DEVELOPMENT AND APPLICATION OF
THE FINITE ELEMENT METHOD TO
THE VIBRATION OF BEAMS

A Thesis for the Degree of
Doctor of Philosophy
in the
Faculty of Engineering
of the
Surrey University.
by
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The First Part of this thesis is concerned with the theoretical formulation, critical examination and practical applications of the Finite Element approximation in the vibration problems of elastic beams idealising the actual turbine blades.

The method is presented in a generality encompassing a large body of useful linear vibration theories of beams. The basic and refined element models are introduced, and the convergence properties of each model studied. A rigorous proof of the convergence of the Finite Element solutions to the exact values is included in the case of torsional vibrations.

The use of refined models is advantageous, especially in cases where the modal curves are of some complexity, e.g., vibrations at higher modes and beams with certain boundary conditions.

The general non-dimensional forms of element dynamic-stiffness matrices for linearly pre-twisted beams are given. Numerical results obtained with polynomial approximation in bending-bending and bending-bending-torsion vibrations of beams, where the effects of shear and rotatory inertia may not be negligible and linear taper may be present, show good convergence characteristics. Satisfactory results are obtained with the use of only a few number of elements.

The application of the method to vibrations of beams carrying concentrated masses and supported on elastic springs is briefly demonstrated. Representative results show good agreement with the exact values.

The Finite Element Method provides strong and stable convergence characteristics. It is simple to apply and ideally suited to digital computers. On the other hand, the versatility of the method, in the treatment of boundary conditions, alone is a considerable advantage over the other more conventional numerical methods.

In the Second Part of the thesis the Finite Element Method is applied to investigate the vibrational characteristics of uniform and tapered slender beams with, or without, pre-twist, and having various boundary conditions. The first four frequency ratios and modal shapes are presented for the following ranges of problem parameters:

Pre-twist = 0 to 90 degrees, ratio of flexural rigidities = 1 to 256,
depth and width taper parameters = -0.5 to 0.5. The boundary conditions considered are those obtainable from combinations of free, pinned and clamped conditions at the ends of a beam.

Theoretical results agree closely with experimental results presented in the thesis.
ACKNOWLEDGEMENTS

The author would like to express his appreciation to his supervisor J. Thomas, B.Sc., B.Sc.(Eng), for his suggestions and encouragement that guided the preparation of this thesis; also to Professor J. M. Zarek and Professor W. Carnegie for permission to carry out the work in the Department of Mechanical Engineering at the University of Surrey.

The investigation presented in the thesis is a part of the turbine blade vibration study being carried out at the University of Surrey under the leadership of Professor W. Carnegie.
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Finite Element Analysis of Vibration Problems of Beams

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**NOMENCLATURE**

(The symbols that are not included in the following list are defined in the text)

- \( a_j \)  
  Taper constants for area, \( j = 0, 1, 2 \).

- \( A \)  
  Diagonal matrix operator for strain energy.

- \( b_j \)  
  Taper constants for \( I_{G_X} \), \( j = 0, 1, 2, 3, 4 \).

- \( B \)  
  Diagonal matrix operator for kinetic energy.

- \( c_j \)  
  Taper constants for \( I_{G_Y} \), \( j = 0, 1, 2, 3, 4 \).

- \( c_x, c_y \)  
  \( e_x / h_0 \), \( e_y / h_0 \), respectively.

- \( C \)  
  Displacement transformation matrix (this is a supermatrix with elements \( c_{ij}, j = 1, 2, \ldots, N \)).

- \( C_j \)  
  Square matrix with partitioned rows \( C^r_j, r = 1, 2, \ldots, R_j \).

- \( G^r_j \)  
  Matrix with partitioned rows \( g^r_{j,k}, k = 0, 1, \ldots, G_{j,k} \).

- \( D_j, D \)  
  Reciprocals of matrices \( C_j \) and \( C \), respectively.

- \( e \)  
  Supervector with elements \( e_j, j = 1, 2, \ldots, N \).

- \( e_j \)  
  Vector with elements \( e_{ji}, i = 1, 2, \ldots, m_j \).

- \( e_{ji} \)  
  Undetermined coefficients associated with \( g_{ji} \).

- \( e_x, e_y \)  
  Co-ordinates of the shear centre in the xy frame.

- \( e_x, e_y \)  
  Co-ordinates of the shear centre in the XY frame.

- \( E \)  
  Modulus of elasticity

- \( f_{ji} \)  
  The \( i \)th function extending the unit displacement functions associated with the argument function \( \sigma_j \).

- \( g_{ji} \)  
  The \( i \)th element of a set of functions approximating the argument function \( \sigma_j, i = 1, 2, \ldots, m_j \).

- \( g_{ji} \)  
  The \( i \)th unit displacement function.

- \( G \)  
  Supermatrix with elements \( g_{ji}, j = 1, 2, \ldots, N \).

- \( G_j \)  
  Modulus of rigidity.

- \( G_{j} \)  
  Row matrix with elements \( g_{ji} \).

- \( G_{j,k} \)  
  The value of the \( k \)th derivative of \( G_j \) at node \( r \).
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Undetermined constants associated with \( f_{ji} \).

Characteristic dimensions of sections in X and Y directions, respectively.

Values of \( h_x \) and \( h_y \) at the node \( i \), respectively.

Values of \( h_x \) and \( h_y \) at the root.

Values of \( h_x \) and \( h_y \) at the tip.

Hamiltonian.

Element taper parameters. \((=\frac{h^x_{i+1}}{h^x_i}, \frac{h^y_{i+1}}{h^y_i})\)

Beam taper parameters. \((=\frac{h^x_{n}}{h^x_0}, \frac{h^y_{n}}{h^y_0})\)

Integers.

A cross-sectional constant defined as \( I_T - I_{P_0}^2 / C_A \).

Second moments of area about x and y axes, respectively.

