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Regulations for Higher Degrees: Copyright

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A MODIFICATION IN THE METHODOLOGY OF MATHEMATICS TEACHING AND LEARNING BASED ON AN ALGORITHMIC APPROACH

By

ADEL ABDUL-KAREEM YASEEN,
B.Sc. (1956), M.Sc. (1973)

Thesis Submitted For The Degree Of Doctor Of Philosophy (Ph.D)

University of Surrey
United Kingdom

(February, 1982)
To the memory of my Mother, who was
the inspiration in my quest for knowledge

and

To my Wife for her constant support and encouragement
ABSTRACT

Countries in all stages of development have long expressed concern over what they perceive to be a low level of achievement attained by students of mathematics.

The concern over mathematical deficiencies has general educational import viz. application of the mathematics that has been learnt. "They can do a thing in mathematics but cannot do the same thing in physics". Through consideration of this situation, I formulated my research problem: the search for a feasible improvement, however partial, in this achievement through modifying the current methodology of mathematics teaching-learning situations. I have confined this study to secondary school mathematics in Kuwait. This notwithstanding, if this study appears to be convincing, then implications could be drawn for curriculum design, methods of teaching and the setting of tests and examinations as well. This is at both secondary school and university levels of education.

In satisfying the research problem, it was indicated that, if the construct 'achievement' is assigned a numerical value A, then A is a function of the independent variable R consisting of the variant component of the 'contents' of a teaching-learning task.* R was identified as a set of instances i.e. examples, exercises and simple problems designed to illustrate, enhance and reinforce attainment of knowledge related to the task. Nevertheless, it was viewed that a true change in R is due to a change in the 'structure' S underlying R. However, a modification may result from the interaction of S with another structure H or some part, U, of H in underlying the

*(A Glossary containing these and other such terms is found in Appendix A)
methodology of mathematics teaching-learning situations. S was identified as a set of instances which do not explicitly make use of a constant of implicit form, 'a' say, other than the variables if any. H on the other hand contains only such instances that make use of one or more of such constants. Empirical studies showed that behaviour is not necessarily consistent in two instances - governed by a unique behavioural objective - one in S and the other in H. In this context a concept of space of knowledge (W,0) was introduced consisting of a set W of structured knowledge - identified by a finite set of rules - which is expected to diffuse a particular piece of knowledge through a certain normal population 0. It was also noted that a space of knowledge (S,0) - the learning space of current methodologies - is influential but insufficient for diffusing the required knowledge either in achievement or application. However, it was argued that a part U (of H) - identified by instances in H which include a unique constant - could feasibly contribute, with S, to a modification in the methodology. Furthermore, it was viewed that learnability in U should be subject to certain conditions that make U 'docile' i.e. applicable.

Finally, a modification in the methodology of teaching-learning situations was proposed which relied on S + U as a learning space in the form of a model based on an algorithm.

Evidence based on experimentation supports the notion that the achievement of students in mathematics under a technique based on S + U is superior to the conventional based on S.
ACKNOWLEDGEMENT

I would like to thank Professor L. R. B. Elton for his continuous encouragement and supervision of this research, and for his humanitarian attitude. I should also state that any clarity found in this thesis is due to his keen views and proposals.

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CHAPTER ONE

THE RESEARCH PROBLEM: ASPECTS OF

THE RESEARCH STRUCTURE
1. THE RESEARCH PROBLEM: ASPECTS OF THE RESEARCH STRUCTURE

1.0. Introduction

It is rational to think that educational research should be based on a general plan derived from the aim of identifying the progress in achievement of individuals in teaching-learning tasks. Accordingly, the aims behind this research were to look for improvement in mathematics achievement. The research problem was identified as the search for a feasible development in the methodology of teaching-learning situations which would fulfill these aims.

In this connection, a researcher conducting his research will inevitably encounter the need of both identifying his terms clearly, and establishing a set of fundamental propositions relevant to his study so that he can rely on a more rigid plan of work. In addition, I believe that a researcher in the educational field should be aware of the following two points:

(i) In the absence of a principle of invariance (see Appendix A) in the psycho-educational field, we indeed are still unable to provide a set of proper conditions for consistency which is a fundamental assumption made by behavioural scientists - that the behaviour of an organism is lawful and predictable (The Law of Universal Determinism)(see Appendix A). This law cannot therefore provide in behavioural sciences (unlike in other sciences, e.g. chemistry) a respectable reliability in findings and results, which thus seem to stand always on shaky ground.
Research is considered to be valuable in so far as it emerges from reality and in so far as it purports to represent it.

Generally, the development of this research was based, as in an experimental science, on tracing a phenomenon which could provide a functional relation between the achievement as a dependent variable and other relevant independent variable(s), provided that the independent variable(s) could be controlled experimentally in order to study variation in the achievement. The idea behind this conception of the research problem inevitable affects the research propositions and the research design.

In the following, I present general aspects of the research structure by identifying the research problem and by reviewing briefly the fundamental research propositions and other relevant points.

1.1. The Identification Of The Research Problem

An increasing number of those involved in mathematics education in Kuwait have been expressing growing concern about the achievement of students in school mathematics. This study was conducted for the purpose of looking for a feasible solution - albeit partial - to the problem that is embodied in the question:

How can we help both ordinary teachers and ordinary students to cope with mathematics better?

As a matter of fact, the problem behind this question is not only significant but also long standing, and under continued study by researchers in
all countries acquainted with educational research. Hence, the study of the problem in this thesis was reduced to the following three points.

1) Secondary school mathematics was the area of the study (although the study was reinforced by extension to students in Kuwait University). The mathematics of this area can be traditionally classified as algebra and introduction to calculus.

2) The term 'better' in the above question was understood in the sense that the achievement of students in mathematics under a proposed development showing an accepted progress compared with the achievement in the current situation. Here a mathematics situation is any possible observable event in the methodology that could be relevant to current achievement, e.g. teaching method, textbook, general examinations and performance in problem-solving. In contrast, the construct achievement should be operationally defined such that two achievements could be comparable in the sense that one could be 'better' in the above sense.

3) A clear-cut solution is not asked for, and a partial promising solution will be satisfactory. It is believed that complete answers to educational problems are in general very elusive because of the human factors involved, which have proved to be very strongly intractable. The implication of such difficulties are found in the uncertainty of results of findings. Generally, those difficulties in my opinion are due to the facts that:
   (i) our knowledge about psychological factors, e.g. memory or ability is very limited.
(ii) we still do not know a principle of invariance in the psycho-educational field which the results of findings might rest on, as most forms of reasoning - whether logical or physical - do rest on the principle of invariance of quantities (see Appendix A, Glossary of Terms).

 Accordingly, Ary et al. (1972) could be understood when they note that scientific investigations seek not absolute truth, but rather theories that explain and predict phenomena in a reliable manner.

 The first two restrictions delimit the study of the research problem in this thesis, while the third one explains why the solutions could be expected to be partial in the educational problems. Furthermore, I chose the area of the secondary schools in Kuwait, since I have had experience of this area for 25 years as a teacher, senior teacher and inspector of mathematics.

 As for tackling the problem, I believed that experience based on specific information about the problematic consequences affecting the achievement of students e.g. methods of teaching or the textbooks, reviewed literature and a more deliberate study of actually 'doing' mathematics, all could experimentally serve for the development of the study.

 Hence the research problem, aims and area were:
Two questions would then arise; how can this development take place so these aims are fulfilled? ... could achievement be taken as dependent on one independent variable such that a variation in this independent variable results in a desired development in the achievement? From both my experience and the review of the literature, it was feasible to expect a particular phenomenon might relate change in achievement to change in other variables. This is briefly discussed in 1.3 and more extensively in Chapter 6. Chapter 4 presents the basic discussion of such a phenomenon called the presentation-phenomenon.

In what follows, I shall present an outline of education in Kuwait as well as the fundamental research propositions. Furthermore, I shall show how relevant terminology will be introduced.

1.2. Education In Kuwait

It is natural now to present some simple information about education in Kuwait and secondary school mathematics.

(a) School Education

Pre-University education in Kuwait consists of 12 grades (years) and divided into three stages, each of four grades:
1. Primary stage from Grade 1 to Grade 4 (ages 6-10 years).
2. Intermediate stage from Grade 5 to Grade 8 (ages 11-14 years)
3. Secondary stage from Grade 9 to Grade 12 (ages 15-18 years).

(b) Types Of Pre-University Specialisation
Students usually specialise either in arts or science subjects after the tenth grade (age 16). Those specialising in science take mathematics for 7 out of the 20 periods in the week in the eleventh grade, and 6 out of the 20 in the twelfth (final) grade; while students specialising in arts take mathematics for only one period a week in the eleventh grade.

(c) Secondary Schools And Relevant Populations
In the academic year 1979/1980, there were in Kuwait 34 secondary schools for boys (24711 students) and 32 secondary schools for girls (22565 students). At least two-thirds of the total number of boys and girls who completed the 10th grade specialised in science. Girls in secondary schools proved in general achievement tests to be as competent in science especially in mathematics as boys. Consequently, there will be no discrimination between boys and girls in this study. Finally, it is to be noted that the total population of Kuwait is about one million.

(d) The New Curriculum: Modern Mathematics
UNESCO, in co-operation with mathematicians and mathematics educators in Arab States contributed to a project of modern mathematics to be applied firstly in the secondary stage. Kuwait followed the project in 1970/1971 in two test schools; one for boys and one for girls. Subsequently, the new curriculum has prevailed in the secondary school since 1974/1975. Lastly, all stages of school in Kuwait have been following a new curriculum based on modern mathematics since 1978/1979.
However, the project of modern maths has proved not to be the solution for problems of low achievement of students in mathematics. We still have the same complaints we used to hear about traditional mathematics in that the achievement of students is still in need of improvement. In addition, there is a certain concern for deficiencies in computational skills as well as in problem-solving. The criticism raised is similar to that of Hammersley (1968) who reported similar deficiencies in mathematical skills.

(e) **Main Source For Mathematical Achievement: The Textbook.**
In mathematics classes in Kuwait, both teachers and students rely mainly on one textbook for each grade in the school, written under the full responsibility of, and authorised by, the Ministry of Education to be the unique 'source' of mathematics. Hence, it is acknowledged that the textbook is a main regulator of mathematics teaching-learning situations. Furthermore, the examinations, whether general or not, play a further role in giving a dominance to the text since the Ministry directs that all examinations should be restricted to the material found in the texts. As teachers like their students to pass the examination (as well as students preferring this) the textbook can then be understood to be a most influential event in mathematics teaching-learning processes.

In this Kuwait is not an anomaly. Howson et al. (1981) tell us that other countries, e.g. East Germany, also follow one single textbook for each grade. In addition, they report that the textbooks continue to have a major influence on the classroom: in many cases it still effectively determines the curriculum.
The written material for a task $k$ in the textbook in Kuwait will be referred to, in this thesis as the contents of a task, since the text serves just the purpose signified by this term.

Here, I now present Table 1.1, which indicates the main topics of secondary school mathematics in Kuwait.

<table>
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<tr>
<th>1st Grade - Age</th>
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Table 1.1
Main Topics Of Modern Mathematics In Secondary Schools In Kuwait (1979/1980).

1.3. The Research Proposals

1.3.0. Introduction

The research problem of looking for a feasible development in methodology for better achievement in secondary school mathematics in Kuwait, is indeed a special sub-problem of the well-known problem in education which can be found in the context of Elton (1979).

Elton with the aims of making students more independent in learning processes and affecting a teaching of understanding, has raised this problem in the following two questions:
(i) How can we help teachers to teach better?
(ii) How can we help students to learn better?

In this context, the word 'better' is employed to mean that the declared aims are more likely to be achieved.

In addition, the size of the research problem could be better understood in the light of the problematic consequences that could have emerged in the educational systems of most developing countries as a result of the sudden increase in the number of schools and students in the last two decades. In Kuwait, this has resulted in the recruitment of mathematics teachers without the possibility of adequate selection. And despite a radical change in the curriculum - the shift towards modern mathematics - and intensive courses of training for all the teachers of mathematics, the complaints of the low achievement in secondary school mathematics are still found to be raised by all of those who are involved in the field, e.g. parents and specialists. More specifically, complaints of deficiencies in the achievement of mathematical knowledge highlight two basic parts of mathematical knowledge, computational knowledge, i.e. the knowledge required to use mathematics as a tool, e.g. applying a certain rule or solving an equation, and comprehensonal knowledge, i.e. the knowledge required to understand properly and identify what should be used to go on with a problem.

My views (based on certain convictions dealt with in Chapter 6) were that as long as we are unable to identify a principle of invariance on which findings and results in the psycho-educational field can consistently rely, an optimization for completely removing such deficiencies is unlikely. However, it is probable that we might minimize the effects of
such deficiencies to the extent that research studies in the field are grounded on reality. This would mean that a researcher could learn more fully about such deficiencies and their real effects, and throughout I have been at pains to take into account the gap between theory and practice. In this connection, Walker (1980) finds that:

"Most educational research simply does not directly connect with the world it purports to study" (P.46).

This is also in line with Becher (1980) who gloomily stated that:

"It must be acknowledged at the outset that very little of the educational research that is done appears to have any noticeable impact on the ordinary teacher and his work. It is not easy to get round this difficulty by saying that research consists in the pursuit of knowledge for its own sake. In the long term, if it is not to be seen as an expensive self-indulgence, there must be a pay-off of some kind. So it is, I think, right for all of us - teachers, policy-makers and the tax-paying public as well as researchers themselves - to look for ways in which the present situation can be improved" (P.64).

In other words, research should not be denied its direction towards development of new knowledge, but equally it should also demonstrate a regard for the people who need its conclusions in real teaching-learning situations.

As discussed in Chapter 6, the work in this thesis was planned to be
conducted in three stages, a pre-proposition stage which facilitated in its turn the proposition stage, which introduced a set of four fundamental propositions considered as a basis for the development of the research work in the final stage. In the following section, I endeavour to throw some light on these fundamental propositions of the research to help give a general view of the research development.

1.3.1. The Fundamental Research Propositions

Any researcher, in conducting his work will inevitably encounter numerous problems such as defining terms clearly, using them consistently and identifying the research proposals for a convenient development of the research. One of the major difficulties I found - resulting from the personal construct of grounding the study on the basis of the experimental sciences - was in identifying a functional relation between the achievement as a dependent variable and one (or more) independent variable that could be governed (controlled) in order to study the variation of the achievement. Hence, I found that I had to match the construct achievement with a number 'A' under certain conditions and to trace a phenomenon that might help in defining a functional relation as

\[ A = g(x_j) \]

with \( x_j \) as one of the controllable independent variables on which A depended. Different factors (events), e.g. reviewed literature and experience helped in identifying such a phenomenon called the presentation-phenomenon. This phenomenon, discussed in Chapter 4, stems primarily from a remarkable study by Mayer and Greeno (1972) which generally indicates that differences in learning outcomes could be due to differences in material contents. Hence, it was believed that this independent variable \( x_j \) is strongly related to the material contents or, more briefly, to the contents. And since the textbook in the area of this study plays
the main role in teaching-learning of tasks (see 'e' in 1.2) then it was viewed that the contents of teaching-learning tasks could operationally coincide with the written material in the text, (see 4.2.1 and 4.2.2).

Consequently, based on this phenomenon as well as on the set of convictions in 6.1.0. dictating the role of theory in development, I presented the following three general propositions:

(Proposition 6.1): A phenomenon that indicates a change in the achievement of students under a certain change in a well-defined teaching-learning situation, is significant as a starting point for development in current teaching-learning situations.

(Proposition 6.2): There exists a theoretical framework on which a development in current teaching-learning situations can rest.

(Proposition 6.3): If a proposed development (reform) in current teaching-learning situations (a) can rest on a theoretical framework and (b) can be operationally applicable, i.e. well-defined and learnable, then such development is expected to be experimentally supported.

It is to be noted that I intended to present such propositions in broad terms, in order to encourage a wider interpretation of them for other fields of study.
In a further step towards identifying such an $x_j$, it was viewed that $x_j$ should be related to the contents, i.e. the written material in the textbook. In addition, it was argued in 6.2.2 that the contents of a teaching-learning task has two main components (a) an invariant component which comprises theoretical aspects of the task, e.g. a definition or a proof of a theorem, and (b) a possibly varying part $R$ which constitutes a set of instances as examples, exercises and simple problems that are supposed to reinforce and enhance the required knowledge of the task. $R$ was called the presentation of the task which indeed constitutes the core of the task. Accordingly, the contents could be viewed to be equivalent to $R$ in the sense that a true change in the contents is genuinely reflected in $R$. In addition, in discussing a true change in $R$, it was viewed that this change should take place in the structure say 'S' underlying $R$ and any modification $K$ would rely on the contribution of another structure 'H', say. Moreover, it was viewed that theory should discriminate between the structures underlying different contents (presentations) in such a way that this discrimination should be directed towards demonstrating how far the knowledge that is achieved in different contents is efficient. Accordingly, a measure called the efficiency (see Glossary of terms, Appendix A) was introduced for such a discrimination. Simply, the efficiency for a problem $P$ is the probability that an individual $x$ is capable of $P$, provided he is completely capable of all the elements of a set called a kernel for $P$ which tests for basic and necessary knowledge for $P$.

Consequently, it was viewed in 6.2.2 that $x_j$ involves both the presentation $R$ and the underlying structure $S$ i.e. $x_j = R(S)$. Hence $A = g(R(S))$ or more briefly $A = g(R)$, provided that $S$ is implicitly involved. Theory was also expected to play its role on the basis of the efficiency.
Lastly, the fourth and final proposition was introduced in Chapter 6 which assumed the existence of structures such as S and H as well as conceptualising the function of theory in the research field.

(Proposition 6.4): (i) There exists a well-defined structure 'S' underlying current presentation of mathematical tasks.
(ii) There exists a structure 'H' that is possible to contribute in the modification of current presentation.
(iii) Theory in Proposition 6.2 could possibly identify a certain discrimination of the efficient knowledge that achieved in 'S' and 'H'.

Further details for the establishment of these four fundamental propositions are found in Chapter 6.

1.4. Final Stage Of The Research Study

In this stage, I identified the two structures 'S' and 'H' (Chapter 7) and also developed a theoretical framework on the basis of a conceptual structure called the space of knowledge (Chapter 5). Further on (Chapter 8) I employed theory on the basis of efficiency to find out what advantages S or U (a part of H) could demonstrate in achievement. In Chapter 9, I finally proposed on the basis of the empirical studies in Chapter 8, an algorithmic approach to a modification \( \hat{R} \) in the methodology of teaching-learning situations. In Chapter 9, I also tested the modification on the basis of relevant experimental method through two experiments. This empirical evidence gave a considerable
support to such a trend of modification. Finally, Chapter 10 presents a summary to the main study as well as it presents a further outlook.

The following Figure 1.1 provides a general scope of the research development based on the fundamental research propositions.

![Diagram Illustrating The General Scope Of the Research Development.](image)

**Figure 1.1:**
Diagram Illustrating The General Scope Of the Research Development.

1. Research problem: To modify the current methodology of mathematics teaching-learning situations with the aim of improving current mathematics achievement.

2. General Propositions 6.1, 6.2 and 6.3 are expected to act by way of the specific Proposition 6.4 through both S and H to give the required modification.
1.5. Terminology

First: Use Of Terms

One of the most important advancements in a research work is found in clarity of terms employed. The clearer the terms are, the easier the study can be developed and grasped. Therefore presentation and use of terms could be one of the numerous problems that a researcher encounters. A difficulty in clarity of usage of terms lies in the fact that any attempt to define all terms in a particular field of knowledge normally proves impractical. Such an attempt will inevitably encounter either a case of circular definition where, for example, an A is defined by B, B is defined by C, and subsequently C is defined by A, or a case of an infinite-sequence definition where A is defined by B, B by C, C by D, etc.; and neither case provides any satisfactory state.

In connection with mathematics, the definition-problem is solved by accepting a state of terminology called 'undefined terms'. In this state, one accepts some primitive terms without definitions in the expectation that the user should formulate certain knowledge of such terms through using them. But this trend seems to have no current parallel in non-mathematical fields, e.g. mathematics education, where I observe that the mathematical procedure of using undefined terms, is not favoured and held in a very poor light.

Such being the case, I shall attempt a sort of compromise: I shall not use the term 'undefined terms' nor shall I define all terms. I shall
introduce terms "which I do not define" by discussing or explaining them operationally, e.g. advantage of achievement. A Glossary of the terms that were found most useful is found in Appendix A.

Second: Proposition And Hypothesis

The term 'proposition' in this thesis will refer to a statement expected to be true. It depicts such hunches that could have been affected by a combination of personal experience, readings, introspection and speculation on the research work and results. As a consequence of such a situation, a researcher would build-up what he believes to be proper idea(s) about the existence of such relationship(s) among the variables in his research. It is more or less an axiom.

As for the term 'hypothesis' it will in this thesis refer to a testable statement. A hypothesis will be treated as a tentative proposition which is subject to verification through subsequent investigation. It is indeed a specific derivative of a proposition; in the sense that events in propositions are expected to be broadly introduced, e.g. mathematics situation, whilst those events in a hypothesis are expected to be operationally defined, i.e. defined in observable terms.

1.6. Summary

This chapter shows the general aspects of the research structure. It states the view that educational research should be based on a general plan derived from the aims of identifying a certain progress of individuals in achieving teaching-learning tasks. It is further believed
that a researcher should be aware of the absence of a principle of invariance in the field so that results and findings that stem from applying the law of universal determinism in this behavioural science can be consistent. Moreover, it was viewed that research is of value to the extent that it emerges from reality and in so far as it represents it. The development of the research was based, as in the experimental sciences, on tracing a certain phenomenon that might provide a functional relationship between a dependent variable (the achievement) and one or more independent variable(s). Later, Section 1.1 presents an identification of the research problem in "looking for a feasible development in methodology of mathematics teaching-learning situations" with the aims of "improving the achievement of mathematics students in secondary schools in Kuwait". Section 1.2 presents general aspects of education in Kuwait as well as the new curriculum of mathematics. It also reveals the dominant role of the textbook as the main and unique source for mathematics achievement. Section 1.3 discusses the research proposals, and presents the basis of adopting the fundamental research propositions. These initially arose from the remarkable phenomenon called the presentation-phenomenon derived from findings of Mayer and Greeno (1972) who reported that differences in learning outcomes could be due to differences in material contents. This phenomenon encouraged me to think of the achievement A as $g(x_j)$ where $x_j$ was preconceived to be strongly related to the contents. In this way were the first three general fundamentals put forward.

A further step required the identification of $x_j$ on the basis of viewing a relationship between $x_j$ and the contents. It was supposed that the contents of a teaching-learning task include in general two components; an invariant one, which constitutes the theoretical part
of the task and another component R, a set of instances, e.g. examples or exercises. A change in the contents is thus reflected in a change in R and vice versa. In addition, it was argued that a true change in R should take part in the structure 'S' that underlies R. However, a modification R should rely on contribution of another structure H. Moreover, it was viewed that the theory should discriminate between the structures that underlie different contents (presentations). A measure called the efficiency was suggested to take part in such a discrimination. Accordingly, the fourth and final fundamental proposition was put forward and assumes the existence of structures such as 'S' and 'H' as well as describing a role for theory based on the efficiency measure.

Finally, Section 1.4 reviews the final stage of study and Section 1.5 discusses basis of introducing terminology.
CHAPTER TWO

THE RESEARCH PROCEDURE
2. THE RESEARCH PROCEDURE

2.0. Introduction

This chapter is introduced mainly to discuss the methodology used in achieving the study in this research, as well as to present an outline of the chapters of this thesis. Nevertheless, it was believed that it is essential to briefly discuss the sources of approaches to knowledge in order eventually to focus on the so-called modern approach, or the scientific method. In this connection, the role of theory - supposed to present a set of interrelated constructs (concepts), definitions, and propositions in a systematic view of certain phenomena - should not be ignored. Further, since this research is supposed to be an educational research, then it would be expected that the approach should be viewed, by the people who are involved in the field, as indeed relying on the scientific method. In the following, I shall try to cover the following general objectives:

(1) To discuss the general approach in educational research and relevant methodologies.

(2) To focus on the research in this thesis

(3) To outline the thesis and each chapter involved

(4) To demonstrate the methodology used in achieving this research. This considers sampling, definition of the tests involved and the two major attributes of instruments, i.e. validity and reliability.
2.1. The Approach In Educational Research

2.1.0. Introduction

There is still no universally accepted and inflexible meaning of research, but under whatever set of conditions research occurs, there is one element that seems common to most research activities in that it has been described in literature as a systematic study (Verma and Beard, 1981). In this connection, the sources of knowledge were categorized by Ary et al. (1972) under five forms: experience, authority, deductive reasoning, inductive reasoning and scientific method. Experience is a familiar and well-used source of knowledge. And, although such a reliance on personal experience has obvious limitations, its usefulness in decision making should not be ignored or denied. However, this experience could only be fertile if it functions systematically, giving reliable knowledge having logical roots.

Authority could also be helpful in providing help or advice by other experts who had experience in the field. And despite its weakness as a source of reliable information, a total rejection of such a source of gleaning knowledge is imprudent (Verma and Beard, 1981). Furthermore, the first approach to reasoning goes back indeed to Aristotle who, with his followers, introduced the use of the deductive method. This method of acquiring knowledge involves a thought process in which one proceeds from broad assumption to specific statements, using prescribed rules of logic. The process starts with recognition of a universal law and applying it to an interpretation of particular phenomena. It operates through a series of statements called "syllogism" which attempts to establish relationship between a major premise, considered to be true,
a main premise, and a conclusion. An example of this syllogism is the well-known one: All humans are mortal; Socrates is a human; hence Socrates is mortal. However, the main weakness of the classical deductive method is found in its reliance on the major premise which might be false. Thus the major criticism of this method by scientists is found in its inability to verify the truth of major premises (Verma and Beard, 1981).

Such being the case, Francis Bacon (1561-1621) advocated a new approach that came to be known as the inductive method. This process requires the application of direct observation of phenomena, arriving at a conclusion through evidence by observing many cases. Bacon held that a man is enslaved by accepting premises handed down by authority as absolute truth. Instead, he believed that an investigator should establish general conclusions on the basis of the facts gathered through direct observation. Nevertheless, this method has its own limitation since a perfect induction should be established on observation of all the examples involved in a class of objects or events, which is in practice impossible. In addition, the induction based on the regularities in mathematics could not possibly apply similar regularities in other fields, especially human behaviour.

Hence, as also the inductive method could not present an adequate technique for many problems, it was natural therefore to try to integrate the relevant aspects of the induction and deductive methods into a more useful method of acquiring reliable and significant knowledge, constituting the so-called modern or scientific method (Verma and Beard, 1981).

Dewey (1933) described five main stages that one passes through when attempting to acquire new knowledge:
(i) recognition and definition of the problem under study,
(ii) observation, collection and classification of data relevant to the problem,
(iii) formulation of a tentative hypothesis concerning these observations or phenomena,
(iv) verification of this hypothesis against all the known facts which might modify the original hypothesis, and
(v) formulation of conclusion(s) in terms of general principles concerning the problem.

Indeed, it was Dewey who gave this so-called scientific method an impetus of major importance in modern times (Verma and Beard, 1981). Yet, although these five steps present a useful pattern for development and implementation of some methods of research, a researcher in practice does not follow them rigidly. Consequently, different researchers will have different strategies of acquiring the required knowledge within the general framework of the scientific method. For example, Ary et al. (1972) suggest that the five steps in the scientific method could be:

(i) definition of the problem,
(ii) statement of a hypothesis,
(iii) deductive reasoning,
(iv) collection and analysis of data, and
(v) confirming or rejecting the hypothesis.

However, Fox (1969) suggests 17 stages divided into three main parts,

(a) designing the research plan,
(b) implementing the research plan, and
(c) implementing the results.
Nevertheless, whatever the suggestions are for implementing this method, the backbone is that this method is a systematic way of reasoning, whereby observations or experiments are made, hypotheses are put forward and tested, and if necessary amendments are made to the original hypotheses in the light of evidence obtained and then conclusions are drawn.

In connection with the discussion of the scientific method, it is worthwhile referring to controversy about this method. Kuhn (1962), for example, finds that scientists who want to break new ground cannot afford to restrict themselves to any arbitrary set of methodological techniques. He finds that the only rule that can guide the selection of a new technique is "that it works". Thus the good scientist does not follow a prescribed method, but on the contrary, discovers methods that work. This does not imply rejection of existing methodologies, but it simply states that scientists should learn many methodologies and use each as a tool rather than as a religion (Lochhead, 1979). Feyerabend (1975) in the context of this controversy finds that the idea of a fixed method or a fixed theory or rationality, rests on too naive a view of man and his social surroundings and:

"there is only one principle that can be defended under all circumstances and in all stages of human development. It is the principle: 'anything goes'" (P.28).

He indeed arrives at such a controversial conclusion by simple extension of a point on which most modern philosophers agree, i.e. knowledge can never be certain. Hence, as Feyerabend claims, the key to scientific progress has not been adherence to a well defined methodology. Verma and
Beard (1981) in this context state that:

"Although controversy exists with regard to the nature and use of the 'scientific' method, it is important to keep in mind that this method represents one way of describing and studying the world around us and that there are various other legitimate approaches which can be adopted." (P.8).

Such discussion of the approach in research fields inevitably requires a discussion of scientific theory. A brief discussion of this point follows.

2.1.1. Scientific Theory

The last aspect of the concept of science to be considered is the construction of theory. A scientist, through his empirical investigations, gathers many facts that need integration, organisation and classification in order to make the isolated findings meaningful. Generally speaking, good theories are built upon facts or evidence and not on mere speculation. Theories consist of a number of statements about variables and about relationships between variables. In other words, theories consist of generalizations, usually called as laws in the physical sciences, and constructs, where a construct is a type of concept used in scientific research to describe events that share similar elements (Borg and Gall, 1971).

The role of theory is essential, in that, it summarises and puts in order the existing knowledge in a particular area as well as providing meaning
to dispersed fragments of empirical knowledge. In addition, theory by specifying relations among variables, can possibly help explain and predict related events in the area of the study. Moreover, it can stimulate new discoveries.

2.1.2. Educational Research

Since both 'education' and 'research' do not have commonly shared definitions, then it is naturally extremely difficult to give the term 'educational research' a definition that would be accepted by all those concerned with educational decision-making and practice. Lovell and Lawson (1970) find that it is virtually impossible to give a definition of the term that would command universal acceptance, as there are innumerable meanings that can be given to the word 'education'. Meanwhile, Travers (1964) defines educational research as:

"an activity directed toward the development of an organised body of scientific knowledge about the events with which educators are concerned" (P.5).

Ary et al. (1972) state that:

"When the scientific method is applied to the study of educational problems, educational research is the result" (P.21).

Whatever the goal of educational research, the common element in all the differing opinions is found in the application of systematic methods to
the study of educational problems so that the search for meaningful and trustworthy knowledge becomes central in the whole process (Verma and Beard, 1981).

But, despite the great necessity for development of education through educational research, it must be said that educational research has not accelerated educational practice to the extent many had hoped for. The progress of research in education has been slower than other research in other fields (Verma and Beard, 1981).

As a matter of fact, education in particular has suffered from an absence of a theoretical orientation towards feasible developments in teaching-learning situations. In this sense, Walker (1980) states that:

"Most educational research does not directly connect with the world it purports to study" (P.46).

In addition, Becher (1980) notes that very little of the educational research that is done appears to have presented any valuable help to the ordinary teacher in this work. Although we do not underestimate the value of knowledge sought for its own sake, neither should we expect people apart from researchers who are involved in the educational field, to innovate and produce the required developments in the field.

2.1.3. Research Methodologies In Education

Research method refers to the general strategy followed in gathering and analysing the data necessary in the search for answers to the research problem. And although the literature does not seem to suggest that there
is a generally accepted scheme for classifying educational research studies, one might find the following four categories of Ary et al. (1972) to be sensible.

1. **Experimental:** which is an approach that is based on a scientific investigation in which the investigator manipulates and controls one or more independent variables and observes the dependent variable(s) for variation concomitant to the manipulation of the independent variable(s). Its major purpose is to determine "what may be".

2. **Ex Post Facto:** which is similar to experimental research except that the investigator is not able directly to manipulate independent variables.

3. **Descriptive:** which is primarily concerned with portraying the present. It is concerned with conditions or relationships that exist e.g. prevailing practices or beliefs. Its major purpose is to say "what is". There are several subcategories of this method, e.g. case study and survey.

4. **Historical:** which involves an understanding of the present in the light of past events and trends. Its major purpose is to say "what was".

In this connection, none of these categories is essentially superior to the others, and further, a piece of research may fall into more than one of the categories.
2.2. This Research

The research in this thesis, in the sense of Travers (1964) and Ary et al. (1972) could be considered as an activity based on the scientific method to be applied in seeking a development of an educational problem. Hence, in this sense, this research is an educational research. An educational research, as a science, uses investigative methods which are consistent with the basic procedure and operating conceptions of science. Generally, a research is conducted on the basis of a set of general and typical stages that can be found in:

(a) selection of a problem,
(b) analysis of logical aspects of the problem and definition of relevant terms,
(c) selection of an adequate method and relevant instruments of measurement,
(d) collection, analysis and interpretation of the relevant data, and finally,
(e) the reporting of results.

In this connection, the set of three stages that I followed in conducting this research is more or less equivalent to the set of general and typical stages previously mentioned. Here, I present a brief discussion of these stages which will demonstrate the structure of research in this thesis.

1. A pre-proposition stage: It was expected that this stage (reported in 6.1) would involve the identification of the research problem, relevant terminology and possible pilot instruments in order to adopt a certain set of fundamental research propositions which would constitute the next stage i.e. the proposition stage. Though the research problem was borne in mind throughout, I was also influenced and convinced by Piaget (1972b), who
proposes that a researcher would better state at the outset three to ten of the most important of his metaphysical theses in the most explicit form. Such a suggestion is sensible in the absence of a clear and reliable vision of the research development. Consequently, a set of five such convictions is presented in 6.1.0 to help as a general guideline in the strategy for achieving the study in this thesis. Hence, this stage (see 6.1) was made up of the following steps:

(i) A set of convictions: I stated five convictions (6.1.0); the first four commonly held in the field of education, while the fifth was a personal conviction, preconceived to help development of study.

(ii) The research problem: I proposed the research problem (6.1.1) as "looking for a feasible development in the methodology of mathematics teaching-learning situations". The aim was specifically "to improve the achievement of students in mathematics" and the area of this study was taken as secondary schools in Kuwait. Details are found in 1.1 and 1.2 of Chapter 1.

(iii) Pilot investigations for the proposition stage: Just as the proposition stage would inevitably influence the course of study in the final stage, so in turn, the proposition stage is influenced by the preproposition stage. Generally, as in experimental science, I viewed that conducting of this study should firstly be based on identifying a functional relation that introduces the dependent variable 'achievement' as a function of another independent variable, controlable under certain experimental methods, in order to study the corresponding variation of the achievement. Nevertheless, based on the belief of a supportive theory for the development, I viewed that the proposition for modification in the final stage should stem from the proposition stage. This in turn, would rely on the conception that the modification should be established on the basis of a comparative study of the advantages
of the achievements in both the current and the proposed teaching-learning situations. This is discussed in 6.1.2.

Accordingly, a discussion of terms is introduced, in 6.1.2.0, as well as two pilot instruments, a questionnaire (6.1.2.1) and a test $T_0$ (6.1.2.2) were introduced to throw light on the next stage. It is to be noted that $T_0$ could indicated that the advances in achievement regarding a problem $P$ could rely on a measure called the **efficiency**. This was based on the probability that an individual $x$ in a representative sample of a certain normal population is able to solve $P$, provided that he has accurately performed in all elements of a set $X_0$ - called a kernel for $P$ - supposed to test attainment of knowledge in necessary prerequisites of $P$ (6.1.2.2.3). This measure, i.e. the efficiency will be eminent in this thesis.

2. **A proposition stage:** As the construct achievement was assigned a number 'A' in terminology in 6.1.2.0, then the study in this stage was directed towards:

   (a) identifying a functional relation $A = g(x_j)$ where $x_j$ is an independent variable that could be controlled under experimental conditions;

   (b) proposing the role of theory in study; and

   (c) identifying how a change from $x_j$ to $\dot{x}_j$ would bring about a relevant change in $A$ as well as proposing, by using theory, how to discriminate between $A = g(x_j)$ and $\dot{A} = g(\dot{x}_j)$. This might then bring to light a proposed modification that would have a theoretical support.

The first part (a) was indeed derived on the basis of a remarkable phenomenon
that is found in the Mayer and Greeno Study of 1972, which generally could indicate that differences in learning outcomes could be induced by differences in material contents. This phenomenon was called the presentation-phenomenon and detailed study of this phenomenon is found in Chapter 4. As a matter of fact, the reviewed literature in Chapter 3 as well as introspection also identified such a phenomenon. I accordingly assumed that such an 'x_j' in A = g(x_j) should be an observable and well-defined independent variable relevant to the material contents. Hence, the first three general propositions were put forward satisfying both (a) and (b). As for (c), it was argued that the presentation R is the core in the contents of a teaching-learning task, since these contents in general include two parts, a theoretical part and a set R of instances as examples, exercises and simple problems. It was also argued that a true change in the contents is found in a true change in R. In connection with conviction C5 in 6.1.0, I assumed the existence of a structure 'S' underlying such an R, i.e. R = R(S) and a true change in R should take place in a change in S. Furthermore, I assumed that x_j = R(S), or simply R, provided that S is understood, and hence A = g(R) was considered as the required functional relation. Later, I assumed the existence of another structure 'H' which could possibly contribute in modification. In addition, I proposed that the use of theory to discriminate between A = g(R) and \( \tilde{A} = g(\tilde{R}) \) would take place on the basis of the 'efficiency' measure discussed in \( T_0 \) (6.1.2.2) - later developed in Chapter 5. Consequently, I proposed the fourth and final fundamental proposition.

3. The final stage: This final stage was, as usual, expected to provide some solutions, albeit partial, to the research problem. Partial, since the behaviour of an organism cannot maintain lawful regularity, despite many behavioural scientists assuming such ordered regularity (Ary et al.;
1972), and despite my own assumption of regularity in some places in this study - as in the pseudo-reality in 8.2.0.

In connection with this stage, I found that what I had to do was:

(a) to identify two structures: S, which underlies the current presentation of mathematical teaching-learning tasks, and H, which is supposed to contribute in modification. (This was satisfied in 7.1).

(b) to develop a theoretical framework in line with the fundamental research propositions, such a theory being expected to have its roots in experiment. (This theory was initiated in 7.3 and developed in Chapter 5 of this study).

(c) to implement further empirical study arising from the theoretical framework on the basis of the research methodology reported later in this chapter. (This was satisfied in Chapter 8).

(d) to propose an algorithmic approach for modifying methodology. This was satisfied in Proposition 9.1 of Chapter 9).

(e) to implement relevant instruments which would provide a comparative study of achievements of students under current situations and related situations under the proposed modification in methodology. (This was satisfied in two experiments in 9.2).

(f) To report the study

One final word is to be said. The literature review in Chapter 3 in relation to Chapter 4 strongly shared both the first two stages: Chapter
2 however, helped in planning the whole study. Chapter 10 finally summarises the main work and findings in this study as well as suggesting further works in the context of this study.

Here, I present Figure 2.1 that illustrates development of study in these stages.

![Figure 2.1: The Development Of The Study](image-url)
2.3 Outline of Thesis

Chapter 1 - The Research Problem: Aspects of the Research Structure

This chapter presents the research problem "To look for a feasible development in methodology of teaching-learning situations" arising from the question "How can we help ordinary teachers and ordinary students to cope with mathematics better?" It further discusses education in Kuwait and reviews the basis of the presentation of the fundamental research propositions. It also briefly states general aspects of the final stage of the research study. In addition, there is a discussion of the basis for introducing relevant terminology.

Chapter 2 - The Research Procedure

This chapter studies approaches to educational research passing through discussion of the definition of research and sources of knowledge and finally focusing on the scientific method. Further on, it discusses in a general way the role of theory in research and relevant methodologies. In addition, this particular piece of research is considered in its stages of development and relevant methodology as based on experiment.

Chapter 3 - Literature Review

It was considered that a review of the relevant literature is a fruitful source which helps the researcher to relate his work to the existing knowledge and draw benefit from it. The review first discusses the philosophy behind the approach to mathematics and then aspects of the debate as to the merit of an early introduction of an approach based on
It further discusses some aspects of the methodology and approach in curricula and other instructional methods. Moreover, in a general discussion, critical points for the approach and relevant methodology are outlined. Consequently, it was concluded that, given an approach which is axiomatic, the controversy about the achievement of students is rooted in methodology of the approach. Finally, it was stated that, deficiencies still obtain in computational skills or knowledge. This also pointed to a phenomenon which links changes in learning outcomes with changes in material contents. (This phenomenon was called the presentation-phenomenon, dealt with fully in Chapter 4).

Chapter 4 - Scope Of The Presentation Phenomenon

The presentation-phenomenon originates mainly from a remarkable study by Mayer and Greeno that indicated "differences in learning outcomes could be induced by differences in material contents". It was firstly viewed that the presentation R, a set of instances - examples, exercises and simple problems - constitutes the core of the material contents. Hence a true change in the contents is due to a change in R. Later, two relevant studies were reported, the Mayer and Greeno study of 1972, and the Ehrenpries and Scandura study of 1974. In addition, the Chapter discusses the presentation-phenomenon and reality through:
(a) personal experience,
(b) the textbook, and
(c) the teacher.

The conclusion in 4.3 is that the presentation-phenomenon can be useful as a starting point in the sense of the fundamental proposition 6.1.
Chapter 5 - Space Of Knowledge: A Conceptual Structure

The main conception of space of knowledge \( (W,0) \) is that any particular knowledge is structured into a set \( W \) of initial (simple) states of knowledge by an underlying structure identified (ruled) by a finite set of rules - called the underlying structure of \( W \). \( 'O' \) is a certain normal population, in which \( W \) is supposed to diffuse a required knowledge under a source of interaction between \( W \) and \( O \). A space of knowledge \( (S,0) \) was identified in Chapter 7 that constitutes the current diffusion of knowledge. Further on, it was viewed that a modification in space of knowledge could be based on a term called the 'efficiency'. This term was introduced as the probability that an individual 'x' is capable of solving a problem P, provided x has demonstrated complete ability in all the elements of a set of stimuli (called a kernel for P) expected to test attainment of necessary knowledge for P. Thus the area of problem-solving was involved. Accordingly, an 'axiom of sufficiency' for problem solving was introduced which simply assumes for the existence of a criterion \( X \) that can induce sufficiency for P. Three properties for such an \( X \) were assumed and in this connection, a problem-space and kernel were defined. It was necessary then to maintain a certain reliability for the efficiency such that this measure can be more relevant to the assumption of the "law of universal determinism". This was achieved by defining a 'uniform operational' for a problem P which is simply an individual who is completely capable of all the elements of all kernels for a problem P in a certain space of knowledge. Consequently a proposition of modification, \( (5.1) \) was introduced on the basis of discrimination with regard to efficiency between two spaces of knowledge.
Chapter 6 - The Research Stages

This chapter forms the core of this study, since it provides the general plan guiding the development of the research. It presents the fundamental research propositions on the basis of a set of convictions, the definition of terminology and the implementation and discussion of two pilot instruments designed to throw light on the introduction of these propositions. The reviewed literature, as well as introspection were also employed in this aspect. The propositions assume:

(i) that a starting point for study could be found in a phenomenon that relates achievement to another situation;

(ii) the existence of a theoretical framework for study;

(iii) the role of theory in development and in proposition;

(iv) the existence of a structure S, that underlies current mathematics situation, and of another structure H that underlies a possible modification in the methodology of teaching-learning situations and involves the use of theory.

Chapter 7 - The Two Structures S And H: Further Implications

The study in this chapter identifies an underlying structure S of current presentations as well as another structure H, which is ignored in current situations. Simply, S is the set of instances which do not contain any constant, 'a' say, while H is the set of instances which contain at least one constant. Hence \( S \cap H = \emptyset \) (the null set). Two instrumental indicators were introduced with the aim of finding any indications as to

(i) how computational knowledge usually gained in S could interact with H, and

(ii) how performance in a problem could be related to relevant knowledge in S and H.

It was found that computational knowledge gained in S is not necessarily
satisfactory in H. Also, mathematical knowledge in H correlates with
ability in problem solving more strongly than that in S. Further,
in discussing the 'pros and cons' of S and H, it was viewed that S is
indispensable, whilst H makes a useful contribution, not as a
whole, but only in part 'U'. Moreover, study in this chapter helped
extensively in throwing light on the conception of space of knowledge
as discussed in Chapter 5.

Chapter 8 - Experimental Studies

It was viewed in Chapter 7 that a part U of H could possibly contribute
to a modification. Hence U was introduced as the subset of H that
contains all instances which contain only one constant a. Moreover,
Proposition 8.2 for learnability in U was put forward to facilitate
learnability in U, and was based on choice of elements in U as associates
of corresponding elements in S. In addition, and in order to employ
theory in any modification, the chapter argues the difficulty in
distinguishing uniform operationals (Definition 5.6) in reality and there­
fore introduces the concept of pseudo-reality, and identifies a uniform
operational for a problem P in pseudo-reality to be the one who is capable
in only one kernel in the relevant space of knowledge. Results and
findings in pseudo-reality were expected to be extrapolated probabilist­
ically in reality. Further, a testing-model for the efficiency in pseudo­
reality was set out along with an empirical study in a set T3 of three
tests each in form $S_0 + U_0 + P$ where $P$ is a problem, and $S_0$ and $U_0$ two
kernels for $P$ in S and U respectively.

From the implementation of these, it was concluded that U could contribute
to a modification on the basis of Proposition 5.1. In addition, to study
the implications of this test-form, the components $S_0$, $U_0$ and $P$ were arranged into six tests: $S_0 + P$, $P + S_0$, $U_0 + P$, $P + U_0$, $S_0 + U_0 + P$, and $P + S_0 + U_0$. Six groups were tested, however the last two groups were tested under three relevant hypotheses. It was concluded that competency in $P$ is not affected by the test-form and a learning space $S + U$ could be promising for the improvement of achievement.

Chapter 9 - Towards Modification In Methodology: $S+U$, A Learning Space

In this chapter, it was viewed that a desired modification would naturally result in raising the value of $A$, a numerical value assigned to the achievement. A Proposition 9.1 was then provided - on the basis of Proposition 5.1 and relevant results drawn from Chapter 8 - to suggest a modification in the methodology of mathematics teaching-learning situations. It was firstly discussed how a mathematical task is achieved through the conventional presentation $R$ in the current methodology and secondly a modified presentation $R'$ based on an algorithmic approach was introduced in the forementioned Proposition. Simply $R'$ is formed of ordered pairs of elements, the first from $S$ and the second from $U$ chosen under Proposition 8.2 of learnability in $U$. Empirical evidence in two experiments was discussed in support of the new learning space $S + U$. It was concluded that a methodology based on $S + U$ under Proposition 9.1 is superior for mathematics achievements to the current methodology based on $S$.

Chapter 10 - Summary And Further Outlook

The final chapter summarises the work involved for the research development.
It also includes general findings in the study. Finally, a further outlook, basically derived from this study, was suggested to contribute for further development in the context of this thesis.

2.4. Methodology

A research methodology cannot be preconceived as detached from the underlying constructs of the researcher when conducting his research, provided that such constructs are derived from the reviewed literature, experience and introspection, which are expected to guide the research plan.

If methodology indicated in general the overall design of a research study including techniques used for data collection, e.g. questionnaire or a test, then a choice of method is a major part and those constructs constitute another major part, in that they affect the design of the techniques used.

In other words, since I tried to develop the study on the basis of constructs, so that a rational development could be finally directed towards identifying a relational function between the achievement A and some other independent variable such as R (the presentation) - i.e. $A = g(R)$ - then this will have an effect on the relevant test design. Furthermore, I believed (as another construct) - as did Scandura (1977) - that if the ability to transfer mathematical knowledge not only can be taught, but also can be tied in with problem-solving, so much the better. Naturally, this second construct would direct the design of techniques so that they are closely linked with problem-solving. Eventually, this construct may coincide with cognitive strategies of Gagne and Briggs (1979) who state that:
"A cognitive strategy is an internally organised skill that selects and guides the internal processes involved in defining and solving novel problems." (P.72).

Hence, novel problems could be expected to be involved in the study. And since assessment of cognitive strategies has not as yet been done as they suggest, then such assessments would initially require (as a third construct), for development of theory in the study, that basic pre-requisite information and skills relevant to the problem should be confirmed beforehand. Such a construct would emphasise that in designing of relevant techniques, a problem and its basic pre-requisites should both be involved. Hence, problem-solving was expected to be a central point on which a modification would rest.

Here, I present a set of general guidelines for describing methods, as well as maintaining both content validity and a basis of analysis for the research techniques.

(i) **Description of method:** This was based on:

(a) material that presents the design of instruments on the basis of a set of basic behavioural objectives that define the relevant specification arising from the aims of the instrument. An exception is found in the first instrument, i.e. the questionnaire (Chapter 6) which was constructed directly on the basis of its aims;

(b) **Subjects and setting** that discuss the place of subjects in the academic schedule as well as their numbers; and

(c) **procedure** that describes how the instrument was employed and the relevant duration.
(ii) **A panel:** A panel of six people, of whom I was one, contributed to the discussion and construction of all the instruments, except in the first instrument, i.e. the questionnaire. This panel was different for the university tests. The panel could indeed maintain the content validity of tests.

(iii) **Discussion of results:** This involved tabling the results on the basis of the supposed constructs behind the instruments. Hence, analysis of results was mainly directed towards the study of constructs.

2.4.0. The Research Instruments

The research instruments or techniques are divided into four categories, where each category was supposed to serve specific purposes in the research development. In addition, each test was established on the basis of these purposes. All the tests, except $T_0$, were essay type tests, ($T_0$ was objective, of multiple choice form) (Appendix B).

2.4.0.1. First Category: Two Pilot Instruments

In discussing an improvement (reform) for the achievement of students, (6.1.2), I viewed that this would do better to rely on an understanding of the advances in the achievement. I further suggested the term "efficiency" (see Glossary in Appendix A) as a measure for such advantages. Accordingly as reported in 6.1.2, I implemented the following two pilot instruments in the pre-proposition stage in order to throw light on a relevant basis for the proposition stage.
(1) **A questionnaire:** This instrument was introduced in 6.1.2.1 with the aim of building up a general view of how people who work in the field think of the advantages (see Appendix A) in mathematics achieved by students in the current situation. The questionnaire comprised only two items that were explained directly by me to those being questioned, who were asked to reply straight away.

(2) **A test \( T_0 \):** This second pilot instrument is found in Appendix B as well as being fully discussed in 6.1.2.2. As the study of results of the questionnaire indicated a low efficiency for the current achievement of students in mathematical knowledge, \( T_0 \) was aimed at learning through a paper and pencil test \( T_0 \) how far the efficiency measure could be reliable for further use.

\( T_0 \) was an **objective** test of twenty items in a multiple-choice form. It was designed for students in the final year of school. \( T_0 \) comprised:

1. Three problems called items 18, 19 and 20;
2. A set \( Y_0 \) of items 1, 2, ..., 10 which was established to test general fundamentals in mathematical knowledge based on ten different deficiencies of students reported by the Inspectorate of Mathematics for the General Examination Board in the Ministry of Education in Kuwait;
3. A set of three kernels (see Appendix A) \( X_0(18), X_0(19) \) and \( X_0(20) \) was constructed for problems 18, 19 and 20 respectively. Each kernel was expected to test the attainment of the necessary mathematical knowledge for each problem.
2.4.0.2. Second Category: Two Indicators

As the study in Chapter 7 would scrutinize two structures, S supposed to underlie current presentation of mathematical tasks and H supposed to be related to the area, despite being almost totally ignored, two indicators were implemented as tools; the purpose of these was:

(i) to learn how far the current achievement, supposed to be taking place in S, could apply in the necessary relevant knowledge in H; and

(ii) to learn about any properties that might imply indications of effective knowledge found to be gained in S and H.

This category was reported in 7.2.

(1) First Indicator: Computational Knowledge - $T_1$: A Battery of Three Tests

The battery $T_1$ (Appendix D) consisted of three tests $t_1$, $t_2$ and $t_3$ with the purpose of studying how far transferability of computational knowledge in S can be carried out in H through relevant events in physics, as well as in mathematics. This study was fully reported in 7.2.0.

(i) $t_3$ was first constructed on the basis of choosing four formulas in novel forms that are learnt in physics.

(ii) $t_1$ was constructed as exercises in H by changing symbols that are used in physics by other symbols that are mostly used in mathematics.

(iii) $t_2$ was constructed as exercises in S as in $t_1$ with some changes in
the symbols. A set of numbers to be substituted was attached to each item so that the instance could then belong to S.

(2) Second Indicator: Mathematical Knowledge - \( T_2 \): A Set Of Three Tests

The set \( T_2 \) (Appendix E) consisted of three tests \( t_1, t_2, \) and \( t_3 \) whose purpose was to reinforce such reliability of the efficiency for further studies on the basis of studying the relationship of the particular mathematical knowledge in S and H related to a well defined problem P. In connection with the construction of these tests, four fundamental phases were suggested:

(i) a choice of a problem P;

(ii) producing a set B of behavioural objectives necessary for P;

(iii) constructing a kernel \( S_0 \) in S on the basis that each \( B_i \) in B is satisfied by a unique \( s_i \) in \( S_0 \) for all \( B_i \) in B; and

(iv) similarly constructing another kernel \( H_0 \) in H.

Here is a brief introduction of those tests in \( T_2 \) reported in 7.2.1.

(1) \( t_1 \) was designed for students studying in the final grade of the secondary school in Kuwait. It comprises 9 items: the first four items are the kernel \( S_0 \) and the next four items are the kernel \( H_0 \), and item 9 is the problem P.

(2) \( t_2 \) was designed for students who are attending the first course in mathematics (called 101) in Kuwait University. It comprises 9 items arranged in a way similar to the items in \( t_1 \).
(3) $t_3$ was designed for students who attend a second course in mathematics (called 102) succeeding 101, in Kuwait University. It comprises 9 items and arranged as in $t_1$.

2.4.0.3. Third Category : Two Empirical Studies

The study reported in 8.3, comprised two sets of tests, $T_3$ (Appendix F). One comprised three different tests, $t_1$, $t_2$, and $t_3$ where each has the same test-form, while the other set comprised six tests supposed to throw light on the effects of the test-form.

(1) An Empirical Study : A Set $T_3$ Of Three Tests Based On The Test-Form $S_0 + U_0 + P$

Based on the proposition of modification 5.1 which set out, on the basis of the efficiency, the choice of a space of knowledge to contribute to a modification, the set $T_3$ was run in an empirical study with the aims of studying the efficiencies in the two spaces of knowledge: $S$ and $U$ which was proposed for the relevant modification. $T_3$, as it was stated, included the following three tests:

(1) $t_1$ which was designed for students in the first course in mathematics at Kuwait University. It comprises 11 question-items: a kernel $S_0$ for $P$ in $S$ which makes up the first five items, a kernel $U_0$ for $P$ in $U$ which makes up the next five items, and the problem $P$ as item 11.

(2) $t_2$ which was designed for students in their second course in mathematics at Kuwait University. It comprises 9 items: a kernel $S_0$ of $P$ in $S$ making up the first four items, a kernel $U_0$ of $P$ in $U$
making up the next four and the problem $P$ as item 9.

(3) $t_3$ which was designed for students studying in the final grade of secondary school in Kuwait. It comprises 9 items arranged similarly to those in $t_2$.

Each of the three tests was administered in the test-form $S_o + U_o + P$, and a detailed study is found in 8.3.

(2) Contingent Implications Of The Test-Form $S_o + U_o + P$: A Set $T_4$ Of Six Tests

This empirical study, reported in detail in 8.3.5, was implemented with the aim of studying both how ability in $P$ is affected by $S_o$ or $U_o$ in the test-form $S_o + U_o + P$ that was prevalent in $T_3$ and the effects of $S_o$ in $U_o$. Hence a problem $P$, suitable for the final grade of the secondary school was chosen. Then two kernels $S_o$ and $U_o$ each of four items were constructed for $P$. Then the set $\{P, S_o, U_o\}$ was split into a set $T_4$ of six tests that were implemented on the following basis.

\begin{itemize}
  \item[(1)] $S_o + P$ against $P + S_o$
  \item[(2)] $U_o + P$ against $P + U_o$
  \item[(3)] $S_o + U_o + P$ against $P + S_o + U_o$.
\end{itemize}

2.4.0.4. Fourth Category: Empirical Evidence: Mathematics Achievement

In $S$ Against $S + U$

The results of the empirical studies in the third category could, on the basis of proposition of modification 5.1, bring about an algorithmic approach for the modification in methodology in Proposition 9.1.
Accordingly, two experiments based on the experimental method, were implemented with the aim of studying indications of the achievement of students from the modified presentation R as against the conventional presentation R.

(1) Experiment I

This experiment, fully reported in 9.2.0, was carried out within the second grade of secondary school in Kuwait. Two successive tasks $K_1$ and $K_2$ were chosen from the textbook and each was given two treatments, the modified and the conventional. Moreover, two groups $G_1$ and $G_2$ in the second grade were chosen to be the subjects of the experiment. An achievement test was constructed for each task in order to test the achievement of students after completion of the task. All the treatments $R_1$ and $\hat{R}_1$ for $K_1$, as well as $R_2$ and $\hat{R}_2$ for $K_2$, are found in Appendix G. Furthermore, the appendix includes the two achievement tests $T(K_1)$ and $T(K_2)$.

The following table 2.1 illustrates the experiment.

<table>
<thead>
<tr>
<th>Group</th>
<th>Type</th>
<th>Task</th>
<th>Treatment</th>
<th>Test</th>
<th>Type</th>
<th>Task</th>
<th>Treatment</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>Experimental</td>
<td>$K_1$</td>
<td>$\hat{R}_1$</td>
<td>$T(K_1)$</td>
<td>Con.</td>
<td>$K_2$</td>
<td>$R_2$</td>
<td>$T(K_2)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>Control</td>
<td>$K_1$</td>
<td>$R_1$</td>
<td>$T(K_1)$</td>
<td>Exp.</td>
<td>$K_2$</td>
<td>$\hat{R}_2$</td>
<td>$T(K_2)$</td>
</tr>
</tbody>
</table>

Table 2.1

Illustrative Procedure In Experiment I.
(2) **Experiment II**

Since it is commonly held that 'ordinary' students make up the majority of the student population, then this experiment is aimed at developing a study of certain hypotheses in demonstrating that the teaching-learning technique based on the learning space $S + U$ under Proposition 9.1 is superior for ordinary students (identified in Definition 9.1) to the conventional technique based on the learning space $S$.

The technique was established in choosing two tasks: $K$ from the mathematics of the final grade; and $\hat{K}$ from the second grade of the secondary school in Kuwait. Each task was presented in two treatments, the modified $\hat{R}$ and the conventional $R$. Two groups $G_1$ and $G_2$ were chosen for $K$ and two more groups $\hat{G}_1$ and $\hat{G}_2$ for $\hat{K}$. In addition, two achievement tests $T(K)$ and $T(\hat{K})$ were constructed to test the achievement of students after completion of the tasks $K$ and $\hat{K}$ respectively. The study is reported in 9.2.1.

The following table illustrates the experiment:

<table>
<thead>
<tr>
<th>Final Grade</th>
<th>Task</th>
<th>Treatment</th>
<th>Test</th>
<th>Group</th>
<th>Type</th>
<th>Task</th>
<th>Treatment</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$K$</td>
<td>$\hat{R}$</td>
<td>$T(K)$</td>
<td>$\hat{G}_1$</td>
<td>Exp. $\hat{K}$</td>
<td>$\hat{R}$</td>
<td>$T(\hat{K})$</td>
<td></td>
</tr>
<tr>
<td>$G_2$</td>
<td>$K$</td>
<td>$R$</td>
<td>$T(K)$</td>
<td>$\hat{G}_2$</td>
<td>Con. $\hat{K}$</td>
<td>$R$</td>
<td>$T(\hat{K})$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.2**

*Illustrative Procedure In Experiment II.*
2.4.1. Sampling

Sampling means selecting a given number of subjects from a defined population as representative of that population where the term population refers to the set of all real or hypothetical persons or objects that might be involved in the study (Borg, 1971). In this connection, a representative sample is that which satisfies a certain characteristic or measure that the population satisfies. Generally, the larger the sample is, the more likely it is to be representative, but in practice, a large sample is not available to the researcher. Hence in choosing a representative sample, I kept to the following rule, except in two cases: the questionnaire in 2.4.0.1 and the study of the implications of the test-form in 2.4.0.3 (both discussed later).

Representative samples in general were identified according to regulations provided by the Ministry of Education in Kuwait. These regulations admit the existence of a grade in a secondary school if the students that could join this grade constitute two classes, i.e. approximately 50 students. In addition, all students in any grade in a secondary school are randomly distributed into classes. All the mathematics teachers in the secondary stage must have a university degree in mathematics. Moreover, all students in the same grade in all secondary schools use the same textbook. This is also true for students in each course in mathematics at Kuwait University.

Hence, I considered any set of students who constitute two classes in the same grade to be a representative sample for the population of the grade. In addition, I considered the sample of size not less than 50 to be representative of the population of a mathematics course in Kuwait.
University since the size of the relevant population is small. I did not pay attention to differences in sex, since female students in Kuwait have always proved to be no less efficient in mathematics than the male students, whether in school or university.

As for the questionnaire, mentioned in 2.4.0.1, I chose all the inspectors of mathematics in the secondary stage, 10 (reduced to 9) senior mathematics teachers at random and ten mathematics teachers in five secondary schools at random. This was the only possible situation for me to conduct this instrument under the available circumstances; details are found in 6.1.2.1.

In connection with the instrument $T_4$ in 2.4.0.3, I chose only one class for each of the six tests that constitute this instrument, since it was impossible to choose two classes because of the conditions that were imposed on them. Details are found in 8.3.5.

Finally, it is to be noted that subjects of the second grade of secondary school were aged on average at least 15-16 years, while subjects of the final grade were aged on average at least 17-18 years. However, subjects of the samples in Kuwait University were on average at least 18 years old. Further details of the numbers in each sample of the tests are found in Table 2.3.

2.4.2. Some Attributes Of A Research Technique

One of the most important foundational skills needed for research is in the ability of the researcher to appraise the procedure he intends to use
for the collection and analysis of data (Fox, 1969). In my view, this means the researcher should firstly be able to define the technique used on the basis of certain constructs he has formed for the research development. Secondly, the researcher should pay substantial attention to the credibility of the measure used, so that he can rely on more confident data as well as results.

2.4.2.0. Definition Of Tests

The procedures in conducting this research generally stem from the construct that a modification for the improvement of the achievement of students in mathematics could result from a study of the truth value of the conditional statement: $x$ is capable of $P_0$ $\Rightarrow$ $x$ is able in $P$. Here $P$ is a well defined problem and $P_0$ is a set of stimuli constructed on the basis of a certain criterion relevant to $P$: $x$ is capable of $P_0$ if he is completely able in all elements of $P_0$.

The truth value of this conditional statement was viewed to present the advantages of the achievement in $P_0$ for $P$ and was called the efficiency (see Appendix A). Further development of this measure suggested $P_0 = X_0$ as a kernel for $P$ in the sense that $X_0$ tests the ability of students in the basic and necessary components of knowledge of $P$ that are based on basic behavioural objectives of $P$. Yet this, according to a certain trend in this thesis based on the conception of space of knowledge (Chapter 5), entailed that a study of the truth value 'r' of the previously mentioned conditional statement should be reliable. By this is meant that r should demonstrate change in the problem, as in $T_0$ (6.1.2.2), as well as being compatible with other measures, such as that compatibility of $r$ in $S$ and $U$ in $T_3$ (8.3) with the correlations in $T_2$ (7.2.1). Nevertheless, the
The tests in the four categories could be called purely research tests in the sense of Krustests (1976) who has run a system of experimental problems for investigating school children's mathematical ability. Each of these tests could, in general, be defined as:

A research test based on revealing hidden phenomena related to a modification in methodology aimed at improving the achievement of students in mathematics, whose results can be expressed quantitatively and can undergo statistical treatment.

The hidden phenomena can be found in the two constructs of improving the achievement on the basis of the personal constructs of efficiency and space of knowledge (see Appendix A).

2.4.2.1. Validity And Reliability

These two terms are two essential attributes of tests or measurements. Klausmeir and Goodwin (1975) advocate a third attribute usability which includes aspects of skills, cost and time, particularly in the case of a researcher relying on published tests. They claim that these are three desirable attributes of educational and psychological measuring devices. Usability is indeed included from aspects of skills and time. Hence I paid attention to the other two attributes, as in what follows:

Generally, as Morris and Fitz-Gibben (1978) state, judgements of validity
involve the question: Is the test an appropriate one for what is to be measured? Reliability indicators answer the question: Does the test yield consistent results? In line with this, according to Gronlund (1968), validity information indicates the degree to which the test is capable of achieving certain aims, while reliability indicates the accuracy (consistency and stability) of its results.

Here, I explain how those two attributes were employed in this study.

2.4.2.1.0. Discussion Of Validity

Since all the tests in the last three categories were essay type tests, they will suffer from lower validity - especially decreased content validity as Pyne (1974) reports. He explains that the behaviour elicited by the test is not hoped for by the instructor or dictated by the table of specifications. Accordingly, I paid special attention to maintain an acceptable level of content validity. In addition, I found the construct validity to be of prime importance, since it may be considered as a statistical extension of content validity, as Fox (1969) states.

a) Content validity: This may be taken as the degree to which the sample of test items represent the content that the test is designed to measure (Borg and Gall, 1971).

