \langle \hat{u}, \hat{v} \rangle + \omega^2 \langle u, v \rangle \rangle \ dt.
\]

As for \( J_2 \) we first note that:

\[
    \langle x_p^\alpha, u \rangle \langle x_p^\alpha, v \rangle = \langle (x_p^\alpha, u) x_p^\alpha, v \rangle
\]

because \( \langle x_p^\alpha, u \rangle \) is a scalar. So:

\[
    J_2 = \int_0^T dt \left( \alpha(\alpha + 2) \langle x_p^\alpha, u \rangle \langle x_p^\alpha, v \rangle \left( \frac{\omega^2}{\alpha} \right)^{\frac{\alpha+2}{\alpha+4}} \right).
\]
Therefore we have:

\[ d^2 A_T^\alpha(x_p^\alpha)(u, v) = \int_0^T dt(-\ddot{u} - \omega^2 u + \alpha(\alpha + 2) \left( \frac{\omega^2}{\alpha} \right)^{\frac{\alpha+2}{\alpha+4}} \langle x_p^\alpha, u \rangle x_p^\alpha, v). \]

Now since

\[ \ker d^2 A_T^\alpha(x_p^\alpha) = \{ u : d^2 A_T^\alpha(x_p^\alpha)(u, v) = 0 \text{ for all } v \}, \]

we have that elements \( u \) of \( \ker d^2 A_T^\alpha(x_p^\alpha) \) are precisely the solutions of the 'kernel equation':

\[ -\ddot{u} - \omega^2 u + \alpha(\alpha + 2)(x_p^\alpha, u)x_p^\alpha \left( \frac{\omega^2}{\alpha} \right)^{\frac{\alpha+2}{\alpha+4}} = 0, \quad u(T) = u(0). \tag{2.30} \]

We search for solutions of (2.30). Firstly let us define:

\[ y(t) = (\cos(\omega t), \sin(\omega t)), \]

so that \( x(t) = py(t), \) and also take:

\[ z(t) = (-\sin(\omega t), \cos(\omega t)). \]

Note that \( \langle y(t), z(t) \rangle = 0 \) for all \( t. \) We aim to decompose \( u(t) \) as:

\[ u(t) = a(t)y(t) + b(t)z(t), \]

where \( a, b : [0, T] \to \mathbb{R}. \) Noting that:

\[ \dot{z}(t) = -\omega y(t), \quad \dot{y}(t) = \omega z(t), \]

we readily derive that:

\[ \dot{u}(t) = (a\omega + b)z + (\ddot{a} - b\omega)y, \quad \ddot{u}(t) = (2\omega \ddot{a} + \ddot{b} - \omega^2)z + (\ddot{a} - 2\ddot{b} \omega - \omega^2)y. \]

Without loss of generality we can take \( x_p^\alpha = py. \) We then have

\[ \langle x_p^\alpha, u \rangle = pa. \]

The kernel equation (2.30) becomes:

\[ -(2\omega \ddot{a} + \ddot{b} - \omega^2)z - (\ddot{a} - 2\ddot{b} \omega - \omega^2)y - \omega^2(ay + bz) + \alpha(\alpha + 2)\rho^2 ay \left( \frac{\omega^2}{\alpha} \right)^{\frac{\alpha+2}{\alpha+4}} = 0. \]

Both the coefficients of \( z \) and of \( y \) must be zero. After a small amount of manipulation we obtain:

\[ \ddot{b} + 2\omega \dot{a} = 0, \quad -\ddot{a} + 2\ddot{b} \omega + \omega^2(\alpha + 2)a = 0. \]

Let us write \( p_a = \dot{a}, \quad p_b = \ddot{b} \) and let \( u = \begin{pmatrix} a \\ b \\ p_a \\ p_b \end{pmatrix}. \) Then \( \dot{v} = M(\alpha, \omega)v, \) where

\[ M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2(\alpha + 2) & 0 & 0 & 2\omega \\ 0 & 0 & -2\omega & 0 \end{pmatrix}. \]
The eigenvalues \( \lambda \) of this matrix are given by \( \det(M - \lambda I) = 0 \). One can easily check that this gives:

\[
\lambda^2(\lambda^2 + (2 - \alpha)\omega^2) = 0.
\]

This has two complex conjugate roots if and only if \( \alpha < 2 \). If \( \alpha < 2 \) we have eigenvalues \( 0, \pm \omega \sqrt{2 - \alpha} \).

The eigenspace corresponding to the eigenvalue 0 is \( E_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \) which corresponds to \( a(t) = 0 \ \forall t \) and \( b(t) = c \ \forall t \) where \( c \) is a constant. This gives

\[
u(t) = cx(t) = c(-\sin(\omega t), \cos(\omega t)).
\]

This variational direction corresponds precisely to phase-shifts of the \( p \)-circles orbit. To see this note that

\[
\frac{d}{dt} \rho \begin{pmatrix} \cos(\omega t + \epsilon) \\ \sin(\omega t + \epsilon) \end{pmatrix} = \rho \omega z(t).
\]

Imaginary eigenvalues give rise to periodic \( a(t) \) and \( b(t) \). However additional conditions on \( a(t) \) and \( b(t) \) need to be met because \( u(t) \) must be periodic of period \( T \). The eigenvalues \( \pm \omega \sqrt{2 - \alpha} \) give rise to two linearly independent solutions of angular speed \( \omega \sqrt{2 - \alpha} \). These are periodic with a period \( T \) (not necessarily minimal) if and only if

\[
\frac{2\pi}{\omega \sqrt{2 - \alpha}} = \frac{T}{j}, \quad j = 1, 2, 3, ...
\]

But here \( T = \frac{2\pi \rho}{\omega} \). We therefore obtain that

\[
\alpha = 2 - \left( \frac{j}{\rho} \right)^2.
\]

This agrees with the values of \( \alpha \) we expected bifurcations to take place at, see (2.22).

**Remark 2.3.12.** In the case \( \alpha = 1 \) i.e. the Kepler case the system is superintegrable. In this case we have \( j = p \). The period of the variations is \( \frac{T}{p} \) which is the same as the amount of time taken to perform a single circular winding. This is what we should expect as all bounded motions trace out ellipses in configuration space. Keplerian orbits of a given period form a three-dimensional manifold of loops. All the loops in this three-dimensional manifold have the same action. There are three parameters involved in describing the 3-dimensional manifold: rotation, eccentricity and phase.

Finding the eigenvectors corresponding to the imaginary eigenvalues yields:

\[
\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \sqrt{2 - \alpha} \sin(\omega t \sqrt{2 - \alpha}) \\ 2 \cos(\omega t \sqrt{2 - \alpha}) \end{pmatrix}, \quad \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \sqrt{2 - \alpha} \cos(\omega t \sqrt{2 - \alpha}) \\ -2 \sin(\omega t \sqrt{2 - \alpha}) \end{pmatrix}.
\]

We find that the two linearly independent solutions of (2.30) not corresponding to phase-shifts are given by:

\[
u_1(t) = \sqrt{2 - \alpha} \sin(\omega t \sqrt{2 - \alpha}) \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} + 2 \cos(\omega t \sqrt{2 - \alpha}) \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}
\]
and
\[ u_2(t) = \sqrt{2 - \alpha} \cos(\omega t \sqrt{2 - \alpha}) \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} - 2 \sin(\omega t \sqrt{2 - \alpha}) \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}. \]

### 2.3.13.2 Morse Indices

Let \( G \) be the group acting on \( \Omega_T^p \) \((p \neq 0)\) by phase-shifts. Note \( G \cong S^1 \). Define \( \Omega_{T,G}^p \equiv \Omega_T^p / G \). Let \([x]_G = \{ y \in \Omega_T^p : x = g \cdot y \text{ for some } g \in G \} \in \Omega_{T,G}^p \). We define \( A_{T,G}^p : \Omega_{T,G}^p \to \mathbb{R} \) by:

\[ A_{T,G}^p([x]_G) \equiv A^p(x). \]

This is well defined since \( A^p \) is \( G \)-invariant.

Note that if \( x_p^\alpha \) is a circular orbit in \( \Omega_T^p \) then \([x_p^\alpha]_G\) is a single point in \( \Omega_{T,G}^p \). The splitting lemma (see [45]) says that in a neighbourhood of \([x_p^\alpha]_G\) the function \( A_{T,G}^p \) can be split into a 'Morse' (meaning nondegenerate) piece on one set of variables and a degenerate piece on a different set of variables whose number is equal to \( \dim \ker d^2 A_{T,G}^p \). More precisely, there exists a neighbourhood \( N \) of \([x]_G \) and a local coordinate system \((y_i)_{i \in \mathbb{N}}\) such that \( y_i = 0 \) for all \( i \) corresponds to \([x_p^\alpha]_G\) and:

\[ A_{T,G}^p = f_\alpha(y_1, y_2) + \sum_{i \geq 3} \pm y_i^2 \]

holds throughout \( N \). We have that \( f_\alpha : \mathbb{R}^2 \to \mathbb{R} \) with:

\[ \frac{d^2}{dy_i^2} f_\alpha(y_1, y_2)|_{(0,0)} \begin{cases} = 0 & \text{for } i = 1, 2 \text{ when } \alpha = 2 - \left(\frac{j}{p}\right)^2 \text{ for some } j \in \{1, \ldots, p\} \\ \neq 0 & \text{for } i = 1, 2 \text{ otherwise.} \end{cases} \]

Tracking \( x_p^\alpha \) as \( \alpha \) is varied, for the Morse index to change it is necessary that \( \dim \ker d^2 A_{T,G}^p(x_p^\alpha) > 0 \). We have proved that this only occurs at bifurcation values. Therefore, as \( \alpha \) is varied, we have proved that inbetween bifurcations the Morse indices of the ‘\( p \)-circles’ orbits are constant.

So suppose that \( \alpha_c \) is a bifurcation value for \( \alpha \). We consider what happens to the function \( f_\alpha \) as we pass through \( \alpha_c \).

- For \( \alpha \) just less than \( \alpha_c \) we have two critical values of \( f_\alpha \): one for the ‘\( p \)-circles’ orbit and one for the ‘\( \xi \)’ orbit. Note that the circular orbits occupy a single point in the domain of \( f_\alpha \), namely the origin. On the other hand the ‘\( \xi \)’ orbits occupy a circle in \( \mathbb{R}^2 \), the corresponding periodic orbits being related by rotations. Our numerical evidence tells us that the critical value of the ‘\( \xi \)’ orbits is lower than that of the circular orbit*.

- For \( \alpha \) just greater than \( \alpha_c \) we just have one critical point of \( f_\alpha \) corresponding to \([x_p^\alpha]_G\).

In figure 2.8 we have depicted what happens to the local structure of \( f_\alpha \) as \( \alpha \) passes through \( \alpha_c \). We can see from this that there is an ‘index exchange’ of 2.

**Remark 2.3.13.** *A rigorous proof of statement may be possible using Taylor series expansions.*

We have found a way of tracking indices. However in order to actually label all the ‘\( p \)-circles’ orbits in \( \Omega_T^p \) with their Morse indices we need ‘somewhere to start’ i.e. know the Morse index of a particular orbit.
Lemma 2.3.8. When $\alpha = 2$ the circular orbits in $\Omega_T^p$ have Morse index 0.

Proof. We have seen that when $\alpha \geq 2$ we have a strong force problem - the action functional diverges to $+\infty$ as we approach collisions. Furthermore there is only one circle of critical points in each homotopy class; these critical points are constant-speed circular motions related by phase-shifts, see remark 2.3.1. They are precisely the critical points found by minimizing the action functional on $\Omega_T^p$ (i.e. the direct method). It follows that the ‘p-circles’ orbit must have Morse index 0 when $\alpha > 2$. 

Starting from lemma 2.3.8 and tracking the Morse index we can label all the ‘p-circles’ orbits in a homotopy class $\Omega_T^p$ with their Morse indices. If we further assume that the ‘$q$’ orbits are nondegenerate up to symmetries of the action functional (namely phase-shifts and rotations about the origin) then the Morse indices of ‘$q$’ orbits are constant along their branches and we can label them with their Morse index, see figure 2.7.

Definition 2.3.9. Let $q_{\min}$ be the smallest integer greater than $p(2 - \alpha)$. Let $q_{\max}$ be the largest integer smaller than $p\sqrt{2 - \alpha}$.

Conjecture 2.3.2. Fixing $\alpha$ and $T$ the periodic orbits in $\Omega_p$ of type ‘$q$’, in descending order of action, are:

‘$p$-circles’, ‘$\frac{p}{q_{\min}}$’ orbit, ‘$\frac{p}{q_{\min}+1}$’ orbit, ..., ‘$\frac{p}{q_{\max}}$’ orbit.

Further, the Morse index of a ‘$q$’ orbit is $2(q - 1)$ and the Morse index of the ‘$p$-circles’ orbit, say $m$, is given by $m = 2j$ if $2 - \left(\frac{j+1}{p}\right)^2 \leq \alpha < 2 - \left(\frac{j}{p}\right)^2$ for $j = 0, 1, ..., p - 1$.

As a corollary to conjecture 2.3.2 we have that the Kepler orbit has Morse index $2(p - 1)$.

Figure 2.8: Index exchange at a bifurcations: plots of $z = f_\alpha$. 
2.4 Conclusion

The central results of this chapter are:

- For $1 < \alpha < 2$ the list of those homotopy classes that do not contain a periodic solution of prime-period is of finite length. However the length of the list diverges as $\alpha \to 1^+$ or $\alpha \to 2^-$. 

- If we fix $T$ (the period), $1 < \alpha < 2$ and the homotopy class $\Omega_T^p$ then the orbits in that homotopy class in order of increasing action are:

$$\frac{p}{q_{\text{min}}} , \frac{p}{q_{\text{min}}+1} , \ldots , \frac{p}{q_{\text{max}}} \text{ 'p-circles'}$$

where $q_{\text{min}} = \lfloor p(2 - \alpha) \rfloor$ and $q_{\text{max}} = \lceil p\sqrt{2 - \alpha} \rceil$.

- We showed that the Morse index of a ‘q’ orbit is $2(q-1)$. We also showed how the Morse index of the ‘p-circles’ orbit varied with $\alpha$ and showed that the Morse index of a Keplerian orbit in $\Omega_T^p$ is $2(p-1)$.

This chapter raises a series of questions and aims:

- Prove the monotonicity conjecture for $\Delta_\alpha(c,h)$.

- Demonstrate the claims regarding the action spectrum mathematically, not just numerically. This will also complete the proof for the values of the Morse indices.

In chapter 4 we develop the analogous theory for the 2-centre problem.
Chapter 3

The Molecular 2-Body Problem

*God does not care about our mathematical difficulties; He integrates empirically.*
- Albert Einstein

Abstract

In this chapter we study the molecular 2-body problem with Lennard-Jones potential. We:

- Find explicitly $T_1 : \mathbb{N}^2 \to \mathbb{R}$ such that if $p, q \in \mathbb{N}$ are such that $\gcd(p, q) = 1$ then there do not exist any $\ell_q$ orbits for $0 < T < T_1(p, q)$ but there does exist $\eta > 0$ such that there are $\ell_q$ orbits for $T_1(p, q) < T < T_1(p, q) + \eta$ (see theorem 3.3.1).
- Give numerical evidence that the set of values of the periods of orbits of a type $\ell_q$ is bounded above.
- Find the region of energy-momentum space that supports the existence of periodic or quasiperiodic motion (see theorem 3.3.2).
- Give numerical evidence that for all non-null homotopy classes the least period on the set of all possible periodic motions in that class is given by a circular orbit.
- Show there exists $m \in \mathbb{N}$ such that for all $q \geq m$ there are periodic orbits in $\Omega^p \cup_{T>0} \Omega^p_T \ (p \in \mathbb{Z} - \{0\})$ with $q$ 'radial oscillations' in one period.

Some of the results are subject to assumptions motivated by numerical computations.

3.1 Introduction

In this chapter we initiate our study of the molecular $N$-body problem by studying the case $N = 2$, which is integrable. Indeed the molecular two body is readily shown to be equivalent to a central force problem with Lennard-Jones potential. It is integrable and is interesting to investigate in a similar manner to chapter 2. Apart from providing some interesting results this chapter also serves as an introduction to chapter 5 which investigates the existence of periodic solutions of the molecular $N$-body problem from a variational viewpoint.
3.1.1 The Lennard-Jones potential

The 2-body problem with identical bodies is equivalent to a central force problem. The Lennard-Jones potential of the central force problem is

$$V(x) = \frac{1}{\|x\|^{\beta}} - \frac{1}{\|x\|^{\alpha}}, \quad \beta > \alpha \geq 2.$$ 

Usually the values $\beta = 12$ and $\alpha = 6$ are used in modelling the motion of molecules. The potential has an attractive tail for large $\|x\|$, is strongly repulsive at short distances and for the $\beta = 12, \alpha = 6$ case reaches a minimum around 1.122. One of the key differences between the Lennard-Jones potential and the $-\frac{1}{\|x\|^{\alpha}}$ potential is that the Lennard-Jones potential is nonhomogeneous and does not enjoy any scaling symmetries. As we shall see consequences of this include the fact that the existence of periodic solutions depends on the value of the period (unlike for $-\frac{1}{\|x\|^{\alpha}}$).

3.1.2 Some notation

We will use polar coordinates $(r, \theta)$. We will denote by $p$ the number of $\theta$-oscillations in one period and by $q$ the number of $r$-oscillations in one period. An orbit with $p$ $\theta$-oscillations and $q$ $r$-oscillations will be called an orbit of type $\frac{p}{q}$. We have $\Delta = \frac{2\pi}{p} = \frac{2\pi}{q}$. The orbit is of prime-period if and only if $p$ and $q$ are coprime. A collisionless periodic orbit with $p$ oscillations is in a homotopy class $\Omega^p_T$.

3.2 Numerical observations

We consider the expressions for $\Delta$ and $T_\theta$ respectively given in (2.4) of chapter 2, taking $V(r) = r^{-12} - r^{-6}$. As was the case for $V(r) = -\frac{1}{r^6}$ the integrals are difficult to analyze analytically. Instead we plot $\Delta$ and $T_\theta$ as functions of $c$ and $h$. In figures 3.1 and 3.2 we have given contour plots of $\Delta(c, h)$ and $T_\theta(c, h)$. We have named the two curves shown in figure 3.1 'edge (I)' and 'edge (II)'. In figure 3.1 the contours are such that the closer the contour's shape is to that of edge (II), the higher the value of $\Delta$ that contour represents. In figure 3.2 the contours closer to the $h$-axis represent higher values of $T_\theta$. We make some observations:

- **Observation 1/Assumption (A):** For fixed $\Delta$, $T_\theta$ is minimized on edge (I).

- **Observation 2/Assumption (B):** The least value of $T_\theta$ is approximately 9.03 and occurs around $(c, h) \approx (1.05, 0.16)$, which is on edge (I).

An immediate consequence of Assumption (B) is that given any homotopy class $\Omega^p_T$, $p \neq 0$ we have that for sufficiently small periods (namely $0 < T < 9.03p$) there are no periodic solutions in the homotopy class $\Omega^p_T$. In chapter 5 this result is rendered superfluous: we prove the nonexistence of periodic solutions of the $N$-body molecular problem in any given tied homotopy class of loops for sufficiently small periods.
Figure 3.1: Contour Plot of $\Delta(c,h)$ for $\alpha = 6$, $\beta = 12$.

Figure 3.2: Contour Plot of $T_0(c,h)$ for $\alpha = 6$, $\beta = 12$. 
3.3 Bounds

3.3.1 A lower bound on the period for \( \frac{\alpha}{q} \) orbits.

Lemma 3.3.1. On edge (I), which corresponds to circular solutions, we have that

\[
\Delta^2 = \frac{\ar^\beta - \beta}{\beta(\beta - 2) - \alpha(\alpha - 2)r_0^{\beta - \alpha}}
\]  

(3.1)

where \( r_0 \) is the radius of the corresponding circle.

Proof. That edge (I) corresponds to circular orbits will become evident later (in our analysis of the function denoted by \( \cdot' \)). Using equation (2.14) of chapter 2, which applies to circular orbits, we have that

\[
\Delta = \frac{\omega_0}{\omega_r} = \sqrt{\frac{V'(r_0)}{r_0V''(r_0) + 3V'(r_0)}}.
\]  

(3.2)

Then using \( V(r_0) = r_0^\beta - r_0^{-\alpha} \) we get the desired result. □

Theorem 3.3.1. Subject to assumption (A),

\[
T_1(p, q) = \frac{2\pi}{\sqrt{\beta(\beta - \alpha)}} \left( \frac{\beta}{\alpha} \right)^{\frac{\beta+2}{2(\beta - \alpha)}} (q^2 + (\beta - 2)p^2)^{\frac{\beta+2}{2(\beta - \alpha)}} (q^2 + (\alpha - 2)p^2)^{-\frac{\beta+2}{2(\beta - \alpha)}}
\]  

(3.3)

is such that there do not exist any \( T \)-periodic \( \frac{\alpha}{q} \) orbits for \( 0 < T < T_1(p, q) \) but there exists \( \eta > 0 \) such that there are \( T \)-periodic \( \frac{\alpha}{q} \) orbits for all \( T \) such that \( T_1(p, q) < T < T_1(p, q) + \eta \).

Proof. We first calculate \( T_\theta \) as a function of \( \Delta \) along edge (I). To do this we first note that for circular orbits you can equate the centripetal force, \( V(r) \), with \( \frac{v^2}{r} \) where \( v \) is the constant speed of the particle. Doing this for the Lennard-Jones potential yields, after a small amount of working:

\[
T_\theta = \frac{4\pi^2 r_0^{\beta + 2}}{\alpha r_0^{\beta - \alpha} - \beta},
\]  

(3.4)

where \( r_0 \) is the radius of the circle. Now letting \( x = r_0^{\beta - \alpha} \) we have from equation (3.1) that:

\[
x = \frac{\beta(1 + (\beta - 2)\Delta^2)}{\alpha(1 + (\alpha - 2)\Delta^2)}.
\]  

(3.5)

Also from equation (3.4), we get that:

\[
T_\theta = \frac{4\pi^2 \alpha^{\beta+2}}{\alpha x - \beta}.
\]  

(3.6)

Substituting equation (3.5) into equation (3.6) gives:

\[
T_\theta = \frac{2\pi}{\Delta} \sqrt{\frac{1 + (\alpha - 2)\Delta^2}{\beta(\beta - \alpha)}} \left( \frac{\beta(1 + (\beta - 2)\Delta^2)}{\alpha(1 + (\alpha - 2)\Delta^2)} \right)^{\frac{\beta+2}{2(\beta - \alpha)}}.
\]  

(3.7)

Using \( \Delta = \frac{\alpha}{q} \), \( T = pT_\theta \) and assumption (A) we obtain (3.3). □
3.3.2 Edges (I) and (II)

In this subsection we calculate analytically the region of energy-momentum space that supports quasiperiodic or periodic motion. Integration of the Lennard-Jones CFP yields:

\[ p_r = \dot{r} = \pm r^{-\beta} \sqrt{2 hr^\beta - c^2 r^{\beta-2} + 2r^{\beta-\alpha} - 2}. \]

Define

\[ f(r) = 2hr^\beta - c^2 r^{\beta-2} + 2r^{\beta-\alpha} - 2. \]

Differentiating (3.8) we get:

\[ f'(r) = r^{\beta-1} \left( 2h r^{\alpha} - c^2 (2 - \alpha) r^{\alpha-2} + 2(\beta - \alpha) \right). \]

Consider

\[ g(r) = 2hr^{\alpha} - c^2 (2 - \alpha) r^{\alpha-2} + 2(\beta - \alpha). \]

Differentiating (3.9) we get:

\[ g'(r) = 2h \alpha r^{\alpha-1} - c^2 (\alpha - 2) (\beta - 2) r^{\alpha-3}. \]

So \( g'(r) = 0 \) if and only if \( r = c \sqrt{\frac{(\alpha - 2)(\beta - 2)}{2ha\alpha}} \).

3.3.2.1 Case: \( h < 0 \).

If \( h < 0 \) then \( g \) is monotonic since the only possible stationary point of \( g \) is at imaginary \( r \). Further noting that \( g(0) = 2(\beta - \alpha) \) and that \( g(r) \rightarrow -\infty \) as \( r \rightarrow \infty \) we see that there is only one value of \( r \) at which \( g(r) = 0 \) \( \iff \) \( f'(r) = 0 \). Therefore \( f \) has only one turning point.

Now \( f(0) = -2 \) and \( f \rightarrow -\infty \) as \( r \rightarrow \infty \), so we have three cases to consider:

(i) \( f \) doesn’t ever cut the \( r \)-axis in which case there is no orbit.

(ii) \( f \) touches the \( r \)-axis tangentially in one place in which case we have only circular orbits.

(iii) \( f \) cuts the \( r \)-axis in two places in which case we get a generic orbit.

We now try to find the equation of edge (I) for \( h < 0 \). To find this we need to analyze the critical case (ii). For circular orbits we have that, since \( \dot{r} = 0 \):

\[ 2hr^\beta - c^2 r^{\beta-2} + 2r^{\beta-\alpha} - 2 = 0, \]

and that, since \( \dot{\rho} = -V'_{\text{eff}}(r) = 0 \), we have:

\[ c^2 r^{\beta-2} - cr^{\beta-\alpha} + \beta = 0. \]

Adding equations (3.10) and (3.11) yields:

\[ 2hr^\beta - (\alpha - 2)r^{\beta-\alpha} + \beta - 2 = 0. \]
Equation (3.12) is in general difficult to solve analytically. However if we take the typical values of \( \alpha \) and \( \beta \) used in the Lennard-Jones model, namely \( \alpha = 6 \), \( \beta = 12 \) then we obtain a quadratic in \( r^6 \). This is readily solved and we get that

\[
r^6 = \frac{1 \pm \sqrt{1 - 5h}}{h}.
\]

(3.13)

Now, since we are dealing with the case \( h < 0 \) here we have that \( 1 - 5h > 1 \) and therefore the negative root of equation (3.13) is not possible for real \( r \). Hence the radius, \( r_0 \), say, of the circle is:

\[
r_0 = \left(1 - \frac{\sqrt{1 - 5h}}{h}\right)^{\frac{1}{6}}.
\]

(3.14)

The energy, \( h \), of the circular orbit is:

\[
h = \frac{c^2}{2r_0^2} + \frac{1}{6} - \frac{1}{r_0^6},
\]

(3.15)

since \( r = 0 \) for circular orbits. Manipulating equation (3.15) we readily obtain:

\[
c^2 = \frac{2}{16} (h r_0^2 + r_0^6 - 2).
\]

(3.16)

Substituting \( r_0 \) given by equation (3.14) into equation (3.16) then gives:

\[
c = \left(\frac{h}{1 - \sqrt{1 - 5h}}\right)^{\frac{1}{6}} \sqrt{\frac{6}{h}(1 - \sqrt{1 - 5h} - 2h)},
\]

(3.17)

which is the equation of edge (1) for \( h < 0 \).

**Remark 3.3.1.** Note that as \( h \to 0 \), \( \frac{h}{1 - \sqrt{1 - 5h}} = \frac{h}{1 - (1 - 2.5h + O(h^2))} \to 0.4 \). Therefore, from equation (3.17), \( c \to (\frac{2}{5})^\frac{1}{6} \sqrt{3} \approx 0.807 \), which agrees with the numerical plot of the edge. Next we find the value of \( h \) on edge (1) when \( c = 0 \). Since \( h \neq 0 \), we must have \( 1 - \sqrt{1 - 5h} - 2h = 0 \). This implies \( h = -0.25 \), which also agrees with the numerical plot. This is reassuring and tells us that the minimum possible energy of a periodic orbit in the LJ-potential is \( -0.25 \).

**3.3.2.2 Case: \( h > 0 \).**

In the following when we refer to the function \( f \) we will implicitly mean \( f \) restricted to the domain \( r > 0 \). If, for example, we say \( f \) has two turning points what we will actually mean is that \( f \) has two turning points for \( r > 0 \); there may be turning points for \( r < 0 \) but they are not of interest to us as they have no physical meaning.

If \( h > 0 \) then we do have a real value of \( r \), say \( r^* \) at which \( g'(r^*) = 0 \). In fact

\[
r^* = c \sqrt{\frac{(\alpha - 2)(\beta - 2)}{2h\alpha\beta}}.
\]

You can easily show \( g''(r^*) = 2 \left(2h\alpha\beta \right)^{-\frac{3}{2}} \left(c^2(\alpha - 2)(\beta - 2)\right)^{\frac{1}{2}} > 0 \) and so \( r^* \) is a minimum of \( g \).

We also know that \( g(0) = 2(\beta - \alpha) > 0 \) and \( g \to \infty \) as \( r \to \infty \). There are three cases to consider:
(i) $g(r^*) < 0$: then $g$ and so $f'$ has two roots and hence $f$ has two turning points.

(ii) $g(r^*) = 0$: then $g$ and so $f'$ has just one root and hence $f$ has one turning point.

(iii) $g(r^*) > 0$: then $g$ and so $f'$ has no roots and hence $f$ is monotonic. (In fact monotonically increasing).

We recall that $f(0) < 0$.

Clearly case (iii) does not yield bounded orbits. So let's look at the generic case (i) and consider what happens to the graph as we vary $c$ whilst keeping $h$ fixed.

Figure 3.3: Sketch of the function $f$ for $r > 0$

For a typical bounded periodic orbit with $h > 0$, $f$ will look like figure 3.3(a). Increasing $c$ to a ‘critical’ state, the picture will look like figure 3.3(b). In this situation the only possible bounded periodic orbit is the circular orbit. It will have radius $r_0$, where $r_0^2 = \frac{1 - \sqrt{1 - 5h}}{h}$. At $r = r_0$, $\dot{r} = 0$ and plugging this into the hamiltonian gives equation (3.17) again, corresponding to the largest possible value of $c$ for a given $h$. Therefore equation (3.17) holds on edge (I) for both $h < 0$ and $h \geq 0$.