Second moments of area about X and Y axes, respectively.

First product of area in the xy plane.

Values of \( I_{G_x} \) and \( I_{G_y} \) at the node \( i \), respectively.

Values of \( I_{G_x} \) and \( I_{G_y} \) at the root.

Taper functions for \( I_{G_x} \) and \( I_{G_y} \), respectively.

Polar moment of area about the shear centre.

A cross-sectional constant \((= \int_{\partial A}(u^2 + v^2)^2 \, d\Gamma)\).

Cross-sectional constants \((= \int_{\partial A}(u^2 + v^2) \, d\Gamma, \int_{\partial A} V(u^2 + v^2) \, d\Omega)\).

Cross-sectional constants \((= e_x I_{P_0} + I_{T_U}, e_x I_{P_0} + I_{T_U})\).

St.-Venant torsion constant.

A square matrix defined by the equation \((1.12)\).

Timoshenko shear coefficient.

Stiffness matrix.

Supermatrix formed from all element stiffness matrices.

Length of beam element.

Length of the beam.
A square matrix defined by the equation (1.13).

$m_j$ Number of undetermined constants $\epsilon_{ji}$ occurring in the linear combination approximating argument function $\Theta_j$.

$M$ Inertia matrix.

$M$ Supermatrix formed from all element inertia matrices.

$n$ Number of elements.

$N$ Number of coupling co-ordinates.

The number of degrees of freedom given to the node $r$ for $\Theta_j$.

$\overline{q}_{jr}$ Value of $\overline{q}_{jr}$ for a basic node.

$q_j$ Vector of element nodal forces.

$\overline{q}$ Supervector of element force vectors.

$r$ Integer.

$R$ Rotory inertia parameter ($=\frac{1}{G_X}/\alpha l^2$).

$R_j$ Number of nodes associated with the displacement $\Theta_j$.

$s$ Spring modulus.

$S$ Shear displacement parameter ($=k G l^2/\alpha l^2$).

$t$ Time.

$T$ Kinetic energy.

$u,v,z$ Fixed co-ordinate frame through the shear centre.

$U,V,z$ Principal co-ordinate frame through the shear centre.

$u_b,u_s$ Bending and shear deflections in the $u$ direction.

$U_b,U_s$ Bending and shear deflections in the $U$ direction.

$v_b,v_s$ Bending and shear deflections in the $v$ direction.

$V_b,V_s$ Bending and shear deflections in the $V$ direction.

$W$ Work done by external forces.

$x,y,z$ Fixed co-ordinate frame through the centroid.

$X,Y,z$ Principal co-ordinate frame through the centroid.

$Z$ Dynamic-stiffness matrix.

$Z$ Supermatrix of all element dynamic-stiffness matrices.

$\alpha, \beta$ Transformation matrices for strain and kinetic energy, respectively.

$\alpha$ Cross-sectional area.

$\alpha_i$ Cross-sectional area at node $i$.

$\alpha_0$ Cross-sectional area at the root.
$\sigma^*$ Area taper function

$\delta_0, \delta_0'$ A multiple of the element frequency parameter.

$\varepsilon$ Variational operator.

$\varepsilon_x, \varepsilon_y$ Position of concentrated mass.

$e_{x}/\ell, e_{y}/\ell$, respectively.

$\Phi$ Angle of twist per unit length.

$\Phi_1, \Phi_2$ Angle of twist of an element.

$\Phi_3, \Phi_4$ Angle of twist of the beam.

$\eta, \xi, \kappa$ Non-dimensional principal frame ($=V/\ell, U/\ell, Z/\ell$)

$\zeta, \psi$ Frequency parameters for a uniform element ($=\omega^2 \mu \ell^4/\nu_0 G_X$).

$\lambda$ Frequency parameter for a uniform beam ($=\omega^2 \mu \ell^4/\nu_0 G_X$).

$\lambda_1$ Frequency parameter for a tapered element ($=\omega^2 \mu \ell^4/\nu_0 G_X$).

$\lambda_\theta$ Frequency parameter for a tapered beam ($=\omega^2 \mu \ell^4/\nu_0 G_X$).

$\rho$ Torsional element frequency parameter ($=\omega^2 \mu \ell^2/\nu_0 G_J$).

$\mu$ Density.

$\omega$ Circular frequency.

$\omega^2 / \nu_0$ Frequency parameter for a uniform beam ($=\omega^2 \mu \ell^4/\nu_0 G_X$).

$\Omega^4 / \nu_0$ Frequency parameter for a tapered beam ($=\omega^2 \mu \ell^4/\nu_0 G_X$).

$\Omega^{2}_\theta$ Torsional beam frequency parameter ($=\omega^2 \mu \ell^2/\nu_0 G_J$).

$\Sigma$ Transformation matrix.

$3.1415927\ldots$: Also, position of spring support.

$\pi$ Polar radius of gyration about shear centre ($=(R/\ell)^2$).

$\pi_0$ Polar radius of gyration about centroid ($=(R^2/\ell^2-2\mu \ell^2/\nu_0 G_J)^{1/2}$).

$\rho$ Torsional stiffness parameter ($=\mu \ell^2/\nu_0 G_J$).

$\Theta$ Angle of twist at point $z$.

$\theta$ Torsional displacement.

$\sigma_j$ Argument functions occurring in the energy equations, $j=1, 2, \ldots, N$.

$\sigma_{r, k}$ The $k$th derivative of $\sigma_j$ at node $r$.

$\sigma_j^r$ Vector of argument functions $\sigma_j = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$.

$\mathcal{U}$ Strain energy.
\( \nu \) Ratio of flexural rigidities for a uniform beam.

\( \nu_i \) Ratio of flexural rigidities at the \( i \)th node.

\( \nu_0 \) Ratio of flexural rigidities at the root.