Pyne (1974) claims that in assessing content validity, one seeks:

"an answer to this question: How adequately do the items of this test measure the objectives, in terms of both subject
matter and cognitive skills, that I want them to measure?
Content validity rests on the specification of the universe
of behaviour to be sampled" (pp. 255 - 256).

Therefore, with the aims of maintaining the content validity of the test,
I formed a panel (of whom I was one) of people who work in the area being
studied to construct the tests. The panel was always of six members,
and a judgement was considered to take place if at least four members
agreed on it. (In case of split vote, I had an extra casting vote,
although there was consensus in most cases). The specifications for
constructing a test were always identified by a set of behavioural
objectives.

b) Construct validity: This is defined (in general) as the ability of
the instrument to distinguish between groups known to behave differently
with regard to the variable or construct under study (Fox, 1969). Verma
and Beard (1981) define construct validity in that:

"It addresses itself to the problem of determining to what
extent a test or tool measures a particular, theoretically
defined aspect of the variable being considered" (P.181).

An important factor of this type of validity is found in its relevance
to measuring hypothetical constructs that constitute unobservable internal
processes, e.g. creativity.

In my turn, I have in general used construct validity to reflect my
constructs of the students' behaviour in spaces S and H. In addition, I
have presented these constructs as testable hypotheses in other places. In this sense, Borg and Gall (1971) assign a considerable role to construct validity in planning a research study that proposes to test a hypothesis.

2.4.2.1.1. Discussion Of Reliability

The term reliability refers in general to the accuracy of the data in the sense of stability, repeatability or precision (Fox, 1969). A perfectly reliable data-collection instrument is one which, if administered twice under the same conditions, would be expected to provide identical data. However, if theoretically it could be assumed that behaviour would be in general consistent, the problem usually amounts to the reliability in marking those tests which are of an assay type in the last three categories discussed in 2.4.0. In this context Tuckman (1975) reports that

"the teacher as an essay response scorer is an imperfect measuring instrument, subject - as any human being is - to fatigue, biases, expectations and other sources of influence. However, because the teacher is a measuring instrument steps must be taken to insure the greatest degree of objectivity (or consistency or accuracy) or reliability. To this end, multiple judgements are recommended" (P.131).

Hence, in order to maintain the reliability of scoring procedure known as the inter-rater reliability or inter-judge reliability (Tuckman, 1975), I allowed a period of time, not less than 12 months, to elapse after correcting the answer sheets. Then I chose 10 answer sheets from each
test in the tests of the last three categories mentioned before, and
corrected them again under the method of scoring suggested for each
test. It could easily be seen that there was concordance between the past
and present marking for each item, except in two cases in two items in
T4 (reported in 8.3.5). As a matter of fact Thorsland and Novak (1974),
in discussing the instrument reliability in an experiment based on
interview tape-analysis, have maintained an accordance of ranking on the
basis of coefficient of accordance value. However, I was satisfied by
the first sight suggested by Tuckman.

As for T0 - reported in 6.1.2.2 - this was a multiple choice test and I
firstly used the method of 'rational form' found in the Kuder-Richardson
21 in order not to overestimate the reliability coefficient. This was
0.66.

A worthwhile word is that the inter-rater reliability is the only method
used in Kuwait secondary schools, as well as in general examinations for
satisfying the reliability of such instruments.

Table 2.3 overleaf, shows the content validity and reliability for the
four categories of tests.
<table>
<thead>
<tr>
<th>Category</th>
<th>Instrument</th>
<th>No.of subjects in the sample</th>
<th>Content Validity</th>
<th>Reliability</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>T₀</td>
<td>71</td>
<td>Judged by a panel</td>
<td>The Kuder-Richardson 21</td>
</tr>
<tr>
<td>Second</td>
<td>T₁: t₁, t₂, t₃</td>
<td>196, 196, 196</td>
<td>&quot;</td>
<td>Inter-Rater</td>
</tr>
<tr>
<td></td>
<td>T₂: t₁, t₂, t₃</td>
<td>73, 84, 53</td>
<td>&quot;</td>
<td>Inter-Rater</td>
</tr>
<tr>
<td>Third</td>
<td>T₃: t₁, t₂, t₃</td>
<td>144, 52, 274</td>
<td>&quot;</td>
<td>Inter-Rater</td>
</tr>
</tbody>
</table>
|          | T₄: 6 tests  | 1) 24 vs 22  
              |                  | 2) 22 vs 19  
              |                  | 3) 23 vs 20   |
| Fourth   | T(K₁)      | 87 vs 87                    | "                | Inter-Rater |
|          | T(K₂)      | 87 vs 87                    | "                | "           |
|          | T(K)       | 33 vs 28                    | "                | "           |
|          | T(ƙ)       | 32 vs 33                    | "                | "           |

Table 2.3

Content Validity And Reliability For The Four Categories Of Tests.

2.5. Summary

This chapter reviews the sources of knowledge in experience, authority, deduction, induction and the scientific method. It also outlines some views that are restricted to the scientific method, such as Kuhn's and Feyerband's. In addition, it tries to throw light on scientific theory which is essential in explaining and predicting related events in the area of study.

In addition, the chapter discusses the controversy over the definition of
educational research. Lovell and Lason (1970) find it difficult to adopt a universal definition, whereas others, such as Ary et al. (1972), propose that educational research is a result of applying the scientific method to the study of an educational problem. It further outlines research methodologies in education through the experimental, ex post facto, descriptive and historical methods. As for this research, the researcher views that it is educational in the sense of Ary et al.

The chapter further discusses the three stages of the research so as to demonstrate the structure in the thesis. In a further step, the thesis was outlined chapter by chapter.

The research methodology was then considered through a discussion of the instruments employed, the samples of the populations involved, the definition of tests that are employed in the study, and the two major attributes of a test, i.e. validity and reliability.
CHAPTER THREE

LITERATURE REVIEW
3. LITERATURE REVIEW

3.0. Introduction

Published literature is a fruitful source for a researcher in highlighting his problems. It also helps him to relate his work to the existing knowledge of a desired system. The literature reviewed in this chapter will have an important function in that it is also supposed to help to identify a certain phenomenon in mathematics education. This phenomenon can serve as a starting-point for the development of this study, as was proposed in proposition 6.1 which I restate:

(Proposition 6.1): A phenomenon that indicates a change in the achievement of students under a certain change in a teaching-learning situation is significant as a starting-point for development in current teaching-learning situations.

It is therefore natural, in looking for such phenomena, that I should review such problematic consequences as are usually discussed in mathematics education. The literature is a rich source and "there exists, in the literature, a solid body of information about mathematics education" (Begle, 1979, P.156).

In discussing current views on approach, and the problematic
consequences found in methods or techniques that are adopted to serve for the approach, one phenomenon attracted my attention. This phenomenon is found in the efficiency of materials, i.e. presentations of tasks, that are set in teaching-learning situations. Frase (1975) tells us that manipulation of learning materials is efficient to the degree that it reduces unnecessary learner processing. He also states that

"There is some evidence that if the learner does not apply certain specific skills, materials can be altered in compensating ways" (P.58).

which shows that there is a certain relationship between the material (contents) and the product of teaching-learning. Another sign is found in the work of Myers et al. (1973), who found that recall was better when sentences were grouped according to 'attributes' than when they were grouped according to 'names'. This indicates that there are two structures, and the change in recall is due to the change between these structures.

Other studies in mathematics, notably those of Mayer and Greeno (1972), found different learning outcomes were obtained when setting a task according to different presentations.

In the following pages, I will give a general outline followed by a simple survey of some mathematical events for the purpose of tracing such phenomena that could possibly help support relating change in mathematics achievement to a relevant change in a mathematical situation either in the material contents or in any other possible event.
3.1. General Overview

The movement towards reform in mathematics education does have purely educational purposes relevant to the growth of knowledge in developmental psychology and its implications for the teaching and learning of mathematics, in addition to its other, social, aims. But it must be said that research into the psychology of education is still limited, and that there is a marked dependence upon introspection (Child, 1973). The cognitive psychology has not made the desired progress, in the opinion of Lovell (1981):

"Cognitive psychology - which reckons to study intelligence or how people think - has made a little progress over the last 30 years, although some of its findings have been known intuitively long before".

As for learning theories, Child (1973) points out that

"no one learning theory provides us with all the answers. Furthermore, all theories put together do not provide us with all the answers. The only course we can justifiably take is a pragmatic one - choosing amongst the experimental findings the points of clear relevance to our task. In most cases, psychologists are not
really arguing about the findings so much as
the interpretation of those findings" (P.107).

The previous points throw light on the intricacy of research in
the educational field. Nevertheless there may be other points
that could help direct us towards a phenomenon that might be found
amongst reported literature: a phenomenon, of clear relevance to the
problem that would fulfill the aims of proposition 6.1.

As Bruner proclaimed in the late 1950's:

"something new was stirring in the land"
(Bruner, 1966, P.vii).

This stirring produced a rash of new curricula in U.S.A. and
most developed countries. Freyberg (1980) notes that

"what these curricula purported to have in
common was a new approach to the organisation
based on "conceptual structure" rather than
historical development, level of complexity,
children's interests, everyday relevance and
so on. There was much talk about fundamental
or basic concepts, and the need for these to
provide the framework on which the particular
curriculum would be built".
The theme of these statements was that the structure of a subject-matter should be the central point of educational development. Bruner (1966), in defending this theme, states that

"Good teaching that emphasizes the structure of a subject is probably even more valuable for the less able student than for the gifted one, for it is the former rather than the latter who is most easily thrown off the track" (P.9).

Howson et al. (1981) tell us that the structuralist approach is based on investigations conducted by genetic epistemology theorists into the processes of concept formation, from which Bruner has developed his theory of the "structure of the disciplines". Howson et al. (1981) in illustrating the broad outline of the theory wrote:

"Cognitive structures are combinations of acquired concepts and thinking abilities. Simple structures, made of a few concepts, are developed into more elaborate ones through the addition of new concepts. At their highest stage of development, cognitive structures correspond to the structure of the sciences, taken as the essence of all concepts and processes contained in them. These structures are so complex and they include all insights, concepts..."
and procedures of the sciences, but they are
simultaneously so easy to formulate that they
can be transmitted at a lower level of cognition.
The purpose of the transmission of the structures
of the scientific disciplines is not the
acquisition of the knowledge of these structures
by pupils; it is not so much a matter of trans­
mitting in an elementary manner the structures as
a content of education, as of displaying their
process character. This, to a large extent, is
the basis for the correspondence between
scientific and cognitive structures, and a means
of promoting cognitive processes in pupils" (P.108).

The structuralists could also have been convinced (or, perhaps
tempted) by the findings of those of the Bourbaki School who found
that the structure of mathematics is founded on three fundamental
structures: (i) the algebraic structures based on the concept of
operation, (ii) the ordering structures based on relations and
(iii) the topological structures based on such concepts as continuity
and proximity. The Bourbaki group, further has offered a systematic
description of mathematics, reorganised so as to emphasise structural
considerations and presented in a uniform language with great
precision. Whatevsoever the case is, the structuralist approach is
based on the idea that:
"operating within the structures of the sciences reinforces the processes by which new concepts are assimilated by a given stage of cognitive development. The learner is thus given the opportunity to gain familiarity with the structures and to obtain a better grasp of their complexity which, in turn, will further the acquisition of new concepts, etc., until a complete correspondence between the learner's cognitive structures and those of the science is achieved, when the learner will have become a scientist" (Howson et al., 1981, P. 108).

In this connection, there has been a belief that structural learning experiences should receive the predominant emphasis at all stages in the school. This, it was believed would lead to unity in the mathematics being used, as well as to better achievement in mathematics. (Lamon, 1972).

The critical point of the structuralist approach is how to transmit these scientific structures to pupils endowed with lower cognitive structures, however, these goals were believed to be achieved through 'discovery learning' (Howson et al., 1981).

Hence, based on Gestalt psychology, appreciation and conceptual development of structure are enhanced in practice by such a methodology that uses learning situations that encourage students
to discover interrelationships of parts as well as of the whole (Lamon, 1971).

It was therefore logical to find a demand for creative mathematical learning situations to be established with the intention of helping in the development of the ability to understand commonly occurring relationships between concepts and ideas, which in turn develop proficiency and skill in manipulation and calculation of mathematical ideas. (Lamon, 1972).

In the schools of various countries, among them Kuwait, all (or most) current mathematics curricula based on the points previously mentioned are constructed for the purpose of satisfying the theme of structure. This theme emphasizes the concept of structure and breaks down the artificial barriers erected for a long time between the different parts of mathematics. This has indeed happened, in a shift from so-called 'traditional mathematics' to 'modern' or 'contemporary mathematics'. The new mathematical textbooks in most Arab States who followed this shift, were designed on the basis of learning by discovery. The 'why' has always been found to help the teacher to follow 'discovery teaching'.

Unfortunately, not all of these changes have been successful in meeting the demands for development in the area. Teachers, students and parents have been complaining about the low achievement in
mathematics. Other complaints are raised about certain weakness in applying mathematics in non-mathematics fields, e.g. physics. The traditional approaches to mathematics have proved to be inappropriate, and modern approaches demonstrate distinct inadequacies in presenting the accomplishments that past methodologies produced and in improving upon past failures (Lamon, 1972).

This being the case, Lamon (1972) gloomily pointed out that

"It should not be surprising that people are expressing dissatisfaction. Mathematics professor, education and psychological professors, as well as practising teachers, question seriously the validity of current pedagogy and the wisdom of teaching youngsters the so-called 'New Math' " (P. 8).

Begle (1979) expressed growing concern about the situation, stating that:

"little new knowledge about mathematics education emerged in the last decade, and we still lack theoretical structure to support our research ... we have no established theory to provide a basis for our discussion. Deductive reasoning now plays no useful role in mathematics education since educators have, in general, nothing to deduce from ..."
... I predict that future international congress
will be much like the past ones, new faces, new opinions,
but very little new knowledge" (Ps. xiv, xix and 156).

These statements, and similar other ones, show the magnitude of
the depression felt by those involved in mathematics education.

Now if we accept that achieving mathematics should be based on
certain approaches, then an optimizing phenomenon for development
of this study in the sense of proposition 6.1. could be found
through discussing different aspects of educational situations
that are used to serve for the approach. It was believed that
a reading in the literature that discusses the approach and the
effectiveness of the different aspects of the events that are
found in the 'methodology for the approach' will disclose a
certain phenomenon that merits study. The term 'methodology
for the approach' in this thesis is used to include all possible
factors that genuinely serve for the approach, e.g. the curriculum,
methods of teaching, etc.

In the following pages I shall discuss some aspects of the current
approach as well as the 'methodology for the approach'.

3.2. The Current Approach: The Structure

To Anthony, an approach is axiomatic. It describes the nature
of the subject-matter to be taught. On this point Anthony (1963)
wrote:

"It states a point of view, a philosophy, an article of faith - something which one believes but cannot necessarily prove. It is often unarguable except in terms of the effectiveness of the methods which grow out of it".

The current theme of many psychologists and many mathematicians is that a mathematical approach should be based on structures. Fischbein (1973) tells us that, to Piaget, a structure in its broadest term is a system, and this system is a totality that has laws and properties that are characteristic of it as a totality. On the other hand he introduces mathematical structure as a set of elements between which there are certain relations. And it is well-known that for the Bourbaki, there are three types of structures, algebraic, order and topological. This indeed means that any branch of mathematics is related to one of these structures, and consequently all different branches of mathematics can be classified and brought together to form an architectural unity.

Skemp (1971) uses "schema" as a mental structure in broadest psychological term. He finds a schema to have two main functions:

"It integrates existing knowledge, and it is a mental tool for the acquisition of new knowledge" (P.39).

The polemic evoked by mathematical approach polarises into two main
schools as Fischbein, (1973) states: One should leave the general schemas of thought to form themselves gradually, by a sort of natural generalization, after a student has assimilated a fairly considerably amount of mathematical knowledge.

The view of this school, is that acquaintance with these (mental) structures after a good preparation in mathematics would naturally help in assimilating generalizations, i.e. structures. This school considers that:

1) The natural course in the knowledge-development is from the particular to the general, from the concrete to the abstract.

2) Mathematical structures are in nature of an extremely high degree of abstraction and, consequently, they could not be understood or assimilated until the intelligence has reached the stage of formal operation, i.e. the stage of formal operation, i.e. the stage of final equilibrium.

3) These structures arose out of the confrontation of diverse mathematical domains. The pupil therefore should have the opportunity on the capability of understanding mathematics on a fairly broad mathematical basis in arithmetic, algebra and geometry.

The view of this school was the predominant one in the so-called 'traditional mathematics'. As for the second school, it views that:

It is better for children to be brought up with mathematical structures. This helps in the development of schemas which in turn help to form
mathematical thought.

This school considers that these structures:

1) express the fundamental structures and general schemas of intelligence and are not just a means of giving information or a manner of proceeding in a particular instant.

2) should be allowed to function during the period of the growth of intelligence.

3) should be introduced in the earlier stages to help during the period of the final equilibrium of formal operations.

The second school could be considered to be the contemporary one. It is closely linked to contemporary developmental psychology.

Piaget (1972a) has indicated that there is a close relationship between the important mathematical structures and the organisation of human intelligence. According to him, mathematical structures are closely related to the main operational structures of intelligence gradually built up during the subjects' ontogenesis, and group structure is at the core of this relationship. The operation is one of main features of mathematical structures, and Piaget (1964) thinks of operation as the essence of knowledge when stating that:

"An operation is an interiorised action. But in addition, it is a reversible action; that is, it can
take place in both directions, for instance, adding or subtracting, joining or separating. So it is a particular type of action which makes up logical structures.

And an operation is never isolated. It is always linked to other operations, and as a result it is always part of a total structure. An operation could consist of joining objects in a class to construct a classification, or of adding, or putting things in a series; or an operation could consist of counting or of measuring.

He then points out that a logical class does not exist in isolation; what exists is the total structure of classification. A number does not exist in isolation. What exists is the series of numbers which constitute a structure, an exceedingly rich structure whose various properties have been revealed by mathematicians. Piaget (1964) writes that these operational structures are what "seem to me to constitute the basis of knowledge, the natural psychological reality, in terms of which we must understand the development of knowledge. And the central problem of development is to understand the formation, elaboration, organization, and functioning of these structures."

Piaget considers the three Bourbaki logical structures of mathematics (the algebraic, order and topological structures) to be in 'correspondence' to elementary structures of the intellect, and claims that
"In reality, if the edifice of mathematics rests on 'structures' that correspond to the structures of the intellect, it is on the progressive organisation of these operational structures that mathematical teaching must be based" (Lamon, 1972, P.136).

However, Skemp (1971), who presents intelligence 'B' in the terms of Vernon as

"the cumulative total of the schemata or mental plans built up through the individual's interaction with his environment, in so far as his constitutional equipment allows" (P.16).

argues that psychologists, who are interested in intelligent learning, find that studying the learning and understanding of mathematics is:

"studying the functioning of intelligence in what is at once a particularly pure, and also a widely available form" (P.16).

Skemp also argues that mathematics is a particularly good example of intelligence 'B' since (i) the learning of mathematics affords many clear examples of the development of schemas which constitute intelligence 'B' as described by Vernon and (ii) the applications of mathematics in different fields of knowledge is so powerful that it appears to be the most highly developed mental tool available to us for dealing with the physical environment.
Lastly, Skemp points out that

"The study of the structures themselves is an important part of mathematics: and the study of the ways in which they are built up, and function, is at the very core of the psychology of learning mathematics" (P.39).

Bruner (1966), who is one of the outstanding figures in the area of the school, introduced the term 'fundamental' for an idea to be a tautology, in the sense that it has wide, as well as powerful applicability. In recommending for teaching the fundamental structure of a subject, Bruner claims that:

(1) Understanding the fundamentals makes a subject more comprehensible.

(2) This approach relates to human memory, since the most basic thing that can be said about memory after a century of intensive research, is that

"unless detail is placed into a structured pattern, it is rapidly forgotten. Detailed material is conserved in memory by the use of simplified ways of representing it. These simplified representations have what may be called a 'regenerative character' " (P.24).

For example, Bruner's view is that a scientist does not try to remember the distance travelled by falling bodies in different
gravitational fields over different periods of time. What he does carry in his memory is a 'structured pattern' for such events, i.e. the formula $s = \frac{1}{2}gt^2$. The detailed material is conserved in the memory by the use of simplified ways of representing it i.e. by a fundamental structure,

(3) An understanding of the fundamental structures of principles and ideas is adequate to 'transfer of training'.

(4) The emphasis on structure in teaching will result in re-examining the material taught in all grades of school for its fundamental character. This will help in narrowing the gap between 'advanced' knowledge and 'elementary' knowledge, since "part of the difficulty now found in the progression from primary school through high school to college is that material learned earlier is either out of date or misleading by virtue of its lagging too far behind developments in a field" (P.26).

According to the views held by eminent individuals in the field of psychology, the 'structure' was considered to be the central approach for all subject matters, especially mathematics and other sciences.

These views are part of the theoretical framework of the current approach.

The psychological and pedagogical implications of the approach underlie such methods and techniques as are employed to serve for it.
To Anthony (1963) and Wardhaugh (1969), as will be discussed later, methods meant plans of curriculum and teaching which derived from approach. These two factors are expected to be based on the particular kind of strategy that derives from an approach. Technique, to both, meant exactly how to do what you decide to do, the specific kinds of practice that one chooses to employ in a specific classroom.

Upon reading the literature, I felt that those who advance the approach were also aware of its implications in methods and techniques.

As a matter of fact, Piaget (1964) who seemed to foresee a possible confusion of the harmony between the approach through structures and methodology, states:

"Now, I should like to show that learning is possible in the case of these logical structures, but one condition—that is, the structure which you want to teach the subject can be supported by simpler, more elementary, logical-mathematical structures."

He then explains that

"In other words, learning is possible if you base the more complex structure on simpler structures, that is, when there is a natural relationship and development of structures and not simply an external reinforcement."
He concludes that learning of structures seems to obey the same laws as the natural development of the structures of the intellect. Thus, learning is subordinated to development, not vice-versa. In addition, we must be aware of the spontaneous operations which we present at the outset and the operational level which has been achieved after the learning experience.

Skemp (1971) also, who as an aware mathematician considered the term 'concept' to be undefined, finds that the formation of a concept requires experiences which have something in common. An appropriate formation of concept will be useful since, as he states:

"A concept is a way of possessing data which enables the user to bring past experience usefully to bear on the present situation" (P.28).

As for the appropriateness of early schemas, Skemp (1971) believes that:

"The central importance of the schema as a tool of learning means that inappropriate early schemas will make the assimilation of later ideas much more difficult, perhaps impossible." (P.51).

In an attempt to develop his ideas, Skemp (1971) presented a set of models (pp. 308-313), based on conceptual analysis of the inner relations between mathematical topics.
In addition, from his work in psychology of learning, Bruner (1966) finds that an abstract structure can be transmitted and assimilated as such, remaining unaltered by means of various embodiments. He suggested (1964) that an idea might be represented in the understanding of the learner on three levels: (i) enactive, where materials are manipulated directly, (ii) iconic, in which mental images of objects are formed, and (iii) symbolic in the sense that images are replaced by symbols that can be manipulated. Enactive, iconic and symbolic representations can serve as vehicles for an embodiment and transmit the same mathematical structure (Fischbein, 1973). There may be more than one embodiment which might help in understanding the structure. A subject will gradually grasp a structure by this means.

Dienes (1963) called this procedure 'the principle of perceptual variability' or in its most general form 'the principle of multiple embodiment'.

Dienes (1972) also reported that mathematics learning situations are of an abstract nature which are difficult to develop if learning is ineffective. He considers a mathematical structure as a set of relations, where the student should first be familiarized with the concrete materials to be used and then presenting different concrete embodiments of the structure, with exercises to encourage the student's awareness of the correlations between the embodiments and the abstractions from the situations. In explaining a
structural learning, Dienes states:

"We started with the interaction of the child with the environment through concrete materials. We went on to identify a mathematical structure by likening one situation to another in a precise way. After that, we tried to present structures spatially to enable us to handle them, look at them and think about them. After that, we described this image, and the description gave us the germs of an axiomatic system. As soon as we established the rules of the 'proof game', we reached a formal system in which we could prove theorems. Needless to say, it is very seldom that mathematics is learned in this rational manner" (Lamon, 1972, P.61).

Suggesting the possibility of establishing fully creative mathematical learning situations in all stages of mathematics-learning, Dienes (1971) writes:

"When a child has effectively formed a concept from his own experiences, he has really created something that was not there before, and this something is built into his personality in the psychological sense in the same way as essential substances in his food are built into his body" (P.17).

There are genuine ideas behind what those psychologists who work on the learning of mathematics produce, reflecting in fact;
(i) their awareness of traits which might lead to becoming stranded in the methodology as well as (ii) the belief that good psychology is good mathematics (Skemp, 1971).

Bruner (1966), who also seemed aware of the difficulty for providing a proper methodology for the approach, stated that

"Given the importance of this scheme, much too little is known about how to teach fundamental structure effectively or how to provide learning conditions that foster it", (P.12).

This would inevitably lead to questions, such as: how effectively can the current methodology for the mathematical approach meet the views of those who work on the psychology of mathematics learning? And what other problems could possibly be involved? The area of such questions as well as other relevant ones constitute the majority of problems in mathematics education, and in the following pages I shall discuss what I believe to be relevant to this study.

3.3. Summary I

Published literature is invaluable for a researcher. It helps him to discover a worthwhile problem, and it highlights important points in his field of research. In looking for a phenomenon that could be a starting-point for the development of this study, evidence
was found to indicate that if materials are changed, then learning outcomes could also change. Hence the efficiency of materials, i.e. presentations of tasks, could be a valuable phenomenon for this study on the basis of findings of Mayer and Greeno (1972).

A general survey of the area showed that research into the psychology of education is still limited and dependent upon introspection. Cognitive psychology has also made limited progress, and learning theories can not provide the answers we need. There has been a trend towards the theme of 'the structure' in the last two decades, and this trend resulted in a shift to modern mathematics. Unfortunately, it did not result in solving the problems of teaching-learning of mathematics. Hence, in looking for other possible phenomenon for this study, it was believed that a survey would be necessary, and that this should be in the field of the belief behind current mathematical approach and some aspects of its implications in the whole methodology that is relevant to the approach.

In approach, there are two main schools of thought; the first believing that mathematical approach should not be earlier based on structure, while the second school believes that it should be. It is believed for example by the Piaget school, that there is a correspondence between mathematical structures and elementary structures of the intellect. Hence earlier learning of mathematical structures will aid adequate growth of the structures of the intellect, which will in turn help grasp mathematics. This also helps learning to be
more comprehensive and memory to conserve the detailed material in simpler presentation. Psychologists, who were also aware of the implications of the approach in methodology, proposed proper learning situations. Piaget suggests that learning of complex structures could be based on simple ones; Skemp stresses the formation of concepts, Bruner proposes enactive, iconic and symbolic representation to serve as tools, and Dienes finds that a child should be helped in forming concepts from his experience. The awareness was also clearly stated by Bruner (1966) who claimed that the learning conditions which will foster effective teaching of fundamental structure are still elusive.

3.4. Some Aspects Of The 'Methodology For Approach'.

3.4.0. Introduction.

In the following section I shall try to throw light on the implications of approach in teaching-learning situations. This could be observed by applying the approach in reality. Since an approach is based on a certain belief (axiomatic), then a phenomenon for development in this study might be found in such events as may emerge from the outcome of applying the approach.

Anthony (1963) introduces an organizational key for the three terms, approach, method and technique, in that:

"techniques carry out a method which is consistent with an approach".
And there may be different methods that share the same approach. Wardhaugh (1969) who shares Anthony's ideas, sees 'method' as plans for curriculum and teaching. He writes:

"Method is then the particular kind of strategy that derives from an approach. It is the overall plan that we have in mind for teaching the subject in a particular set of circumstances."

Technique, to both of them, meant exactly how to do what you have decided to do: the specific kinds of practices that one chooses to employ in a specific class-room. To Wardhaugh (1969), this is just where much of the interest of the class-room teacher lies.

In connection with this thesis, the term 'methodology for approach' will mean all possible factors that contribute towards a particular approach. It naturally includes methods and techniques besides the constructs of those who contribute for fulfilling the demands of the approach, e.g. a curriculum designer removes his belief of teaching into the curriculum and vice versa. Here I present some aspects of the methodology that is implemented to serve for the approach. The two main terrains for this study will be in the curriculum and instruction.

3.4.1. The Curriculum: Definition And Construction

I. Tanner and Tanner (1975) present different views about defining the curriculum. They argue, despite what some curriculum writers
contend, that:

"unless a fixed, universally agreed-upon definition of curriculum is developed, there can be little progress in the field" (P.48).

However, scientists have not agreed on a fixed definition of science, and yet this has not impeded scientific progress. The Tanners suggested what they called a comprehensive and tentative definition in which they see the curriculum

"as the planned and guided learning experiences and intended learning outcomes, formulated through the systematic reconstruction of knowledge and experience, under the auspices of the school, for the learner's continuous and willful growth in personal-social competence" (P.45).

Richmond (1971) listed some definitions found in the English school. These definitions might be seen as (a) all learning which is planned or guided by the school, (b) the curriculum consists of the content, teaching methods and purposes, (c) a program of activities designed so that pupils will obtain certain education and (d) the contrived activity and experience - organized and systematic - that life, unaided, would not provide.

It is not intended in this study to define curriculum, but it has been found that it is not good for such a study completely to ignore this very important factor in education. Nevertheless,
in this study, curriculum will refer only to the mathematical objects which we intend our students to absorb, the contents of the instructional programs devoted to mathematics, and the structure underlying the presentation of mathematical tasks. This inevitably reflects either implicitly or explicitly the constructs of people who made contributions in the area.

In recommending the important role of the curriculum, Bruner (1966) wrote that

"The first and most obvious problem is how to construct curricula that can be taught by ordinary teachers to ordinary students and that at the same time reflect clearly the basic or underlying principle of various fields of inquiry" (P.18).

The radical changes in mathematics curricula are found in the so-called 'New', 'Modern', or 'Contemporary Mathematics' that prevails in the current curricula in many countries, including Kuwait. It was believed that the new mathematics, which is generally based on principles of mathematical logic, together with the concept of set and what might be generated from them for example, relation, function, operation, ..., is the adequate body for building up mathematics based on mathematical structure. It also seemed that there was sometimes an underlying trend for the approach, in that earlier abstraction in the methodology for approach would serve better. The influence of this trend is easily observed in the presentation of
mathematical tasks which heavily depend on definitions, few abstract examples and almost a complete abandonment of what traditionally was called applied mathematics. This trend, I believe, could be related to (a) the belief of some pure mathematicians in abstraction i.e. mathematics is an abstraction that is built on abstraction, and it is therefore best introduced through continuous abstraction, (b) the influence of the psychologists who work in mathematics learning who seem not to have been able to stress sufficiently the methodology for approach to be of more practical use and (c) the inertia which is found in any erected system like the education system, and which has resulted in people who oppose any radical change not co-operating efficiently in the development. In drawing some indications from personal experience in the field in my own country, I believe that the conflict between groups in (a) and (c) where the group in (b) could not efficiently interact, has led to the present situation when we find the majority in (c) simply criticises the approach and the concept of set in the new curriculum. They did not efficiently contribute in discussing what underlies the methodology for approach, and how the approach can be maintained in providing proper conditions of learning.

As for applied mathematics, Scott (1972) in turn criticises in general the separation between pure and applied mathematics in early school mathematics. He finds that the bias favouring pure mathematics has ignored the value of mathematics in gaining knowledge about the natural world. This, he states, devalues the knowledge of scientific applications in amplifying mathematics.
Accordingly, he writes:

"Since the introduction of applied mathematics with its practical situations and its inherent concreteness, also introduces features that have considerable consonance with the characteristic features of children's learning, the continuing neglect of applied mathematics seems indefensible."

(P.30)

Begle (1979) in general considers the textbook as one of those educational variables that are directly concerned with the curriculum. He finds however that the differing contents of textbooks resulted in different mathematical learning outcomes.

In addition, Skemp (1971) presents two principles of the learning of mathematics and recommends that the communicator of mathematical ideas, not the recipient, should be aware of them. But Skemp finds that most textbooks are the first to depart from them.

As regards the development of the mathematics curricula in Kuwait, Al-Dhahir and Yaseen (1980b) reported:

"It is clear that there has been a global trend towards abstraction in the new curricula, to the extent that the computational skills, as well as the applications of mathematics in reality - traditionally mechanics - were almost abandoned. This resulted in curricula apparently of a completely abstract structure, useless to the other sciences. We call for curricula that harmonise structures,
computational skills and applications of mathematics in other fields."

A first impression then is that a starting-point for discussion could be the evidence of Begle (1979), who reported that differing contents resulted in different achievement. This phenomenon was found to be fruitful for further purposes of this study.

II. Another main problem of curricula is their construction. In curriculum construction, the inevitable question that arises is what should be the objectives of the curriculum? The controversy about this problem has mainly two schools (Scandura, 1972).

1) The first school which has many educational psychologists as proponents, such as Mager (1964), Tyler (1964), Lipson (1967), Popham (1969) and Gagne (1970), find that stating objectives is of little use as far as educational planning is concerned, unless the objectives are stated unambiguously and in operational terms. To Popham and Baker (1970), who see the curriculum as the all planned learning for which the school is responsible, curriculum questions revolve around consideration of ends, i.e. the objectives that an educational system hopes its learners will achieve.

They then point out that

"A properly stated behavioural objective must describe without ambiguity the nature of learner behaviour or product to be measured " (P.37).
In addition Gagne, who defines curriculum as:

"a sequence of content units arranged in such a way that the learning of each unit may be accomplished as a single fact provided the capabilities described by specific prior units (in the sequence) have already been learned by the teacher" (Oliver, 1977, P.5),

finds that a course or curriculum requires decision about the sequencing of objectives, since not all objectives can be taught at once (Gagne and Briggs, 1979).

In answering the question of how to find the basis for a correct sequencing of the entire set of topics for a course, Gagne and Briggs (1979) consider that it is difficult to find such basis other than in

"a kind of 'common-sense' logical ordering" (P.146).

Tanner and Tanner (1975) criticize the doctrine of behavioural objectives, in that it

"appears to be based upon the notion that the broken egg can be put back together again. All it takes after the proper analysis, is the reassemblage of the constituent parts." (P.29).
A second point of criticism was put forward by Ehrenpreis and Scandura (1974) in two parts: (a) this approach deals only with observable behaviour and says nothing about how the behaviour is generated and (b) it provides no systematic way of dealing with inter-relationships among the identified objectives or the equivalents, of building transfer into a curriculum, since it is clearly an impossible task to teach the learner explicitly everything the constructor of the curriculum wants him to learn.

As a matter of fact Gagne (1970) and Resnick, Wang and Kaplan (1970) were aware of the second point by making use of task analysis for the approach based on learning hierarchies, which thus might present a better solution to Scandura's next point, i.e. the attention to inter-relationships among the identified objectives (Scandura, 1977).

Tanner and Tanner (1975) criticize Popham and Baker (1970), who put forward the idea that curriculum is the end and instruction is the means, stating that:

"Thus the teacher is regarded as a sort of mechanic whose job it is to see to it that the curriculum and other components of the technological process of production yield the sufficient quantity of products under the necessary quality controls" (P.28).

2) The other school whose main proponents like Eisner (1967), Atkin (1968) and Ebel (1970) raised caution concerning the possibility of
specifying the objectives. They would argue that the most trivial objectives could actually be specified. This school, it seems, is based on mathematics and mathematics education, as is apparent from their comments on the ability to make intelligent guesses, the ability to think mathematically, etc. (Scandura, 1972).

They argue that such comments could be considered to be main objectives of mathematical education and other people should contribute in developing mathematical abilities. (Scandura, 1972).

Scandura (1972), in criticizing both schools and presenting his own opinion, writes that:

"My own view is somewhere in between. On the one hand, I feel that complete reliance on operationally defined objectives has led some to fragmental curriculum - a curriculum based on discrete bits of knowledge with little or no attention to basic relationships that may exist in the subject matter - or to general processing skills that are important in learning and doing mathematics. On the other hand, I feel that the nonobjectivists have not gone as far as possible in pinning down the vague and nonoperationa aims of mathematics education that they propose." (P.141).

Scandura (1977) looks on curricula as simply 'what is to be learned' and proposes an algorithmic approach to curriculum construction when, in this sense, he writes that:
"The algorithmic approach is basically a method for constructing curricula based on behavioural objectives and characterized in terms of rules and higher-order rules" (P.392).

By higher-order rules, he means that if R1 and R2 are two rules, then if there is a relationship between R1 and R2 such that this relationship could induce a rule R to play the roles of both R1 and R2, then R is called a higher-order rule to R1 and R2.

"If rules are operating on other rules, they are referred to as being relatively higher-order; if they are being acted on, they are said to be of relatively lower-order" (Scandura, 1977, P.545).

Gagne and Briggs (1974), in discussing the higher-order rules, have described them in the example of adding the two polynomials

\[ 2x + 3x^2 + 1 \]
\[ 2 + 3x + 4x^2 \]

When the student is asked:

"What is the sum of these two expressions?"

If he knows the rule that a variable y if added to the variable \( y^2 \) results in the sum \( y + y^2 \), then he can use it as a subordinate rule to the problem. If he also knows the rule that \( 2y^2 + 3y^2 = 5y^2 \) (this is another subordinate rule to the problem), the student then combines such rules into a more complex rule which is the solution of the problem. In this case the subject as they state:
"has recalled relevant rules and combined them to form a new higher-order rule " (P.47).

Another example of the higher-order rule can be found in different presentations of the equation of a circle. The equations when (a) the center is on the x-axis, (b) the centre is on the y-axis and (c) the ends of a diameter are known. One rule can replace all these rules, i.e. $(x-x_0)^2 + (y-y_0)^2 = r^2$. This rule could be understood as a higher-order rule for rules in (a), (b), and (c). Further discussion of higher-order rules is introduced in Chapter 4.

In a critical argument to the situation, Begle (1979), who began his career as a mathematician and later turned to curriculum development, believes that there is little hope for any further substantial improvement in mathematics education until;

"we turn mathematics education into an experimental science, until we abandon our reliance on philosophical discussion based on dubious assumptions " (P.xi).

He asks instead for a carefully constructed pattern of observation and speculation, similar to those of the physical and natural scientists:

"We need to follow the procedure used by our colleagues in physics, chemistry, biology, etc. in order to build up a theory of mathematics education ... We need to start with extensive, careful, empirical observations of mathematics education " (P.xi).
Discussion of the curriculum as one of the main problems in the 'methodology for approach' reveals the difficulties to be encountered in applying theory in reality. Bruner (1966) in discussing the question of how to enlist the aid of our most able scholars and scientists in designing curricula for primary and secondary schools states that

"There is at least one major matter that is left unsettled even by a large-scale revision of curricula in the direction indicated. Mastery of the fundamental ideas of a field involves not only the grasping of general principles, but also the development of an attitude towards learning and enquiry, toward guessing and hunches, toward the possibility of solving problems on one's own " (P.20).

I believe that Bruner asked then for numerous intricate achievements, and it is expected that he will ask for them now.

3.4.2. Instruction

Gagne and Briggs (1979) look upon instruction as a human undertaking whose purpose is to help other people learn, while learning could happen without any instruction. They view teaching as only one form of instruction, albeit a signally important one. Other forms may be by a television program or by a combination of physical objects, among other things.

"Of course, a teacher may play an essential role in the arrangement of any of these events. Or, as already
mentioned, the learners may be able to manage instructional events themselves " (Gagne and Briggs, P.3).

Begle (1979) in turn finds that learning depends also on methods of teaching:

"What students learn about mathematics depends not only on the characteristic of their teacher, the curriculum which they follow, their own characteristic, and their environment but also on the way in which they are taught " (P.113).

Tanner and Tanner (1975) in responding to those who contrast instruction with curriculum (Popham and Baker, 1970), state that:

"It is also clear that the theoretical separation of teaching from what is to be taught has led to an undue emphasis on the former and neglect of the latter. This is dilemma of the curriculum field ... Yet the dilemma has worsened with the recent esoteric interest in teaching as an abstract phenomenon. Hopefully, with the heightening of interest in teacher participation in curriculum improvement, the tide will turn away from the study of teaching in abstraction and begin to turn to teaching as an integral part of the curriculum field " (P.601).

In this sense, the Tanners give a very important role to the teacher in that they see teaching as an essential and integral part of the curriculum. In this connection Oliver (1977), who considers the
ultimate goal of curriculum planning as improvement of the learner's behaviour, finds that

"good teaching will carry out the intent of curriculum planners " (P.262).

In discussing instructional variables, Begle (1979) notes that there are different variables that affect instruction:
(i) some of these variables are related to the teacher, e.g. affective characteristic, knowledge of mathematics, effectiveness.

Begle reports that the area of mathematics education is the more fortunate one, in that a considerable amount of substantial reviews of the empirical studies as well as much information about teachers has been put together, organized and made easily available.

Begle then concludes:

"Probably the most important generalization which can be drawn from this body of information is that many of our common beliefs about teachers are false, or at the very best rest on shaky foundations. Thus, for example, there are no experts who can distinguish the effective from the ineffective teacher merely on the basis of easily observable teacher characteristics. Similarly, the effect of a teacher's subject matter, knowledge and attitudes on students' learning seem to be far less powerful than most of us had realized " (P.54).
(ii) Other variables are related to learning theories and teaching children: motivation, whole or part learning, learning by exposition or discovery, learning by insight, individualized learning, etc.

Here a brief mention of learning by discovery and of individualized methods of instruction is presented. The first is generally advocated by those who support the approach through structures e.g. Bruner (1966). Child (1973) comments on the place of learning by discovery, writing that this method:

"has been a hot-potato in educational psychology particularly since the Second World War " (P.110)

The second, i.e. individualized instruction, has strong theoretical support for presenting the optimum programme for learning. It simply states that, since no two students are alike, it is logical to implement individualized programmes that fit the particular abilities and interest of each student.

My choice of both methods is based on:
(1) The demand, in Kuwait, for following the first method, which is well based psychologically and appreciated in opposition to the well-known and prevalent expository method,
(2) individualized learning also has logical and psychological roots, and is also promising since it pays more attention to individual differences; however despite these virtues it is almost ignored in Kuwaiti schools.
(3) The first method usually lays the burden on the teacher in thinking carefully so as to present the subject-matter through
adequate and simple questions that lead to the target, while the second lays the burden on the student in looking for progress on his own but under the sensible guidance of the teacher. Hence I chose these two methods on the one hand for the above mentioned reasons and on the other to begin discussion about instruction which should not be ignored in any educational research.

3.4.2.0. Discovery Instruction.

This sort of learning is characterized by the comparison of the favoured method - discovery - to an unfavoured method labelled variously as expository teaching, receptive teaching, rote teaching and verbatimism. This method is considered to be a disciplinary effort to teach children to think like scientists instead of children. This discovery instruction, in which the student is asked to explore a situation in his own way, is invaluable in developing creative and independent thinking in the individual. (Howson et al., 1981). Such a method as several enquiry-discovery courses follow the 'inductive approach' where the student induces abstractions or generalizations from specific cases. Thus, 'enquiry' is equated with induction (Tanner and Tanner, 1975).

Bruner (1966) as one of the strong proponents of this sort of learning, believes that learning the 'fundamental structure' of a subject is the better if learned through discovery. In this sense he writes:
"... whether one speaks to mathematicians or physicists or historians, one encounters repeatedly an expression of faith in the powerful effects that come from permitting the student to put things together for himself, to be his own discoverer."

In this connection, Watson (1976) states that no discussion of 'psychological insight' can omit mentioning the issue 'learning by discovery' of Bruner (1961). In addition, the whole concept of learning by discovery has been the subject of intense debate. But experimental studies have revealed that attempts

"to demonstrate the superiority of one or the other method have been generally inconclusive" (Watson, 1976, P.32).

Others, like Lavetelli et al. (1972), find that discovery methods of teaching are based on the premise that children learn to solve problems and acquire concepts on the basis of inductive reasoning from empirical data, i.e. this method fosters scientific method. But Conant (1964) states that the so called "scientific method" is a misnomer, since scientific knowledge has not advanced from one single method. He then states that the notion that scientific knowledge today is the result of interconnection of generalizations arrived at by only the empirical-inductive method, is false.

According to Kagan (1966), the discovery approach requires more involvement by the learner, and results therefore in his giving maximum attention to a task. But Kagan points out also that there were some compelling arguments against the discovery method.
Cronbach, (1966) deals with the problem thus:

"I have no faith in any generalization upholding one teaching technique against another, whether that preferred method be audiovisual aids, programmed instruction, learning by doing, inductive teaching, or whatever. A particular education tactic is part of an instructional system; a proper educational design calls upon that tactic at a certain point in the sequence, for a certain period of time, following and preceding certain other tactics. No conclusion can be drawn about the tactic considered by itself" (P.77).

Tanner and Tanner (1975) in turn point out that

"the research on discovery teaching has, thus far, failed to produce generalizations about the place of discovery teaching in the curriculum because this question has not been asked" (P.360).

Decidedly, the debate on discovery teaching is never-ending. In commenting on the work of Wertheimer (1959), Gardiner (1980) writes, in this sense, that

"discovery teaching is much more challenging than expository teaching ... Expository teaching is often the result of rote learning by teachers. They are simply parroting their teacher's procedures as their students, in turn, parrot their procedures" (P.254).

He then states that:
"Discovery teachers are also encouraged by the 'eurekas' they hear from their students ... since the word is so new to infants, they are making discoveries all the time. Some developmental psychologists have argued that the smile is an infant's way of saying 'eureka'. Look for those sudden smiles" (P.255).

Wittrock (1966) finds, in turn, that discovery takes time and

"there are probably several processes involved in discovery. Induction has no exclusive identity with discovery learning" (P.33).

3.4.2.1. Individualized Learning

Individualized learning is considered as a solution to individual difference, hence its importance. A student is allowed to work on a unit as long as he needs to master it. He works independently, using self-instruction, self-correction, and well-developed material. After each self-test, the child is given a check test by his teacher. In connection to programmed instruction, Tanner and Tanner (1975) find that:

"often we hear the terms 'individualized instruction' and 'programmed instruction' used interchangeably, and many proponents of programmed instruction believe erroneously that to use programmed materials is automatically to 'individualize' instruction" (P.266).
While Gardiner (1980) states that advocates of programmed instruction claim that:

"'Programmed instruction' is individualized instruction. Students can proceed ... at different rates and through ... different routes." (P.78).

In addition, Gagne and Briggs (1979) introduce the 'module' for individualized learning, based on performance objectives, learning hierarchies, sequencing and employment of appropriate instruction events and conditions of learning. In commenting on this system, they write:

"Classroom control problems are usually less in individualized instruction than in conventional instruction. Teachers usually need special training in the managements of such systems. Once they master the necessary routines, they after prefer individualized to conventional methods." (P.284).

The earlier strategy for individualized learning goes back to Pressey in the 1920's. This strategy was first used at the college level. Later, several other individualized methods were developed for the college-level student as the Keller Plan (Keller, 1966; Ryan, 1974) and the Audio-Tutorial Approach (Postlethwait, Novak and Murray, 1969). Work in this field has been extended to the school level by such people as Weisgerber (1971) and Talmage (1975).

The 'Mathematics for the Majority Project, Assignment Systems' (1970-4) has strongly recommended the use of individualized
assignments, providing examples of work on cards and giving suggestion for their use. Watson (1976) finds that these

"are also helpful in dealing with problems caused by absence " (P.23).

In connection, Dienes (1971) writes:

"To allow for children's individual differences most of the learning should take place individually or in small groups of twos and threes " (P.37).

Previously, he wrote in 1960 that

"It would probably be necessary to abolish almost completely the present method of class teaching with the teacher pontificating from a central position of power, and to replace this by individualized learning or learning in small groups from concrete material and written instruction, with the teacher acting as guide and counsellor." (P.29).

Elton (1979) who raised questions involving the intentions of (i) making students more independent in their learning and (ii) teaching for understanding, believes that innovative methods which use group discussion and individualized learning - such as the Keller Plan course - should reinforce and encourage good habits in learning, and that teachers might develop their teaching methods if they learnt about current developments in methods.

In discussing "How far we shall succeed in our aims", Elton says,
"It is at present too early to tell".

The role of the teacher in this method seems to be elusive. For example, Wheeler (1970) claims in watching a teacher, that he teaches the whole class or a group of the whole class much of the time, adding that

"It isn't easy to describe the way in which this teacher works."

In connection, Watson (1976) quotes from Wheeler:

"If the children are all working as individuals or in pairs, he will find it impossible to spend much time with each small group, and the opportunity for him to intervene in their thinking is correspondingly reduced. There was, as we realise, much concealed inattention and inactivity in the traditional classroom, even when all the children were apparently watching the teacher in his demonstration, explanation, or questioning. Individualized methods may only make some of this inactivity, or useless activity, the more apparent, but the fundamental dilemma remains; how can the teacher effectively distribute the resource of experience and stimulus which he, and he alone, can provide, so that it is made maximally available to his pupils " (P.28).

Further study-reviews of the empirical studies which compare this method to traditional ones tell us that:
(i) Miller (1976) reported that half the studies he reviewed had shown no significant difference between this method and the traditional one. The other half generally favoured the individualized rather than the traditional one. He also found that more than three-quarters of his study reported no significant differences on affective outcomes. One interesting trend which he reports is that the longer the experiment lasted, the smaller the advantage of individualization.

(ii) Schoen (1976) in two studies found that this method has positive effects, especially on affective outcomes, for the primary grades, but that it is deleterious for grades 5-8. He noted that all the individualized programmes which have been tried out made severe demands on the teacher's time as well as imposing additional financial burdens on the school.

In commenting on the situation, Begle (1979) writes:

"unfortunately, we seem not, as yet, to be able to take advantage of it to improve mathematics instruction in any substantial way " (P.122).

3.4.3. Summary II

In tracing a mathematical phenomenon that could help the development of this study, it was natural to discuss some factors that contribute towards a certain approach e.g. the curricula and instruction. The term 'methodology for approach' was introduced to include all such factors.
(i) In discussing the curriculum it was found that there is no agreement on the definition of curriculum. The radical changes in mathematics curricula came with 'modern mathematics' and the belief that this mathematics can adequately serve for the structure. In addition, there was in many cases no place in the new curriculum for applications of mathematics in reality, i.e. traditional mechanics. This could have happened because of the influence of pure mathematicians, the opposition of other mathematicians as well as educators of mathematics teachers to contribute to any change and finally the weak influence of psychologists working on mathematics learning on designing the curriculum or textbooks. The first impression of a phenomenon as a relevant starting point for this study was in that, "content differences resulted in different learning outcomes."

As for construction of the curriculum the study refers to different views that defend complete reliance on objectives and others who expressed concern regarding the possibility of specifying the objectives.

(ii) As for instruction, Begle (1979) finds that there are no experts who can distinguish the effective from the ineffective teacher merely on the basis of easily observable teacher characteristics. Other variables concerned with instruction could be related to learning theories such as: motivation, learning by exposition or discovery, ..., individualized learning. The study discusses discovery learning as a favoured method as well as the individualised learning as a promising method especially for individual differences. The literature reveals that there is no clear-cut evidence either in favour or against any of these methods.
3.5. Further Discussion

3.5.0. Introduction

The previous review of mathematics education and its relation to the psychology of learning shows that development in the area does not establish a well-established and solid base for continuous progress. Scandura (1977) in describing the situation writes:

"In spite of the so-called paradigm shift toward cognition generally, and information processing in particular, theories of complex human behaviour still tend to be highly restrictive ... The vast bulk of research and theory construction in behavioural science today is still very much a matter of building-up of 'brick-laying' " (P.9).

Begle (1979) on a gloomy note, expresses depression for the little new knowledge that emerged in mathematics education in the last two decades. However, he believes that the literature includes a solid body of information about mathematics education and calls for extracting and organizing this information from the scattered literature, otherwise he predicts future international congresses will be much like the past ones:

"new faces, new opinions, but very little new knowledge " (P.157).

But, although these statements might depict the situation in gloomy tones, the fact is that progress in any field is developed by the
failure or success of others. In this sense, Koestler (1959) describes the agonies and the ecstasies of the many brilliant thinkers who spent their lives trying to find the laws which govern planetary motion. As a result of their efforts we find the brilliant Newton modestly claiming that:

"he could see far only because he stood on the shoulders of giants" (Gardiner, 1980, P.259).

Accordingly, a phenomenon that could be noted by one researcher can benefit another worker in conducting his research. Thereupon, I tried to discuss some different ideas in the approach and methodology for approach as a way of gaining some knowledge about a certain phenomenon that might be of use.

I believed a phenomenon could be helpful if:

(i) it interacts with the experience of the researcher.

(ii) It intrigues the researcher and this interest persists in him.

(iii) the literature review, about the phenomenon, indicates that there could be a tangible change in learning outcomes due to a tangible change in a teaching-learning situation.

The literature is very essential since if theoretical literature is excluded then it is not easy, for example, to know anything of how teachers teach (Freudenthal, 1973).
In the next two sections, I present a simple discussion of the points that were raised in this chapter with the purpose of learning more about phenomena in the field.

3.5.1. First: The Approach.

The approach, as was stated, is axiomatic. And the current approach that some eminent psychologists (e.g. Piaget) advocate, meets with the favour of a fair number of mathematicians, who believe that mathematics is better introduced through structures, e.g. the Bourbaki school. Piaget built his idea about the approach on the basis of the structure of the nervous system. As a matter of fact, developmental psychology is concerned with the structure of the nervous system, whereas biology is concerned with its function and epistemology with its contents (Gardiner, 1980). Piaget argues that we inherit a way of intellectual functioning which operates interactively with the environment, leading to a progressive development of intellectual structure (McNally, 1977). And there is a correspondence between the elementary structures of the intellect and the mathematical structures, in the sense that teaching mathematical structures from the start of the child at school, will help in the consistency of these structures, which, in turn, will then help the child to cope better with mathematics.

The new approach has also made emphasis on understanding, discovery, patterns and structure. These could be considered as basic characteristics of the claims behind the new development.

Three questions fundamentally rise in looking for feasible answers:
(1) Is there a promising change in achievement in mathematics under the current theoretical vision of the approach?

(2) In the case of the answer to the previous question being negative, then what conclusion can be drawn?

(3) What problematic events are encountered in the area? And how can proper learning conditions be fostered to improve achievement in mathematics?

In discussing these questions, especially the first two, I believe that it is still too early to tell whether the current approach, based on structures, is adequate or not. No development will necessarily stand firm from its first application, especially in education. It will require modification as well as application for a considerable period of time.

However, it is true to say that the present situation of achievement in mathematics is not as satisfactory as that which was generally expected, when modern mathematics—based on structure—was introduced into the system of mathematics education in schools. One could depict the current situation as that stated by Skemp (1971):

"I became increasingly concerned with the problem of those pupils who, though intelligent and hard-working, couldn't do mathematics." (P.15).

Watson (1976) also finds that role of techniques and practice was perhaps underestimated when writing that:
"Practice without the power of mathematical thinking leads nowhere; the power of mathematical thinking without practice is like knowing what to do but not having the skill to do it" (P.128).

The situation is that evidence that has been gathered has pointed to structure as facilitating mathematical achievement, and there has been a decline in computational skills in students in modern programmes, but there has, at the same time, been an increase in problem-solving. (Begle, 1979). However, Begle adds:

"It is hard to derive any useful generalizations because such a wide variety of modern instructional programs was involved ... it is clear that further studies are called for" (P.69, P.77).

As a matter of fact, Bruner (1966) seemed to have expected such a situation when stating that:

"In point of fact, drill need not be rote and, alas, emphasis on understanding may lead students to a certain verbal glibness" (P.29).

Further, Boys (1972) in reporting about the new programmes of mathematics in Kuwait, states that:

"I recognise some advantages in the sound logical development of the subject, but too often it seems that the language takes over from the ideas".
Then he continues:

"The exercises seem varied and interesting, but not giving very generous amounts of practice ... I was disappointed in the 'problems'."

In most cases there was found similar complaints of incompetence of students in computational skills within the new curricula.

Moreover, if Piaget's four stages of intellectual development, i.e. sensorimotor, preoperational, concrete operational and formal operational, are accepted, then we might find different factual studies which reveal that a large number of students who are expected, in the past as well as in the present, to be formal operationals did not enter this stage.

For example Herren (1977) reports that:

(i) Lovell in testing a number of students in England has found that only between 23% and 37% of a sample composed of 39 grammar school pupils, 10 training college students, and 3 adults demonstrated formal thought;

(ii) Dale, in a study in Australia, has found that only 25% of the 15 year old students in his sample were able to solve completely a task designed to measure formal thought;

(iii) McKinnon and Renner, have found, in a widely publicized study undertaken at the University of Oklahoma, that 50% of the College
freshmen tested were functioning completely at Piaget's concrete operational level and that only 25% of the sample could be considered fully formal in their thought.

(iv) Herren (1977) has estimated that roughly half of the non-science students in college failed to exhibit formal operational thought.

In a recent study in 1980, I also ran a test in Kuwait to identify the knowledge of students in two different structural patterns and found similar results. The students were 16 and 18 years old. This study is reported in (P.7-8).

If we notice that not all samples of students in the previous five examples were taught in the modern mathematics based on the current approach of the structure, then we certainly cannot make a pre-judgement that the approach in the past or in the present is the culprit. Such an answer would be a very naive one to a very complicated situation. But there must be something in these processes which is lacking or not adequate. This could lead to raising the following major question:

Are we sure that the mathematics we introduce is effective and at the same time meets the demands of the approach?

This question is indeed in the heart of this study and it inevitably entails a discussion of some methodological points that were reviewed.
3.5.2. Second: Methodological Points

(I) The curriculum: One problem is the construction of a curriculum which is still unsolved. For example: do we follow the behavioural objective-model or not? If the answer is negative, then the situation will be ambiguous in the sense that we might be unable to identify what is to be presented. If the answer is positive, then this might result in fragmenting the subject-matter (Scandura, 1972).

Furthermore, Scandura (1977) in describing the situation reports:

"Curriculum construction has traditionally been artistic endeavor. Even today, the vast majority of texts and new curricula are developed almost exclusively on the basis of the curriculum constructors' subject-matter knowledge and professional know-how" (P.389).

Another problem is found in sequencing of the topics of the subject matter. This problem has indeed implications in the psychology of learning, e.g. the place of fraction, ratio, limit, ...

A third problem is also found in different presentations of a definite mathematical task. I would like to present two examples concerning this third problem from the conference of BSPLM (April, 1981) at the University of Leeds.

The first example was in how a text book better presents solving simultaneous equations of the first degree. We could begin with
so that students are then guided to add or subtract in the case where the absolute value of the coefficient of one unknown is the same in both equations. Or should we do better by beginning with

$$5x + 2y = 29 \quad \text{and} \quad 2x + 3y = 16$$

and discuss the solution by guiding to the main idea, in that a student should comprehend that the solution is embodied in producing an equation in one unknown from the two equations?

The second example was in introducing the limit of a function or sequence. Some educators prefer introducing the idea by stating that: \( f(x) \) tends towards a limit "1" as \( x \) tends towards "a" say. Some prefer to say that the limit is unreachable, whence the anomaly of a constant function \( f(x) = 1 \) which reaches its limit for all values of \( x \).

The third problem covering the curriculum was in presenting a certain task in the textbook. This, in my opinion, is the fundamental one, since the phenomenon most directly related to the student in this thesis is primarily concerned with the presentation of mathematical tasks. The textbook is indeed a 'main variable' in the whole process of mathematics education. I therefore draw the distinction between the textbook as opposed to the main heading of the curricula, since: (i) the textbook is a factual projection of the constructs that underlie the author's interpretation of the curricula; and (ii) it is still true that curricula are mostly vague and usually admit of great variety of interpretation (Freudenthal, 1973). The weight of the text could not therefore be ignored when discussing the
curriculum as a real factor in methodology which could have the same weight as the curriculum itself, despite being part of it.

(II) The textbook: Education as a human activity is usually reflected in textbooks, and a textbook can be considered to establish realistic interpretation of a curriculum. Historically, the textbook has been the key medium through which the subject-matter has been organized for instruction. It undoubtedly has influence on the educational system, especially in some developing countries, where the textbook is the main source both for the student and the teacher.

Oliver (1977) refers that whether or not a state or a city dictates the one text to be used, most classes use one or several texts, but:

"some teachers slavishly follow the text chosen; others utilize it as a basic guide or as a point of departure. In any case, the written material exerts a powerful influence on what is and what is not taught. Editors in the various publishing houses may have a hand in the matter, too, as they sometimes select the authors who are to write the books, and their editorial policies may affect what is presented and the manner of presentation" (P.178).

He continues his argument in a way which indeed might accurately describe the situation in developing countries:

"when the selected material is organized and presented in a way that enhances learning, the value of the printed word from authority becomes even greater" (P.198).
In praising the use of the textbook by students, Begle (1979) reported that NLSMA provided considerable evidence that the mathematical topics and the mathematical emphasis that are contained in a textbook do get through to the student. It was found that textbooks of secondary as well as elementary levels have an effect on student learning and evidently should not be ignored (Begle, 1979).

The most interesting findings were clearly demonstrated by NLSMA, in that, content differences in textbooks resulted in differences in student achievement (Begle, 1979). This phenomenon is at the heart of this research problem, and consequently it will be discussed in detail in the following Chapter 4.

The influence of the textbook in mathematics teaching and learning in Kuwait will also be discussed in Chapter 4 as it was briefly discussed in (d) of 1.2.

(III) Reliance on Abstraction: There was also a trend towards abstraction in presenting mathematical concepts. This trend has almost led to a complete disregard of the content and methodology by which traditional mathematical pedagogy has been communicated. This was not only naive, but also a

"regrettable misunderstanding of what the learning and teaching of mathematics is really all about." (Lamon, 1972, P.4).

This has resulted in a shifting to a view that the approach to
mathematics should be based on abstraction. Dieudonné (1969) as an eminent figure in the mathematics of the French school, thinks of mathematics as a study of structure, and that:

"the pupil must be taught to be aware of the absolute necessity of an axiomatic approach to mathematics. At the earliest possible moment, the pupil must get used to the constant dealing with abstract concept." (Watson, 1976, P.124).

In neglecting the concrete applications, there has almost been a total ignoring of the advice of psychologists working on mathematics learning about concept formation. The greater emphasis on the abstraction of the structure has resulted in overlooking the fact that the greater emphasis on structure should lead to greater appreciation of the need for skills useful in applying that structure to practical problems (Lamon, 1972).

(IV) The teacher: Changing a curriculum does not imply changing other factors that are involved in the educational system. This system is complex and has a high inertial mass (Watson, 1976). For example, despite the change in the curriculum and the claims that teaching should be based on understanding of the fundamentals, I found, as an inspector of mathematics, that a number of teachers still work on the rules: (a) teaching is telling; (b) memorizing is learning and (c) being able to repeat something in an examination is evidence of understanding.

In describing similar situations, Scandura (1977) writes:

"In spite of continuing attention to CAI, computer-based
curricula, and other technological approaches to education, most schools and college instruction still takes place in the classroom. Instructors rely for the most part not on theories of learning, problem-solving, or instruction, but rather on professional 'know-how' " (P.553).

Skemp (1971) in turn finds a great responsibility of the teacher in the early stages of learning and he therefore guides the teacher:
(i) to make sure that schematic learning is taking place adequately;
(ii) to distinguish which stages require only straightforward assimilation and which need accommodation; and (iii) to plan schemas that will be adaptable to future as well as present needs. But some teachers do not follow such advice and some fall upon one of those colourless textbooks written without any previous analysis of the subject matter and lacking any didactical background (Freudenthal, 1973). A teacher in this group applies what he might believe to be true, irrespective of its educational value. Skemp (1971) depicts the situation in stating:

"to know mathematics is one thing, and to be able to teach it - to communicate it to those at a lower conceptual level - is quite another; and I believe that it is the latter which is most lacking at the moment. As a result, many people acquire at school a life-long dislike, even fear, of mathematics " (P.36).

(V) Teaching methods: One of the methods that has been highly recommended is the discovery teaching-learning method. But this method has either been naively misunderstood or overestimated.
Teachers who misinterpret this method usually understand the method as embedded in posing any questions relevant to the subject-matter, whether it works or not. I watched a teacher who was discussing the definition of a 'sequence', suddenly ask the students about the possible definition of the limit of a sequence. A second teacher asked students what they 'imagine' the derivative of a function is about, before presenting any relevant idea of this concept. Another teacher asked once about the number of the sides of a triangle, though his students were in the secondary stage, and had learnt about the concept of the triangle and its implications for three years at least - he was asking just any question.

Others gave a sort of overestimation of this method. At a conference on discovery learning, Shulman and Keisler (1966) reported that some experiments showed preference of this method to expository methods, while others proved the contrary. Unfortunately, the situation is no different today (Begle, 1979). While Karplus et al. (1981) report that:

"While student autonomy is necessary for all development of reasoning, students cannot discover on their own all of the mathematics that may be useful for their problem solving and other activities" (P iii).

In connection with this idea, Lamon (1972) states that:

"I, for one, have never had the privilege of noticing any student 'discover' a mathematical abstraction without being prompted by an enormous number of clues. One reason for
this is that in most instances, the cognitive equipment or
capacity for coping with the demands of a specified
cognitive task is almost non-existent in our students." (P.5).

Freudenthal (1973), in turn, finds that theoretical teaching research
often has the character of telling other people how they have to
teach. Then it is programmatic, and often even unrealistic;

"It is backed by firm belief or by the actual teaching
experience of the man who is presenting his idea. He
claims his method has proved successful, though others may
doubt whether his success is not due more to his personal
qualities rather than to the method itself." (P.160).