Alternatively, returning to figure 3.3(a), decreasing $c$ to a critical state the picture will look like figure 3.3(c) where $r_1^2 = \frac{1 + \sqrt{1 - 5h}}{h}$. At $r = r_1 = (\frac{1 + \sqrt{1 - 5h}}{h})^{1/4}$ we have that $\dot{r} = 0$. Substituting this into $h = \frac{1}{2}r^2 + \frac{c^2}{2r^2} + \frac{1}{r^2} - \frac{1}{r^4}$ gives, after a small amount of working that:

$$c^2 = \frac{6h^3}{(1 + \sqrt{1 - 5h})^{3/2}} (1 - 2h + \sqrt{1 - 5h}).$$

This gives the relation between $c$ and $h$ when $c$ is as small as possible for a given $h$. It therefore is the equation of edge (II) in figures 3.1 and 3.2.
Remark 3.3.2. Note that in (3.18) as $h \to 0$, $1 + \sqrt{1 - 5h} \to 2$ and therefore $c^2$ and hence $c$ tends to 0. Therefore edge (II) goes through the origin, which is consistent with figures 3.1 and 3.2. Also note that when $h = 0.2$ we get that $c^2 = 6(0.2)^{3/2}(0.6) \Rightarrow c \approx 1.11$ which agrees with figures 3.1 and 3.2.

We obtain the following theorem:

**Theorem 3.3.2.** All periodic and quasiperiodic orbits have values $(c, h)$ in the region of energy-momentum space $R$ defined by:

$$R = R_1 \cup R_2,$$

where

$$R_1 = \{(c, h) : -0.25 < h \leq 0, \quad 0 < c^2 < \left( \frac{h}{1 - \sqrt{1 - 5h} - 2h} \right)^{3/2} \frac{6}{h}(1 - \sqrt{1 - 5h} - 2h)\}$$

and

$$R_2 = \{(c, h) : 0 < h < 0.2, \quad \frac{6h^{3/2}}{(1 + \sqrt{1 - 5h})^{3/2}}(2h + \sqrt{1 - 5h}) < c^2 < \left( \frac{h}{1 - \sqrt{1 - 5h}} \right)^{3/2} \frac{6}{h}(1 - \sqrt{1 - 5h} - 2h)\}.$$

Moreover given any $(c, h) \in R$ there exists a (quasi)periodic orbit with these values of $c$ and $h$.

3.3.3. *Minimizing $T_q$*

Based on our numerical evidence we shall assume that $T_q(c, h)|_R$ is minimized by a point on edge (I). Along edge (I) we have that:

$$T = T_q = \frac{2\pi p}{\theta} = \frac{2\pi r^2}{c}.$$

We know $c(h)$ for all $-0.25 < h \leq 0.2$; this is given by equation (3.17). We know $r(h) = \left( \frac{1 - \sqrt{1 - 5h}}{h} \right)^{1/2}$ for $h < 0$. Also $r(0) = \lim_{h \to 0} \left( \frac{1 - \sqrt{1 - 5h}}{h} \right)^{1/2} \approx 1.165$. For $h > 0$ as we are interested in the least period we take the smaller value of the two values $\left( \frac{1 - \sqrt{1 - 5h}}{h} \right)^{1/2}$ which, since we are assuming $h > 0$, still corresponds to taking $r = \left( \frac{1 - \sqrt{1 - 5h}}{h} \right)^{1/2}$. We therefore have the following expression for $T(h)$ that applies for $-0.25 \leq h \leq 0.2$:

$$T(h) = 2\pi p \left( \frac{1 - \sqrt{1 - 5h}}{h} \right)^{7/6} \left( \frac{6}{h} \left( 1 - \sqrt{1 - 5h} \right) - 12 \right)^{-1/2}. \quad (3.19)$$

Minimizing this with respect to $h$ gives a minimum value of $T(h)$ of 9.032p (to 3 decimal places) when $h = 0.163$ (to 3 decimal places). Substituting this into equation (3.17) gives $c = 1.056$ (to 3 d.p.). Note that these values match very closely the values found in Observation 2.
3.3.4 Number of radial oscillations in one period.

Lemma 3.3.2. We have \( \inf \{ \Delta(c, h) : (c, h) \in R \} = 0 \).

Proof. By (3.7) we have on edge (I) that:
\[
T_0 = \frac{2\pi}{\Delta} \sqrt{\frac{1 + 4\Delta^2}{72} F(\Delta)}
\]  \tag{3.20}

where
\[
F : (0, \infty) \to \mathbb{R}; \quad \Delta \mapsto \frac{2(1 + 10\Delta^2)}{1 + 4\Delta^2}.
\]

Note that \( F \) is continuous and that
\[
F(\Delta) \to \begin{cases} 5 & \text{as } \Delta \to \infty \\ 2 & \text{as } \Delta \to 0. \end{cases}
\]

Therefore \( F \) is bounded above. Thus from (3.20) there exists \( k > 0 \) such that:
\[
T_0 < k \sqrt{\frac{1}{\Delta^2 + 4}}
\]

which implies that
\[
\Delta < \frac{1}{\sqrt{72T_0^2/k^2}} - 4.
\]  \tag{3.21}

However from (3.19) along edge (I) we have
\[
T_0 = \frac{T(h)}{p} \to \infty \quad \text{as } h \to -0.25.
\]

Therefore by (3.21) we have on edge (I) that \( \Delta \to 0 \) as \( h \to -0.25 \).

Definition 3.3.1. Let \( \Omega_p = \bigcup_{T > 0} \Omega_p^T \) where \( \Omega_p^T \) is as defined in subsection 2.2.5.

Corollary 3.3.1. There exists \( m \in \mathbb{N} \) such that there are periodic orbits in \( \Omega_p \) with \( q \in \mathbb{N} \) radial oscillations for all \( q > m \).

Proof. Given \( p \in \mathbb{N} \) and any \( d > 0 \) we have \( \frac{p}{q} \in (0, d) \) for all \( q \in \mathbb{N} \cap \left( \frac{p}{d}, \infty \right) \). \qed

Remark 3.3.3. The numerical evidence suggests:

- \( \sup \{ \Delta(c, h) : (c, h) \in R \} \approx 0.9306 \). Thus a periodic orbit with \( q \) \( \tau \)-oscillations in \( \Omega_p \) has \( \frac{p}{q} < 0.9306 \). We can take \( q \in \mathbb{N} \cap (p/0.9306, \infty) = \mathbb{N} \cap (1.0746p, \infty) \) i.e. \( q \in \{ [1.0746p], [1.0746p] + 1, \ldots \} \). So we can take \( m = [1.0746p] \) in corollary 3.3.1.

- The number of different types of \( \frac{p}{q} \) orbits in \( \Omega_p^T \) is finite for any given \( T > 0 \). This is because \( p \) is fixed (by the homotopy class), \( T_0 = T/p \) is fixed by the period and \( p \), and for fixed \( T_0 \) our numerics give that \( \Delta \) is bounded below by a strictly positive number, say \( c_T > 0 \). However the number of different types of \( \frac{p}{q} \) orbits in \( \bigcup_{T \leq \tau} \Omega_p^T \) diverges as \( \tau \to \infty \) because \( c_T \to 0 \) as \( T \to \infty \) by the proof of lemma 3.3.2.
3.3.5 Existence of an upper bound on the period of a $i^E_q$ orbit

Note from equation (3.19) that $\lim_{h \to -0.25} T(h) = \infty$ and that $T(h)$ is finite for $-0.25 < h \leq 0.2$. Using figure 3.2 we can use the contours to infer that along edge (II) we have that $T \to \infty$ as we approach the origin but is finite everywhere along edge (II) apart from the origin. Combining this with the evidence from figure 3.1 we see that for a given $i^E_q$ orbit there is a finite upper bound on the periods supported as well as a lower bound. In other words, fixing the homotopy class, i.e. fixing $p$, we have that $i^E_q$ type orbits exist if and only if

$$T_1(p, q) < T < T_2(p, q)$$

where $T_2(p, q) < \infty$ but $T_2(p, q) \to \infty$ as $q \to \infty$. 
Chapter 4

Aspects of the Planar Symmetric Newtonian Two Centre Problem

Abstract

In this chapter we look at various aspects of the planar Newtonian two centre problem. In particular, we classify all of its periodic solutions into homology classes of loops.

4.1 Introduction

To begin with we give a brief background to the Newtonian two centre problem:

• The study of the motion of a point mass moving in the gravitational field of two fixed attracting centres in the plane is a problem proposed and integrated in the 18th century by Euler in a series of three papers, see [32], [33] and [34]. Almost a century later Jacobi showed the potential is also separable in the full 3-dimensional case, see [42]. In 1949 Erikson and Hill found explicitly the third integral of motion of the 3-dimensional case, i.e. the constant of motion that was not the energy nor the angular momentum about the axis passing through both centres, see [31].

• Since then the problem has been used as a model of many systems, including:
  – satellite trajectories, see [49] and [50].
  – the quantum mechanical problem of the $H_2^+$ ion, see for example [30] and [68]. In the Born-Oppenheimer approximation (BOA) the two centres are atomic nuclei and the test particle is an electron.

• Recently in [72] a fairly comprehensive analysis of the problem was given both classically and quantum mechanically. However aspects such as the classification of orbits into homotopy or homology classes of loops in the planar case or the ordering of orbits within homotopy classes by action were not addressed.

There are some basic facts we need to know before describing the key elements of this chapter:
• In this chapter we restrict to the planar two centre problem. The configuration space $Q$ is the plane $\mathbb{R}^2$ with two distinct points removed. This has fundamental group $F_2$ - the free group on two generators. As generators we will take $\alpha_1$ and $\alpha_2$ where $\alpha_j$ is a single clockwise loop about the $j^{th}$ centre. The connected components of loop space (or 'free homotopy classes') correspond to the conjugacy classes of $F_2$. Collision loops are considered to exist on the boundaries of the free homotopy classes.

• The planar Newtonian two centre problem is integrable in elliptic coordinates $(\xi, \eta) = (r_1 + r_2, r_1 - r_2)$ where $r_1$ and $r_2$ are the distances of the test particle from the left and right hand centres respectively. There are two corresponding constants of motion denoted by $H$ (the energy) and $C$.

• Periodic motion corresponds to the ratio of angular frequencies $\Delta = \frac{\omega_1}{\omega_2}$ being rational. We call a prime-period orbit with $\Delta = \frac{p}{q}$, $\gcd(p, q) = 1$, an orbit of type $\frac{p}{q}$. An orbit is said to be of type $\frac{p}{q}$ where $\gcd(p, q) = 1$ if it is $r$ repeats of a $\frac{p}{q}$ orbit.

• The energy-momentum map is defined as $P : T^*Q \to \mathbb{R}^2$ with $P = (C, H)$ so that $(q, p) \mapsto (C, H)(q, p) = (c, h)$.

• Periodic solutions correspond to critical points of the action functionals

$$A_T = \int_0^T L dt, \quad T > 0$$

on loop space. Here $L = K.E. - P.E.$ is the Lagrangian.

The key elements of this chapter are:

• Following Charlier (1902), Deprit (1962) and Strand and Reinhardt (1979) we show in how the orbits can be classified into three 'types.' We shall give precise descriptions. However, loosely speaking, the different types are defined by the properties that points on the orbits all

- $P1$: lie in an elliptic annulus encircling the two centres.
- $P2$: lie within a simply connected region containing both centres.
- $P3$: lie within one of two disjoint regions, each region containing only one of the centres.

• We extend the basic classification of periodic orbits into $P1$, $P2$ and $P3$ types further. We show that the integrability of the system places strong constraints on which free homotopy classes contain periodic solutions, see theorem 4.4.1. This is analogous to the approach taken in chapter 2 but more complicated.

• The existence results for orbits in classes so far discussed implicitly have an unrestricted choice of the period $T > 0$. In section 4.6, we explore the relationship between period and existence for $P1$ orbits. For a given period we list those $P1$ orbits that exist in a class and order them according to their action. We call this the 'action spectrum.' The situation is described carefully with the aid of plots but no proofs are made. We also conjecture the values of the Morse indices of $P1$ orbits by using bifurcation theory and the index exchange technique used in chapter 2.
4.2 Integration

4.2.1 Coordinate system

The symmetric planar NTCP (Newtonian two centre problem) is integrable in elliptic coordinates. Firstly we need to understand this coordinate system. Elliptic coordinates $\xi$ and $\eta$ are defined as

$$\xi = r_1 + r_2, \quad \eta = r_1 - r_2$$

where $r_1$ and $r_2$ are the distances to the left-hand centre and right-hand centres respectively. Therefore

$$r_1 = \frac{1}{2}(\xi + \eta), \quad r_2 = \frac{1}{2}(\xi - \eta).$$

Consider Cartesian coordinates $(x,y)$ where the left-centre is at $(-1,0)$ and the right-centre is at $(1,0)$. Then we have that

$$r_1^2 = (x + 1)^2 + y^2, \quad r_2^2 = (x - 1)^2 + y^2.$$  

Rearranging for $x$ and $y$ gives

$$x = \frac{1}{4} \xi \eta, \quad y = \pm \frac{1}{4} \sqrt{(4 - \eta^2)(\xi^2 - 4)}.$$

Remark 4.2.1. Whilst specifying $\xi$ and $\eta$ uniquely determines $x$, it only determines $y$ up to a sign.

Definition 4.2.1. Borrowing notation from atomic physics we will sometimes call

- the centres protons
- the sets

$$\{ (\xi, \eta) : \eta = -2 \} = \{(x,y) : y = 0, x \leq -1\}, \quad \{ (\xi, \eta) : \eta = +2 \} = \{(x,y) : y = 0, x \geq +1\}$$

the extraproton axes, and

- the set

$$\{ (\xi, \eta) : \xi = 2 \} = \{(x,y) : y = 0, -1 \leq x \leq 1\}$$

the interproton axis.

4.2.2 Separability

We aim to express the Hamiltonian function in elliptic coordinates. Here we follow V. I. Arnold’s exposition, see [4].

Definition 4.2.2. Let $ds$ be the Euclidean distance metric in configuration space $Q$.

Lemma 4.2.1.

$$ds(\xi, \eta)^2 = \frac{(\xi^2 - \eta^2)}{4(\xi^2 - 4)} d\xi^2 + \frac{(\xi^2 - \eta^2)}{4(4 - \eta^2)} d\eta^2.$$
**Figure 4.1:**

**Proof.** Lines of constant $\xi$ are ellipses with foci at the centres. The lines of constant $\eta$ are hyperbolae with the same foci. They are mutually orthogonal.

We therefore have that

$$ds^2 = a^2 d\xi^2 + b^2 d\eta^2$$

for some $a$ and $b$ which we will find. For motion along the ellipse we have that $dr_1 = ds \cos \alpha$ and $dr_2 = -ds \cos \alpha$. Now $d\xi = 0$ because we are moving along an ellipse where $\xi$ is constant. Therefore $d\eta = dr_1 - dr_2 = 2 \cos(\alpha)ds$ and so we have that $b = \frac{1}{2 \cos(\alpha)}$. So:

$$b^2 = \frac{1}{4 \cos^2(\alpha)}. \hspace{1cm} (4.1)$$

For motion along the hyperbola we have that $dr_1 = ds \sin(\alpha)$ and $dr_2 = ds \sin(\alpha)$, see figure 4.1. Now $d\eta = 0$ because we are moving along a hyperbola where $\eta$ is constant. Therefore $d\xi = dr_1 + dr_2 = 2 \sin(\alpha)ds$ and so we have that $a = \frac{1}{2 \sin(\alpha)}$ and thus

$$a^2 = \frac{1}{4 \sin^2(\alpha)}. \hspace{1cm} (4.2)$$

From the triangle $F_1PF_2$ we have that $2^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\pi - 2\alpha)$ from which we deduce:

$$\cos^2(\alpha) - \sin^2(\alpha) = \frac{4 - r_1^2 - r_2^2}{2r_1r_2} \hspace{1cm} (4.3)$$

However, we also have the identity:

$$\cos^2(\alpha) + \sin^2(\alpha) = \frac{2r_1r_2}{2r_1r_2}. \hspace{1cm} (4.4)$$

Solving equations (4.3) and (4.4) for $\cos^2(\alpha)$ and $\sin^2(\alpha)$ gives:

$$\cos^2(\alpha) = \frac{4 - \eta^2}{\xi^2 - \eta^2}, \quad \sin^2(\alpha) = \frac{\xi^2 - 4}{\xi^2 - \eta^2}. \hspace{1cm} (4.5)$$
Using equations (4.4) with equations (4.2) and (4.1) we obtain that:

\[
a^2 = \frac{(\xi^2 - \eta^2)}{4(\xi^2 - 4)}, \quad b^2 = \frac{(\xi^2 - \eta^2)}{4(4 - \eta^2)},
\]

from which the result follows. \(\square\)

Next, we quote the following result (see e.g. [4]):

**Lemma 4.2.2.** If \(ds^2 = \sum a^2_i dq_i^2\), then:

\[
T = \frac{1}{2} \sum a^2_i \frac{dq_i}{dt}, \quad p_i = a^2_i \frac{dq_i}{dt}, \quad H = \sum \frac{p_i^2}{2a^2_i} + V
\]

where \(T\) is the kinetic energy, \(V\) is the potential energy, \(H\) is the energy and \(p_i\) are the conjugate momenta to the coordinates \(q_i\).

Applying lemma 4.2.2 to our two-centre problem with the values of \(a^2\) and \(b^2\) from (4.6) and using \(V = -\frac{1}{r_i} - \frac{1}{r_j} - \xi \eta / (\xi^2 - \eta^2)\), we deduce:

\[
H = \frac{2(\xi^2 - 4)}{\xi^2 - \eta^2} p^2_\xi + \frac{2(4 - \eta^2)}{\xi^2 - \eta^2} p^2_\eta - \frac{4 \xi}{\xi^2 - \eta^2}.
\]

Fixing the energy \(H = h\) we get from equation (4.7) that

\[
h \xi^2 + 4 \xi - 2(\xi^2 - 4)p^2_\xi = h \eta^2 + 2(4 - \eta^2)p^2_\eta.
\]

The left-hand side is an expression in \(\xi\) only and the right-hand side \(\eta\) only so \(LHS = RHS = -c\), say, where \(c\) is a constant. From \(LHS = -c = RHS\) we deduce that:

\[
p_\xi = \pm \sqrt{\frac{h \xi^2 + 4 \xi + c}{2(\xi^2 - 4)}}, \quad p_\eta = \pm \sqrt{\frac{-h \eta^2 - c}{2(4 - \eta^2)}}.
\]

The quantity \(c\) is the second constant of motion. The corresponding map \(T^*Q \mapsto \mathbb{R}\) is called \(C\).

**Definition 4.2.3.** We will define the energy-momentum mapping, \(P\), to be the map:

\[
P : T^*Q \mapsto \mathbb{R}^2; \quad (q, p) \mapsto (C, H)(q, p) = (c, h).
\]

Here \(Q = \mathbb{R}^2 \setminus \{(-1,0), (1,0)\}\) is the configuration space, \(q = (\xi, \eta)\) are the configuration space variables and \(p = (p_\xi, p_\eta)\) are the conjugate momenta.

### 4.3 Classification into ‘Types’

**Definition 4.3.1.** We define an orbit to be of type

- **P1** if the the interproton axis is never crossed during the motion. The orbit loops around both centres.
- **P2** if both of the extraproton axes and the interproton axis are all crossed during the particle’s motion.
• P3 if the perpendicular bisector of the centres is never crossed during the motion. Motion is therefore localised around one of the centres and only one of the extraproton axes is crossed.

Definition 4.3.2. We define the following three regions:
• \( R_1 = \{(c, h) : c < 0, h < 0, -\frac{4}{c} < h < -\frac{4}{4} - 2\} \).
• \( R_2 = \{(c, h) : c < 0, -\frac{2}{4} - 2 < h < 0\} \).
• \( R_3 = \{(c, h) : c > 0, -\frac{4}{4} - 2 < h < -\frac{4}{4}\} \).

Lemma 4.3.1.
• All collisionless periodic orbits at regular values of the energy-momentum map are of P1, P2 or P3 type.
• Those orbits with values of \((c, h)\) in region \( R_j \) with \( \Delta \) rational are precisely the \( P_j \) orbits for \( j = 1, 2, 3 \).

Proof. We have by equations (4.8) that:
\[
\frac{p_1}{\sqrt{}} = \pm \sqrt{\frac{\text{POLY}_1(\xi)}{2(\xi^2 - 4)}}, \quad p_\eta = \pm \sqrt{\frac{-\text{POLY}_2(\eta)}{2(4 - \eta^2)}}
\]
where \( \text{POLY}_1(\xi) = h\xi^2 + 4\xi + c \) and \( \text{POLY}_2(\eta) = \eta^2 + c \). Let the roots of \( \text{POLY}_j \) be \( \alpha_j \) and \( \beta_j \) for \( j = 1, 2 \). So
\[
\alpha_1 = \frac{-2 + \sqrt{4 - hc}}{h}, \quad \beta_1 = \frac{-2 - \sqrt{4 - hc}}{h}, \quad \alpha_2 = -\sqrt{\frac{-c}{h}}, \quad \beta_2 = \sqrt{\frac{-c}{h}}.
\]

First we find the conditions that all periodic orbits must satisfy:
• For bounded orbits we must have \( h < 0 \) in order that \( \text{POLY}_1(\xi) \) is upside-down U-shaped parabola.
• For \( \alpha_1 \) and \( \beta_1 \) to be real we require, from (4.9), that \( hc \leq 4 \).
• We require that \( \beta_1 \geq 2 \) because if we had \( \beta_1 < 2 \) this would contradict the fact that \( \xi(t) = r_1 + r_2 \geq 2 \) for all \( t \). The condition \( \beta_1 \geq 2 \) is equivalent to \( \sqrt{4 - hc} \geq -2(h + 1) \). If \( h > -1 \) then this is automatically satisfied. If \( h \leq -1 \) then we obtain the additional condition that \( -2(h + 1) \geq 0 \), i.e. \( h \geq \frac{-4}{h} - 2 \).
• Noting that \( \eta(t) \geq 2 \) and the fact that \( -\text{POLY}_2(\eta) \) is a U-shaped parabola that cuts the vertical axis at \(-c\) we see that if \( c > 0 \) we must have \( \sqrt{\frac{-c}{h}} < 2 \). Thus \( h < -c/4 \).

We now find the conditions that determine whether or not we cross the interproton axis and whether or not we cross the \( y \)-axis:
• Note that \( \xi^2 - 4 \geq 0 \) because \( \xi \geq 2 \). Therefore \( \text{POLY}_1 \geq 0 \) for allowed values of \( \xi \). The polynomial \( \text{POLY}_1(\xi) \) has an upside-down U-shape because \( h < 0 \). Therefore \( \alpha_1 \leq \xi \leq \beta_1 \). Suppose that we had that \( \alpha_1 < 2 < \beta_1 \). In other words at some point in the orbit the line \( \xi = 2 \), i.e. the interproton line, is crossed. Now \( \alpha_1 < 2 \iff \sqrt{4 - hc} > 2h + 2 \) and \( \beta_1 > 2 \iff \sqrt{4 - hc} > -2h - 2 \). Therefore \( \sqrt{4 - hc} > |2h + 2| \iff h > \frac{-4}{4} + 2 \). We see that periodic orbits cross \( \xi = 2 \) if and only if \( h > \frac{c}{4} + 2 \).
• Suppose that we cross the line $\eta = 0$, i.e. the perpendicular bisector of the two protons. Examining $-\text{POLY}_2$ we see that we cross the line $\eta = 0$ if and only if $c < 0$.

To obtain the lemma we examine the regions we have effectively defined in the $(c, h)$-plane.

4.3.1 Phase Portraits

From the equations (4.8) we can draw phase portraits for the $\xi$ and $\eta$ behaviour. The qualitative appearance of the $(\xi, p_\xi)$ and $(\eta, p_\eta)$ phase portraits depends on the type and hence on the values of $c$ and $h$. The phase portraits for P1, P2 and P3 motion are shown in figure 4.2.

**Remark 4.3.1.** The ‘singularities’ that occur in our phase-portraits at $\xi = 2$ and $\eta = \pm 2$ can be removed by a process of regularisation. See, for example, [71].

**Remark 4.3.2.**

- For P1 orbits the particle’s motion is confined between two ellipses. The events ‘cross $\eta = -2$’ and ‘cross $\eta = +2$’ alternate and so the particle simply loops around both centres.

- For P2 orbits we are not confined between two ellipses but the events ‘cross $\eta = -2$’ and ‘cross $\eta = +2$’ still alternate.
4.3.2 Orbits on $\partial R_i$

On the boundaries of the regions $R_i$ we are not at regular values in the energy-momentum space. The functionals $C$ and $H$ are dependent and $P^{-1}(c,h)$ is an invariant manifold that has fewer than 2 dimensions. In fact for all points on the boundaries other than the origin, $P^{-1}(c,h)$ is diffeomorphic to a circle. At the origin $P^{-1}(c,h)$ degenerates to a point corresponding to the equilibrium orbit: the particle is stationary midway between the centres for all time. One can describe those orbits on the boundaries of the regions $R_1$, $R_2$ and $R_3$; for details we refer the reader to [30]. Here we quickly and crudely describe them as they are useful to know. The proofs are omitted; the results can be deduced from studying the phase-portraits in the limit as we approach the boundary in question.

- **AC.** As we approach the line AC from higher energies we tend towards an ‘elliptic orbit’. This has constant $\xi$-value for all $t$.

- **BCE.** As we approach the line $h = \frac{c^2}{4} - 2$ from above in the P2 zone we tend to the ‘interproton orbit.’ This is motion such that $y(t) = 0, \quad -1 \leq x(t) \leq 1$ for all $t$. We collide with both left and right-hand centres.

- **DE.** The limiting or the boundary orbit here is the periodic orbit perpendicular to the interproton axis that passes through $x = y = 0$.

- **EG.** We get an $x$-axis orbit that is contained within the interproton axis. This is a collision orbit; there is a collision with just one of the centres. As we tend towards $G$ our orbits become shorter and we tend towards the stationary state of our particle sitting at the right-hand centre for all time.

- **DF.** We get $x$-axis orbits to the right of the right-centre which collide with the right-hand centre.

- **AB and BD.** As we approach edges AB and BD we tend towards unbounded motions.
4.3.3 Action-angle variables

The separating coordinate system \((\xi, \eta)\) gives us a natural choice of action variables:

\[
I_\xi = \frac{1}{2\pi} \oint \xi \, dp, \quad I_\eta = \frac{1}{2\pi} \oint \eta \, dq
\]

with corresponding angle variables \(\phi_\xi\) and \(\phi_\eta\). Here \(I_\xi\) and \(I_\eta\) are the projections of the orbits in phase-space to the \(\xi\) and \(\eta\)-axes respectively.

The functions \(\xi(t)\) and \(\eta(t)\) are always periodic. However this does not necessarily mean that the orbit itself is periodic - it could be quasi-periodic.

**Definition 4.3.3.** Let \(T_\xi(c, h)\) be the minimal period of \(\xi(t)\) with \(C = c, H = h\). Similarly, let \(T_\eta(c, h)\) be the minimal period of \(\eta(t)\) with \(C = c, H = h\).

Note that \(T_\xi(c, h), T_\eta(c, h) \leq T\). The functions \(T_\xi\) and \(T_\eta\) are well-defined as functions of \(c\) and \(h\) from standard results on action-angle variables, see e.g. [4]. We have that:

\[
T_\xi(c, h) = \frac{2\pi}{\phi_\xi} = \frac{2\pi}{\omega_\xi(c, h)}, \quad T_\eta(c, h) = \frac{2\pi}{\phi_\eta} = \frac{2\pi}{\omega_\eta(c, h)}.
\]

For any particular orbit \(c\) and \(h\) are constant and therefore so are \(T_\xi(c, h)\) and \(T_\eta(c, h)\).

**Remark 4.3.3.** Collision orbits are not only restricted to singular values of \((c, h)\). Suppose \(\Delta(c, h) \in \mathbb{Q}\). We will see that collision orbits form part of the 2-tori \(P^{-1}(c, h)\) if \((c, h) \in \mathbb{R}^2\) or if \((c, h) \in \mathbb{R}_3\) and a relation between \(p\) and \(q\) holds.

4.3.4 Calculating \(T_\xi(c, h)\) and \(T_\eta(c, h)\)

The appropriate identity required for the calculation of times is given by the following lemma:

**Lemma 4.3.2.** We have

\[
\left( \begin{array}{c} 1 \\ 0 \end{array} \right) = (\omega_\xi, \omega_\eta) \left( \begin{array}{c} \frac{\partial}{\partial I_\xi} \left( I_\xi, I_\eta \right) \\ \frac{\partial}{\partial h} \left( I_\xi, I_\eta \right) \end{array} \right),
\]

where

\[
\frac{\partial}{\partial I_\xi} \left( I_\xi, I_\eta \right) = \left( \frac{\partial I_\xi}{\partial I_\xi}, \frac{\partial I_\xi}{\partial I_\eta}, \frac{\partial I_\xi}{\partial h}, \frac{\partial I_\xi}{\partial c} \right).
\]

**Proof.** This follows from the chain rule using the fact that \(\omega_j = \frac{\partial H}{\partial I_j}\) for \(j \in \{\xi, \eta\}\).