\( \psi_U, \psi_V, \psi_T \) Cross-sectional parameters \( = \frac{I_U}{I_o G_x}, \frac{I_V}{I_o G_x}, \frac{I}{I_o G_x} \).

\( \psi_{j,k}^T \) Approximated values of \( \psi_{j,k}^T \).

\( \psi_j^T \) Vector having elements \( \psi_{j,k}^T, k=0,1,\ldots, j \).

\( \psi_j \) Supervector with elements \( \psi_j^T, r=1,2,\ldots, R_j \).

\( \psi_j \) Supervector with elements \( \psi_j^T, j=1,2,\ldots, N \).

\( \odot \) Symmetric coupling matrix for strain energy.

\( \mathcal{U} \) Symmetric coupling matrix for kinetic energy.

\( \Xi \) Equivalent transfer matrix.

\{ \} Denotes column matrix.

[ ] Denotes row matrix.

\| \| Denotes diagonal matrix.

(\_\_\_\_) Denotes matrix transposition.

(\_\_\_\_\_) Denotes matrix inversion.

(\_\_\_\_\_\_) Denotes differentiation with respect to time.

(\_\_\_\_\_\_\_) Denotes differentiation with respect to \( z \), or \( K \).

(\_\_\_\_\_\_\_) Denotes differentiation values.

Notes: i. Right-subscripts \( Y \), \( X \) and \( \Theta \) are used with element matrices to indicate the nature of vibration, e.g., \( (\_\_\_\_)_Y \) implies independent bending vibrations in the \( Y \) direction, and \( (\_\_\_\_)_{XX} \) implies that vibrations in the \( Y \) and \( X \) directions are coupled.

ii. Left-subscripts \( R, S \) and \( SR \) denote that the relevant matrices take into account the effect of rotatory inertia, shear deflection, and both, respectively.

iii. \( (~) \) denotes that the effect of linear taper is considered.

iv. A superscript in brackets denotes the number of internal nodes considered.
INTRODUCTION

The ability to determine the modes of vibration of turbine and compressor blading is of great importance. Knowledge of the natural vibration characteristics of blades is required in the design stage in predicting the likelihood of resonant vibrations.

The blades used in practice are of non-uniform cross-section and pre-twisted, and have non-symmetric cross-sections, usually of the aerofoil type. The moving blades, on the other hand, are sometimes banded in packets by shrouding or lacing, and subject to centrifugal forces. Further, the blade vibration is also influenced by the type of root fixing, and the disc vibration. These factors considerably complicate the determination of the theoretical natural vibration characteristics of blading, and pose problems that need to be investigated separately.

The subject of the present thesis is concerned with a particular branch of this field, namely the free vibrations of single blades where the centrifugal effects are negligible.

A linearly pre-twisted beam of non-symmetric and gradually varying cross-sections represents an adequate model of the actual blades. The dynamic motion of such beams is usually referred to as bending—bending—torsion vibrations, for the bending in the two principal directions are coupled due to the presence of initial twist, and the torsional vibrations are coupled with the bending vibrations due to the non-collinearity of the centroidal and shear centre axes. The mathematical theory of bending—bending—torsion vibrations covers a large class of beams of simpler geometries.

The increasing complexity of the turbine blades and the accuracy demanded in practical applications at the present time require a comprehensive analysis of the vibration problems of the pre-twisted beams. Specifically, the shear and rotatory inertia effects in bending, and the longitudinal stress and inertia corrections in torsion, and cross-sectional non-uniformities have to be taken into account. On the other hand, a necessity for examining various boundary conditions arises from the fact that many turbine blades are restrained at the tip by shrouding, and the effect of this can be approximated by considering that end as pinned or clamped. Also, at low speeds the root fixing is of the type pinned rather than clamped for certain blade designs. The effect of lacing should also be brought
into the analysis. As a first approximation, this can be done by considering the effect of lacing as that of a concentrated mass or an elastic spring, according to the packet modes.

The object of this thesis is the development of a Finite Element procedure for the numerical treatment of the general coupled vibration problems of pre-twisted beams, and the investigation of the effects of various boundary conditions on the natural vibrational characteristics of pre-twisted symmetric beams with or without linear taper. The scope of the thesis, however, extends to the analysis of the effects of shear and rotatory inertia in bending-bending and bending-bending-torsion vibrations of uniform and non-uniform beams, and also includes a brief demonstration of the application of the present method to beams carrying concentrated masses and supported on elastic springs.

The study of the subject in general falls into three distinct parts, namely, 1) the establishment of the continuum theory, 2) the numerical solution, 3) the experimental confirmation, and has received the attention of a number of investigators in the case of cantilever beams.

The equations of bending-bending vibrations are derived by several authors. DiPrima and Handelman (1)* choose a moving triad, in the directions of the principal axes of cross-sections and the longitudinal axis of the beam, whose rate of rotation per unit length is equal to the pre-twist of the beam per unit length. Then, the linear angular distortions of the beam are judiciously expressed as the scalar product of the corresponding forces and certain dyadics, whose terms are found by inspection from the theories of untwisted beams. Substitution of these relations into the vectorial force and moment equilibrium equations gives the required equations of bending-bending vibrations in the principal frame. Troesch, Anliker and Ziegler (2) presented a similar** but more intricate method of derivation. Martin (3), on the other hand, pointed out that the

* List of references is given in the Bibliography.
** In the sense that a direct use of the constitutive equations of the elementary beam theory rather than the fundamental principles and assumptions leading to such equations is made.
relevant equations could easily be obtained from the Kirchoff-Clebsch theory of bending of naturally twisted beams. The general Kirchoff-Clebsch theory is discussed in detail in the Chapter 18 of Love's "Treatise on the Mathematical Theory of Elasticity", and assumes that the components of the flexural moment in the principal directions are connected with the corresponding curvatures of the centroidal line of the beam as in the Bernoulli-Euler theory.