He then points to an extraordinary event in that he states:

"I must also confess that there was a very high positive
correlation between the most pretentious modern mathematics
and the most archaic pedagogies." (P.160).

(VI) The examinations: Whateoever is said about virtues and vices
of the examinations or tests, they decidedly have their deep influence
in the teaching and learning processes. I have noticed that if
examinations were interested in fundamentals, the teaching as well
as the learning tended to deal with fundamentals. Examinations can
also be bad in the sense of emphasizing trivial aspects of the subject-
matter and this encourages teaching in a disconnected fashion and
rote learning (Bruner, 1966).
The previous six points could refer to several phenomena in the educational system. Further questions could refer to other phenomena in the field which merit discussion. For example, Gagne (1972) poses questions such as:

(i) How can the learning of mathematics be arranged so as to prevent the occurrence of obstacles and emotional blocks to learning?

(ii) How can instruction be arranged so that a higher proportion of students can cope with certain specified mathematics topics.

(iii) Suppose a student has just learned to derive a mathematical expression. How many examples must now be provided before we can be sure he knows how to do this? Two? Five? And does this number vary with the kind of problem or with the nature of the student's past experience?

These questions are among the enormous number which still constitute problematic areas in the body of mathematics education. One interesting statement by Gagne and Briggs (1979) might depict the situation:

"Is there a structure of intellectual skills which presents the 'path of greatest learning efficiency' for every subject in the curriculum? In theory, there is. Do we know what this structure is? Only vaguely, as yet. After all, teachers, curriculum specialists, and textbook writers 'try' to represent structure in their lessons and curriculum plans, and have been trying for many years. Nevertheless, on the whole their efforts must be characterized as partial and inadequate" (P.74).
Therefore in answering a major question which was posed, we restate again: Are we sure that the mathematics we introduce is effective and meets at the same time the demands of the approach? We find in the light of factual evidence and complaints of those who are directly or indirectly involved in the mathematics situation (i.e. teachers and parents), that the effectiveness of what we teach is not satisfactory and the previous discussion proves that the area abounds in enormous problems that concern the approach. All answers for these problematic events could be contained in the answers to the third question (on P. 3-52).

The situation now, is almost similar to that which Bruner (1966) has stated:

"much too little is known about how to teach fundamental structures effectively or how to provide learning conditions that foster it " (P.12).

Nevertheless, the second part of the question is relevant to this study. This is the belief that in accordance with proposition 6.1 a study of the phenomenon "different presentations of a mathematical task result in different learning outcomes" could be worthwhile in improving the current situation. The importance of this phenomenon lies in the change of one variable (learning outcomes) under the change of another variable (presentation), where presentation, as that discussed in Chapter 4, is considered in this thesis to constitute the core of the contents of a certain teaching-learning task.

Lastly, I shall present in the following general conclusion what might give some partial answers to those questions posed on P. 3-52.
3.5.3. General Conclusion 3.1

If the essential psycho-educational problem is: as mathematical structures are by their nature abstractions of extreme generality, then should or should not these structures be presented early (in the educational process) to help develop general schemas of thought? Then this problem is not yet solved. Thus, if we conceive the approach to be axiomatic, we have no strong evidence against the current approach. The controversy about improving the achievement in mathematics could better be shifted towards methodology that involved teaching-learning situations so that we might improve the achievement of both teachers and students in mathematics.

One of the phenomena that could be promising for modification in methodology for better improvement could be included in some studies which indicate that, a change in the achievement of students could result under change in the material contents.

3.5.4. Summary III

There is still no well-established base for substantial progress in mathematics education or in the psychology of learning. The situation seemed gloomy to some mathematics educators as well as to some who work in the psychology of learning mathematics. In looking for a phenomenon to discuss both the approach and relevant methodology in mathematics education, I was content that this phenomenon should (i) integrate with experience, (ii) survive within researcher, and (iii) the literature reveals that learning outcomes
could change under that phenomenon. A discussion in two parts, the first for the approach and the second for some methodological points, was conducted for the purpose of learning more about such phenomena in the field. As for the approach, it was indicated that it is supported by many eminent psychologists as well as mathematicians. The idea was that teaching mathematics through structures would benefit the intellectual structures of individuals as well as helping learners to grasp mathematics better. The answer to the question whether there is a promising change, is that it is too early to tell. The approach is axiomatic and consequently we have no evidence that it is the culprit for any low achievement. A major question was raised, whether the mathematics we introduce is effective and in harmony with the approach, and this question was discussed in the second part of this study through discussing problematic consequences under six headings. (I) The curriculum: where studies reveal that there are still problems in construction, sequencing of topics and different presentations of tasks in textbooks. The last area here was attractive and therefore the second point was (II) The textbook: This point revealed that the textbook is basic and effective on student learning and content differences in textbooks resulted in differences in student achievement. (III) Reliance on abstraction: It was found that there was also a strong trend towards abstraction in the current mathematics situation. This resulted in ignoring that the emphasis on structure should not neglect the concrete application building up concept formation. (IV) The teacher: Changing of the curriculum did not imply changing the teachers' attitudes towards the whole process of change. Many teachers still believe that teaching is telling, memorizing is learning and working on examination papers is
understanding. It was also reported that teachers do not rely on theories of learning, but rather on professional "know-how".

(V) Teaching methods: The discovery method is one of the methods that was highly recommended to be included in the changes in the curriculum. But this method was naively understood or over-estimated

(VI) The examination: The controversy about examinations never ceases. Nevertheless, examinations could monitor to a certain extent trends in teaching-learning situations.

The previous points indicate different problematic events which could underlie different phenomena. And all of these problematic events could underlie different research work since the answers that are presented for the "path of greatest learning efficiency" could be characterized as partial and inadequate. A phenomenon, in the sense of proposition 6.1, was believed to be in "different presentations of a mathematical task result in different learning outcomes" which denote a change in one variable as a function of change in another. This phenomenon was called the presentation-phenomenon and is fully discussed in Chapter 4.

A general conclusion was that we still have no strong evidence against the approach and the controversy of mathematics achievement could better be shifted towards methodology.

3.6. General Summary

In looking for a certain phenomenon that could be used as a starting point for the development of this study, the literature review was monitored towards studying problematic consequences that could be found
in the approach and the methodology for approach. The current opinion is generally accepted in that a mathematics approach is better if based on structures. This approach has psychological roots. If the approach is accepted as an axiomatic one, then these problematic phenomena could be found in methodological procedures. These problems, as was found, could be in the curriculum (in the contents) as well as in the instruction. It was found that textbook's contents are influential tools and should not be ignored. The basic phenomenon that was attractive as a starting point in this study was "content differences could result in different learning outcomes."

3.7. A Personal Point Of View

The vast range of criticism of school mathematics requires a knowledge of available indications in reality. Here I list some of these points which I could perceive whilst working as a teacher, senior teacher and inspector of mathematics for a quarter century in all stages of the school, though mostly secondary.

(I) In mathematics:

(i) The subject-matter proves to be not easily grasped by the majority.
(ii) It is an abstract edifice, where each abstraction is based on another, and we still need more intelligible tools in order to help students grasp this abstraction.
(II) The teacher:

(i) Not all teachers are creative and the majority in most developing countries are not. Most of the teachers in these countries are not content professionally; they work in this profession since they have no better opportunity in another one.

(ii) Teachers, in general, have not enough time, because of the heavy burden they have on their shoulders, to reform their methods, even if they wished to do so.

(iii) The facilities that are available to teachers to introduce a better task are limited. For example, we believe in the idea that preparing for presenting abstraction requires drawing from intelligible concrete models of the abstract idea, before launching into the abstraction itself. Yet we find that in general most teachers have neither the ability nor the time to do this.

(iv) The current approach seemed to have created two different streams: (a) mathematicians, who are seemingly dominant, believe in the possibility of introducing mathematics through abstraction. Their influence is clearly seen in the textbooks which are usually exalted by teachers for whom it is also simple to present mathematical tasks accordingly; and (b) some educators of mathematics together with psychologists who, on the other hand, work on mathematics learning, could not yet exert a stronger influence in the methodology of approach; in the sense that the theoretical framework they present is not easily applicable, and they also cannot influence many textbooks with their ideas. The professionals and pure mathematicians are inversely dominant.

My point of view is that, the solutions to such problematic points are not easy, but primarily a reform requires building-up teaching models
where each model consists of two sub-models: (a) one presenting what we might call the pre-abstract prerequisites of the task, i.e. exemplifying intelligibly and concretely for abstraction, and (b) one presenting a presentation-model for specific mathematical tasks. The structure of this model is better if based on an algorithm. Such a model should suggest what proper instances - examples, exercises and simple problems - might be presented by a normal teacher for a better mathematical situation.

One overt word is to be said. Teachers, as I could observe, are mostly opposed to such statements as: "mathematize your students, let them think critically, think better of Wallas (1926), carefully study Polya (1948) ... always present more intelligible and concrete illustrations for a concept formation." They always want to see these pieces of advice made more intelligible, concrete and easily applied in practice.

One wonders if we can have such a "good teacher" as that prescribed by Wardhaugh (1969)

"A good teacher probably should know how to use a tape recorder, an overhead projector, and some of the media effectively, but a good teacher is not just a technician. A good teacher is someone who continually examines what he does, continually strives to arrive at new understanding of his discipline, and continually tries to steer a course between doubt and dogma. Good teaching practice is based on good theoretical understanding".

The role of the teacher should not be underestimated, but, generally speaking, we should accept that those "good teachers" are few and we
should be pragmatic, in that we should do something to help other teachers, who are the majority, by presenting them "something" that differs from advice. This "something", as I believe, could be the discussion of the "contents" of mathematical tasks in a more deep and systematic way.

Hence, according to these personal views, it was to be expected that I should try to discuss a modification in the contents as a part of the methodology of mathematics teaching-learning in the area of this study. Moreover, this discussion would involve looking for a possible algorithm (set of rules) in the methodology.

Finally, it is worth saying that the discussion in this chapter could seem to be mostly critical. This, in my view, should be valid as long as we admit the present state of students' low achievement in mathematics, and we have no clear-cut teaching-learning situation that may provide proper conditions for better achievement.
CHAPTER FOUR

SCOPE OF THE PRESENTATION - PHENOMENON
4. SCOPE OF THE PRESENTATION-PHENOMENON

4.0. Preamble

This chapter discusses the relationship between achievement (learning outcomes) and material contents (textbooks). The discussion is mainly derived from the work of others and, as such, may be considered to be a natural extension of the literature review (Chapter 3). I have separated it in order to emphasize or clarify the important role my conception of aspects of this relationship played in the generation of propositions underlying the proposition stage discussed in 6.2, which in turn guided the development or conduct of the main study.

In this preamble, I summarize the relevance of the presentation-phenomenon to the work discussed in other chapters of this study. There is an indication that different teaching treatments affect the type of questions students are best able to answer and the method of solution they employ (Lovell, 1981). This finding lends strong support to the notions advanced by Mayer and Greeno (1972) and others. In commenting on their study (ibid), Greeno (1972) claims that:

"Mayer's findings show something about the kind of learning outcomes that can be produced by different methods of presentations" (P.375).

This study suggests that differences in the material contents are related to differences in learning outcomes. This relation is supported by
Begle (1979) who concludes that "content differences in textbooks resulted in differences in student achievement" (see P.3-59).

The findings from studies, such as those described above, and my own experiences led me to think of possible solutions to the research problem (1.1) in the direction discussed in 6.2: If we assume a number 'A' assigned to the achievement of students in a teaching-learning task 'K', then this A could have a certain functional relation with an independent variable 'x_j' where this x_j was believed to be strongly related to the contents of k, i.e. A = g(x_j). This begs a twofold question:

How can we identify the contents of k? And how can we make a true change in the contents for inducing true changes in x_j?

From an investigation of the contents of a teaching-learning task, I emerged convinced of the validity of the earlier assertions (e,1.2, Chapter 1) as well as the conclusion 4.2 advanced later. Thus I equated the contents of k with the written material in the textbook for the task. In addition, on the basis of an argument advanced in 6.2.2, it is viewed that if B is a set of a well-defined behavioural objectives of k, then the content of k mainly embodies two parts:

(i) a theoretical part, e.g. a definition or a proof of theorem; and
(ii) a set R of instances, i.e. examples, exercises or simple (routine novel) problems where each instance is intended to help students grasp mathematical knowledge relevant to a certain B_i in B. The tendency to stability shown by school mathematics means that the theoretical component of the task may be regarded as more or less invariant. Consequently, any true change in the contents will be embodied in a true change in R. This R was denoted as the presentation of k. The following two statements were
then formulated:

(a) Differences in learning outcomes could be induced by differences in the material contents.

(b) Differences in the achievement \( A \) of a teaching-learning task \( k \) could be induced by differences in the presentation \( R \) of \( k \).

In the overall context of the foregoing, statement (b) above may be regarded as a special case of statement (a). However, both statements are in fact equivalent since the learning outcomes represent -and are a product of - achievement, while \( R \) represents the material contents.

(a) and (b) embody a phenomenon which relates achievement (learning outcomes) to the presentation \( R \) (material contents). I called this phenomenon the 'presentation-phenomenon'.

In studying the variation of \( A \) under variation of \( x_j \) in \( A = g(x_j) \), it was found that \( x_j \) should be identified in terms of \( R \). This invites inquiry of the relationship between \( R \) and \( x_j \). Consequently, it was argued in 6.2.2 that \( R \) does not change under mere replacement of an instance, for example, "solve for \( x: x^2 - 1 = 0 \)" by "solve for \( x: x^2 - 9 = 0 \)." It was argued (P.6-39) that such a replacement induces a superficial change and not a true change in \( R \): A true change in \( R \) is dependent on a change in the structure underlying \( R \). This necessarily leads to an inquiry into the existence of such a structure \('S'\) underlying \( R \) as well as in the identification of such a structure. I believed that the argument above and the belief in the role of theory in development (proposition 6.2, P.6-37) justify, and partially explain, the following two propositions:
(Proposition 6.1): A phenomenon that indicates a change in the achievement of students under a certain change in a well-defined teaching-learning situation is significant as a starting point for development in current teaching-learning situations. (P.6-36).

(Proposition 6.4): (i) There exists a well-defined structure 'S' underlying current presentation of mathematical teaching-learning tasks.
(ii) There exists a structure 'H' that is possible to contribute in underlying modification of current presentation.
(iii) Theory in Proposition 6.2 could possibly identify a certain discrimination of the efficient knowledge that achieved in 'S' and 'H'. (P.6-40).

I therefore equated \(x_j\) with \(R(S)\). This means that \(x_j\), with respect to a teaching-learning task \(k\), constitutes both the presentation \(R\) of \(K\) and the structure 'S' underlying \(R\). In this sense and for simplicity, I introduced the achievement \(A\) as \(A = g(R)\) where \(R\) in this functional relation is extended to involve its underlying structure 'S', i.e. \(R = R(S)\).

As for the identification of 'S' and 'H', it was argued in 7.1 that with respect to this study \(S\) constitutes the set of instances were each of them does not contain any constant of implicit form, for example 'a', while \(H\) constitutes the set of instances where each of them contains at least one constant of implicit form, for example 'a'.
Thus, if $B_i$ is a certain behavioural objective and $B_i(S)$ is the class of instances that only contains the instances in $S$, where each is governed by $B_i$ uniquely, (see Definition 5.4), and if $B_i(H)$ is likewise defined in $H$, then it can be deduced that there is no common element between the two classes $B_i(S)$, and $B_i(H)$. This is because an instance in this sense contains or does not contain such a constant 'a'. Hence if $S$ and $H$ contain all such classes $B_i(S)$ and $B_i(H)$ for all $B_i$ in the area of this study, then it is possible to identify:

$$S \cap H = \emptyset \text{ (the null set)} \text{ (see Remark 1, P.7-9).}$$

In the introduction 4.1.0 which follows, I consider some opinions advanced in previous studies concerned with the presentation-phenomenon. In sections 4.1.1 and 4.1.2, I shall outline two experimental studies by Mayer and Greeno (1972) and Ehrenpreis and Scandura (1974) respectively. In section 4.2, I shall discuss the presentation-phenomenon and reality for the purpose of learning about the possible factors that might have basic effects on $R$ as well as how far this phenomenon can be compatible with Proposition 6.1.

4.1. Two Experimental Studies Relevant To The Presentation-Phenomenon

4.1.0. Introduction

From the viewpoint adopted, the presentation $R$ of a mathematical task 'k' is a set of such instances - examples, exercises and simple problems - that is usually considered to constitute the core relied upon for coping with $k$. Generally, this $R$ can be found in the written material - as in
Kuwait - or in a teaching procedure for the same task k. Any differences in the contents, whether written or in a teaching procedure, would be eventually found through such an R. As a matter of fact, the difference between a discovery method of teaching and an expository method is revealed through such an R. In each case R may or may not differ, regardless of the 'label' attached to the procedural method.

The question that inevitably arises is: What effects could be due to such an R, provided that it is presumed that there is a structure - possibly identified - which underlies R where a true change in R is due to a true change in this structure? On the theme of this question, Wittrock (1963) pointed out that subjects who learn with different procedures (i.e. different presentations in the context of this thesis) produce different responses in the process, and different responses can be expected to produce different learning outcomes. In addition, Roughhead and Scandura (1968) argued that the demands of a particular task could lead to the subjects learning different systematic patterns of behaviour, or rules, because the use of some rules may be acquired in order to complete one instructional programme but not another. In this sense, Scandura (1977) comments that, what is learned is making simple discoveries that can be presented in expository form with equivalent results. He also states that such difference in learning outcomes as a result of different contents in the presentation of the task, appear to follow directly from an analysis of what was taught. Scandura (1977) goes on to state that:

"To the extent that content is a primary variable in future studies, therefore, more detailed structural analysis would seem to be almost mandatory" (P.507).
Roughead and Scandura (1968) have indeed found that different teaching methods, e.g. discovery and expository may give the same results. Therefore, the task of identifying "what must be learned" for new heuristics is far from trivial (Scandura, 1977).

In addition, Lovell (1981) reports that there are many studies which deal with the effect of different teaching treatments on specific learning outcomes. As an example, he finds that a representation of a teaching material in statics - formal treatment versus a treatment diagrammatically related to experience - affects the type of question students are able to answer and the method of solution students employ. He finds that this was in line with the findings of Mayer and Greeno (1972), who presented the binomial probability in two instructional sequences called 'Formula' and 'Concept' (discussed later in this Chapter). All of these studies signify the importance of the contents presented for a certain task. Frase (1975) in turn finds that the influence of manipulation in presenting the contents of learning material is beneficial to the degree that this reduces necessary learner processing. Moreover, if enough is understood about a learning task, so that material contents can be modified to aid the learning, then it should be possible to make a learner perform those alterations himself. Frase also finds in the current reviews of curriculum developments, in the area of the natural sciences (Shulman and Tamir, 1973) as well as mathematics (Dessart and Frandsen, 1973; Riedesel and Burns, 1973), support for the need for sharpening the system of analysis in the broader, conventional classroom context. Also, in referring to issues that confirm the importance of the contents presented for a task, he reports that there is some evidence that if the learner does not apply certain specific skills, the material contents can be altered in compensatory ways.
In connection with the foregoing discussion, the material contents, or the presentation R in the context of this thesis, could be optimised to result in a desirable behaviour in learning outcomes. The question would then be: If the behaviours necessary for a task are well-defined, then how can we discriminate such an R in order that a learner could acquire them?

This thesis indeed sets out to discuss this question which strongly relates to the improvement of mathematics achievement. Here, I shall present some aspects of the phenomenon in introducing two remarkable studies that discuss how learning outcomes differ with the change in the contents. Nevertheless, these two studies do not discuss the changes in learning outcomes on the basis of the structured knowledge and the underlying different structure which yield such differences in contents. I further will discuss how this phenomenon relates to reality through discussion of the relationship embodied in the presentation-phenomenon and a whole teaching-learning situation, as in the classroom.

4.1.1. I. The Mayer And Greeno Study 1972

Greeno (1972) asserts that the Mayer and Greeno (1972) study yielded information concerning the nature of general structures: What kind of learning, rather than how much. In his view, the main experimental variable in this study was the sequencing of information given to the subjects.

In this experimental study, Mayer and Greeno used two instructional sequences called Formula and Concept. These sequences were incorporated
in teaching booklets which facilitated the learning of the formula of
binomial probability with the notation:

\[ P(x = r/N) = \binom{N}{r} p^r (1-p)^{N-r}, \]

where \( P(X = r/N) \) is
the probability that a certain outcome occurs \( r \) times
in \( N \) trials.

The Formula sequence had the character of a set of instructions that
could be linked to a computer programme for finding the formula, while
the Concept sequence included more discussion of concepts.

For example, the two approaches to dealing with \( \binom{N}{r} \) were:

**Formula:** Instructions to finding \( \binom{N}{r} \) using a formula given in terms of
factorials and explanation of these terms.

**Concept:** The number of ways to get \( r \) successes in trials given as \( \binom{N}{r} \),
with an explanation of how to calculate the value of this
coefficient.

To conduct the experiment, two groups of subjects, one each for Formula
and Concept, were selected so that their prior knowledge and skills were
almost the same. They were tested mainly according to four categories
of test questions.

Examples of these items are:

(i) **Familiar:** \( N = 6, r = 3, p = \frac{1}{2} \). Find \( P(X = r/N) \)

(Ans. \( \frac{5}{16} \))
(ii) Transformed: \( p = \frac{1}{3} \), \( N-P = \frac{2}{3} \), \( N-r = 0 \). Find \( P(X = r/N) \).

(Ans. \( \frac{1}{3} \)).

(iii) Unanswerable: \( N = 2 \), \( r = 3 \), \( p = \frac{1}{3} \). Find \( P(X = r/N) \).

(No answer, \( N < r \))

(iv) Question: Can \( r \) be greater than \( N \)?

(Ans. No).

The test results are summarised in the following table, which indicates the proportion of correct responses by type of item and instructional method (Greeno, 1972).

<table>
<thead>
<tr>
<th>Instructional Sequence</th>
<th>Type of Item</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Familiar</td>
</tr>
<tr>
<td>Formula</td>
<td>0.75</td>
</tr>
<tr>
<td>Concept</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Table 4.1
Propotion Of Correct Responses By Type of Item And Instructional Method.

Mayer and Greeno interpret the results as showing that in the subjects who received the formula emphasis, the new ideas were assimilated as schemas involving calculational techniques, while for subjects receiving emphasis on general concepts, the new material was assimilated as ideas of a more general kind, involving the subjects' experience with random events.

Greeno (1972) was inclined to the idea that subjects of the Formula Sequence probably connected the material about binomial probability with their prior knowledge about arithmetic—multiplying, raising to powers
and so on - while subjects who received the Concept Sequence probably connected the material about binomial probability with their prior knowledge and experience with tossed coins and some general properties of physical systems with random outcomes. Greeno (1973) interprets these results in terms of the degree to which knowledge is "internally or externally connected". As a matter of fact, Mayer and Greeno (1972) illustrated these two terms thus: p and r are internally connected by the operation of exponentiation - p is some quantity that is raised to the rth power - while they are externally connected by the relationships that a subject realises between these variables and other concepts (which is to some extent similar to the usage of internal and external operations in mathematics). In this sense, Greeno (1972) reports that internal connectedness refers to the extent to which the components of a cognitive structure are connected to each other while external connectedness refers to the extent to which the components of a structure are connected to other contents of the person's mind. Hence they view, as a hypothetical structural property of acquired knowledge, that the components of a cognitive structure have to be connected (internally or externally), but that they can be connected more or less strongly. Hence, their interpretation is that subjects receiving the Concept Sequence tended to acquire structures with relatively strong external connectedness but rather weak internal connectedness, while the opposite held for the subjects receiving the Formula Sequence. Their interpretation was based on the idea that in simple mathematical structures connectedness between components may consist of the arithmetic operations that link the variables. They argue that if this is reasonable, then the emphasis on using the formula given in the Formula Sequence would produce relatively strong connectedness between the components of the structure, i.e. a strong internal connectedness.
On the other hand, the Concept Sequence gave less emphasis to
calculations and more to the meanings of various components, which means
that this would have produced stronger connectedness between concepts
in the new structure and other concepts the subject knew about, i.e.
strong external connectedness. Accordingly, they interpret that subjects
who received the Formula Sequence excelled on problems that were
essentially the same as those used during the learning period, and these
involved simple calculations involving the formula, while subjects of
the Concept Sequence excelled in answering questions about the formula.

4.1.2. II. The Ehrenpreis And Scandura Study 1974

While discussing curriculum construction in mathematics, Scandura (1972)
referred to 'higher-order rules' as useful for making predictions about
higher achievement and for saving time. A higher-order rule is simply
a rule that can be applied on other rules. The following examples might
help, before discussing the study.

If a student has learnt addition, multiplication, additive inverse and
possible multiplicative inverse of real numbers, then the rules of sub-
traction (A) and division (B) can be replaced by one higher rule (H')
as follows:

Rule A: \[ m - n = m + \bar{n}, \quad \bar{n} \text{ is the additive inverse of } n. \]

Rule B: \[ m \div n = m \times n^{-1}, \quad n^{-1} \text{ is the multiplicative inverse of } n \quad (n \neq 0). \]

High order rule \( H' \): \[ mn = m \times n^{-1}, \quad \text{provided that:} \]
(i) if \( o = - \), then \( * = + \) and \( n^{-1} = n \)
(ii) if \( o = \div \), then \( * = \times \) and \( n^{-1} = n \), \( n \neq 0 \).

The implications of this theme are also found in traditional mathematics.
where students are taught that the distance 'S' travelled by bodies moving under gravity is given to the following two rules:

\[ A^* : S = V_o t - \frac{1}{2}gt^2 \]  
if the body is initially moving upwards,

\[ B^* : S = V_o t + \frac{1}{2}gt^2 \]  
if the body is initially moving downwards,

But one rule \( \hat{H} \) can serve for both rules, i.e.,

\[ \hat{H} : S = \frac{1}{2}gt^2 \]  
with \( g = \mp 9.8 \) according to the case.

The following study was conducted by Ehrenpreis and Scandura (1974) and labelled "Algorithmic Approach to Curriculum Construction". Specifically, the approach is built on the assumption that each behavioural objective corresponds directly to a class of tasks that can be computed (solved) by applying a rule or algorithm. It was further assumed that curricula (i.e. "what is to be learned") can be represented in terms of finite sets of rules, including higher-order rules that operate on rules (Ehrenpreis and Scandura, 1974).

The authors conducted this study by using two curricula, D and H. D was characterised in terms of a list of discrete tasks and rules for solving these tasks, one rule for each task. It consisted of 303 tasks and rules.

H was characterised by reducing the rules in D, by means of higher-order rules, to 174 rules. H indeed included the higher-order rules and all curriculum D tasks and rules except those that could be derived by application of the higher-order rules to other rules found in D.

Evidently, curriculum D consisted of 303 tasks and rules, whereas curriculum H of 174 tasks and rules (Scandura, 1977).

The authors stated that the purpose of this study was to determine:
(a) whether making rules completely explicit provides a viable basis for instruction in the classroom and (b) whether the introduction of higher-order rules provides a suitable basis for improving the ability of students to transfer.

Furthermore, they hypothesised that (a) making rules explicit provides a viable basis for class-room instruction, (b) the H-Students would perform as well as the D-students on those curriculum-D tasks that had been eliminated from curriculum-H, and (c) the H-Students would perform significantly better on tasks not in curriculum D that could be solved in the same way. No difference in performance was expected on tasks included in both curricula. The first author was the instructor of both groups following curriculum D or H in the experiment. The post test consisted of 32 task exercises from each of the following categories:

(a) Tasks found in curriculum D and H;
(b) higher-order tasks found only in H;
(c) Tasks found only in D; and
(d) tasks that were found in neither D or H, but theoretically could be derived from rules found in curriculum H.

In discussing their results they find that:

(i) The D-Students were successful on post test tasks for which they had been trained, giving a mastery level of 94%. The number of test items on which each subject had been trained was estimated by examination of his classwork book. The H-Students were successful on lower-order tasks for which they had been trained, gaining a mastery level of 96%.
It would appear that making rules explicit does provide a viable basis for instruction.

However, on the higher tasks, the H-Students performed, as expected at a significantly higher level (71.5%) than the D-Students (53.4%).

It is interesting to note that the D-Students, although not trained on the higher-order tasks, did perform successfully 53.4% of the time. This would suggest either that some of the D-Students may have known these higher-order rules prior to the experiment, or that they may have been able to deduce them as they worked through curriculum D. In other words, some of the D-Students could move to level H without having learnt through H. In any case, the performance level of the D-Students did not approach that of the H-Students.

(ii) In tasks of transfer, as expected, the H-students performed as well on tasks found only in curriculum D as did the curriculum D students. The H and D students were successful in 88% and 93% respectively of the tasks. The difference between these two proportions is not significant.

In addition, the H-Students as predicted, performed at a significantly higher level than the D-Students on tasks beyond the scope of either curriculum. The H-Students were successful in 68% of the transfer opportunities, while the D-Students were only successful in 51.3%.

The authors find that these results clearly show that rules provide a viable and explicit basis for instruction and transfer. The curriculum H-Students not only had fewer rules to learn than did the
D-students, but they were also able to solve tasks that the other students could not. They suggest that the algorithmic approach should be given serious consideration in planning future curriculum development. They confined their study to certain parts of mathematics - as that found in this study - and state that:

"Further research is needed to determine the feasibility of the approach with other subject matter".

4.1.3. General Discussion I

Both studies in I and II confirm the relationship between the material contents and learning outcomes, although neither study discussed what structures might underlie the contents. As for the interpretation of the first study in terms of internal and external connectedness, Scandura (1977) comments that this explanation is simple and intuitively appealing. He further finds that the study presents an important advance over many earlier studies in the degree of emphasis it places on task variability; however, it has a major limitation in that the task involved does not provide the particular competencies necessary and sufficient for successful performance. Nevertheless, the following is worthwhile noticing.

(i) The Formula-students in the Mayer and Greeno study in table 4.1 responded correctly in Familiar and Transformed Items (75% and 57%), while the Concept-students responded correctly in 48% and 40% of these Items respectively. The contrary occurred in the Unanswerable Items (43% vs. 63%) and the Question Items (43% vs. 83%). In my view, the
cognitive structure was linked to the underlying structure of the Formula more strongly than to that of the Concept. The converse was true for the Unanswerable and Question Items. A central question which must surely arise is as follows: If the underlying structures of both the Formula and the Concept can be identified, then is it possible that the cognitive structure might benefit from both?

(ii) Both the D-Students and the H-Students in the Ehrenpreis and Scandura study responded correctly to the same degree in what had been learned through both contents. Furthermore, the D-Students were successful in 51.3% of the tasks beyond both D and H, whilst the H-Students did significantly better, since they were successful in 68%. This could be expected since the H-Students had experienced an abstraction and generality in H that was not available in D.

Generally, the H-Students who benefited from both the computational and comprehensical factors of mathematical knowledge (see Glossary) demonstrated this by exhibiting better performance in such items. This can be interpreted in terms of this thesis, that the efficiency (see Glossary) in H that more or less includes D, is higher than that of D. Furthermore, the performance of the H-Students may provide some answers to the previous question.

4.1.3.0. Conclusion 4.1

Cognitive structure appears to be affected by material content. The former may be said to reflect the latter. This implies a certain relationship between the achievement in a teaching-learning task and the concomitant material contents.
4.2. The Presentation-Phenomenon And Reality

4.2.0. Introduction

In conducting this study, I have always tried to ground my research in reality by deliberately studying what is happening in the classroom and how the presentation R of the task-contents takes over in a true teaching-learning situation. In this, I intended to learn more fully how R could be governed for a realistic modification. It was then believed that such aims could be achieved by tracing answers to the following questions: (i) If k is a mathematical task, then how can a presentation R be identified? (ii) Is such an R affected by the teacher, the textbook or some other factor?

In looking for relevant answers, I believed that personal experience, informal investigations and the reviewed literature could be helpful. It was found that in Kuwait the textbook of mathematics is a main governor for R; moreover, the reviewed literature revealed that the influence of the textbooks on teaching-learning processes cannot be ignored (as in 4.1.1 and 4.1.2). Moreover, although no one can deny the role of the teacher on these processes, the issues concerning effective teaching are still dubious (Begle, 1979). One point stands out firmly, the presentation R in the textbook is dominant in such teaching-learning situations and further processes.

In what follows, I will present a discussion of the presentation-phenomenon through three dimensions; the personal experience, the textbook and the teacher.
4.2.1. Personal Experience

The role of experience cannot be ignored in educational research. Perhaps the most primitive, and yet the most fundamental source of the solution to individuals' many problems in personal experience. Despite its limitation, it is a basic part of the foundation on which science rests. It is a prerequisite - a necessary, if not a sufficient condition to intelligent behaviour and a tool in the discovery of truth (Mouly, 1970).

In my work for 25 years as a teacher, senior teacher and inspector of mathematics in Kuwait schools, I experienced the following points that pertain to this phenomenon and prevail in teaching-learning situations:

(1) Teachers of mathematics, mostly follow the textbook as the main, if not the only, resource for presenting mathematical tasks.

(2) Teachers usually select the simpler examples that are found in the textbooks for illustrating mathematical behaviours that the task demands.

(3) Teachers generally ask students to solve the exercises or problems that are found in the textbook. They also demonstrate interest in examination items, which are usually very similar to those in the textbooks.

(4) An interesting point which is well known in the field, is that most teachers who teach two classes in the same grade, usually introduce the same presentation R of a certain teaching-learning task to both classes. Moreover, a teacher who introduces a certain presentation R of a task k in one academic year, usually introduces
This same R of the task to the same grade in other successive years as long as the textbook remains unchanged. This is very clear from the notes of teachers for the presentation of mathematical tasks in such successive years.

(5) Students usually cope with what is presented, i.e. with the textbook, as with some professional texts of the "know-how" sort. The latter are usually commercial texts, and are interested in the solution of the general examination questions, and in methods such as: "for this type of problem, see the following examples".

Furthermore, I indirectly gathered relevant information by posing the following questions: "How is a mathematical task achieved in the classroom?" to colleagues either in the staff of the Inspectorate of Mathematics, or the teaching staff in different secondary schools. I did not explain my intention in asking the question. Here are typical answers from those I gathered.

(1) Inspector: Teachers quickly mention the concept as it is stated in the text, then they shift to the instances - examples, exercises and simple problems concerning the behavioural objectives of the task that are found in the text. Complicated problems are usually given as homework.

(2) Inspector: Teachers usually do not attempt to expand their knowledge out of the texts or examinations.

(3) Inspector: Look! I once contributed to writing a textbook in mathematics. A number was mis-printed in one of the problems in the
text. The correct number was 0.3 and the misprint 3. Though the results from the number 3 do not work for the problem, some teachers in one secondary school interpreted these results in an unbelievable situation, since the text is 'sacrosanct' and therefore it should be followed literally.

(4) Teacher: It is easier and safer to follow the text. Why shouldn't we do that, since the Ministry of Education asks for it?

(5) Teacher: The textbook is my guide and my only source. It is well-designed since the Ministry of Education usually asks the best minds to contribute in writing such texts.

(6) Teacher: I have no time to look for sources other than the textbook. I also believe that I cannot present better than the text presents.

(7) Teacher: The examinations are compatible with the material contents in the texts. I want my students to pass, and therefore the text is the best.

If we realise that, for every grade in the Kuwait school, there is only one authorized textbook which is prepared under the full responsibility of the Ministry of Education, we might derive from the preceding statements the following conclusion, which depicts current teaching-learning situations of mathematical tasks in Kuwait secondary schools.

4.2.1.0. Conclusion 4.2.

In the area of the study, the textbook is a main governor of the contents of mathematics teaching-learning tasks. Therefore
it is reasonable to equate the contents of the task with the relevant contents of the textbook.

The conclusion denotes how far the textbook is dominant in the teaching-learning processes in Kuwait secondary schools. It is natural to read more about this situation in the literature.

4.2.2. The Textbook

The textbook here means the text that is written to be used by students. In Kuwait, there is only one textbook for each grade in school, and this text is used by all the students in the same grade.

The following study confirms the phenomenon through the textbooks, clearly showing that different texts entail different achievement. Besides that, it also shows how great is the influence of the text on teaching-learning situations in different countries.

In connection with the question: "Would an outline to emphasise concepts and syllabus be sufficient for teachers to achieve mathematics?" It was found that an affirmative answer to this question was considered wishful thinking by a number of mathematics educators since it gives more freedom to teachers to teach what they believe to be appropriate to their students, apart from what is in the textbook (Begle, 1979). But, as it happens, it has been found that these mathematics educators are mistaken, and a study by the NLSMA (reported by Begle, 1979) concludes that there is considerable evidence that mathematical topics and mathematical emphases included in the textbooks do get through to the students.
The analysis of the previous study was based on studying the achievement of two groups of students, $G_1$ and $G_2$, where $G_1$ followed three conventional textbook series for grades 4, 5, and 6, while $G_2$ followed the texts of the SMSG series for grades 4, 5, and 6. The conventional texts emphasised computation more than the SMSG texts, while the latter emphasised understanding and concepts more than the conventional texts did.

Students, at the end of grade 6, who entered grade 4 at the beginning of the NLSMA study, were tested in four topics: (i) multiplication of fractions, (ii) division of fractions, (iii) whole number structure and (iv) algorithms. The test was designed according to two main scales: (a) for (i) and (ii), the scale was based on cognitive level of computation; and (b) for (iii) and (iv), the scale was based on cognitive level of comprehension. The latter scale indeed measured understanding of both the properties of whole numbers and the rationales of computational algorithms.

As a result of this study, Begle (1979) who had contributed to it, reported that, students in $G_1$ who used the three conventional series scored higher than students in $G_2$ who used the SMSG texts in (i) and (ii), while students in $G_2$ scored higher than the conventional students in $G_1$ in (iii) and (iv).

Begle (1979) found that differences in emphasis between the two textbook series were clearly reflected in student achievement. And this indeed goes in line with the Mayer and Greeno (1972) study. Furthermore, he concludes that textbooks do have an effect on students' learning at secondary and elementary level, and the content differences in textbooks resulted in differences in student achievement.
This phenomenon of the textbook reflects what has long been known by people involved in mathematics teaching-learning situations such as Shubring (1980) who stated that:

"textbooks are doubtlessly a bottle neck, although there are quite a number of textbooks ... but not comprehensive ones".

The real awareness might come from the fact that (i) the textbooks are very influential and (ii) the mathematical goals in these textbooks are not achieved by a genuine co-operation among mathematicians, educators of mathematics and psychologists working on mathematics learning.

Oakes (1965) and Harding (1968) in turn have shown how great the influence of mathematics educators - together with classroom teachers - is on the goals of mathematics in the professional literature. The BSPLM in Britain could be an adequate movement to provide grounding for such co-operation between psychologists working on mathematics learning and others in the hope that they would reflect this co-operation in the texts.

Begle (1979) who recognizes the situation, asks that:

"State departments of education, in some of our states, can and do exert an even stronger influence on mathematics education goals by specifying what textbooks may be used in the public schools" (P.10).

He believes that legislation can also exert influence, since he finds that a substantial number of mathematics textbooks, particularly in secondary schools, are written by mathematics educators and:
"textbooks are a powerful, if somewhat indirect, influence on mathematics education goals" (P.9).

However, if many teachers slavishly follow the textbook (Oliver, 1977) then we can perceive the size of the situation in the phenomenon of the presentation of mathematical tasks in the textbooks.

In this connection, Skemp (1971) observes, having presented two principles for proper learning; that the textbooks break the first of these, intrinsic to learning mathematical concepts, which states that "concepts of a higher order than those which a person already has cannot be communicated to him by definition, but only by arranging for him to encounter a suitable collection of examples". He observes that:

"The first of these principles is broken by the vast majority of textbooks, past and present. Nearly everywhere we see now topics introduced, not by examples, but by definition: of the most admirable brevity and exactitude for the teacher (who already has the concepts to which they refer), but unintelligible to the student" (P.32).

Deterline (1971) reflects indeed our concerns about the role of the textbook which is both essential and crucial by recommending that the textbooks in his country should be prepared by:

"the joint committee of the American Educational Research Association, the American Psychological Association, and the Department of Audiovisual Instruction of the National Educational Association."
4.2.3. The Teacher

The situation in the following discussion can be understood in what is embodied in the two-fold question: "Do teachers differ in the presentation of a definite mathematical task?" and "How can we evaluate teaching effectiveness?".

No doubt, the role of the teacher cannot - indeed should not - be ignored. And the "good" teacher is one of the aspirations of mathematics education. Unfortunately, this dream is beyond our reach at the moment. There are good teachers, we hear of them, we know some of them, but the issues state that we cannot either select or create them using scientific means. The difficulty in creating good teachers can be found in the specifications for a good teacher proposed by Wardhaugh (1968), (P.3-71).

In my own experience, I found that presentation of mathematical tasks is almost the same for most teachers, who usually or completely depend on the presentation in the textbooks. However, some "good" teachers (although we cannot give a scientific definition of them) may well vary from the textbook presentation.

Not all teachers are creative, and indeed the majority are not, especially in developing countries where the recruitment of teachers of necessity does not follow logical or sensible rules. Moreover, the problematic debate about 'mastery' and 'minimal competency' for a mathematics teaching-learning task does not provide teachers with sensible guidelines to help them cope better with their job.

Since the "good" teacher in undefinable, we cautiously would look to
Pophan (1970) who recommends that a curriculum should be an 'end' and the teachers should only be a 'means'. In connection with the selection of good teachers, Begle (1979) states that:

"we have to admit that we do not at present know of any way of selecting, in advance, the effective teacher" (P.54).

He goes so far as to call for the abandonment of discussion in this arid field. But Marques et al. (1979) in turn, still hope that the goal of future research will be the more adequate measure of this construct since as they claim:

"The present findings suggest the need for a more eclectic approach to the evaluation of teaching effectiveness."

4.3. General Discussion II

The study presented in this chapter shows that the presentation-phenomenon can be in line with the first Proposition 6.1 in the sense that there could be differences in achievements under relevant changes in the contents or presentation in the context of this thesis. Whilst agreeing with Mayer and Greeno (1972) study, who consider the main variable in their work to be the sequencing of the information given to the subjects, I in contrast suspected the main variable to be the underlying structure of the presentation. I explicate my meaning of "underlying structure" in Proposition 6.4 and what further stretches from it. Nevertheless, I find the second study of Ehrenpreis and Scandura (1974) to be clearer in relevance to the idea of the underlying structure of the contents.
This can be more or less shown in construction of D and H curricula as well as in the theme behind them. Whatever the differences and similarities of emphasis between my ideas and theirs, I believe we meet in the two statements (a) and (b), (P. 4-3), though I tried to identify how differences in contents may clearly occur in order that compatibility may hold between the presentation-phenomenon and Proposition 6.1. Furthermore, the dubious state of the effectiveness of teachers gives more reasons for focussing our efforts to improve the contents identified by such an R to improve the mathematics achievement of students. This trend could also meet with Salomon (1979) who confirms the role of the learning material contents by claiming that:

"Learning seems to be more affected by what is delivered than by the delivery system."

In this connection we might understand Al-Dhahir and Yaseen (1980a) who have stated in their report "Concerning Mathematics in Kuwait" to ICME that:

"What we believe that we might be able to do, is to spot the main problems that teachers usually encounter ... Our experience in the field has convinced us that there are two main points en route to reform. They are the school textbook and method of testing ... As for the textbook, we think that it is not well-built on research and objectives together with basic consideration regarding graded introduction of concepts and examples."
4.3.0. Conclusion 4.3

The presentation-phenomenon can be accepted as a starting point in the sense of Proposition 6.1.

4.4. Summary

The study in this chapter introduces in the preamble the relationship between achievement and material contents based on prior works arrived at by others. It considers the study in this chapter as a natural extension of the literature review in Chapter 3. Despite that it was submitted in a single Chapter to emphasise and clarify the importance of the role of this relationship. In summarizing the relevance of this phenomenon in other chapters, it was reported that the indications that different learning material contents affect the type of the question that students are able to answer, go in line with the findings of Mayer and Greeno (1972). These findings have affected direction in the proposition stage, which in turn has affected execution of the study. Furthermore this relationship helped in assuming the achievement \( A \) as a functional relation \( A = g(x_j) \) where \( x_j \) is assumed to be in relation with the material contents. The (material) contents of a task 'k' was viewed to constitute two components (i) a theoretical component which was argued to be invariant, and (ii) a set \( R \) of instances called the presentation, e.g. examples or exercises. It was further argued that differences in the contents are due to true changes in \( R \). Hence the presentation-phenomenon was stated in that "differences in the achievement \( A \) of a teaching-learning task \( k \) could be induced by differences in the presentation \( R \) of \( k \)." In studying
the variation of $A$ under variation of $x_j$ in $A = g(x_j)$ it was hoped that this study should identify $x_j$ in terms of $R$. It was also denoted that argument in 6.2.2 shows that a true change in $R$ should be related to the structure underlying $R$. In this context, the two Propositions 6.1 and 6.4 were stated. Proposition 6.1 suggests that a starting point in the study could be found in a phenomenon that indicates a change in the achievement under a relevant change in a teaching-learning situation, while Proposition 6.4 assumes the existence of such a structure underlying $R$ and another structure supposed to contribute in modification, as well as further suggesting the use of theory in modification. Hence it was viewed that $x_j$ constitutes $R$ and the structure, say 'S' underlying $R$ i.e. $x_j = R(S)$.

In further discussion of the phenomenon, it was reported that subjects who learn with different procedures (i.e. different presentations in the context of this thesis) produce different responses in the process (Wittrock, 1963). Scandura (1977) comments that, what is learned in making simple discoveries could be achieved in expository form and that contents should be a primary variable in future studies. The presentation phenomenon is indeed due to the remarkable works of Mayer and Greeno (1972) as well as Ehrenpries and Scandura (1974).

(I) Mayer and Greeno used two instructional sequences called Formula and Concept for learning the formula of binomial probability. The main variable in their study was the sequencing of information. The Formula had the character of set instructions that could be linked to a computer programme while the Concept Sequence included more discussion of concepts. The two groups in the trials were tested on the basis of four categories: Familiar, Transformed, Unanswerable and Question. The first two could be considered as more likely to be linked to the
Formula, while the other two were more likely to be linked to the Concept. The two groups responded relevantly strongly to the nature of the contents they had learnt.

(II) Ehrenpries and Scandura used two curricula D and H where H was based on the basis of the notion of higher order rules which operate on other rules. This study also found that learning outcomes link to what is taught. The study demonstrated that the H-Students could work better than the D-Students in items beyond the two curricula. This could be interpreted, in the terminology of this thesis, to be due to differences in the efficiencies of the structures underlying D and H.

In addition, the presentation-phenomenon was considered in relation to experience through discussing the real situation in the classroom. This examined personal experience, the textbook and the teacher. It was found that the presentation of a task found in the textbook was the prevalent one in teaching-learning situations. Teachers find that following the textbook is helpful and safer. Moreover, it was found that the textbook has great influence on learning outcomes. In addition, it was found that a measure of effective teaching is still far off. Yet, the presentation-phenomenon remains essential, in that the variation of contents can lead to further developments. It was finally concluded that the presentation phenomenon can be helpful as a starting point in the sense of Proposition 6.1.
CHAPTER FIVE

SPACE OF KNOWLEDGE: A CONCEPTUAL STRUCTURE
5. SPACE OF KNOWLEDGE : A CONCEPTUAL STRUCTURE

5.0. Introduction

This chapter was conceived as comprising a certain theoretical framework having roots in the fundamental propositions of Chapter 6 as well as in 7.3 of Chapter 7. As a matter of fact, I believe that theory is invaluable for any development in the sense that such development could rest on more reliable and systematic situations. In this connection, I share the views of Wardhaugh (1968) who believes that good teaching practice is based on good theoretical understanding and that there is indeed nothing so practical as good theory. In this sense, one can deal with my views on developing the following three fundamental propositions that are established in Chapter 6 which I state here.

(Proposition 6.2): There exists a theoretical framework on which a development in current teaching-learning situations can rest.

(Proposition 6.3): If a proposed development (reform) in current teaching-learning situations (a) could rest on a theoretical framework and (b) can be operationally applicable, i.e. well-defined and learnable, then such development is expected to be experimentally supported.

(Proposition 6.4): (i) There exists a well-defined structure 'S' underlying current presentation of mathematical tasks. (ii) There exists a structure 'H' that is possible to
contribute in underlying modification of current presentation.

(iii) Theory in Proposition 6.2 could possibly identify a certain discrimination of the efficient knowledge that achieved in 'S' and 'H'.

In achieving such aims, it was preconceived in 6.2.2 that if a number 'A' could be assigned to match with the achievement of students in a certain mathematical task, then this 'A' could be a function: $A = g(R)$ (P.6-40), where R involves the presentation of the mathematical task as well as a structured knowledge S underlying it. Simply; R consists of certain instances found in examples, exercises and simple (routine) problems in the material contents of the task, put there in the expectation that those instances would enhance and reinforce mathematical knowledge that is relevant to the task. Two examples of such R are found in appendix (C).

In this connection, a development in achievement was taken to be a modification of the methodology within the presentation R which would result in improving the achievement A. It was then argued that a true change in R, for modification could be found by the contribution of another structure such as 'H', given that such contribution should be based on theory described in the three previously mentioned fundamental propositions.

Furthermore, it was argued in 7.2.2.0 that a structure, such as 'S' has certain limitations in diffusing mathematical knowledge, despite such an S being indispensable for any modification. However, H taken as a whole would be very complex and a part, say 'U' of H could help in contribution in modification. Such optimization arises from studies of the two indicators in Chapter 7.
A first indicator, which discussed the transferability of computational knowledge, i.e. transferring successfully certain available competency in one item to identical one has indicated that the diffusion of knowledge related to S for transferability was not necessarily totally satisfactory, whilst the potential diffusion of knowledge related to H has demonstrated better perspective for such transferability as from H-Maths into H-Physics in 7.2.0. A second indicator which discussed creativity i.e. problem-solving, would seem to demonstrate a higher correlation for H than for S—7.2.1.

A conception of 'a space of knowledge' as that initially discussed in 7.3, was then formulated in which teaching-learning situations possibly rely on a certain structured knowledge 'W'. This is identified by a set of initial (simple) states of knowledge that are supposed to diffuse a required knowledge within a certain normal population O, under certain interaction between W and O. Such W could have a limitation such as that in S, while such H could diffuse knowledge related to transferability or creativity more efficiently than S. The question that then arose was how to discover how such a construct as 'diffusion of knowledge' could be indicated by a measure helpful in a modification. It was then preconceived the efficiency could probably indicate how far the knowledge has been successfully diffused for further purpose. Simply, the efficiency of the achievement of the knowledge relevant to a problem P, is the probability that an individual is able to solve P, provided that the individual has satisfied certain competency in a set $X_0$, which forms a test for necessary competency for P. Consequently, it was proposed that if W is related to a certain teaching-learning situation, then W could contribute to a modification if the efficiency in W related to a problem P was greater than the corresponding efficiency in W for P.
Here, I present a theoretical framework that is based on conception of space of knowledge in the hope that this would both provide the relevant theory for this study and benefit further fields of study.

5.1. Some Aspects Of Theory

A child usually learns his own language through certain interaction within the environment. He learns the language through interaction with simple states of knowledge that help him grasp and comprehend the underlying structure of the language. He consequently begins to state "I play" and "he plays".

It was viewed that such learning of language is achieved within a space of knowledge that is preconceived to constitute a basis of knowledge where initially; simpler states of knowledge are employed to diffuse certain relevant knowledge within the child through his interaction with these states of knowledge. Furthermore, it was preconceived that such a space of knowledge also has physical effects as well as emotional ones. For example, a child in Wales can easily pronounce a certain alphabet, while another child in England will find that physically difficult. An emotional effect could also be comprehended in the rhythm of a line of poetry apparent to an individual in his language, but not to another foreign to the language. An interesting example of this was one girl I knew who had an Arabic father and English mother and had spent almost all of her first six years in England. She used to add the English ending "ed" to Arabic verbs to denote the past. As a matter of fact, no one could have told her about this ending "ed" in the English language, since such terms as 'present' or 'past' would have been beyond her understanding, and this ending "ed" has no equivalent meaning in Arabic. This girl could indeed comprehend to
a certain extent the structure that underlies the English language and besides; this structure was dominant in transferability to the other language. The question arises as to why the space of knowledge of language is more efficient than that of mathematics; in the sense that almost everyone learns a language while many fewer are proficient in learning mathematics?

To such a genuine question, I do not imagine at the moment, we can find a plausible answer, even though it is worthwhile discussing such a space of knowledge of mathematics with which we, the teachers, and students cope. Firstly, it is essential to note that mathematics learning has not in general two distinct spaces of knowledge, an in-school and an out-of-school one, as other sciences might have. For example, a child learns about force within two spaces of knowledge, the in-school space of physics classes, books, etc. and the out-of-school space, i.e. an untaught space of knowledge such as that of the language.

The two spaces might conflict and consequently this could give rise to "blocks" in learning. In this connection, Viennot (1979) states that:

"We all share a common explanatory scheme of "intuitive physics", which, although we were not taught it at school, represents a common and self-consistent stock of concepts and which, however wrong it may be, resists attempts to change or modify it."

Lovell (1981), in this sense reported that A-level or equivalent students of physics as well as university undergraduates and graduates are often pre-Newtonian in their thinking. For example, force is associated with velocity rather than acceleration. Overcoming these misconceptions and
alternative ways of explaining events becomes a difficult task for the teacher.

These indicators suggest how far a certain space of knowledge can be influential and dominant. Thus as Viennot (1979) finds, that if this intuitive physics is to be replaced or overcome, a major teaching exercise is needed, which goes far beyond the conventional teaching of the Newtonian scheme alone.

Fortunately, mathematics is not subject to such out-of-school space of knowledge. But it was found that there exists a space of knowledge 'S' identified by a structured knowledge S that provides this knowledge. Simply, S consists of the set of all instances that do not contain a constant say 'a' (see 7.1 of Chapter 7). However, it was also found that such space has also side effects such as:

(i) Insufficiency: S is not satisfactory with regard to transferability of computational knowledge in mathematics or non-mathematics fields (as in the first indicator 7.2.0 of Chapter 7), and, further with regard to creativity (as the second indicator 7.2.1 demonstrates) S correlates less with problem-solving than another space H.

(ii) Influential dominance: S may not be of help to students in coping with situations which have a form different from that form that they are familiar with. For example, I asked a group of 15-16 year old Kuwaiti secondary school children, two colleagues in the Inspectorate of Mathematics in Kuwait, a recently graduated engineer, and PhD students in the University of Surrey to do the same test. They were given two equations and asked to write them on the same line if both of them have similar easiness,
otherwise they were asked to write the easier in the first line and the more difficult one below the first. The equations first:

Solve for x: \(2x + a = 5\) (a being constant) and \(2x + 1 = 5\).

Surprisingly, all of them without any exception, chose \(2x + 1 = 5\) to be the easier one. And in asking for reasoning of choice, all of them found that the existence of such 'a' led them directly to choose the second to be the easier. Furthermore, they were surprised when it was explained the first was arguably easier since they did not have to trouble to calculate the difference \((5-a)\) or the quotient \(\frac{5-a}{2}\) as they had to calculate \(5-1 = 4\) and \(\frac{4}{2} = 2\).

I think that this influential dominance could interact with this insufficiency to result in the limitation of dependence in learning on only one structure. In this connection, if we notice that non-mathematical fields such as physics are likely to involve the use of constants, we might understand why students cannot always respond properly in such fields, whereas they are said to be competent in mathematics as shown in the first indicator in 7.2.0.

Another example of knowledge being diffused through different spaces of knowledge can be found in the work of Ehrenpreis and Scandura (1974) reported in Chapter 4. Here the D-students and the H-students were successful in 51.3% and 68% respectively of the exercises that were beyond their direct field of knowledge, despite these exercises having roots in what both groups had learned.

In Chapter 7 of this study, I could indeed discriminate (in 7.1.2 of Chapter 7)
two structures S and H (other than the H-curriculum reported in Chapter 4) that could constitute two spaces of knowledge. S is dominant, while H, although more related to non-mathematical fields, is almost abandoned. Moreover, it was found under precise conditions that \( S \cap H = \emptyset \) "the null set" (P. 7-9).

The theory in this chapter is expected, therefore, to throw light on a space of knowledge as a conceptual structure, and on how we might employ this so that teaching-learning situations may benefit from this concept.

5.2. Space Of Knowledge

It was argued in (P.6-37) that a mathematical task 'k' usually is based on (i) a theoretical part e.g. a definition or a proof of a theorem as well as (ii) a set \( R \) that constitutes initial states of knowledge identified by examples, exercises or simple problems. \( R \) is called the presentation of \( k \). It is designed to enhance and develop a better understanding of \( k \) and it is usually supposed that a student who can deal successfully with \( R \) will be competent in using the relevant knowledge in \( k \) in further related situations, e.g. solving a problem. Mathematics is known to be a structured knowledge in the sense that it can be reduced to the three well-known structures, algebraic, order and topological - identified by the Bourbaki school. Hence, it was viewed that school mathematics is also structured in a broader use of the word "structure". For example, it can be said that the presentation \( R \) is dependent on a set \( S \) of instances such that each \( S \)-instance does not contain a constant 'a' of implicit form, while there is another set \( H \) such that each \( H \)-instance contains at least one constant of implicit form. Hence \( S \) constitutes a certain structured knowledge under the relevant rule, while \( H \) constitutes another one. (see 7.1).
Here is an example of how students deal with a task \( k \) and what is expected from them.

Let the initial states of knowledge for \( k \) be identified by the following and similar instances:

(i) Find \( \frac{d}{dx} (x^2 + 2x) \)

(ii) The slope of the tangent line at \( x_0 \) is \( f'(x_0) \). Find the slope at \( x = 2 \) for \( f(x) = x^2 - x \).

(iii) If \((2,y)\) belongs to a curve \( f: f(x) = x^2 - x \), then find \( y \).

A student is then expected to solve the following problem \( P \): If the line \( n: y = h - 2x \) is tangent to \( f: x \rightarrow x^2 \), then find \( h \). It is preconceived that \( S \) in the sense of 7.1 would eventually diffuse a certain knowledge through such initial states of knowledge within the students such that they would become able to solve \( P \).

As a matter of fact the first indicator in 7.2.0 would suggest that in simple states of knowledge ability in \( S \) is not necessarily transferable to a similar situation in \( H \). In other words, the structured knowledge in \( S \) could not diffuse the attained knowledge into \( H \). This would mean that diffusion of knowledge in \( S \) is limited. Furthermore, the structured knowledge is dominant, in that one might not easily skip to another structured knowledge, as shown both in the first indicator (7.2.0) and in what Viennot (1979) reports about the 'intuitive physics'.

Based on the foregoing argument, I tried to formulate a space of knowledge as a conceptual structure. This conception simply views that knowledge is
structured into $W$ through a finite set of rules such that $W$ diffuses this knowledge within a certain population 'O' through interaction between $W$ and $O$. Hence; the following definition:

5.2.0. Definition 5.1

A space of knowledge is an ordered pair $(W, O)$ where:

(i) $W$ is a set of structured knowledge in a specific field that is identified by simple (initial) states of knowledge under a certain finite set of rules that constitute an underlying structure of $W$.

(ii) $O$ is a certain normal population that is supposed to acquire particular knowledge in that field, through the knowledge that is diffused from $W$ within $O$ under source of interaction between $W$ and $O$.

Consequently, it was viewed that learning would happen through interaction between $W$ and $O$. An example is seen in 7.3, where the space of knowledge $(S, O)$ constitutes current mathematics teaching-learning situations. The processes of interaction are preconceived to help diffuse a certain knowledge of $W$ with $O$, thus developing changes in behaviour for acquiring of learning.

In addition, I believed that the conception of space of knowledge could be useful in modification, if it was possible that we might develop our knowledge about the advantages in a space of knowledge. To simplify the use of symbols, I shall refer to a space of knowledge $(W, O)$ by 'W' only, provided that the population 'O' is clearly identified. Furthermore, I believed that modification of a space of knowledge should be based on studying the advantages of the achievement in that space, as discussed in 6.1.2.0, by observing how far the knowledge diffused from $W$ within 'O'
is efficient. Here efficiency was simply taken as a measure that indicates probability that an individual is able to solve a problem $P$ if he can demonstrate a proper ability in all elements of a set of stimuli that test attainment of the knowledge necessary for $P$.

I therefore found that I needed to achieve the following two general objectives:

I: To identify the employment of problem-solving in the study.

II: To identify such substantial properties that could improve efficiency in preserving a certain reliability for prospective modification.

But, before continuing this argument, I should like to emphasise that this study is only concerned in the improvement of the achievement of students in mathematics whereas this achievement is basically identified by simple behaviour that constitutes a basis for achieving learning tasks, and not by improving abilities in problem-solving. However, I will try to benefit from the area of problem-solving in discussing this improvement in achievement. Nevertheless, I believe that the area of problem-solving stands as the most essential area in mathematics education, although it is far reaching and beyond the scope of this study.

5.2.1. I. Problem-Solving

5.2.1.0. Introduction

Despite the importance of this area, it is acknowledged that much can be written or said about it, with very little gain. The fact is that a
variety of studies can be found in this area ranging from Polya (1948), Wickelgren (1974) and Rubistein (1975) who emphasise, for scientific problem-solving, the acquisition of general skills through interdisciplinary courses, to Feyerabend (1975) who suggests that creative scientists do not follow well-defined methodologies but rather generate methods appropriate for each new situation. Lochhead (1979) who was influenced by Feyerabend claims in his turn that progress in this area is not necessarily dependent on a well-defined methodology and that epistemological anarchism has advantages too. Furthermore, the intricacy in this area has led Perry (1970) to think that the transferability of general problem-solving skills depends on seeing similarities among superficial situations, although those situations might be more appropriate for college students. In connection with this view, Schoenfeld (1979) views that the ability to select useful problem-solving strategies depends on having a sufficient domain of specific knowledge and therefore he recommends that teaching problem-solving techniques should be at the highest grade of college. These last views which suggest that students learn late - if at all - of such techniques lead many to be sceptical about the existence of general skills and consequently they do not take general interdisciplinary skills seriously (Lochhead, 1979).

But, whatever the difficulty that exists, the ability to solve a problem stands as the most valuable profit of learning mathematics (Polya, 1962). This indeed is in the line of the Commission on Secondary Curriculum (1940) who found problem-solving as the basic raison d'être for mathematics in the secondary school. In this connection, Thorsland and Novak (1974) believe that problem-solving is basic for developing an understanding of the processes as well as the content of science, and also that:

"a better understanding of the role of problem-solving abilities
in the learning process would enable one more fully to comprehend the nature of learning."

But despite the importance of this area, Lovell (1981) finds that problem-solving seems more messy, more complex than we would like, and, alas cognitive psychology so far cannot provide a great deal in this area. Furthermore, Lovell reports that general theories of problem-solving such as information processing models, the Gestalt approach, or Stimulus-Response theory are somewhat sketchy, incomplete, or vague and do not offer much help.

5.2.1.1. Sufficiency In Problem-Solving

Scandura (1977), in discussing the area of problem-solving raises the question: Why is it that some people can solve a given problem for which they have all the necessary competent skills, whereas others cannot? This, as he finds, has puzzled scholars almost from the beginning of civilization. The statement of Scandura is much like the common saying: Why is it that some students can do mathematics, but not physics? This was indeed the question discussed in the first indicator $T_1$ in 7.2.0. It was then noted that students do mathematics in a space of knowledge 'S', but in some cases most students could not transfer their mathematical knowledge in 'S' to another space 'H' whether in mathematics or physics. In my view, the question that could be related to Scandura's is: What criterion should one use to identify whether an individual has or has not the necessary competency skills for the problem? More precisely, the major question should be: If $P$ is a well-defined problem, then is there a criterion $X$ which could indicate competency which will satisfy $P$? However, this question is a long standing one, and answers to it would also answer the eternal
search to discover if it is possible to teach problem-solving. But, could such an 'X' exist? ... be identified? The answer is not easy, but although X might not be identified, we do not have any evidence against the existence of such X, and therefore the following axiom is possible.

5.2.1.1.0. Axiom Of Sufficiency 5.1

If P is a well-defined problem, then there exists a certain criterion 'X' that can induce sufficiency for P.

In connection with the axiom, it was also my view that such X would satisfy a set of properties, of which the following three properties are believed to be elements:

(X1) Consistency: in the sense that X provides an algorithm (set of rules) that defines a certain $X_0(X)$ that tests the necessary competency for P.

(X2) Uniformity: in the sense that someone who is capable of P and demonstrates comprehension of such ability should be expected beforehand to be completely capable in $X_0(X)$.

(X3) Completeness: in the sense the competency in $X_0(X)$ implies competency in P.

It is to be noted that in X1 the type of behaviour in $X_0(X)$ is expected to be generally simpler than the more complex behaviour in P. Also X2 and X3 could play the role of the inverse to each other. The uniformity in X2
implies that a student who is both capable of $P$ and able to demonstrate full understanding of what he is performing, will inevitably be able in $X_0(X)$. This does not imply that all of those who are capable of $P$ are also able in $X_0(X)$ since there could be some individual who has known this or a similar problem without obtaining the necessary understanding of the knowledge involved. Also, there could be those instrumental operationals who can perform without having the corresponding understanding. Finally, $X_3$ could be the most powerful property in the sense that it pre-determines any transfer of ability in $X_0(X)$ to $P$. This property indicates that if a student is competent in $X_0(X)$ identified by $X$, then the probability that he will be able in $P$ is 1 i.e. the ideal probability. The inverse might however be untrue, because of possible instrumental operationals.