Let

\[
M = \frac{\partial}{\partial I_\xi} \left( I_\xi, I_\eta \right).
\]

Then lemma 4.3.2 tells us that:

\[
\left( \begin{array}{c} \omega_\xi \\ \omega_\eta \end{array} \right) = (M^t)^{-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]
We want to calculate each of the four quantities \( \frac{\partial I_x}{\partial c}, \frac{\partial I_x}{\partial h}, \frac{\partial I_y}{\partial c}, \) and \( \frac{\partial I_y}{\partial h}. \) We can do this numerically. We take the expressions
\[
2\pi I_x = 2 \int_{\max(2, a_1)}^{b_1} p_\xi d\xi \quad \text{and} \quad 2\pi I_y = 2 \int_{\gamma} p_\eta d\eta.
\]
Exchanging \( \frac{\partial}{\partial u} \) with \( \int \) where \( u \in \{c, h\} \) gives us the following expressions:
\[
2\pi \frac{\partial I_x}{\partial h} = 2 \int_{\max(2, a_1)}^{b_1} \frac{\partial p_\xi}{\partial h} d\xi,
\]
\[
2\pi \frac{\partial I_x}{\partial c} = 2 \int_{\max(2, a_1)}^{b_1} \frac{\partial p_\xi}{\partial c} d\xi,
\]
\[
2\pi \frac{\partial I_y}{\partial h} = 2 \int_{\gamma} \frac{\partial p_\eta}{\partial h} d\eta,
\]
\[
2\pi \frac{\partial I_y}{\partial c} = 2 \int_{\gamma} \frac{\partial p_\eta}{\partial c} d\eta.
\]
From our expressions for \( p_\xi(\xi) \) and \( p_\eta(\eta) \) given by the equations (4.8) we derive that:
\[
\frac{\partial p_\xi}{\partial h} = \frac{1}{2} \left( \frac{b\xi^2 + 4\xi + c}{2(\xi^2 - 4)} \right)^{-\frac{1}{2}} \left( \frac{\xi^2}{2(\xi^2 - 4)} \right),
\]
\[
\frac{\partial p_\xi}{\partial c} = \frac{1}{2} \left( \frac{b\xi^2 + 4\xi + c}{2(\xi^2 - 4)} \right)^{-\frac{1}{2}} \left( \frac{1}{2(\xi^2 - 4)} \right),
\]
\[
\frac{\partial p_\eta}{\partial h} = \frac{1}{2} \left( \frac{-h\eta^2 - c}{2(4 - \eta^2)} \right)^{-\frac{1}{2}} \left( \frac{-\eta^2}{2(4 - \eta^2)} \right),
\]
\[
\frac{\partial p_\eta}{\partial c} = \frac{1}{2} \left( \frac{-h\eta^2 - c}{2(4 - \eta^2)} \right)^{-\frac{1}{2}} \left( \frac{-1}{2(4 - \eta^2)} \right).
\]
Thus given a pair of values \( (c, h) \) we can calculate the matrix \( M \) by doing four numerical integrations. From this we can compute \( \omega_1 \) and \( \omega_2 \), and hence \( T_x \) and \( T_y \). We can do this for all allowed values of \( c \) and \( h \) and so obtain a contour plot of \( \Delta(c, h) \) where \( \Delta(c, h) = \frac{T_x(c, h)}{T_y(c, h)} \), see figure 4.4.
The numerics of \( \Delta(c, h) \) (graphed in figure 4.4) yield the following set of observations:

Observation (I):
\( \Delta(c, h) \) is continuous on \( R_i \) for \( i = 1, 2, 3 \) and, defining \( \Delta(R_i) = \{ \Delta(c, h) : (c, h) \in R_i \} \) for \( i \in \{1, 2, 3\} \), we have

(i) \( \Delta(R_1) = (1, \infty) \),

(ii) \( \Delta(R_2) = (0, \infty) \),

(iii) \( \Delta(R_3) = (0, 1) \).

Remark 4.3.4. Observation (I) implies that:

- For all \( d \in (1, \infty) \cap \mathbb{Q} \) there exists a \( P1 \) orbit with \( \Delta = d \).
- For all \( d \in (0, \infty) \cap \mathbb{Q} \) there exists a \( P2 \) orbit with \( \Delta = d \).
Figure 4.4: Contour Plot of $\Delta(c, h)$. 
• For all \( d \in (0,1) \cap \mathbb{Q} \) there exists a \( P3 \) orbit with \( \Delta = d \).

We provide no proof of observation (I) above but comment that:

• The numerical evidence to support it is very strong.

• The observations agree with the paper [30] in which the above observations are implicit.

• We will see that observation (I) is not required to be able to describe the set of all homology classes of loops containing periodic solutions. However the observations on the range of \( \Delta|\mathbb{R} \) will allow us to more precisely describe the set of all pairs \( (p,q) \) such that \( P1 \) orbits with \( \Delta = \frac{p}{q} \) lie in a particular topological class of loops. Namely we can have any \( (p,q) \in \mathbb{N}^2 \) such that \( p > q \) for \( P1 \) orbits, any \( (p,q) \in \mathbb{N}^2 \) for \( P2 \) orbits and any \( (p,q) \in \mathbb{N}^2 \) such that \( p < q \) for \( P3 \) orbits.

### 4.4 Homotopy and Homology Classes of Loops

**Definition 4.4.1.** We say an orbit is of prime-period if it is a periodic orbit with minimal period \( T > 0 \).

**Definition 4.4.2.** We define the modulo operation \( \text{mod} \) by

\[
\text{mod}(a,m) = a - m\lfloor \frac{a}{m} \rfloor,
\]

where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). That is \( \text{mod}(a,n) \) is the remainder after numerical division of \( a \) by \( n \).

We think of our particle moving in the complex plane with position \( z(t) \in \mathbb{C} \) for all \( t \). The centres are at \( \pm 1 \). We will write \( z(t) = x(t) + iy(t) \) so the real axis corresponds to the \( x \)-axis and the imaginary axis corresponds to the \( y \)-axis. We denote the complex conjugate of \( z(t) \) by \( \bar{z}(t) = x(t) - iy(t) \).

**Definition 4.4.3.** Call the loop \( z^{(j)} : [0,jT) \to \mathbb{C} \) the loop \( z : [0,T) \to \mathbb{C} \) repeated \( j \) times, or \( j \) multiples of the loop \( z \), if

\[
z^{(j)}(t) = z(\text{mod}(t,T)) \quad \forall \ t \in [0,jT).
\]

**Remark 4.4.1.** Note that if \( z(t) \) is a periodic orbit then \( z^{(j)}(t) \) where \( j \in \mathbb{N} \) is also a periodic orbit. Any periodic orbit \( w(t) \) can be written as \( w(t) = z^{(j)}(t) \) where \( z(t) \) is of prime-period and \( j \in \mathbb{N} \).

**Definition 4.4.4.** We say a prime-period periodic orbit is of type \( \frac{p}{q} \) ' if \( \Delta = \frac{p}{q} \) with \( \gcd(p,q) = 1 \). We say a non-prime-period periodic orbit is of type \( \frac{r\mathbb{Z}}{q} \), where \( \gcd(p,q) = 1 \), if it is \( r \) multiples of a \( \frac{p}{q} \) orbit.

**Definition 4.4.5.** Given a prime-period periodic orbit of type \( \frac{p}{q} \), we define \( \tau \) for that orbit by

\[
\tau = pT_\eta = qT_\xi.
\]
Figure 4.5: A P2 ‘1/2’ orbit.

Figure 4.6: A P2 ‘1/4’ orbit.
A crucial point that we make is that, by remark 4.2.1, fixing $\xi$ and $\eta$ only fixes a point in $\mathbb{R}^2$ up to reflection in the real axis. Therefore a prime-period orbit of type $[\frac{p}{q}]$ does not necessarily have period $\tau$. This is because after time $\tau$ we may be in a state with conjugate position and velocity to when we started. In such a case we have to let the system run for a further $\tau$ time units before we return to our initial state. This process will be called ‘doubling-up.’

**Definition 4.4.6.** We define the symmetry $(S)$ by

$$ (S): \quad z(t + \frac{T}{2}) = \bar{z}(t) \quad \text{for all } t. $$

**Lemma 4.4.1.** Prime-period $P_2$ orbits of type $[\frac{p}{q}]$ with $q$ odd and prime-period $P_3$ orbits with $p+q$ odd have the symmetry $(S)$ and have period $2\tau$. All other prime-period periodic orbits have period $\tau$.

**Proof.** Let us consider the prime-period orbit of type $[\frac{p}{q}]$. The idea is to count the number of crossings of the $x$-axis in time $\tau$. If this number is odd then at the end of time $\tau$ we are at a position (and velocity) conjugate to when we started. Consequently, we will need to double-up our orbit. If the number of crossings is even then we must have returned to our original state.

To count the number of crossings of the $x$-axis we recall that the $x$-axis is the union of the interproton and extraproton axes and examine the phase-portraits. We see that:

(i) For $P_1$ orbits every $T_0$ time units we cross the $x$-axis twice (the line $\eta = -2$ once and $\eta = +2$ once). We therefore cross the $x$-axis $2p$ times in time $\tau$.

(ii) For $P_2$ orbits we cross the $x$-axis $2p + q$ times in time $\tau$ - the lines $\eta = \pm 2$ each get crossed $p$ times each and the line $\xi = 2$ gets crossed $q$ times.

(iii) For $P_3$ orbits we cross the $x$-axis $p + q$ times - the line $\eta = 2$ gets crossed $p$ times and the line $\xi = 2$ gets crossed $q$ times.

□

**Definition 4.4.7.** Say a periodic orbit $z(.)$ has homology $(a, b) \in \mathbb{Z}^2$ if it winds clockwise $a$ times around the left hand centre and $b$ times around the right hand centre in one period.

**Definition 4.4.8.** Loop space is defined by:

$$ \Omega = \bigcup_{\tau > 0} \Omega_{\tau}, $$

where $\Omega_{\tau} = H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2\setminus\{\pm 1\})$.

Loop space, $\Omega$, is topologically equivalent to the fundamental group of configuration space i.e. $\pi_1(\mathbb{R}^2\setminus\{\pm 1\})$. We know that $\pi_1(\mathbb{R}^2\setminus\{\pm 1\}) = F_2$, the free group on the two generators $\alpha_1$ and $\alpha_2$ where $\alpha_1$ denotes a single clockwise loop around the left hand centre and $\alpha_2$ denotes a single clockwise loop around the right hand centre. The free homotopy classes correspond bijectively to the conjugacy classes of $F_2$ (see, for example, [54]). An element of $F_2$ is said to be in reduced form if there are no occurrences of $\alpha_1\alpha_1^{-1}$ or $\alpha_1^{-1}\alpha_1$ in the word. We will denote by $id$ the homotopy class of loops that can be contracted to a point.
Remark 4.4.2. If a and b are reduced words in the same conjugacy class of \( F_2 \) then they are cyclic permutations of each other. A conjugacy class is often referred to by one of its elements. A free homotopy class is reduced if and only if it is in the form

\[ a_1^{N_1}a_2^{N_2} \ldots a_2^{N_{2k}}. \]

Note that the homology of this class is

\[ \left( \sum_{i \text{ odd}} N_i, \sum_{i \text{ even}} N_i \right). \]

Definition 4.4.9. Let \( H_i \) be the set of all free homotopy classes containing prime-period periodic orbits of type \( P_i \), for \( i = 1, 2, 3 \).

Remark 4.4.3. If we define \( H_i^* \) to be the set of homotopy classes of loops containing \( P_i \) orbits (not necessarily of prime-period) then we have

\[ H_i^* = \{ \beta^k : \beta \in H_i \text{ and } k \in \mathbb{N} \}. \]

4.4.1 Main Theorem

A central goal is to prove the following theorem:

Theorem 4.4.1.

(i) \( P_1 \) Orbits

(a) \[ H_1 = \{ (a_1a_2)^a : a \in \mathbb{Z} \setminus \{0\} \}. \]

(b) The set of types of prime-period \( P_1 \) orbits in \( \alpha = (a_1a_2)^a \in H_1 \) is a subset of

\[ \{ \frac{p}{q} : p = |a|, \gcd(p, q) = 1 \} \]

and assuming observation (I) is precisely

\[ \{ \frac{p}{q} : p = |a|, 1 < q < p, \gcd(p, q) = 1 \}. \]

(ii) \( P_2 \) Orbits

(a) \[ H_2 \subset \{ a_1^{N_1} \ldots a_2^{N_{2k}} : N_i \in \{\pm1\}, \left( \sum_{i \text{ odd}} N_i, \sum_{i \text{ even}} N_i \right) \in \{(0, 0), (1, 1), (1, -1)\} \}. \]

(b) Assuming observation (I), given any \( p, q \in \mathbb{N} \) with \( \gcd(p, q) = 1 \) there exists a \( P_2 \) prime-period orbit of type \( \frac{p}{q} \).

(c) Those \( P_2 \) prime-period orbits of type \( \frac{p}{q} \) with \( q \) odd have homology \((0, 0)\) whereas those with \( q \) even have homology \( \pm(1, \pm1) \).
(d) Any $P2$ prime-period orbit of type $\frac{p}{q}$ in $\alpha_1^{N_1}...\alpha_2^{N_2} \in H_2$ has
\[
p = \begin{cases} 
k/2 & \text{if } \sum_i \text{ odd } N_i = \sum_i \text{ even } N_i = 0 \\k & \text{otherwise.} \end{cases}
\]

(iii) P3 Orbits

(a) \[H_3 = \{\alpha_1^{\pm 1}, \alpha_2^{\pm 1}\} \cup \{id\}.\]

(b) The set of types of prime-period P3 orbits in $\alpha_1 \cup \alpha_1^{-1} \cup \alpha_2 \cup \alpha_2^{-1}$ is a subset of
\[
\{ \frac{p}{q} : p, q \in \mathbb{N}, \gcd(p, q) = 1, \mod(p + q, 2) = 0 \}
\]
and assuming observation (I) is precisely
\[
\{ \frac{p}{q} : p, q \in \mathbb{N}, q > p, \gcd(p, q) = 1, \mod(p + q, 2) = 0 \}.
\]

(c) The set of types of prime-period P3 orbits in id is a subset of
\[
\{ \frac{p}{q} : p, q \in \mathbb{N}, \gcd(p, q) = 1, \mod(p + q, 2) = 1 \}
\]
and assuming observation (I) is precisely
\[
\{ \frac{p}{q} : p, q \in \mathbb{N}, q > p, \gcd(p, q) = 1, \mod(p + q, 2) = 1 \}.
\]

Remark 4.4.4. Note that the results for prime-period $P2$ orbits are not as strong as the results for prime-period $P1$ or $P3$ orbits because we do not describe $H_2$ completely. Nevertheless the constraint given in (ii)(a) of theorem 4.4.1 is highly restrictive.

Remark 4.4.5. Note that $H_1 = H_1^*$ but $H_j$ is a strict subset of $H_j^*$ for $j \in \{2, 3\}$.

Remark 4.4.6. In part (ii) of theorem 4.4.1, $q$ is not uniquely fixed by the homotopy class. An interesting problem would be to describe the set of all such possible $q$.

To prove theorem 4.4.1 we introduce the notion of a ‘letter-sequence’.

4.4.1.1 Letter-sequences

Definition 4.4.10. Let $A_+$ denote the event of crossing the line $\eta = -2$ from below to above the real axis. Let $A_-$ denote the event of crossing the line $\eta = -2$ from above to below the real axis. Similarly define $B_+$ and $B_-$ with respect to the line $\xi = 2$ and $C_+, C_-$ with respect to the line $\eta = +2$.

Remark 4.4.7. Note that the periodic orbit touching the line $\eta = \pm 2$ or $\xi = 2$ at time $t$ - but not crossing - is not counted as an event at time $t$. As $t$ increases from 0 to $T$ any particular periodic orbit executes an ordered sequence of the events $A_{\pm}, B_{\pm}$ and $C_{\pm}$.
Definition 4.4.11. The letter sequence of a periodic orbit of period $T$ is the string $X^1X^2 ... X^l$ such that:

- $X^i \in \{A_\pm, B_\pm, C_\pm\}$ for all $i \in \{1,...,l\}$.
- The $X^i$ occurs at time $t_i$ for all $i \in \{1,...,l\}$, where $0 < t_1 < t_2 < ... < t_i < T$.
- At times in the set $[0,T) \setminus \{t_1,...,t_l\}$ none of the events $A_\pm, B_\pm$ or $C_\pm$ occur.

We say that a periodic orbit generates a letter-sequence. Phase-shifts of a periodic orbit lead to cyclic permutations of the generated letter-sequence. We call a letter sequence with the subscripts omitted an unsigned letter sequence.

Lemma 4.4.2. The letter-sequence must obey the following two rules:

- The letter-sequence must have alternating $+$ and $-$ subscripts,
- An even number of letters.

Proof. Both of these properties follow directly from the continuity of our trajectory in configuration space. □

Remark 4.4.8. For convenience we will sometimes drop the subscripts on the $A_\pm, B_\pm, C_\pm$.

Lemma 4.4.2 holds for any planar two centre problem. But what about the Newtonian case - what are the additional restrictions arising from integrability? We note that:

- The phase portrait for $\eta$ for $P_2$ orbits implies that the events cross $\eta = -2$ and cross $\eta = +2$ alternate. The implication for our letter-sequence is that, ignoring the $B$’s, we have a sequence of alternating $A$’s and $C$’s.
- For any particular periodic orbit $T_\xi$ is fixed implying the time intervals between successive $B$’s are constant. Also $T_\eta$ is fixed implying equal time intervals between successive $A$’s and successive $C$’s.

4.4.1.2 From letter-sequence to free homotopy class.

We want a prescription for moving from letter-sequence to free homotopy class. First we define a map $\theta$ from the set of all letter sequences to $F_2$.

Suppose we are given a letter-sequence $V = W^1 ... W^m$ where each $W^i$ for $1 \leq i \leq m$ consists of two signed letters.

Definition 4.4.12. Let

$$\theta(W^1 ... W^m) = \phi(W^1) ... \phi(W^m) = \phi(W^1) \cdot \phi(W^2) \cdot ... \cdot \phi(W^m)$$

where $\cdot$ denotes usual product of the fundamental group and $\phi$ given by the table below:
Each of the entries in the right-hand column of this table is a pointed homotopy class.

The free homotopy class we assign to the letter sequence $V$ is the conjugacy class of $\theta(V)$ in $F_2$. We write/represent this simply as $\theta(V)$.

We want to show that our prescription is well-defined and 'works' (i.e. gives the correct homotopy class for a given letter sequence):

Lemma 4.4.3.

(i) Given any letter sequence $\lambda$ there exists a loop (not necessarily a periodic solution) with letter-sequence $\lambda$ and pointed homotopy class $\theta(\lambda)$.

(ii) Any two loops with the same letter sequences are homotopic.

Proof. To see point (i), suppose our letter sequence starts with a subscript +. Fix the base point to be a point below the x-axis, call it $b$. Then for each pair of letters we form a closed loop that starts and ends at $b$. The concatenation of these loops in order gives a loop with the correct letter sequence and has pointed homotopy class given by our prescription. To understand this we refer the reader to figure 4.7. A similar argument holds if the subscript starts with a −. Point (ii) follows from the observation that one can continuously deform any two collisionless loops with the same letter sequence into one another without passing through loops with collisions.

An important point is that we can 'cancel' adjacent identical letters e.g. $BB$ in the unsigned letter sequence without affecting the pointed or free homotopy class. We prove this and some other useful lemmas in the following subsection.

4.4.1.3 Useful Lemmas

Lemma 4.4.4. $\theta(W^I W^J) = \theta(W^I)\theta(W^J)$ where both $W^I$ and $W^J$ consist of an even number of letters and start with the same subscript.
Proof. Suppose $W^I = W^1 W^2 \ldots W^r$ and $W^J = W^{r+1} \ldots W^k$ where each $W^j$ for $1 \leq j \leq k$ consists of two letters. Then $\theta(W^I W^J) = \theta(W^1 W^2 \ldots W^r W^{r+1} \ldots W^k) = \phi(W^1) \ldots \phi(W^{r-1} W^r) \phi(W^{r+1}) \ldots \phi(W^k) = \theta(W^I) \theta(W^J)$. □

Lemma 4.4.5. $\theta(B_+ W_+ B_-) = \theta(W)$ where $W$ is even in length and starts with a + subscript. Similarly $\theta(B_+ W_- B_) = \theta(W)$ where $W$ is even in length and starts with a − subscript.

Proof. Let $W = W^1 W^2 \ldots W^r$ be the decomposition of $W$ into pairs (each $W^i$ consists of two letters). Note that

$$\theta(B_+ W_- B_) = \theta(B_+ W^1 \ldots W^r B_-) = \theta(B_+ W^1 B_-) \theta(B_+ W^2 B_-) \ldots \theta(B_+ W^r B_-).$$

Now we check all the pairs, see the table below:

<table>
<thead>
<tr>
<th>$\theta(W)$ = $\phi(W)$</th>
<th>$\theta(B_+ W_+ B_-)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(A_+ B_-) = \alpha_1$</td>
<td>$\theta(B_+ A_+ B_- B_+)$ = $\phi(B_+ A_+ B_- B_+)$ = $\alpha_1 \cdot id$ = $\alpha_1$</td>
</tr>
<tr>
<td>$\phi(A_+ C_-) = \alpha_1 \alpha_2$</td>
<td>$\theta(B_+ A_+ C_- B_+)$ = $\phi(B_+ A_+ C_- B_+)$ = $\alpha_1 \alpha_2$</td>
</tr>
<tr>
<td>$\phi(B_+ A_-) = \alpha_1^{-1}$</td>
<td>$\theta(B_+ B_+ A_- B_+)$ = $\phi(B_+ B_+ A_- B_+)$ = $\alpha_1^{-1}$</td>
</tr>
<tr>
<td>$\phi(B_+ C_-) = \alpha_2$</td>
<td>$\theta(B_+ B_+ C_- B_+)$ = $\phi(B_+ B_+ C_- B_+)$ = id.$\alpha_2$ = $\alpha_2$</td>
</tr>
<tr>
<td>$\phi(C_+ A_-) = \alpha_2^{-1} \alpha_1^{-1}$</td>
<td>$\theta(B_+ C_+ A_- B_+)$ = $\phi(B_+ C_+ A_- B_+)$ = id.$\alpha_1^{-1}$ = $\alpha_1^{-1}$</td>
</tr>
<tr>
<td>$\phi(C_+ B_-) = \alpha_2^{-1}$</td>
<td>$\theta(B_+ C_+ B_- B_+)$ = $\phi(B_+ C_+ B_- B_+)$ = $\alpha_2^{-1}$</td>
</tr>
<tr>
<td>$\phi(A_+ A_-)$ = id</td>
<td>$\theta(B_+ A_+ A_- B_+)$ = $\phi(B_+ A_+ A_- B_+)$ = id.$\alpha_1^{-1}$ = id</td>
</tr>
<tr>
<td>$\phi(B_+ B_+ B_-)$ = id</td>
<td>$\theta(B_+ B_+ B_- B_+)$ = $\phi(B_+ B_+ B_- B_+)$ = id.$id$ = id</td>
</tr>
<tr>
<td>$\phi(C_+ C_-)$ = id</td>
<td>$\theta(B_+ C_+ C_- B_+)$ = $\phi(B_+ C_+ C_- B_+)$ = id.$\alpha_2$ = $\alpha_2$</td>
</tr>
</tbody>
</table>

One can perform a similar check for all two letter words with subscript starting with a −. □

Lemma 4.4.6. $\theta(W^I B_+ B_- W^J) = \theta(W^I W^J)$ where $W^I$ and $W^J$ are both odd in length or $W^I$ and $W^J$ are both even in length.

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Proof. Suppose that \( W^I \) and \( W^J \) are both even in length. Then by lemma 4.4.4 \( \theta(W^I B_+ B_- W^J) = \theta(W^I) \theta(B_+ B_-) \theta(W^J) = \theta(W^I) \text{id} \theta(W^J) = \theta(W^I) \theta(W^J) = \theta(W^IW^J) \).

Suppose that \( W^I \) and \( W^J \) are both odd in length. Without loss of generality suppose that \( W^I \) starts and ends with a \(-\) subscript and \( W^J \) starts and ends with a \(+\) subscript. Write \( W^I = H^I X_- \) and \( W^J = Y_+ H^J \) where \( X_- \) and \( Y_+ \) are single letters and both \( H^I \) and \( H^J \) are even in length. Then:

\[
\theta(W^I B_+ B_- W^J) = \theta(H^I X_- B_+ B_+ Y_+ H^J) = \theta(H^I) \theta(X_- B_+ B_- Y_+ H^J) \]

using lemma 4.4.4. However we have from lemma 4.4.5 that:

\[
\theta(X_- B_+ B_- Y_+) = \theta(B_+ X_-) \theta(B_+ B_-) \theta(Y_+ B_-) = \theta(B_+ X_- B_-) \theta(Y_+ B_-) = \theta(B_+ X_- B_- Y_+),
\]

the final equality following from lemma 4.4.5. Thus:

\[
\theta(X_- B_+ B_- Y_+) = \theta(B_+ X_-) \theta(B_+ B_-) \theta(Y_+ B_-) = \theta(B_+ X_- Y_+), \tag{4.11}
\]

the final equality following from lemma 4.4.5 again. Substituting equation (4.11) into (4.10) gives:

\[
\theta(W^I B_+ B_- W^J) = \theta(H^I) \theta(X_- Y_+) \theta(H^J) = \theta(H^I X_- Y_+ H^J) = \theta(W^I W^J).
\]

\( \square \)

4.4.1.4 Null homology

Lemma 4.4.7. Prime-period \( P^2 \) orbits of type \( \frac{p}{q} \) with \( q \) odd and \( P^3 \) orbits with \( p + q \) odd have homology \((0,0)\).

Proof. The result of ‘doubling-up’ is a letter-sequence of the form

\[
X = (X_1^1 X_2^2 ... X_{2p+q}^2) (X_1^- X_2^- ... X_{2p+q}^-)
\]

because the motion of the body in the time interval \( \left[ \frac{T}{2}, T \right) \) is the motion in the time interval \( \left[ 0, \frac{T}{2} \right) \) conjugated (i.e. reflected in the \( x \)-axis). By lemma 4.4.6 we have:

\[
\theta(X) = \theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_- B_+ X_1^- X_2^- ... X_{2p+q}^-)
\]

which, by lemma 4.4.4, equals:

\[
\theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_-) \theta(B_+ X_1^- X_2^- ... X_{2p+q}^-) \theta(B_- B_+).
\]

Now using lemma 4.4.5 this equals:

\[
\theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_-) \theta(B_- B_+ X_1^- X_2^- ... X_{2p+q}^-) \theta(B_+).
\]

Using lemma 4.4.6 again gives:

\[
\theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_-) \theta(B_- B_+) \theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_+) = \theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_-) \theta(X_1^1 X_2^2 ... X_{2p+q}^2 B_+).
\]

Referring to our prescription we see that for all letter pairs \( XY \) we have that \( \theta(X Y_-) \) is related to \( \theta(X_- Y_+) \) by the map of generators \( \alpha_i \mapsto \alpha_i^{-1} \) for \( i = 1, 2 \). Therefore the homotopy class is of the form

\[
(\alpha_1^{N_1} ... \alpha_2^{N_2}) (\alpha_1^{-N_1} ... \alpha_2^{-N_2})^a, \quad a \in \mathbb{N}.
\]

In particular the homology class is \((0,0)\). \( \square \)

Remark 4.4.9. Note that \( P^3 \) orbits in the homology class \((0,0)\) are null-homotopic. However we shall see in the following subsection that the same is not true for \( P^2 \) orbits.
4.4.1.5 Powers of the generators of the free homotopy classes containing P2 orbits

We wish to prove the claim in part (ii) of theorem 4.4.1 that all superscripts of all elements of $H_2$ are $\pm 1$. Essentially this follows from the fact that A's and C's alternate. We demonstrate the result rigorously below. Suppose we take the letter-sequence generated by a prime-period orbit and cancel all pairs of adjacent B's. This does not affect the free homotopy class that we get from the letter sequence. We call the sequence we end up after cancelling adjacent B's a reduced-letter-sequence.

Lemma 4.4.8. Any free homotopy class of a P2 orbit has the form: $\alpha_1^{N_1}\alpha_2^{N_2}...\alpha_2^{N_{2k}}$ where $N_j = \pm 1$ for all $j \in \{1, ..., 2k\}$.

Proof. Firstly we find all those letter pairs $W$ that can immediately follow the letter pair $A_+B_-$ in the reduced-letter-sequence. From the restrictions that:

- + and − subscripts alternate,
- ignoring B's, the A's and C's alternate,
- the reduced-letter-sequence has no adjacent B's,

we deduce that:

$W \in \{C_+A_-, C_+B_-, A_+B_-\}$,

so:

$A_+B_-W \in \{A_+B_-C_+A_-, A_+B_-C_+B_-\}$

and

$\theta(A_+B_-W) \in \{\alpha_1\alpha_2^{-1}\alpha_1^{-1}, \alpha_1\alpha_2^{-1}\}$.

One can check for all other possible pairs of letters in the reduced letter sequence ($A_+B_-$ was just an example) that the restrictions have the effect of forcing the powers of the $\alpha_i$ ($i = 1, 2$) to be $\pm 1$; immediately following an $\alpha_i^1$ we must have an $\alpha_i^{-1}$ and vice versa. $\Box$

4.4.1.6 Length of a homotopy class

Definition 4.4.13. Define the length of $\alpha_1^{N_1}\alpha_2^{N_2} \in F_2$ to be $2l$.

Lemma 4.4.9. Any P2 prime-period $\frac{p}{q}$ orbit in $\alpha_1^{N_1}...\alpha_2^{N_{2k}} \in H_2$ has

$p = \begin{cases} 
\frac{k}{2} & \text{if } q \text{ is odd} \\
k & \text{if } q \text{ is even}
\end{cases}$

Proof. A P2 prime-period $\frac{p}{q}$ orbit has up A's and up C's in its reduced letter sequence where $v = 2$ if we have ‘doubled-up’ and $v = 1$ otherwise. Doubling-up occurs if and only if $q$ is odd because we cross the $x$-axis $2p + q$ times in time $\tau$. By the proof of lemma 4.4.8 when we apply the prescription of subsection 4.4.1.2 to a reduced-letter-sequence of a P2 orbit we get a free homotopy class that is in reduced form. Examining the table given in definition 4.4.12 we see that each A and each C contributes $+1$ to the length of the homotopy class and each B contributes $0$. Therefore $2k = 2vp$. $\Box$

The remainder of the content of theorem 4.4.1 amounts to finishing the classification of prime-period orbits into homology classes.
4.4.2 Homotopy Classes of P1 Orbits

A P1 prime-period orbit of type $\text{I}_p^q$ has $p \eta$-oscillations in one period and we never cross the set $\xi = 2$. The letter sequence is therefore

$$(A_+ C_-)^p = A_+ C_- ... A_+ C_- \quad \text{or} \quad (A_- C_+)^p.$$ 

There are no B's because we never cross the set $\xi = 2$. The prescription of subsection 4.4.1.2 gives homotopy classes $(\alpha_1 \alpha_2)^{\pm p}$. The homological classes are therefore $\pm(p, p)$.

4.4.3 Homotopy Classes of P3 Orbits

There are two cases: $p + q$ odd or $p + q$ even.

4.4.3.1 $p + q$ odd.

If $p + q$ is odd then by lemma 4.4.7 the orbit is in the homology class $(0, 0)$. Since the motion of a P3 orbit is localised around a single centre this implies that we are in the homotopy class $\text{id}$.

4.4.3.2 $p + q$ even.

We concentrate on prime-period P3 orbits localized around the left hand centre. Any results for the left hand centre will also apply to the right hand centre by symmetry. Suppose we look at a prime-period orbit with $A = \text{I}_p^q$. This will have $p \eta$-oscillations and $q \xi$-oscillations. From the phase portraits we see that this gives us $p$ A's and $q$ B's in our letter-sequence. Moreover, the A's and B's are mixed as 'evenly as possible' owing to the fact that $T_\xi$ and $T_\eta$ are constant. More precisely up to cyclic permutations the letter sequence is such that there are

$$\left\lfloor \frac{rq}{p} \right\rfloor - \left\lfloor \frac{(r - 1)q}{p} \right\rfloor$$

B's between the $r^{th}$ and $(r + 1)^{th}$ A for $r \in \{1, ..., p\}$.