The ordinary approximate theory of bending-bending-torsion vibrations could also be deduced from the classical Kirchoff-Clebsch theory, where the torsional moment is connected to the twist as in the St.-Venant theory. However, the effect of the twisted beam fibres on the torsional stiffness of the beam, as well as the effect of the cross-sectional shapes must be considered. Such a derivation, in relation to pre-twisted beams with no axis of symmetry, is given by Carnegie, who employs the Euler equations in performing the variations indicated in the application of the Hamilton's Principle.

More rigorous theories of bending-bending-torsion vibrations of slender beams are usually based on finding an expression for the longitudinal strain in a deformed beam fibre. The tangents to the space curves representing the beam fibres are assumed to be normal to the beam sections before and after deformation. By vector algebra it can be shown for a helix, representing a linearly twisted beam fibre, that such a condition is approximately true before the deformation if \( c^2 \Phi^2 \ll 1 \), where \( c \) is the maximum dimension of a cross-section and \( \Phi \) is the pre-twist per unit length of the beam. Subsequent analysis then indicate that this assumption also linearises the expression for the longitudinal strain. Houbolt and Brooks considered pre-twisted (and rotating) beam having one axis of symmetry in this way, and showed that the effect of the twisted beam fibres on the bending-bending-torsion motion is not only to increase the torsional stiffness of the beam, but also to introduce an elastic coupling between the flexural and torsional deformations. Recently, Montoya derived the equations of the bending-bending-torsion vibrations for beams having no axis of symmetry, essentially on the similar lines.

In the works mentioned in the preceding paragraph the equations of motion are obtained by the substitution of the derived inertia-loadings, and the elastic forces and moments, obtained from the integration of the longitudinal strain expression, into the general force and moment equilibrium equations of a beam element. A more
compact analysis can be carried out by the application of Hamilton's Principle*. The strain energy and kinetic energy defining the bending-bending-torsion vibrations of slender beams are derived in Appendix 1. The expression for the longitudinal strain coincides with that given by Montoya(6). The variational procedure employed in the application of the Hamilton's Principle is illustrated in Appendix 2 in relation to general forms** of energy equations occurring in the vibration problems of beams. The actual differential equations are, however, not required in the present analysis.

It is well known that the Bernoulli-Euler theory of bending vibrations is inadequate for higher modes, and also inadequate where the effects of cross-sectional dimensions on the beam frequencies cannot be neglected. Towards this end, a number of higher order theories have been suggested by various authors. Rayleigh(6) introduced the effect of rotatory inertia, and Timoshenko(9) extended it to include the effect of transverse shear displacements. Prescott(10) and Volterra(11) suggested, by independent reasoning, various Timoshenko type beam models. At the present, the Timoshenko theory is considered satisfactory due to the close agreement it gives with the exact solutions obtained by the Pochammer theory(12). However, it should be noted that in the Timoshenko theory neither a quantitative measure of thickness, nor a rigorous definition of the shear coefficient is available. Recently, Cowper(13) deduced the Timoshenko beam equations by the integration of the three-dimensional elasticity equations through the use of the concepts of average displacements and average rotations over the beam sections***. Thus, a new formula emerged for the shear coefficient, which did not agree with its conventional definition, namely, the ratio of the average shear stress to the maximum shear stress at the centroid, as given from a parabolic shear stress distribution. For example, for a rectangular section Cowper's calculations gave $k=10(1+\mu)/(12+11\mu)$, where $k$ is the shear coefficient and $\mu$ the Poisson's ratio. However, this rigorous work

* Courant and Hilbert(7) state that "the laws of motion are most simply formulated in terms of Hamilton's variational principle. Carnegie also advocates a similar approach.

** See Article 1.1.

***These concepts were first used, to the author's knowledge, by Prescott(10). Later, it became widely known in the derivation of Reissner(62) type plate theories.
does not answer the point put forward by various authors\(^{(14)}\), namely, the dependence of \(k\) on the modes of vibration. This is clearly indicated, for the shear constant, in Cowper's type derivations, emerges as a result of assumed distributions of static shear stress, which in the dynamic case is not the same for all modes. For vibrations of low frequencies Prescott shows that the shear stress distribution is parabolic. The value of \(k\) corresponding to parabolic shear stress distribution can be obtained from the formulas given in Ref.\(^{(15)}\) by putting Poisson's ratio equal to zero, e.g., for rectangular beams \(k=5/6\). It is also interesting to note that this value agrees closely with that based on high frequency vibration modes, namely \(k=\pi^2/12\) (see Ref.\(^{(15)}\)). Hence, it is hoped that the inequality \(\pi^2/12 \leq k \leq 5/6\) brackets the correct value of the shear coefficient for rectangular beams.

A number of correction terms can also be added to the classical St.-Venant theory of torsional beam vibrations. For a cantilever beam the St.-Venant theory predicts that the \(n\)th frequency is \((2n+1)\) times greater than the fundamental frequency. But, Barr\(^{(15)}\) points out that the experimental results obtained by a number of authors indicate a departure from this proportionality; generally the \(n\)th frequency is found to be greater than \((2n+1)\) times the fundamental frequency. This is attributed to the fact that St.-Venant theory takes into account only of shear stresses and inertia forces acting in the plane of cross-sections, and the longitudinal inertia and stress arising from the warping motion are neglected. The effect of longitudinal stress was first considered by Timoshenko in relation to I-section thin walled beams\(^*\). Barr\(^{(15)}\) extended this to beams other than thin walled sections, and also added the effect of longitudinal inertia correction. The resulting theory, stated to be a counterpart of the Timoshenko theory of flexure, yielded good agreement between the theoretical and experimental results for the rectangular section beams considered.