5.2.1.2. The Problem-Space

Greeno (1976) reported that all problems given in high school mathematics are well-defined; in the sense, that they present a specific set of premises or data to work from. Accordingly, all the problems that I will discuss in this study are well-defined whether this is specifically stated or not.

In this connection, the term behavioural objective is usually used to identify the desired behaviour to achieve a relevant objective. But some behavioural objectives could be considered to be more necessary or relevant than others. These will be called basic and are better identified as follows:
5.2.1.2.0. Definition 5.2

A behavioural objective concerning a mathematical task is defined to be basic if a panel of people who are interested and involved in the field accept it as such.

It is to be noted that it was intended that the definition should be loose so that this would help the panel to choose such basic behavioural objectives without any pressure of predetermined criteria.

Based on the previous definition, we might reach the following definition:

5.2.1.2.1. Definition 5.3

If P is a well defined problem, then the problem-space of P is defined to be the ordered triplet \((D, B, X)\) where:

(i) \(D\) is the set of the data and specific goal(s) of P.

(ii) \(B\) is a set of basic behavioural objectives, necessary for P.

(iii) \(X\) is the criterion for P in the sense of the axiom of sufficiency.

\(X\) was introduced as a third coordinate in order to maintain the essential role of such \(X\) in any further studies. In this study, it was intended to look for an approximation for such an \(X\) to help the theoretical development of this study.

5.2.1.3. A Problem-Kernel

This term rises from what commonly held, that a student, who is to be
exposed to a problem P, should firstly be able to demonstrate his competency in the necessary mathematical knowledge related to P. Simply, a kernel is a set of simple stimuli which test the attainment of such necessary knowledge. A definition of such a kernel will be given after the following definition.

5.2.1.3.0 Definition 5.4

If \( W \) is a space of knowledge and \( B_i \) is a well-defined behavioural objective, then, the class \( B_i(W) \) is defined to be the set of all stimuli in \( W \) such that each stimulus is governed (satisfied) by \( B_i \), i.e., \( B_i(W) = \{ w : w \text{ is a stimulus in } W \text{ that is uniquely governed by } B_i \} \).

Any \( w_j \) in \( B_i(W) \) will be called a representative for the class \( B_i(W) \) or more simply, a representative for \( B_i \) in \( W \). Furthermore, a behavioural objective is well-defined as long as it admits a fair choice for any representative in the sense that, the behaviour in a certain representative does not depend on the choice of this representative and could be consistent for two representatives.

5.2.1.3.1. Definition 5.5

Let \( W \) be a space of knowledge and let:

(i) \( P \) is a well-defined problem in \( W \);

(ii) \( B \) is a set of well-defined basic behavioural objectives necessary for \( P \);

(iii) \( B_i(W), \ldots, B_n(W) \) are the classes \( B_i(W) \) for all \( B_i \) in \( B \);

Then if we choose only one representative \( w_i \) from each class
The set of all those representatives will be defined to be a kernel for $P$ in $W$ under $B$, and will be denoted as $W_0$.

In set-notation $W_0 = \{w_i : w_i \text{ in } W \text{ uniquely governed by } B_i \forall B_i \text{ in } B\}$, which means that a kernel tests the attainment of the knowledge rising from the basic prerequisites of $P$ and identified by $B$.

The definition indicates that a kernel $W_0$ is not unique since it is dependent on both the choice of representatives and on $B$.

Figure 5.1 illustrates the definition.

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Figure 5.1: Illustrative Diagram Of Choice Of A Kernel $W_0$ In $W$ Regarding A Well-Defined Problem $P$. 
5.2.1.4. A Conceptual Process In Problem-Solving

It was believed that on the basis of the foregoing one would be likely to present a conceptual path to conceptualising how a student undertakes problem-solving. As a matter of fact, the chief difficulty in studying the processes in problem-solving is that it is a private activity inaccessible to observation, and all that we can observe is to be found in the results of the processes rather than the processes themselves (Wason and Johnson-Laird, 1968). So in this sense one can more fully understand Scandura (1977) who states that relatively little is known about how to teach people to solve problems, or how to programme computers to do so. Scandura continues arguing that, in regard to cognition, it is agreed that a successful problem solver should be able (i) to understand the problem, (ii) to identify suitable subgoals of the problem, (iii) to retrieve relevant information from memory, (iv) to derive a solution procedure for each subgoal, and (iv) to carry out these procedures correctly and verify both the intermediate and terminal results. But, beyond these general outlines, he finds that, there is relatively little agreement on the actual nature of the underlying processes. What does it mean to say that a person understands a problem? How does he go about it? How does he form subgoals?, retrieve relevant information?, derive solution procedures?, or finally use them? Further, in this connection, Lochhead (1979) reports that experts might agree on the merits of various techniques, e.g. breaking the problem into parts or examining special cases, but there is no consensus as to the best overall strategy.

In this thesis, it was preconceived that a problem solver might employ the following conceptual path in process of solving a problem.
(i) He defines D in the problem-space (D, B, X) of P.

(ii) He decomposes P into simple parts i.e. subproblems or relations that might lie either on the surface or deeper in the structure of P. Such simple parts P₁, P₂, ..., Pₙ define a certain B for him.

(iii) He draws analogies by matching those simple parts with similar ones in the set of his aggregate knowledge within the space of knowledge W which he can deal with.

(iv) He proceeds with the solution of P as he evaluates the individual solutions to the sub-problems.

This conceptual path could be expected to match each Pᵢ with a corresponding wᵢ, - expressed Pᵢ ↔ wᵢ - where wᵢ is an element in his aggregate knowledge relevant to Pᵢ. However, it was thought that both Pᵢ and wᵢ are governed by the particular basic behavioural objective Bᵢ he chooses. Consequently a problem-solver would match \{P₁, P₂, ..., Pₙ\} of P through (his) B to a certain kernel W₀ in W for P.

Figure 5.2 illustrates the conceptual process.

![Figure 5.2: Illustrative Diagram Of A Conceptual Process In Problem-Solving.](image-url)
Simply, it is foreseen that in such a conceptual process, the problem solver identifies a certain B through breaking down P into \( \{P_1, P_2, \ldots, P_n\} \). He then matches the latter set with a kernel for P through B.

5.2.2. II. The Efficiency In A Space Of Knowledge

The term *efficiency*, as it sounds, would be expected to indicate how far the achievement of a particular task 'k' is efficient when such achievement is put to work on a situation relevant to k. In other words, if a student has properly achieved k, then the question of efficiency would ask what is the probability of this student being able to solve a problem P in domain of k. In this connection, if B is a set of basic behavioural objectives necessary for k, then success in mathematics is usually obtained through emphasising mathematical knowledge related to simple behaviours, e.g. \( B_i \) in B. Thus the efficiency would indicate how far a student can benefit from acquiring knowledge on basis of such simple behaviours, when he encounters a problem P in the domain of k. This assumes that the necessary behaviour for P is complex in that it constitutes more than one of those simple behaviours \( B_i \) in B. Hence, such a measure could throw light on the advantages that a certain space of knowledge can provide in teaching-learning situations, as long as it could demonstrate reliability.

In earlier works for this study (P.6-11), the efficiency was explained in general terms as a measure that is identified by the probability that a student can solve a problem P, provided that he has demonstrated ability in a set \( P_0 \) that is relevant to P. Further on, in a pilot study based on a general conclusion of a test \( T_0 \) (in 6.1.2.2.3), it was proposed that such a study of efficiency could be maintained on \( p_0 = X_0 \) as a kernel of P.
Here, I shall discuss such a measure of efficiency by identifying \( P_o \) on basis of the three properties of \( X \) in the axiom of sufficiency, i.e. consistency, uniformity and completeness. I believed that the degree of relation between \( P_o \) and \( X \) would indicate the reliability of this measure.

(i) **Consistency:** Here, there should be identified an algorithm that provides for the construction of a \( P_o \) related to \( X_o(X) \) in the axiom of sufficiency. But \( X_o(X) \) in the axiom of sufficiency was supposed to test the necessary competency for \( P \), however a kernel in a space \( W \) i.e. \( W_o \) would, by definition, serve for this purpose. Hence, one could take \( P_o \) as a kernel \( W_o \), and the algorithm for construction of a kernel could serve for consistency. This choice would also be in line with the general conclusion in 6.1.2.2.3 in that efficiency can be maintained on \( P_o \) as a kernel \( X_o \) for \( P \).

It is to be noted that \( X_o(X) \) has not been identified and therefore could be completely different from \( W_o \). Furthermore, we are not sure whether \( X_o(X) \) is a kernel in some space or not, although I would accept \( W_o \) as an acceptable approximation for \( X_o(X) \) in \( W \).

(ii) **Uniformity:** Here someone solving \( P \) should, in general, be able to develop a proper mathematical knowledge in the necessary competency for \( P \) that was identified by a kernel \( W_o \) of \( P \), i.e. a solver of \( P \) should be able accurately to perform in each element in the kernel for all the elements. This proper knowledge was preconceived to be independent of the choice of \( W_o \), i.e. the uniformity here implies that the study should be orientated towards those who are able in all kernels \( W_o \) in \( W \) for \( P \). A member who satisfies such proper ability in all the elements of a kernel and for all the kernels of \( P \) will be called a uniform operational. However, the
5.2.2.0. Definition 5.6

If \((D, B, X)\) is the problem-space of a certain problem \(P\) in a space of knowledge \(W\) with a certain normal population '0', then a uniform operational for \(P\) in '0' is supposed to be completely able, for all kernels of \(P\), in all elements of any kernel \(W_0\).

Accordingly, all such kernels \(W_0\) are identical to uniform operationals, and the kernel \(W_0\) will have, in this sense, the corresponding meaning in the ordinary language. In this connection, those uniform operationals who are expected to be ideal ones imply a certain reliability for an adequate development of the study, based on the law of 'universal determination'. In behavioural science this is founded on the assumption that the behaviour of an organism is lawful and hence exhibits consistent behaviour. As a matter of fact in the absence of a principle of invariance in the psycho-educational field which might provide such consistency in behaviour under a given set of conditions, we might expect results and findings to differ, which clearly is found to be so in repetition of research studies.

(iii) Completeness: Here we would require that the efficiency provide an indication of how far the knowledge achieved by uniform operationals is complete for \(P\), i.e. satisfies \(P\) in the sense of the axiom of sufficiency. In other words, the efficiency is expected to 'measure' how far, a space of knowledge \(W\) approximates to the ideal situation involved in the axiom of sufficiency.

Hence the following definition was found to be plausible.
5.2.2.1. Definition 5.7

Let \((W, O)\) be a space of knowledge such that a certain normal population 'O' in a certain grade of school is seeking to gain a mathematical knowledge through a structure of knowledge \(W\) and let:

(i) \((D, B, X)\) be a problem-space of a certain problem \(P\):
(ii) \(W_o\) is any kernel for \(P\) in \(W\) identified by \(B\);

then the efficiency in \(W\) with respect to \(P\) is defined to be the probability '\(r\)' that a uniform operational in \(O\) for \(P\) is capable of \(P\).

5.2.2.2. Basic Remarks

(1) \(r = 1\) for \(X_o(X)\) in the axiom of sufficiency.
(2) The definition of efficiency is theoretically restricted to those uniform operationals who satisfy uniformity.
(3) The efficiency '\(r\)' is expected to change with \(P\) in the same \(W\).
(4) '\(r\)' is mainly dependent on \(W\) under consistent interaction between \(W\) and \(O\).

A textbook that prevails in such an interaction could possibly induce a consistency i.e. \(r = r(W)\). (The teacher's role is never denied, but since this is a probabilistic study that intends to deal with different subjects who have different teachers, I supposed the textbook to be the more markedly efficient factor in these processes). (see Conclusion 4.2).

(5) If \(N(W_o)\) denotes the number of uniform operationals in \(O\) and \(N(W_o + P)\)
denotes the number of those who are both uniform operationals and capable of P, then \( r(W) \) or \( r = \frac{N(W_o + P)}{N(W_o)} \) could be a fair measure of efficiency. This 'r' may be supposed to be invariant under choice of \( W_o \) because of the ideal property of uniform operationals.

![Figure 5.3: Illustrative Diagram For Calculation Of Efficiency.](image)

Furthermore, representative simples of 'O' could be expected to induce the same value of r.

5.2.3. General Discussion

(a) It was suggested in these previous basic remarks that r is mainly dependent on \( W \) under the same conditions of interaction with \( O \). Suppose then \( G \) and \( \tilde{G} \) are two representative groups of \( O \) and both are exposed to a task \( k \) through the two different spaces of knowledge \( W \) and \( \tilde{W} \) respectively under the same conditions of interact. Then for a certain problem P, it is not necessarily expected that "r in \( W \) through \( G \)" is equivalent to
"r" in $\hat{W}$ through $\hat{G}$. This expectation, indeed, is in line with the second indicator $T_2$ in 7.2.1 that demonstrated different correlations in the two spaces of knowledge $S$ and $H$ for the same problem $P$.

Hence the following conclusion:

5.2.3.0. Conclusion 5.1

In a space of knowledge $(W, 0)$ the efficiency of the gained knowledge in $W$ with respect to a problem $P$ would differ with a change of $W$.

The conclusion provides confirmation of the statement differences in contents could result in different learning outcomes. In this connection, this confirmation goes in line of the works of Mayer and Greeno (1972) and Ehrenpries and Scandura (1974), however those works in turn would cast further reliability over this measure, i.e. the efficiency. Moreover, this measure of efficiency can in turn indicate that cognitive abilities can meet different influences in different spaces of knowledge. These influences were previously discussed in 5.1. Hence different learning outcomes could be expected to be induced by different underlying structures of learning.

(b) According to completeness of (iii) in 5.2.2 (P.5-23), it could be conceptualised that a kernel $W_0$ for a problem $P$, could be in proximity of $X_0(X)$ in the axiom of sufficiency, in so far as the efficiency 'r' in $W$ is close to the ideal probability, i.e. $r = 1$. It was indicated in the completeness property of $X$ in the axiom of efficiency that any student who is able in $X_0(X)$ will be capable of $P$, i.e. $r = 1$ for all who satisfy $X_0(X)$, (see
Remark 1 in 5.2.2.2). Hence if \( W_0 \) and \( \tilde{W}_0 \) are two kernels of \( P \) under \( B \) in two spaces of knowledge \( W \) and \( \tilde{W} \) respectively and corresponding efficiencies in \( W \) and \( \tilde{W} \) are \( r \) and \( \tilde{r} \) respectively, such that \( r < \tilde{r} < 1 \), then \( \tilde{W}_0 \) is assumed to be in closer proximity to \( X_0(X) \) than \( W_0 \), whereas \( X_0(X) \) always stands as the desired target for modification. Hence, we obtain the next general proposition of modification in the methodology of teaching-learning situations expected to preserve the reliability of the efficiency measure.

5.2.3.1. Proposition 5.1

If '0' is a certain normal population and each of \( W \) and \( \tilde{W} \) is a structured knowledge in the sense of Definition 5.1, and if:

(i) \( (D, B, X) \) is a problem-space for a problem \( P \);
(ii) \( r \) and \( \tilde{r} \) are the two efficiencies in \( W \) and \( \tilde{W} \) respectively for '0' and \( P \);
(iii) \( r < \tilde{r} < 1 \);
then \( \tilde{W} \) could contribute in whole or in part to a modification in the methodology of teaching-learning situations, provided that the part or whole of \( \tilde{W} \) is applicable in the sense of Proposition 6.3.

The following diagram 5.4 illustrates the proposition.
Figure 5.4: Illustrative Diagram For Discriminating Efficiencies.

Key:

(i) \( X_0(X) \) is that which satisfies the axiom of sufficiency. Hence \( g_0 \) is that group in \( O \) who can properly respond in \( X_0(X) \) i.e. satisfy \( X \). Hence \( r = 1 \) i.e. all those who are able in \( X_0(X) \) are consequently able in \( P \). (The completeness property in the axiom of sufficiency.)

(ii) \( g \) and \( g' \) are uniform operationals in \( W \) and \( W' \) respectively under \( W'_0 \) and \( W'_0' \) which identify two relevant kernels of \( P \) under the same set \( B \) of basic behavioural objectives necessary for \( P \). \( r \) and \( r' \) are the two corresponding efficiencies in \( W \) and \( W' \).

(iii) If \( r < r' < 1 \) then \( W'_0 \) is expected to be in closer proximity to \( X_0(X) \) than \( W'_0 \).

(iv) Proposition 5.1 of modification suggests that \( W' \) could contribute in whole or in part to a modification in the methodology of teaching-learning situations.
5.3. Summary

The chapter was to contain a certain theoretical framework that could arise from Chapters 6 and 7, and personal experience. It is introduced to fulfill the fundamental research propositions 6.2, 6.3 and 6.4 that were established in Chapter 6. It was preconceived that an 'efficiency' could be a measure that indicates how far the diffused knowledge is effective. Simply, the efficiency of the achievement of the knowledge related to a problem P would be measured by the probability that an individual is capable of P, provided that he has properly satisfied some related achievement.

In discussing some aspects of the theory, it was viewed that a child acquires language in a space of knowledge through simple states of knowledge that diffuse knowledge within the child which helps him to comprehend the underlying structure of the language, such as when he begins to demonstrate "I play" and "he plays". This space of knowledge could have physical effects, in that, a child who has learnt in this space, e.g. a domestic situation in Wales, can pronounce an alphabet which can't be easily spelt by another child in a domestic situation in England. Furthermore, the space could have certain emotional effects such as the perception of the rhythm of a line of poetry. In this connection, Viennot (1979) finds by using different terms that there is an untaught 'intuitive physics' (a space of knowledge in terms of this thesis), and a 'taught physics' (a 'school' space of knowledge in physics), such that the first is a hindrance to change or modification in the second. Fortunately, mathematics does not have two such spaces of knowledge, an out-of-school space and an in-school space, but rather there is a space 'S' that underlies mathematical knowledge. However, it is insufficient and of 'influential dominance', in that it results in limitation of the knowledge gained in S. It was found that there is another space of knowledge, H, which is ignored, although it is more relevant to non-
mathematical fields, e.g. physics and $S \cap H = \emptyset$ (the null set) under a certain precise content.

A space of knowledge was identified to be an ordered pair $(W, O)$ where $W$ is a set of structured knowledge identified by all initial states of knowledge that are well-defined by a finite set of rules. This $W$ diffuses particular knowledge within a certain normal population $'O'$ under the interaction between $W$ and $O$. It was viewed that the conception of space of knowledge could be helpful in identifying a modification based on the advantages that $W$ could provide in problem-solving by means of the reasonable measure, efficiency. Hence the study was expected to discuss (I) problem solving and (II) the use of "efficiency" in that area.

(I) It was seen that the problem-solving area is very important in school mathematics despite its own intricacy and our limited knowledge of it. In this connection, the axiom of sufficiency was introduced to posit the existence of an 'X' which satisfies a problem $P$. Such an $X$, while still unidentified, was seen as having a set of properties such as:

(i) consistency, in the sense that $X$ could constitute an algorithm that identifies $X_0(X)$ which tests the necessary competency for $P$,
(ii) uniformity, in the sense that an individual capable of $P$, who also understands what he demonstrates, should be capable in $X_0(X)$ and,
(iii) completeness in the sense that anyone who satisfies $X_0(X)$ will be able to complete $P$ properly.

Further on a problem space $(D, B, X)$ was introduced, where $D$ is the set of data and goals of a problem $P$, $B$ is a set of basic behavioural objectives necessary for $P$, and $X$ is the criterion for $P$ in the sense of the axiom of
sufficiency. Hence, a kernel \( W_0 \) was defined as a set of stimuli such that only one stimulus is uniquely governed by each \( B_i \) for all \( B_i \) in \( B \). In this connection, it was preconceived that a process for problem-solving could use the fact that a person solving a problem matches his work in the problem to a particular kernel \( W_0 \) that he constructs through this solution.

(II) "The efficiency" was introduced as a measure based on an approximation to \( X'_0(X) \) of \( P \). It was argued that \( X'_0(X) \) might be approximated to a kernel \( W_0 \). The measure was supposed to take effect through those 'uniform operationals' defined as those capable in all kernels and hence the efficiency would be independent of the choice of kernel. Consequently, the efficiency was introduced as the probability 'r' that a uniform operational in a certain normal population is able to solve \( P \). In further discussion, it was concluded that the efficiency of gained knowledge in a space \( W \) would differ from that in a space \( \hat{W} \). This is in line with Mayer and Greeno and others in that "differences in contents could result in differences in learning outcomes." Finally, a proposition 5.1 of modification was presented. It simply notes that if \( r \) and \( \hat{r} \) are two efficiencies of the same problem \( P \) in two spaces of knowledge \( W \) and \( \hat{W} \) respectively and if \( r < \hat{r} \) then \( \hat{W} \) might either in whole or in part contribute towards a modification in the methodology of teaching-learning situations.
CHAPTER SIX

THE RESEARCH STAGES
6. THE RESEARCH STAGES

6.0. Introduction

In conducting research, the first and most complex problem that a researcher encounters is usually found in formulating the research problem simply, clearly and completely (Kerlinger, 1973). Furthermore, work in an educational research is more or less as in science, in that it is not basically a mere fact-gathering activity. It is as Cohen (1956) states:

"There is ... no genuine progress in scientific insight through the Baconian method of accumulation of empirical facts without hypotheses or anticipation of nature. Without some guiding idea we do not know what facts to gather ... we cannot determine what is relevant and what is irrelevant " (P.148).

In this connection, Poincaré (1952) states that the experiments of the research should be based on some preconceived ideas, otherwise the experiments would be fruitless.

A considerable account of these ideas has led me to believe that, having formulated the problem clearly in the first step, the second step should be towards putting forward a set of fundamental propositions that might guide and support a plan of work in gathering the relevant data for development of the research.

In fulfilling such aims, a researcher will inevitably encounter a set of
problematic situations, mainly relevant to the identification of the research problem, the research proposal, and the plan of work suggested for the research development. In general, I found that firstly, the research problem and relevant terminology should be clearly identified and secondly, the research fundamental proposition should be clearly set on the basis of a mental argument that could be illustrated by a pilot work in the field, as well as on introspection that is based on both experience and the reviewed literature. But, in the absence of such clear guidelines for the research, I found Piaget (1972b) helpful when in more or less similar situations he suggests that a researcher should:

"state in the most explicit form possible (in terms of a set of hypotheses or axioms) three to ten of the most important of his metaphysical theses" (P.64).

In this study, I classified the research stages into three: pre-proposition, proposition, and final stages. Here I shall fully discuss the first two stages, whilst I shall briefly refer to the third and final stage.

(I) In the pre-proposition stage I presented, in the sense of Piaget but in my own words, a set of five convictions most of which are prevalent in the research field. I also identified the research problem. In this connection, I believed that an improvement of the achievement should be better based on a better understanding of the advantages of the current achievement. This, consequently implied clarity in terminology, and the implementing two pilot instruments. As for (II) the second stage of the research, the proposition stage, I employed such available factors - experience, reviewed literature, pilot instruments and relevant terminology - as well as introspection, for developing the fundamental research propositions that were supposed to establish a logical basis for the development of the final stage (III) of study. This final stage is an
extension of the proposition stage, in that it was guided by these fundamental propositions in development of theory, as well as relevant empirical studies that led to modification.

Figure 6.1 illustrates general aspects of these stages.
Fig. 6.1: Stages in Research Development.

The double arrow \( \longleftrightarrow \) indicates that effects are in both directions, whilst a single arrow \( \rightarrow \) denotes effects only in the direction of the arrow.
6.1. I. A Pre-Proposition Stage

In this stage I intended to put forward a set of convictions, which could initially be used as general guidelines for helping to clarify the research problem in the propositional stage. The presentation here is deliberately intended to be loose, since I wished to encourage broad interpretation of those beliefs in the convictions.

6.1.0. General Convictions

C1: The current teaching-learning situation in mathematics is not satisfactory.

C2: Not all mathematics teachers are creative; perhaps the majority are not. Hence, reform should not be considered to be a main duty of teachers who are already burdened with heavy instructional duties.

C3: Theory is not necessarily inimical to reform, and a feasible reform is necessarily based on good theory.

C4: There is no known principle of invariance in the behavioural sciences - as the principle of invariance of quantities in the physical
sciences - on which the results and findings can consistently rest. Only a partial solution (reform) can therefore be expected.

C5: Modification of a certain situation would be more sure if (i) a structure exists and can be identified, which constitutes rules that might govern the situation and (ii) it is possible to identify another structure which constitutes a new set of rules that might contribute towards governing the new situation.

The first four convictions indeed are seen to prevail in the field, while C5 is a personal conviction. Furthermore, C1 justifies the existing research, and it could have influenced my tackling the following research problem.

6.1.1. The Research Problem

The area of mathematics education abounds with numerous problems that still need answers, e.g. curriculum or methods of teaching. A researcher usually looks for a problem that will prove to be a true one in the area, i.e. worthwhile for research; and the problem would be better if it were rooted in the researcher's experience and survived within his thoughts. As for the research aims, I believe that any basic aim of mathematics education could not - indeed should not - by any means be in disagreement with the aims in this research, which seeks an improvement of the achievement of students in mathematics. This aim is indeed under continuing debate, since all those who are involved in the field express growing concern for the low achievement of students in mathematics (which constituted the basis for conviction C1). Furthermore my experience and the reviewed literature could indeed indicate that improvement of the
achievement is a worthwhile study to be conducted. In speculating on a desired improvement, whether in the current approach (based on the philosophy of the structure) or in the methodology for the approach which comprises all possible factors that implemented for a certain approach, e.g. curriculum (Chapter 3), I was convinced that the research problem was more relevant to such methodology. Consequently, such an aim for improving the achievement was believed to be found in:

"looking for a feasible development in methodology of mathematics teaching-learning situations"

and this constituted the research problem. Further discussion of the research problem is found in Chapter 1.

6.1.2. Pilot Investigations For The Proposition Stage

The proposition stage would inevitably influence the design and relevant method(s) that would be implemented in the research development. However, the proposition stage would also be influenced by preconceptions in this first stage. In general, as in an experimental science, I set out to discuss the research problem through identifying a functional relation that introduces the achievement as a function mainly dependent on an independent variable which could be experimentally governed (controlled) in a study of the variation of the achievement. Moreover, broadly speaking, it was viewed that a modification of a situation z into \( \hat{z} \) should rely on recognizing through some criterion the advantages of both z and \( \hat{z} \). Accordingly, it was preconceived that the advantages of the achievement would be better introduced by search for a modification. It was also believed that this view would enrich the theory.
This inevitably raises questions of terminology such as achievement and improvement of achievement as well as questions of a measure of advantage and the reliability of this measure.

In connection with the foregoing, I firstly shall discuss relevant terminology on the basis of 1.4, as well as implementing two pilot instruments to investigate the compatibility of the advantage measure with the study and hence to throw light on the introduction of the proposition stage.

6.1.2.0. Discussion Of Relevant Terminology

Here, I shall present discussion of some relevant terms; however, a glossary in Appendix A will involve most essential terms that are discussed in this thesis.

a) The achievement: This construct refers to what the student has learned in a certain task. Usually, in Kuwait, the achievement of a certain learning task 'k' takes place on the basis of a task-analysis, in terms of a set 'B' of behavioural objectives found necessary for 'k': A student usually copes with the available knowledge relevant to B of k. Consequently, the achievement is measured by an achievement test, which is supposed to measure what the student has learned to do (Thorndike and Hagen, 1969).

Such an achievement test is constructed on the basis of such a B and graded under a certain criterion that matches a grade to the achieved knowledge under certain rules. For example, the criterion might suggest
two simple stimuli in the form of question-items to measure the achievement of a student in each $B_i$ of $B$. A score, (say '+1') would be matched with the achievement of the student in a relevant $B_i$, if the student could accurately perform in at least one of the two items; otherwise 'zero'. Consequently, the sum 'A' of these scores is defined to be the grade of the student that matches with his achievement in $k$. For a group of students, this $A$ could be the average grade of all the grades of all the students in the group. The achievement in this sense provides a certain indicator of mastery in the task. For convenience, $A$ will refer to the construct achievement that is relevant to a student or group of students. This makes the study of this construct more tangible, and it helps to introduce the achievement as a measuratic term; hence two achievements could legitimately be compared.

b) Improvement of the achievement: Such improvement was conceived to be possibly acquired on the basis of measurability and comparability of two achievements. Hence, if $A$ and $\hat{A}$ are two achievements (in the sense discussed in (a)) and if $A < \hat{A}$, then the achievement $\hat{A}$ is said to be better than $A$ and improvement of $A$ could possibly be proposed on the basis of knowledge of attaining $\hat{A}$.

c) Advantage of the achievement: This term was conceived to be a basic one, in the sense that the development was based on the theme that to develop a certain situation 'z' relevant to certain aims, one should learn about the advantages of (i) $z$ in satisfying such aims, (ii) $\hat{z}$ that could be proposed to replace in the development, and further (iii) to compare advantages of $z$ and $\hat{z}$.

Accordingly, based on such views as Polya (1962), who suggests that the
ability to solve a problem might be the most valuable gain of learning mathematics, as well as the report of the Committee on the Function of Mathematics in General Education (1940) that used "problem solving as the basic raison d'etre for mathematics in the secondary school", I believed that a considerable development of the current achievement of students in mathematics is better based on a study that helps in understanding of advantages of such achievement in the area of problem solving.

Simply, this study is expected to explain the advantages of the achievement of a task 'k' that supposedly could help the students who have properly achieved the task k when they are exposed to a problem 'P' in domain of k. Accordingly, I believed that a development for the achievement could maintain consistency if based on the theme of the advantage of the achievement, not on mere raising A into A.

In discussion of how we might employ a study for learning about such advantages, I found that the following terms should be better explained.

\(\text{(c1): computational knowledge:}\) This term will refer in this thesis to the part of mathematical knowledge that is found in applying a certain response in a specific goal that genuinely is relevant to that response e.g. applying the rule \(\frac{d}{dx} (ax^n)\) in finding \(\frac{d}{dx} (3x^2)\). A behavioural objective in this part of mathematical knowledge could initially be "applying the knowledge."

\(\text{(c2): comprehensational knowledge:}\) This term will refer in this thesis to the other part of mathematical knowledge that is found in identifying a certain response that is relevant to a specific goal that genuinely is relevant to that response, e.g. 'identifying that the rule \(\frac{d}{dx} (ax^n)\) is
relevant to finding \( \frac{d}{dx}(3x^2) \). A behavioural objective in this part of knowledge could initially be "identifying the knowledge ...".

(c3) **Mathematical knowledge:** This term is expected now to refer both to c1 and c2. As a matter of fact, it was generally believed that the existence of such a sharp line between c1 and c2 was not useful in this thesis and therefore the behavioural objectives will in this thesis mostly begin "to identify and apply the knowledge ..."

(c4) **A general fundamental:** This term is employed in Kuwait to mean a certain behavioural objective in mathematical knowledge, whether computational or comprehensonal, that is considered by a certain panel of people who work in the field to be basic in the achievement of mathematics. This term will be used for limited purposes that are found in the second pilot instrument in this chapter.

(c5) **The efficiency:** This term is most important in this study since it is supposed to identify the advantages of an achievement. This simply is based on studying how far the achievement that is tested by some \( P_0 \), which is relevant under certain conditions to a problem \( P \), indicates a probable effectivity in solving \( P \).

Accordingly, if '0' is a certain normal population studying a specific field of mathematical knowledge, and if:

(i) \( P \) is a certain well-defined problem in this field.

(ii) \( P_0 \) is a certain set of simple stimuli presented such that each is constructed on the basis of testing achievement of knowledge in the field
relevant to only one behavioural objective. \( P_0 \) is expected in general to be strictly relevant to \( P \) for convenience.

(iii) \( G \) is a representative sample of '0'.

then the efficiency of the mathematics achieved in \( P_0 \) for \( P \) is measured by the probability \( 'r' \) that an 'x' in \( G \) is able to solve \( P \), provided that he has accurately performed in all elements of \( P_0 \).

Simply the efficiency is the probability that the conditional statement: 'x is able in \( P_0 \) \( \Rightarrow \) x is able in \( P \)' is true, provided that the first statement is true.

Remarks:

(i) \( P_0 \) will be denoted by \( Y_0 \) if \( P_0 \) is concerned with general fundamentals in the field. This \( Y_0 \) will be used in this thesis only in the second pilot instrument \( T_0 \) of this study.

(ii) \( P_0 \) will be denoted by \( X_0 \) if \( P_0 \) is concerned with basic prerequisites of \( P \). These basic prerequisites of \( P \) will be identified on the basis of a certain set \( B \) of behavioural objectives found to be basic and necessary for \( P \). This \( X_0 \) will be dominant in all tests of this thesis. And further, it will be called a kernel for \( P \).

(iii) A representative sample refers to a certain sample that could represent a normal population in the sense that this sample could satisfy a certain characteristic or measure that the population might satisfy.

In Kuwait this sample was identified according to certain regulations provided by the Ministry of Education. These regulations admit that a grade (school year) in a secondary school could exist if the students that would join this grade constitute two classes, i.e. approximately 50 students.
Provided that all students in the secondary stage in Kuwait, i.e. the field of this study, are randomly distributed, and all the teachers of mathematics should have a University degree in mathematics, then a representative sample for such population was considered to be the students in any two classes of the grade, whether the classes were in the same school or not. Accordingly, any two classes could be considered to be equivalent since we have had no evidence against that and hence any choice for the two classes is accepted as satisfying the random method, although the larger the number of classes is, the more reliable the efficiency measure could be expected.

(iv) If \( N(P_o) \neq 0 \) denotes the number of students in \( G \) who could accurately perform in all items of \( P_o \) and if \( N(P_o + P) \) denotes the number of those students in \( G \) who could accurately perform in all the elements of both \( P_o \) as well as \( P \), then \[ r = \frac{N(P_o + P)}{N(P_o)} \] could be expected to be a fair estimation for the efficiency \( r \), regarding \( P_o \) and \( P \).

It is to be noted that if \( N(P_o) = 0 \), then construction of \( P_o \) is not valid since this could mean that either \( P_o \) is not in some sense relevant to \( P \) or the sample is not representative.

Finally, it was preconceived that the advantages of a certain achievement that is relevant to problem \( P \), could be supposed to be identified in studying the efficiency of the achievement in \( P_o \) that is relevant to the problem \( P \).

Accordingly, I tried to learn about such advantages in order to throw light on presentations of the research proposals in the proposition stage on a relevant basis. Two pilot instruments were implemented for such purposes.
6.1.2.1. First Pilot Instrument: A Questionnaire

A questionnaire could be an efficient tool for gathering data if it could be made to be clear, brief and attractive, so that it persuades the respondent to answer readily, provided he was able to. I have tried to satisfy these conditions when implementing this instrument, especially by explaining clearly the procedure, purpose and terms of this questionnaire directly to the respondents.

6.1.2.1.0. Description Of Method

The main aim of this instrument was to build up a general knowledge of how people who work in the field think of the advantages of the mathematics of achievement of students in the current situation. It hence aimed to find how those people generally think of the efficiency 'r' of the current achievement.

A description of method in this instrument as well as in other ones is illustrated in the research methodology (Chapter 2). Simply, such description will be discussed though three main steps:

(i) the material that describes setting of the items of the test;
(ii) subjects and setting that describe people who participate in the instrument and the setting; and
(iii) the procedure that describes how the instrument was employed.

(a) The Material

The questionnaire consisted of the following two question-items:
Item I: Suppose that a student has been proved to have attained the proper computational knowledge relevant to a fixed problem P. What do you think of the probability that he (or she) is able to solve that problem.

Please choose one of the following measures for this probability from the set: {1/10, 1/4, 1/3, 1/2, 1} and write it in the following box:

☐

Item II: Suppose that a student has attained both the proper computational and comprehensional knowledge relevant to a fixed problem P. What do you think of the probability that he (or she) is able to solve that problem.

Please choose one of the following measures for this probability from the set: {1/8, 1/4, 1/3, 1/2, 1} and write it in the following box:

☐

The measures of probability in the two items were indeed gathered from people who work in the field. I gathered such measures through informal discussion with colleagues and teachers in the field. It is to be noted that all the participants were well-acquainted and familiar with the concept of probability.
(b) Subjects And Setting

The subjects who participated in answering the questionnaire could be categorized into three categories according to their official posts:

(i) 10 inspectors of mathematics (8 male and 2 female)

(ii) 9 senior teachers of mathematics (6 male and 3 female) in 9 different secondary schools for boys and girls

(iii) 10 teachers of mathematics (6 male and 4 female) in 5 different secondary schools for boys and girls.

The participants in the first category account for all the staff of inspectors of mathematics (except for myself) who deal with the secondary stage of education in Kuwait. All the members of this category met at a fixed time. I then administered the sheets of questionnaires to them and consequently recorded their responses as explained in the procedure.

The second category (originally comprised of 10 members, as the number in the first category), of senior teachers who were randomly chosen from ten secondary schools in Kuwait. They were asked for an appointment as with the first category. One male senior teacher could not participate, and consequently, the number of members in this category was then nine. It is to be noted that there were fewer female senior teachers merely because of transport difficulties in attending the centre appointed for the meeting.

The third group was chosen with the help of those senior teachers who participated. I asked for five volunteer senior teachers from the second group and got three males and two females. Consequently, I asked each
volunteer senior teacher to choose two members of the mathematics staff in his school to participate in the questionnaire on condition that each of the two members should have been teaching mathematics of the final grade for at least the previous year. This guaranteed that the chosen members were well-acquainted with the mathematics of the secondary stage, since a teacher in the secondary stage is not allowed to teach in the final grade unless he has taught all previous grades in the secondary stage. An appointment was then arranged for each meeting with the members in the five schools, where the questionnaire was explained and administered to them as described.

(c) The Procedure

There were seven meetings, one for the inspectors of mathematics, one for the senior teachers and five for the teachers in the five secondary schools. The following steps describe the procedure.

(1) I explained that I was conducting a research work that could provide suggestions for better achievement of mathematics. I explained what is generally meant by achievement of a task that is discussed in detail in 6.1.2.0. As for the better achievement, I used broad terms to demonstrate that the better achievement could be found in raising the average score of a group of students in an achievement test.

(2) I administered the questionnaire to the members of the group in the meeting and asked them firstly to read the questions.

(3) I explained simply that (i) computational knowledge (skill) refers
to the knowledge that is relevant to applying a certain response in a specific mathematical goal, e.g. solving an equation or applying a certain rule, but comprehensional knowledge refers to the knowledge of perceiving a proper response for a relevant stimulus, e.g. choice of a rule that is relevant to only one behavioural objective, like an exercise or a simple problem.

(4) A problem is a novel stimulus that is well-defined but not simple.

(5) The participants were asked to respond directly after discussion and their responses were collected at the time.

6.1.2.1.1. A Questionnaire Construct

Optimisation in the second item is expected to significantly differ with that for the first.

6.1.2.1.2. Discussion Of Results

If the validity of the questionnaire data depends on the ability and the willingness of the respondent to provide the information requested as Mouly (1970) believes, then I believe that such validity holds, since the questionnaire poses continuing cozy problems that survive within them. Furthermore, the questionnaire was very short, and the appointments of meetings were not burdensome to the participants. Moreover, I felt that all of them could clearly understand the questions, since none of them raised any noticeable questions. The possibility of the misinterpretation of any of the questions was excluded, since all of them responded in almost
the same time, as was clear when the sheets were returned.

The following two tables classify responses in each item, according to the probability scale.

<table>
<thead>
<tr>
<th>ITEM I</th>
<th>Probability 'r_1'</th>
<th>1/10</th>
<th>1/4</th>
<th>1/3</th>
<th>1/2</th>
<th>1</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Teachers</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>No. of Senior Teachers</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>No. of Inspectors</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>21</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>29</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ITEM II</th>
<th>Probability 'r_2'</th>
<th>1/8</th>
<th>1/4</th>
<th>1/3</th>
<th>1/2</th>
<th>1</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Teachers</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>No. of Senior Teachers</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>No. of Inspectors</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>9</td>
<td>14</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>29</td>
<td></td>
</tr>
</tbody>
</table>

Tables 6.1 and 6.2 demonstrate by and large that people who work in the field are not satisfied with the mathematical knowledge whether (i)
computational in table 6.1 or (ii) both computational and comprehensonal, that achieved by students. For 25 out of 29 participants (86%) $r_1 \leq \frac{1}{4}$ whilst for an almost equal number (23 out of 29 or 79%) $r_2 \leq \frac{1}{4}$.

Such a value of $r_1$ was not surprising, for as Lovell (1981) states:

"we all know that performance in computation and problem work often correlates poorly."

The surprising situation could be found in table 6.2 by the higher accumulation of opinions in the range of the probability scale that corresponds to $r \leq \frac{1}{4}$, where 79% of all the respondents were not sure of the efficiency of mathematical knowledge that constitutes the current school mathematics achievement. $r_1$ and $r_2$ did not genuinely differ significantly for the majority of opinions which should stand against the questionnaire construct.

As a general view, I could observe that "people who work in the field think that the current achievement of mathematical knowledge is not efficient in problem solving."

Such opinions indeed support a call for a change in current situations of achievement.
6.1.2.1.3. Summary Of Questionnaire

<table>
<thead>
<tr>
<th>TITLE</th>
<th>A questionnaire - Ministry of Education - Kuwait</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIMS</td>
<td>The main aim behind this questionnaire was to find out how people who work in the field think of the advantages of the achievement of mathematics under current situations. It was based on learning how these people think of the efficiency that is relevant to a certain problem P, in cases (i) $P_0$ tests the necessary computational knowledge for P, and (ii) $P_0$ tests the necessary mathematical knowledge for P.</td>
</tr>
<tr>
<td>MATERIAL</td>
<td>The questionnaire comprised only two items.</td>
</tr>
<tr>
<td>SUBJECT AND SETTING</td>
<td>The subjects were 29 divided into three main categories: (i) 10 inspectors of mathematics. (ii) 9 senior teachers of mathematics. (iii) 10 teachers of mathematics. All were involved in the secondary stage of education in Kuwait schools. They were divided into seven groups for ease of procedure.</td>
</tr>
<tr>
<td>PROCEDURE</td>
<td>(i) It was explained that the questionnaire is part of a research study and better achievement was operationally defined, (ii) the sheet was administered to them, and (iii) technical terms, such as computational or comprehensonal knowledge, simple or routine problem, and problem were explained. Consequently, the respondents answered the questionnaire and gave it back in time.</td>
</tr>
</tbody>
</table>
6.1.2.2. A Second Pilot Instrument: A Test $T_0$

The study of the rough results in the questionnaire revealed a low efficiency in the current achievement of mathematical knowledge in students. I therefore tried to learn more fully about this achievement of mathematical knowledge in students by a paper-pencil test $T_0$. (Appendix B). I further aimed to learn how far the efficiency measure could be reliable for further use, on the basis of any compatibility with views that usually describe situations in the field about problem solving. For example, ability in problem solving differs from one problem to another, despite having the ability in the basic prerequisites that are necessary for different problems. Moreover, I wanted to learn of how this efficiency could be better employed in discriminating achievements.

It is to be noted that a panel of six people, myself included, discussed the construction of $T_0$ and approved its material contents with the aim of
preserving content validity of the text (Chapter 2). As for the
construct validity discussed in the research methodology in Chapter 2,
I stated the test-constructs which reflect my beliefs based on
experience and the reviewed literature. The later analysis of the test
was based on discussing such constructs.

6.1.2.2.0. Discussion Of Method
(a) The Material

$T_0$ was constructed in a multiple-choice form of 20
items, designed for students in the final year of school in Kuwait. The
responses were four for each, and extracted from the answers of students
in four different classes in the final year by exposing five items of $T_0$
to each class in the form of open-ended questions, i.e. essay form.
None of the students in these four classes participated in $T_0$.

(a_1) Three problems were chosen to constitute items 18, 19 and 20 in
$T_0$.

(a_2) In construction $P_0 = Y_0$, a set of general fundamentals, I intended
to find through the efficiency measure, how far competency in general
fundamentals could be employed in problem-solving. The difficulty was in
finding such general fundamentals that could be significant and relevant
to the problems chosen. As an acceptable solution, I chose those general
fundamentals that are both relevant to the area and reported by the
Inspectorate of Mathematics as deficiencies of students in general
fundamentals. Those deficiencies are, for example, found in solving
equations of certain types or in identifying a relationship between limit
and continuity.
Accordingly, I could identify ten deficiencies that were relevant to what the students have learnt. Hence \( Y_o \) was constructed on the basis of one question-item for each deficiency. It consisted of the first ten items in \( T_o \), where items 1, 2, 3, 4, 6, 7 and 8 related to computational knowledge, based on using mathematics as a tool, while 5, 9 and 10 were related to comprehension. Here, I present three of those general fundamentals:

(1) The first general fundamental corresponds to deficiency in solving equations of the first degree with one unknown, say 'x' in the denominator. The corresponding item in \( T_o \) is:

Item 1: If \( \frac{x-2}{2x-1} = 5 \), then x equals:

\[3 \quad \square, \quad -3 \quad \square, \quad -\frac{1}{3} \quad \square, \quad \frac{1}{3} \quad \square.\]

(2) The 6th general fundamental corresponds to deficiency in finding the limit of \( \frac{f(x)}{x} \) as \( x \to 0 \).

Item 6: \( \lim_{x \to 0} \frac{x^2 + x}{x} \) as \( x \to 0 \) is:

zero \( \square \), non-existing \( \square \), 1 \( \square \), none of the proceeding \( \square \).

(3) The 10th general fundamental corresponds to deficiency in identifying the relationship between a certain two of the following: continuity of a function \( f \) at 'a', existence of \( f \) at 'a', limit of \( f \) at 'a', the existence of the derivative of \( f \) at 'a'.

Item 10: If the function \( f \) is continuous at \( x = a \) then:

\( f \) is defined at 'a' \( \square \), \( f \) has a derivative at 'a' \( \square \), \( f \) may have no limit at 'a' \( \square \), \( f \) may be undefined at 'a' \( \square \).
(a3) As for construction of $P_0 = X_0$, it could be expected that there would be one $X_o$ for each problem in items 18, 19 and 20. For convenience, these will be denoted by $X_0(18)$, $X_0(19)$ and $X_0(20)$ where:

(i) $X_0(18)$ consists of items 11, 12 and 14 of $T_0$.
(ii) $X_0(19)$ consists of items 15 and 16.
(iii) $X_0(20)$ consists of items 13 and 17.

It could be noted that $X_0(18)$ is a kernel for problem $P=18$ ($P$)
Similarly $X_0(19)$ and $X_0(20)$. Here I illustrate the construction of $X_0(18)$ that corresponds to the problem in item 18.

**Item 18:** If the line $y = x-h$ is a tangent line to the curve $f(x) = x^2 + 5x$, then $h$ equals:

- $4$ □  
- $-4$ □  
- $7$ □  
- $-7$ □

In construction of $B$, the set of basic behavioural objectives that are necessary for any problem $P$, the term basic was used to mean that the behavioural objective $B_i$ in $B$ was accepted by the panel to be necessary (see Glossary in Appendix A). Hence not all behavioural objectives that I had stated were found to be necessary by the panel.

The panel agreed indeed on three basic behavioural objectives for $B$ that identify basic prerequisites for $P$.

$B_1$: To identify and apply the knowledge that a point belongs to a certain curve $f$.

$B_2$: To identify and apply the knowledge that the slope of a tangent line to a curve $f$ at $a$ equals $f(a)$; provided that $f(a)$ exists.

$B_3$: To identify and apply the knowledge of the derivative of a constant symbolic term in a function $f$. In all cases, $f$ was taken to be a polynomial of the second degree.
Hence the corresponding items respectively were:

**Item 11:** If \( T(x) = x - x^2 \), then one of the following points belongs to \( T \):

\[
(1,2), \quad (-1,0), \quad (2,2), \quad (2,-2).
\]

**Item 12:** If \( f(x) = x + x^2 \), then the slope of the tangent line to \( f \) at \( x = -2 \) is:

\[-3, \quad 2, \quad 5, \quad -4.\]

**Item 14:** If \( T(x) = x^2 + h^2 \), \( h \) being constant, then \( T(x) \) is:

\[2x + h^2, \quad 2x, \quad x^2 + 2h, \quad 2x + 2h.\]

(b) Subjects And Setting

A representative sample was chosen to constitute three classes in the final grade from three different secondary schools. The total number of students who participated in answering the test was 71 made up as follows:

(b1) Fhahil Secondary School for Girls: The final grade comprised four classes of approximately equal size. A class was randomly chosen that comprised 25 girls.

(b2) Rabia Secondary School for Girls: The final grade comprised seven classes. The chosen class comprised 22 girls.

(b3) Khitan Secondary School for Boys: The final grade comprised eight classes. The chosen class comprised 24 boys.

It is to be noted that girls in Kuwaiti secondary schools have always proved, in general examinations of all the grades, that they are no less competent in sciences, especially in mathematics, than boys. Hence I
shall not consider any differences of sex in this study.

(c) The Procedure

(c1) According to the regulations of the Ministry, all students in the same grade should in any week of the academic year be learning in the same task, although being one or two periods ahead or behind does not matter. It was ascertained that all students of $T_0$ had covered the materials contained in the test.

(c2) The teachers of the experimental classes administered the test to the students, as a normal school procedure for assessing the student's achievement.

(c3) The duration of the test was 60 minutes, and all the students gave the sheets back in time.

(c4) The answers were written on the sheet of the test by ticking (✓) in the appropriate box.

(c5) $T_0$ for each class was corrected first by the teacher of the class, then by the senior teacher of mathematics in the same school and lastly by me. A score of +1 was given for each correct answer and 'zero' otherwise. Scoring of the answered sheets was performed by the guidance of an answering sheet for correct responses. This sheet is attached with $T_0$ in Appendix B.

6.1.2.2.1. The $T_0$-Constructs

The following constructs were supposed to identify the construct validity of the test.
(1) The efficiency differs with the problem, i.e. any two problems have not necessarily the same efficiency in $Y_0$ as in $X_0$.

(2) Generally, students are more competent in items of computation than in those of comprehension.

(3) The efficiency regarding a certain 'P' relevant to general fundamentals (i.e. in $Y_0$) is almost equivalent to the efficiency in the set $B$ of basic behavioural objectives that are necessary for $P$ (i.e. in $X_0$).

6.1.2.2.2. Discussion Of The Results

(1) The final grade was chosen for $T_0$ since it is widely acknowledged that students in this grade are both mature enough and studious, due to the strong competition to enter the fields of Medicine, Engineering and Science Faculties. Besides the total mark in mathematics (120) is a high proportion of the total mark (410) for all subjects in the General Examination of the final year, usually authorized by the Board in the Ministry. Hence, it was expected that those students will work hard in the test, which is a desired aim for the researcher trying to study efficiency as reliably as possible.

(2) The following table 6.3 denotes the number (n) of correct responses to each item of $T_0$:

| Item | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | Total No. of Students |
|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----------------------|
| n    | 60| 40| 51| 58| 34| 51| 42| 39| 30| 32| 50| 62| 26| 28| 46| 41| 30| 22| 41| 19| 71                     |

Table 6.3
Number Of Correct Responses In Items Of $T_0$. 
It could be noted that students in general were less able in general fundamentals that pertain to comprehension (items 5, 9 and 10) than in general fundamentals that pertain to computation (items 1, 2, 3, 4, 6, 7 and 8). This was indeed an expected construct (Construct 2).

(3) For studying the efficiency that is relevant to \( Y_0 \) let:

(i) \( N(Y_0) \) denote the number of students who were completely able in all items of \( Y_0 \);
(ii) \( N(Y_0 + P = 18) \) denote the number of students who were completely able in both \( Y_0 \) and item 18.
(iii) \( N(Y_0 + P = 19) \) and \( N(Y_0 + P = 20) \) be defined similarly.
(iv) the efficiency \( r(Y_0, 18) = \frac{N(Y_0 + P = 18)}{N(Y_0)} \) denote the probability that a student 'x' is able in \( P = 18 \), provided he is able in \( Y_0 \).
\( r(Y_0, 19) \) and \( r(Y_0, 20) \) are defined similarly.

The following table 6.4 illustrates the number of the able students in \( Z \):

<table>
<thead>
<tr>
<th>( Z )</th>
<th>( Y_0 )</th>
<th>( (Y_0 + P = 18) )</th>
<th>( (Y_0 + P = 19) )</th>
<th>( (Y_0 + P = 20) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(Z) )</td>
<td>31</td>
<td>16</td>
<td>24</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 6.4

Number Of Able Students In General Fundamentals \( Y_0 \) And \( Y_0 + P \).

Hence the efficiencies regarding \( Y_0 \) and the three problems are:

\[
\begin{align*}
r(Y_0, 18) &= \frac{16}{31} \approx 0.52 \\
r(Y_0, 19) &= \frac{24}{31} \approx 0.77 \\
r(Y_0, 20) &= \frac{14}{31} \approx 0.45
\end{align*}
\]

The efficiency of the gained knowledge in \( Y_0 \) is not the same for different
problems, since different problems differ in difficulty. This construct was also expected. (Construct 1).

(4) If, in a way similar to that discussed in (3), \(N(X_0(18))\) denotes the number of students who were able in \(X_0(18)\) (Items 11, 12 and 14), and \(N(X_0(18) + P = 18)\) denotes the numbers of students who were able in both \(X_0(18)\) and \(P=18\), then:

\[
r(X_0(18), 18) = \frac{N(X_0(18) + P = 18)}{N(X_0(18))}
\]

and consequently, we obtain the table 6.5 which discusses the efficiency regarding a problem \(P\) and its kernel \(X_0(P)\).

<table>
<thead>
<tr>
<th>Z</th>
<th>(X_0(18))</th>
<th>(X_0(18) + P = 18)</th>
<th>(X_0(19))</th>
<th>(X_0(19) + P = 19)</th>
<th>(X_0(20))</th>
<th>(X_0(20) + P = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N(Z))</td>
<td>24</td>
<td>12</td>
<td>38</td>
<td>29</td>
<td>18</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 6.5

Number Of Able Students In a Kernel \(X_0(P)\) For A Problem \(P\)
And In Both \(X_0(P)\) And \(P\).

Hence:

\[
r(X_0(18), 18) = \frac{12}{24} = 0.50
\]

\[
r(X_0(18), 19) = \frac{29}{38} = 0.76
\]

\[
r(X_0(20), 20) = \frac{11}{18} = 0.61
\]

The efficiency shows that the gained knowledge in \(X_0\) of \(P\) could also differ according to \(P\), which stands with the generally recognised knowledge in problem-solving (Construct 1). The construct of efficiency can almost
stand with such beliefs that are found in Construct 1.

(5) In comparing the efficiency relevant to the same \( P \) with that relevant to the gained knowledge in \( Y_0 \) or \( X_0(P) \):

(i) \( 0.52 \) vs. \( 0.50 \)
(ii) \( 0.77 \) vs. \( 0.76 \)
(iii) \( 0.45 \) vs. \( 0.61 \)

we notice that in (i) and (ii) the two efficiencies are approximately equivalent, while in (iii) the efficiency that is relevant to \( X_0(20) \) was higher than that of \( Y_0 \). This differs slightly with the third construct that expected the two efficiencies to be approximately equivalent for the same \( P \), hence this difference is in favour of \( X_0 \). In addition choice of \( X_0 \) regarding a problem \( P \) is more precise than \( Y_0 \), which could be vague since those general fundamentals are broad terms and not easily identified. \( X_0 \) could be favoured to \( Y_0 \) since it is more precise and more related to \( P \) as well as any proposed improvement will inevitably involve \( Y_0 \). A study of efficiency regarding \( P \) could be dependent on \( X_0 \) of \( P \).

6.1.2.2.3. Summary Of \( T_0 \)

<table>
<thead>
<tr>
<th>TITLE</th>
<th>A Test ((T_0)) for students in the final grade of Secondary School in Kuwait. (Appendix B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIMS</td>
<td>Primarily to learn about the reliability of the efficiency measure as well as how to better employ it.</td>
</tr>
<tr>
<td>MATERIAL</td>
<td>( T_0 ) was a multiple-choice test of 20 items. It contained</td>
</tr>
</tbody>
</table>
(i) Three problems in items 18, 19 and 20;
(ii) $Y_q$, a set of general fundamentals in items 1, 2, ..., 10; and
(iii) Three sets $X_q(18)$, $X_q(19)$ and $X_q(20)$ constructed to assess the attainment of knowledge in basic pre-requisites for the problems 18, 19 and 20 respectively.

$X_q(18) =$ items 11, 12 and 14,
$X_q(19) =$ items 15 and 16,
$X_q(20) =$ items 13 and 17.

SUBJECTS AND SETTING

The subjects were 71 students in three classes of the final grade in three different secondary schools.

PROCEDURE

The test was administered by the class teacher and the duration was 60 minutes.

VALIDITY

(a) Content validity was satisfied by a panel who helped in the construction of the test.
(b) Construct validity was identified through discussion of the results.

RELIABILITY

0.66 (The Kuder-Richardson 21).

GENERAL VIEW (CONCLUSION)

(i) The efficiency that could be compatible with such beliefs (Construct 1) might be a reliable measure for discriminating the advantages of the achievements.
(ii) A study of the advantage of an achievement could be based on the efficiency that maintains on $X_q$ for a certain $P$. 
Such $X_0$ is supposed to be constructed on the basis of basic pre-requisites for $P$ identified by a set $B$ of basic behavioural objectives that are necessary for $P$. ($X_0$ is called a kernel for $P$).

6.2. II. The Proposition Stage

6.2.0. Introduction

The previous study in the pre-proposition stage was expected to help develop this second stage. Furthermore, it might throw light on my thoughts of achieving the study, in that I would occasionally resort to introspection that was based on experience as well as further reading. So, I also wanted to employ what I think to be convenient experimental work in order to base my study on more reliable situations. In presenting my convictions in 6.1.0. about current achievement, I tried to depict what could be involved in discussing a certain modification in the methodology for improving the achievement of students in mathematics. Furthermore, they reflect my belief of the role of theory as well as the fifth conviction, which is more personal in how I think a modification could be conducted. As for terminology, it was provided to help in consistency of study, as well as it helped in suggesting tools for development that were based on the idea that a change from $z$ into $\bar{z}$ should be founded on studying the advantages of achievements in both $z$ and $\bar{z}$. This could have led to the two pilot instruments that demonstrated how people who work in the field think of such advantages and how such measure of efficiency relevant to study of advantages could be reliable in further applications.
In this stage, I indeed mainly employed introspection, based mainly on experiencing readings and beliefs that are to be found in the previous set of convictions. The study was also influenced by the efficiency as a measure that could be supposed to be reliable on the basis of the second pilot instrument in \( T_0 \), in the expectation that this measure would be useful in achieving this study. The main theme in modification found in Proposition 6.4 stretches from the fifth conviction as well as the reliability of the efficiency-measure that originates from \( T_0 \).

The proposition stage is supposed to benefit from the previous stage in discussing the establishment of certain fundamental propositions that provide a basis for a systematic study for development. There were established four of such fundamental propositions; the first three were called general in the sense that they were expected to serve in further research fields, while the fourth was called specific since it basically was orientated to serve progress in this research. It stems mainly from the pre-proposition stage.

6.2.1. Three General Propositions

In an attempt to establish basis for the research development, I firstly found that such development should be directed towards enquiring how such an improvement in the mathematics achievement could in general be pre-conceived, and secondly how theory might interact with the development.

In dealing with the improvement, I tried indeed to employ a methodology that was more or less similar to that in experimental sciences, where a
scientist in developing certain knowledge for a phenomenon, tries to look for a certain functional relation, e.g. \( f(y, x_1, x_2, \ldots, x_n) = c \) that could govern one or more desired variables in terms of other relevant variables in the field of his study. If for example \( y \) is a desired variable, then the lower the number of such variables, \( x_1, x_2, \ldots \) the better is the situation for the scientist.

In connection with this study, I believed that if 'A', the number assigned to the achievement of a group of students in a mathematical task, could be governed by a certain functional relation that might define \( A \) as a function of different variables and if a change in one or more of these variables could be governed such that the change results in an increase in \( A \), then the improvement could in principle be promising. For this I presumed that \( A = f(x_1, x_2, \ldots, x_i, \ldots) \) with \( x_i \) as a possible factor could have effects on 'A', e.g. memory. Furthermore, I presumed that a study of a change in \( A \) could be particularly feasible if it is possible to minimize the number of those variables to only one, say \( x_j \) - under certain conditions - that could possibly minimize, indeed neutralise, the effects of all \( x_i \) where \( i \neq j \). Accordingly, \( A \) could almost be conceived to be a function of an independent variable \( x_j \), i.e. \( A = g(x_j) \), and consequently a prospective modification for raising \( A \) could be found in modifying \( x_j \).

Such being the case, in looking for such a relational function \( A = g(x_j) \), a question could have arisen in that: could there be a phenomenon in the psycho-educational field that could possibly be such an 'A' in a certain teaching-learning situation, in that it would be mainly dependent on such an \( x_j \)? As a matter of fact, I was influenced in such a procedure of thought by different works in the field that drew my attention to the
fact that the achievement of students changed with changes in the contents of teaching-learning tasks. A remarkable phenomenon in this sense could be found in Mayer and Greeno (1972). Their work, that reported in 4.4.1, could show how learning outcomes could change under relevant changes in the contents. I called this the presentation-phenomenon and details of this phenomenon are found in Chapter 4.

Generally, I was convinced, if such a phenomenon could - as a starting point - exhibit changes in A to be mainly dependent on changes of a likely, observable and well-defined independent variable \( x_j \) (e.g. the content of mathematical courses), then this could contribute to the development of this study and perhaps in other further studies.

And this accordingly encouraged the following, first general proposition.

6.2.1.0. **Proposition 6.1**

A phenomenon that indicates a change in the achievement of students under a certain change in a well-defined teaching-learning situation, is significant as a starting point for development in current teaching-learning situations.

As for further enquiry for the existence of such a theory that might theoretically support and help for development, I was indeed aware of the polemic about the gap between practice and theory which was mentioned in Chapter 1 through such statements of Becher (1980) and Walker (1980). Hence, based on such general convictions C3 and C4 with the expectation that partial reform could possibly be found, I proposed the following two general propositions that pursue the possible existence of such a theory. This was supposed to be reliable for, generally, a development which can rest on such a theory suggests experimental support to such
development if it could prove to be applicable.

6.2.1.1. Proposition 6.2

There exists a theoretical framework on which a development in current teaching-learning situations can rest.

6.2.1.2. Proposition 6.3

If a proposed development (reform) in current teaching-learning situations (a) can rest on a theoretical framework and (b) can be operationally applicable i.e. well-defined and learnable, then such development is expected to be experimentally supported.

It is to be noted that I intended to present such general propositions in broad terms, in order to encourage a wider interpretation of them, so that they might be useful in other different research work. Besides that, they provide a general conception of how progress in the study might start (Prop.6.1) and further, the last two reflect my belief in the existence of such a possible relationship between practice and theory.

6.2.2. The Presentation-Phenomenon: The Identification Of $x_j$

In setting these three general propositions, the next step was expected in grounding them in the study. The identification of such $x_j$ in $A = g(x_j)$ would inevitably help in identifying such a starting point.
The studies of Mayer and Greeno (1972), Ehreinpries and Scandura (1974), and Begle (1979), reported in Chapter 4 of this study, have indicated that "Differences in learning outcomes could be induced by differences in the material contents" whereas I referred to this as the presentation-phenomenon. In this connection, Lovell (1981) reports that many studies which deal with the effects of different teaching treatments (contents) on learning outcomes have shown that those treatments affect the type of question students are best able to answer and the method of solution they employ, and;

"This is in line with the findings of Mayer and Greeno (1972)" (Lovell, 1981).

Accordingly, I viewed that such an x_j should have relationship with the contents. But the 'contents' of a teaching-learning task remains vague. Does it mean the written material? ... All that could be written or said for the task? ... Both the written material plus the construct of underlying competency?

In this thesis, the contents will refer to the written material in the textbook, given the influence of the textbook in Kuwait on teaching-learning situations (see 'd' in Section 2 of Chapter 1). The genuine question that was raised: How can differences in contents be defined? For this, I found that one should learn how the contents of a mathematical task, say 'k' is presented.

Generally, such 'k' is analysed so that a set B of behavioural objectives is identified to satisfy k. Accordingly the contents of 'k' is supposed to include (i) a theoretical part, e.g. a proof of theorem, and (ii) a set 'R' of such instances as illustrative examples, exercises and simple
problems that are supposed to help the learner develop basic mathematical knowledge relevant to 'k' so that he will be able to understand the theoretical part better as well as to solve problems that are in domain of k. Now any changes in the contents are expected indeed to be in R and not in the theoretical part, which could be invariant under changes in the contents, however, the theoretical part of school mathematics is seen to be almost stable in most texts. Consequently, I thought of $x_j$ to have a strong relationship with R, which I called the presentation of the task. Hence the same question which poses the definition of the differences in contents could possibly be found in defining the differences in the presentation R. In discussing how such change could be genuine, I believed that replacing an instance of an equation of the first degree $2x + 1 = 5$, by $3x - 2 = 4$ say, in some R, does not yield a true change in R, but only a superficial and trivial one. A change in R should be different from such a trivial change and I believed that such true change could be based on introspection arising from the fifth conviction in 6.1.0. Consequently, I posed the following multifold structural question: 

$q_1$: Is there a structure 'S' that underlies R such that a true change in R is due to such a change in 'S'? ... And is it possible to identify another structure 'H' that could contribute to an underlying modification?

Furthermore, if supposedly such structures as S and H could be identified to induce a true change in R into $\tilde{R}$, then another genuine question would have arisen.

$q_2$: How theory might interact - theoretically at least - to give $\tilde{R}$ relevant to a modification in R?