Example 4.4.1. If $p = 5$ and $q = 7$ then the (unsigned) letter sequence could be

$ABB$ $AB$ $ABB$ $AB$ $AB$,

but could not be

$ABB$ $AB$ $ABB$ $AB$ $AB$ $AB$.

Alternating $\pm$ subscripts are then attached to the letters. Since $p + q$ is even we do not double-up our letter sequence. We will drop the subscripts of $\pm$ on the letters as they are not useful for now. Let $\mu_A(1) = p$ denote the number of A's initially and let $\mu_B(1) = q$ denote the number of B's in the letter-sequence.

Definition 4.4.14. We define the function $F$ in the following way. Let

$$F(x, y) = \begin{cases} \mod(x, y) & \text{if } \left\lfloor \frac{x}{y} \right\rfloor \text{ is even,} \\ y - \mod(x, y) & \text{if } \left\lfloor \frac{x}{y} \right\rfloor \text{ is odd.} \end{cases}$$

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Remark 4.4.10. Observe that $F(x,y) \leq y$. Also note that if $x < y$ then $F(x,y) = x$.

To begin with we 'cancel' any pairs of adjacent B's. This gives us a new letter sequence with $\mu_A(1)$ A's and $\mu_B(2)$ B's where $\mu_B(2)$ is given by:

$$\mu_B(2) = F(\mu_B(1), \mu_A(1)).$$

We note from the definition of $F$ that $\mu_B(2) \leq \mu_A(1)$ i.e. we now have at least as many A's as B's. The A's and B's are still 'evenly mixed'. Next we cancel adjacent pairs of A's to form a new letter sequence with $\mu_A(2)$ A's and $\mu_B(2)$ B's:

$$\mu_A(2) = F(\mu_A(1), \mu_B(2)).$$

Note that $\mu_B(2) \geq \mu_A(2)$.

We continue in this fashion until $\mu_A(i)$ and $\mu_B(i)$ no longer change because there are no more adjacent identical letters to cancel. We can then infer the homotopy class from the corresponding 'ultimately-reduced' letter-sequence.

Example 4.4.2. Suppose $p = 5$ and $q = 7$ (note $p+q = 12$ is even). We have 5 A's and 7 B's mixed as evenly as possible: ABABBABABABB. We cancel adjacent B's to give: ABAABABA. Then we cancel adjacent A's to give: BBAB. Then we cancel adjacent B's to give: AB. The sequence AB is the ultimately reduced letter-sequence. The homotopy class is therefore either $\alpha_1$ or $\alpha_1^{-1}$.

More generally we have the algorithm:

**Lemma 4.4.10.** An algorithm that gives the number of letter A's, $\mu_A(i)$, and number of letter B's, $\mu_B(i)$, following $(i - 1)$ reductions, is:

(i) Set $\mu_A(1) = p$ and $\mu_B(1) = q$ and $i = 1$

(ii) Let $\mu_B(i + 1) = F(\mu_B(i), \mu_A(i))$ and $\mu_A(i + 1) = F(\mu_A(i), \mu_B(i + 1))$

(iii) If $\mu_A(i) = \mu_A(i - 1)$ and $\mu_B(i) = \mu_B(i - 1)$ then stop. Otherwise increase $i$ by 1 and go back to step (ii).

**Lemma 4.4.11.** $gcd(x, y) = gcd(y, F(x, y))$.

*Proof.* Suppose $r|\mu_A$ and $r|\mu_B$. Then we have that

$$F(x, y) = \begin{cases} x - ky & \text{for some } k \in \mathbb{Z} \\ y - (x - ky) = (k + 1)x - x & \text{for some } k \in \mathbb{Z} \end{cases}$$

if $\frac{x}{y}$ is even, and

if $\frac{x}{y}$ is odd.

In both cases we have that $r|F(x, y)$. Thus $gcd(x, y)$ is a common divisor of $y$ and $F(x, y)$ and hence $gcd(x, y)|gcd(y, F(x, y))$.

Conversely suppose that $r|\mu_A$ and $r|\mu_B$. There are two cases:

- If $\frac{x}{y}$ is even then we have that $F(x, y) = \text{mod}(x, y) = x - ky = rl$ for some $k$ and some $l$ and $y = tr$ for some $t$. Then $x = rl + ky = rl + ktr = r(l + kt)$ so $r|x$.

- If $\frac{x}{y}$ is odd then we have $F(x, y) = (k + 1)y - x = ts$ for some $k$ and some $s$ and $y = tr$ for some $t$. Thus $x = r((k + 1)t - s)$ so again $r|x$.  

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Thus \( \gcd(y, F(x, y)) \) is a common divisor of \( y \) and \( x \) and hence \( \gcd(y, F(x, y)) \mid \gcd(x, y) \). □

We will use lemma 4.4.11 to prove the following result:

**Lemma 4.4.12.** We have that:

\[
\gcd(\mu_A(i + 1), \mu_B(i + 1)) = \gcd(\mu_A(i), \mu_B(i)) \quad \text{for all } i \in \mathbb{N}.
\] (4.12)

**Proof.** From our algorithm we know that:

\[
\gcd(\mu_A(i + 1), \mu_B(i + 1)) = \gcd(F(\mu_A(i), \mu_B(i + 1)), \mu_B(i + 1)).
\]

However, by lemma 4.4.11,

\[
\gcd(F(\mu_A(i), \mu_B(i + 1)), \mu_B(i + 1)) = \gcd(\mu_A(i), \mu_B(i + 1)).
\]

This equals \( \gcd(\mu_A(i), F(\mu_B(i), \mu_A(i))) \) from our algorithm which equals \( \gcd(\mu_A(i), \mu_B(i)) \) by lemma 4.4.11 again. □

We now address the question of when the algorithm stops. In other words when does \( \mu_A(i + 1) = \mu_A(i) \) and \( \mu_B(i + 1) = \mu_B(i) \)? Suppose this happens when \( i = j \). From our algorithm we see this implies \( \mu_B(j) = F(\mu_B(j), \mu_A(j)) \) and \( \mu_A(j) = F(\mu_A(j), \mu_B(j)) \) from which we conclude \( \mu_A(j) = \mu_B(j) \). We deduce that \( \gcd(p, q) = \gcd(\mu_A(1), \mu_B(1)) = \gcd(\mu_A(j), \mu_B(j)) = \mu_A(j) = \mu_B(j) \). This gives us the following proposition:

**Proposition 4.4.1.** Suppose we have a prime-period \( P^3 \) orbit of type \( \frac{p}{q} \) where \( p+q \) is even. Then the orbit lies in one of the homotopy classes in the set \( \{\alpha_i^{\pm 1} : i = 1, 2\} \).

**Proof.** Note that if an orbit is of prime period then \( h = \gcd(p, q) = 1 \). It follows then when the algorithm given in lemma 4.4.10 stops at \( i = j \) we have \( \mu_A(j) = \mu_B(j) = 1 \). Therefore we have precisely one letter A and one letter B in our ultimately reduced letter sequence. For the right hand centre we'd have one B and one C. The result follows from inferring the free homotopy class from the letter sequences \( A + B - \), \( A - B + \), \( B + C - \) and \( C + A - \). □

**Corollary 4.4.1.** Suppose we have a prime-period \( P^3 \) orbit of type \( \frac{p}{q} \). If \( p + q \) is even then we have \( \gcd(p, q) = 1 \). Then the orbit lies in one of the homology classes \( \{(\pm 1, 0), (0, \pm 1)\} \).

### 4.4.4 Homology Classes of \( P^2 \) Orbits

We already know from lemma 4.4.7 that \( P^2 \) prime-period orbits of type \( \frac{p}{q} \) where \( \text{mod}(q, 2) = 1 \) have homology \( (0, 0) \).

**Proposition 4.4.2.** Prime-period \( P^2 \) orbits of type \( \frac{p}{q} \) have homology \( \pm(1, \pm 1) \) if \( \text{mod}(q, 2) = 0 \).

**Proof.** Lets try and find the winding number around the left hand centre. Consider the letter sequence of the \( P^2 \) orbit. We perform the operation on the letter sequence of replacing all C's with B's. The point is that this preserves the winding number around the left hand centre (but not the right hand centre - the winding number around the right hand centre becomes 0). Our new letter sequence for this purpose then has \( p + q \) B and \( p \) A. Between any two successive A's there is:

- one B that was originally a C, and
• \( \lfloor \frac{p}{r} \rfloor \) or \( \lceil \frac{p}{r} \rceil \) + 1 other B's from intersections with \( \xi = 2 \).

As in subsection 4.4.3.2 the constancy of \( T \xi \) and \( T \eta \) implies that the B's are spread out as 'evenly as possible' amongst the A's.

We then effectively have the P3 problem of subsection 4.4.3.2 because:

• we have A's and B's evenly mixed with a total of \( p + p + q = 2p + q \) (an even number) letters in our sequence.
• \( \gcd(p + q, p) = \gcd(p, q) = 1 \).

The winding number around the left hand centre is therefore \( \pm 1 \). A similar analysis applies to the right hand centre. □

We have now proved all the separate parts of theorem 4.4.1.

4.5 Some implications of our homological classification

4.5.1 Reducibility

Definition 4.5.1. Call a free homotopy class \( \gamma \) reducible if it can be written in the form \( \gamma = \beta^r \) for some \( r > 1 \). If a free homotopy class is not reducible we will call it irreducible.

In analogy to chapter 2 we ask whether reducible homotopy classes contain prime-period orbits. Firstly note that non-prime-period periodic orbits necessarily lie in reducible homotopy classes. The converse statement, that reducible homotopy classes only contain non-prime-period periodic orbits, is, in general, false. For example, the homotopy class \( (\alpha_1 \alpha_2)^3 \) is reducible but contains P1 '\( \frac{3}{2} \)' orbits, which are of prime-period. Indeed all elements of \( H_1 \) other than \( \alpha_1 \alpha_2 \) contain prime-period orbits and non prime-period orbits. However we do have a partial converse:

Proposition 4.5.1. There do not exist prime-period orbits of type P2 or P3 in reducible homotopy classes.

Proof. A reducible homotopy class say \( \gamma = \beta^r \) containing P2 orbits will have homology \( \pm (r, \pm r) \) where \( r > 1 \). But prime-period P2 orbits have homology \( \pm (1, \pm 1) \). All prime-period P3 orbits lie in one of the four homotopy classes \( \alpha_{1,2}^{\pm 1} \) which are all irreducible. □

4.5.2 'Density' of homology classes containing periodic orbits

Here we consider all periodic orbits, not just the prime-period ones. Consider the set of all homology classes with winding numbers less than \( R \) in size. There are \( |\{(a, b) \in \mathbb{Z}^2 : -R \leq a, b \leq R\}| = (2R + 1)^2 \) of these. However only the homology classes \( \{(r, \pm r), (r, 0), (0, r) : -R \leq r \leq R\} \) can contain periodic orbits. This has size \( 8R + 1 \). Note that \( \frac{8R + 1}{(2R + 1)^2} \to 0 \) as \( R \to \infty \). In this sense 'very few' homology classes contain periodic orbits. This can be contrasted with the strong force planar 2-centre problem for which every non-null homology (and homotopy) class contains periodic solutions (see, for example, [54]). Furthermore in the strong force case existence results for periodic solutions are independent of the value of the period. In section 4.6 we shall see that the same is not true for the Newtonian problem. Note that up to this point we have not placed any restrictions on the value of \( T \).
4.6 Action Spectra, Morse Indices: A Preliminary Study

We emphasize that this section is only a preliminary study. There are no proofs and many of the statements made are based only on numerical evidence.

4.6.1 Calculating Action and Period

The action and period of a periodic orbit of type \( \frac{b}{q} \) is:

\[
A_T = \int_0^T L \, dt = \int_0^T (p\dot{q} - H) \, dt = \int_{t=0}^{t=T} p\dot{q} \, d\xi + \int_{t=0}^{t=T} p\dot{q} \, d\eta - hT \tag{4.13}
\]

\[
A_T = 2qv \int_{\max\{2,\alpha_1\}}^{\beta_1} p\dot{q} \, d\xi + 2pv \int_\gamma^{t=0} p\dot{q} \, d\eta - hT, \tag{4.14}
\]

\[
T = 2qv \int_{\xi_{\min}}^{\xi_{\text{max}}} \frac{dt}{\dot{q}} = 2qv \int_{\max\{2,\alpha_1\}}^{\beta_1} \frac{\xi^2 - \eta^2}{4(\xi^2 - 4)p\xi}, \tag{4.15}
\]

where \( u = \begin{cases} 
2 & \text{if the orbit is homologically null} \\
1 & \text{otherwise.}
\end{cases} \)

and \( \gamma = \begin{cases} 
-2 & \text{if } c < 0 \\
\sqrt{-\frac{c}{4}} & \text{if } c > 0.
\end{cases} \)

4.6.2 A Procedure for producing an ‘Action Spectrum’ of P1 orbits

Let \( \lambda_2 \doteq \{ x \in \Omega : \xi(x(t)) = 2 \text{ for some } t \} \).

Let \( \lambda_1 \doteq \Omega \setminus \lambda_2 \).

Given a homotopy class \( \beta \in H_1^+ \), \( \beta \) may contain P1 and P2 orbits. We define \( g_j(\beta) = \beta \cap \lambda_j \) for \( j = 1, 2 \). We have \( \beta = g_1(\beta) \cup g_2(\beta) \). The set \( g_j(\beta) \) only contains orbits of type \( P_j \).

Remark 4.6.1. The interior of \( g_j(\beta) \) is open in \( H^1(\mathbb{R}/\mathbb{T}^2, \mathbb{Q}) \). Therefore if we restrict to such a set then local properties are preserved.

The procedure is:

- For any given \( \beta \in H_1^+ \) we use theorem 4.4.1 to find all those periodic orbits in \( g_1(\beta) \); given \( p \in \mathbb{Z}\setminus\{0\} \) the set \( g_1((\alpha_1,\alpha_2)p) \) contains periodic orbits of type \( \frac{b}{q} \) for all \( q \in \{1, ..., p-1\} \).

- Take one of these P1 orbits in \( g_1(\beta) \). Suppose it is of type \( \frac{b}{q} \). Consider the set:

\[
l(p, q) = \{(c, h) \in R_1 : \Delta(c, h) = \frac{p}{q}\}.
\]

These are contours of figure 4.4. They are found numerically.

- \( A_T \) and \( T \) can be numerically calculated along \( l(p, q) \) using equations (4.14) and (4.15). We can then plot \( A_T \) as a function of \( T \) for P1 orbits of type \( \frac{b}{q} \).

- Do this for all the P1 orbits in \( \beta \) and include singular orbits of significance in our plot (i.e. elliptical orbits). We then obtain the ‘action spectrum’ for P1 orbits in that class.
4.6.2.1 Action Spectra of P1 Orbits

Consider $g_1((\alpha_1 \alpha_2)^p)$. This contains the P1 orbits of type \('p'\) where $q \in \{1, \ldots, p-1\}$ and elliptical orbits that wind $p$ times clockwise around both centres. In figure 4.8 we have graphed three of the orbits of the action spectrum for the case $p = 7$. Thinking of $T$ as a bifurcation parameter the orbits bifurcate off the elliptical orbit that winds 7 times around both centres.

![Figure 4.8: Plot of $\mathcal{A}_T$ for '7-ellipses', '7/6' and '7/5' orbits.](image)

4.6.2.2 Conjecture: general description of action spectrum of $g_1((\alpha_1 \alpha_2)^p)$

Non-elliptical orbits in $g_1((\alpha_1 \alpha_2)^p)$ have $q \in \{1, \ldots, p-1\}$. These bifurcate off the 'p-ellipses' orbit. The 'p-ellipses' orbit has values of $c$ and $h$ such that $ch = 4$, $c < -4$, $h < 0$. The period of the 'p-ellipses' orbit increases as $c$ decreases. Our numerical data for $\Delta|_{R_1}$ tells us that $\Delta(R_1) = (1, \infty)$. Further it says that for fixed $h$, $\Delta|_{R_1}$ monotonically increases as $c$ increases with $\Delta|_{R_1} \to 1^+$ as $c \to -\infty$ and $\Delta \to \infty$ as we approach the line $h = \frac{-5}{4} - 2$. The contour $\Delta = \frac{p-1}{p}$ is the leftmost in $R_1$ with numerator $p$, the contour $\Delta = \frac{p}{p-2}$ is the second leftmost with numerator $p$ and so on until $\Delta = \frac{p}{q}$. As $T$ decreases from $+\infty$ (and $c$ increases along $ch = 4$) these bifurcations occur off 'p-ellipses' in order of decreasing denominator. Letting $T_q$ for $q \in \{1, \ldots, p-1\}$ be the value of $T$ at which an orbit with $\Delta = \frac{p}{q}$ bifurcates off the 'p-ellipses' orbit. We have:

$$T_1 < \ldots < T_{p-1}.$$

Furthermore, if we fix the period to be $T$, then:

- there exists $T_0 > 0$ such that $T_0 < T_1$ and there exist 'p-ellipses' orbits if and only if $T > T_0$.
  (Numerics indicate that $T_0$ is approximately 2.014$p$).
there exist $\frac{p}{q}$ orbits if and only if $T > T_q$.

The numerics indicate that:

- the 'p-ellipses' orbit always has the highest action of all the orbits in $g_1((\alpha_1\alpha_2)^p)$.
- the 'p-ellipses' orbit always has the highest action of all the orbits in $g_1((\alpha_1\alpha_2)^p)$.
- the $\frac{p}{q}$ orbit has lower action than the $\frac{q}{q+1}$ for all values of $T$ such that both orbits exist.

Figure 4.9: Crude sketch of action spectrum in $g_1((\alpha_1\alpha_2)^p)$, annotated with Morse indices.

This has been summarized in figure 4.9.

4.6.3 Morse indices of $P1$ type orbits

The Morse index of a periodic orbit $x$ is defined to be the maximal positive integer $m$ such that the Hessian of the action functional at $x$ is negative on an $m$-dimensional subspace of $x$.

4.6.3.1 Index of 'p-ellipses' for large $T$

Suppose we move 'left' along the curve $ch = 4$ ($h \to 0^-$ and $c \to -\infty$). Then, up to scalings, our orbits tends towards a circular orbit of the Kepler problem. From chapter 2 (see conjecture 2.3.2) the Morse index of an orbit in such a limit is expected to be $2(p - 1)$.
4.6.3.2 Tracking Morse Indices

To calculate the index exchange at a bifurcation we use the same idea as for the one centre problem with potentials $-\|z\|^2$, see subsection 2.3.13.2. Namely, we track the index exchange at bifurcations. The planar Newtonian 2-centre problem and the central force problem with potential $-\|z\|^p$ studied in chapter 2 are similar in that they are both planar, integrable and have two constants of motion. Here we have elliptical orbits rather than circular ones and we are using $T$ as a bifurcation parameter rather than $a$. In this initial study we assume that the index exchange behaviour is the same as in chapter 2. In particular, if $x^T_p$ denotes an elliptical orbit of period $T$ in the homotopy class $(a_1, a_2)^p$ then we assume an analogue of proposition 2.3.4, namely:

$$\dim \ker d^2A_T(x^T_p) = \begin{cases} 
1 & \text{if } T \text{ is not at a bifurcation value,} \\
3 & \text{if we are at a bifurcation value.}
\end{cases} \quad (4.16)$$

One of these dimensions comes from $A_T$ being an $S^1$-invariant function under phase-shifts. It may be possible to prove (4.16) in a similar way to chapter 2 but the calculations are long and messy and are not attempted here. Further we assume that the critical orbits that bifurcate off the ‘$p$-ellipses’ periodic orbits are nondegenerate, so that their Morse indices are constant along the ‘$p$’ branches of our bifurcation diagram 4.9.

If we look at our action spectrum for $(a_1, a_2)^p$ we see that there are no bifurcations for $T \in (T_{p-1}, \infty)$. There is only an index exchange at bifurcations. Therefore the ‘$p$-ellipses’ orbit has index $2(p-1)$ for $T > T_{p-1}$. By tracking the index we can label the branches in figure 4.7 with Morse indices. From this we formulate the conjecture that:

- the Morse Index of a $P_1$ ‘$q$’ orbit is $2(q - 1)$ for those values of $T$ for which it exists i.e. $T > T_q$.

- the ‘$p$-ellipses’ orbit has a Morse index depending on $T$. For $T_q \leq T < T_{q+1}$ the ‘$p$-ellipses’ orbit has Morse index $2q$ for $q \in \{0, 1, ..., p - 2\}$. For $T \geq T_{p-1}$ the ‘$p$-ellipses’ orbit has Morse index $2(p - 1)$.
Chapter 5

Periodic Solutions of The Molecular $N$-body Problem

_The universe is made of stories, not atoms._
- Roger Penrose

The material presented in this chapter is a joint research project with Dr. Luca Sbano.

Abstract

In this chapter we consider a system of $N$ identical particles interacting through a potential of Lennard-Jones (LJ) type. We consider any subset of loop space that satisfies a few basic conditions, one of which corresponds to the notion of 'tiedness' introduced by Gordon in [41]. We then show, by means of critical point theory, that this system admits periodic solutions in every homotopy class of this subset of loop space. More precisely we show that every homotopy class contains at least two periodic solutions for sufficiently large periods. One of these solutions is a local minimum and the other is a mountain-pass critical point of the action functional. We also prove that given a homotopy class of one of these subsets there do not exist any periodic solutions in it for sufficiently small periods. Our results have wide applicability. For example, one can consider the space of choreographies and prove existence results for choreographical solutions. Our existence proof relies upon an assumption that global minimizers of standard strong force potentials on suitable spaces are nondegenerate up to some symmetries. To finish the chapter we consider the problem of finding periodic solutions in classes of loops that do not satisfy any tiedness condition. In particular we use a result in [27] to construct a periodic solution of the restricted spatial $(2N + 2)$-body problem.

5.1 Introduction

Variational methods and critical point theory are the chief non-perturbative methods used to prove the existence of periodic orbits of the $N$-body problem; for a general overview see [1]. Of central interest in the literature are so called choreographic motions - these are periodic solutions in which all bodies travel along the same closed curve in the plane. In this chapter we shall consider more
general classes of loops and will treat choreographical motions as a corollary. For \( N \in \{3, 4\} \) choreographic orbits have been found by combining the symmetrical structure of the \( N \)-body problem for equal masses with methods of the calculus of variations. In finding periodic solutions of \( N \)-body systems the usual difficulty is avoiding collisions i.e. showing that the critical point is not a solution along which two or more bodies collide. In the cases considered in [14, 15, 19, 20] the symmetries allowed the identification of a class over which one could minimize the action functional. It was then shown that the minimum obtained was collision-free. Very recently, in [3], mountain pass choreographic solutions (possibly with collisions) of the gravitational 3-body problem in a rotating frame have been found. However all other periodic solutions that have been found for the \( N \)-body problem so far have been minimizers of the action functional on a suitable class of symmetric loops. This minimizing approach is also used in numerical studies of \( N \)-body systems (see [18]). In 1996 C.Marchal [48] made a breakthrough in understanding the mechanism involved in avoiding singularities of the gravitational problem for which the action functional does not necessarily diverge on collisions. Using Marchal’s idea D.Ferrario and S.Terracini very recently constructed a general strategy to prove the existence of periodic solutions with prescribed symmetries in the gravitational \( N \)-body problem, see [36]. In their approach, Ferrario and Terracini gave a geometric characterization of classes of choreographies that are obtained as collisionless minimizers of the action functional in the \( N \)-body problem.

No such direct minimizing approach can be employed in systems with potentials of Lennard-Jones type. This is because the action functional, denoted by \( B^n \), is not bounded from below and in fact diverges to \(-\infty\) on any sequence of loops tending towards a loop with a collision. Indeed we point out that, to date, there are no results on the existence of periodic solutions for \( N \)-body systems with \( N > 2 \) and a potential different from the homogeneous one of the form \(-1/r^a\) with \( \alpha > 0 \). In this chapter we shall consider a system of \( N \) identical particles in \( \mathbb{R}^2 \) interacting through a Lennard-Jones potential:

\[
V(x_1, \ldots, x_N) = \sum_{i \neq j}^N \left[ \frac{1}{\|x_i - x_j\|^\beta} - \frac{1}{\|x_i - x_j\|^\alpha} \right], \alpha, \beta \in \mathbb{N} \text{ and } \beta > \alpha \geq 2.
\]

The standard Lennard-Jones potential has \( \alpha = 6 \) and \( \beta = 12 \), see [65]. Such a potential is often used to model molecular systems. The Lennard-Jones potential has two distinguishing features. Firstly the action functional is not bounded from below. Secondly, the action satisfies the strong force condition (see [1, 6, 27, 63]). The first property implies that, in general, we cannot use a direct minimizing approach and the second property guarantees that the action functional diverges to \(-\infty\) if we approach a loop with collisions.

A variant of this type of problem can be found in [1, 27], in which a system is considered that consists of a single particle in \( \mathbb{R}^n \) acted on by a potential that is like the Lennard-Jones central force potential near the origin. Such a system is not necessarily integrable as it is only central force in a small neighbourhood of the origin. In [27] Morse theory was employed and, in particular, the homology of the level sets of the action functional were estimated. In [1] the existence of periodic solutions was proved by constructing a mountain-pass on the whole loop space for the action functional of the Lennard-Jones type system. We remark that in both these papers no topological constraints were imposed when proving the existence of periodic solutions and the space of loops they considered lacked coercivity. Interestingly coercivity can be an obstruction to existence proofs
because the non-existence of low action orbits at large distances, although normally useful, is actually a hinderance if one is trying to construct a mountain-pass geometry. For example these works cannot be readily adapted to choreographies, where the action functional is necessarily coercive.

We are interested in the study of critical points of the action $B^\eta$ on connected components (called homotopy classes) of a space loops, denoted by $A_s^*$ satisfying certain conditions (specified in definition 5.2.6). Firstly we give a presentation of the key properties of the system. Next we show that given any homotopy class there are no periodic solutions in that homotopy class for sufficiently small periods. We do this by showing that for small periods the action functional has nonzero gradient along scaling directions for all points in the homotopy class. To tackle existence for large periods we show that there is a link between the problem of finding critical points of the Lennard-Jones system when the period, $T$, is set to be very large and the standard strong force (SF) system with period 1. This can be understood by the following argument. Define $x(t)$ to be the locations of the bodies as a function of time so that $x : [0, T] \rightarrow \mathbb{R}^{2N}$. We define $y : [0, 1] \rightarrow \mathbb{R}^{2N}$ by the equation:

$$x(t) = T^{\frac{1}{\alpha+2}} y(t/T).$$

The equation (5.1) is well defined on any class of functions. In particular we can consider only those paths in $C^2$ and show that the equations of motion written in terms of $y$ are:

$$\ddot{y}_k(s) = -\sum_{k>j}^N \alpha \frac{(y_k(s) - y_j(s))}{\|y_k(t) - y_j(t)\|^{\alpha+2}} + \eta \sum_{k>j}^N \frac{\beta (y_k(s) - y_j(s))}{\|y_k(t) - y_j(t)\|^{\beta+2}}$$

where

$$\eta = T^{-\frac{\beta(\beta-\alpha)}{\alpha+2}}.$$

At $\eta = 0$, namely $T = \infty$, the system becomes a standard strong force system. In this way one can consider the Lennard-Jones system at large values of $T$ to be a perturbation of a SF system. To do so it is convenient to look at a variational principle whose Euler-Lagrange equations are given by (5.2). One such action functional is given by:

$$B^\eta[y] = B_\alpha[y] - \eta I_\beta[y]$$

where

$$I_\beta[y] = -\int_0^1 ds \sum_{k>j}^N \frac{1}{\|y_k(s) - y_j(s)\|^\beta} \quad \text{and}$$

$$B_\alpha[y] = \frac{1}{2} \int_0^1 ds \sum_k^N \|y_k(s)\|^2 + \int_0^1 \sum_{k>j}^N \frac{1}{\|y_k(s) - y_j(s)\|^\alpha}.$$

Periodic solutions can be found by finding critical points of $B^\eta$ and then using equation (5.1) to find the corresponding solution of the original system. The functional $B_\alpha[y]$ is a strong-force action functional. Critical points of $B_\alpha$ are collisionless; the existence of choreographic critical points has been established by Chenciner, Gerver, Montgomery and Simó in [18]. They are also degenerate due to symmetries of $B_\alpha$: namely phase-shifts and rotations of the loops. Let $\Sigma_0$ be the manifold of critical points. Assuming that $\Sigma_0$ is a nondegenerate manifold with respect to $B_\alpha$ we show that critical points persist to the functional $B^\eta$ for small $\eta > 0$. The existence of this minimum then
allows us to also prove the existence of a mountain-pass critical point by using the fact that the action functional diverges to $-\infty$ if we approach a loop with collisions. This proof is essentially the mountain-pass theorem due to Rabinowitz in [64]. To finish the chapter we consider the problem of finding periodic solutions in classes of loops that do not satisfy condition (5.12) of definition 5.2.6. In particular we use a result by Coti Zelati in [27] to construct a periodic solution of the restricted spatial $(2N + 2)$-body problem.

5.2 Setup

5.2.1 Equations of motion

Consider $N$ identical particles with positions $x_i(t) \in \mathbb{R}^2$ moving under a potential

$$
\frac{1}{\|x_i - x_j\|^\beta} - \frac{1}{\|x_i - x_j\|^\alpha}, \quad \beta > \alpha \geq 2.
$$

The configuration space is:

$$
\mathcal{M} = \{x = (x_1, \ldots, x_N) \in (\mathbb{R}^2)^N : \sum_{i=1}^{N} x_i = 0\} \setminus K_c,
$$

where $K_c$ is the diagonal:

$$
K_c = \{x \in (\mathbb{R}^2)^N : \sum_{i=1}^{N} x_i = 0 \text{ and there exists } i \neq j \text{ such that } x_i = x_j\}.
$$

We shall denote the Euclidean scalar product by $(\cdot, \cdot)$, and the associated norm by $\|\cdot\|$. Note that we have fixed the centre of mass at the origin.