The consideration of the rotatory inertia and shear displacement effects in flexure, and longitudinal stress and inertia effects in torsion enables a more accurate prediction of the frequencies and mode shapes of slender beams, and at the same time extends the range of applicability of the corresponding elementary theories to beams which cannot be classified as slender. However, it must be emphasised that

\* See footnote 2 in Ref.\(^{(16)}\)
a quantitative measure of "thickness" of beams with which the higher order theories are physically associated cannot be predicted theoretically at the present time.

The Timoshenko type flexure and torsion theories of bending-bending-torsion vibrations are presented by Carnegie\(^{(17,18,19)}\). The rotatory inertia term was also given in the work of Houbolt and Brooks\(^{(5)}\) in relation to pre-twisted (and rotating) beams having one axis of symmetry. The cross sections were assumed to rotate about the shear centre by an arbitrary small angle. This differs from Carnegie's assumptions\(^{(17)}\), namely, that the cross-sections rotate about the centroid by angles given by the first derivatives of the centroidal deflections, the shear displacements having no rotational components. In Appendix 1 of the present thesis the kinetic energy of the beam due to rotatory motion of its cross-sections are also derived, yet by another set of assumptions that the cross-sections rotate about the centroid by angles given by the first derivatives of the flexural displacements of the shear centre. The potential energy due to shear displacements coincides with that given by Carnegie. The effects of longitudinal stress and inertia in torsion are neglected in the present analysis.

The general vibration problem of pre-twisted beams is clearly a complex one, and an electronic digital computer has to be employed in order to get numerical results. Therefore it is important to select a method of solution which is particularly suitable for machine computation, and also the problem under consideration.

The application of a number of standard methods of numerical solution to particular cases of the coupled vibration problems of beams have been presented by several authors. The bending-bending vibration of uniform cantilever beams has received most of the attention.

The fundamental frequencies can be obtained within the accuracy demanded in engineering practice by the Rayleigh's Principle. Di Prima and Handelman\(^{(1)}\) calculated the fundamental frequencies for cantilever beams of ratio of rigidities 48, 64, 144 and pre-twist \(\pi/16, \pi/8, \pi/2\). Fourth order polynomials, satisfying zero deflection and slope conditions at the clamped end, were chosen as trial functions for the displacements in the principal directions. The number of undetermined constants was then reduced to one by certain assumptions. Carnegie\(^{(4)}\) obtained explicit formulae for the fundamental frequency of bending-bending and bending-bending-torsion vibrations of uniform cantilever beams through
the use of exact static deflection curves (20).

The application of the Myklestadt method is indicated by Rosard (21). Frequencies of cantilever beams of ratio of flexural rigidities in the range 16 to 144 and pre-twist angles up to 40° were obtained and compared with the experimental values, which are subject to criticism (22).

The Myklestadt method (23) is essentially an extension of the well-known Holzer's method, devised for the analysis of torsional systems composed of a number of flywheels on massless flexible shafts, to flexural vibrations. It has enjoyed considerable amount of popularity before the advent of digital computers, due to its suitability to hand computation. The modern version of the Holzer-Myklestadt method is known as the Transfer Matrix method, which is essentially a matrix formulation of the former, but it is not restricted to lumped mass idealization. A comprehensive account of the Transfer Matrix method is given by Pestel and Leckie (24). Briefly, it consists of segmenting the actual beam into a number of elements, and expressing the forces and displacements at one end of an element in terms of respective quantities at the other end through a "transfer" matrix. When this is done for all elements, the transfer relation between the ends of the whole beam is found by the multiplication of the element transfer matrices in a prescribed order. The imposition of the boundary conditions then results in a frequency determinant, which is solved by trial and error. This procedure was applied by Targoff (25) to bending-bending vibration of uniform clamped-free beams. In the finitisation procedure, masses were lumped at the centres of elements which were considered as having no twist across their length, their orientation being those of the tangent planes at mass stations. The flexural rigidity of an element was considered constant on each side of the mass station. It was stated that, the use of transfer matrices obtained by this idealization yields the first and second mode frequencies within ±2% and ±4% error, respectively, the number of elements being 10.

Martin (5) presented solutions to the equations of bending-bending motion of cantilever beams, treating the pre-twist as a first order perturbation. Since the solutions of the pre-twisted beam equations will exist at zero pre-twist, the eigenvalues and eigenvectors can be expanded by Taylor's series about this point. The eigenvalues will be unchanged if the sense of the initial twist is changed.
Hence it is concluded that the eigenvalues of the pre-twisted beams are even functions of the pre-twist angle. Similarly, the eigenfunctions corresponding to the broadside and edgewise displacements are taken as even and odd functions of the pre-twist, respectively. By the substitution of these expansions into the differential equations of motion and the boundary equations, and equating the like powers of the pre-twist angle a new set of equations are obtained. From the solutions of these equations, the constant functions appearing in the Taylor expansion formula can be determined. The calculations are straightforward but laborious and long. It is shown that the derived frequency ratio expression becomes indeterminate for critical values of the ratio of flexural rigidities, i.e., those values equal to the ratio of any two eigenvalues of the untwisted beam, and in the region of these values care should be taken in the numerical calculations for securing accuracy. Although only cantilever beams were considered in Ref. (3), the method is clearly applicable to other boundary conditions, too. However, in a search for a versatile method of numerical solution this approach should be dismissed.