In the absence of a clear vision, one indeed could possibly employ introspection, and in my case, I believed that a true change in R should
be based on identifying a structure 'S' - if it exists - that could underlie R, as well as a theory which could be employed in providing certain discrimination of advantages of the achievements in both S and H. In this connection the term efficiency was proper to help in theory. I therefore found a modification that could be rooted in the presentation-phenomenon could progress in the following specific proposition.

6.2.2.0. Proposition 6.4

(i) There exists a well-defined structure 'S' underlying current presentation of mathematical tasks.
(ii) There exists a structure 'H' that is possible to contribute in underlying modification of current presentation.
(iii) Theory in proposition 6.2 could possibly identify a certain discrimination of the efficient knowledge that achieved in 'S' and 'H'.

Consequently, I thought of $x_j$ to be of the form $R(S)$ whereas a change in $R$ is mainly dependent on a change of $S$. Furthermore, I found it convenient for simplicity to replace $A = g(x_j)$ by $A = g(R)$ provided that such $R$ involves the structure $S$ that underlies $R$. The proposition was called specific since it uses terms that are directly relevant to this specific study of research.

Hence, proposition 6.4 confirms for a prospective progress, the existence of structures such as $S$ and $H$ as well as monitoring the construction of relevant theory in describing how such theory could be employed. Moreover, the presentation-phenomenon could be expected as a starting point, in the
sense of proposition 6.1, that would help in the research development.

The questions that could have then arisen were: How can we identify such an 'S', ... such an 'H'? And how should such theory be constructed and employed? And how can modification take place then?

6.3. III. The Final Stage

The answers to such questions are expected to be found in the third and final stage of this study. Chapter 7 develops identification of both S and H as well as it provides an elementary view of theory that developed in Chapter 5 based on the conception of space of knowledge. Chapter 8 presents empirical studies established on a certain criteria that genuinely stretch from theory of space of knowledge in Chapter 5, and consequently Chapter 9 provides basis for modification that is empirically supported. Finally, Chapter 10 presents a summary of findings in this study as well as presenting a further outlook.

6.4. Summary

The study in this chapter was mainly conducted in the hope of building up a basis in a certain systematic procedure for the research development that could be indicated in figure 6.1. It was viewed that in the absence of a clear vision, a researcher could rely on a certain set of convictions (beliefs) as well as introspection that could possibly help him for the research progress. Furthermore, it was suggested that initially study in this chapter could be mainly conducted in two stages. In the first stage, i.e. I, the pre-proposition stage, I set five general convictions of which
most are prevalent in the field; besides this they reflected my own belief. I also formulated the research problem that aimed at "looking for a feasible development in the methodology of mathematics teaching-learning situations" such that "the achievement of students could be improved". As a consequence, the clarity of terms should have arisen for pilot investigations that might provide better understanding of the current situations. It was seen that an improvement in the achievement would serve to define operationally the construct achievement and how such an improvement generally takes place. Furthermore, it was thought that a change in the achievement should be based on identifying such advantages in the current and desired achievements. Consequently, it was thought that if a number 'A' could be assigned to the achievement, then improvement could be conceived to be increasing such A into Ā. But it was also seen that this development is better if based on discussion of the advantages in A for a prospective Ā. The increase in achievement was based on a relevant term called the efficiency, which is simply supposed to describe how reliable the achievement of students is when they are exposed to a problem in the domain of what they have achieved. Two instruments, a questionnaire and a test $T_0$, were employed to learn (i) how people who work in the field think of such efficiency (questionnaire) and (ii) if this efficiency measure could be reliable by providing compatibility with some beliefs that belong to problem-solving ($T_0$). Furthermore, $T_0$ could throw light on employing this efficiency to be studied on the basis of a problem $P$ and a proper achievement in a set $X_0$ called a kernel for $P$. This kernel is constructed on the basis of basic prerequisites of $P$ that are identified by a set $B$ of basic behavioural objectives that are necessary for $P$. As for the second stage i.e. II, the proposition-stage and in an attempt to identify those fundamental propositions of the research, I indeed tried to employ a method more or less similar to that of the experimental sciences. This was by finding a possible functional relation that might
present the achievement $A$ as a function mainly dependent on one variable $x_j$, i.e. $A = g(x_j)$. This function indeed could have roots in Mayer and Greeno (1972) work and others who found that differences in learning outcomes could be due to differences in the material contents. My personal consideration that such phenomenon of learning outcomes and material contents could be a starting point for the research development, as well as my convictions about the role of theory in practice helped in formulating the first three fundamental propositions. I indeed called them general since I intended to formulate them in broad terms in order to encourage wider interpretation for them in the hope that they could be used in different research fields.

In an identification of such $x_j$ in $A = g(x_j)$, I viewed this $x_j$ has a strong relationship with the contents on the basis of the presentation-phenomenon drawn out from the remarkable works of Mayer and Greeno (1972) as well as others. I operationally defined the contents as the written material in the textbook. Furthermore, I viewed that the contents contain genuinely two parts, a theoretical set and another set $R$ which includes instances such as examples or exercises. Also a change in the contents should be a true change in $R$ and not a trivial one. I therefore viewed that such $R$ should rely on a structure 'S' and a true change in $R$ should pass through a change in $S$. In this connection, I believed that another structure such as 'H' could contribute in the modification. However, theory was expected to discriminate between the achievement in $S$ and $H$ on basis of the efficiency measure that could have a basis of reliability on basis of the second pilot instrument $T_0$. Consequently, I proposed the fourth and last fundamental proposition which I named as specific since its wording is more relevant to this study, than to others.
Further on, I viewed $x_j$ to involve $R$, which was called the presentation part of the contents of a task, as well as the structure $S$ that is underlying $R$. However, for simplicity, I replaced $A = g(x_j)$ by $A = g(R)$, while taking into consideration that such $R$ is $R(S)$.

A final stage in the research development was achieved by guidance of the proposition-stage.
CHAPTER SEVEN

THE STRUCTURES S AND H: FURTHER IMPLICATIONS
7. THE STRUCTURES S AND H: FURTHER IMPLICATIONS

7.0. Introduction

In setting the four fundamental research propositions in Chapter 6, it was viewed that a modification for an improvement in the achievement of the students in mathematics could find roots in the presentation-phenomenon that denotes for a possible change in the achievement - which is relevant to learning outcomes - under changes in the material contents. In this connection, the presentation \( R \) i.e. one of two main parts of the contents (see Appendix A) was found to be the core in changes of the contents. It was argued that a true change in \( R \) is due to a change in a structure 'S' that underlies \( R \), however another structure 'H' was viewed to contribute to the underlying presentation. But, it was also viewed that raising the achievement \( A \) is not a mere process for finding another \( A' \) where \( A < A' \), but this should be based on the advantages of the achievements. Hence, a theory was proposed which possibly might interact with the research development on basis of the efficiency measure (see Appendix A) which was supposed to help identify such advantages of the achievements. A pilot study in a test \( T_0 \) in 6.1.2.2 could point to some reliability of this measure.

Consequently, the questions arise of identifying such \( S, H \) and theory for developing the research. Hence, this chapter intends to discuss the main objectives that could be embodied in the following questions:

I: How could structures such as 'S' and 'H' proposed in Proposition 6.4 be identified?

II: What primitive implications could possibly be drawn from such structures?
III: How could theory be constructed and employed for the required development?

The study was then expected to be developed in three parts through discussion of I, II and III.

As a matter of fact, the first part was developed through a suggestion from an excellent girl student in a classroom discussion. The second part was based on employing two indicators. (i) The first mainly aimed at discussing general implications of the achievement of mathematics in S when applied in a non-mathematics field, e.g. physics and relevant relationships with H. (ii) The second indicator was expected to throw more light on the reliance of the efficiency measure by correlating knowledge in S and H regarding certain problems. The efficiency was conceived to be reliable if it stands with certain constructs of relationships among the problem and relevant necessary knowledge in S and H. Furthermore, in discussing the 'pros and cons' of both S and H, it was argued that such H need not completely replace S, however a part say U of H could help in contributing in modification.

As for the third part III, this was indeed rooted in the foregoing study whereas a concept of space of knowledge was introduced as a conceptual structure constituting a basis for an abstract study that presented in Chapter 5 as theory for development mainly in this field and further in different ones.

7.1. I: Identification Of Structures: S and H

7.1.0. Introduction

In searching for a structure such as 'S' underlying current presentation 'R',
a brilliant girl student drew my attention to an invaluable suggestion for
the study. I should admit that I had worked for 25 years in the educational
field, contributed in writing about 22 textbooks and revised more than 10
textbooks in the area of the mathematics of the secondary stage without
observing that most of the instances e.g. examples or exercises that we
present to our students belong to a well-defined learning structure 'S'.
This hint has emerged through a discussion(in a class in the final grade
of secondary school of girls), of the following question:

Let $f$ be the curve: $x \rightarrow x^2 + x$. Find the equation of the tangent line
to $f$ at $(a,2)$ with $a < 0$.

This girl - who was very proficient in mathematics - could not work out the
problem and I felt that she was worried. When I asked her what was wrong,
she answered that she "worries very much over the existence of constant,
say 'a', in any mathematical problem". She also added "Why don't you use
'(x,2) with $x < 0'$ instead of '(a,2) with $a < 0'$? When I asked her what
difference this made the surprising reply was "Why don't mathematics text-
books abandon using constants as such? Leave this to physics, we know we
need them there."

In an attempt to show how the theme of the 'constant' appears to the school
students, I asked 38 students in two different classes (one for boys and
one for girls) in the final year of school just to say which was the easier
question in each of the following:

solve for $x$

(i) $4x + 12 = 51$ and (ii) $4x = 5a$

solve for $x$

(i) $x^2 + bx + 5 = 0$ and (ii) $x^2 - 17x - 54 = 0$

find the slope of the curve:

(i) $y = x^2$ at $x = 2a$ and (ii) $y = x^3 + 5x^2 - 2$
at $x = 23$. 
Surprisingly, all of them chose such instances which do not contain the constants 'a' or 'b'. As a matter of fact I received the same answers as those of the students when I posed two similar items in the first row to colleagues in the Inspectorate of Mathematics and PhD students in physics. None of them could indeed observe similarity in structure, and further I should have expected that a student aware of the identity in structure, would have chosen the others (with the constant) because of the simpler calculation (see P.5-7).

Consequently, this guided me to study the nature of those instances that define the presentation 'R' of mathematical tasks whereas such 'R' is dominant in coping with mathematics.

7.1.1. The Nature Of Instances In Textbooks

In a careful study of such mathematics instances, I found that most of them (not less than 95%) do not contain any constants. Here are some examples from the contents of the textbooks of mathematics in the secondary stage in Kuwait:

(i) solve for x: \[ 2x + 5 = 7, \quad x - 2 = \frac{x - 1}{4} \]
\[ x^2 + 2x - 1 = 0, \quad 2x = 7(x - 1) \]

(ii) find the solution set of:
\[ 2x - 1 < 4x + 3, \quad |x - 1| < 2 \]
\[ x^2 - 4x + 3 < 0, \quad x^2 < 9 \]
(iii) find the centre and radius of each of the following circles:

\[ M_1: \quad x^2 + y^2 = 10 \]
\[ M_2: \quad x^2 + y^2 + 2x - 4y + 1 = 0 \]
\[ M_3: \quad (x-1)^2 + y^2 = 16 \]
\[ M_4: \quad 2x^2 + 2y^2 + x + y - 6 = 0 \]

(iv) use \((\varepsilon, \delta)\) to prove that:

\[ \lim_{x \to 2} [x] = 2 \quad \text{as} \quad x \to 2 + 0. \]
\[ \lim_{x \to 0} \frac{x^2 + x}{x} = 1 \quad \text{as} \quad x \to 0 \]

(v) find \(\frac{dy}{dx}\) if:

\[ y = x^2 + x, \quad y = \ln(x^2 + 2), \quad y = \frac{2x - 1}{x + 5} \]
\[ y = \sin2x, \quad y = \exp(x^2)(x + \cos x) \]

(vi) find:

\[ \int (2x + 1) \, dx, \quad \int \frac{3(\frac{1}{x} + x)}{2} \, dx, \quad \int x \sin x \, dx \]

By extending my investigations to some available texts in some foreign countries, such as Howes (1975) and Alwin (1971) in the U.S.A., or David and Hicks (1975) and Book 1 of SMP (1973) in the U.K., I could observe that most instances are similar to those instances of the Kuwait textbooks of mathematics in that they mostly do not attempt to make use of constants.

I therefore assumed that the presentation 'R' of mathematical tasks is almost dependent on a certain learning structure 'S' that only includes such instances that do not contain usual constants which do not denote specific numerical values.
7.1.2. Two Structures 'S' And 'H'

In defining those structures, it was found that the term constant should be clear. In this thesis a constant will refer to any fixed number, real or complex that is denoted by an implicit form, e.g. $a$, $2b$, $a + 2i$ with $i = \sqrt{-1}$. Hence, other numbers such as $2$, $\frac{3}{2}$, $\sqrt{5}$, $\pi$, $e$ are said to be constants of explicit forms.

Another point worthy of note is that the mathematics that is discussed here is the mathematics based on set-language. Hence an algebraic expression such as $x + y$ is indeed a function as: $f(x) = x + y$ ($y$ being treated as constant), $f(y) = x + y$ ($x$ being a constant), or $f(x, y) = x + y$ (no constants). Such assumptions are necessary for further development.

7.1.2.0. Definition 7.1

(i) An instance that does not contain any constant, say 'a', besides the variable(s) - if any - will be called an 'S-instance'.

(ii) An instance that contains at least one constant, say 'a', besides the variable(s) - if any - will be called an 'H-instance'.

For example, the following are S-instances:

(1) Solve for $x$: $2x + 1 = 5$
(2) Allocate the point (2,3).
(3) Find $\frac{dy}{dx}$ if $y = x + 1$
(4) Find $\int (3x + 2) \, dx$
While the following are H-instances:

(1) Solve for x: \(2x + a = 5\) "a being a constant"
(2) Solve for x: \(2x + a = b\) "a, b being constants"
(3) Find: \(\int(2x + a)\,dx\) "a being a constant"

It could be noticed that no instance could be both an S-instance and an H-instance, provided that a constant is employed explicitly.

Accordingly, it could be noticed that both the teachers and students mainly cope with mathematics through such S-instances. The H-instances are rarely used in presentation of mathematical tasks in the texts. It is then expected that the S-structure comprises only S-instances while the H-structure comprises only H-instances.

7.1.2.1. Definition 7.2

The set of all S-instances in the contents of the field of this study will be called the S-structure, (S, briefly).

7.1.2.2. Definition 7.3

The set of all H-instances in the contents of the field of this study will be called the H-structure, (H, briefly).

In deliberately studying the school mathematics in Kuwait that are being coped with in the field of this study, we find that the S-structure is the dominant learning structure, while H very rarely figures in the textbooks, which continue to have a major influence in the classroom. According to
Howson et al. (1981) who find that in many cases textbooks still effectively determine the curriculum, we can see just how far the text is effective.

Such influence of the texts could be understood in the following two short experimental cases, which show how far the 'S'-structure formulates and delimits their mathematical knowledge.

Case 1: In a short test, I asked 21 students in the 2nd Grade of a secondary school (ages: 15-16) to say whether the following three equations in $x$: $2x + 1 = 5$, $2x + \frac{1}{3} = 7$ and $2x + a = 5$ can be solved in the same way or not, and to distinguish the different patterns of equations. I found that only 4 students out of 21 could observe that the three items belong to the same pattern, while the other 17 (81%) could not find a place for $2x + a = 5$ in the pattern. It is to be said that these students had been meeting as well as practicing such items for three years at least.

Case 2: Similarly 43 students in the 4th (final) grade of the secondary school were asked similar questions for the inequalities in $x$: $x^2 > 4, x^2 > a$ (with $a > 0$) and $x^2 > 5$, and only 6 of them could respond correctly, while the other 37 (88%) could not find a place for $x^2 > a$ in the same pattern with the other two.

It is to be noted that students in this case had also been meeting and practising corresponding items for at least three years.
7.1.2.3. Some Basic Remarks

1) Any instance in set-language is usually well defined in that a constant or a variable is clearly identified. Consequently, any such instance cannot at the same time both contain and not contain a constant. An instance therefore cannot, indeed should not, be isolated from its context despite the fact that a mathematical form of an instance could be employed in both \( S \) and \( H \), but under different contexts. For example, the equation \( x + y = 2 \) of the two unknowns \( x, y \), is an \( S \)-instance, but when it is employed in finding \( 'x' \) in terms of \( 'y' \) then such \( 'y' \) is treated as a constant and under such employment it is an \( H \)-instance. Hence it is to be understood that a mere general form of an instance should not classify this instance as to \( S \) or \( H \) in isolation from the context that explains how this instance should be employed. An instance in terms of its employment is only either in \( S \) or \( H \) and in this sense we might accept that \( S \cap H = \emptyset \) (the null set), (see P.4-5).

2) \( H \) contains all basic and standard forms that constitute a basis for deriving formulae or rules, either in mathematical or in non-mathematical fields since such a generalization is genuinely based on constants e.g. \( ax^2 + bx + c = 0 \) is a standard form for deriving a general rule (formula) for solving \( x \) in such equations.

3) \( S \) is dominant in most textbooks, although \( H \) does make appearances in texts of traditional mathematics. It also did appear in the earliest shift to modern mathematics texts in Kuwait in the beginning of the last decade, but complaints of students and teachers of the difficulties that were encountered in dealing with such \( H \)-instances have eventually resulted in an unconscious shift to \( S \).
The dominance of the S-structure (space) in the task-presentation can be seen in Appendix C which provides two examples from current textbooks in Kuwait.

7.2. II. Primitive Implications Of S and H: Two Indicators

In this part, I generally intended to learn (i) how far the current achievement of mathematics that takes place in S, could work in necessary knowledge relevant in H, and (ii) about any properties that might imply indications of effectiveness of knowledge gained in S or H. Hence, I found that a first indicator could possibly satisfy (i) if this indicator is based on identifying how far the computational knowledge achieved in S satisfies transferability into H in two cases: one relevant to a non-mathematical field, e.g. physics and the other relevant to mathematical form. This indicator is therefore supposed to discuss transferability of computational knowledge from S into H-Physics as well as H-Maths. Thus, the indicator could be expected to constitute a battery of three tests in S, H-Maths and H-Physics, so that a certain behaviour concerning computational knowledge is tested under the three tests. By H-Physics and H-Mathematics, I only wanted to differentiate between the instances used in both in that one is taken from physics text and the other is not.

As for a second indicator that expected to satisfy (ii), I found that this indicator is better based on finding such general properties of S and H through learning how mathematical knowledge in S and correspondingly in H could correlate with problem-solving. As a far aim of this indicator, I indeed intended to support reliance on the efficiency measure for further purposes.
7.2.0. First Indicator/Computational Knowledge - $T_1$: A Battery Of Three Tests

This indicator, by employing both $S$ and $H$ tries genuinely to throw light on what is commonly said: "students can do it in mathematics but not in a non-mathematical field, e.g. physics". The indicator is therefore expected to discuss "how students can do it in mathematics, but not in physics and what they actually can do in mathematics". As a consequence, a battery $T_1$ of three tests was constructed with the aim of identifying how computational knowledge primarily achieved in $S$, could be demonstrated through $H$ in both physics and mathematics (see Appendix D).

7.2.0.1. Description Of Method.

The aims behind the battery $T_1$ throw light on the construction of $T_1$. It generally aims to find out how the computational knowledge that is achieved in $S$ could in general function in $H$, whether in maths or physics.

(a) The Material

$T_1$ consists of three tests $t_1$, $t_2$ and $t_3$ where $t_1$ and $t_2$ were constructed in $H$ and $S$ spaces on basis of $t_3$ which is based on instances that are both used in physics and belong to $H$. (I avoided in $t_1$, $t_2$ and $t_3$ numbering from zero to avoid complexity in numbering '00' in some places).

All items in $t_1$, $t_2$ and $t_3$ were constructed to satisfy the computational behavioural objective "to apply knowledge of finding a subject in an equation".
(a₁) $t_3$ was first constructed on basis of choosing four formulae that are learnt in physics: $t_3 = q_{31}, q_{32}, q_{33},$ and $q_{34}$. ($t_3$ consists of items included in the physics texts and therefore it will be denoted as $H$-physics).

(a₂) $t_1$ was constructed by replacing the symbols that are used in physics by other symbols that are most used in mathematics: $t_1 = q_{11}, q_{12}, q_{13},$ and $q_{14}$. ($t_1$ consists of usual items of mathematics and therefore it will be denoted as $H$-maths).

(a₃) $t_2$ was constructed on the same basis as $t_3$, but in addition, each item was equipped with a set of numbers to be substituted so that the instance could then belong to $S$: $t_2 = q_{21}, q_{22}, q_{23},$ and $q_{24}$. ($t_2$ consists of usual items of mathematics and it will be called $S$-maths).

Hence for a fixed 'j' the three items of $q_{ij}$ with $i = 1, 2, 3$, have the same mathematical form and are similar or identical from a behavioural point of view. Here is an example for $q_{11}, q_{21},$ and $q_{31}$ in $t_1, t_2,$ and $t_3$ respectively.

$q_{11} (t_1)$: If $z = \frac{1}{h} \cdot \frac{x \cdot y}{x - y}$, then find $x$ in terms of the other elements $z, y,$ and $h$.

$q_{21} (t_2)$: If $z = \frac{1}{x} \cdot \frac{y \cdot n}{y - n}$. Find $y$ when $z = 3, x = 2$ and $n = 5$.

$q_{31} (t_3)$: The capacity 'C' (Farad) of a special condenser is given by $c = \frac{1}{k} \cdot \frac{R \cdot r}{R - r}$ where $k$ is a constant, and $R, r \ (m)$ are the measures of the large and small radii of the condenser. Find $R$ in terms of $c, k,$ and $r$.

The panel (see P. 2-24 ) which participated in discussion of the test
confirmed that \( q_{21} \) will be trivial if the substitution is introduced beforehand for such students in the final year, however, the students are acquainted with such procedures in their work, and besides the panel believed that, as a test-construct, students will begin by substitution in \( t_2 \) to make such instances in their pure form of S-instances.

(b) Subjects And Setting

The subjects who generally participated in the tests were 207 students in 8 classes in the final grade of four secondary schools, two classes from each school. The schools were two for boys and two for girls.

(c) The Procedure

The three tests \( t_1, t_2 \) and \( t_3 \) were administered to the students on three successive days respectively. The first two were administered by the mathematics teachers who teach the experimental classes while the third was administered by the physics teacher or the senior teacher of physics in the school.

None of those who administered the tests has had any knowledge about the nature of the tests. Furthermore the answers of the students were to be written on the question paper. The duration was 30 minutes for each test and all students finished in time.

In marking, the student was given (+1) for both perfect procedure and answer; otherwise 'zero'. The tests were marked by those who helped in
administration of the tests.

7.2.0.2. The Battery-Constructs

(1) Students in working in $t_2(S)$ would most begin by substitution.

(2) Ability in $t_2(S)$ is not necessarily an indicator for transferability in $t_1$ or $t_3(H)$.

(3) Ability in $t_1$ (H-maths) could be an indicator for transferability in $t_3$ (H-physics).

(4) Ability in $t_3$ would surpass ability in $t_1$ (familiarity).

7.2.0.3. Discussion Of Results.

In excluding the students who could not participate in the three tests, it was found that 196 students could participate in all the three tests.

(1) Tables

(i) Table 7.1 denotes the number of correct responses in each item of the three tests, as well as relevant percentages, while table 7.2 helps in identifying the correct responses in the three identical items from a behavioural point of view in the three tests, given in percentages.
Table 7.1

Numbers And Percentages Of Correct Responses In Each Item Of The Three Tests.

<table>
<thead>
<tr>
<th>Structure</th>
<th>H (maths): $t_1$</th>
<th>S(math): $t_2$</th>
<th>H(physics): $t_3$</th>
<th>No of correct responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item</td>
<td>$q_{11}$</td>
<td>$q_{12}$</td>
<td>$q_{13}$</td>
<td>$q_{14}$</td>
</tr>
<tr>
<td>$t_1$</td>
<td>29</td>
<td>119</td>
<td>75</td>
<td>122</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.15</td>
<td>0.61</td>
<td>0.38</td>
<td>0.62</td>
</tr>
<tr>
<td>$t_3$</td>
<td>0.20</td>
<td>0.69</td>
<td>0.44</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Table 7.2

Direct Comparative Percentages Of Correct Responses In Each Item In Any Test And Its Corresponding Similar (Identical) Items In The Two Other Tests.

<table>
<thead>
<tr>
<th>+ Test</th>
<th>$q_{1j}$</th>
<th>$q_{2j}$</th>
<th>$q_{3j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>0.15</td>
<td>0.55</td>
<td>0.20</td>
</tr>
<tr>
<td>$t_2$</td>
<td>0.61</td>
<td>0.75</td>
<td>0.69</td>
</tr>
<tr>
<td>$t_3$</td>
<td>0.38</td>
<td>0.52</td>
<td>0.44</td>
</tr>
</tbody>
</table>

(ii) The following Table 7.3 is considered to be the indicative one since it provides a comparative knowledge of the performance of students in the three similar items. This comparative knowledge is introduced according
to eight matrices that indicate for such knowledge. For example \([111]\) denotes that the student has correctly performed in the three identical items in the three tests, while \([010]\) denotes that the student has correctly performed in an item in \(t_2\), but he failed in the similar ones in \(t_1\) and \(t_3\). Furthermore, each number in a cell of 7.3 denotes the number of students who acted accordingly.

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{Total} & 196 & 196 & 196 & 196 & 196 & 196 & 196 & 196 & 196 \\
\text{[q_{11}, q_{21}, q_{31}] } & 14 & 9 & 1 & 14 & 5 & 71 & 10 & 72 \\
\text{[q_{12}, q_{22}, q_{32}] } & 100 & 10 & 6 & 15 & 3 & 22 & 15 & 25 \\
\text{[q_{13}, q_{23}, q_{33}] } & 55 & 11 & 3 & 10 & 6 & 26 & 18 & 67 \\
\text{[q_{14}, q_{24}, q_{34}] } & 96 & 10 & 8 & 12 & 8 & 27 & 11 & 24 \\
\end{array}
\]

Table 7.3

Number Of Both Correct And Wrong Responses In Any Three Identical Items In The Three Tests \(t_1\), \(t_2\) And \(t_3\) Under Any Corresponding Combination Of Three Elements Chosen From '1' And '0' Where '1' Refers To Correct Response While '0' Refers To Wrong Response.

2. **Analysis Of Results**

The analysis will be directed towards discussion of the constructs in 7.2.0.2.
(i) It was observed that most students in working in $t_2$ began by substitution. For example in $q_{21}(t_2)$, a student began in: $3 = \frac{1}{2} \cdot \frac{y \times 5}{y - 5}$

This is in line with Construct 1.

(ii) The third column in table 7.2 shows how competency in $S$ could not predict for competency in $H$(maths) or $H$(physics) (i.e. 0.15, 0.55 and 0.20).

Furthermore: Table 7.1 indicates that 108 students could solve $q_{21}$ while:

Table 7.3 indicates that $14 + 9 = 23$ students could solve both $q_{11}$ and $q_{21}$.

Hence $23/108 = 21\%$ of those who demonstrated competency in $S$ were competent in $H$-maths.

Similarly: $28/108 = 30\%$ of those who demonstrated competency in $S$ were competent in $H$-physics of $q_{31}$.

This in my belief, interprets the statement that "students can do something in mathematics but not in physics" while the students themselves can't do it in $H$-maths. Hence the genuine question to be asked could be "what is it that they can do in maths?" A student who is able in such items as $q_{21}$ in $S$ is not necessarily able to transfer this knowledge into $H$, whether in maths or in physics. Evidence could be found in Columns 5, 6 and 7 of table 7.3, which refers to such disparity in performances in $S$ and $H$.

Table 7.2 refers also to such differences in performances in $S$ and $H$.

(Construct 2).

(iii) Columns 1 and 3 in table 7.3 together with table 7.1 show that:

\[
\frac{14}{29} + \frac{1}{6} = 52\%, \quad \frac{100}{119} + \frac{6}{7} = 89\%, \quad \frac{58}{75} = 77\%, \quad \text{and} \quad \frac{104}{122} = 85\% \text{ of those who demonstrated competency in } t_1 (H\text{-maths}) \text{ could do it in } (H\text{-physics}), \text{ at least for identical items.}
\]
Such percentages could be optimistic since students do not learn in H, however some of them might have skipped to H-maths through S-or H-physics (Construct 3).

(iv) Table 7.2 shows that ability in performance of students in H-physics surpasses corresponding ability in performance in H-maths. This was to be expected since students are familiar with the use of the same items of $t_3$ in their physics lessons (Construct 4).

7.2.0.4. Conclusion 7.1

Mathematical computational knowledge gained in the S-structure is not necessarily satisfactory in functioning for a desired complete transferability to non-mathematical fields, e.g. physics. However, such computational knowledge as could be gained in the H-structure, could be expected to be functioning better.
### 7.2.0.5 Summary Of The First Indicator $T_1$

<table>
<thead>
<tr>
<th>TITLE</th>
<th>First Indicator: Computational Knowledge - Battery $T_1$ of three Tests: $t_1$, $t_2$, and $t_3$. (Appendix D).</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIMS</td>
<td>To find out how computational knowledge usually gained in $S$, could interact with $H$.</td>
</tr>
<tr>
<td>MATERIAL</td>
<td>The battery consists of three tests $t_1$, $t_2$, and $t_3$, however $t_1$ and $t_2$ were constructed on the basis of $t_3$. $t_3$ was constructed by choosing four formulae from physics and the computational skill was based on finding a 'subject' in each formula in terms of the other elements in the formula. Hence $t_1$ was constructed for the same purposes in 'H' as well as $t_2$ in $S$.</td>
</tr>
<tr>
<td>SUBJECTS AND SETTING</td>
<td>196 out of 207 students in 8 classes in the final grade of school could participate in the three tests.</td>
</tr>
<tr>
<td>PROCEDURE</td>
<td>$t_1$, $t_2$, and $t_3$ were given to the students in three successive days respectively. Duration was 30 minutes for each test.</td>
</tr>
<tr>
<td>VALIDITY</td>
<td>(a) Content validity was satisfied by a panel who discussed construction.</td>
</tr>
<tr>
<td></td>
<td>(b) Construct validity was satisfied through the analysis of results. Such validity mainly assumed that, ability in $S$ is not necessarily an indicator for relevant ability in $H$.</td>
</tr>
</tbody>
</table>
**RELIABILITY**

This was satisfied by the inter-rater reliability that is discussed in 2.4.2.1.1 of Chapter 2.

**GENERAL CONCLUSION**

Mathematical computational knowledge gained in 'S' cannot satisfactorily satisfy relevant knowledge in 'H' whether this relevant knowledge is found in mathematics or non-mathematics fields.

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7.2.1. Second Indicator/Mathematical Knowledge - $T_2$: A Set of Three Tests

The test $T_0$ that is reported in Chapter 6, had shown that the efficiency measure could be reliable in studying the advantages in gaining knowledge in S and H. Furthermore, with the aim of reinforcing such reliability of this measure, $T_2$ was proposed for studying the relationship of certain mathematical knowledge in S and H related to a well-defined problem P. Thence, the measure could be more reliable if it could in further studies coincide with this relationship. It was seen that a comparative correlation of S and P as well as that of H and P could induce a greater convenience in further studies. The two correlations will be based on the general conclusion in 6.1.2.2.3 in that all the tests in $T_2$ will be based on the following four fundamental phases.

---

7.1.2.0 Four Fundamental Phases In Construction Of Tests

(i) Choose a well-defined problem P in the domain of what students have learned. P should be new to the students; at least different from that available in the textbook.
(ii) Produce a set B of behavioural objectives which are necessary for P, i.e. B identifies basic prerequisites of P. (It is advisable that B is identified by the help of a panel).

(iii) Construct a kernel \( X_0 = S_0 \) of question-items in S such that each \( B_i \) in B is satisfied (tested) by a unique \( s_i \) in \( S_0 \). \( S_0 \) is a kernel in S for P (under such a B).

(iv) Construct a kernel \( X_0 = H_0 \) in H similar to \( S_0 \). \( H_0 \) is a kernel in H for P (under such a B).

7.2.1.1. Description Of Method

The indicator \( T_2 \) (Appendix E) consists of three tests \( t_1, t_2 \) and \( t_3 \).

(a) **The Material**

(a1) \( t_1 \) consisted of 9 question-items. Here I present a brief discussion of construction that is based on the four phases.

(i) \( P = \text{item 9:} \) Find the area of the region that is bounded by \( f(x) = x^2 - x - 2, \ x = -2, \ x = 1 \) and the x-axis.

(ii) In solving P, four basic behavioural objectives were identified for B. For example:

\( B_1: \) To identify and apply the knowledge of the relationship between a function and its primitive, provided that the primitive is a polynomial of the third degree.

(iii) \( S_0 \) consequently consisted of four elements \( s_1, s_2, s_3 \) and \( s_4 \) where each one of them satisfies a unique \( B_i \) in B. \( S_0 \) = items 1, 2, 3 and 4 that respectively correspond to \( s_1, s_2, s_3 \) and \( s_4 \). For example:

\( s_1 = \text{Item 1:} \) Let \( f(x) = x^3 - 2x^2 + 5 \) be a primitive of a function g. Find g. (Here \( g(x) = f(x) \).)
(iv) \( H_0 \) is similarly constructed in \( H \). \( H_0 = \) Items 5, 6, 7 and 8 that respectively correspond to \( h_1, h_2, h_3 \) and \( h_4 \). For example:

\[ h_1 = \text{Item 5. If } T(x) = a^2x^3 + 3b^2 \text{ (a and b being constants) is a primitive of a function 'g'. Find 'g'.} \]

\((a_2)\) \( t_2 \) consisted also of 9 question-items where:

(i) \( P = \) Item 9: Let \( D(x) = x^2 - 4x \) and the line \( L: y = 6x + c \).

Find \( c \) when the line \( L \) is tangent to the curve \( D \).

(ii) Four basic behavioural objectives were identified for \( B \). For example: \( B_2 \): To identify and apply knowledge of the slope of a line.

(iii) A set \( S_0 \) in \( S \) was constructed on the basis of \( B \). \( S_0 = \) Items 1, 2, 3 and 4. For example:

\[ s_2: \text{Find the slope of the line } L: y = 4 - 3x. \]

(iv) A set \( H_0 \) in \( H \) was constructed similarly as in \( S_0 \). \( H_0 = \) Items 5, 6, 7 and 8. For example:

\[ h_2: \text{Find the slope of the line } m: 4by = 4c - 3ax \text{ (a, b and c being constants, } b \neq 0). \]

\((a_3)\) \( t_3 \) consisted of another 9 question-items where:

(i) \( P = \) Item 9: The point \( M = (x, f(x)) \) moves on the curve, \( f(x) = x\sqrt{x} \). Let \( N = (8, 0) \). Find the position(s) of \( M \) when the distance \( D \) between \( M \) and \( N \) is least.

(ii) Four basic behavioural objectives were identified for \( B \). For example:

\[ B_3: \text{To identify and apply knowledge of finding a point on a curve that corresponds to a critical value.} \]

(iii) \( S_0 \) was constructed on the basis of \( B \). \( S_0 = \) items 1, 2, 3 and 4 corresponding to \( s_1, s_2, s_3 \) and \( s_4 \) respectively. For example:
Let \( f(x) = \frac{1}{3}x^3 - 4x \). Find the \( x \) co-ordinates when \( f \) has critical values.

(Hint: such critical values here correspond to \( f(x) = 0 \)).

(iv) \( H_0 \) similarly contained Items 5, 6, 7 and 8 that correspond to \( h_1, h_2, h_3 \) and \( h_4 \) respectively. For example:

\( h_3: \) Given \( f(x) = a^2x^3 - 3x^2 \), \( a \) being constant \( \neq 0 \). Find the \( x \) co-ordinates when \( f \) has critical values.

(b) Subjects And Setting.

(b_1) The test \( t_1 \) was presented to 73 students in a secondary school for girls in three classes in the final grade.

(b_2) The test \( t_2 \) was presented to 84 students in Kuwait University. The students were reading for their first semester in the University, directly after they had finished the final year of secondary school in Kuwait in 1977-1978. All the students were specializing in sciences and engineering, but not in Maths, also they all were reading a course called '101' in introductory calculus in the University. It is to be noted that only 218 students were following this course in the University.

(b_3) The test \( t_3 \) was presented to 53 students in Kuwait University who were reading for their second semester in the University after leaving school in 1977-1978. These students had finished reading course '101' and were reading the next course '102' in more advanced elementary calculus. Moreover, all of those were specializing in sciences as well as in engineering but none of them in maths. Furthermore, 171 students were following the course '102' in University.

(c) The Procedure

(c_1) \( t_1 \) was presented to the 73 students in these three classes as a
usual achievement test at the same time. Students' answers were to be written on the test paper. The duration of the test was 45 minutes and all the students gave back the sheets in time.

\((c_2)\) \(\hat{t}_2\) was administered to the 84 students in course '101' of mathematics in Kuwait University by 4 lecturers who lecture to them in mathematics, on two different days. The answers were on the same sheet and the duration of the test was 50 minutes and all the students gave back the sheets in time.

\((c_3)\) \(\hat{t}_3\) was administered to the 53 students who attend the course '102' of mathematics in Kuwait University by 3 of their lecturers of mathematics, on three different days. The procedure for answers and duration was the same as in \(c_2\).

(d) **Scoring Of Test**

In marking the tests +1 was given to either a perfect procedure with correct answer, or with a perfect procedure with simple miscalculations; otherwise a mark of 'zero' was given. \(\hat{t}_1\) was corrected with the help of the mathematics staff in the School while I only corrected \(\hat{t}_2\) and \(\hat{t}_3\).

7.2.1.2. \(T_1\)-Constructs

(1) The relationship between \(S_0\) and \(P\) is less strong than that of \(H_0\) and \(P\).

(2) \(S_0\) correlates with \(H_0\) in school less than that in the University.
7.2.1.3. Discussion Of Results

1. Tables

The analysis of the results will be based on discussing such constructs as are relevant to the test aims which are embodied in the constructs. The following table 7.4 illustrates the main aspects of the results.

<table>
<thead>
<tr>
<th>Test</th>
<th>No. of Students</th>
<th>Level</th>
<th>Mean</th>
<th>Cor. ($S_o$, $H_o$)</th>
<th>Cor. ($S_o$, $P$)</th>
<th>Cor. ($H_o$, $P$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>73</td>
<td>Final Grade of School</td>
<td>2.32</td>
<td>0.58</td>
<td>0.41</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_2$</td>
<td>84</td>
<td>Kuwait Univ. Course 101</td>
<td>2.31</td>
<td>0.52</td>
<td>0.37</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.46</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_3$</td>
<td>53</td>
<td>Kuwait Univ. Course 102</td>
<td>2.75</td>
<td>0.58</td>
<td>0.37</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.87</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.4

Illustrative Table For Competency-Mean And Corresponding Correlations In $P$, $S_o$ And $H_o$.

2. Analysis Of Results

Table 7.4 shows that the first construct is reasonable. While the second construct was not compatible with results in the table. I believe that this is due to the fact that students in the final year take any test more seriously than any other students, especially those who have passed this stage and found a place in the University. Furthermore, I learnt that some lecturers in the University told the students about the nature of the test, which could have had some effect on their enthusiasm to do better.
Nevertheless, the first construct, which is of most interest to this study, is genuinely satisfied.

7.2.1.4. Conclusion 7.2

Mathematical knowledge from H has a stronger relationship with problem-solving than that of S. This could reinforce prospective reliance on the efficiency measure, if it could support such relationships.
### 7.2.1.5. Summary Of The Second Indicator $T_2$

<table>
<thead>
<tr>
<th>TITLE</th>
<th>Second Indicator: Mathematical knowledge - A set $T_2$ of three tests $t_1$, $t_2$ and $t_3$, (Appendix E).</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIMS</td>
<td>The study aimed to identify a general view of relationships of performance in problem-solving and relevant knowledge in $S$ and $H$.</td>
</tr>
<tr>
<td>FOUR FUNDAMENTAL PHASES</td>
<td>Four instructions were proposed for such a study that might satisfy aims. (i) to choose a problem $P$, (ii) to identify $B$ of $P$, (iii) to construct a set $S_0$ in 'S' that possibly assesses $P$ and (iv) to construct in $H$ a set $H_0$ similar to $S_0$.</td>
</tr>
<tr>
<td>MATERIAL</td>
<td>Three tests $t_1$, $t_2$ and $t_3$ were constructed on the basis of the four fundamental phases.</td>
</tr>
</tbody>
</table>
| SUBJECTS AND SETTING | (1) For $t_1$: 73 students in three classes of the final year of school participated in the test.  
(2) For $t_2$: 84 students who were following '101' course in Kuwait University participated in the test.  
(3) For $t_3$: 53 students who were following '102' course in Kuwait University participated in the test. |
| PROCEDURE | The tests were mostly run by those who teach or lecture the subjects of the tests. |
| RELIABILITY | This was satisfied by the inter-rater reliability in 2.4.2.1.1 of Chapter 2. |
### CONCLUSION

| The relationship between mathematical knowledge in H and problem-solving is stronger than that in S. The efficiency measure is more reliable in standing with such relationships. |

7.2.2. General Discussion: Implications of S And H

It was found that it is reasonable now to present a brief summary of the foregoing in throwing light on what might be helpful.

The study in this Chapter demonstrated that there is an almost dominant structure $S$ underlying current task-presentation $R$ in the mathematics of the field of this study. Another structure $H$ was recognised to be essential in mathematics and non-mathematical fields. However, this $H$ has a poor appearance in current mathematics teaching-learning situations. The effects of the limitation of building-up mathematical knowledge only in such a space as $S$ can perhaps be observed when 81% of a sample of students in the tenth grade in secondary school could not find a place for $2x + a = 5$ in a related pattern with $2x + 1 = 5$ and $2x + \frac{1}{3} = 7$ (Case 1, P.7-8), as well as when 88% of a sample of students in the final grade in secondary school could not find a place for $x^2 > a$ (with $a > 0$) in a similar pattern to $x^2 > 4$ and $x^2 > 5$ (Case 2, P. 7-8). Furthermore, the first indicator 7.2.0 demonstrates how computational knowledge gained in $S$ was not satisfactorily transferred to related knowledge in $H$ whether in physics or mathematics instances. However, this does throw light on what is commonly held about the weakness of students in computational knowledge in physics, despite the fact that such students are thought to have attained such knowledge properly.
Moreover, the second indicator has demonstrated that mathematical knowledge in H correlates with problem solving more adequately than that in S. It was then believed that this indicator could be helpful in providing more reliance for the efficiency measure.

As a matter of fact, these implications could be expected since the H-structure could be described as one involving, by its nature, more abstraction than S because of the existence of such constants of implicit form; whereas gaining knowledge by any means in an abstract level would logically be expected to promote higher mathematical abilities. A question arises: "Does this mean that a replacement of S by H would answer the problem of development in this research?" However, the answer is not as simple as it might appear.

Here, in analysing the 'pros and cons' of S and H, I shall primarily be discussing this answer although with genuine further implications.

7.2.2.0. The 'Pros And Cons' of S and H

If the term learning structure refers to such a structure as underlies the presentation of mathematical tasks, then S could be, in this sense, a learning structure, while H at the moment is not. Here I shall look at some properties of S and H to discuss how far H could possibly be a learning structure.

(1) The S-structure is certainly indispensable, in that, it gives the chance to understand easily, in a concrete way, the concepts or rules to
be included, while \( H \) is of no use in this sense. Illustrative examples in \( H \) for a concept or rule are expected not to be as easily grasped as those in \( S \). Besides that such examples in \( H \) could possibly cause frustration in children trying to cope with mathematics.

(2) All basic instances for rules or formulae are found in \( H \), while similar instances in \( S \) might help to understand relevant ones in \( H \). But this does not go against the fact that students still find difficulties in working through such basic instances in \( H \). Teachers, therefore, in most cases identify the rule or formula in \( H \), and then quickly shift to relevant instances in \( S \). They feel that this is fair enough for coping with the rule or formula. For example, they show the student how to find a formula for \( x \) in \( ax^2 + bx + c = 0 \), but, for instance, I mostly did not find any questions which asked the students to find this formula whether in a test or general examination. Such procedures leave the foundations unstable since this could have the effect that students become instrumental operationalists in the sense that they would miss basic fundamentals by mere applications of rules.

(3) Mathematics is an abstract subject, and building for abstraction cannot be done adequately through \( S \). Those students who were in the majority in Case 1 (P. 7-8) and could not place \( 2x + a = 5 \) in a pattern with \( 2x + 1 = 5 \) and \( 2x + \frac{1}{3} = 7 \), and the others in the final year in Case 2 (P. 7-8) - also in the majority - who could not find a place for \( x^2 > a \) (with \( a > 0 \)) with the structural pattern that includes \( x^2 > 4 \) and \( x^2 > 5 \), are not expected to have achieved the required behaviour for the pattern if they could only satisfy such \( S \)-instances.

(4) Moreover, in discussing the battery of tests \( T_1 \), it was found that
students who could satisfy a desired behaviour in S could not demonstrate a related competency in other similar instances in H, whether those instances were in mathematics or in physics (7.2.0).

(5) A lack of abstraction and promoting transferability is found in S (as shown in T₁). Further, S correlates with problem-solving less strongly than H (as shown in T₂). But it is fair to say that instances of H create, as in the past, 'blocks' which might hinder coping with mathematics.

Clear evidence for the difficulty of the H-instances was widely found - and consequently complaints were raised - when the UNESCO project was first applied in Kuwait secondary schools in 1970/1971, with appearance of such instances in the textbooks of the project.

In connection with the foregoing, I believe that a part, say 'U' of H must contribute with S and must be introduced systematically by taking these 'blocks' into consideration. I also believe on the basis of experience as well as the battery T₁, that the barriers to coping with mathematics in a non-mathematics field, say physics, result from almost completely missing out H as an important factor in presenting mathematical tasks. The learning structure S is essential in that it helps students first to understand tangibly, or apply, a certain mathematical concept, but the mathematical knowledge should be reinforced by that related to H. This view could possibly go along with Klein:

"It is my opinion that in teaching it is not only admissible but absolutely necessary to be less abstract at the start, to have constant regard to applications, and so refer to refinements gradually as the student becomes able to understand them". (Lamon, 1972, P. 28.).
As a consequence to the past discussion, I found the following general conclusion possibly to be relevant.

7.2.2.1. General Conclusion 7.3

(1) The S-structure is indispensable and necessary, but not sufficient for building-up mathematical knowledge as a complete learning structure.

(2) The H-structure is difficult in general. Instances of H do not help for a concrete grasp of mathematical knowledge, but H could not be ignored in contributing for better abstraction and competency.

(3) If L could be introduced as a learning structure proposed by a modification in the methodology, then:
   (i) S should be a part of L
   (ii) A part 'U' of H could be a part of L in the sense of Proposition 6.3, in that U should be operationally applicable, i.e. well-defined and learnable.

Finally, a last word is to be said about the almost total shift from H into S under current situations. One wonders if some educators have justified this shift by interpretation of Piaget (1964) who stated that:

"... I should like to show that learning is possible in the case of those logical mathematical structures, but one condition - that is that the structure which you want to teach to the subjects can
be supported by simpler, more elementary, logical mathematical structure. ... in other words, learning is possible if you base the more complex structure on simpler structure, that is, when there is a natural relationship and development of structures and not simply an external reinforcement."

Could this have been the message that has been taken to justify a complete reliance on S whilst almost completely ignoring H?


In a discussion of relevant theory to this study, it was generally seen that interaction with knowledge takes place towards targets in knowledge by structuring and introducing it in simple statements or instances, e.g. examples. It is expected then that such simple statements of knowledge would diffuse required information for the target-knowledge within a certain population. One of the main purposes could be expected, in that, an individual in the population would become able to use this knowledge appropriately. Simply, a mathematical task mainly involves a theoretical part that constitutes basic information of a certain knowledge, as well as another part R called the presentation (see Appendix A) that enhances and reinforces this information, so that a student, who attempts such a task, would be able to identify and apply this knowledge adequately in further related mathematical situations, e.g. problem-solving.

It was then preconceived that an abstraction of this view can be found in conceiving a structure called a space of knowledge. I saw a space of knowledge as a conceptual structure that is mainly interested in studying
how far a structured knowledge \( W \), in a certain field of study, could be effective when it interacts with a certain normal population 'O' in which the knowledge is supposed to diffuse. Such a \( W \) was taken to be identified - under a certain finite set of rules - by a set of simple states of knowledge, e.g. instances of the presentation \( R \). The diffusion of knowledge within \( O \) would result through any possible interaction between \( W \) and \( O \). If such interaction depends on almost invariant factors, e.g. the textbook, then the diffusion of knowledge would mainly depend on such a \( W \) as would identify such factors. Hence, a space of knowledge was taken as the ordered pair \((W, O)\) or \( W \) briefly if \( O \) is identified, e.g. \((S, O)\) or \( S \) is a space of knowledge.

I also viewed that such diffusion of knowledge through such \( W \) would have an influence in development of the structure of the intellect. For example, a learning structure as a space of knowledge has a certain limitation on diffusion of a certain knowledge. This appears in effects such as those of train-effects which usually induce inside reluctance to any change in a certain familiar form. But, it is to be noted that those effects are not identical in all individuals, otherwise all individuals would be identical. Accordingly, I believed that such a learning structure as \( S \) could have such a limitation as that in Cases 1 and 2 (P. 7-8) and also in the battery \( T_1 \). Hence if, as it was thought learning happens under a change in behaviour, then I saw that better achievement could also be related to a change in learning structure, i.e. space of knowledge, provided that this change should rest on a systematic and applicable modification.

Based on general conclusion 7.2.2.1 (P. 7-32) and Proposition 6.4, I viewed that if \( S \) was indispensable in being a part of a proposed learning structure \( L \), and \( U \) as a part of \( H \) that might be introduced to be applicable
in contribution, then the efficiency that related to U for a problem P should be shown to be higher than that of S. In more abstract form, I viewed that if W and W are two spaces of knowledge that could be related to a certain teaching-learning situation for a certain normal population, then if the efficiencies related to a certain well-defined problem P in the domain of W and W under certain conditions could be assumed to preserve a certain uniformity, then the higher efficiency would possibly suggest that the corresponding space would contribute in a desired modification. Accordingly, and for such uniformity, a concept of uniform operational was introduced. The concept of uniform operational was introduced to maintain certain consistency of results, because of the absence of a principle of invariance (see c, in 6.1.0) that could maintain consistency of results in the psycho-educational field. Simply, if W is related to a space of knowledge and P is a well-defined problem in W, then a uniform operational in W regarding P is able to respond accurately in all kernels \( X_0 = W_0 \) of P in W.

Now, if it is accepted that abstraction of theory has, in general, metaphorical roots, more or less, in experience of reality, then the theory of space of knowledge basically rises from studies in Chapters 6 and 7 as well as additional personal conceptualisation derived from experience.

A development of theory of the conceptual structure i.e. space of knowledge is found in Chapter 5. However, Chapters 8 and 9 are expected to make further progress in developing the research which could be expected to also rest on a theoretical framework such as that in Chapter 5.
7.4. Summary

Chapter 7 set out to discuss three main objectives that rise from three questions that were relevant to the fundamental research propositions in Chapter 6. The questions ask for (I) the identification of two structures: S which underlies the current presentation of mathematical tasks as well as another structure H that could contribute in underlying a possible modification (Proposition 6.4), (II) the implications that could possibly be drawn from such structures and (III) a conception of the construction and employment of theory in development.

(I) In examining the current presentation R of mathematical tasks, it was observed that most instances in R do not contain any constant of an implicit form, say 'a'. This could distinguish two structures: S identified as the set of instances that do not contain constants of implicit form, and H identified as the set of all instances that do contain such constants. It was argued that S is the dominant structure in underlying current presentation while H is almost ignored. However, H could be basic in that it contains all standard mathematical forms that are relevant to both mathematical and non-mathematical fields and furthermore $S \cap H = \emptyset$ (the null set).

(II) The study in this part was directed towards identifying (i) how the achievement of computational knowledge could function in transferability in similar situations of knowledge in H and (ii) how mathematical knowledge in S and its twin in H could correlate with problem-solving. As a first indicator for the first case, it was suggested to constitute a battery $T_1$ of three tests $t_1$, $t_2$ and $t_3$ based on studying how far computational knowledge achieved in $t_2$ of S could be successfully transferred into $t_3$ of H-Physics, as well as into $t_1$ of H-Maths. As a conclusion to this
study, it was found that students were competent in $S$, but they might be competent neither in $H$-Physics nor in $H$-Maths.

The second indicator, was a set $T_2$ of three tests, $t_1$, $t_2$ and $t_3$, where construction of each test was based on four fundamental phases identified as (i) choosing a problem $P$, (ii) producing a set $B$ of basic behavioural objectives that are necessary for $P$, (iii) constructing a kernel $X_0 = S_0$ in $S$ where $S_0$ comprises a set of question-items such that each $B_i$ in $B$ is satisfied (tested) by one $s_i$ in $S_0$ and (iv) constructing a kernel $X_0 = H_0$ in $H$ so that $H_0$ serves in the same way as $S_0$. The three tests were administered to three groups of students at University and in the final year of secondary school in Kuwait. As a general conclusion, it was found that $H$ correlates higher than $S$ with problem-solving. It was hence suggested that the efficiency measure could be more reliable if it stands with such correlation in relevant situations.

Furthermore, in discussing the 'pros and cons' of $S$ and $H$, it was argued that $S$ is dominant and indispensable for any modification, while $H$ is more difficult to replace $S$; however, a part 'U' of $H$ was recommended to contribute to a modification in methodology, provided that such a U could be identified as learnable.

(III) In discussing relevant theory, it was generally considered that interaction with knowledge takes place towards targets in knowledge identifiable as being introduced through simple statements or instances. It is then expected that such simple statements of knowledge would diffuse required information for the target-knowledge within a certain population so that an individual would become able to use this knowledge properly. It was viewed that this constituted a basis for identifying a conceptual
structure, the space of knowledge, identified by the ordered pair \((W, 0)\). Here \(W\) is a set of structured knowledge identified - under a certain finite set of rules - by a set of simple states of knowledge which are supposed to diffuse a certain and required knowledge throughout a certain normal population '0' through interaction between \(W\) and 0. Moreover, it was seen that if \(W\) and \(\hat{W}\) are relevant to two spaces of knowledge, then one of them, say \(\hat{W}\), could possibly contribute in a desired modification if the efficiency (see Glossary) related to \(\hat{W}\), was demonstrated to be higher than that of \(W\).

A development of this theory based on the space of knowledge as a conceptual structure is reported in Chapter 5.
CHAPTER EIGHT

EXPERIMENTAL STUDIES
8. EXPERIMENTAL STUDIES

8.0. Introduction

In this introduction, I shall summarize what I think essential for a better understanding of the follow-up to the study in the two chapters 8 and 9, which include the basic findings and provide for a further outlook. In this connection is discussed the research problem of seeking a feasible development in methodology of teaching-learning situations in order to improve the achievement of students in mathematics. For this it was viewed that if under certain conditions, a number A is assigned to the achievement, then such A would be mainly dependent on a certain independent variable R. Here this R involves those basic instances, examples and exercises that constitute the core of the material contents of a mathematical task, as well as a structure 'S' that underlies such contents, i.e. \( A = g(R) \). Moreover, a true change in R to modify A was believed basically to be related to changes in S in the sense of 6.2.2. It was also viewed that S is indispensable for any modification. However, a part, say 'U' of another structure 'H' was also supposed to take part in modifications in the methodology (7.2.2.1). Consequently, the resultant view regarding modification could be expected to be based on a learning structure based on both S and U. But I believe such modification is better:

(i) if it would help mathematics education in the sense that such a modification in the methodology of teaching-learning situations could be made to rely on an algorithm;
(ii) if it is based on a certain acceptable argument that could be found in a theoretical framework arising mainly from reviewed literature, experience and introspection, based on empirical studies, and

(iii) if it links theory and practice, in that the modification in the methodology should be monitored by a derived theory and that further and relevant empirical works based on theory should give support to such a derived modification.

As a consequence, I found that this chapter should mainly discuss the following general objectives:

I: To propose both the indentification of U and an approach in U for a possible learning space in the sense of general conclusion 7.3, (P.7-32).

II: To discuss the employment of theory in studying modifications.

III: To implement the relevant empirical instruments on the basis of I and II.

It was then viewed that in achieving these general objectives, a step towards the proposition of a modification could be expected.

8.1. I. The U-Structure

In discussing the nature of a 'U' that could be concomitant to S for
modification (7.2.2.1), I believed that such 'U' should be identified and used, so that a teaching-learning situation which implements U would not induce such "blocks" that could result in "matho-phobia". I viewed that such blocks were involved without identification of their sources when there was an almost total abandonment of the essential structure H (Definition 7.3) in the achievement of mathematics, whether this abandonment was planned or not. As far as I know, this abandonment in Kuwait was actually planned on the basis of the growing complaints by teachers and students of those H-instances that coincided with S-instances (Definition 7.1) in the first version of the textbooks of modern mathematics of the UNESCO Project for Arab States in 1970. But it is fair to note that implementation of such H-instances did not, to my knowledge, rest on any systematic study besides abandoning them.

8.1.0. Identification Of U

In an attempt to identify and employ such a 'U', I indeed drew on introspection on relevant knowledge. For example, if instances in S such as \(2x + 1 = 5, 3x - 2 = 4\) are to be employed for a certain behavioural objective, then the inquiry about U would range through discussing related instances in H. These could be in instances such as: \(ax + b = c, a^2x - b = cx\). And as a matter of fact, there might be no genuine reasons against this choice. But here we will also encounter the same dilemma of the unsystematic presentation of H-instances. I then found that possible search for such a systematic presentation could be directed towards the number of constants of the implicit form in H-instances, so that we might maintain a certain level of abstraction found in H through them. Would there be only one, two, three? A smaller number of constants could avoid such 'blocks' and
make the task more learnable.

According to this argument, I proposed a U-structure as follows:

8.1.0.0. Proposition 8.1

'U' is the set of all instances in 'H' which each contain one and only one constant of implicit form, other than the variable(s) if one or more variable exists.

Accordingly, a U-instance is an H-instance that contains only one constant, say 'a'. It is evident that S ∩ U = ∅ (the null set) in the sense of remark 1 in 7.1.2.4.

Here are some illustrative examples.

(i) Solve for x: x + a = 1, a being constant. :U-instance
(ii) Solve for x: x + 2 = 1. :S-instance
(iii) Solve for x: √ax + 2 = a² - a, a being constant a > 0 :U-instance
(iv) Solve for x: ax + b = 1, a, b being constants. :H-instance, but not U-instance.
(v) f(x) = sin(ax) + exp(ax), a being constant. :U-instance
(vi) Describe allocation of (2,y). :S-instance
(vii) Describe allocation of (2,a) where a is a negative constant. :U-instance

It can be noted that allocation in (vi) produces the line x = 2, while
the allocation in (vii) produces a fixed point on this line below the x-axis.

8.1.1. Towards Learnability in U

In discussing an adequate learning approach in U - as a concomitant space of knowledge to S - which could be employed in modifications, I viewed that this approach should avoid any possible 'blocks' that might distract from a proper behaviour in S. In other words, if we choose an S-instance, then the choice of a corresponding U-instance should be manipulated such that a proper behaviour in the first instance could maintain consistency in its associate the U-instance. Hence, if \( s_i \) is an S-instance, then its associate \( u_i \) in U is supposed to preserve behaviour consistent with that in \( s_i \), so that \( u_i \) could be learnable as \( s_i \) is.

For example, let \( s_i \), solve for \( x: x^2 - 4 = 0 \) be an S-instance, then a relevant question is: How can we choose a learnable associate \( u_i \) (of \( s_i \)) in U so that behaviour could be consistent in both \( s_i \) and \( u_i \)? Furthermore, could there be an algorithm, i.e. a set of rules, that could generate a learning associate \( u_i \) for \( s_i \)?

In connection with these questions, I viewed then that both \( s_i \) and \( u_i \) should maintain a certain homogeneity in form so that both could induce identical form in their results. For me this meant that:

(i) \( s_i \) and \( u_i \) should stem from the same general mathematical form and serve the same purpose of a well-defined behavioural objective, e.g. both could be expressed by one general form of a function,
(ii) the behavioural demands in $u_i$ should yield identical results as that in $s_i$, e.g. if the results in $s_i$ do not contain irrationals, then it is logical that results in $u_i$ should not contain an extended form of irrationals in $U$, e.g. $\sqrt{a}$; and

(iii) the results in $u_i$ should maintain the constant, if possible, in order to preserve the abstraction in $U$ conceived to be found in the use of constants, in the results.

Hence, the following proposition of learnability in $U$:

8.1.1.0 Proposition 8.2

If $s_i$ is an instance, chosen in $S$ to satisfy a well-defined behavioural objective $B_i$ then the generation of a (learning) associate $u_i$ (of $s_i$) in $U$ could be based on:

(i) Any fixed number in $s_i$ is replaced by a suitable functional form of a constant, 'a' say, e.g. $F(a)$.

(ii) This replacement should satisfy:

(a) $u_i$ should stem as $s_i$ from the same general mathematical form, so that $u_i$ - precisely as $s_i$ - satisfies $B_i$;

(b) $u_i$ gives an identical form of results to that of $s_i$ under $B_i$.

(c) The results in $u_i$ contain, if possible, some functional form of the same constant 'a' e.g. $g(a)$.

It is to be noted that 'any' in (i) refers to 'one' at least, while 'if possible' needs to be inserted in (c) since it is not possible in some
cases to preserve the functional form of the constant in results, as will be found in example (4) of the following illustrative examples:

(1) Let $s_i$ be: solve for $x : x^2 - 4 = 0$.
Then "solve for $x : x^2 - a^2 = 0"$ is a valid choice for $u_i$.
Others may be: $a^2x^2 - 4 = 0$, or $x^2 - 4a^2 = 0$.
But $u_i$ : "solve for $x : x^2 - a = 0"$ is not valid since $x = \pm 2$ in $s$; but $x = \pm \sqrt{a}$ (provided $a > 0$) differs in form as a result from $s_i$ since it contains an irrational form $\sqrt{a}$. Hence $u_i$ is an associate to $s_i$, but $u_i$ is not, in the sense of Proposition 8.2.

(2) Let $s_i = "x^2 - 2 = 0"
Then $u_i = "x^2 - a^2 = 0"$ is not valid, because there is an irrational form of numbers in the results of $s_i$, i.e. $\sqrt{a}$, but $\sqrt{a^2} = |a|$ does not denote an irrational form explicitly as $\sqrt{a}$ would do.
Therefore, it is reasonable that the corresponding associate is:
$u_i : x^2 - a = 0$ (with $a > 0$)
Hence $u_i$ is not an associate, but $u_i$ is.

(3) Let $s_i = \frac{d}{dx} (x^2 + 1)$
Then $u_i = \frac{d}{dx} (x^2 + a)$ is not valid, since a functional form $f(a)$ is not preserved in the results while:
$u_i = \frac{d}{dx} (ax^2 + 1)$ is valid.

(4) Let $s_i = \frac{d}{dx} (2)^n$,
then $u_i = \frac{d}{dx} (a)^n$ is valid, since it is not possible to obtain a functional form in the results by using sensibly the proposition of learnability 8.2 in any other appropriate way. Hence the term if possible was introduced in (c) in the Proposition.
Finally, the U-space (structure) under the proposition of learnability 8.2 was preconceived to be introduced as a learnable structure contributing to a suitable modification.

8.2. II. Theory In Modification

The fundamental research propositions 6.2, 6.3 and 6.4 in Chapter 6 proposed the essential role of theory in any modification, and predicted (as shown in Proposition 6.3) that if a development is learnable and rests on a theoretical framework, then it can be experimentally supported. Further on Proposition 6.4 described how theory could be employed by maintaining a certain discrimination of efficient knowledge gained in different spaces of knowledge. In this connection, the development of a theoretical framework that benefited from studies in Chapters 6 and 7, was introduced on the basis of a conceptual structure called space of knowledge (Chapter 5). The theory in Chapter 5 suggested that, on the basis of efficiency in Proposition 5.1, if S is a learning space of knowledge and U another space of knowledge manipulated to be learnable, then U could be proposed as contributing, with the indispensable space S viewed in 7.2.2.1, to a modification if the efficiency in U, regarding a certain problem P, surpasses that in S. But, the efficiency in Definition 5.7 was based on those uniform operationals which are preconceived to be ideal; and not easily identified in reality. One might suggest that a uniform operational could be accepted as the one that can properly perform in a certain set of kernels for P. The question would be: how many kernels could there be; one, two, three, ...? The problem would be as that of mastery, the larger the number, the higher might be the reliability in realizing them. But this could be
tedious to the participants in experiments as well as time consuming, even if such time could be available. Hence, I looked for solutions to this problem in the following.

8.2.0. Pseudo-Reality

Based on the previous argument of the difficulty, if not impossibility of identifying in reality uniform operations, I therefore tried to remove such difficulty by implementing the study in a so-called pseudo-reality, where the following holds.

8.2.0.0. Definition 8.1

If $W_0$ is a kernel for 'P' in a space of knowledge 'W', then an individual who is capable of all the elements of this kernel will be considered to be a uniform operational for P in the pseudo-reality.