The equations of motion are:

$$
\ddot{x}_i(t) = \sum_{k > i} \left[ -\frac{\alpha(x_i(t) - x_k(t))}{\|x_i(t) - x_k(t)\|^{\alpha+2}} + \frac{\beta(x_i(t) - x_k(t))}{\|x_i(t) - x_k(t)\|^{\beta+2}} \right]. \quad (5.4)
$$

Let $x(t) = (x_1(t), \ldots, x_N(t))$ for all $t$. Given $x : \mathbb{R} \to \mathcal{M}$ we define the map $\xi_T$ by:

$$
\xi_T(x) = y; \quad y(s) = T^{-\frac{\beta}{\alpha+2}} x(sT). \quad (5.5)
$$

The equations satisfied by $y = \xi_T(x)$ are:

$$
\dot{y}_i(s) = \sum_{k > i} \left[ -\frac{\alpha(y_i(s) - y_k(s))}{\|y_i(s) - y_k(s)\|^{\alpha+2}} + T^{-\frac{2\beta}{\alpha+2}} \frac{\beta(y_i(s) - y_k(s))}{\|y_i(s) - y_k(s)\|^{\beta+2}} \right]. \quad (5.6)
$$

5.2.2 The Action Principle

We are interested in obtaining solutions of (5.4) that satisfy $x(t + T) = x(t)$ for all $t$. I.e. we want to find solutions of period $T$. Equivalently we want to find solutions $y = \xi_T(x)$ of (5.6) such that $y(s) = y(s + 1)$ for all $s$. 

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Definition 5.2.1. Let $\Sigma^\infty(W) \subseteq \mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}, W)$ where $W \subseteq (\mathbb{R}^2)^N$. Let $\Sigma^1(W)$ denote the completion of $\Sigma^\infty$ with respect to the norm $\|\|_1$ defined by:

$$
\|y\|_1^2 = \sum_{i=1}^N \int_0^1 \dot{y}_i(s)^2 \, ds + \sum_{i=1}^N \int_0^1 \ddot{y}_i(s)^2 \, ds.
$$

We define

$$
\Lambda^1 \triangleq \Sigma^1(M).
$$

Remark 5.2.1. Note that:

- $\Lambda^1$ is an open subset of the Hilbert space $\Sigma^1(M \cup K_C)$.

- Sobolev’s theorem says that weak $\Sigma^1$-convergence implies $C^0$ convergence. Hence the functions belonging to $\Sigma^1$ are continuous. For a deeper discussion of the function-analytic preliminaries we refer the reader to section 5 of [41].

Definition 5.2.2. Let $\eta \triangleq T^{-2(\alpha-\alpha_2)}$.

The action principle says that solutions of (5.6) are critical points of the action functional:

$$
B^\eta : \Lambda^1 \to \mathbb{R},
$$

$$
B^\eta[y] \triangleq \frac{1}{2} \int_0^1 ds \sum_k \|\dot{y}_k(s)\|^2 + \int_0^1 ds \sum_{k>j} \frac{1}{\|y_k(s) - y_j(s)\|^\alpha} - \eta \int_0^1 ds \sum_{k>j} \frac{1}{\|y_k(s) - y_j(s)\|^\beta}.
$$

Equations (5.6) are the Euler-Lagrange equations of (5.7). We will write $B^\eta$ in three different ways:

(i)

$$
B^\eta[y] = \frac{1}{2} \int_0^1 ds \sum_k \|\dot{y}_k(s)\|^2 - \int_0^1 ds V_\eta(y(s)).
$$

where

$$
V_\eta(y) \triangleq \sum_{i>j} \left[ \frac{\eta}{\|y_i - y_j\|^\beta} - \frac{1}{\|y_i - y_j\|^\alpha} \right].
$$

The form of $B^\eta$ given in (5.8) will be useful for the proof of the Palais-Smale condition.

(ii)

$$
B^\eta[y] = I_1[y] + I_\alpha[y] - \eta I_\beta[y],
$$

where:

Definition 5.2.3.

$$
I_1[y] \triangleq \frac{1}{2} \int_0^1 ds \sum_k \|\dot{y}_k(s)\|^2,
$$

and

$$
I_m[y] \triangleq \int_0^1 \sum_{k>j} \frac{1}{\|y_k(s) - y_j(s)\|^m} \text{ for } m \in \{\alpha, \beta\}.
$$
The form of $B^\eta$ given in (5.9) will be useful for our non-existence proof for small periods.

\[(5.10) \quad B^\eta[y] = B_\alpha[y] - \eta I_\beta[y]\]

where:

\[(5.11) \quad B_\alpha[y] = I_1[y] + I_\alpha[y].\]

The form of $B^\eta$ given in (5.10) will be useful for our existence proof for large periods.

**Definition 5.2.4.** For $\lambda \in \mathbb{R}\setminus\{0\}$, we define an action on $\Lambda^1$ by:

\[\lambda \cdot y(.) \mapsto \lambda \cdot y(.); \quad (\lambda \cdot y_i)(s) = \lambda y_i(s) \text{ for all } i \text{ and all } s.\]

**Definition 5.2.5.** Let:

- $\delta(y) = \inf_s \min_{k,l} \|y_k(s) - y_l(s)\|.$
- $\Delta(y) = \sup_s \max_{k,l} \|y_k(s) - y_l(s)\|.$

**Definition 5.2.6.** We will consider an open subset $\Lambda^1_*$ of $\Lambda^1$ such that:

(i) Any critical point of $B^\eta|_{\Lambda^1_*}$ is a critical point of $B^\eta$.

(ii) $\Lambda^1_*$ is invariant under scalings. That is, if $\lambda > 0$ and $y \in \Lambda^1_*$ then $\lambda \cdot y \in \Lambda^1_*$. 

(iii) there exists $c > 0$ such that

\[(5.12) \quad I_1(y) > c\Delta(y)^2 \text{ for all } y \in \Lambda^1_*.\]

In section 5.5 we will give examples of such sets that meet the criteria of definition 5.2.6. For example the so called ‘space of choreographies’ satisfies the criteria of definition 5.2.6.

### 5.2.3 Properties of $V_\eta$

We now investigate properties of $V_\eta$. These will be used in the proof of the Palais-Smale condition.

**Proposition 5.2.1.** For all $\eta > 0$ the function $V_\eta$ satisfies:

(i) There exists $m > 0$ such that: $V_\eta(y) \geq -m$ for all $y \in \mathcal{M}$.

(ii) For any sequence $y_n \in \mathcal{M}$ such that $y_n \to K_c$ as $n \to \infty$ satisfies:

\[\lim_{n \to \infty} V_\eta(y_n) = +\infty.\]

(iii) The strong force condition: there exists $U \in C^1(\mathbb{R}^{2N}\setminus K_c, \mathbb{R})$, a neighbourhood $\Xi$ of $K_c$, and $c_1 \geq 0$ such that:

\[\lim_{y \to K_c} U(y) = +\infty, \quad V_\eta(y) \geq \|\nabla U(y)\|^2 - c_1 \quad \forall y \in \Xi.\]
(iv) There exists $c_2 > 0$ such that every $y \in \mathcal{M}$ satisfies:

$$\langle \nabla V_\eta(y), y \rangle \leq c_2.$$ 

Proof. (i) Let

$$v_\eta : [0, \infty) \to \mathbb{R} : \quad r \mapsto \eta r^{\beta} - \frac{1}{r^\alpha}.$$ 

Since $\beta > \alpha$ for $v(r)$ we have that $v(r)$ satisfies:

- $v_\eta(r) < 0$ for $r > 1$,
- $\lim_{r \to 0^+} v_\eta(r) = +\infty$,
- $\lim_{r \to +\infty} v_\eta(r) = 0^-$,
- $v_\eta((\eta \beta / \alpha) \frac{1}{\alpha - \beta}) = \inf_{r > 0} v_\eta(r) < 0$.

The potential $V_\eta(y)$ can be written as:

$$V_\eta(y) = \sum_{i < j}^N v_\eta(\|y_i - y_j\|).$$

Therefore

$$V_\eta(y) \geq \frac{N(N - 1)}{2} \inf_{r > 0} v_\eta(r) = \frac{N(N - 1)}{2} v_\eta((\eta \beta / \alpha) \frac{1}{\alpha - \beta}) \geq -m, \quad \text{for all } y \in \mathcal{M}.$$

(ii) The potential $V_\eta(y)$ is:

$$V_\eta(y) = \sum_{i < j}^N v_\eta(\|y_i - y_j\|).$$

if $y \to K_e$ then there exists $(i, j)$ with $i < j$ such that $\|y_i - y_j\| \to 0^+$. This implies $v_\eta(\|x_i - x_j\|) \to +\infty$ and, since $V_\eta$ is bounded from below, we obtain:

$$\lim_{y \to K_e} V_\eta(y) = +\infty.$$

(iii) Consider the function:

$$U(y) = -\sum_{i > j} \ln(\|y_i - y_j\|).$$

Then

$$\nabla_\eta U(y) = \sum_{j \neq i} \frac{(y_i - y_j)}{\|y_i - y_j\|^2}.$$ 

Now

$$\|\nabla U(y)\|^2 = \sum_i \langle \nabla_\eta U(y) \cdot \nabla_\eta U(y) \rangle = \sum_i \left( \sum_{j \neq i} \sum_{l \neq i} \frac{(y_i - y_j) \cdot (y_l - y_i)}{\|y_i - y_j\|^2 \cdot \|y_l - y_i\|^2} \right).$$
This sum can be written as:

\[
\sum_{i} \sum_{j \neq i} \| y_i - y_j \|^2 + \sum_{i} \sum_{j \neq i} \sum_{l \neq i, l \neq j} (y_i - y_j) \cdot (y_l - y_i) \cdot (y_l - y_l).
\]

Finally using Cauchy-Schwarz inequality we obtain:

\[
\| \nabla U(y) \|^2 = \sum_{i > j} \| y_i - y_j \|^2 + \sum_{i} \sum_{j \neq i} \sum_{l \neq i, l \neq j} (y_i - y_j) \cdot (y_l - y_i) \cdot (y_l - y_l) \leq \sum_{i > j} \| y_i - y_j \|^2 + \sum_{i} \sum_{j \neq i} \sum_{l \neq i, l \neq j} \frac{1}{\| y_i - y_j \|^2} \| y_i - y_j \|^2.
\]  \hspace{1cm} (5.13)

Let \( d(y) = \min_{i,j} \| y_i - y_j \| \). Then from (5.13) we have that that:

\[
\| \nabla U(y) \|^2 \leq \frac{C_1}{d^2(y)}
\]  \hspace{1cm} (5.14)

for some \( C_1 > 0 \). Choose \( \ell \) and \( m \) such that

\[
d(y) = \min_{i,j} \| y_i - y_j \| = \| y_1 - y_m \|.
\]

Then we have:

\[
V_\eta(y) = \sum_{i,j} \left( \frac{\eta}{\| y_i - y_j \|^\beta} - \frac{1}{\| y_i - y_j \|^\alpha} \right) = \frac{\eta}{d^\beta(y)} - \frac{1}{d^\alpha(y)} + \sum_{i < j, (i,j) \neq (l,m)} \left( \frac{\eta}{\| y_i - y_j \|^\beta} - \frac{1}{\| y_i - y_j \|^\alpha} \right) \geq \frac{\eta}{d^\beta(y)} - \frac{1}{d^\alpha(y)} + \sum_{i < j, (i,j) \neq (l,m)} \frac{1}{\| y_i - y_j \|^\alpha}.
\]

The latter inequality can be rewritten as follows:

\[
V_\eta(y) \geq \frac{1}{d^\beta(y)} \left( \eta - d^{\beta-\alpha}(y) - d^{\beta-\alpha}(y) \sum_{i < j, (i,j) \neq (l,m)} \frac{d^\alpha(y)}{\| y_i - y_j \|^\alpha} \right)
\]

which implies

\[
V_\eta(y) > \frac{1}{d^\beta(y)} \left( \eta - d^{\beta-\alpha}(y) - \frac{(N+1)(N-2)}{2} d^{\beta-\alpha}(y) \right)
\]  \hspace{1cm} (5.15)

where we have noted that the number of terms in the set

\[
\{ (i,j) : 1 \leq i < j \leq N, \; (i,j) \neq (l,m) \}
\]

is \( \frac{N(N-1)}{2} - 1 = \frac{(N+1)(N-2)}{2} \).

From equations (5.14) and (5.15) we see that to obtain the inequality

\[
V_\eta(y) \geq \| \nabla U(y) \|^2
\]

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it is sufficient that the following inequality is satisfied:
\[
\frac{1}{d^\beta(y)} \left( \eta - d^{\beta-\alpha}(y) - \frac{(N+1)(N-2)}{2} d^{\beta-\alpha}(y) \right) > \frac{1}{d^\beta(y)} C_1 d^{\beta-2}(y).
\] (5.16)

Namely:
\[
\eta - d^{\beta-\alpha}(y) - \frac{(N+1)(N-2)}{2} d^{\beta-\alpha}(y) > C_1 d^{\beta-2}(y).
\] (5.17)

We define
\[
\Xi_\epsilon(K_\alpha) = \{ y \in \mathcal{M} : d(y) \leq \epsilon \}.
\]

Note that if \( y \in \Xi_\epsilon(K_\alpha) \) then:
\[
\eta - d^{\beta-\alpha}(y) - \frac{(N+1)(N-2)}{2} d^{\beta-\alpha}(y) > \eta - \epsilon^{\beta-\alpha}(y) - \frac{(N+1)(N-2)}{2} \epsilon^{\beta-\alpha}
\]
and
\[
C_1 d^{\beta-2}(y) < C_1 \epsilon^{\beta-2}(y),
\]
so for (5.17) to hold it is sufficient to take
\[
\eta - \epsilon^{\beta-\alpha}(y) - \frac{(N+1)(N-2)}{2} \epsilon^{\beta-\alpha}(y) > C_1 \epsilon^{\beta-2}(y).
\] (5.18)

Finally inequality (5.18) holds for sufficiently small \( \epsilon \) since \( \beta > \alpha \geq 2 \).

(iv) We have
\[
V_\eta(y) = \sum_{i < j} \left( \frac{\eta}{\|y_i - y_j\|^\beta} - \frac{1}{\|y_i - y_j\|^\alpha} \right).
\]

Therefore
\[
\frac{\partial V_\eta}{\partial y_i} = \sum_{j \neq i} \frac{\partial}{\partial y_i} \left( \eta(y_i - y_j, y_j - y_j)^{-\frac{\beta}{2}} - (y_i - y_j, y_i - y_j)^{-\frac{\beta}{2}} \right)
\]
which equals
\[
\sum_{j \neq i} \left( -\eta \beta(y_i - y_j) \|y_i - y_j\|^{\beta+2} - \alpha(y_i - y_j) \|y_i - y_j\|^{\alpha+2} \right)
\]
Then
\[
\langle \nabla V_\eta(y), y \rangle = \sum_i \left( \frac{\partial V_\eta}{\partial y_i}, y_i \right) = \sum_{i, i \neq j} \frac{\alpha(\|y_i\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\alpha+2}} - \sum_i \sum_{j \neq i} \frac{\beta(\|y_i\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\beta+2}}.
\]

This equals:
\[
2 \sum_{i < j} \frac{\alpha(\|y_i\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\alpha+2}} + 2 \sum_{j < i} \frac{\alpha(\|y_j\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\alpha+2}} - 2 \sum_{i < j} \frac{\beta(\|y_i\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\beta+2}} - 2 \sum_{j < i} \frac{\beta(\|y_j\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\beta+2}}.
\]

By swapping the dummy indices \( i \) and \( j \) on the second and fourth sums we see this equals:
\[
2 \sum_{i < j} \frac{\alpha(\|y_i\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\alpha+2}} + 2 \sum_{j < i} \frac{\alpha(\|y_j\|^2 - \langle y_i, y_j \rangle)}{\|y_j - y_i\|^{\alpha+2}} - 2 \sum_{i < j} \frac{\beta(\|y_i\|^2 - \langle y_i, y_j \rangle)}{\|y_i - y_j\|^{\beta+2}} - 2 \sum_{j < i} \frac{\beta(\|y_j\|^2 - \langle y_i, y_j \rangle)}{\|y_j - y_i\|^{\beta+2}}.
\]
Therefore:

$$\langle \nabla V_\eta(y), y \rangle = 2 \sum_{i<j} \alpha \Vert y_i - y_j \Vert^2 - 2 \sum_{i<j} \beta \Vert y_i - y_j \Vert^2 \leq 2 \sum_{i<j} \beta \eta \Vert y_i - y_j \Vert^2 - 2 \sum_{i<j} \beta \eta \Vert y_i - y_j \Vert^{\beta+2}. \quad (5.19)$$

We then note from (i) that the RHS of (5.19) equals $-2\beta V_\eta(y) \leq 2 m \beta = c_2$.

### 5.2.4 Scaling properties of the Action

We will study the action functional on the connected components of $\Lambda^1_\alpha$, called homotopy classes. To begin with we study properties of the action functional with respect to the scalings given in definition 5.2.4. By definition 5.2.6 if $y \in \Lambda^1_\alpha$ then $\lambda \cdot y \in \Lambda^1_\alpha$ for all $\lambda > 0$. Therefore scaling transformations do not introduce collisions and therefore do not change the homotopy class that our loop is in. We note the following scaling properties of $I_1$, $I_\alpha$ and $I_\beta$:

$$I_1[\lambda \cdot y] = \lambda^2 I_1[y] \quad \text{and} \quad I_m[\lambda \cdot y] = \lambda^{-m} I_m[y] \quad \text{for} \quad m \in \{\alpha, \beta\}. \quad (5.20)$$

**Definition 5.2.7.** We define the (standard) norm $\|\cdot\|_2$ by

$$\|y\|_2^2 = \sum_{i=1}^N \int_0^1 \|y_i(s)\|^2 ds.$$ 

The following lemma is a consequence of condition (5.12) of definition 5.2.6. It will be of crucial importance to the proof of the Palais-Smale condition.

**Lemma 5.2.1.** Suppose $y \in \Lambda^1_\alpha$. Then

$$\|\dot{y}\|_2^2 \leq \|y\|_1 \leq \left( 1 + \frac{N}{2c} \right) \|\dot{y}\|_2^2.$$ 

**Proof.** Firstly we show that $\Delta(y) \geq D(y)$ where $D(y) = \sup_i \max_j \|y_i(s)\|$ and $\Delta$ is as in definition 5.2.5. Suppose that $D(y) = \|y_{i_0}(s_0)\|$ for some $i_0 \in \{1, \ldots, N\}$ and $s_0 \in \mathbb{R}/\mathbb{Z}$. Note that

$$\Delta = \sup_i \max_j \|y_i(s) - y_j(s)\| \geq \max_j \|y_{i_0}(s_0) - y_j(s_0)\|. \quad (5.21)$$

Suppose, for contradiction, that $\|y_{i_0}(s_0) - y_j(s_0)\| < \|y_{i_0}(s_0)\|$ for all $j$. Then all $y_j(s_0)$ for $j \in \{1, \ldots, N\}$ lie inside a circle in $\mathbb{R}^2$ with centre $y_{i_0}(s_0)$ and radius $\|y_{i_0}(s_0)\|$. Therefore $\sum_j y_j(s_0) \neq 0$. This is a contradiction because the centre of mass is fixed at the origin. Thus there exists $j_0$ such that $\|y_{i_0}(s_0) - y_{j_0}(s_0)\| \geq \|y_{i_0}(s_0)\| = D(y)$. So from (5.21) we deduce that

$$\Delta(y) \geq D(y). \quad (5.22)$$

Using (5.22) with condition (5.12) we have that:

$$I_1(y) \geq cD(y)^2.$$ 

Note from definitions 5.2.3 and 5.2.7 that $I_1(y) = \frac{1}{2} \|\dot{y}\|_2^2$, so:

$$\|\dot{y}\|_2^2 \geq 2cD(y)^2. \quad (5.23)$$
Now $D(y)^2N \geq \sum_{i=1}^{N} \int_0^1 \|y_i(s)\|^2 ds$, so:

$$D(y)^2 \geq \frac{1}{N} \sum_{i=1}^{N} \int_0^1 \|y_i(s)\|^2 ds.$$  \hspace{1cm} (5.24)

Using (5.24) with (5.23) we get that

$$\|\dot{y}\|^2 \geq \frac{2c}{N} \sum_{i=1}^{N} \int_0^1 \|y_i(s)\|^2 ds = \frac{2c}{N} \|y\|^2_2$$

so that

$$\|\dot{y}\|^2 \leq \frac{N}{2c} \|\dot{y}\|^2_2.$$  \hspace{1cm} (5.25)

Finally we use $\|y\|^2_1 = \|y\|^2 + \|\dot{y}\|^2$ with (5.25) to get

$$\|\dot{y}\|^2 \leq \|y\|^2_1 \leq (1 + \frac{N}{2c}) \|\dot{y}\|^2_2.$$  

Remark 5.2.2. Note that the proof of lemma 5.2.1 uses the fact that we have fixed the centre of mass at the origin.

Definition 5.2.8. Given $y \in \Lambda_1^+$, let:

$$f_y(\lambda) = B^y[\lambda \cdot y].$$

By definition 5.2.8 and the scaling properties given in (5.20) we have that

$$f_y(\lambda) = \lambda^2 I_1[y] + \lambda^{-\alpha} I_\alpha[y] - \lambda^{-\beta} I_\beta[y].$$

Definition 5.2.9. Let

$$g[w] = C(\alpha, \beta) \eta^{\frac{\alpha+2}{\beta-\alpha}} I_\beta[w]^{\frac{\alpha+2}{\beta-\alpha}} I_\alpha[w]^{\frac{\alpha+2}{\beta-\alpha}} I_1[w]^{-1},$$  \hspace{1cm} (5.26)

where

$$C(\alpha, \beta) = \frac{1}{2} \frac{\alpha}{\beta} \left( \frac{\alpha + 2}{\beta + 2} \right)^{\frac{\alpha+2}{\beta-\alpha}} - \frac{1}{2} \frac{\alpha}{\beta} \left( \frac{\alpha + 2}{\beta + 2} \right)^{\frac{\alpha+2}{\beta-\alpha}}.$$

Lemma 5.2.2. The function $f_y : [0, \infty) \to \mathbb{R}$ has zero turning points for $g[y] < 1$, one turning point for $g[y] = 1$ and two turning points for $g[y] > 1$.

Proof. Consider:

$$\frac{\partial f_y}{\partial \lambda} = \lambda \left[ 2I_1(z) + \eta\beta I_\beta(z) \frac{\alpha I_\alpha(z)}{\chi^{\alpha+2}} - \frac{\alpha I_\alpha(z)}{\chi^{\alpha+2}} \right] = 0.$$  \hspace{1cm} (5.27)

Note that the function

$$F(\lambda) = -\eta\beta I_\beta(z) \frac{\alpha I_\alpha(z)}{\chi^{\alpha+2}} + \frac{\alpha I_\alpha(z)}{\chi^{\alpha+2}}$$

is such that

$$F(\lambda) \to -\infty \text{ for } \lambda \to 0^+, \text{ and } F(\lambda) \to 0 \text{ for } \lambda \to \infty$$

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and that it has only one positive zero at:

\[ \lambda_0 = \left[ \frac{\eta \beta I_\beta(z)}{\alpha I_\alpha(z)} \right]^{\frac{1}{\beta-\alpha}}. \]

The function \( F(\lambda) \) has only one positive local maximum at:

\[ \lambda_m = \left[ \frac{\beta (\beta + 2) I_\beta(z)}{\alpha (\alpha + 2) I_\alpha(z)} \right]^{\frac{1}{\beta-\alpha}} > \lambda_0. \]  

(5.29)

This is also a global maximum. We deduce that equation (5.27) has two solutions for \( 2I_1 < F(\lambda_m) \), one solution for \( 2I_1 = F(\lambda_m) \) and zero solutions for \( 2I_1 > F(\lambda_m) \). Substituting (5.29) into (5.28) to get an expression for \( F(\lambda_m) \) then yields the result. □

5.3 Nonexistence of periodic solutions for small periods

We can split each of \( I_m(y) \) for \( m = \alpha, \beta \) into two parts:

- **Part \( P_m(y) \):** This is defined as the sum of all pairs and integral over all scaled times (i.e. in \([0,1]\)) such that the mutual separation of bodies in the pair is less than 1. We define:

  \[ p_{kl} = \{ 0 \leq s \leq 1 : \|y_k(s) - y_l(s)\| < 1 \} \]

  and

  \[ P_m(y) = \sum_{k \neq l} \int_{p_{kl}} \frac{ds}{\|y_k(s) - y_l(s)\|^m}, \quad m \in \{\alpha, \beta\}. \]

- **Part \( Q_m(y) \):** This is defined as the sum of all pairs and integral over all times such that the mutual separation of pair in question is more than 1. We define:

  \[ q_{kl} = \{ 0 \leq s \leq 1 : \|y_k(s) - y_l(s)\| \geq 1 \} \]

  and

  \[ Q_m(y) = \sum_{k \neq l} \int_{q_{kl}} \frac{ds}{\|y_k(s) - y_l(s)\|^m}, \quad m \in \{\alpha, \beta\}. \]

We have \( I_m(y) = P_m(y) + Q_m(y) \) for \( m = \alpha, \beta \).

**Lemma 5.3.1.** The quantities \( \sup_y \{Q_m(y)\} \) are finite for \( m \in \{\alpha, \beta\} \).

**Proof.** From the definition of \( Q_m \) we have that for any \( s \in q_{kl} \):

\[ \frac{1}{\|y_k(s) - y_l(s)\|^m} \leq 1 \]

so

\[ Q_m \leq \sum_{k \neq l} \int_{q_{kl}} 1 \, ds \leq \frac{N(N-1)}{2} \]

because there are \( \frac{N(N-1)}{2} \) couples \( k, l \) and the measure of each \( q_{kl} \) is at most 1. □
Lemma 5.3.2. We have the inequality \( P_{\alpha}(y) < P_{\beta}(y) \) for all \( y \in \Lambda_1^1 \).

Proof. We are summing over the same pairs and integrating over the same intervals but the integrands are different. For every each \( s \in \rho_{kl} \) we have \( \|y_k(s) - y_l(s)\| < 1 \) so

\[
\frac{1}{\|y_k(s) - y_l(s)\|^\beta} > \frac{1}{\|y_k(s) - y_l(s)\|^\alpha}
\]

because \( \beta > \alpha \). Therefore \( P_{\beta}(z) > P_{\alpha}(z) \).

Proposition 5.3.1. Suppose we have a sequence of loops \( y^{(n)} \in \Lambda_1^1 \) such that \( I_\alpha(y^{(n)}) \to \infty \). Then \( I_\beta(y^{(n)}) \to \infty \).

Proof. Suppose that on a sequence of loops \( y^{(n)} \) we have that \( I_\alpha(y^{(n)}) \to \infty \). Then \( P_\alpha(y^{(n)}) \to \infty \) because \( Q_\alpha(y^{(n)}) \) is a bounded sequence. By lemma 5.3.2 this implies \( P_\beta(y^{(n)}) \) diverges. Now \( Q_\beta(y^{(n)}) \geq 0 \). Thus \( I_\beta(y^{(n)}) = P_\beta(y^{(n)}) + Q_\beta(y^{(n)}) \) diverges.

Proposition 5.3.2. Given a connected component \( \theta \) of \( \Lambda_1^1 \), there exists \( T^*(\theta) > 0 \) such that there do not exist periodic solutions of period \( T < T^*(\theta) \) in \( \theta \).

Proof. Suppose we are given a homotopy class \( \theta \). A loop \( y \) is a critical point in that class if

\[
Df[y](u) = 0 \quad \text{for all } u.
\]

We now show that, for sufficiently small \( T \), given any loop \( y \) in the homotopy class there is a direction \( u \) for which \( Df[y](u) \neq 0 \).

We consider the set:

\[
\Theta = \{ y \in \theta : I_\beta(y) = 1 \}.
\]

Note that the functional \( g \) defined in (5.26) is invariant under scaling. Therefore we have an equivalence of sets:

\[
g(\theta) = g(\Theta). \tag{5.30}
\]

Firstly we find an upper bound for \( g \) in \( \Theta \) and hence for \( g \) in \( \theta \). By proposition 5.3.1 there exists a constant \( K_1 \) such that

\[
I_\alpha(y) < K_1 \quad \text{for all } y \in \Theta. \tag{5.31}
\]

Recalling the inequality (5.12) and noting that \( \frac{1}{\sqrt{2}} \leq I_\beta(y) \leq 1 \) for \( y \in \Theta \), we deduce that there exists \( c > 0 \) such that:

\[
I_1(y) \geq c \quad \text{for all } y \in \Theta. \tag{5.32}
\]

Using inequalities (5.31) and (5.32), the equivalence of sets (5.30) and our definition of \( g \) given in (5.26) we deduce that:

\[
g[y] < C(\alpha, \beta) \eta^{-\left(\frac{\beta+2}{\beta-\alpha}\right)} K_1^{\frac{\beta+2}{\beta-\alpha}} / c \quad \text{for all } y \in \theta.
\]

If \( \eta^{-\left(\frac{\beta+2}{\beta-\alpha}\right)} < cK_1^{\frac{\beta+2}{\beta-\alpha}} C(\alpha, \beta)^{-1} = K_2^2 \) i.e. \( T < K_2 \) then \( g[y] < 1 \) so, by lemma 5.2.2, \( f_y(\lambda) \) has no turning points as a function of \( \lambda \). This condition corresponds to the nonexistence of turning points of \( B^0[\lambda \cdot y] \) with respect to \( \lambda \). Now set \( \lambda = 1 + \phi \) and note that:

\[
\frac{d}{d\lambda} B^0[\lambda \cdot y]_{\lambda=1} = \frac{d}{d\phi} B^0[(1 + \phi) \cdot y]_{\phi=0} \neq 0
\]

which implies that, for all \( y \in \Lambda_1^1 \), along variations \( y \mapsto y + \phi \cdot y \), the action \( B^0 \) is not stationary. Therefore no \( y \) in the class can be a critical point of the action functional.
In the following subsection we prove, subject to an assumption, the existence of periodic solutions in every connected component of $\Lambda_1^1$.

5.4 Existence for Large Periods

5.4.1 Local minimizer

We recall the following result (see [18], [59]):

Theorem 5.4.1. The planar N-body problem with action $B_\alpha$ ($\alpha \geq 2$) attains a collisionless minimizer $y_0$ in each connected component of $\Lambda_1^1$.

Remark 5.4.1. The point is that condition (5.12) ensures coercivity of $B_\alpha$. The functional $B_\alpha$ is bounded below by $0$ and, furthermore, standard strong force action functionals are such that the action diverges to $+\infty$ as we approach a loop with collisions. The solutions of theorem 5.4.1 known to exist correspond to the global minimizers of $B_\alpha$ on $\Lambda_1^1$.