As the digital computers become widely available, more comprehensive studies of the bending-bending vibrational characteristics of uniform beams appeared in the literature. Slyper (27) derived the relevant orthogonality condition by the application of the Reciprocal Theorem, and hence formulated a Stodola method of mode iteration. Frequency ratios of cantilever beams of ratio of flexural rigidities 4 to 256, and pre-twist up to 180° were presented and compared with the experimental values obtained by the author and also by other investigators. It was concluded that by allowing at most 25 iterations for each mode the first 6 frequencies could be calculated to about 5% of the values determined experimentally. The first five modal patterns of a beam of ratio of flexural rigidity 64, and angle of twist 90.9° were given and compared with the experimental results. In the application of the Stodola method, one starts from a trial deflection curve and then calculates the deflection resulting from the inertia loading corresponding to an arbitrary unit frequency. In this way one obtains a sequence of deflections, which can be shown to converge to the fundamental mode shape of the beam, and the ratio of the nth derived deflection curve to the (n+1)th derived curve converges in some average sense to the square of the corresponding frequency. In the determination of the higher modes
the use of the orthogonality relation is made for starting the iteration. The frequencies and mode shapes of the beam are thus obtained in ascending order. The sequence of improved deflections are obtained by the numerical integration of the moment equations resulting from the loading on the beam under the appropriate boundary conditions, and the substitution of these into the integral forms of the constitutive equations. However, under this formulation the Stodola method is not particularly suitable to machine computation, and the convergence is usually slow. For an account of the modern version of the Stodola method the reader is referred to Ref.28.

Analytical solutions were given by Anlíker and Troesch29, who also considered a number of boundary conditions apart from the clamped-free case. The imposition of the boundary conditions at both ends of the beam results in a 8x8 frequency determinant whose terms depend in a very complicated way through the roots of an eighth order equation, on the angle of twist, the ratio of principal flexural rigidities, and the frequency parameter. Since the roots of the characteristic equation associated with the differential equations were complex, it was found convenient to employ complex arithmetic in the iterative calculation of the zeros of frequency determinant. First four frequency values were plotted against the angles of pre-twist up to 450° for ratio of flexural rigidities 4, 25, and 256. The case of infinite ratio of flexural rigidities was presented previously by Troesch, Anlíker and Ziegler2.

For a uniform beam the exact frequency determinant can theoretically be derived, since the associated differential equations of motion are of the type with constant coefficients. However, the elements of this determinant are highly complicated hyperbolic and trigonometric functions, whose arguments depend on the frequency values and the physical beam parameters through the roots of the characteristic equation, which is, effectively, of the fourth order in bending–bending, and of the fifth order in bending–bending–torsion. The analysis of the bending–bending vibrations is, therefore, relatively simple, for analytical formulae exist for the determination of the roots of biquadratic equations. However, it is well known that no such explicit formulae can be found for fifth and higher order equations. Therefore, an additional iterative scheme has to be included in the calculations. On the other hand, a sophisticated iterative method has to be employed in the determination of the zeros of the frequency determinant. The elementary evaluation of the determinant
is unsatisfactory for higher modes of vibration, for then exponential functions with large arguments appear, and precautions have to be taken to prevent loss of accuracy. This fact is also noted in Ref. (24).

The analytical solutions of vibration problems of uniform beams can be considerably simplified by the incorporation of formulae for the roots of cubic and biquadratic equations in the cases of bending-torsion and bending-bending vibrations, respectively. Although these formulae are classical in the theory of equations (30), their use are not mentioned in works where they could have been applied. Mendelson and Gendler (31) proceed to solve the bending-torsion equations by first transforming them into six first order equations, and then determining the elements of the matrizeant through the use of Sylvester's Theorem. In this process the characteristic equation of the problem appears, and its roots are left arbitrary. The resulting frequency determinant for a cantilever beam is then very complex. In Appendix 3 of the present thesis a much simpler determinant is presented. The simplification is brought about by the use of Descartes' Rule of Signs (30) and the Cardan's formulae (50) for the roots of a cubic equation. These enable a complete determination of the nature and form of the roots of the characteristic equation. In the bending-bending vibrations, on the other hand, similar considerations through the use of Ferrari's method (30) for biquadratic equations may be valuable. Such an analysis, however, is not mentioned in Ref. (29).

The bending-bending-torsion vibrations of cantilever beams have recently occupied the attention of a few investigators in this field. Carnegie, Dawson, and Thomas (32) presented the application of the Runge-Kutta numerical integration method to a uniform cantilever beam. This method, briefly, consists of transforming the three coupled differential equations of the motion to ten first order equations, which are then integrated by the Runge-Kutta procedure, for an assumed value of the frequency, under the five imposed boundary conditions at the clamped end of the beam, and giving the remaining five functions the values of unity in turn while the other four are set to zero. Then, the five boundary values thus found may be combined linearly to obtain the values of the functions at the free end for the corresponding assumed frequency. Since these must satisfy the boundary conditions at the free end, a 5x5 determinant is obtained. This determinant must vanish for the correct value of the frequency. Once the latter is determined, the corresponding values of the functions along
the beam can be obtained by back substitution. The authors considered an aerofoil section beam of pre-twist angles in the range $0^\circ$ to $90^\circ$, for which the first five frequencies were determined and compared with experimental values. Montoya also formulated independently, a similar method of solution, and presented the first seven theoretical frequencies of an actual turbine blade which was highly twisted and also non-uniform. In the solutions the effect of the elastic coupling of bending and torsional deformations due to the twisted nature of the beam fibres was taken into account. Agreement with the experimental frequencies was shown to be satisfactory.

The Runge-Kutta method of the numerical integration of systems of first (or, second) order linear differential equations by frequency iteration is, in general, preferred to the Stodola type of mode shape iteration. The accuracy of the higher frequencies are not dependent on the previously calculated modes, and the frequencies, hence the mode shapes, can be calculated in any order desired. The step size required to get satisfactory results is usually small, and must be adjusted for a particular problem and mode of vibration.

The effects of shear and rotatory inertia on vibrational characteristics of uniform classical beams have been thoroughly examined. A large number of authors contributed to this end. A comprehensive exact treatment is given by Huang, who derived the frequency and mode shape equations of Timoshenko beams for the six standard boundary conditions. First five frequencies and mode shapes (also, moments and shear forces) were tabulated for a wide range of shear and rotatory inertia parameters.