The definition, on the one hand weakens the property of 'uniform operational', but it is convenient for this study to use such a space of pseudo-reality, where results from the pseudo-reality could be expected to be extrapolated into reality. This definition springs from putting the fundamental assumption of the law of determinism to work in pseudo-reality. Thus, the fact of the behaviour of an organism being lawful and predictable is reflected by a successful behaviour in a kernel $W_0$ in W being considered to be consistent for all other kernels of P in W.
8.2.0.1. Testing Model For The Efficiency In Pseudo-Reality

Based on study of efficiency in Chapter 5 (P.5-24), the four fundamental phases in construction of the tests in 7.2.1.0 (P.7-20) and the concept of the pseudo-reality, I suggested the following six steps for a testing model for efficiency, based on open-ended tests

(E1) Choose $S$ and $U$ to be two spaces of mathematical knowledge for mathematical tasks.

(E2) Choose a well-defined problem $P$ which is relevant to what the individuals in the study are to learn, but which is not related to any past experience of those individuals, so that at least $P$, or any similar task, is not to be found in the textbooks of the individuals.

(E3) Let a panel study $P$ carefully for presenting a well-defined $B$ in the problem-space $(D, B, X)$ of $P$ (see Definition 5.3).

(E4) Let the panel choose a kernel $S_0$ in $S$ for $P$.

(E5) Let the panel choose an associate kernel $U_0$ in $U$ for $P$. This kernel comprises the associates of $S_0$ in $U$ under proposition of learnability in $U$, i.e. Proposition 8.2.

(E6) Administer a test that comprises $S_0$, $U_0$ and $P$ as $S_0 + U_0 + P$ to a representative sample for the population that is under study.

It is to be noted that restriction of the construction $U_0$ from $S_0$ yields a different kernel $U_0$ from the construction of $H_0$ in the tests of the second indicator $T_1$ in 7.2.1. The construction of $H_0$ was based only on general consideration involved on relevant behavioural objectives, while construction of $U_0$ is more precise, which is desirable for the development of the study.
In this way, it is expected that the efficiency will be manipulated on the basis of the uniform operationals in the pseudo-reality space. It is to be noted that the efficiency as a measure helps to minimize, or even exclude the influence of factors which depend on memory, knowledge and skills that affect the ability in problem-solving, since such factors are almost maintained in the pre-determined ability in $S_0$ or $U_0$ as the efficiency measure demands. Accordingly, the efficiency in the pseudo-reality can be identified as the probability that the conditional statement: $x$ is able in a kernel $\Rightarrow x$ is able in $P$ is true, provided the first statement is true. It is also to be noted that the invaluable help of the panel (see ii of 2.4) is found in maintaining the content validity of the test.

8.3. III. An Empirical Study: A Set $T_3$ Of Three Tests Based On The Test-Form $S_0 + U_0 + P$

The aims behind the study of the three tests in $T_3$ (Appendix F) were to observe how efficiency could change relatively in $S$ and $U$, so that the proposition for modification based on Proposition 5.1 would have some roots in experimentation.

8.3.0 Description Of Method

$T_3$ consists of three tests $t_1$, $t_2$, and $t_3$ where each of them was constructed on the basis of the previous six stages proposed for testing of efficiency in the pseudo-reality in 8.2.0.1. The method will be presented as usual by discussing: (a) the materials, (b) the subjects and setting, and (c) the procedure. And in order that the situation may be clearer to the reader,
I shall present a and b for each test directly while the procedure for all the tests will be presented afterwards.

The Material And The Subjects And Setting.

(a1): t1 consisted of 11 question-items, where item 11 was the problem P.

P: A particle begins penetrating a sandy soil at t = 0.
If the velocity \( V(\text{cm/sec}) \) of the particle after t seconds is given by \( V = 16 - t^2 \), find the distance travelled by this particle until it stops, provided that the whole motion was in a straight line.

Solution: 
\[
\frac{dx}{dt} = V = 16 - t^2 \quad (1)
\]
\[
=> x = 16t - \frac{1}{3}t^3 + c \quad (2)
\]
Zero conditions: \( V = 16 - t^2 = 0 \) \( (3) \)
\[
=> t = 4
\]
\[
x = \left[ 16t - \frac{1}{3}t^3 \right]_0^4 = \frac{128}{3} \quad (4) \text{ and } (5)
\]

Hence the basic behavioural objectives accepted by the panel were five.

B1: To identify and apply the knowledge of the position \( x(t) \) of a particle moving in a straight line with a given velocity \( V(t) \), i.e. \( \frac{dx}{dt} = V \) \( \ldots (1) \)

B2: To identify and apply the knowledge of finding the integral of a function of the second degree. \( \ldots (2) \).

B3: To identify and apply the knowledge of the zero conditions of a particle moving in a straight line. \( \ldots (3) \).
B₄: To identify and apply the knowledge of finding the numerical value of a definite integral of a function of the second degree. ... (4).

B₅: To identify and apply the knowledge of finding the distance travelled by a particle moving in a straight line between two instances \( t_1 \) and \( t_2 \), provided that the particle does not change direction of motion in this period of time. ... (5).

Hence, each of the two kernels \( S_0 \) and \( U_0 \) contains five elements, although elements of \( U_0 \) are constructed as associates of relevant ones in \( S_0 \) on the basis of Proposition 8.2.

\( S_0 \) contains items 1, 2, 3, 4 and 5 while \( U_0 \) contains 6, 7, 8, 9 and 10 of \( t \). Here are some elements from both \( S_0 \) and \( U_0 \).

For \( B_1 \): \( s_1 \) = item 1: A particle moves in a straight line with \( V = t^2 - 4t + 2 \). Find its position at 't'.

\( u_1 \) = item 6: A particle moves in a straight line with \( V = at^2 - 4t + 2 \). Find the position of this particle at t, (a is a constant).

For \( B_3 \): \( s_3 \) = item 3: A particle moves in a straight line where, after \( t \) sec \( V = (t - 4)(t + 3) \). When does this particle stop?

\( u_3 \) = item 8: Let \( V = (t + 3)(t - b) \) with \( b > 0 \); define the velocity of a particle moving in a straight line after 't' sec. When does this particle stop?
For $b_4$: $s_4 =$ item 4: Find $\int_2^5 (4x - 3x^2) \, dx$.

$u_4 =$ item 9: Find $\int_2^a (ax - 3x^2) \, dx$ (a being constant).

(b1) The subjects of the first test ($t_1$) were 144 students (male and female) in Kuwait University, who were studying for a degree in engineering or other sciences, but none of them majoring in mathematics. All of them were attending a course called 101 in introductory calculus in their first semester. The total number in this course in the first semester in the academic year 1979/1980 was 266 students.

(a2) $t_2$ consists of 9 items where item 9 is the problem $P$.

$P$: Two particles $M_1$ and $M_2$ start moving together in the same straight line. The position of $M_1$ at 't' is given by $x_1 = t^3 - 4t$. The second particle $M_2$ moves with a constant velocity 8 (cm/sec) starting from $x_2 = a$. Determine 'a' so that both $M_1$ and $M_2$ have the same velocity when they meet.

Solution:

$V(M_1) = \frac{dx_1}{dt} = 3t^2 - 4$. \hspace{1cm} (1)

$M_1$ and $M_2$ meet at $x_1(t) = x_2(t)$ \hspace{1cm} (2)

$x_2 = v_0t + a = 8t + a$ \hspace{1cm} (3)

$3t^2 - 4 = 8 \Rightarrow t = 2$

Hence $x_1(2) = x_2(2)$ \hspace{1cm} (4)

$\Rightarrow 8 - 8 = 16 + a$, or $a = -16$

Here, the panel proposed the following basic behavioural objectives for $P$. 
B₁: To identify and apply the knowledge of the velocity of a particle moving in a straight line; provided that the position function is given, i.e. $V = \frac{dx}{dt}$. ... (1)

B₂: To identify and apply the knowledge that two particles with the position-functions $x_1(t)$ and $x_2(t)$ meet when $x_1(t) = x_2(t)$. ... (2)

B₃: To identify and apply the knowledge that $x = V_0 t + x_0$ is the position-function of a particle starting moving from 'x₀' in a straight line with a constant velocity 'V₀'. ... (3)

B₄: To identify and apply the knowledge that if two particles move in a straight line with position-functions $x_1(t)$ and $x_2(t)$, then they meet at $t_0$ if $x_1(t_0) = x_2(t_0)$. ... (4)

Hence, each of $S_0$ and $U_0$ contain four items: $S_0$ = items 1, 2, 3 and 4 and $U_0$ = items 5, 6, 7 and 8.

Here are some items of $S_0$ and $U_0$.

For $B_2$: $s_2$ = item 2: Let $x_1 = t^3 + 10$ and $x_2 = t^3 + t^2 + 1$, define for $t > 0$ the positions of two particles moving in a straight line. When do the two particles meet?

$u_2$ = item 6: If $x_1 = t^3 + 7a^2$ and $x_2 = t^3 + t^2 - 2a^2$ define for $t > 0$ the positions of two particles moving in a straight line. When do they meet? (a is a constant).
For $B_4$: $s_4 = \text{item 4}$: Let $x_1 = t^3 + 4t^2$ and $x_2 = 3t + 1$.

define at 't' the positions of two particles moving in a straight line. Do they meet at $t = 2$?

$u_4 = \text{item 8}$: Let $x_1 = t^3 + ct^2$ and $x_2 = c^2t + c^3$,

define the positions at $t$ of two particles moving in a straight line. Do they meet at $t = c$? ($c > 0$)

(b2) The subjects of the second test ($t_2$) were 52 students (male and female) in Kuwait University who did not major in mathematics and attended a more advanced course called 102 in calculus that normally follows the first course 101 of the subjects of $t_1$. Those students were attending this course (102) in their second semester in the University in the academic year 1979/1980. The total attendance at this course that year was 218 students.

(a3) $t_3$ consists of 9 items arranged in the same way as in $t_2$ where item 9 is the problem P.

P: Two particles moving in a straight line have at 't' the displacements $x_1 = 2t^3 + 3t$ and $x_2 = 3t^3 + a$ ($a$ being constant). Find 'a' if the two particles have the same velocity when they meet.

Solution: $V_1 = 6t^2 + 3$

$V_2 = 9t^2$  \( \ldots (1) \) & \( \ldots (2) \)

$9t^2 = 6t^2 + 3$

$\Rightarrow t = 1$

$x_1(1) = x_2(1)$  \( \ldots (3) \) & \( \ldots (4) \)

$5 = 3 + a$ or $a = 2$

The basic behavioural objectives were identified by the panel as follows:
B₁: To identify and apply the knowledge of the derivative of a polynomial of the third degree ... (1)

B₂: To identify and apply the knowledge of the velocity of a particle moving in a straight line when the displacement (position) function is given, and this function is a polynomial of the third degree. ... (2)

B₃: To identify and apply the knowledge that if two particles with \( x_1(t) \) and \( x_2(t) \) move in a straight line, then they meet at \( x_1(t) = x_2(t) \). ... (3)

B₄: To identify and apply the knowledge that if two particles with \( x_1(t) \) and \( x_2(t) \) move in a straight line, then they meet at \( t_0 \) if \( x_1(t_0) = x_2(t_0) \). ... (4)

The kernel \( S_0 \) consists of items 1, 2, 3 and 4 and \( U_0 \) consists of their associates (under Proposition 8.2) 5, 6, 7 and 8 respectively.

Here are some items of \( S_0 \) and \( U_0 \).

For \( B₁ \): \( s_1 \) = item 1: Find \( \frac{dy}{dx} \) if \( y = 2x^3 + 5x - 3 \)
\( u₁ \) = item 5: Find \( \frac{dy}{dx} \) if \( y = 2ax^3 + 5x - 3 \) (a being a constant).

For \( B₃ \): \( s_3 \) = item 3: The displacements \( x_1 \) and \( x_2 \) with respect to an origin of two particles moving in a straight line are given by: \( x_1 = t^2 + 2t \) and \( x_2 = t^2 + 8 \). When do the two particles meet?
The positions of two particles moving in a straight line are defined at 't' by 
\[ x_1 = t^2 + 2at \]
and 
\[ x_2 = t^2 + a, \] 
'a' being a constant \( \neq 0 \). When do they meet?

(b3) The subjects of \( t_3 \) were 274 students (boys and girls) in the final year of six secondary schools in Kuwait. The total number of students in this grade was 3500 approximately.

(c) **The Procedure**

The design of the tests was such that the students should reply on the same sheet as the question. All the subjects were told the duration of the three tests was not fixed. Nevertheless, all of them gave the test sheets back within 60 minutes.

(c1) Both \( t_1 \) and \( t_2 \) were administered to the relevant students by their lecturers on two different days.

(c2) \( t_3 \) was administered to the relevant students by their teachers on three different days.

(d) **Scoring**

An item was given '+1' for a correct procedure even with simple miscalculations, and '0' otherwise. I personally marked all the sheets of \( t_1 \) and \( t_2 \), but was helped in marking \( t_3 \) by the senior teachers and
teachers of mathematics of the classes who participated in the experiment.

Lastly, a point to be noted is that in the whole study I will not discuss any differences in sex, since girls in Kuwait have proved to be no less competent in mathematics than boys, if not more competent.

8.3.1. The $T_3$ - Constructs

The $T_3$ -Constructs were taken to be:

1. The performance in an element $s_i$ in $S_o$ would be superior to performance in the associate $u_i$ in $U_o$.

2. The number of students who are completely capable of all the items in $S_o$ will be larger than the number of students who are completely capable in all items in $U_o$.

3. The efficiency '$r$' in $S$ is less than '$r$' in $U$ regarding the same problem $P$, i.e. $r(S) < r(U)$.

4. The efficiency $r(U)$ for a sample of university students would be higher than that of a sample of school children.

8.3.2. Discussion Of Results

1. Tables: The following tables illustrate aspects of the results.
Table 8.1 demonstrates the numbers of correct responses and relevant percentages for each item $s_i$ in kernel $S_0$ and $u_i$ in kernel $U_0$ as well as in the problem $P$.

<table>
<thead>
<tr>
<th>Test</th>
<th>Item</th>
<th>$S_0$</th>
<th>$U_0$</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$s_1$</td>
<td>120</td>
<td>119</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>140</td>
<td>126</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>$s_3$</td>
<td>101</td>
<td>104</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$s_4$</td>
<td>115</td>
<td>115</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$s_5$</td>
<td>89</td>
<td>0.62</td>
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<tr>
<td></td>
<td></td>
<td>0.89</td>
<td>0.83</td>
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<td></td>
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<td>0.97</td>
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<td></td>
<td></td>
<td>0.70</td>
<td>0.72</td>
<td></td>
</tr>
<tr>
<td>$t_2$</td>
<td></td>
<td>0.80</td>
<td>0.80</td>
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<td></td>
<td></td>
<td>0.62</td>
<td>0.33</td>
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<td>0.83</td>
<td>0.31</td>
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<td></td>
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<td>0.72</td>
<td>0.54</td>
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<tr>
<td>$t_3$</td>
<td></td>
<td>0.80</td>
<td>0.50</td>
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<td>0.71</td>
<td>0.21</td>
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<td>0.87</td>
<td>0.54</td>
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<td>0.62</td>
<td>0.67</td>
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<td></td>
<td></td>
<td>0.31</td>
<td>0.01</td>
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</tr>
</tbody>
</table>

Table 8.1
Numbers And Percentages Of Correct Responses In $P$, $S_0$ and $U_0$ For All Tests.

Table 8.2 provides the efficiencies in $S$ and $U$ for $P$. It is to be noted that if $N(S_0)$ and $N(S_0 + P)$ are the numbers of uniform operationals and the number of uniform operationals capable of $P$ respectively in the representative sample, then the efficiency 'r' in $S$ for $P$ is $r(S) = \frac{N(S_0 + P)}{N(S_0)}$, and that in $U$ for $P$ is $r(U) = \frac{N(U_0 + P)}{N(U_0)}$ (5.2.2.2).
Table 8.2
Efficiencies In The Two Spaces of Knowledge S
And U Regarding A Problem P.

Furthermore, one might like to know the number of those students who were uniform operationals in both S and U for P, i.e. the number $N(S_0 + U_0)$ of those who were capable of all items $S_0 + U_0$. This number demonstrates those who could preserve constant behaviour in both $s_i$ and its associate $u_i$, for all $s_i$ in $S_0$ and $u_i$ in $U_0$. Moreover, if $N(S_0 + U_0 + P)$ denotes the number of those who were able in $S_0$, $U_0$ and P, and if we extend the efficiency 'r' in the space $S + U$ to be

$$r(S+U) = \frac{N(S_0+U_0+P)}{N(S_0+U_0)}$$

in $S + U$.

* If X and Y are two disjoint sets then $X + Y$ denotes the union of the two sets i.e. $XUY$. 
Table 8.3

Efficiency In The Space $S + U$ Regarding A

Problem $P$ And $S_0 + U_0$

<table>
<thead>
<tr>
<th>Test</th>
<th>No. of Students</th>
<th>$N(S_0 + U_0)$</th>
<th>$N(S_0 + U_0 + P)$</th>
<th>$r(S + U)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>144</td>
<td>30</td>
<td>29</td>
<td>0.97</td>
</tr>
<tr>
<td>$t_2$</td>
<td>52</td>
<td>8</td>
<td>5</td>
<td>0.63</td>
</tr>
<tr>
<td>$t_3$</td>
<td>274</td>
<td>101</td>
<td>72</td>
<td>0.71</td>
</tr>
</tbody>
</table>

(2) Analysis Of Results

The analysis of results will be as usual discussed on the basis of the $T_3$-constructs. But, it is firstly worthwhile to note that the efficiency measure would gain some reliability in that it demonstrated compatibility with the second indicator $T_2$ in 7.2.1 which indeed constituted the basis for the second construct of $T_3$ which will be discussed in what follows.

(i) Students are expected, in general, to perform in $S$ better than in $U$, since they have behavioural familiarity primarily with $S$, which is also supported by means of $S_0$ and $H_0$ in table 7.4. Table 8.1 illustrates the expected construct (1) in 8.3.1. Consequently, the subjects are expected in general to be more capable of the kernel $S_0$ than of the kernel $U_0$ as shown in table 8.2 (construct 2).

(ii) The second indicator in 7.2.1 has demonstrated that performance in $H$ (which includes $U$) in a set $H_0$ as a kernel to a problem $P$ correlates
with the performance in P higher than that of $S_0$ in S and P. Hence it was expected in construct 3 that $r(S)$ would be demonstrated to be less than $r(U)$ regarding the same problem P. Table 8.2 agrees with this construct and also supports the reliance on the efficiency-measure by agreeing with the conclusion 7.2 (P.7-26). Thus the extremely important construct 3 might provide on optimisation on the basis of the proposition of modification 5.1, in proposing a modification in the methodology of mathematics teaching-learning situations.

(iii) As for the final construct (4), it was also expected that students in the University - who have a greater opportunity of working in H, which includes U, than school students - would demonstrate $r(U)$ higher than the school students. Table 8.2 indeed demonstrates $r(U)$ for University students, i.e. 0.98 and 0.70 to be higher than $r(U)$ for the school students, i.e. 0.66. This indeed supports that the more the students are familiar with a certain space of knowledge, the better the efficiency, which seems quite logical in the case of those tests in $T^3$ which include general mathematical concepts in common.

(iv) Table 8.3 demonstrates how successful behaviour in both $S_0$ and $U_0$ could induce higher efficiency in $S + U$ than in S (shown in table 8.2). And such an indication may suggest that a learning space $S + U$, taking into consideration Proposition 8.2, would possibly be superior to the learning space S.

Finally, a point worthy of note is that, despite the poor performance in $T^2_2$ observed in tables 8.1, 8.2 and 8.3, and whatever the reasons for this may be, the two efficiencies $r(S)$ and $r(U)$ are in line with the most important construct (i.e. construct 3).
8.3.3. Conclusion 8.1

If U is accepted to be an adequate learning space under the proposition of learnability 8.2, then U could be promising for contributing to the indispensable learning space S in the sense of general conclusion 7.3 and thus introduce a modification in the methodology of mathematics teaching-learning situations on the basis of proposition 5.1.

8.3.4. Summary Of $T_3$

<table>
<thead>
<tr>
<th>TITLE</th>
<th>A set of three tests based on the test-form $S_0 + U_0 + P$. (App.F).</th>
</tr>
</thead>
<tbody>
<tr>
<td>THE TESTS</td>
<td>$\xi_1$</td>
</tr>
<tr>
<td>PURPOSE</td>
<td>A study of efficiency in S and U for proposing a modification.</td>
</tr>
<tr>
<td>MATERIAL</td>
<td>An open-ended test of 11 question items. Item 11 = Problem P Kernel $S_0 = \text{Items 1, 2, 3, 4 and 5 in S.}$ Kernel $U_0 = \text{Items 6, 7, 8, 9 and 10 in U.}$</td>
</tr>
<tr>
<td></td>
<td>An open-ended test of 9 question items Item 9 = Problem P Kernel $S_0 = \text{Items 1, 2, 3 and 4 in S.}$ Kernel $U_0 = \text{Items 5, 6, 7 and 8 in U.}$</td>
</tr>
<tr>
<td></td>
<td>Design very similar to $\xi_2$</td>
</tr>
<tr>
<td>SUBJECTS AND SETTING</td>
<td>144 students in Kuwait University not majoring in Maths and taking an introductory course 101 in calculus in their first semester at University.</td>
</tr>
<tr>
<td></td>
<td>52 students in Kuwait University not majoring in Maths taking a more advanced course 102 in calculus.</td>
</tr>
<tr>
<td></td>
<td>274 students in the final year of school in Kuwait taking an introductory course in calculus which succeeds 101. They were in second semester at University.</td>
</tr>
<tr>
<td><strong>PROCEDURE</strong></td>
<td>The test was administered by the students' lecturers for course 101. Duration was not fixed; but all gave the sheets back within 60 minutes.</td>
</tr>
<tr>
<td>---------------</td>
<td>------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td><strong>CONTENT VALIDITY</strong></td>
<td>It was determined by a panel of 6 people who helped in test construction (see ii of 2.4, Chapter 2).</td>
</tr>
</tbody>
</table>
| **CONSTRUCT VALIDITY** | The $T_3$-constructs were: 
(1) performance in an $s_i$ in $S_0$ would be better than in an associate $u_i$ in $U_0$;  
(2) performance in $S_0$ would be higher than that in $U_0$;  
(3) $r(S) < r(U)$;  
(4) $r(U)$ in University would be higher than that in school. |
| **RELIABILITY** | This was satisfied by inter-rater reliability discussed in 2.4.2.1.1 of Chapter 2. |
| **EFFICIENCY** | $r(S) = 0.58$ and $r(U) = 0.98$  
$r(S) = 0.56$ and $r(U) = 0.70$  
$r(S) = 0.53$ and $r(U) = 0.66$ |
| **CONCLUSION** | U could be promising under the proposition of learnability 8.2 in contributing with $s_i$ towards satisfying a modification in the methodology of mathematics teaching-learning situations, based on proposition 5.1. |
8.3.5. Contingent Implications Of the Test-Form $S_0 + U_0 + P$:  
A Set $T_4$ Of Six Tests

The following empirical study in $T_4$ was conducted mainly to discover any contingent implications of competency in $P$ and hence of the efficiency that could be affected by the test-form in $T_3$, i.e. in the form $S_0 + U_0 + P$. Could $P$ be influenced by $S_0$ or $U_0$ before $P$, given that necessary behavioural objectives for $P$ are also necessary for each of $S_0$ and $U_0$. Hence, it was found that it is essential to study competency in $P$ of $S_0 + U_0 + P$ against that of $P$ only. In the context of the study, I extended the second test concerning $P$ into the form $P + S_0 + U_0$ with the purpose of studying the consistency of behaviour in both $S_0$ and $U_0$ before and after $P$, as well as studying if competency in $U_0$ could be affected by behaviour in $P$ through $S_0$ for $U_0$, i.e. the behaviour in $U_0$ in $P + S_0 + U_0$.

In attempting to satisfy such aims, I implemented a set $T_4$ of derived from splitting a set that comprises a problem $P$ as well as two kernels $S_0$ and $U_0$ for $P$, i.e. $\{P, S_0, U_0\}$. $T_4$ was consequently put into operation on the basis of the following forms.

1. $S_0 + P$ vs. (2) $P + S_0$. (3) $U_0 + P$ vs. (4) $P + U_0$.
2. (5) $S_0 + U_0 + P$ vs. (6) $P + S_0 + U_0$.

I supposed that the first four tests would provide some useful indications suggesting certain constructs that could monitor certain hypotheses that would increase understanding about the effects of the test-form on $P$ and about possible events involved in the transferability from $S_0$ into $U_0$. 
8.3.5.0. Construction Of $P$, $S_0$ and $U_0$

First: The Problem

$P$: Let $f(x) = x^2 + 4x$ and let $M$ be the region defined by the curve $f$, the $x$-axis and the two lines $x = -3$ and $x = 1$. Find the area of $M$.

Solution:

A student, in solving such a problem, is always advised to follow the following steps:

(i) To find the points common to both $f$ and the $x$-axis.

(ii) To sketch a rough curve of $f$ and to identify what parts of $f$ lie under or above the $x$-axis.

(iii) To state the relevant area in definite integrals of $f$.

(iv) To use the previous steps to work out the solution.

Accordingly, the panel proposed the following solution:

$P_1$: $f(x) = 0 \implies f$ cuts the $x$-axis at $x = -4$ and $x = 0$.

$P_2$: $f$ is below the $x$-axis between $x = -4$ and $x = 0$ and above it elsewhere (Figure 8.1).

$P_3$: The region $M$ is the shaded one in Figure 8.1. The area:

$$A(M) = -\int_{-3}^{0} f + \int_{-3}^{1} f \text{ or } \int_{-3}^{0} |f| + \int_{0}^{1} |f|$$

$P_4$: $A(M) = -\left[\frac{1}{3}x^3 + 2x^2\right]_{-3}^{0} + \left[\frac{1}{3}x^3 + 2x^2\right]_{0}^{1} = 29 \frac{1}{3}$ (Area Units)

Figure 8.1: Curve of $f: x \to x^2 + 4x$. 

[Figure of curve $f: x \to x^2 + 4x$]
Hence a set $B$ of **basic behavioural objectives** were identified for $P$ as follows:

**$B_1$:** To identify and apply the knowledge of the common points between a curve and the $x$-axis.

**$B_2$:** To identify and apply the knowledge of sketching a rough curve of the second degree (the bell-curve).

**$B_3$:** To identify and apply the knowledge of the area of a region in general terms of definite integrals such that the region lies under and above the $x$-axis.

**$B_4$:** To identify and apply the knowledge of finding a definite integral to a function of the second degree.

**Second:** Construction of two kernels $S_0$ and $U_0$

<table>
<thead>
<tr>
<th>$S_0$ in $S$</th>
<th>$U_0$ in $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$: let $f(x) = x^2 - 2x - 3$ find the common points between $f$ and the $x$-axis.</td>
<td>$u_1$: Let $f(x) = (x + 1) (x - 3h)$, $h$ being constant, find the common points between $f$ and the $x$-axis.</td>
</tr>
<tr>
<td>$s_2$: sketch a rough graph of $f(x) = x^2 - x$.</td>
<td>$u_2$: sketch a rough graph of $f(x) = x^2 - ax$, $a &gt; 0$.</td>
</tr>
</tbody>
</table>
8.3.5.1. The Scoring Of The Test

An essential part of scoring the solution to P should be to score according to the steps that a student ought to identify in P. This scoring would then serve the purpose of implementing this testing so that one could match performance in S or U and the related performance in the steps of P. In the following I present such a step-scoring:

(i) A student was given '+1' for P₁ if he wrote \( f(x) = 0 \), or \( x^2 + 4x = 0 \). No attention was paid to incorrect manipulations. Otherwise he was given zero.

(ii) A student was given '+1' for P₂ if he sketched the bell-curve such that the portion of the curve between the two roots of...
f(x) = 0 is under the x-axis, otherwise he was given zero.

(iii) A student was given '+1' for $P_3$ if he could identify that $A(M) = -3\int_0^0 f + 0\int_1^1 f$ or $|-3\int_0^0 f| + |0\int_1^1 f|$; otherwise he was given zero.

(iv) A student was given '+1' in the final step $P_4$ if he could identify both $\int f = F$ and $\int_{x_1}^{x_2} f = F(x_2) - F(x_1)$. No attention was paid to mere numerical calculations. A student was given '+1' if he made an error in the numerical coefficients in $\int f = F$. Otherwise he was given zero.

A score was given for any step if it implicitly embodied or led to the next one. Scoring in $S_0$ and $U_0$ was based on the previous rules of scoring.

Finally, it should be said that there had been controversy amongst the members of the panel, as to whether there should be a basic behavioural objective that identifies the competency of students in manipulation of an integral of a polynomial of the second degree. There was a majority opinion that this basic behavioural objective is trivial since students have been successfully working with more difficult forms of integrals, and since this was involved in $s_4$ and $u_4$ of the two kernels $S_0$ and $U_0$ respectively.

8.3.5.2. Description of Method

a) The material

The previous set $P$, $S_0$ and $U_0$ was arranged in three different sheets and
was categorized in six forms: \((S_0 + P) vs. (P + S_0),(U_0 + P) vs. (P + U_0)\)
and \((S_0 + U_0 + P) vs. (P + S_0 + U_0)\).

b. Subjects and setting

Six groups of students \((g_1, \ldots, g_6)\) were chosen from the final grade in three different secondary schools according to what follows:

\((b_1)\) \(g_1\) (24 students) and \(g_2\) (22 students) from the same school for boys, all having the same teacher of maths.

\((b_2)\) \(g_3\) (22 students) and \(g_4\) (19 students) from the same school for girls and all having the same teacher of maths.

\((b_3)\) \(g_5\) (23 students) and \(g_6\) (20 students) from another school for boys and all having the same teacher of maths.

It is to be noted that it was impossible to choose a representative sample of two classes for each test since in Kuwait there was no teacher who teaches four classes in the same grade of secondary school.

c. The Procedure

\((c_1)\) Each student in \(g_1\) was tested by \(S_0\) first and then given \(P\) on handing back \(S_0\) i.e. \(S_0 + P\). Each student in \(g_2\) was tested by \(P\) first and then given \(S_0\) on handing back \(P\), i.e. \(P + S_0\).

\((c_2)\) Similarly, \(g_3\) was tested first by \(U_0\) and then by \(P\), i.e. \(U_0 + P\).
was tested first by $P$, then by $U_0$, i.e. $P + U_0$.

(c3) $g_5$ was tested first by $S_0$, then by $U_0$ and finally by $P$, i.e. $S_0 + U_0 + P$. $g_6$ was tested first by $P$, then by $S_0$ and finally by $U_0$, i.e. $U_0 + S_0 + P$.

Table 8.4 illustrates the numbers of students in the six groups and the corresponding test forms.

<table>
<thead>
<tr>
<th>Group</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$g_5$</th>
<th>$g_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of students</td>
<td>24</td>
<td>22</td>
<td>22</td>
<td>19</td>
<td>23</td>
<td>20</td>
</tr>
<tr>
<td>Test</td>
<td>$S_0 + P$</td>
<td>$P + S_0$</td>
<td>$U_0 + P$</td>
<td>$P + U_0$</td>
<td>$S_0 + U_0 + P$</td>
<td>$P + S_0 + U_0$</td>
</tr>
</tbody>
</table>

Table 8.4

Number Of Students In The Six Groups And Corresponding Test-Forms.

(c4) An attempt was made to exclude factors depending on students' memory and past skills and relevant abilities in problem solving by bringing these to their attention when testing their achievements in both $S_0$ and $U_0$. All the students were generally activated in the general mathematical knowledge that underlies the set $B$ of basic behavioural objectives for $S_0$, $U_0$ and $P$. For example, they were activated by drawing a curve of first and second degree, relating the area to the integral, and calculating a definite integral. I myself, as an inspector of mathematics, visited all the six classes on three different days and achieved such activation without revealing any of my purposes.
Nevertheless, it is worthwhile to note that all attempts to find mathematics situations which would be determined by ability - of which achievement is part - and which would not depend at all on the available knowledge, skills and habits are doomed to failure, as Krutitski (1976) states. This means that we can only weaken and minimise other influences in order to equalise all examinees in this respect, but we cannot completely remove these influences.

\( (c_5) \) The tests were run in three different days; one day for each school. In this I was helped by the senior teachers of mathematics in the schools, as well as the mathematics teachers of relevant classes. Students were told that duration of the tests was not fixed, but all of them gave the sheets back within 45 minutes.

\( (c_6) \) The teachers and senior teachers who were involved in the experiment helped me in marking the tests.

8.3.5.3. Constructs Of The Test-Form: Three Hypotheses

In this study, I viewed that I had best present the constructs as hypotheses to be tested in order to present statistical support for them. I therefore, as stated before \((P.8-26)\), believed that I should investigate such hypotheses through observation of the results of \( g_1 \) versus \( g_2 \) and \( g_3 \) versus \( g_4 \). And although the groups were under different conditions, nevertheless, they could be considered to behave approximately the same in \( S_0 \) as for relevant behaviour in \( U_0 \). I take these views since:

(i) students are usually randomly distributed in their classes;
(ii) all of them follow the same text as well as the same plan of task achievement; and,
(iii) all the teachers of mathematics in the secondary stages, especially
the final grade, have approximately the same qualifications.

The following table 8.5 illustrates the competency as the percentages of consecutive correct responses in $S_i$, $U_i$, as well as $P_i$, where a consecutive correct response in $S_i$, means a correct response to $s_1$, $s_2$, ..., $s_j$, etc.

<table>
<thead>
<tr>
<th></th>
<th>$g_1$: $S_0 + P$</th>
<th>$g_2$: $P + S_0$</th>
<th>$g_3$: $U_0 + P$</th>
<th>$g_4$: $P + U_0$</th>
<th>$g_1$: $\Sigma S_i$</th>
<th>$g_2$: $\Sigma U_i$</th>
<th>$g_3$: $\Sigma P_i$</th>
<th>$g_4$: $\Sigma P_i$</th>
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<td>1</td>
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<td>47</td>
<td>38</td>
<td>45</td>
<td>55</td>
<td>47</td>
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</tbody>
</table>

Table 8.5

Percentages Of Consecutive Correct Responses In $S_0$, $U_0$, And The Steps Of $P$ For The First Four Groups Of The Six Groups In The Experiment.

Hence, on the basis of the three previous suggestions, I draw from table 8.5 the following points of interest.

(i) Comparing column 5 with column 6 might suggest that competency in $P$ is eventually better if $P$ is given before $S_0$, while comparing column 7
with column 8 might suggest the contrary. Does this mean a student does not learn from $S_0$ anything more for $P$? And does it mean that a student does learn from $U_0$ something for $P$? Or does this mean that $U_0$ might effect processes for $P$ better than $S_0$? Further, could $P$ have effects in $U_0$?

(ii) Comparing column 3 with column 4 might suggest that competency in $U_0$ is not affected by $P$.

(iii) Comparing columns 1 and 2 against 3 and 4 might suggest that competency in $S_0$ does not preserve consistency in $U_0$.

Hence the following hypotheses were suggested to be statistically tested for groups $g_5 (S_0 + U_0 + P)$ and $g_6 (P + S_0 + U_0)$.

$H_1$: Competency in $P$ after $S_0 + U_0$ is significantly different from competency in $P$ only.

$H_2$: Competency in $U_0$ for $S_0 + U_0 + P$ is significantly different from competency in $U_0$ in $P + S_0 + U_0$.

$H_3$: Competency in $S_0$ is significantly higher than competency in $U_0$.

8.3.5.4. Discussion Of Results

1. Tables

Firstly, I will present two tables, 8.6 and 8.7. The first illustrates the whole set of results, while the second (8.7) throws light on the validity of the aforementioned hypotheses $H_1$, $H_2$ and $H_3$. 
<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$U_0$</th>
<th>$P$</th>
<th>$S_0$</th>
<th>$U_0$</th>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>19</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**PART I of**

*Table 8.6*

(PART II overleaf)
The table demonstrates the performances of students in each step. $x_1$, $x_2$ and $x_3$ indicate the total score for each student in $S_0$, $U_0$ and steps of $P$ respectively. It could be noticed that $P_4$ indicates students capable of $P$. Hence 11 students in $g_5$ could solve $P$, while 8 in $g_6$. 

**PART II of**

**Table 8.6**

**Illustrative Table Of Competency Of Each Student**

In The Items Of $S_0$ And $U_0$ And The Steps Of $P$ For Groups $g_5$ And $g_6$. 

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$U_0$</th>
<th>$P$</th>
<th>$S_0$ $U_0$ $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$ $S_2$ $S_3$ $S_4$</td>
<td>$U_1$ $U_2$ $U_3$ $U_4$</td>
<td>$P_1$ $P_2$ $P_3$ $P_4$</td>
<td>$x_1$ $x_2$ $x_3$</td>
</tr>
<tr>
<td>1 1 1 1 1</td>
<td>1 0 1 0</td>
<td>1 0 0 0</td>
<td>4 2 1</td>
</tr>
<tr>
<td>2 1 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>4 4 4</td>
</tr>
<tr>
<td>3 1 1 0 1</td>
<td>1 0 1 0</td>
<td>1 0 0 0</td>
<td>3 2 1</td>
</tr>
<tr>
<td>4 1 1 1 1</td>
<td>1 1 1 0</td>
<td>1 1 0 0</td>
<td>4 3 2</td>
</tr>
<tr>
<td>5 1 1 0 1</td>
<td>1 1 1 0</td>
<td>1 1 1 1</td>
<td>3 3 4</td>
</tr>
<tr>
<td>6 1 1 1 0</td>
<td>1 1 1 0</td>
<td>0 0 0 0</td>
<td>2 3 0</td>
</tr>
<tr>
<td>7 1 1 1 1</td>
<td>1 0 1 0</td>
<td>1 1 1 1</td>
<td>4 4 4</td>
</tr>
<tr>
<td>8 1 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>4 4 4</td>
</tr>
<tr>
<td>9 1 0 1 1</td>
<td>1 0 1 1</td>
<td>0 0 0 0</td>
<td>3 3 0</td>
</tr>
<tr>
<td>10 1 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>4 4 4</td>
</tr>
<tr>
<td>11 1 1 1 1</td>
<td>1 1 1 0</td>
<td>1 0 0 0</td>
<td>4 3 1</td>
</tr>
<tr>
<td>12 1 1 1 1</td>
<td>1 1 1 0</td>
<td>1 0 0 0</td>
<td>4 3 1</td>
</tr>
<tr>
<td>13 0 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>3 4 4</td>
</tr>
<tr>
<td>14 1 1 1 1</td>
<td>1 0 1 0</td>
<td>1 1 0 0</td>
<td>4 2 2</td>
</tr>
<tr>
<td>15 1 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>4 4 4</td>
</tr>
<tr>
<td>16 1 1 1 0</td>
<td>1 0 1 0</td>
<td>1 1 0 0</td>
<td>4 2 2</td>
</tr>
<tr>
<td>17 0 1 1 1</td>
<td>1 0 1 0</td>
<td>0 0 0 0</td>
<td>2 2 0</td>
</tr>
<tr>
<td>18 1 1 1 1</td>
<td>1 1 0 1</td>
<td>1 1 0 0</td>
<td>4 3 2</td>
</tr>
<tr>
<td>19 0 1 1 1</td>
<td>1 0 0 0</td>
<td>1 0 0 0</td>
<td>2 1 1</td>
</tr>
<tr>
<td>20 1 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>4 4 4</td>
</tr>
</tbody>
</table>

| 17 19 17 17 | 20 12 18 8 | 17 12 8 8 |
could solve P.

The following table 8.7 is extracted from table 8.6. It illustrates in each cell the competency of the group as the percentages of correct responses in $E_s, E_u, E_p, E_s + p, E_u + p, E_s + u + p$ provided that the necessary behaviour for $s_i, u_i$ and $p_i$ is the same for all, $i = 1, 2, 3$ and 4.

**Table 8.7**

Percentages Of Consecutive Correct Responses In $S_o, U_o$, $P$, As Well As In $S_o + P$, $U_o + P$, $S_o + U_o$ And $S_o + U_o + P$. 

<table>
<thead>
<tr>
<th>j</th>
<th>$g_5: S_o + U_o + P$</th>
<th>$G_6: P + S_o + U_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$E_{s_1}$</td>
<td>78</td>
</tr>
<tr>
<td>2</td>
<td>$E_{u_1}$</td>
<td>91</td>
</tr>
<tr>
<td>3</td>
<td>$E_{p_1}$</td>
<td>91</td>
</tr>
<tr>
<td>4</td>
<td>$E(s_1 + p_1)$</td>
<td>78</td>
</tr>
<tr>
<td>5</td>
<td>$E(u_1 + p_1)$</td>
<td>91</td>
</tr>
<tr>
<td>6</td>
<td>$E(s_1 + u_1)$</td>
<td>78</td>
</tr>
<tr>
<td>7</td>
<td>$E(s_1 + u_1 + p_1)$</td>
<td>78</td>
</tr>
</tbody>
</table>
The following diagram illustrates the percentages of correct responses in $\Sigma s_j$, $\Sigma u_j$, and $\Sigma p_j$.

Figure 8.2: Illustrative Comparison Of the Consecutive Competencies As Percentages Of Correct Responses In $s_o$, $u_o$ And Steps Of $p$.
2. Analysis Of Results

Before testing the three hypotheses $H_1$, $H_2$, and $H_3$ in 8.3.5.3, I would firstly like to note that table 8.7 seems to indicate that the behaviour of both groups $g_5$ and $g_6$ eventually becomes consistent in access to $\sum s_i$. This could support the equivalence of the two groups, remembering that this equivalence was considered beforehand, since the students in both classes $g_5$ and $g_6$ attended mathematics classes with the same teacher for the whole of the academic year 1979/1980 at the end of which the test was run. Moreover, the behaviour of the four groups $g_1$, $g_2$, $g_5$, and $g_6$ in access to $\sum s_i$ also show some consistency especially for $g_1$, $g_5$ and $g_6$. This in turn throws light on how a dominant space of knowledge, e.g. $S_1$, could diffuse almost the same 'amount' of knowledge in different groups.

In the following, I shall present the calculations I employed for testing the hypotheses.

(i) For testing $H_1$ and $H_2$, I tested the difference between two means $M_1$ and $M_2$ of the two populations of $g_5$ and $g_6$ on basis of the 't' distribution, provided that the samples are independent and small ($n < 30$) and we can assume that the variances in both populations are approximately equal as well as being approximately normal.

Let us consider $g_5$ and $g_6$ as random samples, one from each population of sizes $n_1$ and $n_2$ respectively. Then, if we compute the two relevant means $\bar{y}_1$ and $\bar{y}_2$, as well as the two relevant standard deviations $D_1$ and $D_2$, then the common variance of the two populations is estimated by $D^2$ where:
\[ D^2 = \frac{(n_1 - 1)D_1^2 + (n_2 - 1)D_2^2}{n_1 + n_2 - 2} \]

And
\[ t = \frac{(\bar{y}_1 - \bar{y}_2) - d_0}{\sqrt{\frac{n_1 + n_2}{n_1 - n_2}} \cdot D^2} \]

(Walpole, 1968, P. 228).

is a value of the random variable \( T \) having the \( t \) distribution with
degrees of freedom \( df = n_1 + n_2 - 2 \) and \( d_0 \) some specified value equal
to the difference of the means of the two populations, i.e. \( M_1 - M_2 = d_0 \).

Here, I shall restate the first two hypotheses and relevant calculations
in table 8.8 that is based on the null set \( H_0 \) for the means in both cases
that correspond to \( M_1 - M_2 = 0 = d_0 \).

\( H_1 \): Competency in \( P \) after \( S_0 + U_0 \) is significantly different from
competency in \( P \) only.

\( H_2 \): Competency in \( U_0 \) of \( S_0 + U_0 + P \) is significantly different from
competency in \( U_0 \) in \( P + S_0 + U_0 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( P(g_5) )</th>
<th>( P(g_6) )</th>
<th>( U_0(g_5) )</th>
<th>( U_0(g_6) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>23</td>
<td>2.6087</td>
<td>2.25</td>
<td>2.6957</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>1.4690</td>
<td>1.5853</td>
<td>1.2223</td>
</tr>
<tr>
<td>Mean</td>
<td>(n-1) (S.D)^2</td>
<td>47.4751</td>
<td>47.7503</td>
<td>32.8696</td>
</tr>
<tr>
<td>S.D</td>
<td>( \sqrt{\frac{n_1 + n_2}{n_1 \cdot n_2}} \cdot D^2 )</td>
<td>2.3226</td>
<td>1.1871</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>Common variance ( D^2 )</td>
<td>( \sqrt{\frac{n_1 + n_2}{n_1 \cdot n_2}} \cdot D^2 )</td>
<td>0.7697</td>
<td>0.6133</td>
<td></td>
</tr>
</tbody>
</table>

**Table 8.8**

Table Showing The Specific Calculations Testing Differences Between
The Means Related To The Independent Samples Concerned In Hypotheses
\( H_1 \) And \( H_2 \).
Now let $\alpha = 0.05$ be a level of significance, then the critical region in tables of the percentage points of the $t$ distribution with $df = 23 + 20 -2 = 41$ corresponds to $T < -t_{0.025} = -1.960$ and $T > 1.960$ (Walpole, 1968, P.333).

This means that both $H_1$ and $H_2$ must be rejected against the null set $H_0$ corresponding to each of them. However, in the following section 8.3.5.5. of general discussion, I shall discuss and modify them.

(ii) Here we test $H_3$ which I restate:

$H_3$: Competency in $S_o$ is significantly higher than competency in $U_o$.

I employed the $t$-test for paired measures taken of the same sample. Hence, if $d_1, d_2, \ldots, d_n$, represent the differences of $n$ related pairs of measurements then $t = \frac{\bar{d} - d_o}{\frac{s_d}{\sqrt{n}}}$ (Walpole, 1968, P.237), where $t$ is a value of the random variable $T$ having the $t$ distribution with $df = n-1$, with $\bar{d}$ and $s_d$, the mean and standard deviation of the $d_i$, and $d_o$ is a specific value for the mean of differences for the relevant population.

The following table 8.9 presents the relevant calculations for the null hypotheses $H_o$ of $H_3$ that correspond to $d_o = 0$ for both groups $g_5$ and $g_6$. 

Table 8.9
Table Of Specific Calculations For Testing
Differences In Related Pairs Of Measurement
Concerning Hypothesis $H_3$.

In choosing $\alpha = 0.05$ as a level of significance, then the critical regions
for a one tailed test which corresponds to $d_0 > 0$ and $df = n - 1$, are found
from tables to be:
For $g_5$: $T > T_\alpha = 1.717$
For $g_6$: $T > T_\alpha = 1.729$ \((Walpole, 1968 \text{, P.333})\).

Thus the null set ($H_0 : d_0 = 0$) is significantly rejected and $H_3$ is
accepted.

8.3.5.5. General Discussion

According to testing of the three hypotheses $H_1$, $H_2$ and $H_3$, I will state
them in their final form $\hat{H}_1$, $\hat{H}_2$ and $\hat{H}_3$—by modifying $H_1$ and $H_2$ into $\hat{H}_1$
and $\hat{H}_2$ while $\hat{H}_3 = H_3$. 

<table>
<thead>
<tr>
<th></th>
<th>$S_0(g_5) - U_0(g_5)$</th>
<th>$S_0(g_6) - U_0(g_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>23</td>
<td>20</td>
</tr>
<tr>
<td>$\sqrt{n}$</td>
<td>4.7958</td>
<td>4.4721</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>0.6520</td>
<td>0.6000</td>
</tr>
<tr>
<td>$s_d$</td>
<td>0.8317</td>
<td>0.9403</td>
</tr>
<tr>
<td>$t$</td>
<td>3.7596</td>
<td>2.8536</td>
</tr>
</tbody>
</table>
H$_1$: Competency in P is not affected by the test-form $S_o + U_o + P$.

H$_2$: Competency in $U_o$ is not affected by P.

H$_3$: Competency in $S_o$ is significantly higher than competency in $U_o$.

(a) Hypothesis H$_1$ indicates that there is no statistical significance in the competency for P that could be affected by the test-form $S_o + U_o + P$, which means that competency for P is inherited through learning mathematics and that, in probabilistic terms, it is not affected by possible spontaneous direction found in $S_o$ or $U_o$. But tables 8.5 and 8.7 could possibly indicate that competency in $P_{i}$ is comparatively higher when P is preceded by $U_o$ as in groups $g_3$ and $g_5$. It seems that the processes for P could profit from $U_o$ which means that students can learn something from $U_o$ of use in P which is mostly desirable. In addition it was found that:

For $g_5$ ($S_o + U_o + P$) : $\text{Cor} (U_o, P) = 0.766$.

and for $g_6$ ($P + S_o + U_o$) : $\text{Cor} (U_o, P) = 0.379$.

This means that the relationship between $U_o$ and P is stronger when there is a natural sequence according to behaviour i.e. from simpler to more difficult. This, in turn, could also mean that either processes for P have profited from $U_o$, or processes in $U_o$ were retarded when exposed by a complex state as in P, although it must be said that H$_2$ does not confirm statistically significant differences of competencies in $U_o$ for $g_5$ and $g_6$. The processes might not have been retarded in $S_o$ because of the dominancy of $S$ as a learning space.

The preceding idea was indeed implicitly behind implementing the final test-form $P + S_o + U_o$. This was because I wanted to reassure a
sequence of learning in S and U which agrees with a remarkable idea of Piaget's (1964), who argues that learning is possible if we base the more complex structure on simpler ones which would yield a natural relationship and development of structure. (Figure 8.2 might also refer to better consistency in $(E_{u_i}, E_{p_j})$ in 1 of $g_5$ than in 1 of $g_6$.)

Furthermore, $\text{Cor} (U_0, P) = 0.766$ regarding the form $S_0 + U_0 + P$ is higher than correlations of $(H_0, P)$ regarding the forms $S_0 + H_0 + P$ in Table 7.4. This means that $U_0$ is related more strongly to $P$ than $H_0$ is. $U_0$ is indeed more specific, limited and less difficult - under the proposition of learnability 8.2 - than $H_0$, and further learnability in U could possibly be governed by the algorithm in proposition 8.2 which is also desirable.

(b) Hypothesis $H_3 (= H_3)$ goes in line of the first indicator $T_1$ in 7.2.0. It confirms that competency in S is not necessarily transferred into U which is part of H. This may also indicate that, generally speaking, the limitations of a student's mathematical knowledge, especially in non-mathematical fields such as physics, may be related to the fact that processes are more or less restricted to S. In addition, the following correlations might be noticeable:

(i) For $g_5$: $\text{Cor} (S_0, U_0) = 0.745$ and for $g_6$: $\text{Cor} (S_0, U_0) = 0.601$.

(ii) $\text{Cor} (S_0, H_0) = 0.58, 0.52$ and $0.58$ in $\hat{E}_1, \hat{E}_2$ and $\hat{E}_3$ respectively of table 7.4 (P.7-25).
This could, in general mean that skipping from \(S\) into \(U\) is more possible than from \(S\) into \(H\). This may reinforce conclusion 8.1 suggesting \(U\) as possibly being able to contribute to a desired modification, since \(U\) has stronger relationships with both \(S\) and \(P\) than \(H\), remembering that it was viewed in 7.2.2.0 that learnability in \(H\) as a whole might create such 'blocks' as might hinder progress in learning.

(c) Taking into consideration that competency in \(U_0\) after \(P\) could be retarded as \(\text{Cor}(U_0, P) = 0.379\) for \(g_6\) might suggest that a study of case \(g_5\) in Table 8.7 could be more consistent, since \(g_5\) was tested in a more natural procedure than \(g_6\). Hence, let us compare the competency of \(g_5\) in the following rows in Table 8.7.

row 1 compared with row 4  
row 2 compared with row 5  
row 6 compared with row 7.

We might notice that successful penetration in steps of \(P\) is more stable in the case of successful penetration through \(S_0 + U_0\) (rows 6 and 7), than the other two cases.

This would mean that learning in \(S + U\) may possibly be more efficient than mere learning in \(S\) or \(U\). And achievement in \(S + U\) could have better stability as well as other advantages if it is compared to current advantages of \(S\), provided that \(S\) is supposed to be indispensable. So we find that this learning space is in line with (3) in General Conclusion 7.3 (P.7-32).
8.3.5.5.0. Conclusion 8.2

(i) Competency in a problem $P$ in a test-form $S_0 + U_0 + P$ is not significantly affected statistically by this form. The efficiency regarding this form could be reliable.

(ii) A learning space $S + U$ may provide an optimisation for a modification in the methodology of mathematics teaching-learning situations yielding an improvement in the achievement of students in mathematics.
Contingent implications of the test-form: At set $T_4$ of 6 tests.

The main purpose was to discover if competency in $P$ - hence the efficiency - might have contingent implications because of the test-form in $S_0 + U_0 + P$ studied in $T_3$. A problem $P$ was chosen and two kernels $S_0$ and $U_0$ were constructed in $S$ and $U$ respectively. The set $P, S_0$ and $U_0$ was split into 6 tests: $S_0 + P, P + S_0, U_0 + P, P + U_0, S_0 + U_0 + P$ and $P + S_0 + U_0$. It was viewed that the first four tests would help in building up constructs to be presented as hypotheses to be tested, concerning both the main purpose and other contingent ones, through the last two tests.

Three pairs of classes $(g_1, g_2), (g_3, g_4)$ and $(g_5, g_6)$ from three different schools in the final grade of the secondary schools in Kuwait, were chosen for running this experimental study. Each pair of classes had the same mathematics teacher and came from the same school. The following illustrates the experimental study.

<table>
<thead>
<tr>
<th>Group</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
<th>$g_5$</th>
<th>$g_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of students</td>
<td>24</td>
<td>22</td>
<td>22</td>
<td>19</td>
<td>23</td>
<td>20</td>
</tr>
<tr>
<td>Test</td>
<td>$S_0 + P$</td>
<td>$P + S_0$</td>
<td>$U_0 + P$</td>
<td>$P + U_0$</td>
<td>$S_0 + U_0 + P$</td>
<td>$P + S_0 + U_0$</td>
</tr>
</tbody>
</table>

All the classes were activated before the tests with the necessary knowledge for $P$. Students in the test were given a first sheet, e.g. $S_0$, then the successive sheet was only given when the student gave back the previous sheet.
### SCORING

Duration of test was not fixed.

Scoring in P was identified by steps of P and '+1' was given to each correct step in P whether this appeared explicitly or implicitly in the procedure of the student: '0 was given otherwise. Scoring of $S_0$ and $U_0$ was the same as the correction of explicit steps in P.

### CONTENT VALIDITY

This was achieved by the panel who helped in construction of $T_4$ (see ii of 2.4).

### RELIABILITY

This was satisfied by inter-rater reliability discussed in 2.4.2.1.1 of Chapter 2.

### HYPOTHESES

Three hypotheses were found to be valid under testing:

1. P is not affected by the test-form $S_0 + U_0 + P$.
2. Competency in $U_0$ is not affected by P.
3. Competency in $S_0$ is significantly higher than competency in $U_0$.

### CONCLUSION

Competency in P is not affected by the test-form and a learning space $S + U$ is optimised to provide a modification in the methodology for improving the achievement of students in mathematics.

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#### 8.4. Summary

The chapter begins by summarising what would be essential for following up the study in this chapter. It indicated that the research problem is based on looking for a feasible development in methodology of teaching-learning situations in order to improve the achievement of students in mathematics. It was also noted that there is a structure $S$ that underlies the current presentation of mathematics; however a part, $U$ of $H$, was...
supposed to take part in modification in the methodology of mathematics teaching-learning situations i.e. a modification could be expected to be based on both $S$ and $U$. Furthermore, it was viewed that:

(i) mathematics education could benefit from a modification based on an algorithm;
(ii) modification is based on a theoretical framework that rises mainly from experience, reviewed literature and introspection;
(iii) such modification could link theory and practice.

Hence the study in this chapter was expected to achieve three main objectives:

(I) proposing both the identification of $U$ and an approach (learnability) in $U$;
(II) discussion of the employment of theory in Chapter 5 in searching for a modification; and
(III) implementing empirical studies on the basis of I and II.

(I) In proposing $U$, it was viewed that such a $U$ would not cause 'blocks' in learning if it contained the least number of constants in instances, i.e. if it contained only one constant of implicit form in each $H$-instance. Hence $U$ was proposed to be the set of all $H$-instances that contain only one constant in each. The approach for learnability in $U$ was based on a pre-determined choice of $s_i$ in $S$, such that an associate $u_i$ in $U$ is chosen by replacing any fixed number in $s_i$ by a suitable functional relation $F(a)$ for a constant, 'a' say. This is on condition that such a replacement satisfies the following:

(i) $u_i$ stems from the general mathematical form of $s_i$ so that $u_i$ satisfies the same behavioural objective $B_i$ as $s_i$.
(ii) $u_i$ gives identical form of results as $s_i$.
(iii) the results in $u_i$ eventually contain, if possible, a functional form of the constant $a$. This was satisfied in Proposition 8.2.
(II) For employing theory in modification, it was supposed that this could be achieved through the so-called pseudo-reality. A uniform operational for a problem $P$ in pseudo-reality is defined to be an individual who is able in one kernel for $P$. Hence the efficiency discussed in Chapter 5 was suggested to be calculated on the basis of such uniform operationals in the pseudo-reality. Furthermore, it was suggested that the testing of the efficiency should be based on open-ended tests that were established on the six steps $E_1$, $E_2$, ..., and $E_6$.

(III) An empirical study in a set $T_3$ of three tests was carried out. The aim behind $T_3$ was the study of the efficiencies in $S$ and $U$ for proposing a modification in the methodology. Each test was based on the choice of a problem $P$ as well as constructing two kernels $S_0$ and $U_0$ for $P$ in $S$ and $U$ respectively, where $U_0$ is constructed on basis of Proposition 8.2. It was found that the efficiency $r(S)$ in $S$ is always smaller than the efficiency $r(U)$ in $U$ for each test. Each test was arranged in the form $S_0 + U_0 + P$.

Furthermore, it was suggested that it would be worthwhile to study how the test-form $S_0 + U_0 + P$ could affect ability in $P$ and hence in the efficiency. Accordingly, a set $P$, $S_0$, $U_0$ was split into six tests, $S_0 + P$, $P + S_0$, $U_0 + P$, $P + U_0$, $S_0 + U_0 + P$ and $P + S_0 + U_0$, and administered respectively to the six groups of students, $g_1$, ..., $g_6$. $g_1$ and $g_2$ came from the final grade in one secondary school and had the same mathematics teacher. The same applied for $g_3$ and $g_4$ and for $g_5$ and $g_6$. The first four tests were expected to present constructs for the final two tests. Those constructs were established on the basis of testing hypotheses. The study indicated that:
(i) Competency in $P$ is not affected by the test-form,

(ii) Competency in $U_0$ in general is not affected by $P$.

However, it was argued that competency in $U_0$ could be retarded by $P$ being before $U_0$, which reassures us that learning should proceed from the simpler structure to the more complex one; and

(iii) competency in the space $S$ is higher than relevant competency in $U$, which means that competency in $S$ is not necessarily transferred into $U$.

Finally, it was concluded that the learning space $S + U$ could provide an optimisation to promote the achievement of students in mathematics.
CHAPTER NINE

TOWARDS MODIFICATION IN METHODOLOGY: S + U,
A LEARNING SPACE
9. TOWARDS MODIFICATION IN METHODOLOGY: S + U, A LEARNING SPACE

9.0. Introduction

In studying the research problem which seeks a modification in methodology of teaching-learning situations which would imply improvement in the achievement of students in mathematics, it was argued that if 'A' is a number assigned to the achievement of a mathematical task 'k', then \( A = g(R) \) where \( R \) is the presentation (see Appendix A) of \( k \). Simply, \( R \) constitutes all instances such as examples, exercises and simple problems that are usually introduced, so that each of them is designed to serve one behavioural objective in the set \( B \) of behavioural objectives necessary for \( k \).

Furthermore, it was viewed that a desired modification yielding an improvement of students in mathematics could naturally result in raising \( A \) into \( \hat{A} \).

In other words, it was viewed that if:

(i) \( G_1 \) and \( G_2 \) are two equivalent groups,
(ii) \( G_1 \) achieves a task \( k \) under presentation \( R \), i.e. \( A = g(R) \),
(iii) \( G_2 \) achieves the same task \( k \) under a modification that yields a presentation \( \hat{R} \), i.e. \( \hat{A} = g(\hat{R}) \).

and if \( A \) and \( \hat{A} \) represent the average scores of achievements of \( G_1 \) and \( G_2 \) respectively, then \( A < \hat{A} \).

In this connection, it was argued in 6.2.2 (P.6-37) that \( R = R(S) \)}
i.e. \( R \) is dependent on a structure \( S \) that underlies current presentation. However, it was also argued that a true change in \( R \) should be based on a scrutiny of \( S \) as well as another structure \( H \), possibly contributing to a modification in the methodology under certain considerations (Proposition 6.4). \( S \) and \( H \) were identified in 7.1 and it was argued in 7.2.2.0 (P.7-29) that \( S \) is indispensable whereas \( H \) might contribute through a part \( U \) of \( H \) provided that this \( U \) is presented as 'docile' for learnability, satisfying Proposition 6.3. In addition, it was believed (see C3 P.6-5) that such a modification should rest on a theoretical framework, since theory is invaluable in any modification. This is in the sense that it provides a systematic study for modification as well as providing help in repeating the study in the same or other fields of knowledge. A study, therefore, was presented in Chapter 5, which was a theoretical study arising from an initial discussion about modification in 7.3 (P.7-33). The study was based on the efficiency (see Appendix A) as a measure of the efficient knowledge of what has been achieved in a space of knowledge, when this is applied in a relevant situation, e.g. solving a problem. Accordingly a Proposition 5.1 (P.5-27) of modification in the methodology of teaching-learning situations was introduced which I restate again here, because of its importance:

(Proposition 5.1): If 'O' is a certain normal population and each of \( W \) and \( \dot{W} \) is a structured knowledge in the sense of Definition 5.1, and if:

(i) \( (D, B, X) \) is a problem-space for a problem \( P \),
(ii) \( r \) and \( \dot{r} \) are the two efficiencies in \( W \) and \( \dot{W} \) respectively for \( O \) and \( P \),
(iii) \( r < \dot{r} \leq 1 \),

then \( \dot{W} \) could contribute in whole or in part to modification in the methodology of teaching-learning situations, provided that the part or whole of \( \dot{W} \) is applicable in the sense of Proposition 6.3.
In this connection, it was viewed in general conclusion 7.3 (P.7-32) that \( U \), a part of \( H \), could possibly contribute to a modification in the methodology. Further, Proposition 8.1 proposed \( U \) as the subset of \( H \) that contains \( H \)-instances, involving only one constant of implicit form.

Proposition 8.2 was then introduced to monitor an approach to learnability in \( U \). The empirical study in 8.3 could then discriminate between the efficiencies in \( S \) and \( U \) studied on the basis of the test-form \( S_0 + U_0 + P \), where \( S_0 \) and \( U_0 \) are two kernels for \( P \) in \( S \) and \( U \) respectively. A further study in 8.3.5 could support this discrimination by indicating that competency in \( P \) is not affected by the test-form. This indeed fulfilled Propositions 6.3 and 6.4 and it was accordingly viewed that these results might facilitate the use of Proposition 5.1 for the required modification, as in conclusion 8.2 (P.8-47). Nevertheless, it is worthwhile indicating that the subjects of the four sets of instruments \( T_1 \) and \( T_2 \) (Chapter 7) as well as \( T_3 \) and \( T_4 \) (Chapter 8) were either students in the final grade of school, or students in their first year at Kuwait University, who were still under the influence of the mathematics taught at school. This procedure took over since these subjects are more acquainted with \( H \) (or \( U \)) than those in the first three grades of Kuwait secondary schools, through non-mathematical fields, e.g. physics. In fact, believed that the efficiencies in \( S \) and \( U \) would be more reliable for those subjects. Despite this, it is to be noted that all those instruments, especially \( T_3 \) (reported in 8.3), were implemented in order to draw indications to help propose a modification in the methodology in the relevant area of this study, i.e. the secondary stage of school in Kuwait. Hence, a modification in the methodology was proposed and relevant experimentations were carried out in the main area proposed by the study.
It was natural then to discuss the following two main objectives in this chapter to present a prospective modification to fulfill the aims of this study.

I: To propose the modified presentation $\hat{R}$ in the methodology.

II: To justify this modification by evidence based on experimentation.

9.1. I. A Proposed Model For Modification

A task 'k' is usually achieved in the methodology of teaching-learning situations on the basis of the following steps:

$k_1$: $k$ is analysed into a certain set $B$ of basic behavioural objectives that are considered to be necessary for $k$.

$k_2$: A theoretical part may be identified and introduced, e.g. the proof of a theorem or an argument for a conclusion.

$k_3$: A set $R$, called the presentation of $k$, is introduced as a set of instances in the form of examples, exercises and simple problems, so that each is designed to serve for a $B_i$ in $B$ for all $B_i$ in $B$.

$k_4$: An achievement test is designed on the basis of a certain criterion that assesses the mastery for each objective of $k$. The competency is judged on the basis of performance under this criterion. Furthermore, the criterion might, as is the case in Kuwait, provide rules for grading the achievement, in that it helps assign a number 'A' to indicate for the achievement of $k$ (see (a), P. 6-8).

It could be noticed that $R$ in step $k_3$ is the core of the achievement
under $B$, since the theoretical part in step $k_2$ may be invariant under different methodologies of teaching-learning situations relevant to $k$. This is especially true of the theoretical part in the school mathematics which is almost stable and hence invariant.

It was found that current presentation $R$ mainly depends on $S$ (see 7.1.1 and Appendix C).