Definition 5.4.1. Take a solution, say $y_0$ for the SF action $B_\alpha$. Define the group $G$ to be a symmetry group generated by phase-shifts and rotations. We have $B_\alpha(g \cdot y) = B_\alpha(y)$ for all $g \in G$. The manifold $\Sigma_0$ is defined to be the $G$-orbit of $y_0$.

The group $G$ is a 2-torus generated by phase-shifts and rotations. Note that $\Sigma_0$ is compact because $G$ is compact. Further, $\text{Cat}(\Sigma_0) \geq 1$.

Assumption (I): We assume that $\Sigma_0$ is a nondegenerate critical manifold for $B_\alpha$:

$$T_y \Sigma_0 = \ker D^2 B_\alpha[y]$$

for every $y \in \Sigma_0$.

Let $\text{Cat}(.)$ denote the Lusternik-Schnirelman category. We quote the following result from [13]:

Proposition 5.4.1. Let $M$ be a Hilbert-Riemannian manifold and let $f : [0,1] \times M \to \mathbb{R}$ be a $C^2$ function. Let $\Sigma_0$ be a compact nondegenerate critical manifold for $f^0 = f(0,.)$. Assume that $D^2 f^0(x)$ is a Fredholm operator for every $y \in \Sigma$. Then there exists a $\epsilon$ and a neighbourhood $U$ of $\Sigma_0$ such that for all $0 < \epsilon < \bar{\epsilon}$ the function $f^\epsilon = f(\epsilon,.)$ has at least $\text{Cat}(\Sigma_0)$ critical points in $U$.

In proposition 5.4.1 we take $M = \Lambda_1^1$, define $f^\eta \equiv B^n$ and take $\Sigma_0$ as in definition 5.4.1. It is known (see [2]) that $D^2 B_\alpha$ is Fredholm for $\alpha \geq 2$. Therefore, provided assumption (I) is true, proposition 5.4.1 implies that some critical points of $B_\alpha$ persist to critical points of $B^n$ for small $\eta > 0$.

Remark 5.4.2. The number of critical points of $B^n[.]$ remains infinite when $\eta$ changes from zero to small and strictly positive because if $y$ is a critical point then so is $g \cdot y$ for all $g \in G$.

5.4.2 Mountain Pass

To construct the geometry of a Mountain Pass we require:

Proposition 5.4.2. For $\eta > 0$ small enough there exists a local orbit of minimisers of $B^n[.]$. 

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Proof. Given any connected component of $\Lambda \ast$, $B_\alpha[\cdot]$ has a minimizer, say $y^0_m$, on that class. The minimizer $y^0_m$ is a member of a critical manifold $\Sigma_\eta \doteq G \cdot y^0_m$. It is assumed that $\Sigma_\eta$ satisfies assumption (I). Proposition 5.4.1 implies that the minimizer $y^0_m$ of $B_\alpha$ can be continued into a critical point $y^\eta_m$ of $B^\eta$ for all $\eta > 0$ sufficiently small. Since $B^\eta$ is $G$ invariant, $\Sigma_\eta \doteq G \cdot y^\eta_m$ is a critical manifold of $B^\eta$ with

$$D^2B^\eta[y^\eta_m](v,v) = 0 \text{ for } v \text{ tangent to } \Sigma_\eta.$$  

Assumption (I) guarantees that:

$$D^2B^\eta[y^\eta_m](u,u) = D^2B_\alpha[y^0_m](u,u) > 0 \quad (5.33)$$

for any $u$ transversal to $\Sigma_\eta$.

In [13], as a consequence of the proof in proposition 5.4.1 the manifold $\Sigma_\eta$ is deformed into another smooth manifold $\Sigma_\eta$ such that if $u_\eta$ is transversal to $\Sigma_\eta$ then $u_\eta$ is transversal to $\Sigma_0$ for all $\eta > 0$ sufficiently small. By continuity this and $(5.33)$ imply that:

$$D^2B^\eta[y^\eta_m](u_\eta, u_\eta) > 0$$

for all $\eta > 0$ sufficiently small. □

We shall use this minimizer to construct a mountain pass. To prove our geometry yields critical points we need two major tools:

Definition 5.4.2 (Palais-Smale). $B^\eta[\cdot]$ satisfies the Palais Smale-condition if for any $(y^{(n)}) \subset \Lambda \ast$ such that:

$$\langle B^\eta[y^{(n)}], y^{(n)} \rangle \rightarrow c, \text{ and } \|DB^\eta[y^{(n)}]\|_{-1} \rightarrow 0,$$

we have that $y^{(n)}$ converges up to a subsequence.

Proposition 5.4.3. The Palais-Smale condition holds for $B^\eta[\cdot]$ in $\Lambda \ast$.

Proof. The proof is an adaptation of the proof presented in [27].

We take $(y^{(n)}) \subset \Lambda \ast$ such that:

$$B^\eta[y^{(n)}] \rightarrow c, \text{ and } \|DB^\eta[y^{(n)}]\|_{-1} \rightarrow 0.$$

We want to show that $y^{(n)}$ converges up to a subsequence. We recall that as usual $\|\cdot\|_{-1}$ denotes the norm for the dual of the Sobolev space $\Sigma^1$.

Let us evaluate

$$\langle DB^\eta[y^{(n)}], y^{(n)} \rangle_1 = \sum_{i=1}^N \int_0^1 ds \|y_i^{(n)}(s)\|^2 - \sum_{i=1}^N \int_0^1 ds \langle y_i^{(n)}(s), \nabla_i V_\eta(y^{(n)}(s)) \rangle. \quad (5.34)$$

Here $\nabla_i$ denotes $\frac{\partial}{\partial y_i}$ for all $i \in \{1, \ldots, N\}$. Using property (iv) of $\nabla V_\eta$ in proposition 5.2.1 with $(5.34)$ we get:

$$\|y^{(n)}\|_2^2 = \sum_{i=1}^N \int_0^1 \|y_i^{(n)}(s)\|^2 ds \leq \langle DB^\eta[y^{(n)}], y^{(n)} \rangle_1 + c_2 \leq \|y^{(n)}\|_1 \|DB^\eta[y^{(n)}]\|_{-1} + c_2$$

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for some $c_2 > 0$. By lemma 5.2.1 we know there exists $c_3 > 0$ such that $\|y\|_1^2 \leq c_3\|y\|_2^2$ for all $y \in \Lambda_1^4$. Thus

$$\|y^{(n)}\|_1^2 \leq c_3\|y^{(n)}\|_1 \|DB^n[y^{(n)}]\|_{-1} + c_3 c_2$$

and so

$$0 \leq \|y^{(n)}\|_1 \leq \frac{1}{2} \left[ c_3\|DB^n[y^{(n)}]\|_{-1} + \sqrt{c_3^2\|DB^n[y^{(n)}]\|_{-1}^2 + 4c_3c_2} \right].$$

Since $\|DB^n[y^{(n)}]\|_{-1} \to 0$ this implies that $\|y^{(n)}\|_1$ is bounded. Therefore, by a Sobolev embedding theorem, there exists a subsequence, that we still denote by $(y^{(n)})$, and a loop $y^* \in \Lambda_1^4$ such that such that $y^{(n)} \to y^*$ weakly in $\Sigma^1$ and strongly in $C^0$. Now assume $y^* \in \partial\Lambda_1^4$. Then there exists $s^*$ such that $y^*(s^*) \in K_c$. By the continuity of $y^*(s)$, there exists $\delta > 0$ such that $y^*(s^* + \delta)$ is in a neighbourhood of $K_c$. We can evaluate

$$-U(y^{(n)}(s^* + \delta)) + U(y^{(n)}(s^*)) = -\int_{s^*}^{s^* + \delta} ds\frac{d}{ds}U(y^{(n)}(s)) = \int_{s^*}^{s^* + \delta} ds\{-\nabla U(y^{(n)}(s)), y^{(n)}(s)\}.$$

Therefore, by the Schwarz inequality we have:

$$-U(y^{(n)}(s^* + \delta)) + U(y^{(n)}(s^*)) \leq \int_{s^*}^{s^* + \delta} ds\|\nabla U(y^{(n)}(s))\|\|y^{(n)}(s)\|.$$ 

Using Schwarz's inequality again we obtain:

$$\int_{s^*}^{s^* + \delta} ds\|\nabla U(y^{(n)}(s))\| \|y^{(n)}(s)\| \leq \sqrt{\int_{s^*}^{s^* + \delta} ds\|\nabla U(y^{(n)}(s))\|^2} \sqrt{\int_{s^*}^{s^* + \delta} \|y^{(n)}(s)\|^2} ds.$$

Now using $\|y\|_1 \geq \|y\|_2$ (see lemma 5.2.1) and

$$V^n(y) \geq \|\nabla U(x)\|^2 - c_1 \quad (\text{see part (iii) of proposition 5.2.1})$$

for $y$ in a neighbourhood of $K_c$, we can write:

$$-U(y^{(n)}(s^* + \delta)) + U(y^{(n)}(s^*)) \leq \|y^{(n)}\|_1 \sqrt{\int_{s^*}^{s^* + \delta} dt(V^n(y^{(n)}(s)) + c_1).}$$

We have $y^{(n)}(s^*) \to y^*(s^*) \in K_c$ so $U(y^{(n)}(s^*)) \to +\infty$ whereas $-U(y^{(n)}(s^* + \delta))$ is bounded. Therefore

$$\int_{s^*}^{s^* + \delta} dtV^n(y^{(n)}(s)) \to +\infty$$

since $\|y^{(n)}\|_1$ is bounded. Then, using part (i) of proposition 5.2.1 to estimate the potential in $[s^*, s^* + \delta]$, we have:

$$B^n[y^{(n)}] = \frac{\|y^{(n)}\|_2^2}{2} - \int_{s^*}^{s^* + \delta} dsV^n(y^{(n)}(s)) - \int_{s^*}^{s^* + \delta} dtV^n(y^{(n)}(s)) \leq \frac{\|y^{(n)}\|_2^2}{2} - \int_{s^*}^{s^* + \delta} dtV^n(y^{(n)}(s)) + m(T - \delta) \to -\infty \text{ for } n \to \infty.$$
This would contradict the definition of Palais-Smale sequence. Hence we have that $y^{(n)}(.)$ tends weakly in $\Sigma^1$ to $y^* \notin \partial \Lambda^1_1$. We now show that the convergence of $y^{(n)}$ to $y^*$ is strong in $\Sigma^1$. By lemma 5.2.1 we have

$$
\|y^{(n)} - y^*\|_1 \leq (1 + \frac{N}{2c}) \|y^{(n)} - y^*\|_2 = (1 + \frac{N}{2c}) \sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \ddot{y}^{(n)}_i(s) - \ddot{y}^*_i(s) \rangle,
$$

so that:

$$
\|y^{(n)} - y^*\|_1^2 \leq (1 + \frac{N}{2c}) \sum_{i=1}^{N} \left[ \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \ddot{y}^{(n)}_i(s) - \ddot{y}^*_i(s), \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s) \rangle \right]. \tag{5.35}
$$

Note that (5.35) has a term

$$
(1 + \frac{N}{2c}) \sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \ddot{y}^{(n)}_i(s) - \ddot{y}^*_i(s) \rangle \to 0 \text{ as } n \to \infty \tag{5.36}
$$

because $y^{(n)} \to y^*$ weakly in $\Sigma^1$. The term

$$
(1 + \frac{N}{2c}) \sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \ddot{y}^{(n)}_i(s) \rangle
$$

in (5.35) can be evaluated by observing that we can write $\langle DB^n[y^{(n)}], y^{(n)} - y^* \rangle_1$ as:

$$
\langle DB^n[y^{(n)}], y^{(n)} - y^* \rangle_1 = \sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \ddot{y}^{(n)}_i(s) \rangle - \sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \nabla_i V(y^{(n)}(s)) \rangle. \tag{5.37}
$$

By hypothesis $DB^n[y^{(n)}] \to 0$ as $n \to \infty$. Also, since $y^{(n)}(.)$ converges weakly to $y^*(.) \notin \partial \Lambda^1_1$ as $n \to \infty$, we have:

$$
\sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \nabla_i V(y^{(n)}(s)) \rangle \to 0.
$$

Therefore from (5.37) we deduce

$$
\sum_{i=1}^{N} \int_{0}^{1} ds \langle \dot{y}^{(n)}_i(s) - \dot{y}^*_i(s), \ddot{y}^{(n)}_i(s) \rangle \to 0. \tag{5.38}
$$

Using (5.36) and (5.38) in (5.35) we deduce $\|y^{(n)} - y^*\|_1 \to 0$ as $n \to \infty$. Therefore $y^{(n)}(.)$ converges strongly in $\Sigma^1$. \hfill \square

The second ingredient is the Deformation lemma.

**Definition 5.4.3.** $B^n_c = \{ y \in H : B^n[y] \leq c \}$.

**Lemma 5.4.1 (Deformation lemma).** Let $H$ be a Banach space and $B^n[.] \in C^1[H, R]$. Suppose that $B^n[,.]$ satisfies the PS-condition. If $c$ is not a critical value of $B^n$ then for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ and $\Phi \in C^0([0, 1] \times H, H)$ such that:

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(i) $\Phi(0, y) = y$ if $B^n[y] \in [c - \epsilon, c + \epsilon]$,

(ii) $\Phi(1, B^n_{c + \epsilon}) \subset \Phi(1, B^n_{c - \epsilon})$.

The proof of this lemma can be found in [64].

We now state and prove our main result:

**Theorem 5.4.2.** Subject to assumption (I), every connected component of $\Lambda^1_p$ contains at least two distinct critical orbits of the action functional for sufficiently large $T$. One of these has Morse index $0$ and the other has Morse index $\geq 1$.

**Proof.**

We know that for a fixed homotopy class and $T$ sufficiently large, $B^n[\cdot]$ has a critical point, say $y_m \in \Lambda^1_p$, such that:

$$B^n[y_m] = B^n[g \cdot y_m] > 0 \text{ for all } g \in G.$$  \hspace{1cm} (5.38)

Let $r > 0$ and define:

$$U_r(y_m) = \{ y \in \Lambda^1_p : \inf_{g \in G} ||y - g \cdot y_m||_1 < r \}.$$  \hspace{1cm} (5.39)

The set $U_r(y_m)$ is open in $\Lambda^1_p$ with respect to the $|| \cdot ||_1$-norm. Since $G \cdot y_m$ is a nondegenerate critical manifold and locally minimizes $B^n[\cdot]$ it follows that there exists $r > 0$ such that:

$$0 < B^n[y_m] < \inf_{y \in \partial U_r(y_m)} B^n[y] = c_m.$$  \hspace{1cm} (5.40)

In the same homotopy class as $y_m$ we can construct $y_0$ such that:

$$B^n[y_0] < 0.$$  \hspace{1cm} (5.41)

For example, you could take $y_0 = \mu \cdot y_m$ with $0 < \mu < 1$. To set up a mountain-pass we consider the set of homotopies:

$$\Gamma = \{ \gamma \in C^0([0,1], \Lambda^1_p) : \gamma(0) = y_0 \text{ and } \gamma(1) = y_m \}. \hspace{1cm} (5.39)$$

We define a mountain-pass level as follows:

$$c^* = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} B^n(\gamma(s)). \hspace{1cm} (5.40)$$

Any $\gamma$ joining $y_0$ to $y_m$ has to cross $\partial U_r(y_m)$. We follow the proof given in [64]. We have that:

$$\max_{s \in [0,1]} B^n(\gamma(s)) > \inf_{\partial U_r(y_m)} B^n[y],$$

which implies that $c^* \geq c_m$.

Now one follows the standard argument and uses the Palais-Smale condition and deformation lemma as in [64].

$\square$
5.5 Applications of proposition 5.3.2 and theorem 5.4.2.

5.5.1 Choreographies

Define the space of choreographies by:

\[ \mathcal{X}^1 = \{ y \in \Lambda^1 : y_{n+1}(s + \frac{1}{N}) = y_n(s) \text{ for all } s \in \mathbb{R}/\mathbb{Z} \} \]

Lemma 5.5.1. \( \mathcal{X}^1 \) satisfies the criteria of definition 5.2.6.

Proof. The fact that critical points of \( B^0|_{\mathcal{X}^1} \) are critical points of \( B^0 \) follows from Palais’ principle of symmetric criticality. Scale invariance of \( \mathcal{X}^1 \) is immediate. It remains to show that (5.12) holds. We know there exist \( s_0 \in \mathbb{R}/\mathbb{Z} \) and some \( 1 \leq i_0 \neq j_0 \leq N \) such that

\[ \Delta = \| y_{i_0}(s_0) - y_{j_0}(s_0) \| . \]

Then

\[ \Delta = \| y_{i_0}(s_0) - y_{i_0}(s_0 + \frac{(j_0 - i_0)}{N}) \| . \]

We see that

\[ I_1(y) = \sum_i \int_0^1 \| \dot{y}_i(s) \|^2 ds = N \int_0^1 \| \dot{y}_{i_0}(s) \|^2 ds \geq N \int_{s_0}^{s_0 + \frac{(j_0-i_0)}{N}} \| \dot{y}_{i_0} \|^2 ds , \]

the second equality following from the fact that \( y \in \mathcal{X}^1 \). Using the Cauchy-Schwarz inequality this gives:

\[ I_1(y) \geq \frac{N}{(j_0-i_0) \frac{1}{N}} \left( \int_{s_0}^{s_0 + \frac{(j_0-i_0)}{N}} \| \dot{y}_{i_0}(s) \|^2 ds \right) \geq \frac{N \Delta(y)^2}{(j_0-i_0) \frac{1}{N}} \geq N \Delta(y)^2 , \]

where we have noted that \( (j_0-i_0) \frac{1}{N} \leq 1. \) \( \Box \)

5.5.1.1 Figure Eights

Consider a finite group \( G \) and three representations of \( G \):

- \( \rho : G \to O(\mathbb{R}^2) \),
- \( \sigma : G \to S_N \) and
- \( \tau : G \to \mathbb{R}/\mathbb{Z} \).

For each \( y \in \mathcal{X}^1 \) there is an associated action:

Definition 5.5.1.

\[ g \cdot (y_1, y_2, ..., y_N)(t) = (\rho(g) \cdot y_{\sigma(g^{-1})}(t), ..., \rho(g) \cdot y_{\sigma(g^{-1})}(t)) \]

We consider the space of equivariant loops defined by

Definition 5.5.2.

\[ \mathcal{X}^1_G = \{ y(\cdot) \in \Lambda^1 : (g \cdot y)(s) = y(s) \text{ for all } s \} . \]

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The space $X_q$ satisfies the conditions of definition 5.2.6. As a specific example we consider the symmetry class of the famous Chenciner-Montgomery 'eight' (see [19]). In this example $N = 3$ and

$$G = D_6 = \{g_1, g_2 : g_1^2 = g_2^6 = 1, g_2g_1 = g_1g_2^{-1}\},$$

where:

$$\rho(g_1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(g_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\tau(g_1)(t) = -t, \quad \tau(g_2)(t) = t + \frac{T}{6}$$

and

$$\sigma(g_1) = (2, 3), \quad \sigma(g_2) = (1, 2, 3).$$

### 5.5.2 Tied homotopy classes of $\Lambda^1$

**Definition 5.5.3.** Let us denote arc length by

$$l : \Lambda^1 \to \mathbb{R}, \quad y \to \sum_{i=1}^N \int_0^1 \|\dot{y}_i(s)\|ds.$$ 

We use Gordon's notion of tied (see [41], also [59]):

**Definition 5.5.4.** Say that the connected component $\gamma$ of $\Lambda^1$ is tied to $K_c$ if for all sequences $y^{(n)} \in \gamma$ such that $D(y^{(n)}) \to \infty$ as $n \to \infty$ we have $l(y^{(n)}) \to \infty$.

**Definition 5.5.5.** Given $y \in \Lambda^1$, let $B_i(y) = \{y_i(s) : s \in \mathbb{R}/\mathbb{Z}\}$.

**Lemma 5.5.2.** Suppose $\gamma$ is a tied connected component of $\Lambda^1$. Then for all $y \in \gamma$ and $i \in \{1, ..., N\}$ there exists $j \in \{1, ..., N\} - \{i\}$ such that $B_i(y) \cap B_j(y) \neq \emptyset$.

**Proof.** Suppose there exists $y \in \gamma$ and $i \in \{1, ..., N\}$ such that for all $j \neq i$ we have $B_i(y) \cap B_j(y) = \emptyset$. Then we can move body $i$ off to infinity in one direction, and the rest of the system off to infinity in the opposite direction, without affecting the homotopy class and keeping the centre of mass fixed at the origin. It is possible to do this without the lengths diverging. In other words there exist sequences $y^{(n)} \subset \gamma$ such that $D(y^{(n)}) \to \infty$ but $l(y^{(n)})$ is bounded. Therefore, by definition 5.5.4, $\gamma$ is not tied. \(\square\)

**Remark 5.5.1.** The fact that we have assumed that the centre of mass is fixed at the origin is crucial to lemma 5.5.2. For example $y(s) = (0, e^{2\pi is})$ (where we have identified $\mathbb{R}^2$ with the complex plane) is not a valid counterexample because the centre of mass is at $\frac{1}{2} e^{2\pi is}$ at time $s$.

**Lemma 5.5.3.** If $\gamma$ is tied then $l(y) \geq \Delta(y)$ for all $y \in \gamma$.

**Proof.** Note that:

$$\Delta(y) \leq \sup_{a, b \in \bigcup_i B_i(y)} \|a - b\|. \quad (5.41)$$

Lemma 5.5.2 implies that, since $\gamma$ is tied and $y \in \gamma$, that $\bigcup_i B_i(y) \subset \mathbb{R}^2$ is path connected. Therefore:

$$l(y) \geq \sup_{a, b \in \bigcup_i B_i(y)} \|a - b\|. \quad \square$$

Using this with (5.41) implies the result.

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The following number theory result will be useful, the proof of which can be obtained by induction:

**Lemma 5.5.4.** \[ \left( \sum_{i=1}^{N} a_i \right)^2 \leq 2^{N-1} \sum_{i=1}^{N} a_i^2 \text{ for all } a_i \geq 0. \]

**Lemma 5.5.5.** Suppose \( \gamma \) is a tied homotopy class of \( \Lambda^1 \). Then \( \gamma \) satisfies the conditions of definition 5.2.6.

**Proof.** We note that the connected components of \( \Lambda^1 \) are scale invariant and that critical points in the connected components of \( \Lambda^1 \) are critical points in \( \Lambda^1 \). Now we deal with condition (5.12):

\[
I_1(y) = \frac{1}{2} \sum_{t=1}^{N} \int_{0}^{1} |\dot{y}_t|^2 ds
\]

by definition of \( I_1 \)

\[
\geq \frac{1}{2} \sum_{t=1}^{N} \left( \int_{0}^{1} |\dot{y}_t| ds \right)^2
\]

by Cauchy-Schwarz

\[
\geq \frac{1}{2N} \left( \sum_{t=1}^{N} \int_{0}^{1} |\dot{y}_t| ds \right)^2
\]

using lemma 5.5.4

\[
= \frac{\Delta(y)^2}{2N}
\]

using definition 5.5.3.

Finally we use lemma 5.5.3 to obtain:

\[
I_1(y) \geq \frac{\Delta(y)^2}{2N} \text{ for all } y \in \gamma.
\]

\[\square\]

5.5.3 Hip-hop classes

Up to this point our analysis has been in the plane. In this subsection we show that, subject to assumption (I), the molecular 4-body admits non-planar solutions for large periods. The main points are that:

- We can define a suitable class of paths and employ an action principle as before.
- The hip-hop symmetry class defined in [67] satisfies the conditions of definition 5.2.6.
- All of our results follow as before. However we do not know whether the critical points of \( B_\alpha \) given by theorem 5.4.1 are planar or not.
- To show that the critical points of \( B_\alpha \) given by theorem 5.4.1 are non-planar we generalize Salomone and Xia's proof of the existence of the non-planar (so called 'hip-hop') solutions of the Newtonian 4-body problem to the standard strong force problem with action functional \( B_0 = B_\alpha, \alpha \geq 2 \). We take little credit for this analysis since the generalization of [67] to \( B_\alpha \) is straightforward. Nevertheless as it is crucial to our argument and not completely immediate we have included a concise account of it in this chapter.
- For sufficiently small \( \eta \) the tracked local minimizer of \( B_0 \) is still non-planar.
We now go through these points in more detail.

Definition 5.5.6. Let
\[ \Lambda^1_3 = \{ y \in H^1(\mathbb{R}, (\mathbb{R}^3)^4) : \sum_{i=1}^{4} y_i(s) = 0 \quad \text{for all } s \in \mathbb{R} \}. \]

Note that members of \( \Lambda^1_3 \) are not necessarily periodic.

Let us denote by \((X, Y, Z)\) Cartesian coordinates for \( \mathbb{R}^3 \).

Definition 5.5.7. Let \( Z_i(s) \) denote the Z-component of \( y_i(s) \). Let \( \pi : \mathbb{R}^3 \to \mathbb{R}^2 \), \( \pi((X, Y, Z)) = (X, Y) \).

Definition 5.5.8. Let \( S_d \) be the set of elements of \( \Lambda^1_3 \) such that:

(i) \( \pi(y_i(s + 1)) = R_{2\pi d} \pi(y_i(s)) \),
   where \( R_{2\pi d} \) denotes an oriented rotation in the XY-plane by angle \( 2\pi d \) about the origin.

(ii) \( Z_i(s + 1/2) = -Z_i(s) \),

(iii) \( \pi(y_i(s + \frac{1}{2d})) = \pi(y_{i+1}(s)) \),

for all \( s \in [0,1), 1 \leq i \leq 4 \). We say that paths in \( S_d \) have \( d \)-rotational symmetry with reduced period 1.

Note that we do not find periodic solutions directly. Rather we look at a set of paths with \( d \)-rotational symmetry with reduced period 1. Critical points of the action functional will yield motions in this symmetry class that is not necessarily periodic but, provided \( d \in \mathbb{Q} \), we can then construct a periodic motion from it. Indeed if \( d = \frac{p}{q} \) with \( \gcd(p, q) = 1 \) then given any path in \( S_d \) we can construct a loop of period \( qT \).

The action principle says that periodic solutions in \( \Lambda^1_3 \) correspond to critical points of
\[ B_\alpha : \Lambda^1_3 \to \mathbb{R}, \quad y \mapsto \frac{1}{2} \sum_{i=1}^{4} \int_0^1 \|y_i(s)\|^2 ds + \sum_{1 \leq i < j \leq 4} \int_0^1 \frac{1}{\|y_i(s) - y_j(s)\|^2} ds. \]

Palais' principle of symmetric criticality that any critical points of \( B_\alpha|_{S_d} \) are also critical points of \( B_\alpha|_{\Lambda^1_3} \).

Lemma 5.5.6. The class of symmetric paths \( S_d \) is scale-invariant and, if \( d > \frac{1}{4} \), satisfies condition (5.12).

Proof. Scale invariance of \( S_d \) is immediate from definition 5.5.8. We write \( I_1 \), the kinetic contribution to \( B_\alpha \), as

\[ I_1(y) = I_{XY}(y) + I_Z(y), \]

where
\[ I_{XY}(y) = \frac{1}{2} \sum_{i=1}^{4} \int_0^1 (X_i(s)^2 + Y_i(s)^2) ds, \quad I_Z(y) = \frac{1}{2} \sum_{i=1}^{4} \int_0^1 Z_i^2(s) ds. \]
The first term can be thought of as the kinetic contribution to the action of the loop \( \pi(y(.)) \). From part (iii) of definition 5.5.8 this is part of a choreographic loop of period \( 1/d \). By the same argument used in the proof of lemma 5.5.1 we have that there exists \( c > 0 \) such that

\[
\frac{1}{2} \int_0^1 \left( \dot{X}^2(s) + \dot{Y}^2(s) \right) ds \geq c \Delta_{XY}^2,
\]  

where \( \Delta_{XY} = \sup_s \max_{i,j} \| \pi(y_i(s)) - \pi(y_j(s)) \|. \) Using that the phase-shift between bodies in the choreography is \( 1/4d \) with (5.42) we get:

\[
\frac{1}{2} \int_0^1 \left( \dot{X}^2(s) + \dot{Y}^2(s) \right) ds \geq \frac{1}{4} c \Delta_{XY}^2.
\]

Therefore if \( d > \frac{1}{4} \) we have:

\[
\int_0^1 \left( \dot{X}^2(s) + \dot{Y}^2(s) \right) ds \geq \int_0^1 \left( \dot{X}^2(s) + \dot{Y}^2(s) \right) ds \geq \frac{1}{2} c \Delta_{XY}^2.
\]

Let \( \Delta_2 = \max_{i,j} \sup_s |Z_i(s) - Z_j(s)| \). Then \( \Delta_2 \leq 2 \sup_s \max_i \| Z_i(s) \| = 2 \| Z_0(s_0) \| \) for some \( s_0, t_0 \). We also know that \( Z_i(.) \) is 1-periodic and that \( Z_i(s_0 + \frac{1}{2}) = -Z_i(s_0) \). By Cauchy-Schwarz, \( I_1(y) \geq \frac{1}{2} \int_0^1 Z_i'^2(s) ds \geq \frac{1}{2} \int_{s_0}^{s_0 + \frac{1}{2}} Z_i'^2(s) ds \geq |Z_i(s_0) - Z_i(s_0 + \frac{1}{2})|^2 = 4 |Z_i(s_0)|^2 \geq \Delta_2^2 \). Therefore:

\[
I_1(y) = I_{XY}(y) + I_2(y) \geq \frac{c}{4} \Delta_{XY}^2(y) + \Delta_{Z}^2(y) \geq \min \left\{ \frac{c}{4}, 1 \right\} (\Delta_{XY}(y)^2 + \Delta_2(y)^2) \geq \min \left\{ \frac{c}{4}, 1 \right\} \Delta(y)^2,
\]

where we have used Pythagoras' theorem.

5.5.3.1 Planar Solutions

Definition 5.5.9. Let \( \Omega_2 = \{ y \in S_d : Z_i(s) = 0 \) for all \( s \) and all \( i \} \).

Note that \( \Omega_2 \subset S_d \) is precisely the set of planar choreographies of prime-period \( \frac{1}{4} \). The \( B_\alpha \)-minimizers on \( \Omega_2 \) is known to be the Lagrange solutions, see [7].