The shear and rotatory inertia parameters are defined by the relations

$$ S = k G J / E I X^2, \quad R = I J / G X^2, $$

respectively. The meanings of these symbols are given in the nomenclature. The effect of the independent variations of these quantities was considered by Huang in the range $R = 0, 0.01, \ldots, 0.1$, and $S = 0, 0.1, \ldots, 0.1$. These values obviously prescribe arbitrary values to the shear coefficient, $k$. In general, increase in the value of $R$ for a fixed $S$ results in the decrease of the classical frequency parameters, the amount of reduction being greater at the higher modes of vibration. The decrease of $S$, on the other hand, causes additional reductions which are again greater at the higher modes. This implies that the smaller the values of the shear coefficient, $k$,
the lower are the frequencies.

The analysis of the effects of the shear displacement and rotatory inertia on pre-twisted beam vibrations has received considerably less attention. Dawson(22) examined, using the equations derived by Carnegie(17), the effects of shear and rotatory inertia on the frequencies and mode shapes of rectangular section uniform cantilever beams of ratio of flexural rigidities 16 and 64, and angles of pre-twist in the range 0 to 90 degrees. The width and length of the beam were 1 in. and 6 in., respectively. The theoretical results were obtained by the Rayleigh-Ritz method where the assumed displacement functions were taken as a linear combination of the characteristic functions of the classical clamped-free beams. Recently, Ghosh(34) applied the Runge-Kutta method of numerical integration to this problem. The results of this research are in preparation.

Another aspect that must be taken into account in vibration problems of beams is the effect of cross-sectional non-uniformities. Much work has been done on this subject in relation to clamped-free Bernoulli-Euler beams with linear taper. Exact solutions are available only in a few cases, namely cones, wedges and pyramids. For other cases several authors employed numerical procedures. Martin(26) obtained approximate solutions to the equations of motion by treating the depth and width taper as second order perturbations. The analysis proceeds on the similar lines as that described earlier in relation to this author's work on pre-twisted beams. The derived expression for the frequency ratios was found satisfactory for taper parameters in the range 0 to -0.5. The accuracy of the mode shapes was not discussed. Hasuer and Keightley(35) used a Myklestadt type procedure, where the beam is divided into a number of segments, and the mass included within the half segments on either side of the division points is concentrated at the division points. The flexural rigidity of each segment was assumed to be constant. The first three frequencies and mode shapes could be obtained under this idealisation by using ten elements, provided the depth taper was not excessive. For certain taper values the method yielded very poor results. In such cases refinement of results were made by using the Stodola method, or employing as many as 100 elements. Risson and Williams(36) presented confirmation of the results obtained in Ref.(35), through the application of the
central finite differences approximation. The equation of motion was transformed into a set of simultaneous linear equations by replacing the derivatives of the deflection by central finite differences. This procedure results in a non-symmetric algebraic eigenvalue problem. Satisfactory results were obtained by an extrapolation technique, from the results obtained by using 30 and 40 intervals. Even then, the accuracy of the frequency parameters was slightly less than those given by the Myklestadt and Stodola methods.

The application of a similar finite differences approximation to linearly tapered pre-twisted cantilever beams was developed, independently, by Carnegie, Dawson and Thomas\(^{(32)}\). Extrapolation of eigenvalues and eigenvectors was made from the values obtained by taking 10, 20 and 50 intervals.

The advantage of the matrix formulated version of the finite differences approximation over the methods discussed so far is that the frequencies and mode shapes can be obtained without a trial and error technique based on the evaluation of the zeros of a frequency determinant. This advantage, however, is substantially offset by the non-symmetric character of the eigenvalue problem, and the slow convergence properties. The latter necessitates the use of extrapolation, from the results obtained after a uniform convergence level has been reached, according to an error law\(^{(32,36,37)}\).

The effects of shear and rotatory inertia on the frequencies of the classical clamped-free wedge were calculated by Lee\(^{(38)}\), for a particular beam. Both upper and lower bounds for the first three frequencies were evaluated. Lee and Bisshopp\(^{(39)}\) used the Rayleigh-Ritz and Galerkin methods, and indicated the effects of variations of the shear and rotatory inertia parameters, for a particular value of the shear coefficient, and the effect of variations of the latter for a fixed value of the rotatory inertia parameter. Recently, Gaines and Volterra\(^{(40)}\) considered Timoshenko type clamped-free truncated cones and wedges. Upper bounds were obtained by the Rayleigh-Ritz method, and lower bounds were also presented.

The above discussion is a summary of the great amount of work that has been done in connection with the analysis of vibration problems of elastic beams, which have in general, been limited to the clamped-free end conditions. A need for further investigation of the subject also arises due to the advances made, as a result of the advent of digital computers, in the philosophy of numerical procedures.
The computers, by their very nature, demand numerical methods that are well suited to matrix formulation and that have strong stable convergence characteristics. Most of the methods mentioned in the previous paragraphs were initiated before the invention of digital computers. Therefore, their inherent operational characteristics had to be suitable for hand computation. Recent years have witnessed attempts on the reformulation of these methods in matrix notation. The most valuable fruit of these investigations has been the introduction of a new philosophy in the approximation of continuous media, namely the Finite Element method.

The essential feature of the Finite Element method is the substitution of the actual continuum by an assemblage of discrete structural elements, interconnected at a finite number of points (or, "nodes"), the material properties of the actual system being retained in the elements. The idea of approximation of a continuum by discrete elements is not new. Lumped mass models of Myklestadt, and the Hreniko\'ff models of plates are examples. But, the use of elements which retain the original material properties is new.