The current presentation $R$ in $S$ could be expressed in a model identified in the following matrix:

$$R = \begin{bmatrix}
B_1 & B_2 & \cdots & B_1 & \cdots & B_n \\
S_{11} & S_{21} & \cdots & S_{11} & \cdots & S_{ij} \\
S_{12} & & \ddots & & & \\
\vdots & & & \ddots & & \\
S_{ij} & & & & \ddots & \\
\end{bmatrix} = R(S)$$

For each $B_i$ the set $\{s_{ij}\}_{j}$ is a set of instances such as examples, exercises and simple problems, i.e. novel instances, supposed to serve for a $B_i$ of $k$ in $S$. This $R$ will be called the conventional presentation.

But conclusions 8.1 and 8.2 derived from the empirical studies in 8.3 and 8.3.5 based on the efficiency in pseudo-reality, as well on both Proposition 5.1 and Proposition 8.2 of learnability in $U$, would encourage putting forward the learning space $S + U$. This would provide a model in modification in the methodology, based on a modified presentation $\hat{R}$ for a mathematical teaching-learning task. It was proposed that this model $\hat{R}$ is more practical if it is based on an algorithmic approach. This would
be derived from a task-analysis in terms of behavioural objectives and employ these in S as well as in U under Proposition 8.2 of learnability. The following proposition was put forward to satisfy such an algorithm supposed to provide a certain model for modification in the methodology of mathematics teaching-learning situations.

9.1.0. Proposition 9.1

If B is a set of basic behavioural objectives necessary for a teaching-learning task k, then the achievement of k is possibly improved under the following presentation:

\[ R' = \begin{bmatrix}
B_1 & B_2 & \cdots & B_i & \cdots & B_n \\
(s_{11}, u_{1k}) & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(s_{ij}, u_{ikj}) & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(s_{im}, u_{ikj}) & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix} = R'(S + U) \]

Where the matrix, except in the first row, is made up of ordered pairs \((s_{ij}, u_{ikj})\), and \(u_{ikj}\) is an associate element in U of some \(s_{im}\) in S governed (as \(s_{ij}\)) by \(B_i\), under Proposition 8.2 of learnability.

\( R' \) will be called the modified presentation, or a modified model in the methodology, and \( S + U \) will be called the modified learning space.
This \( R \) demonstrates that dealing with each \( B_i \) should be based on both \( S \) and \( U \). In other words, coping with instances for each \( B_i \) should be in pairs in both \( S \) and \( U \) under Proposition 8.2 in all examples, exercises and simple problems. Furthermore, it is to be noted that the choice of the ordered pair \((s_{ij}, u_{ikj})\) was intended in that a change in form of the associate is desired in order to avoid tedious direct repetitions of instances. Nevertheless, \( u_{ikj} \) can be chosen as the associate in \( U \) for \( s_{ij} \) in \( S \). Hence an instance concerning a behavioural objective \( B_i \) in the proposed learning space is an ordered pair \((x,y)\) with \( x \) as a simple instance in \( S \) and \( y \) as a simple instance in \( U \), introduced under certain conditions that satisfy Proposition 9.1 provided that both \( x \) and \( y \) are governed by \( B_i \).

It is to be noted that the achievement \( A = g(R) \) with \( R = R(S) \) is expected to be empirically studied against \( A = g(\hat{R}) \) with \( \hat{R} = \hat{R}(S + U) \) in the following section.

9.2. II: Empirical Evidence: Mathematics Achievement In S Against S+U

Here I shall present the following empirical evidence, based on two experiments. The first will be based on studying indications of the achievement of students on the basis of the conventional presentation \( R(S) \) against the modified presentation \( \hat{R}(S + U) \). The second experiment will be concerned with achievement under \( R(S) \) against \( \hat{R}(S + U) \) on the basis of hypothesis-testing.
9.2.0. Experiment I

The procedure in this experiment was based on the following steps:

(i) choice of a certain grade in secondary school;
(ii) choice of two representative samples, $G_1$ and $G_2$, of the population of the grade;
(iii) choice of two mathematical tasks, $k_1$ and $k_2$, relevant to the grade;
(iv) preparation of two treatments, one conventional and one modified, for each task: $R_1(S)$ and $\hat{R}_1(S + U)$ respectively for $k_1$, and $R_2(S)$ and $\hat{R}_2(S + U)$ respectively for $k_2$;
(v) planning and conducting work on the basis that one group is experimental (Exp.) and the other is control (Con.) for the first task $k_1$, and vice-versa for $k_2$;
(vi) administration of two achievement tests, $T(k_1)$ and $T(k_2)$, to both groups, $G_1$ and $G_2$, after completion of the relevant tasks $k_1$ and $k_2$ respectively. The test was constructed in $S + U$ under proposition 8.2 in order to study how the achievement of the two groups would be demonstrated in both structures $S$ and $U$.

Table 9.1 illustrates how those steps were put to work.

<table>
<thead>
<tr>
<th>Group</th>
<th>Type</th>
<th>Treatment</th>
<th>Achievement Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>Exp.</td>
<td>$\hat{R}_1$</td>
<td>$T(k_1)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>Con.</td>
<td>$R_1$</td>
<td>$T(k_1)$</td>
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Table 9.1
Illustration of Procedure In The First Experiment.
In this connection, the two tasks $k_1$ and $k_2$ were chosen from the mathematics of the second grade of secondary school in Kuwait (ages 15-16). This choice was due to the fact that this grade is crucial in the secondary stage, since students who successfully finish this grade are given the choice to follow Arts or Sciences in the last two grades of secondary school.

Although there is a formal policy towards attracting more students to sciences rather than arts, a study has revealed that at least one third of students in this grade follow arts mainly because they feel that they cannot do better in mathematics. In addition, $k_2$ directly succeeds $k_1$ in the text, so that the experiment proceeded from $k_1$ into $k_2$ without any detachment.

9.2.0.1. Description Of Method

(a) The material: two treatments and two achievement tests

(al) Task $k_1$: some relations in line and circle (Appendix G)

(all) The set $B(k_1)$ of behavioural objectives for $k_1$ was identified as five elements:

$B_1$: To identify and apply knowledge about the equation of a line with a given slope passing through a given point. (This is a review element to relate the students' past knowledge to the new one).

$B_2$: To identify and apply knowledge about the centre of a given circle with a given equation.
B₃: To identify and apply knowledge about the slope of a certain radius in a given circle.

B₄: To identify and apply knowledge about the slope of a line perpendicular to a line identified by two points on it.

B₅: To identify and apply knowledge about the slope of a tangent to a given circle at a given point.

Accordingly, two treatments, conventional \( R = R_1(S) \) and modified \( \hat{R}_1 = \hat{R}_1(S + U) \) (Appendix G) were prepared for the control group \( G_2 \) and the experimental group \( G_1 \) respectively. Here are some examples from the two treatments concerning \( B_1 \) and \( B_3 \).

<table>
<thead>
<tr>
<th>Conventional ( R_1 = R_1(S) ): Group ( G_2 )</th>
<th>Modified ( \hat{R}_1 = \hat{R}_1(S + U) ): Group ( G_1 )</th>
</tr>
</thead>
</table>

For \( B_1 \): Find the equation of the line that has the slope 'm' and passes through the point \( N \) in each of the following cases:

(i) \( m = 2, N = (3,4) \)

(ii) \( m = \frac{2}{3}, N = (1,-2) \)

(iii) \( m = -\frac{2}{5}, N = (2,-2) \)

(iv) \( m = 3, N = (2,3) \)

(i) \( m = 2, N = (3,4) \)

(ii) \( m = -\frac{2}{3}, N = (1,-2) \)

(iii) \( m = -\frac{2}{5}, N = (2,-2) \) (a being a constant)

(iv) \( m = 3, N = (2b,3) \) (b being a constant)
For $B_3$: Find the slope of the radius which passes through $M$ in each of the following circles.

(i) $A : x^2 + y^2 - 2x + 4y - 4 = 0, \ M = (1,1)$
(ii) $B : x^2 + y^2 + 2x - 1 = 0, \ M = (-2,-1)$
(iii) $C : x^2 + y^2 - 8y - 4 = 0, \ M = (2,8)$
(iv) $D : x^2 + y^2 + 6x - y - 2 = 0, \ M = (-6,2)$

The achievement test $T(k_1)$ (Appendix G) was designed to test attainment of knowledge in the learning space $S + U$ denoted in (vi) of 9.2.0. The test consisted of 10 items such that two items, one in $S$ and the other its associate in $U$ - under Proposition 8.2 - were designed to assess the attainment of mathematical knowledge that is relevant to each behavioural objective $B_i$ in $B(k_1)$. Items 1, 2, 3, 4 and 5 were in $S$; while 6, 7, 8, 9 and 10 were their associates under Proposition 8.2 respectively in $U$.

Here are some examples:

(i) For $B_2$

$s_2$ (item 2): Find the centre of the circle:
$A : x^2 + y^2 + x - 4y - 1 = 0$

The associate item $u_2$ (of $s_2$) in $U$ is:

$u_2$ (item 7): Find the centre of the circle:
$A = x^2 + y^2 + ax - 4y - 2a = 0$ ('a' being a constant).
(i) For $B_5$

$s_5$ (item 5): Let $E = (3, 7)$ be the centre of a circle $A$ and $M = (5, 8)$ belongs to $A$. Find the slope of the tangent to $A$ at $M$.

The associate $u_5$ (of $s_5$) in $U$ is:

$u_5$ (item 10): Let $M = (5, 8a)$ belong to a circle with centre $I = (3, 7a)$. Find the slope of the tangent to this circle at $M$. ('$a$' being a constant $\neq 0$).

(a2) Task $k_2$: Some relations in transformation geometry (Appendix G).

(a21) The set $B(k_2)$ of behavioural objectives was identified by five elements as in $k_1$.

$\hat{B}_1$: To identify and apply knowledge of how to find the image of a point under a transformation.

$\hat{B}_2$: To identify and apply knowledge of how to find the pre-image of an image under a transformation.

$\hat{B}_3$: To identify and apply knowledge of how to find the pre-image of the image $(x, y)$ under a transformation.

$\hat{B}_4$: To identify and apply knowledge of how to describe a line in set-language.

$\hat{B}_5$: To identify and apply knowledge of how to describe the image line of a given line under a transformation.

Accordingly, two treatments as in $k_1$ were prepared, conventional $R_2$ for the new control group $G_1$, and a modified $\hat{R}_2$ for the new experimental group $G_2$.

Here are some examples of the two treatments:
Conventional \( R_2 = R_2(S) \): Group \( G_1 \)

Modified \( \tilde{R}_2 = \tilde{R}_2(S + U) \): Group \( G_2 \)

For \( B_2 \):

If the image of \((x, y)\) under a certain transformation \( T \) is \((4x, -2y)\), then fill in the blanks to show the points which have been transformed under \( T \).

<table>
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<th>(i)</th>
<th>(ii)</th>
<th>(iii)</th>
<th>(iv)</th>
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<td>(..., ...)</td>
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<tr>
<td></td>
<td>( \mapsto (8, -6) )</td>
<td>( \mapsto (-4, 4) )</td>
<td>( \mapsto (10, 6) )</td>
<td>( \mapsto (-8, 5) )</td>
</tr>
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For \( B_5 \):

Describe in set-language the image-line \( \hat{1} \) of a line \( 1 \) under the transformation \( T \) in each of the following cases:

<table>
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<th>(i)</th>
<th>(ii)</th>
<th>(iii)</th>
<th>(iv)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T: (x, y) \to (x, -y) )</td>
<td>( T: (x, y) \to (\frac{1}{2}x, 3y) )</td>
<td>( T: (x, y) \to (-x, 2y) )</td>
<td>( T: (x, y) \to (2x, -3y) )</td>
</tr>
</tbody>
</table>

Illustrative solutions are:

- Instance (iii) of \( B_2 \) in \( R_2 \) may be solved as:
  \( (-\frac{10}{4}, -\frac{5}{2}) \) or \( (\frac{5}{2}, -3) \to (10, 6) \)

while instance (iii) in \( \tilde{R}_2 \) may be solved as:

\( (-\frac{10}{4}, -\frac{6a}{2}) \) or \( (\frac{5}{2}, -3a) \to (10, 6a) \)

- Instance (iv) of \( B_5 \) in \( R_2 \) may be solved as:

\[ \hat{1} = \{(x, y) : (x, y) \text{ is the image of } (\frac{x}{2}, \frac{y}{3}) \text{ on } \hat{1}\} \]

\[ = \{(x, y) : (x, y) \text{ on } \hat{1} \Rightarrow (\frac{x}{2}, -\frac{y}{3}) \text{ on } \hat{1}\} \]
The achievement test $T(k_2)$ (Appendix G) was designed in the same way as $T(k_1)$. It consisted of 10 items in $S + U$ such that elements in $S$ and their associates in $U$ were designed to assess mathematical knowledge attained for each $B_i$ in $B(k_2)$. Items 1, 2, 3, 4 and 5 were in $S$; while items 6, 7, 8, 9 and 10 were respectively their associates in $U$ under Proposition 8.2.

Here are some examples:

For $B_3$:

$s_3$ (item 3): If $T$ is the transformation $: (x, y) \rightarrow (3x, 2y)$, then find the pre-image of $(x, y)$.

$u_3$ (item 8): Let $T$ be the transformation $: (x, y) \rightarrow (3ax, 2ay)$. Find the pre-image of $(x, y)$.

For $B_4$:

$s_4$ (item 4): Use set-language to describe a line which passes through the point $(2, 0)$ with a slope $m = -2$.

$u_4$ (item 9): A line $l$ with slope $2a$ passes through $M = (2a, 0)$. Use set-notation to describe $l$.

It is to be noted that a line, in this case $l$, is described in set-language as:

$$l = \{(x, y) : y - y_0 = m(x - x_0)\}$$

(b) Subjects And Setting

The two groups $G_1$ and $G_2$ with 87 boys and girls in each were chosen from three different schools according to the following rules:
(1) Two classes $x_1$ and $y_1$ of 26 and 28 students respectively, were chosen from a school for boys. Both classes had the same mathematics teacher. Two students were randomly eliminated from $y_1$ so that the classes had equal numbers. This was done to facilitate direct observations.

(2) Two classes $x_2$ and $y_2$ were chosen from a school for girls according to the same criteria as in (1). Each class contained 28 students.

(3) Two classes $x_3$ and $y_3$, of 33 and 34 students respectively, were chosen from a third secondary school for girls under the previous conditions. One student was randomly eliminated from $y_3$ so the classes had equal numbers.

Accordingly, group $G_1$ was made up of the pupils from $x_1$, $x_2$ and $x_3$, while $G_2$ from those of $y_1$, $y_2$ and $y_3$.

\[ G_1 = \{x_1, x_2, x_3\} \quad \text{and} \quad G_2 = \{y_1, y_2, y_3\} \]

Each group could be considered as a representative sample for its parent population, since it contained two classes at least (see Glossary); hence they were considered to be equivalent.

(c) The Procedure

(cl) The teachers were given the sheets of treatments of the two tasks successively in order to guarantee that each class would follow precisely
the relevant task according to the plan (usually, students in the same grade in different classes exchange leaflets given to them by their teachers). A teacher was asked to discuss with each class the two instances (i) and (iii), i.e. (item i, item iii) in each of the five sets of instances of relevant treatment in both tasks $k_1$ and $k_2$. The teacher and the senior teacher of mathematics observed and helped students to cope with instances (ii) and (iv) in each set of instances which were usually introduced by the teacher writing them on the blackboard. I participated in this test by being in the classroom at least once for each task. In addition, there was an intensive follow-up in observing, guiding and helping students in all other instances given as exercises, in order to make sure of the precise completion of the task by all the participants in the experiment.

(c2) Group $G_1$ was experimental and $G_2$ was control for $k_1$. For $k_2$ the roles were vice versa.

(c3) Each task took an average of about three periods of 45 minutes each. Indeed, it took about 4 periods of some classes affected by contingent situations. All students were given new copy books to write in, and these copy books were collected after each session to be given back in the following session, in order to make sure that each class in the experiment is utterly restricted to its treatment.

(c4) Each class in the same school was given the achievement test as they both completed the task or within one or two days. The answers to the questions were to be written in spaces provided on the question sheet. The duration of the test was not fixed, but all students finished the
test in the time of one period, i.e. 45 minutes.

(d) Marking

An item was given '+1' for a completely correct procedure and answer, and '0' otherwise. The sheets were marked by the teachers and senior teachers who took part in the experiment in each school, and myself.

9.2.0.2. The Testing Constructs

Let $A^i_j$ and $A^i_j$ be the two achievements (scores) of students in a control class/group and experimental class/group respectively for items $j = 1, 2, ..., 10$. Also let $\hat{A}$ and $\hat{A}$ refer to the mean achievements of a control class/group and experimental class/group respectively, where the mean is taken as the sum of the scores of a class/group divided by the number of students in the class/group, then the testing constructs are:

1. $A^i_j$ might overlap with $\hat{A}^i_j$ for $j = 1, 2, ..., 10$ in both classes $x^i$ and $y^i$ ($i = 1, 2, 3$).

2. $A < \hat{A}$ for corresponding classes $x^i$ and $y^i$ as well as for the two groups $G_1$ and $G_2$ for the same task.

3. The achievement of a control group in $k_2$ would demonstrate progress if compared to the achievement of the control group in $k_1$. 
9.2.0.3. Discussion Of Results

1. Tables

Table 9.2 demonstrates the number of correct answers for each item in the first achievement test $T(k_1)$, and since the numbers of students of the two classes $x_i$ (Experimental) and $y_i$ (Control) are equal for each school $(i = 1, 2, 3)$, and these numbers are comparatively small, the percentages are not used, whilst they are used in the table with the two groups $G_1$ and $G_2$.

<table>
<thead>
<tr>
<th>School</th>
<th>Class</th>
<th>No. of Students</th>
<th>Type</th>
<th>Items S</th>
<th>Items U</th>
<th>Mean Achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$x_1$</td>
<td>26</td>
<td>Exp.</td>
<td>22</td>
<td>19</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>6</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>$y_1$</td>
<td>26</td>
<td>Con.</td>
<td>16</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>2nd</td>
<td>$x_2$</td>
<td>28</td>
<td>Exp.</td>
<td>24</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>$y_2$</td>
<td>28</td>
<td>Con.</td>
<td>9</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>3rd</td>
<td>$x_3$</td>
<td>33</td>
<td>Exp.</td>
<td>24</td>
<td>25</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>15</td>
<td>19</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 9.2

Achievement In Items In S And U For Test $T(k_1)$ For Experimental (Exp.) Classes And Control (Con.) Classes. Further, The Table Indicates The Mean Achievement For Each Experimental And Control Class.
The following table 9.3 indicates the percentages of correct responses (achievements) in each item of the test for the two groups: the experimental \( G_1 = \{x_1, x_2, x_3\} \) against the control \( G_2 = \{y_1, y_2, y_3\} \) in \( k_1 \).

<table>
<thead>
<tr>
<th>Group</th>
<th>No. of students</th>
<th>Type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Mean Achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>87</td>
<td>Exp.</td>
<td>64</td>
<td>69</td>
<td>78</td>
<td>69</td>
<td>33</td>
<td>66</td>
<td>53</td>
<td>52</td>
<td>6.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( G_2 )</td>
<td>87</td>
<td>Con.</td>
<td>46</td>
<td>49</td>
<td>46</td>
<td>70</td>
<td>31</td>
<td>36</td>
<td>23</td>
<td>39</td>
<td>36</td>
<td>20</td>
<td>3.94</td>
</tr>
</tbody>
</table>

Table 9.3

Percentages Of Correct Responses In Items Of Test \( T(k_1) \) For Experimental Group \( G_1 \) And Control Group \( G_2 \). The Table Also Provides The Mean Achievement Of Each Group.

As for \( k_2 \), the following tables 9.4 and 9.5 are similar to those in 9.2 and 9.3 respectively. In this case, \( G_1 \) was the control group in this task, while \( G_2 \) was the experimental group.
### Table 9.4
Achievement in Items in S and U for Test T(k2) for Experimental and Control Classes Besides the Mean Achievement of Each Class.

<table>
<thead>
<tr>
<th>School</th>
<th>Class No. of stu.</th>
<th>Type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Mean Achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>x1</td>
<td>Con.</td>
<td>25</td>
<td>22</td>
<td>15</td>
<td>21</td>
<td>26</td>
<td>23</td>
<td>15</td>
<td>14</td>
<td>17</td>
<td>24</td>
<td>7.47</td>
</tr>
<tr>
<td></td>
<td>y1</td>
<td>Exp.</td>
<td>24</td>
<td>20</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>22</td>
<td>20</td>
<td>14</td>
<td>18</td>
<td>20</td>
<td>7.62</td>
</tr>
<tr>
<td>2nd</td>
<td>x2</td>
<td>Con.</td>
<td>24</td>
<td>16</td>
<td>11</td>
<td>9</td>
<td>17</td>
<td>20</td>
<td>9</td>
<td>10</td>
<td>9</td>
<td>17</td>
<td>5.07</td>
</tr>
<tr>
<td></td>
<td>y2</td>
<td>Exp.</td>
<td>22</td>
<td>21</td>
<td>20</td>
<td>7</td>
<td>19</td>
<td>22</td>
<td>14</td>
<td>17</td>
<td>8</td>
<td>21</td>
<td>6.11</td>
</tr>
<tr>
<td>3rd</td>
<td>x3</td>
<td>Con.</td>
<td>30</td>
<td>22</td>
<td>17</td>
<td>16</td>
<td>15</td>
<td>18</td>
<td>13</td>
<td>13</td>
<td>16</td>
<td>15</td>
<td>5.30</td>
</tr>
<tr>
<td></td>
<td>y3</td>
<td>Exp.</td>
<td>27</td>
<td>23</td>
<td>18</td>
<td>23</td>
<td>23</td>
<td>26</td>
<td>19</td>
<td>25</td>
<td>20</td>
<td>6.88</td>
<td></td>
</tr>
</tbody>
</table>

### Table 9.5
Percentages of Correct Responses in Items of Test T(k2) for Control Group G1 and Experimental Group G2, Besides the Mean Achievement of Each Group.

<table>
<thead>
<tr>
<th>Group</th>
<th>No. of students</th>
<th>Type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Mean Achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>87</td>
<td>Con.</td>
<td>91</td>
<td>70</td>
<td>49</td>
<td>53</td>
<td>67</td>
<td>70</td>
<td>43</td>
<td>43</td>
<td>48</td>
<td>64</td>
<td>5.69</td>
</tr>
<tr>
<td>G2</td>
<td>87</td>
<td>Exp.</td>
<td>84</td>
<td>74</td>
<td>64</td>
<td>57</td>
<td>74</td>
<td>77</td>
<td>70</td>
<td>57</td>
<td>59</td>
<td>70</td>
<td>6.85</td>
</tr>
</tbody>
</table>

---

*Note: The content is presented as it appears in the image.*
Figure 9.1: Any diagram on the left indicates the achievement $A_j$ of a class $x_i(\text{Exp})$ against $A_j$ of class $y_i(\text{Con})$ in $T(k_1)$. The contrary holds for $T(k_2)$ in the opposite side for all $i = 1, 2$ and 3.
Figure 9.2: Achievement Of Experimental Group Against Control Group Indicated In Percentages In Items $j$ of $T(k_1)$ And $T(k_2)$.

(a) $G_1$ Experimental And $G_2$ Control : $T(k_1)$.

(b) $G_1$ Control and $G_2$ Experimental : $T(k_2)$. 
2. Analysis of Results

This analysis will as usual discuss the tables on the basis of the testing constructs as well as any possible relevant indications that might be involved.

(i) Tables 9.2 and 9.4 and the corresponding diagrams of figure 9.1 illustrate the overlap between the achievement of students in classes $x_1$ and $y_1$ and classes $x_2$ and $y_2$. Such an overlap does not hold for $x_3$ and $y_3$ except in the case of item 1 of figure 9.3.

My expectation for this correlation was based to differing degrees, on the following:

(a) Students, as I have experienced and is commonly acknowledged in Kuwait secondary schools are always aware of examinations, especially in mathematics. And therefore students of one class who are in the same grade and have the same teacher of mathematics usually try to trace what their teacher has set their peers in other classes. Some students in the experimental classes could have re-copied all or some of these instances in U for their own benefit, as they thought, and for their colleagues in the control class. However, I have no accredited evidence for this situation.

(b) Some students in the control classes, as I strongly suspected, were able in $k_1$ to skip into U and they were as competent in U in some cases as in the experimental classes who learnt in U for a short period of time. This is in line with the observation about D-students in the Ehrenpreis and Scandura study of 1974 (P.4-15).
(c) As for the overlap in $T(k_2)$ of figure 9.3, I conjecture that control students in $k_2$ benefited from learning in $U$ in $K_1$, especially where case (a) was not involved.

All these expectations, and perhaps others, go in line with the first construct.

(ii) Despite such correlation expected from Construct 1, tables 9.2, 9.3, 9.4 and 9.5 demonstrate that the mean achievement $\bar{A}$ for each control class/group was less than the mean achievement $\bar{A}$ for the corresponding experimental class/group in both tasks $k_1$ and $k_2$. Such a demonstration of the achievement in $S + U$ may provide accreditable support to Propositions 5.1, 8.2 and 9.1 as well as providing a sort of reliance on the construct efficiency and theory discussed in Chapter 5 on the basis of the space of knowledge as a conceptual structure. Hence the results stand in line with the second construct. Diagrams of figure 9.2 illustrate that the achievement of the experimental group in each item was, in all items except item 1 of $T(k_2)$, superior to the achievement of the control group.

(iii) Furthermore, if we speculate about the mean achievement of the representative groups $G_1$ and $G_2$ demonstrated in Tables 9.3 and 9.4, we observe that:

<table>
<thead>
<tr>
<th></th>
<th>$k_1$</th>
<th>Mean</th>
<th>$k_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>6.29</td>
<td>(Exp.)</td>
<td>5.69</td>
</tr>
<tr>
<td>$G_2$</td>
<td>3.94</td>
<td>(Con.)</td>
<td>6.85</td>
</tr>
</tbody>
</table>

and this indicates that (a) the control group $G_1$ in $k_2$ has benefitted
from the modified treatment received by G₁ as experimental in k₁ compared to the mean achievement of control group G₂ in k₁ which supports the third construct; and (b) the decline of the mean achievement of G₁ from k₁ to k₂, as well as the increase of the mean achievement of G₂ from k₁ to k₂ might suggest that this variation is in some way related to change in the methodology which in its turn is mainly related to the change in learning space in favour of S + U.

9.2.0.4. Conclusion 9.1

(i) There is an indication that the modified treatment-model which group G₁ received in k₁ improved its performance in k₂, as it did not have the modified treatment, if compared to the achievement of G₂ in k₁.

(ii) The achievement of a certain mathematical teaching-learning task under S + U demonstrates itself to be superior to the relevant achievement of the same task under S.
9.2.0.5. Summary Of Experiment I.

**TITLE**
Experiment I

**PURPOSE OF STUDY**
To study the achievement under the conventional learning space $S$ against the achievement in the proposed learning space $S + U$.

**PROCEDURE (i)**
Two successive learning tasks $k_1$ and $k_2$ were chosen from the textbook of mathematics of the second grade, and two treatments: one, $R$, conventional and the other, $\hat{R}$, modified were prepared for each task. (Appendix G).

**PROCEDURE (ii)**
Two representative groups $G_1$ and $G_2$ of equal numbers of students aged 15-16 years from six classes from three different secondary schools in Kuwait.

**PROCEDURE (iii)**
The study was run as follows:

<table>
<thead>
<tr>
<th>Task</th>
<th>(Experimental, Treatment)</th>
<th>(Control, Treatment)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>$(G_1, \hat{R}_1)$</td>
<td>$(G_2, R_1)$</td>
</tr>
<tr>
<td>$k_2$</td>
<td>$(G_2, \hat{R}_2)$</td>
<td>$(G_1, R_2)$</td>
</tr>
</tbody>
</table>

**TESTING OF ACHIEVEMENT**
$T(k_1)$ and $T(k_2)$ were two achievement tests (Appendix G), each of ten items with the first five in $S$ and the other five as their associates in $U$ under Proposition 8.2. $T(k_1)$ and $T(k_2)$ were designed and administered to the two groups after completion of the task. This was judged by the panel that contributed in construction of the test demonstrated in (ii) of 2.4 - Chapter 2.
<table>
<thead>
<tr>
<th>TESTING CONSTRUCTS</th>
</tr>
</thead>
</table>
| (1)              | An overlap in performances of classes that constitute the two groups $G_1$ and $G_2$.  
| (2)              | The mean achievement of a control class/group would be less than the mean achievement of the corresponding experimental class/group.  
| (3)              | The achievement of a control group in $k_2$ would demonstrate progress if compared to the achievement of a control group in $k_1$.  
| RELIABILITY      | This was satisfied on basis of the inter-rater reliability discussed in 2.4.2.1.1 of Chapter 2.  
| CONCLUSION (i)   | The modified treatment $\tilde{R}$, which group $G_1$ received in $k_1$, improved its performance in $k_2$, since it did not have the modified treatment $\tilde{R}_2$ if compared to the achievement of $G_2$ in $k_1$.  
| (ii)             | The achievement of a certain mathematical teaching-learning task under $S + U$ demonstrates superiority to achievement of the task under $S$.  

9.2.1. Experiment II

This experiment was concerned to find out on the basis of a statistical hypothesis* how far the learning space $S + U$ could be beneficial to ordinary students, commonly said to be in the majority. It is therefore necessary to identify a sensible definition of an ordinary student in the

* A statistical hypothesis is an assumption or statement which may or may not be true, concerning one or more population (Walpole, 1968, P.209).
As a matter of fact a student in Kuwait is considered to be excellent at mathematics if he (or she), over a considerable period of the academic year, e.g. 5 or 6 months, obtains an average score of not less than 80%. The student is considered to be weak and hence fails if his/her average score was less than 40%. Accordingly, it was found adequate to define an ordinary student to be neither excellent nor weak. Hence the following definition:

9.2.1.0. Definition 9.1

A student is said to be ordinary in the mathematics of the area of this study, if in a considerable period of the academic year, the student could obtain an average score (A.S.) in mathematics which is less than 80% and greater than or equal to 40%.

For convenience, all total marks of all grades of secondary school in Kuwait will be considered to be 100 and hence the average score A.S. for an ordinary student will satisfy: \(40 \leq \text{A.S.} < 80\).

9.2.1.1. Purpose Of Study

The study was aimed to demonstrate on the basis of hypothesis-testing that teaching-learning techniques based on the learning space \(S + U\) under Proposition 9.1 are superior for ordinary students to that technique conventionally based on the learning space \(S\). In more precise terms this evidence was directed towards studying the following hypothesis regarding ordinary students in the conventional achievements in \(S\). \(S\) was introduced into the study in order to demonstrate that the
achievement in $S + U$, would be superior to the achievement of the $S$
students in their space of knowledge $S$ for a further purpose discussed
in 9.2.1.4.

9.2.1.2. Hypothesis 9.1

The mathematics achievement in a teaching-learning task based
on the learning space $S + U$ in the sense of Proposition 9.1
is superior, for ordinary students in the area of this study,
to the achievement in the teaching-learning situation of the
same task conventionally based on the learning space $S$.

9.2.1.3. Two Tasks And Their Treatments

I chose two tasks $k$ and $k'$ from the mathematics of the fourth (final)
grade and second grade of the secondary school in Kuwait. The choice of
the second secondary grade was discussed in 9.2.0, while the choice
of the final grade of secondary school was related to the fact that the
students in this grade are recognised to be mostly serious and keen to
make all possible efforts in coping with mathematical tasks. This is due
to the high competition for places at a college of high repute. This,
it was thought, would present a more valuable decision about the hypothesis.

Each of the tasks were presented in two treatments, the conventional $R$
in $S$ and the modified, $\hat{R}$, in $S + U$ based on Proposition 9.1, as in the
previous tasks in 9.2.0. Here I present both tasks fully. Each task
was constructed on the basis of a unique basic behavioural objective $B$ for
$k$ and $\hat{B}$ for $k'$. 
Task: k  
Grade: Final grade of the secondary school in Kuwait.

B: To identify and apply the knowledge of solving a simple differential equation of the first degree by using an integrating factor.

<table>
<thead>
<tr>
<th>The Conventional Treatment:</th>
<th>The Modified Treatment:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = R(S)$ or $R(k)$</td>
<td>$R = R(S + U)$ or $R(k)$</td>
</tr>
</tbody>
</table>

i) The following examples were chosen as the same in the textbook.

| Example 1: $\frac{dy}{dx} = \frac{2y}{x} + 1$ (S-Instance) | Example 1: $\frac{dy}{dx} = \frac{2y}{x} + 1$ (S-Instance) |
| Example 2: $\frac{dy}{dx} + y = e^x$ (S-Instance) | Example 2: $\frac{dy}{dx} + ay = e^x$ (a being a constant $\neq 0$) (U-Instance) |
| Example 3: $\frac{dy}{dx} = y + 4$ (S-Instance) | Example 3: $\frac{dy}{dx} = y + 4$ (S-Instance) |
| Example 4: $3\frac{dy}{dx} + y + e^x = 0$ (S-Instance) | Example 4: $3a\frac{dy}{dx} + y + e^x = 0$ (a being a constant $\neq 0$) (U-Instance) |
(ii) The following exercises were chosen from 10 exercises in the textbook (see the Arabic sheet in Appendix G).

1. \( \frac{dy}{dx} = x + 1 \)  
   (S-Instance)

2. \( \frac{dy}{dx} + \frac{1}{x} y = 4 \)  
   (S-Instance)

3. \( \frac{dy}{dx} + 2y = e^{-x} \)  
   (S-Instance)

4. \( 2\frac{dy}{dx} - y = e^{\frac{1}{2}x} \)  
   (S-Instance)

5. \( \frac{dy}{dx} - xy = x \)  
   (S-Instance)

6. \( x\frac{dy}{dx} + (1 + x)y = e^{x} \)  
   (S-Instance)

Here is the solution of Example 1 given in the textbook:

\[ \frac{dy}{dx} - \frac{2}{x} y = 1; \text{ the integrating factor } M = \exp \left( \int \frac{-2}{x} dx \right) = \exp \left( \log \frac{1}{x^{2}} \right) = \frac{1}{x^{2}} \]

Hence:\[ \frac{d}{dx} (M \cdot y) = \frac{1}{x^{2}} \Rightarrow \frac{1}{x^{2}} \cdot y = -\frac{1}{x} + c \]

Then: \( y = cx^{2} - x \) (with \( x \neq 0 \), i.e. the point (0, 0) is excluded).

The treatment of \( k \) was similar to that of \( k \) as follows:
Task: k
Grade: Second grade of the secondary school in Kuwait.

To identify and apply the knowledge of finding the solution set of
\[ x^2 + bx + c \geq 0 \] where \( x^2 + bx + c = (x-x_1)(x-x_2) \) with \( x_1 \) and \( x_2 \) integers (certain considerations are given to \( x_1 \) and \( x_2 \) in \( U \)).

The Conventional treatment:
\[ R = R(S) \text{ or } R(k) \]

The Modified treatment:
\[ \bar{R} = \bar{R}(S + U) \text{ or } \bar{R}(k) \]

Examples: Find the solution set of:

(1) \( x^2 - x > 12 \) (S-Instance)
(2) \( (x-1)(x-4) < 0 \) (S-Instance)
(3) \( x^2 - 4 \geq 0 \) (S-Instance)
(4) \( x^2 + 3x + 2 \leq 0 \) (S-Instance)

(1) \( x^2 - x > 12 \) (S-Instance)
(2) \( (x-a)(x-4) < 0 \) ('a' being constant) (U-Instance)
(3) \( x^2 - 4 \geq 0 \) (S-Instance)
(4) \( x^2 + 3ax + 2a^2 \leq 0 \) ('a' being constant) (U-Instance)

Exercises: Find the solution set of:

(1) \( x^2 + 4x > 0 \)
(2) \( x^2 - 5x > -4 \)
(3) \( (x-1)(x+2) \leq 0 \)
(4) \( x^2 \geq 9 \)
(5) \( 5x - 6 \leq x^2 \)
(6) \( x < 6 - x^2 \)

(1) \( x^2 + 4x > 0 \)
(2) \( x^2 - 5ax > -4a^2 \)
(3) \( (x-1)(x+2) \leq 0 \)
(4) \( x^2 \geq 9a^2 \)
(5) \( 5x - 6 \leq -x^2 \)
(6) \( ax < 6a^2 - x^2 \)
It is to be noted that the examples and exercises of the treatment $R$ were chosen from the textbook for the second grade. Here is an illustrative example of the method of solution also found in the textbook and consequently taught to students.

Ex. 1: $x^2 - x > 12 \Rightarrow (x-4)(x+3) > 0$

$x - 4 \geq 0 \Leftrightarrow x \geq 4$ and $x - 4 \leq 0 \Leftrightarrow x \leq 4$, similarly $x + 3$.

Accordingly, the following table 9.6 is constructed. It assigns the solution set.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x-4$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$x+3$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$(x-4)(x+3)$</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

**Table 9.6**

*Illustrative Table Used For Finding The Solution Set Of $(x-x_1)(x-x_2) \neq 0$.*

Hence the solution set derived from Table 9.6 is \{x: x < -3 or x > 4\}.

9.2.1.4. Description Of Method

Each of the two groups of students were given a relevant achievement test which was in S, since I preferred to confirm on the basis of hypothesis-testing that the achievement in $R$ is superior in $S$ to the conventional achievement in $R$. By this I also intended to demonstrate that such
results might meet a convenient contention for change in the educational system. This I wanted to presume on the basis of my experience that this system is inertially reluctant to radical changes in the learning space $S$. Here, most people who work in this field believe that such a space is the adequate one, although all of them, myself included, do not recognise the existence of such a space.

(a) The Material

Here I present the two achievement tasks $T(k)$ and $T(\hat{k})$ for tasks $k$ and $\hat{k}$ respectively

\begin{tabular}{|l|}
\hline
(a$_1$) $T(k)$: A achievement test for task $k$: final grade. \\
\multicolumn{1}{|c|}{Solve each of the following differential equations by means of the integrating factor.} \\
\hline
(1) $\frac{dy}{dx} + 2y = e^x$  \\
(2) $x \frac{dy}{dx} - xy = x^3$  \\
(3) $3 \frac{dy}{dx} + y = e^{-\frac{1}{3}x}$  \\
(4) $(x+1) \frac{dy}{dx} = x - y$ \\
\hline
\end{tabular}

\begin{tabular}{|l|}
\hline
(a$_2$) $T(\hat{k})$: An achievement test for task $\hat{k}$: second grade. \\
\multicolumn{1}{|c|}{Find the solution set of each of the following open sentences in $x$.} \\
\hline
(1) $x^2 - 4x - 5 > 0$  \\
(2) $x^2 - 2x \leq 0$  \\
(3) $x^2 - 1 < 0$  \\
(4) $x^2 + 3x + 2 \geq 0$ \\
\hline
\end{tabular}
(b) Subjects And Setting

(b_1) For k: Two classes \( x_1 \) and \( y_1 \) who had the same mathematics teacher were chosen from one school for girls, and another two classes \( x_2 \) and \( y_2 \), who had the same mathematics teacher, were chosen from another school for boys. The four classes were in the final grade of secondary school in Kuwait. This number of classes was chosen as any two classes might theoretically constitute a representative sample (see Appendix A) of the total population for a grade. The average age was approximately 18 years.

(b_2) For k: The same procedure for k was adopted for k. Two classes \( x_1 \) and \( y_1 \) were chosen from one school for girls and another two classes \( x_2 \) and \( y_2 \) were chosen from the same school for boys. The two classes in each school had the same teacher of mathematics.

All classes were in the second grade of secondary school in Kuwait. The average age was approximately 16 years.

(b_3) Group \( G_1 = \{ x_1, x_2 \} \) was set as the experimental group, and \( G_2 = \{ y_1, y_2 \} \) as the control for k. Furthermore, group \( \hat{G}_1 = \{ \hat{x}_1, \hat{x}_2 \} \) was the experimental group and \( \hat{G}_2 = \{ \hat{y}_1, \hat{y}_2 \} \) was the control for k.

The following table 9.7 illustrates the number of subjects and setting of them.
<table>
<thead>
<tr>
<th>Task</th>
<th>Experimental Class</th>
<th>No. of subjects</th>
<th>Control Class</th>
<th>No. of Subjects</th>
<th>Experimental Group</th>
<th>Control Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>$x_1$</td>
<td>23</td>
<td>$y_1$</td>
<td>20</td>
<td>$G_1 = {x_1, x_2}$</td>
<td>$G_2 = {y_1, y_2}$</td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>25</td>
<td>$y_2$</td>
<td>24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{k}$</td>
<td>$\hat{x}_1$</td>
<td>28</td>
<td>$\hat{y}_1$</td>
<td>27</td>
<td>$\hat{G}_1 = {\hat{x}_1, \hat{x}_2}$</td>
<td>$\hat{G}_2 = {\hat{y}_1, \hat{y}_2}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{x}_2$</td>
<td>22</td>
<td>$\hat{y}_2$</td>
<td>24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9.7
Classification Of Classes/Groups By Control Or Experimental And Numbers Of Students In Classes.

(It is to be noted that the numbers of ordinary students in $G_1$, $G_2$, $\hat{G}_1$ and $\hat{G}_2$ respectively were 33, 28, 32 and 33, as indicated in Tables 9.8 and 9.9).

(c) The Procedure

(c₁) The R-subjects (control) learnt the task following their textbooks according to the plan for the task, explained in the treatment $R = R(S)$, while the $\hat{R}$-subjects (experimental) were given the elements of $U$, whether examples or exercises by the teacher who wrote them up on the blackboard, along with the elements of $S$ from the text.

(c₂) The teacher firstly discussed the four examples planned in the task to both the control and the experimental classes. Then the teacher, the senior teacher, or at least once for each class, I followed, guided and helped students as they worked through the exercises. It took on average
two periods for the final grade and three periods for the second grade to complete this. The students used their own copy books, and this was intended to reduce the 'unusual' factors during the experiment.

\(c_3\) The achievement test \(T(k)\) or \(T(k')\) was given to the students on the day after completion of the previously described task. The duration of the test was set as one period of 45 minutes and all students could work through most of the exercises in this time. Their answers were written on the same sheet as the questions.

\(d\) In marking the test, '+1' was given to a complete correct procedure in any question-item, even with simple miscalculations; otherwise, '0' was given. The teachers of the two classes and the senior teacher in the same school helped me in correction of the answering sheets.

9.2.1.5. Discussion Of Results

1. Tables

In the following, I present table 9.8 which illustrates the average score 'A.S' for mathematics of each student taken from the school records in the first six months of the academic year 1979/1980. The 'A.S' partitioned each class into excellent, ordinary and weak in the sense of Definition 9.1. The table also contains the testing score 'T.S' of each student in the relevant achievement test.
<table>
<thead>
<tr>
<th>Final (4th Secondary) Grade</th>
<th>2nd Secondary Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Experimental</strong></td>
<td><strong>Control</strong></td>
</tr>
<tr>
<td>Cl. $x_1$</td>
<td>Cl. $y_1$</td>
</tr>
<tr>
<td>AS</td>
<td>TS</td>
</tr>
<tr>
<td>1</td>
<td>94</td>
</tr>
<tr>
<td>2</td>
<td>88</td>
</tr>
<tr>
<td>3</td>
<td>86</td>
</tr>
<tr>
<td>4</td>
<td>82</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
</tr>
<tr>
<td>6</td>
<td>79</td>
</tr>
<tr>
<td>7</td>
<td>75</td>
</tr>
<tr>
<td>8</td>
<td>73</td>
</tr>
<tr>
<td>9</td>
<td>70</td>
</tr>
<tr>
<td>10</td>
<td>68</td>
</tr>
<tr>
<td>11</td>
<td>68</td>
</tr>
<tr>
<td>12</td>
<td>64</td>
</tr>
<tr>
<td>13</td>
<td>62</td>
</tr>
<tr>
<td>14</td>
<td>59</td>
</tr>
<tr>
<td>15</td>
<td>54</td>
</tr>
<tr>
<td>16</td>
<td>53</td>
</tr>
<tr>
<td>17</td>
<td>50</td>
</tr>
<tr>
<td>18</td>
<td>46</td>
</tr>
<tr>
<td>19</td>
<td>43</td>
</tr>
<tr>
<td>20</td>
<td>41</td>
</tr>
<tr>
<td>21</td>
<td>34</td>
</tr>
<tr>
<td>22</td>
<td>28</td>
</tr>
<tr>
<td>23</td>
<td>22</td>
</tr>
</tbody>
</table>

**Table 9.8**

Average Scores (A.S) And Testing Scores (T.S) For The Relevant Test.
The following table 9.9 derived from table 9.8, illustrates the number 'n' of ordinary students in each class/group, as well as the mean of the 'A.S' and 'T.S' for classes/groups.

<table>
<thead>
<tr>
<th>TASK k:</th>
<th>Class/Group</th>
<th>$x_1$</th>
<th>$y_1$</th>
<th>$x_2$</th>
<th>$y_2$</th>
<th>$G_1 = {x_1, x_2}$</th>
<th>$G_2 = {y_1, y_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>16</td>
<td>13</td>
<td>17</td>
<td>15</td>
<td>33</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>Mean of AS</td>
<td>59.88</td>
<td>60.46</td>
<td>62.65</td>
<td>61.66</td>
<td>61.30</td>
<td>61.11</td>
<td></td>
</tr>
<tr>
<td>Mean of TS</td>
<td>2.75</td>
<td>2.23</td>
<td>2.82</td>
<td>2.4</td>
<td>2.79</td>
<td>2.32</td>
<td></td>
</tr>
</tbody>
</table>

| TASK k: |
|---------|--------------|-------|-------|-------|-------|------------------|------------------|
| Class/Group | $x_1$ | $y_1$ | $x_2$ | $y_2$ | $G_1 = \{x_1, x_2\}$ | $G_2 = \{y_1, y_2\}$ |
| n       | 18           | 18    | 14    | 15    | 32    | 33               |                  |
| Mean of AS | 60.5         | 60.28 | 60.86 | 60.93 | 60.66 | 60.58           |                  |
| Mean of TS | 2.72         | 2.22  | 3     | 2.53  | 2.78  | 2.36            |                  |

**Table 9.9**

The Mean Of The Average Scores And The Mean Of Testing Scores Of Ordinary Students In Each Class/Group.

2. **Analysis Of Results**

(i) Table 9.9 indicates the mean of the average scores of ordinary students in different classes or groups is almost equal. The differences are very slight and this provides additional support to the assertion that
all classes in the same grade in the secondary schools in Kuwait are more or less equivalent. This could be due to the fact that:

a) all students in any grade of secondary school are randomly distributed into their classes;

b) all the mathematics teachers have similar qualifications and training; and

c) all students in the same grade follow the same textbook and the same periodical plan for undertaking tasks in the text.

Nevertheless, the mean of the testing scores (T.S) was shown to be higher in the case of the experimental class/group than in the corresponding control class/group. This indication is in line with the results in tables 9.2, 9.3, 9.4 and 9.5, as well as with part (ii) of conclusion 9.1 that can be found in the previous piece of experimental evidence.

(ii) In testing the hypothesis 9.1, it was considered that this would be better only discussed for groups $G_1$ against $G_2$, $G_1$ against $G_\eta$, $G_2$, $G_\eta$, $G_1$, and $G_2$. This is firstly because the numbers in the classes are comparatively small. Another reason is that the groups studied should not belong to different teachers, so that the results under this change may confirm primarily the role of the conception of a space of knowledge in the methodology. This is in the sense that a change of the learning space $S$ to $S + U$ would induce a related change in the achievement.

The testing of hypothesis 9.1 will be similar to the study of the hypotheses $H_1$ and $H_2$ in table 8.8. The following table 9.10, derived from
Table 9.8 is similar to table 8.8 and related to the groups $G_1$, $G_2$, $\hat{G}_1$ and $\hat{G}_2$. It will be assumed that those groups are representative samples (see Appendix A) since they are drawn from two classes in the field. This assumption is supported by assuming that the parent populations of the groups are approximately normal with equivalent variances.

<table>
<thead>
<tr>
<th></th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$\hat{G}_1$</th>
<th>$\hat{G}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>33</td>
<td>28</td>
<td>32</td>
<td>33</td>
</tr>
<tr>
<td>Mean</td>
<td>2.79</td>
<td>2.32</td>
<td>2.78</td>
<td>2.36</td>
</tr>
<tr>
<td>S.D.</td>
<td>0.74</td>
<td>0.90</td>
<td>1.01</td>
<td>1.22</td>
</tr>
<tr>
<td>$(n-1)(S.D.)^2$</td>
<td>17.52</td>
<td>22.11</td>
<td>31.47</td>
<td>47.64</td>
</tr>
<tr>
<td>Common variance $D^2$</td>
<td></td>
<td>0.67</td>
<td>1.26</td>
<td>0.21</td>
</tr>
<tr>
<td>$t$</td>
<td>2.24</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9.10

Specification Table For Testing Of Hypothesis 9.1.

Let $M_1$ and $M_2$ represent the achievement of the experiment and control groups in achievement tests $T(k)$ and $T(\bar{k})$ respectively, and let hypotheses $H_0$ and $H$ be:
$H_0 : M_1 = M_2$ i.e. $d_0 = 0$ - the null hypothesis.

$H : M_1 > M_2$ - a one-sided alternative.

Let $\alpha = 0.05$ be a level of significance.

Then the critical region is identified by the value of the random variable $T$ corresponding to $T > t_\alpha$ for the alternative $H$ (Walpole, 1968, P.228).

Now, in tables of R. A. Fisher, for the percentage points of the $t$ distribution in (Walpole, 1968, P.333), we find:

$t_\alpha = 1.645$ for $G_1$ and $G_2$ with $df = 33 + 28 - 2 = 59$

as well as for groups $G_1$ and $G_2$ with $df = 63$.

After comparing the values of $t$ in table 9.10 we might conclude that:

$H_0$ is significantly rejected against the alternative $H$ in hypothesis 9.1 concerning $G_1$ and $G_2$; nevertheless, $H_0$ is accepted for $G_1$ and $G_2$.

This could mean that the modification in the methodology based on Proposition 9.1 is more adequate and significant for students in the higher grades, such as the final grade, rather than those in the lower grade. Nevertheless, this indication may be related to the fact that students in the final grade are more familiar with the $H$-structure than students in the second grade, since they are used to applying their mathematics in physics or chemistry, which is related with $H$. Hence, they could have benefited significantly more from the learning space $S + U$ than students of the second grade.

However, if we choose a level of significance $\alpha = 0.20$, then:
\[ t_{0.10} = 1.282, \text{ which means that for an approximate 80\% confidence interval for } M_1 - M_2 \text{ for Groups } G_1 \text{ and } G_2 \text{ in table 9.10 we have:} \\
(2.78 - 2.36) - 1.282 (0.28) < M_1 - M_2 < (2.78 - 2.36) + 1.282(0.28) \]

or: \[ 0.06 < M_1 - M_2 < 0.78, \text{ (Walpole, 1968, P.187),} \]

which means that we are 80\% confident that the interval from 0.06 to 0.78 contains the true difference of the average grades for the two learning spaces, \( S + U \) and \( S \), for the second grade. And the fact that both confidence limits are positive indicates that the methodology of learning in \( S + U \) under Proposition 9.1 is superior to the other conventional methodology and hypothesis \( H \) is to be accepted.

Nevertheless, despite the confidence interval for the second grade not being as significant as for those in the final grade, we could optimise the methodology relevant to \( S + U \) in improving mathematics achievement in the sense of conclusion 9.1 related to the first experiment in 9.2 for the second grade, which demonstrated that the \( (S + U) \) group was superior to the \( (S) \) group in both tasks \( k_1 \) and \( k_2 \) under Proposition 9.1.

Hence, it was believed that a general and basic conclusion may be drawn from the evidence found in those two experiments in this Chapter. This conclusion may present a feasible reform in the methodology sought in this study. I believe a reform in this field will remain partial until we obtain a fair amount of knowledge concerning a principle of invariance (see Appendix A). This might then provide proper conditions so that the law of universal determinism (see Appendix A) may stand stronger in the behavioural science.
9.2.1.6. Basic Conclusion 9.2

A methodology in mathematics teaching-learning situations based on the learning space $S + U$, in the sense of the algorithm suggested in proposition 9.1, is superior for the mathematical achievement of students - in the area of this study - to the current methodology that is ultimately based on the learning space $S$.

(N.B. One might ask if the 'A.S.' for normal students could have any indications that may support prediction on such basis. Begle (1979) indeed reports that the score of a student's achievement in pre-algebra school mathematics could be a prediction of the later achievement of the student in algebra. Moreover, and from another point of view, Greeno (1972) tells us that subjects whose abilities were intermediate, i.e. ordinary students as in the context of this thesis, provided the results of greatest interest. He continues, that if these subjects are identified by the scores of MSAT (Mathematics Scholastic Aptitude Test) then their performance in discovery and rule learning is practically indistinguishable. However, as regards the scores obtained in tests of background concepts and arithmetical skills, the two methods used makes quite a difference to subjects with intermediate ability which in general is in line with Begle. The 'A.S' in table 9.8, which indicate normal students, could be accepted as an indicator in the sense of their performance under methodologies in both $S$ and $S + U$.)
9.2.1.7. Summary Of Experiment II

<table>
<thead>
<tr>
<th>TITLE</th>
<th>Experiment II</th>
</tr>
</thead>
<tbody>
<tr>
<td>PURPOSE OF STUDY</td>
<td>If an ordinary student is defined as in the field of this study, this experiment was concerned in testing the following hypothesis: The mathematics achievement in a teaching-learning task based on the learning space ( S + U ) in the sense of Proposition 9.1 is superior, for ordinary students in the area of this study, to the achievement in the teaching-learning situation of the task conventionally based on the learning space ( S ).</td>
</tr>
</tbody>
</table>

In fulfilling such purposes, two learning tasks \( k \) and \( k' \) were chosen from the mathematics of the final (4th) and 2nd grades of secondary school in Kuwait. Each task was produced in two different treatments, the conventional \( R \) in \( S \), and the modified \( R \) in \( S + U \). Then two groups, \( G_1 \) and \( G_2 \), of ordinary students (of average age 18 years in each) were identified in the final grade, while another two groups, \( G_1' \) and \( G_2' \), of ordinary students (of average age 16 years in each) were identified in the 2nd grade. After completion of the tasks under relevant treatments, two achievement tests \( T(k) \) and \( T(k') \) were designed for \( k \) and \( k' \) respectively and administered to the students who participated in the experiment. The study was run as follows:
This was judged by a panel which had contributed to the construction of the tests (see P.1-24).

INTER-RATER (2.4.2.1.1).

This demonstrated that the achievement of the \((S + U)\) students in the final grade was significantly superior to the achievement of the \((S)\) students, while this was not significantly valid for the 2nd grade. Nevertheless, the achievement of the \((S + U)\) students in the 2nd grade was superior to the achievement of the \((S)\) students under an acceptable level of significance, as long as the hypothesis was tested within groups of ordinary students.

In a deliberate study of the evidence in both experiments in 9.2, it was concluded that a methodology in the mathematics teaching-learning situations, based on the learning space \(S + U\), in the sense of proposition 9.1, is superior for improving the mathematics achievement in the area of this study to the current methodology, that is ultimately based on the learning space \(S\).
9.3. Summary

Chapter 9 presents a prospective modification in the methodology of teaching-learning situations that might fulfill the aims of the study in this thesis. This was conducted on the basis of the following two main objectives.

I: Proposing a modified presentation $R$ in the methodology. This was fulfilled on the basis of an algorithm found in proposition 9.1. This algorithm is based on:

(i) a task-analysis which provides a set $B$ of well-defined behavioural objectives relevant to the task;

(ii) the learning space $S + U$; and

(iii) employment of Proposition 8.2 of learnability in $U$.

It proposes that the achievement of students in each $B_i$ in $B$ should rest on instances in $S + U$ introduced as order pairs $(x, y)$ with $x$ as a simple instance in $S$ and $y$ the associate of some simple instance $\hat{x}$ in $S$ under Proposition 8.2, provided that $x$, $\hat{x}$ and $y$ are governed by $B_i$, with $x$ possibly equal to $\hat{x}$.

II. Implementation of empirical studies in support of the proposed modification. This was achieved through evidence based on two experiments on the basis of studying achievement of students in the conventional presentation $R$ in $S$ against the modified presentation $\hat{R}$ in $S + U$ in the sense of proposition 9.1. The first experiment was based on the choice of two successive tasks in the mathematics of the 2nd grade of school in Kuwait. Each task was presented in two treatments, one in $S$ and one in
S + U, as well as an achievement test that was constructed for discussing the achievement under both treatments. Furthermore, two groups \( G_1 \) and \( G_2 \) of students were selected such that in \( k_1 \), \( G_1 \) was experimental and \( G_2 \) was control, while the contrary was followed for \( k_2 \). In both tasks, the experimental group followed the modified presentation in \( S + U \) while the control followed that of \( S \). The evidence demonstrated that the achievement of the \((S + U)\) i.e. the experimental group, was superior to the achievement of the \((S)\) group, i.e. the control.

As for the second experiment, a definition of an ordinary student in the secondary school in Kuwait was introduced. The experiment was based on testing a hypothesis which assumes that the achievement of ordinary students in \( S + U \) under Proposition 9.1 is superior to that in \( S \). It demonstrated that this could be significant for students in the final grade, while the confident interval was 80% for the students in the second grade of school in Kuwait.

Finally, a basic conclusion was drawn from the evidence in both experiments in that:

A methodology in the mathematics teaching-learning situations based on \( S + U \) in the sense of Proposition 9.1 is superior to the conventional one that is based on \( S \).
CHAPTER TEN

SUMMARY AND FURTHER OUTLOOK
10. SUMMARY AND FURTHER OUTLOOK

10.0. Introduction

The problems in this research emerge from the persistent demand for any possible solutions - albeit partial - to the problem of improving the achievement of students in mathematics. The problem of the low achievement of students seems not only to pertain to developing countries, Kuwait is one, but also to some developed countries, such as the United States. In this context, the Times Educational Supplement (11th September, 1981) tells how the Ford Foundation Report claims that there has been a significant decline in mathematics standards over the past few years and adds that the United States in terms of mathematics and science training might almost be called under-developed. And such a decline, combined with falling school rolls, says the Report, threatens to produce a major shortfall in the number of graduates with a strong mathematical background.

The Times Educational Supplement of 11th September 1981, continues that the National Council of Teachers of Mathematics stated at its annual conference in April that the three major problems were:

"- School maths failing to keep pace with changing technologies
- Most pupils not studying maths to the level required for their futures.
- A dramatic increase in the present shortage of qualified maths teachers" (P.21).
Despite different countries having differing reasons for such a decline, the fact that stands out and in which all of them share, is that a major part of students cannot demonstrate satisfactorily the mathematics they are said to have achieved, either in mathematical fields, or in non-mathematical fields which demand adequate mathematical knowledge, e.g. physics.

In manipulating the discussion of a well-defined research problem, solutions which might fulfil - albeit partially - the aims of improving the mathematics achievement of students, I formulated the following as the research problem to be studied.

"To look for a feasible development (modification) in the methodology of mathematics teaching-learning situations."

The study was conducted with reference to secondary school mathematics in Kuwait (see 1.1 of Chapter 1). In addition, students in the first year of Kuwait University were partially involved.

I was, in fact, convinced that the problem of better achievement would centre in the area of 'methodology', since I agreed with Anthony (1963) and others who assume that an 'approach' is a philosophical point of view arising as axiomatic. To me, 'methodology' means all possible observable factors (teaching-learning situations) supposed to bring about mathematical knowledge within students (e.g. curriculum (general), textbook, teaching methods) and modification for improvement of students' achievement in mathematics should be achieved in such a methodology.
As for my line of thought in conducting this research, I tried - as in experimental sciences - to relate 'what is involved in aims' to definite 'means', in the sense that I have tried to relate the achievement (aims) assigned by a number 'A' to an independent variable, say $x_j$, identified in the methodology (means), i.e. $A = g(x_j)$. Hence, in this line, it was reasonable to identify a certain phenomenon in the psycho-educational field that admits such a functional relation.

In the following, I shall summarise and discuss the work and findings involved in this thesis, as well as what I believe is a further outlook for future development in the line of this thesis.

10.1. 1. General Summary

The study in this thesis was developed in three stages; pre-proposition, proposition and final which includes the general findings of the study.

10.1.0. A Pre-Proposition Stage

In this stage, discussed in 6.1, I firstly began by introducing five general convictions (6.1.0) which I believed to constitute general guidance for the research procedure. These convictions reflect a general, as well as a personal belief. They reflect the belief in:

(i) The unsatisfactory nature of current teaching-learning situations;
(ii) The possible limiting role of teachers in development;
(iii) The essential role of theory for a feasible reform (modification);
(iv) The limitation in a reform due to the absence of a certain principle of invariance in the psycho-education field; and

(v) My own personal view of development that should be achieved by identifying an existing structure in current situations, as well as identifying another structure that might contribute towards governing a new situation (modification).

In addition, based on these convictions, a review of literature and personal experience, I could clearly formulate the research problem stated in 10.0. In this connection, and for preserving a level of acceptable consistency, I gave terminology a prior role in development through a pilot investigation for the proposition stage in 6.1.2. This pilot investigation was influenced by my general conception of development based on the functional relation $A = g(x_j)$ mentioned in 10.0.

And, broadly speaking, I foresaw a conception that might stretch from the fifth conviction - that "modification of (any) situation $z$ into another $z'$ should rely on identifying a certain criterion which might judge the advantages of both $z$ and $z'$. I accordingly viewed that the modification of $z$ to $z'$ should be based on identifying the advantages of the achievement under both $z$ and $z'$. Hence, consistency of terminology would arise as was stated above. The first and basic category of terminology was introduced in 6.1.2.0. It discussed the achievement in a certain task, as well as how improvement in achievement could be conceived. It further introduces the term "advantage of the achievement" which should be based on a certain 'measure' called later the 'efficiency' that involved problem-solving. Simply, the efficiency as a measure for the advantage of the achievement of a task $k$ was introduced as the probability that the conditional statement $x$ is capable of $P \Rightarrow x$ is capable of $P$ is true, where $x$ is a member of a normal population in a
school grade relevant to \( k \), \( P_0 \) is a certain set of stimuli related in some definite sense to \( P \), a problem in the domain of \( k \), and the first statement (\( x \) is capable of \( P_0 \)) is true. \( P_0 \) was further clarified and enhanced on the basis of a second pilot instrument \( T_0 \) discussed in 6.1.2.2 to mean precisely a set \( X_0 \) called a kernel (see Appendix A) for \( P \) identified and constructed on the basis of a set \( B \) of basic behavioural objectives necessary for \( P \).

A modification in \( z \) for \( z \) was then assumed to be based on finding out the advantage of the achievement under \( z \) (the efficiency under \( z \)) and the advantage of the achievement under \( z' \) (the efficiency under \( z' \)) regarding the same problem \( P \) in both cases. A Glossary is found in "Appendix A" of the terms I found it essential to expound in this way.

As for the continuation of the study into the second stage involving the research proposals, I implemented two pilot instruments, a questionnaire in 6.1.2.1 and a test \( T_0 \) in 6.1.2.2. These had the purpose of learning how people who work in the field think of such advantage of achievement, and how such an efficiency could be reliable as a measure. A further implicit purpose is found in enhancing the choice of elements of \( P_0 \).

It was generally concluded at this stage, based on these two pilot instruments, that:

(i) a change in the current teaching-learning situation is called for; and

(ii) the efficiency as a measure of the advantage of achievement could rely on \( P_0 = X_0 \) where \( X_0 \) is constructed to satisfy the basic pre-requisites for \( P \).
Simply, this $X_0$ consists of simple stimuli where each is uniquely governed by one $B_i$ in $B$ for all $B_i$ in $B$. $X_0$ was called a kernel for $P$.

Generally, I may state that the literature review and personal experience besides terminology and the two pilot instruments in this stage, as well as introspection, have all co-operated in the achievement of the second stage, i.e. the proposition stage of this study. Figure 6.1 (P.6-4) may be helpful for visualising the stages in research development.

10.1.1. A Proposition Stage

This stage can be taken as the backbone of the work in this thesis. It introduces a set of four fundamental propositions in 6.2 which facilitated and monitored the execution of the final stage of this study. It was substantially directed by my pre-conception of the development through the functional relation $A = g(x_j)$ mentioned in 10.0.

As a matter of fact, I was first attracted to this relation by the remarkable study by Mayer and Greeno (1972) reported in 4.1.1, which points to a possible relationship between the learning outcomes - which can be considered as the product of achievement - and sequencing of information. Furthermore, another study, carried out by Ehrenpreis and Scandura (1974) and reported in 4.1.2, could demonstrate in more precise terms that this relationship holds between the learning outcomes and the curricula. In addition, Begle (1979) claims that content differences in textbooks resulted in differences in student achievement (see P.3-59). Moreover, in discussing different methodological points (pp.3-56, 65), I concluded in P.3-66 that one of the phenomena that might be promising
for modification in methodology leading to an improvement (in achievement), could be found in some studies which indicate that a change in the achievement of students could result under change in the material content. In this context, and in the light of other considerations related to my belief of the role of theory in development (see C3, p.6-5), I put forward the following three fundamental propositions.

(Proposition 6.1):
A phenomenon that indicates a change in the achievement of students under a certain change in a well-defined teaching-learning situation, is significant as a starting point for development in current teaching-learning situations. (P.6-36).

(Proposition 6.2):
There exists a theoretical framework, on which a development in current teaching-learning situations can rest. (P.6-37).

(Proposition 6.3):
If a proposed development (reform) in current teaching-learning situations (a) can rest on a theoretical framework, and (b) can be operationally applicable, i.e. well-defined and learnable, then such development is expected to be experimentally supported. (P.6-37).