We now examine the Lagrange solutions. Let \( d^*(i,j) = \| y_i(s) - y_j(s) \| \) be the distance between bodies \( i \) and \( j \). Since the Lagrange solution is a relative equilibrium this is constant for all \( s \). Suppose that the bodies are positioned around a circle of radius \( R \) in the order 1, 2, 3, 4. Then \( d^*(1, 2) = d^*(2, 3) = d^*(3, 4) = d^*(1, 4) = R \sqrt{2} \), and \( d^*(1, 3) = d^*(2, 4) = 2R \). Therefore:

\[
B_\alpha = 4 \int_0^1 \left( \frac{1}{2} (2 \pi R \alpha) \right)^2 \alpha + 4 \left( \frac{1}{(R \sqrt{2})^\alpha} \right) + 2 \left( \frac{1}{(2R)^\alpha} \right)
\]

i.e.

\[
B_\alpha = \frac{8 \pi^2 R^2 d^2 + (2^{2-\frac{\alpha}{2}} + 2^{1-\alpha}) R^{-\alpha}}{R^{-\alpha}}.
\]

Partial differentiating with respect to \( R \) we obtain:

\[
\frac{\partial B_\alpha}{\partial R} = -4 \pi^2 R d^2 - (2^{2-\frac{\alpha}{2}} + 2^{1-\alpha}) R^{-\alpha-1}.
\]

For relative equilibria we have:

\[
\frac{\partial B_\alpha}{\partial R} = 0.
\]
Therefore the radii of the relative equilibria are:

\[ R_d = \left( \frac{\alpha (2^2 - \frac{2}{3} + 2^1 - \alpha)}{16\pi^2} \right)^{\frac{1}{4}} \left( \frac{1}{d} \right)^{\frac{1}{4}}. \]  

(5.43)

We denote the set of Lagrange solutions by \( L \). We have \( L \simeq S^1 \).

**Proposition 5.5.1.** There exists \( d \in (\frac{4}{5}, 1) \cap \mathbb{Q} \) such that the minimizer of \( B_\alpha \) in \( S_d \) is nonplanar.

**Proof.** To show this we assume the minimizer of \( B_\alpha \) in \( S_d \) is planar and then produce a contradiction by finding a nonplanar loop in \( S_d \) with a lower value of \( B_\alpha \).

We consider \( p^\epsilon(\cdot) \), a one-parameter family of curves in \( S_d \), defined by:

\[ p^\epsilon_k(s) = \left( R_k \cos \left( 2\pi ds + \frac{k\pi}{2} \right), R_k \sin \left( 2\pi ds + \frac{k\pi}{2} \right), \epsilon(-1)^k \sin (2\pi s) \right), \]

and note that \( p^0 \in L \). We seek a power series expansion of the action functional \( B_\alpha(p^\epsilon(s)) \) with respect to \( \epsilon \). Since \( p^0 \) is a critical point of \( B_\alpha \) we have:

\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} B_\alpha(p^\epsilon) = 0. \]

We examine the sign of the second derivative. To compute this, we split \( B_\alpha \) into contributions:

\[ B_\alpha(p^\epsilon) = \int_0^1 K(p^\epsilon(s)) ds + \int_0^1 U(p^\epsilon(s)) ds, \]

where:

\[ K(p^\epsilon(s)) = \frac{1}{2} \sum_{k=1}^4 ||p^\epsilon(s)||^2, \quad U(p^\epsilon(s)) = \sum_{k,j} U_\alpha(||p_k^\epsilon(s) - p_j^\epsilon(s)||), \quad U_\alpha(r) = \frac{1}{r^\alpha}. \]

We can compute the kinetic contribution exactly:

\[ \int_0^1 K(p^\epsilon(s)) ds = \frac{1}{2} \int_0^1 \left( \sum_{i=1}^4 ||p_i^\epsilon(s)||^2 \right) ds = \int_0^1 8\pi^2 (d^2 R_i^2 + \epsilon^2 \cos^2 (2\pi s)) ds = 8\pi^2 R_i^2 d^2 + 4\pi^2 \epsilon^2. \]

The second derivative of (5.44) with respect to \( \epsilon \) is \( 8\pi^2 \). The potential contribution is:

\[ 2 \int_0^1 \frac{1}{(2R_i)^\alpha/2} ds + 4 \int_0^1 \frac{ds}{(2R_i^2 + 4\epsilon^2 \sin^2 (2\pi s))^{2\alpha/3}}. \]

(5.45)

Differentiating (5.45) with respect to \( \epsilon \) kills the first term. As for the second term we note that we can differentiate under the integral sign because the integrand is a uniformly bounded continuous function of \( \epsilon \) on a neighbourhood of \( \epsilon = 0 \). To aid in the calculation consider:

\[ f(\epsilon) = (2R_i^2 + 4\epsilon^2 \sin^2 (2\pi s))^{-\frac{2\alpha}{3}}. \]
Differentiating $f$ twice with respect to $e$ and then setting $e = 0$ we obtain:

$$f''(0) = -4\alpha \sin^2(2\pi e)(2R_e^2)^{-\frac{3}{2}}.$$  

Thus:

$$\frac{d^2}{de^2} \left. U(p^e(s)) \right|_{e=0} \int_0^1 U(p^e(s)) ds = -8\alpha (2R_e^2)^{-\frac{3}{2}}.$$  

So:

$$\frac{d^2}{de^2} \left. B_\alpha(p^e(s)) \right|_{e=0} = 8\pi^2 - 8\alpha (2R_e^2)^{-\frac{3}{2}}.$$  

(5.46)

Substituting (5.43) into (5.46) gives:

$$\frac{d^2}{de^2}_{e=0} B_\alpha = 8\pi^2 \left(1 - \frac{16d^2}{2^{\frac{3}{2}+1}(2^{2-\frac{3}{2}} + 2^{1-\alpha})}\right).$$

Let

$$\kappa \equiv \frac{d^2}{de^2}_{e=0} B_\alpha = 8\pi^2 \left(1 - \frac{16d^2}{2^{\frac{3}{2}+1}(2^{2-\frac{3}{2}} + 2^{1-\alpha})}\right).$$

For $\kappa < 0$ we require that $d^2 > \frac{8+2^{-\frac{1}{2}}}{16}$. Noting that for $\alpha \geq 2$ we have that $\frac{8+2^{-\frac{1}{2}}}{16} \leq \frac{5}{8}$ we see that there is always a $d \in (\frac{1}{2}, 1) \cap \mathbb{Q}$ satisfying these conditions. □

Following section 5.4 we track the solution as ‘$\eta$’ is switched on. We note that for sufficiently small $\eta$ the critical point is:

- still a local minimizer,
- still nonplanar,
- still collisionless.

Remark 5.5.2. The nonexistence result for small periods still applies. For large periods it is not clear whether or not the mountain-pass solutions are planar.

### 5.6 Non-tied classes

In this section we will consider the problem of finding periodic solutions in classes of loops that do not satisfy condition (5.12) but do satisfy the other conditions of definition 5.2.6. For all periods $T > 0$ we construct a simple but nontrivial periodic solution of a spatial restricted $2N + 2$-body problem. We take $2N + 1$ identical particles and 1 test particle. To begin with we find a periodic solution of the planar $2N + 1$ body problem.

#### 5.6.1 A solution of the planar molecular $2N + 1$-body problem

We first take $2N + 1$ identical bodies moving in the plane with their centre of mass fixed at the origin:

$$\sum_{i=1}^{2N+1} x_i(t) = 0 \quad \text{for all } t. \quad (5.47)$$

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\[ x_k(t) = r(t) \exp(i \phi(t) + i \pi k/N), \quad k = 1, \ldots, 2N, \quad x_{2N+1}(t) = 0, \quad \forall t. \quad (5.48) \]

**Proposition 5.6.1.** There exist \( r(t) \) and \( \phi(t) \), with \( r(t) \) nonconstant, such that (5.48) solves the \((2N+1)\)-body planar LJ problem.

**Proof.** First note that the equations of motions can be written as:

\[
\ddot{x}_k = -\frac{\partial V_{LJ}(r_{k,2N+1})}{\partial r_{k,2N+1}} \frac{x_k}{\|x_k - x_{2N+1}\|} - \sum_{l \neq k}^{2N} \frac{\partial V_{LJ}(r_{kl})}{\partial r_{kl}} \frac{(x_k - x_l)}{\|x_k - x_l\|} \quad \text{for} \ k \neq 2N + 1, \\
\ddot{x}_{2N+1} = -\sum_{i=1}^{2N} \frac{\partial V_{LJ}(r_{i,2N+1})}{\partial r_{i,2N+1}} \frac{(x_i - x_{2N+1})}{\|x_i - x_{2N+1}\|},
\]

where \( V_{LJ}(r) = \frac{1}{r_{kl}} - \frac{1}{r_{kl}} \), \( r_{kl} = \|x_k - x_l\| \).

Now, since \( x_{2N+1}(t) = 0 \), we obtain:

\[
\ddot{x}_k = -\frac{\partial V_{LJ}(r_{k,2N+1})}{\partial r} \frac{x_k}{\|x_k\|} - \sum_{l \neq k}^{2N} \frac{\partial V_{LJ}(r_{kl})}{\partial r} \frac{(x_k - x_l)}{\|x_k - x_l\|} \quad \text{for} \ k \neq 2N + 1, \\
0 = -\sum_{l=1}^{2N} \frac{\partial V_{LJ}(\|x_l\|)}{\partial \|x_l\|} \frac{x_l}{\|x_l\|}. \quad (5.49)
\]

Note that (5.49) is identically satisfied by \( x_k(t) = -x_{N+k}(t) \) for all \( k \in \{1, \ldots, N \} \).

We look for solutions \( x_k(t) \) \( k = 1, \ldots, 2N \). To do so we note that there is the following Lagrangian:

\[ L = N (r^2 + r^2 \dot{\phi}^2) - W_{LJ}(r) \]

where

\[ W_{LJ}(r) = \sum_{k \neq l} \left[ \frac{1}{(2r \sin(l-k)\pi/N))^{\beta}} - \frac{1}{(2r \sin(l-k)\pi/N))^{\alpha}} \right] + 2N \left[ \frac{1}{r^{\beta}} - \frac{1}{r^{\alpha}} \right]. \]

The equations of motions for \( r(t), \phi(t) \) are:

\[ \ddot{r} = -\frac{1}{2N} \frac{\partial W_{LJ}}{\partial r} + r \dot{\phi}^2, \quad \ddot{\phi}^2 = J. \]

Thus we have reduced the system to a central-force problem. We know from work on the molecular central force problem (see chapter 3) that there exist periodic solutions to this such that \( x_k \) for \( k = 1, \ldots, 2N \) do not move on a circular path and are not stationary. We choose such a periodic solution. \( \square \)
5.6.2 A solution of the restricted molecular $2N + 2$-body problem.

If this planar motion occurs in the $xy$-plane then we constrain the motion of our $(2N + 2)^{th}$ body - the test particle - to be on the $z$-axis. The motions of bodies $1, \ldots, 2N + 1$ are considered fixed as functions of time. To find a periodic motion of body $2N + 2$, the test particle, we use an action functional:

$$A_T[z] = \frac{1}{2} \int_0^T \| \dot{z} \|^2 dt - \int_0^T \sum_{j=1}^{2N+1} V_j(z(t)) dt.$$ 

where

$$V_j(z(t)) = \frac{1}{\| x_j(t) - z(t) \|^\beta} - \frac{1}{\| x_j(t) - z(t) \|^\alpha} \text{ for } j \in \{1, \ldots, 2N + 1\} \text{ with } \beta > \alpha \geq 2.$$

We make the following points:

- Note that $x_k(t) = -x_{N+k}(t)$ for $k = 1, \ldots, N$.

It follows that if $A_T[\cdot]$ is the action functional for a test particle constrained to the $z$ axis, then $D A_T[z](u) = 0$ for all loops $u$ in the $xy$-plane.

- Since $x_{2N+1}(t) = 0$ for all $t$ we see that any collisionless loop constrained to the $z$-axis will lie on only one half of the $z$-axis for all time.

- It is not trivial that there exists a periodic solution for the test particle because we have set bodies $1, \ldots, 2N$ in a non-circular and non-stationary motion. (In the case of 'circular' motion there is a solution in which the test particle sits stationary at a point on the $z$-axis for all time which is trivial).

We have reduced the problem to a 1-dimensional central force problem with time-dependent potential. We now quote a result from [27]:

Proposition 5.6.2. Suppose $V \in C^2(\mathbb{R}^n - \{0\}, \mathbb{R})$ satisfies:

(i) $V(t + T, x) = V(t, x)$ for every $(t, x) \in (\mathbb{R}, \mathbb{R}^n - \{0\})$.

(ii) $\lim_{x \to 0} V(x, t) = +\infty$, monotonically increasing along the rays as $\|x\|$ small uniformly in $t$.

(iii) $V(t, x) \to 0$ as $\|x\| \to \infty$, monotonically increasing along the rays as $\|x\|$ large, uniformly in $t$. Also, $\nabla V(t, x) \to 0$ as $\|x\| \to \infty$ uniformly with respect to $t$.

(iv) There exists $c_2$ such that $(\nabla V(t, x), x) \leq c_2$ for every $t \in \mathbb{R}$, $x \in \mathbb{R}^n - \{0\}$.

(v) There exists $U \in C(\mathbb{R}^n - \{0\}, \mathbb{R})$ and a neighbourhood $\mathcal{R}$ of $0$ and $c_1 \geq 0$ such that $V(t, x) \geq \|\nabla U(x)\|^2 - c_1$ for all $x \in \mathcal{R} - \{0\}$ and $t \in \mathbb{R}$.
Then
\[ \dot{x} = -\nabla V(t, x), \quad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \]
has at least one solution.

In proposition 5.6.2 we take \( n = 1, x = z \) constrained on the \( z \)-axis, \( V(z) = \sum_{j=1}^{2N+1} V_j(z) \). It is readily verified that all the conditions of proposition 5.6.2 are met; (i) follows from the planar solution of the \((2N+1)\)-bodies being \( T \)-periodic, (ii) follows from there being a body fixed at the origin for all \( t \), (iii) follows from properties of the function \( 1/r^\beta - 1/r^\alpha \) for large \( r \), (iv) follows from the fact that \( V \) is continuous as a function of \( z \) and \( V(z) \to +\infty \) as \( z \to 0 \) and \( V(z) \to 0^- \) as \( z \to \infty \), (v) follows in the same way as point (iii) of proposition 5.2.1. This completes our construction.

Remark 5.6.1. Our nonexistence results are not valid here because the nonexistence proof relies upon (5.12) which doesn't hold here.

Remark 5.6.2. Solutions of the restricted \( 2N + 2 \)-body problem may persist to solutions of the full \( 2N + 2 \)-body problem with potential

\[ V_\zeta(z) = \sum_{1 \leq i < j \leq 2N+1} \left( \frac{1}{\|x_i - x_j\|^\beta} - \frac{1}{\|x_i - x_j\|^\alpha} \right) + \zeta \sum_{i=1}^{2N+1} \left( \frac{1}{\|x_i - x_{2N+2}\|^\beta} - \frac{1}{\|x_i - x_{2N+2}\|^\alpha} \right), \]

where \( 0 < \zeta \ll 1 \).
Chapter 6

A Note on Relative Periodic Orbits of Symmetric Strong-Force $N$-Centre Problems

Abstract

We present a note on McCord, Montaldi, Roberts and Sbano's paper (see [54]) on relative periodic orbits in symmetric Lagrangian systems. For the $N$-centre problem with a strong force potential that is bounded above we find those homotopy classes of relative loops on which the action functional is coercive. We describe the homotopy types of the homotopy classes of relative loops. Under the assumption that the action functional is an $S^1$-invariant Morse function we describe a set of homotopy classes that contain infinitely many periodic orbits. The work can be viewed as an expansion on an example presented at the end of [54] which had $N = 2$; in particular we correct and generalize some assertions made regarding coercivity of the action functional and the centralizers of the $g$-twisted action.

6.1 Introduction

From a topological point of view planar $N$-centre problems can be regarded as simpler analogues of $N$-body problems. The configuration space is $\mathcal{M} = \mathbb{R}^2 - \bigcup_{i=1}^{N} \{P_i\}$ where $P_i$ for $i = 1, ..., N$ are the positions of the centres. This is a 2-plane with $N$ points removed. The fundamental group $\pi_1(\mathcal{M})$ is a free group, rather than a braid group. A free group is 'free' in the sense that it is subject to no relations. This defining property will be of crucial importance to the analysis of relative periodic orbits of the $N$-centre problem and makes the investigation far more straightforward than for the $N$-body problem.

6.2 Notation

We introduce some notation:
We denote the space of relative loops by
\[ A^0(M) = \{ \gamma \in \mathcal{C}([0,T], M) : \gamma(t+T) = g\gamma(t) \}, \]
where \( g \) is a fixed diffeomorphism of \( M \) under which the potential of the system is invariant. Here \( T \), the relative period, is fixed. If \( g \) has order \( |g| \) then the corresponding trajectory is periodic with period \( |g|T \).

Let \( e \) denote the identity diffeomorphism, so \( e \cdot \alpha = \alpha \) for all \( \alpha \in A^0(M) \).

We denote
\[ A^0_m(M) = \{ \gamma \in A^0(M) : \gamma(0) = m \} \subset A^0(M) \]
and
\[ A_m(M) = A^0_m(M). \]

We will write \( x \sim_m y \) to denote that \( x \) and \( y \) are in the same connected component of \( A^0_m(M) \). We will write \( x \sim y \) to denote that \( x \) and \( y \) are in the same connected component of \( A^0(M) \).

We let \([x]_g = \{ y : y \sim x \}\). We let \( e \in F_N \) denote \([\omega_1]_e\) where \( \omega_1 \) is a small loop winding clockwise around centre \( P_i \).

Given \( \delta \in A^0_m, \epsilon \in A^0_{gm}(M) \) with \( \delta(T) = \epsilon(0) \) we define \( \delta \epsilon \in A^0_{gm}(M) \) by
\[ (\delta \epsilon)(t) = \begin{cases} \delta(2t), & \text{if } 0 \leq t < \frac{T}{2}, \\ \epsilon(2t - T), & \text{if } \frac{T}{2} \leq t < T. \end{cases} \]

Suppose \( \gamma \in A^0_m \). Let us fix a particular path \( \omega \in A^0_m(M) \). Then we define a map: \( \Phi_\omega : A^0_m(M) \to A_m(M) \) by:
\[ \Phi_\omega(\gamma) = \gamma \omega^{-1}. \]

If \( m = gm \) then the path \( 1_m \) defined by \( 1_m(t) = m \) for all \( t \) is in \( A^0(M) \). Note that if \( m_1 \neq m_2 \) are such that \( g(m_1) = m_i \) for \( i = 1, 2 \) then it is not necessarily true that \([1_m_1]_g = [1_m_2]_g\). For example for a 2-centre problem in \( C \) with centres at \( \pm 1 \) and \( g \) conjugation we have that \([1_{-2}]_g \neq [1_0]_g\).

Let \( id \) denote the null homotopy class of periodic loops of the \( N \)-centre problem. In particular this class contains the constant loops: \( id = [1_m]_e \) for all \( m \in M \).

\( A_T \) will denote an action functional defined on \( A^0(M) \) corresponding to a potential that is bounded above by zero and is standard strong force, see definition 1.1.2. For example, one could take
\[ A_T(x) = \frac{1}{2} \int_0^T \| \dot{x}(t) \|^2 dt + \sum_{i=1}^N \int_0^T \frac{dt}{\| x(t) - P_i \|^\alpha} \quad \text{with } \alpha \geq 2. \]

We will call a non-null homotopy class of loops composite if it can be expressed in the form \( \phi^k \) for some \( \phi \) and some \( k > 1 \). Here \( \phi^k \) is formed by concatenating the homotopy class \( \phi \) with itself \( k \) times. The homotopy class \( id \) will be defined to not be composite.
• Suppose that $\alpha \in \Lambda^g_m$, then we define $g \cdot \alpha$ by

$$(g \cdot \alpha)(t) = g \cdot (\alpha(t)), \quad \forall t.$$  

Further, for convenience, we define

$$G : \Lambda^g(M) \rightarrow \Lambda^g(M); \quad \gamma \mapsto g \cdot \gamma.$$  

The map $G$ induces a map on free homotopy classes of relative loops defined by

$$G([\alpha]_g) = [G(\alpha)]_g.$$  

For every relative loop $\gamma \in \Lambda^g(M)$ we can associate a full loop $\gamma G(\gamma) \ldots G^{[g]-1}(\gamma)$. In fact given $\gamma \in \Lambda^g$ we define

$$\beta : \Lambda^g(M) \rightarrow \Lambda(M); \quad \gamma \mapsto \gamma G(\gamma) \ldots G^{[g]-1}(\gamma)$$  

and the set

$$S = \{ \delta \in \Lambda^g : [\beta(\delta)]_e = id \}.$$  

• For $\alpha \in \Lambda^g_m(M)$ we define $F(\alpha) = \alpha_g \in \Lambda^g$ by:

$$\alpha \mapsto F(\alpha) = \alpha_g = \omega(g \cdot \alpha)\omega^{-1}.$$  

• The $g$-twisted action of $\pi_1(M,m)$ on itself is given by

$$\alpha \cdot \beta = \alpha^{-1}\beta\alpha_g, \quad \alpha, \beta \in \pi_1(M,m).$$  

6.3 A recap of the central results of [54]

• The number of connected components of the space of relative loops is equal to the number of orbits of the $g$-twisted action of the fundamental group $\pi_1(M)$ on itself. More precisely:

$$\pi_0(\Lambda^g(M)) \cong \pi_1(\Lambda^g(M))^{[g]}$$  

where $\pi_1(\Lambda^g(M))^{[g]}$ is the set of orbits of the $g$-twisted orbits of $\pi_1(M,m)$ on itself.

• If $M$ is a $K(\pi,1)$ then the connected components of the relative loop spaces are also $K(\pi,1)$'s, with fundamental groups isomorphic to the isotropy subgroups of the $g$-twisted action of $\pi_1(M)$ on itself. Explicitly, given $\gamma, \omega \in \Lambda^g_m(M)$ we have

$$\pi_1([\gamma]_g) \cong Z(\gamma) \cong \{ \alpha \in \Lambda_m(M) : \alpha^{-1}\Phi_\omega(\gamma)F(\alpha) \sim_m \Phi_\omega(\gamma) \}.$$  

Remark 6.3.1. The results are independent of the choice of $\omega \in \Lambda^g_m(M)$ and base point $m \in M$.  

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6.4 Coercivity

Definition 6.4.1. Suppose $\gamma \in \Lambda^g(M)$. Let

$$\delta(\gamma) \doteq \min_i \inf_t |\gamma(t) - P_i|, \quad (6.1)$$
$$\Delta(\gamma) \doteq \max_i \sup_t |\gamma(t) - P_i|, \quad (6.2)$$

where $P_i$ for $i \in \{1, \ldots, N\}$ are the positions of the $N$ centres.

For standard strong force problems the following lemma holds:

Lemma 6.4.1. If $\gamma_n \in \Lambda^g$ is a sequence of relative loops such that $\delta(\gamma_n) \to 0$ as $n \to \infty$ then $A_T(\gamma_n) \to \infty$ as $n \to \infty$.

Definition 6.4.2. Let $L_T(\gamma) \doteq \int_0^T |\gamma'(t)| dt$.

Lemma 6.4.2. If $k$ is the order of $g$ and $\gamma \in \Lambda^g(M)$ then

$$\Delta(\beta(\gamma)) = \Delta(\gamma), \quad A_{kT}(\beta(\gamma)) = kA_T(\gamma).$$

Definition 6.4.3. Say that $A_T$ is coercive on a relative homotopy class of loops if for all sequences $\gamma_n$ in that class such that $\Delta(\gamma_n) \to \infty$ as $n \to \infty$ we have $A_T(\gamma_n) \to \infty$.

Remark 6.4.1. Note that coercivity of $A_T$ is defined on connected components of $\Lambda^g$, not on connected components of $\Lambda^g_m$. For any fixed $m$, $A_T$ is coercive on all connected components of $\Lambda^g_m$. However, this does not imply that $A_T$ is coercive on connected components of $\Lambda^g = \bigcup_{m \in \mathbb{M}} \Lambda^g_m$.

We mention here some analytic points. Our action functional is of an $N$-centre problem whose potential is strong force and everywhere negative. It is bounded below and satisfies the Palais-Smale condition; for a proof of this we refer the reader to [1]. If the action functional of the strong force $N$-centre problem is coercive on a homotopy class of relative loops then by the direct method (see subsection 1.1.8) that class contains a solution of the equations of motion. Thus by finding classes on which the action functional is coercive we are also finding classes that contain solutions of the equations of motion.

Lemma 6.4.3. We have that

$$A_T(\gamma) \geq \frac{L_T^2}{2T}.$$  

Proof. This follows from the fact that the potential is negative everywhere and an application of the Cauchy-Schwarz inequality to the kinetic contribution of the action.

Lemma 6.4.4. If the action functional is coercive on the homotopy class $[\beta(\gamma)]_e$ then it is coercive on the relative homotopy class $[\gamma]_g$.

Proof. Suppose $g$ has order $k$, i.e. $g^k = e$. Consider

$$C \doteq \{ x \in [\beta(\gamma)]_e : x = \beta(\phi) \text{ for some } \phi \in \Lambda^g(M) \}.$$  

Since $C \subseteq [\beta(\gamma)]_e$ and $A_{kT}$ is coercive on $[\beta(\gamma)]_e$ it follows that $A_{kT}$ is coercive on $C$. Take a sequence of loops $\gamma_n$ such that $\gamma_n \in [\gamma]_g$ for all $n$ and $\Delta(\gamma_n) \to \infty$ as $n \to \infty$. Then we have a sequence $\beta(\gamma_n) \in C$ with $\Delta(\beta(\gamma_n)) = \Delta(\gamma_n) \to \infty$ as $n \to \infty$. Since $A_{kT}$ is coercive on $C$ it follows that $A_{kT}(\beta(\gamma_n)) = kA_T(\gamma_n) \to \infty$ as $n \to \infty$. Therefore $A_T(\gamma_n) \to \infty$ as $n \to \infty$. \qed
The converse of lemma 6.4.4 is false. For example, consider a 2-centre problem in the plane $\mathbb{C}$ with centres at $\pm 1$ and $m = 0$. Let $g$ denote conjugation. Then the action functional is coercive on the homotopy class $[1_m]_g$ of relative loops. However $\mathcal{A}_T$ is not coercive on the free homotopy class $[1_mG(1_m)]_e = id$ of loops.

The action functional is coercive on all non-null homotopy classes of full loops.

Therefore the condition $\gamma \in S$ is necessary for $\mathcal{A}_T$ to not be coercive on $[\gamma]_g$, but is not sufficient.

If $g$ is an isometry then there are three possible cases:

- $g$ has order 1, so $g = e$.
- $g$ has order 2, so $g$ is a rotation by $\pi$ or a reflection.
- $g$ has order $> 2$, so $g$ is a rotation.

Definition 6.4.4. Let $R = \{m \in \mathcal{M} : gm = m\}$.

If $R \neq \emptyset$ then any element $\gamma \in \Lambda^g$ can be homotoped to an element, say $\gamma^*$, such that $\gamma^* \in [\gamma]_g \cap \Lambda(\mathcal{M})$; in particular $\gamma^*(0) = \gamma^*(T) = m \in R$. Moreover,

$$\gamma^* \in S \iff \gamma \in S.$$ (6.3)

The statement (6.3) follows because we can use the homotopy from $\gamma$ to $\gamma^*$ to construct a homotopy from $\gamma G(\gamma) \cdots G^{[s]-1}(\gamma)$ to $\gamma^* G(\gamma^*) \cdots G^{[s]-1}(\gamma^*)$, so:

$$[\gamma G(\gamma) \cdots G^{[s]-1}(\gamma)]_e = [\gamma^* G(\gamma^*) \cdots G^{[s]-1}(\gamma^*)]_e.$$

Note that for a given $m \in R$ and $\gamma \in \Lambda^g$ there is more than one choice of $\gamma^*$. Furthermore these choices do not necessarily lie in a unique connected component of $\Lambda^e(\mathcal{M})$. Consider the set:

$$O_m(\gamma) = \{h \in \pi_0(\Lambda^g_m(\mathcal{M})) : [\gamma]_g \cap h \neq \emptyset\}.$$

Given any element $h \in O_m(\gamma)$, the set $O_m(\gamma)$ is the $g$-twisted orbit of $h$.

6.4.1 Case $g = e$

If $g = e$ then we are considering periodic loops. The action functional is coercive on all non-null homotopy classes of loops and $S = id$.

6.4.2 Rotations

Proposition 6.4.1. If $g \neq e$ and $g$ is a rotation then

$$S = \begin{cases} \emptyset & \text{if the point we rotate about coincides with a centre} \\ [1_m]_g & \text{if } m \text{ is the point we rotate about and } m \notin \bigcup_{i=1}^N \{R_i\}. \end{cases}$$

The action functional $\mathcal{A}_T$ is coercive on all homotopy classes of $g$-relative loops i.e. all connected components of $\Lambda^g(\mathcal{M})$. 

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Proof. Suppose $\gamma \in \Lambda^0(M)$ where $g$ is a rotation about a centre, say $P_1$. If $g \neq e$ has order $k \in \mathbb{N}\setminus\{1\}$ then $g$ turns through $a + \frac{1}{k}$ revolutions about $P_1$ where $a \in \mathbb{Z}$. Therefore $\beta(\gamma)$ winds $k(a + \frac{1}{k}) = ka + 1$ times around $P_1$. It follows that, since $k \geq 2$ and $a \in \mathbb{Z}$, $\beta(\gamma)$ must wind a nonzero number of times around $P_1$. Therefore $[\beta(\gamma)]_e \neq id$. Thus $\gamma \notin S$. Hence $A_T$ is coercive on all homotopy classes of $\Lambda^0(M)$.

So now suppose that the point we rotate about does not coincide with the position of a centre. Then $R = \{m\}$ where $m = gm$ is the point that we rotate about. Furthermore $G$ cyclically permutes all the generators of $\mathcal{P}_R$. Firstly we shall show that $S = [1m]_g$. Suppose that $\gamma \in \Lambda^0$. Then we know that there exists $\gamma^* \in [\gamma]_g \cap \Lambda(M)$ such that $\gamma^* \in S \iff \gamma \in S$. Suppose that:

$$[\gamma^*]_e = \alpha_{c_1}^{N_1} \cdots \alpha_{c_k}^{N_k}.$$ 

Here, without loss of generality, we assume that $[\gamma^*]_e$ is in reduced form so $c_j \neq c_{j+1}$ for $j \in \{1, \ldots, k - 1\}$ and $N_j \neq 0$ for $j \in \{1, \ldots, k\}$. Consider $\beta(\gamma) = \gamma G(\gamma) \cdots G^{[\gamma]_e^{-1}}(\gamma)$. We have for $\gamma^* \in S$ that $[\beta(\gamma^*)]_e = id$. Therefore:

$$[\alpha_{c_1}^{N_1} \cdots \alpha_{c_k}^{N_k}] [\alpha_{p(1)}^{N_1} \cdots \alpha_{p(k)}^{N_k}] \cdots [\alpha_{p(k)}^{N_1} \cdots \alpha_{p(1)}^{N_k}] = id,$$

where $p$ is the permutation of centres induced by $G$. Notice that $c_j \neq c_{j+1}$ for $j \in \{1, \ldots, k-1\}$ and $p(c_j) \neq p(c_{j+1})$ for $j \in \{1, \ldots, k-1\}$ since $p$ is a cyclic permutation on all points. Therefore, due to the fact that distinct generators do not commute, in order for $[\beta(\gamma)]_e$ to reduce to $id$ we see that there must exist $0 \leq i \leq |g| - 2$ such that $p^i(c_1) = p^{i+1}(c_1)$. Thus $c_k = p(c_1)$ and further $N_1 = -N_k$. Note that each square bracket must contain something of length $\geq 2$. Equation (6.4) then becomes:

$$[\alpha_{c_1}^{N_1} \cdots \alpha_{c_k}^{N_k}] [\alpha_{p(1)}^{N_1} \cdots \alpha_{p(k)}^{N_k}] \cdots [\alpha_{p(k)}^{N_1} \cdots \alpha_{p(1)}^{N_k}] = id.$$ 

Applying the same argument again we see that there exists $0 \leq i \leq |g| - 2$ such that $p^i(c_{k-1}) = p^{i+1}(c_2)$. Thus $p(c_2) = c_{k-1}$ and further $N_2 = -N_{k-1}$. By continuing in this fashion we find that:

$$N_r = -N_{k+1-r}, \quad c_{k+1-r} = p(c_r) \quad r \in \{1, \ldots, k\}.$$ 

This implies that $k$ must be even for otherwise $N_{k+1} = 0$ which contradicts the fact that $[\gamma^*]_e$ was written in reduced form. Therefore $k = 2a$ for some $a \in \mathbb{N}$ and the general form of $[\gamma^*]_e$ is:

$$\alpha_{c_1}^{N_1} \alpha_{c_2}^{N_2} \cdots \alpha_{c_a}^{N_a} \alpha_{p(c_a)}^{-N_a} \cdots \alpha_{p(c_2)}^{-N_2} \alpha_{p(c_1)}^{-N_1}.$$ 

We note that these elements are precisely those elements that can be written as $\alpha^{-1} C(\alpha)$ for some $\alpha$; indeed we can take $\alpha = \alpha_{c_1}^{N_1} \cdots \alpha_{c_a}^{N_a}$. Therefore $S = [1m]_g$. The set $S$ is the set of all candidates for the action functional to be non-coercive on. However the action functional is coercive on the class $[1m]_g$ with $g \neq e$ because if $\gamma_n$ is a sequence in $[1m]_g$ such that $\Delta(\gamma_n) \to \infty$ then $L(\gamma_n) \to \infty$ and hence, by lemma 6.4.3, $A_T(\gamma_n) \to \infty$ as $n \to \infty$. □

6.4.3 Reflections

Proposition 6.4.2. Let $g$ denote reflection in the line $l \subset \mathbb{R}^2$. Suppose we have centres $P_1, P_2, \ldots, P_j$ on $l$ in that order from 'left' to 'right'. Suppose that $m_1$ is a point on $l$ left of centre $P_1$, $m_{j+1}$ is
a point on \( l \) right of centre \( P_j \) and \( m_k \) is a point on \( l \) between centre \( P_{k-1} \) and centre \( P_k \) for all \( k \in \{2, \ldots, j\} \). Then

\[
S = \bigcup_{i=1}^{j+1} [1_{m_i}]_g.
\]

The only homotopy classes of \( \Lambda^g(M) \) that \( \mathcal{A}_T \) is not coercive on are \([1_{m_k}]_g\) for \( k \in \{1, j+1\} \).

**Proof.** To begin with we show that \( S = \bigcup_{i=1}^{j+1} [1_{m_i}]_g \). Suppose \( \gamma \in S \). For any \( m \in l - \bigcup_{i=1}^{j} P_i \) we know that there exists \( \gamma^* \sim \gamma \) with \( \gamma^* \in \Lambda^g_m(M) \cap S \). Suppose that

\[
[\gamma^*]_e = \alpha_{c_1}^{N_1} \cdots \alpha_{c_a}^{N_a}.
\]

Then, for reflections, we have

\[
[\gamma^*G(\gamma^*)]_e = \alpha_{c_1}^{N_1} \cdots \alpha_{c_k}^{N_k} \alpha_{p(c_1)}^{N_1} \cdots \alpha_{p(c_a)}^{N_a}.\]

We look at two different cases:

- **k even.** If \( k = 2a \) then for (6.4.3) to contract to \( id \) we must have:

  \[
  c_{a+i} = p(c_{a+1-i}), \quad \text{and} \quad N_{a+i} = N_{a+1-i} \quad \text{for} \quad i = 1, \ldots, a.
  \]

Then:

\[
[\gamma^*]_e = (\alpha_{c_1}^{N_1} \cdots \alpha_{c_a}^{N_a}) (\alpha_{p(c_1)}^{N_1} \cdots \alpha_{p(c_a)}^{N_a}).
\]

This is of the form \( \alpha^{-1}G(\alpha) \).

- **k odd.** If \( k = 2a + 1 \) then for (6.4.3) to contract to \( id \) we must have:

  \[
  c_{a+1} = p(c_{a+1}) \quad \text{and} \quad c_{a+i} = p(c_{a+2-i}), \quad N_{a+i} = N_{a+2-i} \quad \text{for} \quad i = 2, \ldots, a + 1.
  \]

Then:

\[
[\gamma^*]_e = (\alpha_{c_1}^{N_1} \cdots \alpha_{c_a}^{N_a}) (\alpha_{p(c_1)}^{N_1} \cdots \alpha_{p(c_a)}^{N_a}).
\]

If \( N_{a+1} \) is even then this can be written as:

\[
[\gamma^*]_e = (\alpha_{c_1}^{N_1} \cdots \alpha_{c_a}^{N_a} \alpha_{c_{a+1}}^{N_{a+1}/2}) (\alpha_{p(c_{a+1})}^{N_{a+1}/2} \alpha_{p(c_1)}^{N_{a+1}/2} \alpha_{p(c_a)}^{N_a} \cdots \alpha_{p(c_1)}^{N_a}).
\]

which is of the form \( \alpha^{-1}G(\alpha) \).

If \( N_{a+1} \) is odd then this can be written as:

\[
[\gamma^*]_e = (\alpha_{c_1}^{N_1} \cdots \alpha_{c_a}^{N_a} \alpha_{c_{a+1}}^{N_{a+1}/2}) (\alpha_{p(c_{a+1})}^{N_{a+1}/2} \alpha_{p(c_1)}^{N_{a+1}/2} \alpha_{p(c_a)}^{N_a} \cdots \alpha_{p(c_1)}^{N_a}).
\]

which is of the form \( \alpha^{-1} \alpha_{c_{a+1}} G(\alpha) \) where \( p(c_{a+1}) = c_{a+1} \). Since \( P_{c_{a+1}} \in R \), the relative homotopy class \( \alpha_{c_{a+1}} \) with respect to one base point, say \( m_i \in l - \bigcup_i P_i \), is \([1_{m_i}]_g\) with respect to another point \( m_j \in l - \bigcup_i P_i \) where \( j \in \{i \pm 1\} \).

Examining \([1_{m_k}]_g\) for \( k \in \{1, \ldots, j+1\} \) we see that \( \mathcal{A}_T \) is coercive on these if and only if \( k \neq 1, j+1 \), see figure 6.1.

\[\square\]
Figure 6.1: Note how $A_T$ is coercive on $[1_{m_2}]_g$ because we cannot homotope the relative loop off to infinity without $A_T$ diverging. However $A_T$ is not coercive on $[1_{m_1}]_g$ because we can homotope $1_{m_1}$ off to infinity (simply by moving the loop further and further left) without $A_T$ diverging.
6.5 Homotopy type of homotopy classes of RPOs

In this section we calculate the centralizers of the $g$-twisted action. By using one of the central results of [54] this allows us to infer the homotopy type of the homotopy classes of relative periodic orbits.

Lemma 6.5.1. Given $\gamma \in \Lambda_m^\gamma$ we have that $\alpha \in Z(\gamma)$ if and only if

$$\gamma G(\alpha) \sim_m \alpha \gamma.$$  \hspace{1cm} (6.6)

Proof. We are looking for solutions for $\alpha$ of

$$\Phi_\omega(\gamma)F(\alpha) \sim_m \alpha \Phi_\omega(\gamma).$$

This gives

$$\gamma \omega^{-1} F(\alpha) \sim_m \alpha \gamma \omega^{-1}$$

But we have $F(\alpha) = \omega(g \cdot \alpha) \omega^{-1}$. So we get

$$\gamma (g \cdot \alpha) \sim_m \alpha \gamma$$

i.e.

$$\gamma G(\alpha) \sim_m \alpha \gamma.$$

\[\square\]

Remark 6.5.1. Note that if we have $\alpha \in \Lambda_m(\mathcal{M})$ then $G(\alpha) \in \Lambda_{gm}(\mathcal{M})$.

Lemma 6.5.2. $(\bigcup_{k \in \mathbb{Z}} \{[\beta(\gamma)]_e \}) \cap \Lambda_m(\mathcal{M})$ is a subgroup of $Z(\gamma)$.

Proof. We use equation (6.6). Observe that

$$\gamma G(\beta(\gamma)) = \gamma G(\gamma G(\gamma) \ldots G^{[\omega]^{-1}}(\gamma)) = \gamma G(\gamma) G^2(\gamma) \ldots G^{[\omega]}(\gamma) = \beta(\gamma) \gamma.$$

\[\square\]

Notice that $[\beta(\gamma)]_e$ may be composite:

Example 6.5.1. Consider the problem of two fixed centres with $g$ a rotation by $\pi$ about the midpoint of the two centres, say $m$. Suppose $\gamma \in \Lambda_m(\mathcal{M})$ with $[\gamma]_e = \alpha_1 \alpha_2 \alpha_1$ so that $[G(\gamma)]_e = \alpha_2 \alpha_1 \alpha_2$ and $[\beta]_e = [\gamma]_e [G(\gamma)]_e = (\alpha_1 \alpha_2)^3$, which is composite.

Lemma 6.5.3. Suppose $e \in \Lambda_m(\mathcal{M})$. Then

$$e^k \in Z(\gamma) \text{ if and only if } e \in Z(\gamma).$$

Proof. $e \in Z(\gamma) \Leftrightarrow \gamma G(e) \gamma^{-1} \sim_m e \Leftrightarrow (\gamma G(e) \gamma^{-1})^k \sim_m e^k \Leftrightarrow \gamma G(e^k) \gamma^{-1} \sim_m e^k \Leftrightarrow e^k \in Z(\gamma).$ \[\square\]

Lemmas 6.5.2 and 6.5.3 imply that:

Lemma 6.5.4. $(\bigcup_{k \in \mathbb{Z}} \{e^k\}) \cap \Lambda_m(\mathcal{M})$ is a subgroup of $Z(\gamma)$ where $e$ is a root or power of $[\beta(\gamma)]_e$.
Proposition 6.5.1. Consider $\gamma \in \Lambda_m^n$. If $g = e$ then

$$Z(\gamma) \cong \begin{cases} 
\mathbb{Z} & \text{if } [\gamma]_e \neq \text{id} \\
F_N & \text{if } [\gamma]_e = \text{id}.
\end{cases}$$

If $g \neq e$ then:

$$Z(\gamma) \cong \begin{cases} 
\mathbb{Z} & \text{if } \gamma \in \Lambda_m^n \setminus S, \\
1 & \text{if } \gamma \in S
\end{cases}$$

where 1 denotes the trivial group.

Proof. We have for elements $\alpha \in Z(\gamma)$ that:

$$\alpha^{-1}\gamma G(\alpha) \sim_m \gamma$$

(6.7)

Suppose $g = e$. Then we have:

$$\alpha^{-1}\gamma \sim_m \gamma$$

(6.8)

If $[\gamma]_e = \text{id}$ then (6.8) is satisfied by all $\alpha \in \Lambda_m(M)$ so $Z(\gamma) \cong F_2$. If $[\gamma]_e \neq \text{id}$ then $\alpha$ must be in a homotopy class of loops that is a root or power of $[\gamma]_e$ - otherwise it would imply a nontrivial relation between generators of the free group, so $Z(\gamma) \cong \mathbb{Z}$.

Now suppose that $g \neq e$. Applying $G^i$ for $i = 0, 1, \ldots, |g| - 1$ to both sides of (6.7) we obtain

$$G^i(\alpha^{-1})G^i(\gamma)G^{i+1}(\alpha) \sim_{g^m} G^i(\gamma), \quad i \in \{1, \ldots, |g| - 1\}.$$  

(6.9)

Using (6.9) we obtain, by concatenation of relative loops, that

$$\prod_{i=0}^{|g|-1} G^i(\alpha^{-1})G^i(\gamma)G^{i+1}(\alpha) \sim_m \prod_{i=0}^{|g|-1} G^i(\gamma).$$

(6.10)

The left hand side of (6.10) is:

$$\alpha^{-1}\gamma G(\alpha)G(\gamma)G^2(\alpha)G^2(\gamma)\ldots G^{|g|-1}(\alpha)^{-1}G^{\beta(\gamma)}(\alpha) \sim_m \alpha^{-1}\beta(\gamma)\alpha$$

because $G^i(\alpha^j)G^i(\alpha^{-j}) = G^i(\alpha^j\alpha^{-j}) \sim_{g^m} 1_{g^m}$ for all $i, j \in \mathbb{N}, \alpha \in \Lambda_m(M)$.

The right hand side of (6.10) is $\beta(\gamma)$. Thus we have:

$$\alpha^{-1}\beta(\gamma)\alpha \sim_m \beta(\gamma).$$

(6.11)

There are two cases to examine:

(i) $\gamma \in \Lambda_m^n \setminus S$.

We have $[\beta(\gamma)]_e \neq \text{id}$. It follows that any solution for $\alpha$ is in a homotopy class of loop space that is a root or power of $[\beta(\gamma)]_e$, for otherwise we would have a relation between generators which would contradict the definition of a free group. So using lemma 6.5.4 we know that the solutions of (6.7) are those elements of $\Lambda_m(M)$ that lie in homotopy classes of loops that are roots and powers of $[\beta(\gamma)]_e$. Hence $Z(\gamma) \cong \mathbb{Z}$.

(ii) $\gamma \in S$.

We have $[\beta(\gamma)]_e = \text{id}$. There are two subcases:
(a) Suppose \( g \neq e \) is a rotation. We have \( S \neq \emptyset \) so by proposition 6.4.1 we know that \( g \) is not a rotation about a centre. Without loss of generality we take \( m \) to be the point we rotate about. Again by proposition 6.4.1 we have that \( \gamma \sim_m 1_m \). By (6.7) this gives \( G(\alpha) \sim_m \alpha \), a contradiction unless \( \alpha \sim_m 1_m \), because \( g \neq e \). Thus \( Z(\gamma) \neq 1 \).

(b) Suppose \( g \) is a reflection. Then by proposition 6.4.2 we have that

\[
\gamma \in [1_{m,i}]g \quad \text{for some} \quad i \in \{1, \ldots, j + 1\}.
\]

We take \( m = m_i \). Therefore \( \alpha \sim_m G(\alpha) \) implying \( \alpha \sim_m 1_m \). Thus \( Z(\gamma) \neq 1 \).

\[ \square \]

6.6 Estimate of the number of periodic orbits in some classes

The idea for this was presented in section 3.2.2 of [54] in an example. However the example, in which \( N \) was taken to be 2 and \( g \) was taken to be a reflection, was not fully correct in that:

- coercivity was assumed to hold on some classes that it does not.
- it was stated that for \( g \) a reflection the \( g \)-twisted action is free. However this is not always true. If it were true then all isotropy subgroups would be trivial. However if we examine, for example, \( \gamma = \alpha_1 \alpha_2 \) we find, by lemma 6.5.4, that \( \{ (\alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1})^k : k \in \mathbb{Z} \} \subset Z(\gamma) \), which is certainly nontrivial.

The results of this chapter allow us to:

- amend the detail regarding coercivity,
- give the set of all those \( \gamma \) with trivial isotropy subgroup,
- generalize the result to \( N \) centres.

Proposition 6.6.1. Suppose that

\[ \sigma = \begin{cases} [1_{m}]g & \text{if } g \neq e \text{ is a rotation about a point } m \text{ that does not coincide with a centre} \\ [1_{m_k}]g, & k \in \{2, \ldots, j\} \quad \text{if } g \text{ is a reflection and the } \{m_k\} \text{ are as in proposition 6.4.2}. \end{cases} \]

Then, assuming \( A_T \) is an \( S^1 \)-invariant Morse function, \( \sigma \) contains at least one critical orbit of Morse index \( 2r \) for every \( r \in \mathbb{N} \cup \{0\} \).

Proof. For the result in section 3.2.2 of [54] to work triviality of the isotropy subgroups and coercivity of the action functional are required. Therefore, by proposition 6.5.1, we see that the set of relative homotopy classes that we are interested in is:

\[ \Sigma \subset \{ [\gamma]_g : \gamma \in S \text{ and } A_T \text{ is coercive on } [\gamma]_g \}. \]

If \( g \) is a rotation then by proposition 6.4.1 and its proof, \( \Sigma = \{ [1_{m}]g \} \) where \( m \) is the point that we rotate about.

If \( g \) is a reflection and \( N > 1 \) then by proposition 6.4.2 and its proof, \( \Sigma = \bigcup_{i=2}^{j} \{ [1_{m_i}]g \} \) where \( m_i \) for \( i \in \{2, \ldots, j\} \) are as in proposition 6.4.2.
If $\gamma \in \sigma \in \Sigma$ then $Z(\gamma) \cong 1$ and $\sigma$ is contractible by the central results of [54]. As in [54], we are assuming that $A_T$ is an $S^1$-invariant Morse function on the homotopy class. The critical points form $S^1$ orbits corresponding to performing phase-shifts. Restricting to this component of loop space, let the number of critical points of the action functional with Morse index $i$ be $n_i$. We have that the Poincaré polynomial of $S^1$ is $1 + t$ and that of the component is 1 (because its contractible).

Here the Morse inequalities (1.6) are:

$$\sum_{i=0}^{\infty} n_i t^i (1 + t) - 1 = (1 + t) \sum_{i=0}^{\infty} q_i t^i$$

where $q_i \geq 0$ for all $i$. This implies that

$$\sum_{i=0}^{\infty} n_i t^i = (1 + t)^{-1} + \sum_{i=0}^{\infty} q_i t^i \quad \text{i.e.} \quad \sum_{i=0}^{\infty} n_i t^i = \sum_{i=0}^{\infty} q_i t^i + 1 - t + t^2 - t^3 + \ldots$$

It follows that $n_j \geq (-1)^j$ for all $j \geq 0$. \qed
Chapter 7

Questions, Problems and Projects

In this final chapter we present a selection of questions and problems prompted by this thesis. There are many avenues open for further research.

**Systems with potentials** $V_\alpha(x) = -\frac{1}{\|x\|^\alpha}$

- Prove the monotonicity conjecture for $\Delta_\alpha$ defined on $R_+$.

- Prove via an appropriate Taylor series expansion that the ‘$p$’ orbit bifurcates off the ‘$p$-circles’ orbit such that it has lower action than the ‘$p$-circles’ orbit, at least in the case that we are ‘close’ to the bifurcation.

- Although we have proved that, up to phase-shifts, the action functional is nondegenerate on the $p$-circles orbits we have not proved that ‘$q$’ periodic orbits form nondegenerate critical orbits (with symmetry group generated by phase-shifts and rotations) away from bifurcations; do this or find some other method of showing that the Morse index is constant along these branches in the bifurcation diagrams.

- Studying systems with potentials $-\frac{1}{\|x\|^\alpha}$ proved to be an interesting and educational toy that indirectly shed light onto the Kepler problem and, in particular, the Morse indices of Kepler elliptical orbits. However do the systems with $1 < \alpha < 2$ relate to anything in the physical world?

**Planar Molecular $N$-body problem**

- For the 1-centre/2-body molecular problem we found that there were infinitely many different ‘shaped’ orbits in any given homotopy class of loops (in particular they had different number of radial oscillations) provided one looked at all periods. Do analogous results hold more generally? Can we find any periodic solutions in the Lagrange class of choreographies of the molecular $N$-body problem that are not relative equilibria?

- The numerics in chapter 3 for the molecular 2-body problem implied that the 'critical period', defined as the infimum over all periods for which periodic solutions exist, was achieved by a relative equilibrium. Does an analogous result hold in the Lagrange class of the planar $N$-body problem? I.e. is the least period achieved by a relative equilibrium?
• In this thesis we have proved, subject to a nondegeneracy assumption, the existence of critical points of the Lennard-Jones action functional in suitable tied classes of loops. An obvious aim is to prove that this nondegeneracy condition holds in general, or at least in a few interesting cases. For example can we prove that the minimizer of $\mathcal{B}^0$ in the same symmetry class as the Chenciner-Montgomery Eight is nondegenerate up to some symmetries of $\mathcal{B}^0$?

• An alternative approach to the problem of proving the existence of a mountain-pass critical point in a class $\theta$ is to find a manifold $\Upsilon \subset \theta$ and two loops $z_A, z_B \in \theta$ such that:

(i) the action $\mathcal{B}^\eta$ is bounded below on $\Upsilon$ by some $c > 0$,

(ii) all homotopies from $z_A$ to $z_B$ pass through $\Upsilon$,

(iii) $\max\{\mathcal{B}^\eta(z_A), \mathcal{B}^\eta(z_B)\} < c$.

A version of the mountain-pass theorem then says that a saddle critical point exists (but it is not necessarily in $\Upsilon$). Such a method can be used in the generalized molecular 1-centre problem, see [27]. However it seems difficult to find an $\Upsilon$ that works for nontied homotopy classes of loops when $N > 2$ or for any tied homotopy classes of loops when $N \geq 2$.

There is no recipe for finding a suitable $\Upsilon$. Moreover it is not necessarily true that such an $\Upsilon$ exists even if mountain-pass critical points do. Luca Sbano and the author have attempted but not succeeded in finding such $\Upsilon$'s; however it is possible that more talented/persistent/lucky individuals will succeed where we have failed.

• Recently software has been developed by Claudia Wulff and Andreas Schebesch (see [73]) that can be used for the continuation of periodic solutions as a system parameter is varied. Based on a 'multiple shooting algorithm', the algorithm is able to compute relative periodic orbits persisting from periodic orbits in symmetry breaking bifurcations and is able to numerically continue nondegenerate Hamiltonian relative periodic orbits under certain conditions. They have successfully used it to continue the famous Chenciner-Montgomery Eight choreography of the 3-body system to find 'less symmetric Eights'.

Suppose we consider choreographic solutions of the molecular $N$-body problem (which can be constructed by glueing RPOs together). We know these correspond to critical points of a functional $\mathcal{B}^\eta$ on the space of choreographies. When $\eta = 0$ we have a strong force action functional. Given any homotopy class of choreographies $\sigma$ we can perform minimization of $\mathcal{B}^\eta$ with respect to a Fourier series representation of $\sigma$, similarly to [18]. For example one can use Moore's algorithm, essentially a steepest-descent procedure (see [60]), applied to motions of 3 bodies in the same symmetry class as the Chenciner-Montgomery Eight. One possible project could be to use the Wulff-Schebesch algorithm to continue this solution to critical points of $\mathcal{B}^\eta$ for $\eta > 0$ and hence to periodic solutions of the molecular $N$-body problem. As we approach the critical period from above we must have that $\Delta = \max_{i,j} \sup_{t} \|x_i(t) - x_j(t)\|$ diverges and/or the solution tends towards a loop with collisions. Close to the critical period the solutions may have interesting shapes. Of interest will be periodic solutions in which the bodies pass close to one another and the repulsive nature of the potential at short distances becomes evident in the shape of the solutions.

• Do the results of chapter 5 have any applications in the semiclassical descriptions of molecular systems?
• Find (lower bounds on) the Morse indices of the saddle critical points of the molecular $N$-body problem. The paper [24] might help.

• Which of the periodic solutions are stable?

• Another interesting problem is to classify relative equilibria of the molecular $N$-body problem. This has already been done for $N \in \{1, 2, 3\}$ in [25] and [26]. Can their method be generalized to $N > 3$ or is another approach required?

Planar Symmetric Newtonian 2-centre problem

• Make the treatment of the action spectrum and Morse indices of P1 orbits presented in chapter 4 more rigorous. In particular calculate the dimension of the kernel of the Hessian of the action functional for all elliptical orbits and orbits that bifurcate off it.

• Find the Morse indices of P2 orbits. The difficulty is that although fixing homotopy class fixes $p$ bifurcations occur off 'interproton' collision orbit (rather than elliptical orbits as was the case for P1 orbits).

• Find the Morse indices of P3 orbits. The difficulties are that fixing a homotopy class does not fix $p$ or $q$ and that we have no orbit with a known Morse index to track.

• Do our results on prime-period orbits have any applications, perhaps in connection with the study of the motion of artificial earth satellites?

• Suppose we look at the planar 2-centre problem with potential

$$V(x) = \frac{1}{\|x - P_1\|^{\alpha}} - \frac{1}{\|x - P_2\|^\alpha}.$$  

(7.1)

The Newtonian problem corresponds to $\alpha = 1$. Let $W_R$ denote the set of all homology classes with homology $(a, b) \in \mathbb{Z}^2$ where $\max\{|a|, |b|\} \leq R$. Let $C_R$ denote those elements of $W_R$ that contain at least one periodic solution. We consider the quantity $\nu = \lim_{R \to \infty} |C_R|$. It may be interesting to investigate what happens to $\nu$ as the $\alpha$ in (7.1) is increased from 1 to 2. We know that

$$\nu = \begin{cases} 
0 & \text{when } \alpha = 1, \\
1 & \text{when } \alpha = 2.
\end{cases}$$

However, is $\nu$ continuous as a function of $\alpha$ for $1 \leq \alpha \leq 2$? In particular are there discontinuities of $\nu$ at $\alpha = 1$ or $\alpha = 2$? The fact that $\nu = 0$ when $\alpha = 1$ can be interpreted as a 'sparsity' of homology classes of loops of the Newtonian 2-centre problem that contain periodic solutions. Our analysis in chapter 4 suggests this is due to the integrability of the system. However we also know that it is related to the weakness of the Newtonian potential; $\nu = 1$ when $\alpha \geq 2$ because $N$-centre systems with strong force potentials that are bounded above admit periodic solutions, given by minimizers, in every non-null homotopy class of loops. This suggests a possible proof of non-integrability of strong force $N$-centre problems; if one could show that integrability of a system necessarily results in $\nu < 1$ this would prove nonintegrability.
• Make theorem 4.4.1 sharper. In particular describe $H_2$ precisely rather than describing it as a subset of a larger class. This is possible if it can be shown that the time taken to go from $\eta = -2$ to $\eta = +2$ is the same as from $+2$ to $-2$. Also describe what 'types' of orbits exist in each element of $H_2$.

**Newtonian N-body problem**

• Ferrario and Terracini in [36] found a subset of classes of the Newtonian problem that do contain periodic solutions - can we describe a subset of classes that don't? This 'nonexistence question' is not mentioned in the literature, presumably because either it is considered uninteresting or there are not yet any satisfactory answers. The best that this thesis has managed is some proofs for the planar symmetric Newtonian 2-centre problem (which is integrable and therefore 'easy') and tied classes of loops of the molecular N-body problem for small periods. One strategy for constructing a nonexistence proof is to show that for all $x$ in your space of loops there exists $u$ such that $dA_T[x](u) \neq 0$ (in chapter 5 we took $u = x$).

• We have already seen, in the case $N = 2$, how the existence of periodic solutions in homotopy classes of loops of N-centre problems with Newtonian potentials can depend on the value of the period. Other preliminary studies by the author (not included in this thesis) suggest that low period is, in general, a hindrance to proving the existence of solutions and, in particular, finding collision-free minima. However for large periods the kinetic contribution to action becomes 'less significant' and it appears that a combination of this observation and Marchal's theorem may allow for a proof of periodic solutions in some symmetry/homotopy classes of loops. A possible project is to use this idea to construct periodic solutions of the Newtonian N-centre problem. Can a systematic approach be formulated? How do these results compare with [11]?

• Many more open problems on the Newtonian N-body problem can be found in [58].

**Relative Periodic Orbits**

• The fundamental group of the configuration space of the planar N-body problem is the coloured braid group. It is harder to manage than the free group on $N$ generators and, in particular, centralizers with respect to the $g$-twisted action are difficult to compute. A better understanding is needed; articles such as [10] may help. One possible alternative approach to finding centralizers is to use a software package called 'Magnus', see [8]. Magnus is concerned with experiments and computations on infinite groups which in some cases are known to terminate, while in others are known to be generally recursively unsolvable.

• In [54] the category of a space was used to estimate the number of critical points in homotopy classes of relative loops. As suggested at the end of their paper, $G$-equivariant category theory could be investigated to take into account that in general we are dealing with action functionals that are $G$-invariant (due to phase-shifts, rotations and other symmetries). Is the paper [52] helpful?
Maslov Indices

• In this thesis we have looked at Morse indices as a way of gaining insight into the structure of the action functional. Another important index is the so-called Maslov index. The Maslov index appears in semiclassical Gutzwiller type trace formulæ (see [12]) for describing the quantum energy density of systems exhibiting chaotic Hamiltonian dynamics in the classical limit. Maslov indices are also relevant to classical mechanics as an invariant object on invariant tori (see [5]); they are related to the stability of periodic orbits, see [61] and to singular points of the energy-momentum map, see [37]. The relationship between has been studied for closed geodesics (see [4], [56], [23] and [44]). What is the relationship between Morse and Maslov indices for $N$-body systems?
Bibliography


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