These propositions may indeed reflect the orientation of the course in conducting this study. I accordingly tried to identify $x_j$ - in $A = g(x_j)$ - as a factor of the methodology related to the material contents based on the studies of Mayer and Greeno (1972), Ehrenpreis and Scandura (1974) and Begle (1979). To me, this meant firstly
that I should identify precisely the (material) contents of a teaching-
learning task \( k \) and second that I should relate \( x_j \) unambiguously to
those contents.

I viewed in 6.2.2 (P.6-38) that the contents of a task \( k \) mainly com-
prise two parts:

(i) a theoretical component including the theoretical part \( k \),
e.g. a definition or a proof of a theorem; and
(ii) a set \( R \) of instances as examples, exercises and simple (novel
routine) problems constructed such that each is governed by
one behavioural objective necessary for \( k \). Such an \( R \) is
usually supposed to illustrate and reinforce the mathematical
knowledge related to \( k \).

It was further viewed that the theoretical part is more or less
invariant in school mathematics, while a change in the contents of \( k \) is
substantially a change in \( R \). \( R \) was called the presentation of \( k \). A
phenomenon was then conceived, drawn from the works previously mentioned
and stated thus:

\[
\text{Differences in the achievement } A \text{ of a teaching-learning task } k \text{ can be induced by differences in the presentation } R \text{ of } k
\]
(P.4-3).

This was called the presentation-phenomena and the scope of this
phenomenon is found in Chapter 4. In addition, the (material) contents
of such a \( k \) was equated with the written material for \( k \) in the text-
book of the area of this study in the sense of conclusion 4.2 (P.4-21).

As for discussing how a true change could take place in \( R \), it was argued
in P.6-39 that a replacement of an instance in R, say \(2x + 1 = 5\) by another one as \(3x - 2 = 4\) is merely superficial and does not induce a true change in R. Hence, based on introspection, as well as on the fifth conviction (Chapter 6, P.6-6) it was argued that a true change in R should be due to the structure underlying R and consequently any modification in the methodology relevant to R should take place by interaction with another structure supposed to contribute to such a modification. In this context, the following final fundamental Proposition was stated:

(Proposition 6.4):

(i) There exists a well-defined structure 'S' underlying current presentation of mathematical tasks.
(ii) There exists a structure 'H' that is possible to contribute in underlying modification of current presentation.
(iii) Theory in Proposition 6.2 could possibly identify a certain discrimination of the efficient knowledge that achieved in 'S' and 'H'. (P.6-40).

This proposition helps the work progress in proposing the existence of a structure S underlying the current R as well as such a structure H that might contribute to the modification R of R. It further suggests a basis for the establishment of theory based on the 'efficiency' which matches as a measure the construct "advantage of achievement".

10.1.2. A Final Stage : General Findings

This stage is expected:
(a) to identify two structures 'S' and 'H' in the sense of Proposition 6.4;
(b) to identify a theoretical framework in the line of propositions 6.2, 6.3 and 6.4;
(c) to implement empirical studies on the basis of such theory identified in (b); and
(d) to propose a modification in the methodology of teaching-learning situations, provided that this modification should experimentally support an improvement in the achievement of students in the mathematics of the area of this study.

10.1.2.0. a: Towards Identification Of S And H

In a study in 7.1 for the presentation R in the contents of teaching-learning tasks in the textbooks of the area of this study, it was found that most of the instances (not less than 95%) that make up R do not make use of any constant of implicit form, e.g. 'a', in other words, a constant in the usual use in mathematics - (see Appendix C). Hence, it was viewed that R takes place in a set 'S' called the S-structure, where S could be seen as a set of structured knowledge in form of instances identified (structured) by a rule excluding constants from the instances in S. Consequently, it was assumed there was another structure 'H' being a set of structured knowledge in the form of instances, identified by a rule demanding explicitly the presence of at least one constant in each instance in H. It could be easily noticed that all instances of general forms as those used for rules or formulas, whether in mathematical fields or non-mathematical fields, using mathematics, e.g. physics, are members
of $H$. Hence, if $B_1$ was assumed to be a behavioural objective and $B_1(S)$ and $B_1(H)$ are two classes of instances in $S$ and $H$, so that each member of the two classes is basically governed by $B_1$ only (in the sense of Definition 5.4, P.5-17), then any of these members may contain or not contain a constant 'a' say. For example:

"Solve for $x$: $2x + 1 = 5"$

is a member of $B_1(S)$, while:

"Solve for $x$: $2x + a = 5"$

is a member of $B_1(H)$. In this sense, we might deduce that:

$$B_1(S) \cap B_1(H) = \emptyset$$

provided that the existence of a constant, 'a' say, is clearly identified in the instance.

And further, in assuming that all the instances in $S$ and $H$ constitute such classes as $B_1(S)$ and $B_1(H)$ which explicitly identify any constant of implicit form within them, then in this context we can deduce that $S \cap H = \emptyset$ (see remark 1, P.7-9).

In an attempt to learn about any possible primitive implications of the two structures $S$ and $H$, two instrumental indicators were implemented: A battery $T_1$ of three tests (Appendix D) was employed for the purpose of finding out how mathematical computational knowledge (see Glossary: Appendix A) usually gained in $S$, could interact in $H$. This is reported in 7.2.0.

As a second indicator, a set $T_1$ of three tests (Appendix E) was employed for the purpose of identifying a general view of relationships of performance in problem-solving and relevant mathematical knowledge in $S$ and $H$. This is reported in 7.2.1.
It was concluded that mathematical computational knowledge gained in S is not satisfactorily transferred to non-mathematical fields using mathematics (Conclusion 7.1, P.7-18). In addition, the mathematical knowledge found to be gained in H has a stronger relationship with problem-solving than that of S (Conclusion 7.2, P.7-26).

Furthermore, in the discussion of the 'pros and cons' of S and H, it was concluded in the general conclusion 7.3 (P.7-32) that S is indispensable for any modification and H is generally difficult as a whole, but should not be ignored in its partial contribution for modification. Hence, if L is proposed as a learning structure to underlie a desirable modification in the methodology, then this L should include S and a part 'U' of H, such that this U should be introduced to satisfy learnability in the sense of the fundamental Proposition 6.3.

10.1.2.1. b: Towards A Theory

The above study in (a) could in fact provide help for identification of a theoretical framework in the context of this study. It was argued in 7.3 that interaction with certain definite knowledge takes place towards 'goals of knowledge' such as problem-solving, through simple statements found in previously discussed instances. Hence, it was preconceived that such knowledge is introduced in a structured form, i.e. identified by a set of instances determined (structured) by a finite set of rules. In this context, I introduced 'space of knowledge' as a 'conceptual structure' where details of this are found in Chapter 5.

A space of knowledge was identified in Definition 5.1 (P.5-10) for a
certain normal population 'O' as an ordered pair \((W, 0)\), where \(W\) constitutes a set of structured knowledge identified for some particular knowledge by simple (initial) states of knowledge, governed by an underlying structure which is made up of a finite set of rules. For simplicity, we might denote this space by \(W\) only (see 'space of knowledge' in Appendix A). For the further development of this concept, which is supposed to spring from part (iii) of the fundamental Proposition 6.4, I introduced the axiom of sufficiency (P.5-14), which assumes the existence of a certain criterion 'X' that can induce sufficiency regarding a well-defined problem \(P\). It was also assumed that despite \(X\) being still unidentified, we might assume that \(X\) would satisfy the following three properties; besides others not yet identified. These properties are:

1) **consistency**, which assumes that \(X\) may provide an algorithm (set of rules) that defines a set \(X_0(X)\) which tests necessary competency for \(P\);

2) **uniformity**, which assumes that a person capable of \(P\) who demonstrates comprehension of such ability should be expected to be completely capable of \(X_0(X)\); and

3) **completeness**, in the sense that a person's capability in \(X_0(X)\) implies his being capable of \(P\).

The modification was assumed to look for an approximation for \(X_0(X)\) which illustrates greater advantages than that of the current situation, i.e. the efficiency under the proposed modification should be demonstrated as higher than that of the current situation.
Hence, in order to preserve a certain consistency in results in the absence of a principle of invariance (see Appendix A) in the behavioural sciences which might provide such a consistency under the law of universal determinism (see Appendix A), a concept of a uniform operational was introduced in Definition 5.6. This simply assumes that a uniform operational regarding a problem $P$ in a space of knowledge $W$ is the same one who is capable of all the kernels $W_o$ of $P$ in $W$, where a kernel is a set of stimuli (instances) constructed such that each one satisfies a unique behavioural objective necessary for $P$, (Definition 5.5, P. 5-17).

Hence, the efficiency '$r$' in a space of knowledge $W$ with respect to $P$ is theoretically identified in Definition 5.7 (P.5-24) as the probability that:

"$x$ is a uniform operational for $P'$ =>$x$ is capable of $P$"

is true, provided that the first part of the premise is satisfied.

It was then concluded that the efficiency with respect to $P$ would differ with $W$ (Conclusion 5.1, P.5-26).

Finally, the following proposition for modification in the methodology was introduced:

(Proposition 5.1):

If 'O' is a certain normal population and each $W$ of $W$ is a structured knowledge in the sense of Definition 5.1, and if:
(i) $(D, B, X)$ is a problem-space for a problem $P$;
(ii) $r$ and $r$ are the two efficiencies in $W$ and $W$ respectively
for '0' and P,
(iii) $r < r' \leq 1$,

Then $W$ could contribute in whole or in part to a modification in the methodology of teaching-learning situations, provided that the part or whole of $W$ is applicable in the sense of Proposition 6.3 (P.5-27).

Figure 5.4 (P.5-28) provides a useful explanation of this Proposition.

10.1.2.2. c: Theory At Work: Experimental Studies.

In this part of the final study, I found that I should:

(i) Set out that part $U$ of $H$ which would contribute to a modification in the methodology as was explained in 10.1.2.0;

(ii) Propose the learnability in $U$ such that $U$ could be applicable in the sense of Proposition 6.3;

(iii) Employ the theory previously summarised to test the efficiencies $r(u)$ and $r(s)$ in $U$ and $S$ with respect to a problem $P$ in order to find out if $U$ can contribute to a modification in the methodology in the sense of Proposition 5.1.

As for the first step (i) it was argued in 8.1.0 and suggested in Proposition 8.1 (P.8-4) that $U$ constitutes all the instances in $H$ which contain one and only one constant $a$, other than the variables.

As for the second point (ii), an argument for the possible applicability or learnability in $U$ was introduced in 8.1.1 and, further, the following Proposition 8.2 (P.8-6) was suggested to facilitate an approach for learnability in $U$. 
(Proposition 8.2):
If \( s_i \) is an instance, chosen in \( S \) to satisfy a well-defined behavioural objective \( B_i \), then a generation of a (learning) associate \( u_i \) (of \( s_i \)) in \( U \) could be based on:
(i) Any fixed number in \( s_i \) is replaced by a suitable functional form of a constant 'a' say, e.g. \( F(a) \).
(ii) This replacement should satisfy:
   (a) \( u_i \) should stem as \( s_i \) from the same general mathematical form, so that \( u_i \) - precisely as \( s_i \) - satisfies \( B_i \);
   (b) \( u_i \) gives identical form of result to that of \( s_i \) under \( B_i \).
   (c) The results in \( u_i \) contain, if possible, some functional form of the same constant 'a', e.g. \( g(a) \).

In fulfilling the third step (iii), it was argued in 8.2 that those uniform operational identified theoretically in Definition 5.6 (P.5-23), and introduced in order to identify the efficiency in Definition 5.7, (P.5-24) are ideal ones and not easily - if at all - recognised in reality. Hence, it was found convenient to introduce the concept of pseudo-reality in 8.2.0, so that results in pseudo-reality could be extrapolated probabilistically into reality. A uniform operational in pseudo-reality in a space of knowledge \( W \) for a problem \( P \) was identified in Definition 8.1 (P.8-9) to satisfy ability in all elements of one kernel \( W_0 \) for \( P \). The efficiency and Proposition 5.1 were extended to pseudo-reality in this way.

Accordingly, an empirical study was conducted on basis of a set \( T_3 \) of
three tests (Appendix F) each based on the test-form $S_o + U_o + P$
with $S_o$ and $U_o$ two kernels for $P$ in $S$ and $U$ respectively. The study
is reported in 8.3. In all the three tests, it was found that
$r(S) < r(U)$, table 8.2 (P.8-21). In further study for contingent
implications of the test-form $S_o + U_o + P$, hypothesis-testing reported
in 8.3.5 implies that such a form does not affect the competency in $P$.

10.1.2.3. d: Towards Modification In Methodology: $S + U$, A
Learning Space

In this final part of the final stage of this study, on the basis of the
preceding discussion, I found that I should:

(i) propose a modification in the methodology in the sense of the
presentation-phenomenon discussed in Chapter 4, by putting
forward a modified presentation $R'$ to replace the current
conventional presentation $R$.

(ii) justify such a modification experimentally.

As for (i), I introduced the following Proposition 9.1 (P.9-6) suggested
to provide a desirable model for modification based on an algorithm, in
the methodology of mathematics teaching-learning situations in order to
give improvement in students' achievement in mathematics.

(Proposition 9.1):
If $B$ is a set of basic behavioural objectives necessary for a
teaching-learning task $k$, then achievement of $k$ is possibly
improved under the following presentation:
\[
R' = \begin{bmatrix}
B_1 & B_2 & \cdots & B_{i-1} & B_i & \cdots & B_n \\
(s_1^1, u_1^1) & \cdots & (s_{i-1}^1, u_{i-1}^1) & \cdots & (s_i^1, u_i^1) & \cdots & (s_n^1, u_n^1) \\
\vdots & & \vdots & & \vdots & & \vdots \\
(s_1^j, u_1^j) & \cdots & (s_{i-1}^j, u_{i-1}^j) & \cdots & (s_i^j, u_i^j) & \cdots & (s_n^j, u_n^j)
\end{bmatrix}
= R'(S + U)
\]

where the matrix, except in the first row, is made up of ordered pairs \( (s_{ij}, u_{ikj}) \), and \( u_{ikj} \) is an associate element of \( U \) of some \( s_{im} \) in \( S \), governed (as \( s_{ij} \)) by \( B_i \) under Proposition 8.2 of learnability.

It is to be noted that the conventional \( R \) excludes - in general - \( U \) (as well as \( H \)). In a second step for satisfying (ii) empirical evidence was gathered from two experiments established on the basis of Proposition 9.1. Experiment I, reported in 9.2.0, was based on a comparative study of mathematics achievement in \( S \) (through \( R \)) against \( S + U \) (through \( R' \)).

As for Experiment II, a definition of a normal student in a Kuwait Secondary school was introduced in Definition 9.1, (P. 9-28) and the study is reported in 9.2.1. This was achieved on the basis of a statistical hypothesis concerning mathematics achievement of ordinary students in \( S \) (through \( R \)) against \( S + U \) (through \( R' \)).

As a consequence of the evidence drawn from these two experiments, concerning a modification in the methodology, the following basic Conclusion 9.2 was drawn and which summarises the basic findings in the context of this thesis:
(Basic conclusion 9.2):

A methodology in the mathematics teaching-learning situation based on the learning space $S + U$, in the sense of the algorithm suggested in Proposition 9.1, is superior for improving the mathematical achievement of students - in the area of this study - to the current methodology, that is ultimately based on the learning space $S$ (P.9-44).

10.1.2.4. Further Discussion And General Findings

The move towards an improvement of students' achievement in Kuwait in the early seventies can find an exact description in the terms of Howson et.al. (1981). They view that an improvement in mathematics achievement was believed to be possible from the so-called "Modern Mathematics". In this there is a move from a fragmented arithmetic, algebra and geometry course to one demonstrating the 'unity' of mathematics. Such a move has not yet resulted in the promised and desired situation. In this context, Howson et al. find that we are still unable to ensure that students acquire even simple technical mastery and:

"we must admit that we know little about what really happens in the learning process of either the individual or of a group in the classroom, or about the processes of interaction in the classroom" (P.255).

In a further comment over the disappointment that could result from failure and errors of recent curricula changes, Howson et al. (1981)
state that:

"We must not, however, fear that enthusiasm for improving school education will die out. Alas, it has been demonstrated that innovation and improvement are not necessarily synonymous. Nevertheless, it is only through curriculum development that our goals for school education will be realised." (P.265).

Though the term 'curriculum development' used in this comment remains vague as long as the term "curriculum" stands as such, I believe that development in mathematics education through any factor, x, involved is better based on certain identification of the underlying structure, i.e. the set of rules that identifies such an x.

I feel that Marton and Saljo (1976) seem to be aware of this - without telling that explicitly - by claiming that learning should be described in its contents. Such a description, I believe, should basically identify the underlying structure of the contents. Furthermore, the influence of a dominant structure, e.g. 'S', can be noticed when students as well as others defined as educators of mathematics in the field, have spontaneously responded that solving $2x + 1 = 5$ is easier than solving $2x + a = 5$ (P.5-7). This dominance can be described - more or less as the dominance of the temperature measure - Fahrenheit or Centigrade - where most people who live in a space of one measure can't directly feel the temperature given by the other measure. Kaput (1979) gives another example directly in discussing cognitive meaning of some aspects of inequality, but does not mention any instance in his examples included in U or H. Two of his examples were:
(i) \[ x^2 + 5x + 6 = (x+2)(x+3) \text{ and } (x+2)(x+3) = x^2 + 5x + 6. \]

(ii) \[ \frac{d}{dx}(x^3+5x) = 3x^2 + 5 \text{ and } \int (3x^2 + 5) \, dx = x^3 + 5x + c. \]

It is relevant that the last conference of the BSPLM held at the University of Leeds in April 1981, could not find in their discussions any example of an instance such as:

\[ 2x - 3ay = 1 \text{ and } 2x + 3ay = 2 \]

in presenting mathematical knowledge concerning the solution of two simultaneous equations of this form.

If one accepts Hilgard et al. (1975) who define learning as:

"a relatively permanent change in behaviour that occurs as the result of experience" (P.194),

then in this study I advise that the change in behaviour could be improved by learning if this change were to take place in two different learning spaces, S and U, under the same behavioural objective, provided this is based systematically, as in Proposition 9.1. Learning in one space 'S' would show up a dissatisfaction with this space, as was discussed in the first indicator in the battery T1 reported in 7.2.0.

10.1.2.4.0. General Findings

Here, I now state what can possibly be identified in the context of this study.
1. The (material) contents of a mathematics teaching-learning task can be equated with the written material for the contents in the textbook.

2. There exists a structure $S$ underlying the current contents. Hence the methodology of mathematics teaching-learning situations is fundamentally based on $S$ as a learning space. Simply, $S$ (see Glossary, Appendix A) contains such instances that do not contain any constants of implicit form - other than the variables if any exist.

3. There exists another structure $H$ which is essential, but is more or less completely ignored in the current methodology. $H$ (see Glossary) contains all instances that include at least one constant - other than the variables, if any. Further, it was deduced under definite conditions that $S \cap H = \emptyset$ (the null set).

4. The transferability of mathematical computational knowledge based on $S$ to other fields such as physics, which use mathematics is not satisfactory.

5. Mathematical knowledge in $H$ correlates closer with problem solving than in $S$.

6. A set $U$, as a part of $H$ was put forward and identified, so as to be applicable in learnability under Proposition 8.2. Then, on the basis of the test-form $S_0 + U_0 + P$, with $S_0$ and $U_0$ as two kernels in $S$ and $U$ respectively for $P$, it was found that the efficiency (see Glossary) in $U$ is higher than that in $S$, regarding the same $P$. 
7. Based on the theory introduced in Chapter 5, as well as findings in the previous point (6), an algorithm for a modification in the methodology of mathematics teaching-learning situations was introduced in Proposition 9.1. It was advanced on the basis that $S + U$ presents a model which should underlie such a methodology - based on the algorithm suggested in Proposition 9.1.

In this context, empirical evidence from two experiments appears to support that methodology based on $S + U$ in the sense of Proposition 9.1 is superior to the current methodology based on $S$ for the mathematics achievement in the area of this study.

10.2. II. A Further Outlook

The seminal work in this study can in fact be identified in the proposed modification in the methodology of mathematics teaching-learning situations, based on a model derived by the algorithm advanced in Proposition 9.1.

Mathematics in fact asks for an ability in abstraction and generalization. In this sense, Krutetski (1976) claims that,

"abstraction and generalization constitute the essence of mathematics, and mathematical thinking, therefore, is largely abstract and generalized thinking" (P.86).

Also in this line, Bruner and Kenney (1965) believe that, once abstraction is achieved, the learner can then rely upon a stock of imagery, which might
permit him to work at a level of heuristic means in order to explore problems and relate them to others already mastered.

Hence, we have the belief that children should progress in mathematics from concrete situations to abstract ones, so that, for example, they start with the natural numbers and progress to symbols. Then, in this context, - more or less in line with Bruner (1964), see P.3-20 as well as Deines, see P.3-21, - I have the view that in the area of this study, the S-structure is a zero-abstract dimensional space while the U-structure is a first abstract dimensional space. I believe that in order to cope with symbolism, one should 'act on them' in the terms of Piaget (1964). Hence, I find the algorithm in proposition 9.1 to agree with Bruner (1964) who states,

"if it should seem that I am urging that the growth of symbolic functioning links a unique set of powers to man's capacity, the appearance is quite as it should be"

Whether one would interpret such a statement differently, I believe that a student should always work with instances involving constants of implicit form, whatever the behavioural objective or the approach. These instances contain, in my opinion, the essence of symbolism which helps cognitive growth.

In addition, for a further enquiry along the lines of the study in this thesis, I believe a prospective work could be found in the following:

(i) Repetition of this study, which would be worthwhile to confirm or refine the results and conclusions.
(ii) Identification of another possible algorithm, either to enhance or replace the algorithm in Proposition 9.1.

(iii) Identification of another possible learning structure $S + \hat{U}$, with $\hat{U}$ as another $N$-abstract dimensional space. I recommend $N$ to be the set $\{1, 2\}$ which provides an extension to the term 'dimensional' used in conventional context. By this, I suppose that $\hat{U}$ would contain the set of all instances in $H$ which contains at least one and not more than two constants. This would inevitably result in another algorithm to replace that advanced in Proposition 9.1.

(iv) The area of problem-solving is most essential in mathematics education. In the trend of this study, I believed (as a construct) that the learning space $S + U$ could benefit in problem-solving in the long term. Hence, a comparative study might be fruitful, following up the abilities of two groups of students in problem-solving in the long-term, provided one group follows $S$, while another follows $S + U$ under Proposition 9.1.

(v) I will not add anything new if I mention that I found a strong correlation between achievement in mathematics and physics in 9 classes with different teachers of mathematics and physics in different secondary schools in Kuwait. So, I followed-up the total scores $x$ and $y$ in mathematics and physics respectively obtained by a random sample of 300 students in the General Examinations in Kuwait held for the final grade of the secondary school in the four academic years 1976/77, 1977/78, 1978/79 and 1979/80. I observed that, in not less than 90% of the cases $y > \frac{2}{3}x$ for the students who passed both mathematics and
physics examinations. Such a relation may confirm that improvement of the achievement in mathematics would improve the achievement in other subjects such as physics and engineering, which use mathematics.

Hence, I recommend a comparative study to follow the achievement of two groups in a non-mathematical subject, say physics or engineering, taken over a long period and where one group learns in $S$, while the other group learns in $S + U$, under Proposition 9.1.
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APPENDIX A

A GLOSSARY OF TERMS
A. A GLOSSARY OF TERMS

This glossary of terms includes such terms that commonly occur in the context of this study and following up the work in this thesis. The terms which - to my knowledge - are not found in other sources are more or less personal and the sign (*) is prefixed to such terms to indicate this possibility. Nevertheless, the other terms were modified to suit the context of this study, while a reference is introduced in formal places. The underlined terms indicate that these terms are found in this glossary.

<table>
<thead>
<tr>
<th><strong>A</strong></th>
<th>Symbol referring to a number assigned to the achievement of a student in a certain task. A is usually assigned by an achievement test graded on the basis of a predetermined criterion. For a group of students, such an 'A' could be the mean of the values of A of the students. This A measures the product achieved in the learning process, i.e. the learning outcomes.</th>
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<tr>
<td><strong>Achievement</strong></td>
<td>A construct referring to what a student has learnt in a certain teaching-learning task. Possibly a number A is assigned under certain considerations to match this construct.</td>
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</table>
Achievement test

An instrument designed to measure and appraise what a student has learned to do. It measures the extent to which the student has attained (achieved) certain knowledge in a task $k$ under a specific teaching-learning situation.

Generally, this is designed on the basis of producing a set $B$ of (basic) behavioural objectives found necessary for $k$. The test accordingly is made up sets of simple stimuli (usually in the form of simple question-items) so that each of these sets is provided to measure the achievement in one $B_i$ in $B$. The number of elements in each set is subject to certain considerations defined by related criteria grading.

(*) Advantage of Achievement

This construct refers in this thesis to how far an achievement of a (teaching-) learning task provides the capability to the students of solving a certain problem $P$ in the domain of $k$. A measure called efficiency is introduced as a measure regarding $P$ for this construct.

$B$

A symbol referring to a set of (basic) behavioural objectives necessary for a task $k$. ($k$ could be a certain problem).

Battery of tests

A set of at least two tests designed under certain relationship, usually administered to the same group of students.
A-3

(*) Basic Behavioural Objective

A behavioural objective identified to be most necessary in comparison with others. This is necessarily normally assessed by a panel of people working in the field.

Behavioural Objective

An intended goal stated in a clear and specific way so that its attainment (or lack of it) can be observed or measured. Usually, it is an instructional objective restricting an instructional goal labelled as 'measurable' or 'behavioural' which are relatively interchangeable terms - indeed the outcome will be 'measurable' rather than 'behavioural' - (Tuckman, 1975, P.24).

A statement related to a behavioural objective in the context of this thesis will begin in "To apply (the knowledge) ...", "To identify (the knowledge) ..." or "To identify and apply (the knowledge) ..." in cases of mathematics computational knowledge, mathematics comprehensional knowledge or mathematical knowledge respectively.

Competency

A student is said to have satisfied competency in attaining the knowledge related to a certain behavioural objective $B_i$ if he satisfies certain criteria in the outcomes of $B_i$. The term can also be used to demonstrate capability in solving a problem $P$. 
In this thesis, this term refers to a basic part of mathematical knowledge. It refers to identifying a specific piece of knowledge in the search for a specific goal, e.g. identifying a proper rule related to a response for a behavioural objective. In this case, the relevant behavioural objective will normally begin "To identify ...".

This is a basic part of mathematical knowledge. It refers to the knowledge of how to apply a certain response - e.g. a rule - in order to achieve a goal. A behavioural objective related to this part of knowledge will thus begin in "To apply ...".

The contents of a teaching-learning task k in general consist of the available information about k found in the interaction between the student, the teacher and k. The contents of k in the field of the study in this thesis are equated with the information found in the written material for k in the textbook. Furthermore, it was viewed that the contents of k, basically include two parts, a theoretical part, which is almost invariant and a presentation P where variances in contents take place through P.
Domain of task

If \( B \) is a set of behavioural objectives necessary for a task \( k \) - i.e. the elements of \( B \) govern \( k \) - then a task \( k \) is said to be in domain of \( k \) if \( k \) is governed by a subset of \( B \). Hence, such a domain is identified by the set \( B \) of behavioural objectives of \( k \).

(*)Efficiency

This term can be identified as a measure for the advantage of achievement in a certain task. It was primarily introduced as the probability that an individual 'x' is capable of a problem \( P \), provided that x has demonstrated a predetermined competency in solving all the elements of a set \( P_0 \) of simple stimuli relevant under certain considerations to \( P \). In developing the theory in this research, this term - theoretically - refers to the probability that x is capable of \( P \), provided that x is able (competent) to undertake accurately all the kernels for \( P \) in a certain space of knowledge, i.e. x satisfies the uniform operational property (see Definition 5.7). For conveniency sake in the further development of the research, Definition 5.7 was weakened (see Definition 8.1) by regarding the efficiency in the so-called pseudo-reality as the probability that the conditional statement: x is capable in all the elements of one kernel for \( P \Rightarrow x \) is capable in \( P \) is true, providing that the first premise holds true.
Symbol referring to the H-space (or structure) in the context of this thesis. (H stands in fact, for higher).

Symbol referring to a kernel of a problem P in H-space.

An instance that contains explicitly at least one constant - of implicit form, 'a' say - other than the variables, if any.

This refers to the space of knowledge \((H, O)\) structured by H-instances.

This term refers to the set of all H-instances in the mathematics of the area of this study. (This could also be equated loosely with the H-space).

This refers to a tentative statement proposed to be subject to verification through subsequent investigation (Verma and Beard, 1981, P.84).

The term refers in this thesis to an item in the form of an example, exercise or a simple or routine (novel) problem, governed by only one behavioural objective in domain of a certain task k.
(*kernell) Symbol referring to a (teaching-) learning task.

If \( B \) is a set of behavioural objectives necessary for a problem \( P \), then a kernel for \( P \) in a space of knowledge \( W \) is defined to be a set \( W_o \) of instances in \( W \) such that for all \( B_i \) in \( B \) there exists a unique \( w_i \) in \( W_o \), which is necessarily governed by \( B_i \).

Law (Assumption) of Universal Determinism

This basically refers to a fundamental assumption made by scientists in that the events they investigate follow a law or are ordered so that no event is capricious. Behavioural scientists likewise extend this law into their field, assuming that the behaviour of an organism is lawful and predictable (Ary et al., 1972, P.12).

(*Learning space) A space of knowledge manipulated to be applicable to the teaching-learning situations in a specific area of a particular knowledge, e.g. secondary school mathematics.

(*Mathematical knowledge) This term refers in this thesis to the knowledge that can be identified - comprehensional - or applied - computational - with respect to a certain mathematical response. It consists in this thesis of the two basic parts comprehensional and computational. A behavioural objective for mathematical knowledge will begin "To identify and apply ...".
D

Symbol referring to a certain normal population.

P

Symbol referring to a given novel problem which is not simple, i.e. not an instance in a particular domain.

Proposition

This term refers in this thesis to a certain statement - premise - accepted by the researcher to be true, i.e. more or less an axiom.

(*)Presentation

It is the set of instances introduced in (domain) of the contents of a task k in order to illustrate, enhance and reinforce mathematical knowledge in the achievement of k.

Principle of Invariance

The law of universal determinism in science assumes that under specific conditions certain events will occur in a predictable way (Ary et al., 1972, P.12). A principle that specifies such conditions is called in this sense a principle of invariance. For example (Bruner, 1966, P.41) claims that the most elementary forms of reasoning - whether logical, arithmetical, geometrical or physical - rest on the principle of invariance of quantities: the whole remains, whatever may be the arrangement of its parts, the change of its form, or its displacement in space or time.

contd/........
In this thesis, this principle refers to a possible set of conditions which may preserve consistency and extends the law of universal determinism into a behavioural science.

Pseudo-reality
A hypothetical space of reality assumed for facilitating progress in study (see 8.2.0, P.8-9).

(*)R
A symbol denoting a presentation.

(*)\tilde{R}
A symbol denoting a modification in the methodology for R based on Proposition 9.1.

Representative sample
This term refers to a random sample chosen from a certain normal population 'O' in the sense that this sample is expected to satisfy the same "characteristic" or "measure" which the whole population satisfies.

It was viewed in the context of this thesis that two classes in one grade of secondary school in Kuwait would generally form a representative sample of the population of the grade in the area of this study.

(*)S
A symbol denoting the S-space (structure) in the context of this thesis. (S stands for 'sub' in the sense that this is indispensable; see H).
(*)S-instance

An instance that does not contain any constant - i.e. symbolic constant of an implicit form, 'a' say - other than the variables, if any.

(*)Simple stimuli

This refers to an instance in the context of this thesis.

(*)Space of knowledge

This term is a conceptual structure identified for a certain normal population 'O' by the ordered pain \((W, 0)\) or simply \(W\). This \(W\) was identified as a particular structured knowledge for a particular area of knowledge. \(W\) is considered to be structured (ruled) by a finite set of rules which is called the underlying structure of \(W\). The elements of \(W\) are called initial states of knowledge e.g. instances.

(N.B. Mathematics may enhance this term for further purposes by identifying the space of knowledge as the ordered triplet \((W, F, 0)\) where \(F\) is the underlying structure of \(W\). But \(F\) was not introduced in the context of this study, since it was believed that the term \((W, 0)\) is sufficient and appropriate to other fields of knowledge, provided that the underlying structure of \(W\) is identified within \(W\)).

(*)Structured knowledge

It refers to a certain knowledge introduced as a set \(W\) of instances in a particular area of knowledge - e.g. the area of this study - identified by a finite set of rules, say 'F'. This F is called the under-

contd/.....
lying structure of \( W \) and it could be identified
within \( W \).

(N.B. Mathematics may enhance this term for
further purposes by identifying this term as the
ordered pair \( (W, F) \)).

\((*)\) S-space

The term refers to the space of knowledge \( (S, 0) \)
structured by S-instances.

\((*)\) S-structure

This refers to the set of S-instances in the
mathematics of the area of this study. It is
a structured knowledge identified by a rule
identifying such S-instances. (This could also
be equated loosely with the S-space). (This
structure was identified to underlie current
presentation \( R \) of mathematical teaching-learning
tasks in the sense that \( R \) is a subset of this
structure).

\((*)\) S + U

This refers to a learning space structured under
an algorithm suggested in Proposition 9.1. This
space is introduced to be applied for a modification
in the methodology to be applied in relevant
teaching-learning situations.
| Teaching-learning situation | The term refers in this thesis to a possible observable factor (event) planned overall by a methodology of mathematics teaching-learning, e.g. curriculum, teaching method or a presentation $R$ of a task - a methodology is reflected in such a situation. |
| (*)U | A symbol denoting the U-space in the context of this thesis. ($U$ stands for 'ultra': see $S$ and $H$). |
| (*)U-instance | This is an $H$-instance which includes only one constant, other than the variables, if any. |
| (*)Uniform operational | If $P$ is a problem relevant to a space of knowledge $(W, 0)$, then a uniform operational in 'O' is a person who is able in all kernels of $P$ in $W$ in the sense that this person is completely competent in all the elements of a kernel $W_0$ for all such kernels in $W$ (see Definition 5.7). Nevertheless, this was modified for convenience in the research development to indicate someone in 'O' who is able in only one kernel for $P$ (see Definition 8.1). |
| (*)U-space | The term refers to the space of knowledge $(U, 0)$ structured by U-instances. |
(*)U-structure

The term refers to the set of U-instances in the mathematics of the area of this study. (This could also be equated loosely with the U-space).

(*)W

A symbol in general denoting simply a space of knowledge \((W, 0)\) in the context of this study. Specifically, it is a structured knowledge identified by a finite set of rules in a particular piece of knowledge.
APPENDIX B

A SECOND PILOT INSTRUMENT: A TEST $T_0$. 
A TEST To

For Students In The Final Grade Of Secondary School In Kuwait.

Instruction

In the following test of twenty multiple-choice items, there is only one correct response out of the four responses given with each item. Please tick (✓) in the appropriate box of the correct response. Please do not guess.

-------------------------------

1. If \( \frac{x - 2}{2x - 1} = 5 \), then \( x \) equals:
   - \( 3 \) ➕
   - \( -3 \)
   - \( -\frac{1}{3} \)
   - \( \frac{1}{3} \)

2. If \( |x - 2| = -1 \), then the solution set for \( x \) is:
   - \( \{3, 1\} \)
   - ∅
   - \( \{1\} \)
   - \( -3 \)

3. If \( x^2 - 3x - 5 = 0 \), then the solution set for \( x \) is:
   - ∅
   - \( \{3, 5\} \)
   - \( \frac{3 + \sqrt{29}}{2} \)
   - \( \left\{3, \frac{5}{3}\right\} \)
4. If \( x^2 - 4x < 0 \), then the solution set is identified by:

- \( 0 < x < 4 \)
- \( x < 4 \)
- \( 0 \leq x \leq 4 \)
- \( |x| < 2 \)

5. In the figure at the side, \( \lim_{x \to 1} f(x) \) is:

- 1
- 2
- non-existing
- zero

6. \( \lim_{x \to 0} \frac{x^2 + x}{x} \) as \( x \to 0 \) is:

- Zero
- non-existing
- 1
- none of the preceding

7. If \( T(x) = -\sqrt{x} \), then the domain of \( T \) corresponds to:

- \( x > 0 \)
- \( x \geq 0 \)
- \( x \leq 0 \)
- all real numbers

8. \( \lim_{x \to \infty} \frac{1 - x^2}{2x^2 + 1} = 1 \) as \( x \to \infty \) is:

- \(-\frac{1}{2}\)
- \(\frac{1}{2}\)
- zero
- \(-2\)

9. If the function \( D \) has a limit at \( x = a \), then:

- \( D \) is defined at \( a \)
- \( D \) is continuous at \( a \)
- \( D \) is undefined at \( a \)
- \( D \) might be continuous at \( a \)
10. If the function $f$ is continuous at $x = a$, then:

- $f$ is _ _ _
- $f$ has a derivative _ _ _
- $f$ may have no limit _ _ _
- $f$ may be undefined _ _ _

at a

11. If $T(x) = x - x^2$ then one of the following points belongs to $T$:

- $(1, 2)$ □
- $(-1, 0)$ □
- $(2, 2)$ □
- $(2, -2)$ □

12. If $f(x) = x + x^2$, then the slope of the tangent line to $f$ at $x = -2$ is:

- $-3$ □
- $2$ □
- $5$ □
- $-4$ □

13. If $x = x(t), y = y(t)$ and $x^2 + y^2 = 10$, then:

\[
\begin{align*}
\frac{dy}{dt} &= 10 - 2x \frac{dx}{dt} \quad \square \\
\frac{dy}{dt} + \frac{dx}{dt} &= 0 \quad \square \\
\frac{dx}{dt} + y \frac{dy}{dt} &= 0 \quad \square \\
2x + 2y \frac{dy}{dt} &= 0 \quad \square
\end{align*}
\]

14. If $T(x) = x^2 + h^2$, $h$ being constant, then $\dot{T}(x)$ is:

- $2x + h^2$ □
- $2x$ □
- $x^2 + 2h$ □
- $2x + 2h$ □
15. If \( x = t^2 - t^3 \) is the displacement of a point moving in a straight line, where \( t \) is the time (time units), then the acceleration of this point at \( t \) is denoted by:

\[
2t - 3t^2 \quad \frac{x}{t^2} \quad \frac{\dot{x}}{t} \quad 2 - 6t
\]

16. If \( v = 3t^2 + 2t \) measures the velocity of a point moving in a straight line, \( t \) being the time, then the acceleration at \( t = 0 \) is:

Non-existent \quad Zero \quad 2 \quad None of the preceding

17. The distance between \( A = (2x, x) \) and \( B = (4x, x) \) is given by:

\[
2x \quad \pm 2x \quad -2x \quad 2|x|
\]

18. If the line: \( y = x - h \) is a tangent line to the curve: \( f(x) = x^2 + 5x \), then \( h \) equals:

\[
4 \quad -4 \quad 7 \quad -7
\]

19. The displacement \( x \) of a point moving in a straight line is \( x = t^2 + t \), \( t \) being a number measuring the time. If \( V \) and \( A \) are two numbers measuring the velocity and the acceleration respectively of this point at \( t \), then \( V^2 - 4A = 1 \) when \( t \) is:

\[
2 \quad -1 \quad \frac{1}{2} \quad 1
\]
20. Let \( x = 2t \) and \( y = t + 4 \), define the displacements of the two points \( M_1 \) and \( M_2 \) which move on the x-axis and y-axis respectively, \( t \) being a number measuring time. Then the rate of change of the distance \( M_1M_2 \) when \( y = x \) is:

\[
3\sqrt{2} \quad \square \quad 48 \quad \square \quad \frac{3}{\sqrt{2}} \quad \square \quad \frac{3}{2} \quad \square
\]
CORRECT ANSWERS

A tick (✓) indicates which of the four boxes corresponds to the correct answer.

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APPENDIX C

TWO EXAMPLES OF TASK-PRESENTATION
I. This example of presentation is translated verbatim from the textbook for the final year, entitled:

Contemporary Mathematics: 4th Secondary Year (Final Grade)-

This book was revised by a group (of whom I was one) comprising four people from the mathematics department in Kuwait University and the Department of Mathematics in the Ministry of Education.

The main behavioural objective of the task k is:

"To identify and apply a knowledge of derived function that is defined by more than one rule in relevant domains"

This assumes that the students have already learnt the definition and rules of derivatives.

(1) \( e_1 \): A set of examples (pp.110-112)

(i) If \( f(x) = x|x| \), then find:
\[ \hat{f}(1), \hat{f}(-3), \hat{f}(0) \]
(ii) Find $g(x)$, provided that:

$$g(x) = \begin{cases} 
  x^2 & \text{if } 0 < x \\
  0 & \text{if } -1 < x < 0 \\
  x & \text{if } x < -1 
\end{cases}$$

(2) $e_2$: A set of exercises (P.113).

(i) If $H(y) = \begin{cases} \sqrt[3]{y} & \text{if } 0 \leq y \\
  y^2 & \text{if } y < 0 \end{cases}$

Find $H(1)$, $H(-1)$ and $H(0)$

(ii) If $F(n) = \begin{cases} n^2 & : 2 \leq n \\
  4n-4 & : n < 2 \end{cases}$

Find $F(3)$, $F(-2)$ and $F(2)$

(iii), (iv), ... (other similar instances).

(3) $e_3$: A set of simple or routine problems (P.113).

(i) If $H(n) = \begin{cases} \sqrt[n]{n} & : 0 < n < 2 \\
  n^2 & : 2 < n \\
  4n-1 & : n < 0 \end{cases}$

then find $H(n)$ whenever it exists and discuss when $H(n)$ does not exist.

(ii), (iii), ... (other similar instances).

Then $R = \{e_1, e_2, e_3\}$. Thus it can be easily noticed that all instances which eventually make $R$, belong to the $S$-structure, and no instance belongs to the $H$-structure.
II. This example of presentation is also quoted from the textbook of the second secondary year (10th grade) of secondary school in Kuwait (1979).

The main behavioural objective of the task k is:

"To identify and apply the knowledge of a translation as a transformation in the cartesian plane".

The contents in the text discuss the translation \( T_d : (x,y) \rightarrow (x+d, y) \) where all points \( A = (x,y) \) in the plane are removed to \( A' = (x+d, y) \) such that the vector \( \overrightarrow{AA'} \) has a well-defined direction \( \theta \) and the distance \( |d| \) between \( A \) and its image \( A' \) is always fixed.

The presentation \( R \) of \( k \), as it was defined, was made up of three sets:

1. \( e_1 \): A set of examples (pp. 91 - 94)

   (i) Fill in spaces:
   \[
   \begin{align*}
   X_3 (-1, 2) &= (\ldots, \ldots) \\
   Y_3 (2, -2) &= (\ldots, \ldots) \\
   Y_3 (0, \ldots) &= (\ldots, -3)
   \end{align*}
   \]

   (ii) Find \( d_1 \) and \( d_2 \) in \( T_d = X_{d_1} \circ Y_{d_2} \) where \( T_d (2, 3) = (6, -4) \) and '0' denotes composition of functions.

   Note: \( X_d \) denotes a translation in the X-axis with the positive direction if \( d > 0 \) and in the opposite direction if \( d < 0 \).
   Similarly \( Y_d \).
(iii) Find the equations of the image line to the line 
1 : 2x - 3y + 5 = 0 under the composite function:
Y_3 \rightarrow X_3 = L_d.

(iv), (v), ... (other similar examples).

(2) e_2: A set of exercises (P.95).

(i) Define the translation \( L_d \) in terms of two translations \( X_{d_1} \) and \( Y_{d_2} \) in each of the following:
(2, 5) \( \rightarrow \) (6, 8)
(4, 1) \( \rightarrow \) (5, -3)

(ii), (iii), ... (other similar exercises).

(3) e_3: A set of simple routine problems (P.96).

(i) If (4, -1) is the image of (1, 3) under \( L_d \), then define \( L_d' \).

(ii), (iii), ... (other similar problems).

It could be noticed that as \( R = \{ e_1, e_2, e_3 \} \), all the instances which make up \( R \) fall in 'S' and none of them is in H.
APPENDIX D.

FIRST INDICATOR/COMPUTATIONAL KNOWLEDGE:

$T_1$: A BATTERY OF THREE TESTS
T₁: A BATTERY OF THREE TESTS, t₁, t₂ AND t₃

For The Final Grade Of Secondary School In Kuwait

t₁: (H-Mathematics)

q₁1) If \( z = \frac{1}{h} \cdot \frac{x \cdot y}{x - y} \) then find \( x \) in terms of the other elements \( z, y \) and \( h \).

q₁2) If \( z = \sqrt{\frac{2x^2}{h}} \) then find \( I \) in terms of other elements.

q₁3) If \( z = \sqrt{x^2 + y^2 + 2xy \cos \theta} \) then find \( \cos \theta \) in terms of other elements.

q₁4) If \( y = \frac{1}{k} \cdot \frac{x \cdot g}{z^2} \) with \( z > 0 \) then find \( z \) in terms of other elements.

-----------------------------------------------

t₂: (S-Mathematics)

q₂1) If \( z = \frac{1}{x} \cdot \frac{y \cdot n}{y - n} \) find \( y \) when \( z = 3 \), \( x = 2 \) and \( n = 5 \).

q₂2) If \( f = \sqrt{\frac{2z}{h}} \) find \( h \) when \( f = 4 \), \( z = 2 \) and \( x = 7 \).

q₂3) If \( x = \sqrt{h^2 + 1^2 + 2h \cos y} \), find \( \cos y \) when \( x = 10 \), \( h = 9 \) and \( 1 = 3 \).

q₂4) If \( h = \frac{1}{k} \cdot \frac{y \cdot x}{n^2} \) with \( n > 0 \), find \( n \) when \( h = 5 \), \( k = 3 \), \( x = 1 \) and \( y = 2 \).
The capacity $C$ (Farad) of a spherical condenser is given by
\[ C = \frac{1}{k} \cdot \frac{R \cdot r}{R - r} \]
where $k$ is a constant, and $R$ and $r$ (m) are the measures of the large and small radii of the condenser. Find $R$ in terms of $C$, $k$ and $r$.

Let $V$ be the potential difference between two parallel plates with different charges. If $M$ (kg) and $q$ (coulomb) are the mass and charge of an electron moving between the two plates, and if the electron strikes the positive plate with velocity $v$ (m/sec) then
\[ v = \sqrt{\frac{2qV}{M}}. \]
Find $q$ in terms of the other elements.

If $\theta$ is the angle between two forces, $F_1$ and $F_2$ (Newton) and $R$ (Newton) is their resultant then we have:
\[ R = \sqrt{F_1^2 + F_2^2 + 2F_1 \cdot F_2 \cos \theta}. \]
Find $\cos \theta$ in terms of other elements.

If $m_1$ and $m_2$ (kg) are the masses of two point particles and $d$ (m) is the distance between them, then the Newtonian force of attraction $F$ (Newton) between them is given by:
\[ F = \frac{1}{k} \cdot \frac{m_1 \cdot m_2}{d^2} \]
($k$ being a constant). Find $d$ in terms of the other elements.
APPENDIX E

A SECOND INDICATOR / MATHEMATICAL KNOWLEDGE -

$T_2$: A SET OF THREE TESTS
T_{2}: A SET OF THREE TESTS, \( t_1, t_2 \) AND \( t_3 \)

\( t_1 \): For The Final Grade Of Secondary School In Kuwait

\( S_1 \): 1. Let \( f(x) = x^3 - 2x^2 + 5 \) be a primitive of a function \( g \). Find \( g \). (Here \( g(x) = f(x) \))

\( S_2 \): 2. Find \( \int_{-2}^{2} (2x - 3x^2) \, dx \).

\( S_3 \): 3. Let \( f(x) = x^2 - 2x - 1 \). Find the common points of the curve \( f \) and the \( x \)-axis.

\( S_4 \): 4. The two curves: \( f(x) = x^3 - 2 \) and \( g(x) = 4x^2 - 5x \) have only two common points \((1, -1)\) and \((2, 6)\). Find the area of the region between \( f \) and \( g \).

\( h_1 \): 5. If \( T(x) = a^2x^3 + 3b^2 \) (and and \( b \) being constants) is a primitive of a function \( g \), find \( g \).

\( h_2 \): 6. Find \( \int_{a}^{b} (3bx^2 - 2a^2) \, dx \), \( a \) and \( b \) being constants.

\( h_3 \): 7. Find the common points between the curves \( y = x^2 + bx \) and \( y = bx + c^2 \), when \( b \) and \( c \) are constants.

\( h_4 \): 8. The graph of a curve \( f \) is given in the diagram at the side. How do you find the area of the shaded region?

\( P \): 9. Find the area of the region that is bounded by \( f(x) = x^2 - x - 2 \), \( x = -2 \), \( x = 1 \) and the \( x \)-axis.
1. Let \( f(x) = 3x^2 - x - 1 \). Find a point on the curve \( P \).

2. Find the slope of the line \( L : y = 4 - 3x \).

3. Find the slope of the tangent line to the curve \( H(x) = x - x^3 \) at \( x = 2 \).

4. Let \( f(x) = x^2 + 2x \) and \( A = (1, 3) \) a point on \( f \). Let \( L \) be the line \( y = 4x - 3 \). Is \( L \) a tangent to \( f \) at \( A \)?

5. If \( (2a, z) \) belongs to \( f \) were \( f(x) = x^2 - x \). Find \( z \) in terms of \( a \).

6. Find the slope of the line \( m : 4by = 4c - 3ax \) (\( a, c \) and \( b \) being constants and \( b \neq 0 \)).

7. The slope of line \( h \) is \( (-2c) \). If \( h \) is a tangent line to a curve \( f \) at \( (a, b) \), find \( f(a) \) provided it exists.

8. Let the line \( k : y = 2mx \) and \( f \) the curve \( y = x^2 + m^2 \). Is \( k \) a tangent to \( f \) at \( (m, 2m^2) \)?

9. Let \( D(x) = x^2 - 4x \) and the line \( L : y = 6x + c \). Find \( c \) when the line \( L \) is tangent to the curve \( D \).
1. Find \( \frac{dy}{dx} \) when \( y = \sqrt{x^2 - 5x + 1} \).

2. Let \( f(x) = x^2 \). Find the distance \( d \) between the two points \( A = (2, f(2)) \) and \( B = (-2, 3) \).

3. Let \( f(x) = \frac{1}{3} x^3 - 4x \). Find the \( x \)-co-ordinates when \( f \) has critical values.

4. Let \( f(x) = x^3 - x^2 - x + 2 \). Given that \( f \) has a critical value at \( x = 1 \), determine whether \( f(1) \) is a local maximum or a local minimum.

5. If \( L^2 = y^2 + a^2 \) where \( y = f(x) \), \( a \) being constant, find \( \frac{dy}{dx} \).

6. Let \( f(x) = x^2 \). Find the distance \( d \) between the two points \( A = (a, f(a)) \) and \( B = (1, 3) \).

7. Given \( f(x) = a^2x^3 - 3x^2 \), \( a \) being constant ≠ 0. Find the \( x \)-co-ordinates when \( f \) has critical values.

8. \( T(x) = a^2x^3 - bx \), \( a \) and \( b \) being constants. If \( T \) has a critical value at \( k < 0 \) then determine whether \( T(k) \) is a local maximum or local minimum.

9. The point \( M = (x, f(x)) \) moves on the curve, \( f(x) = xy^\sqrt{x} \). Let \( N = (8, 0) \). Find the position(s) of \( M \) when the distance \( D \) between \( M \) and \( N \) is least.
APPENDIX F

AN EMPIRICAL STUDY: A SET \( T_3 \) OF THREE TESTS BASED ON THE TEST-FORM \( S_0 + U_0 + P \)
A SET OF THREE TESTS $t_1$, $t_2$ and $t_3$

$t_1$: For Students At Kuwait University Studying Course 101*

1. A particle moves in a straight line with $v = t^2 - 4t + 2$. Find its position at $t$.

2. Find $\int (4x - 3x^2) \, dx$.

3. A particle moves in a straight line where, after $t$ sec;
   $v = (t-4)(t+3)$. When does this particle stop?

4. Find $\int_{4}^{5} (4x - 3x^2) \, dx$.

5. Let $v = t^2 - 2t$ define the velocity of a particle moving in a straight line. Find the distance travelled between $t = 3$ and $t = 6$.

6. A particle moves in a straight line with $v = at^2 - 4t + 2$. Find the position of this particle at $t$, ($a$ is a constant).

7. Find $\int (ax - 3x^2) \, dx$, ($a$ is a constant).

8. Let $v = (t + 3)(t - b)$ with $b > 0$; define the velocity of a particle moving in a straight line after $t$ sec. When does this particle stop?

9. Find $\int_{2}^{a} (ax - 3x^2)$ ($a$ being a constant).
u_5): 10. If \( v = t^2 - 2at \) with \( a > 0 \) defines the velocity of a particle moving in a straight line, find the distance travelled between \( t = 3a \) and \( t = 6a \).

(P): 11. A particle begins penetrating a sandy soil at \( t = 0 \). If the velocity \( v(\text{cm/sec}) \) of this particle after \( t \) seconds is given by \( v = 16 - t^2 \), find the distance travelled by this particle until it stops, provided that the whole motion was in a straight line.

* Throughout \( v(\text{cm/sec}) \) is the velocity after \( t \) seconds.
1. Let $x = t^3 + t$ define, at $t$, the position of a particle moving in a straight line. Find the velocity of this particle at $t$.

2. Let $x_1 = t^3 + 10$ and $x_2 = t^3 + t^2 + 1$ define for $t > 0$ the positions of two particles moving in a straight line. When do the two particles meet?

3. A particle starts moving in a straight line from $x = -3$ with a constant velocity $2$ (cm/sec). Find the position $x$ of this particle at $t$.

4. Let $x_1 = t^3 + 4t^2$ and $x_2 = 3t + 1$ define at $t$ the positions of two particles moving in a straight line. Do they meet at $t = 2$?

5. Let $x = t^3 + at$ ($a$ being a constant) define at $t$ the position of a particle moving in a straight line. Find the velocity of this particle at $t$.

6. If $x_1 = t^3 + 7a^2$ and $x_2 = t^3 + t^2 - 2a^2$ define for $t > 0$ the positions of two particles moving in a straight line. When do they meet? ($a$ is a constant).

7. A particle starts moving in a straight line from $x = a$ with constant velocity $2a$ (cm/sec). Find the position $x$ of this particle at $t$. 
u_4): 8. Let \( x_1 = t^3 + ct^2 \) and \( x_2 = c^2t + c^3 \) define the positions at 
t of two particles moving in a straight line. Do they meet 
at \( t = c \)? \((c > 0)\).

(P): 9. Two particles \( M_1 \) and \( M_2 \) start moving together in the same 
straight line. The position of \( M_1 \) at \( t \) is given by \( x_1 = 
3t - 4t \). The second particle \( M_2 \) moves with constant velocity 
8\,(cm/sec) starting from \( x_2 = a \). Determine \( a \) so that both \( M_1 \) and 
\( M_2 \) have the same velocity when they meet.

* Throughout \( \tilde{t}_2 \), \( x(cm) \) defines the position at time \( t \) seconds.
For Students In The Final Grade Of Secondary School In Kuwait

1. Find $\frac{dy}{dx}$ if $y = 2x^3 + 5x - 3$.

2. The displacement of a particle moving in a straight line, with respect to the origin is given at $t$ by the relation $x = 3t^3 - 2t$. Find the velocity of this particle at $t$.

3. The displacements $x_1$ and $x_2$ with respect to an origin of two particles moving in a straight line are given by $x_1 = t^2 + 2t$ and $x_2 = t^2 + 8$. When do the two particles meet?

4. Two particles moving in a straight line have the displacements $x_1$ and $x_2$ from the origin where $x_1 = t^3 - 7t - 1$ and $x_2 = t^2 - 2t$. Do these two points meet at $t = 3$?

5. Find $\frac{dy}{dx}$ when $y = 2ax^3 + 5x - 3$ (a being a constant).

6. The position of a particle moving in a straight line is defined at $t$ by $x = at^3 - 2t$, (a being a constant). Find the velocity of this particle at $t$.

7. The positions of two particles moving in a straight line, are defined at $t$ by $x_1 = t^2 + 2at$ and $x_2 = t^2 + a$, a being a constant $\neq 0$. When do they meet?

8. Two particles moving in a straight line have at $t$ displacements $x_1 = t^3 + 7a^2t - a^3$ and $x_2 = at^2 + 6a^2t$ with $a > 0$. Do they meet at $t = a$?
(P): 9. Two particles moving in a straight line have at t the displacements $x_1 = 2t^3 + 3t$ and $x_2 = 3t^3 + a$ (a being a constant). Find 'a' if the two particles have the same velocity when they meet.

*Throughout t, x(cm) and v(cm/sec) define the position and velocity after t seconds.
APPENDIX G

- EXPERIMENT I: CONVENTIONAL AND MODIFIED TREATMENTS (PRESENTATIONS) AND ACHIEVEMENT TESTS

- EXPERIMENT II: ORIGINAL ARABIC SHEET FOR EXERCISES IN 9.2.1.3
EXPERIMENT I:

**Task: k_1**

Presentations \( R_1 \) and \( \tilde{R}_1 \) For Task \( k_1 \) In S And S+U Respectively

Conventional Presentation \( R_1 = R_1(S) \). Group \( G_2 \) (Control).

Modified Presentation \( \tilde{R}_1 = \tilde{R}_1(S+U) \) Group \( G_1 \) (Experimental).

B_1: 1) Find the equation of the line that has the slope \( m \) and passes through the point \( N \) in each of the following cases.

i) \( m = 2, N = (3, 4) \)
ii) \( m = -\frac{2}{3}, N = (1, -2) \)
iii) \( m = -\frac{2}{5}, N = (2, -2) \)
iv) \( m = 3, N = (2, 3) \)

B_2: 2) Find the centre of each of the following circles:

i) \( A: x^2 + y^2 + x - 2y - 1 = 0 \)
ii) \( B: x^2 + y^2 - 4x + 5y + 8 = 0 \)
iii) \( C: x^2 + y^2 + 4x - y - 1 = 0 \)
iv) \( D: x^2 + y^2 + x - y - 3 = 0 \)
3) Find the slope of the radius passing through \( M \) in each of the following circles:

i) \( A: x^2 + y^2 - 2x + 4y - 4 = 0 \)
   \( M = (1, 1) \)

ii) \( B: x^2 + y^2 + 2x - 1 = 0 \)
    \( M = (-2, -1) \)

iii) \( C: x^2 + y^2 - 8y - 4 = 0 \)
    \( M = (2, 8) \)

iv) \( D: x^2 + y^2 + 6x - y - 2 = 0 \)
    \( M = (-6, 2) \)

4) Find the slope of the line \( l \) which is perpendicular to the line \( l \) which passes through \( M_1 \) and \( M_2 \) in each of the following cases:

i) \( M_1 = (2, 3) \) and \( M_2 = (-1, 4) \)

ii) \( M_1 = (1, -2) \) and \( M_2 = (3, 5) \)

iii) \( M_1 = (3, 3) \) and \( M_2 = (6, -2) \)

iv) \( M_1 = (-2, 3) \) and \( M_2 = (2, 6) \)

5) Find the slope of the tangent to the circle at \( M \) in each of the following cases:

i) \( A: x^2 + y^2 + x + y = 0 \), \( M = (-1, -1) \)

ii) \( B: x^2 + y^2 - 4x - 3y + 3 = 0 \), \( M = (1, 3) \)

iii) \( C: x^2 + y^2 + 2x - 1 = 0 \), \( M = (-2, 1) \)

iv) \( D: x^2 + y^2 - 2x + 2y = 0 \), \( M = (2, -2) \)
Achievement Test: T(k_1)

Solve all the following:

1. Find the equation of a line which passes through \(N = (1,3)\) with slope \(\frac{4}{5}\).

2. Find the centre of the circle \(A: x^2 + y^2 + x - 4y - 1 = 0\).

3. Let the circle \(B: x^2 + y^2 - x - y = 0\) and \(M = (1,1)\) belong to \(B\). Find the slope of the radius of \(B\) which passes through \(M\).

4. Let \(l\) be a line passing through \(M = (2, -5)\) and \(N = (-1, 3)\). Find the slope of \(l\) where \(|l|\).

5. Let \(E = (3, 7)\) be the centre of a circle \(A\), and \(M = (5, 8)\) belongs to \(A\). Find the slope of the tangent to \(A\) at \(M\).

6. A line \(l\), with slope \(\frac{4a}{5}\) passes through \(M = (1, 3a)\). Find the equation of \(l\), (\(a\) being a constant).

7. Find the centre of the circle, \(A: x^2 + y^2 + ax - 4y - 2a = 0\), (\(a\) being a constant).

8. Let \(M = (1, b)\) belong to the circle \(B: x^2 + y^2 - x - by = 0\). Find the slope of the radius which passes through \(M\), (\(b\) being a constant).
9. Find the slope of the line n perpendicular to the line l which passes through \( M = (2a, -5) \) and \( N = (-a, 3) \), (a being a constant \( \neq 0 \)).

10. Let \( M = (5, 8a) \) belong to a circle with centre \( I = (3, 7a) \). Find the slope of the tangent to this circle at \( M \), (a being a constant).
**Task: \( k_2 \)**

Presentations \( R_2 \) and \( R'_2 \) for Task \( k_2 \) in \( S \) and \( S+U \) Respectively

<table>
<thead>
<tr>
<th>Conventional Presentation ( R_2 = R_2(S) )</th>
<th>Modified Presentation ( R'_2 = R'_2(S+U) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group: ( G_1 ) (Control)</td>
<td>Group: ( G_2 ) (Experimental)</td>
</tr>
</tbody>
</table>

\( \hat{B}_1: \) 1) If the image of \((x, y)\) under a certain transformation is \((2x, 3y)\), then for the same transformation, fill in the blanks.

- i) \((2, 3) \rightarrow (\ldots, \ldots)\)
- ii) \((\frac{1}{2}, \frac{1}{3}) \rightarrow (\ldots, \ldots)\)
- iii) \((-2, 6) \rightarrow (\ldots, \ldots)\)
- iv) \((4, 2) \rightarrow (\ldots, \ldots)\)

\( \hat{B}_2: \) 2) If the image of \((x, y)\) under a certain transformation \( T \) is \((4x, -2y)\), then fill in the blanks to show the points which have been transformed under \( T \).

- i) \((\ldots, \ldots) \rightarrow (8, -6)\)
- ii) \((\ldots, \ldots) \rightarrow (-4, 4)\)
- iii) \((\ldots, \ldots) \rightarrow (10, 6)\)
- iv) \((\ldots, \ldots) \rightarrow (-8, 5)\)
3) If $T$ is a transformation, then find in each of the following cases the pre-image of the image $(x, y)$:

i) $T: (x, y) \rightarrow (3x, 4y)$

ii) $T: (x, y) \rightarrow (2x, -y)$

iii) $T: (x, y) \rightarrow (-x, 3y)$

iv) $T: (x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$

iv) $T: (x, y) \rightarrow (ax, 3y)$, $a \neq 0$

iv) $T: (x, y) \rightarrow \left(x, \frac{1}{2}y\right)$, $b \neq 0$

4) Give a general description, using set notation, for each of the following lines. It is sufficient to write the general form: $\{(x, y) : y - y_0 = m(x - x_0)\}$

i) Slope of $l = \frac{2}{3}$ and $(2, 3) \in l$.

ii) The line passes through $(1, 2)$ and $(3, 4)$

iii) The line passes through $(3, -1)$ and is perpendicular to $2x - y - 1 = 0$.

iv) The line passes through $(2, 5)$ and is perpendicular to a line whose slope is $\frac{3}{5}$.

5) Describe in set-language the image line $\tilde{l}$ of a line $l$ under the transformation $T$ in each of the following cases:

i) $T: (x, y) \rightarrow (x, -y)$

ii) $T: (x, y) \rightarrow (\frac{1}{2}x, 3y)$

iii) $T: (x, y) \rightarrow (-x, 2y)$

iv) $T: (x, y) \rightarrow (2x, -3y)$
Achievement Test: $T(k_2)$. Solve all the following:

1) Let the function $T: (x,y) \rightarrow (-2x,y)$ be a transformation. Find the image of $(3,-2)$.

2) Let the function $T: (x,y) \rightarrow (2x,7y)$ be a transformation. Find the pre-image of $(4,10)$.

3) If $T$ is the transformation: $(x,y) \rightarrow (3x,2y)$. Find the pre-image of $(x,y)$.

4) Use set-language to describe a line which passes through the point $(2,0)$ with a slope $m = -2$.

5) Let $l$ be a line and $T$ the transformation $(x,y) \rightarrow (2x,4y)$. If $\hat{l}$ is the image of $l$ under $T$, then fill in spaces: $\hat{l}=((x,y):(\ldots,\ldots)\ldots l)$.

6) Let $T$ be the transformation $(x,y) \rightarrow (-2ax,y)$. Find the image of $(3,a)$.

7) Let $T$ be the transformation $(x,y) \rightarrow (2x,ay)$. Find the pre-image of $(a,1)$. $(a \neq 0)$.

8) Let $T$ be the transformation $(x,y) \rightarrow (3ax,2ay)$. Find the pre-image of $(x,y)$. $(a \neq 0)$. 
9) A line $l$ with slope $2a$ passes through $M = (2a, 0)$. Use set-notation to describe $l$.

10) Let $T$ be the transformation $(x, y) \rightarrow (2ax, 4y)$ and let $\tilde{l}$ be the image of the line $l$ under $T$. Fill in spaces:
$\tilde{l} = \{(x, y): (\ldots, \ldots) \ldots l \}$. ($a \neq 0$).
Original Arabic text from which the items of task k (conventional) in Experiment II were chosen. (Starred items were not included).