The most significant contributions which started the evolution of the Finite Element method have been the early works of Argyris and Clough in relation to static problems of structural and solid mechanics. Argyris profoundly developed comprehensive matrix methods of structural analysis, and Clough introduced the idea of idealization of a continuous system by an assemblage of finite elements, whereby Argyris's methods of structural synthesis could be applied. Since these original works appeared, the method has been applied to a wide variety of problems in solid and structural mechanics with noteworthy success. However, applications to coupled vibration problems of beams have not appeared, the main attention being directed towards two and three dimensional problems of elasticity, and static and dynamic problems of plates and shells of arbitrary configurations. The success enjoyed by the Finite Element method in these problems indicates that a similar approach will possibly provide a more powerful and versatile analysis of vibration problems of pre-twisted beams than the standard numerical methods.

Indication of the better accuracy of the Finite Element method has been given by Archer and Leckie and Lindberg in connection with the uniform Bernoulli-Euler beams, and by Lindberg in the case of linearly tapered Bernoulli-Euler beams. In these works it was
demonstrated that the derived basic element dynamic-stiffness matrices* yielded more accurate frequencies and mode shapes, in terms of convergence properties of the approximate results with the number of elements used and hence the computer time, than more conventional methods such as the finite differences approximation, Myklestadt method, Stodola method and other methods where continuous beam parameters are physically lumped.

The standard procedure in the application of the Finite Element method to free vibration problems of beams proceeds as follows. The beam is divided into a number of elements and each element is segmented from the beam. In order to restore the equilibrium of such an element external forces are applied at the ends of the element. Thus, each element performs forced vibrations, its motion being given by the Virtual Work equation

\[
\text{Virtual work done by external forces} + \text{Virtual work done by inertia forces} + \text{Virtual work done by elastic forces} = 0
\]

Up to this stage the analysis is exact. Approximations are now introduced by assuming a displacement function over the length of an element, with a number of undetermined coefficients equal to the number of external forces applied at the ends of the element. This function must allow rigid body motions as well as straining motions. The undetermined coefficients are then transformed into the generalised displacements of the ends of the element by considering the local geometry of the element. By substituting the new displacement function thus obtained into the Virtual Work equation, an equation of the following type will be obtained

\[
\begin{bmatrix}
\text{Virtual work} \\
\text{done by inertia forces} \\
\text{done by elastic forces}
\end{bmatrix}
= 0
\]

Up to this stage the analysis is exact. Approximations are now introduced by assuming a displacement function over the length of an element, with a number of undetermined coefficients equal to the number of external forces applied at the ends of the element. This function must allow rigid body motions as well as straining motions. The undetermined coefficients are then transformed into the generalised displacements of the ends of the element by considering the local geometry of the element. By substituting the new displacement function thus obtained into the Virtual Work equation, an equation of the following type will be obtained

\[
\begin{bmatrix}
\text{Vector of} \\
\text{element end} \\
\text{forces}
\end{bmatrix}
= 0
\]

If the actual beam is divided into n elements, there will be n equations of the foregoing type. These are assembled together by imposing the conditions that at any junction point continuity of displacements must be satisfied, and the total forces acting at the junction points must be zero. Finally, the imposition of the boundary conditions of the actual beam leads to the simple eigenvalue problem

\[
\begin{bmatrix}
\text{Beam} \\
\text{stiffness} \\
\text{matrix}
\end{bmatrix}
- \omega^2
\begin{bmatrix}
\text{Beam} \\
\text{inertia} \\
\text{matrix}
\end{bmatrix}
= 0
\]

for free vibrations.

The Virtual Work equation actually corresponds to Hamilton's

* See Article 1.2.1.
Principle for non-conservative systems, or to Lagrange's equations of the second kind after the approximations. Thus the element stiffness and inertia matrices can simply be obtained by substitution of the transformed approximate displacement functions into the appropriate potential and kinetic energy expressions and the application of the Lagrange's equations to the resulting finite quadratic forms. This fact was noted by a number of authors \((43, 46, 47)\) and as a result there has been a tendency to describe the Finite Element method based on the approximation of the displacements as a generalisation of the Rayleigh-Ritz method. This interpretation may be misleading, for the principles underlying the approximations in the two methods are different.

The contents of this thesis are contained in two parts. In the first part a formulation, which differs from those appearing in the literature, of a Finite Element procedure is presented, and applications to various important coupled vibration problems of beams are demonstrated. Various finitisation techniques are introduced and their relative merits are discussed. The method is versatile, has strong and stable convergence characteristics, and ideally suited to digital computers. The treatment of the boundary conditions does not present any additional work. This fact alone is a considerable advantage over the more conventional numerical procedures. The second part of the thesis is devoted to the investigation of the vibration characteristics of tapered and pre-twisted beams having various boundary conditions.
PART 1

FINITE ELEMENT ANALYSIS

OF

VIBRATION PROBLEMS OF BEAMS
1. THEORETICAL CONSIDERATIONS

The application of the Hamilton's Principle gives two types of equations (see appendix 2), namely the differential equations of motion (or, field equations), and the boundary equations. The former and the latter express the conditions of dynamic equilibrium in the field and at the boundaries, respectively. The solutions of the field equations give the classes of argument functions, which are uniquely determined by the differential equations (and hence, by the theory that has resulted in the energy equations), but otherwise arbitrary. These functions will be called the "field functions". The field functions which satisfy the essential boundary conditions can be called the Ritz functions, while the Ritz functions which satisfy the natural boundary conditions are usually termed eigen functions. Evidently, eigen functions represent the complete solution of a particular problem of the theory.

In this section, after the introduction of a matrix representation of energy equations occurring in the free vibration problems of beams, a procedure (subsequently called Displacement Method) for the finitization of the exact energy equations is formulated. This method is based on the approximation of the field functions, and therefore, advantages over the Rayleigh-Ritz method, where the Ritz functions are approximated, are expected. The piecewise approximation of the field functions, on the other hand, leads to a Finite Element type representation of beams, where the element properties are determined by the Displacement method.

1.1 Matrix Representation of Energy Equations

In order to present a treatment of the kinetic beam problems in a generality encompassing a large body of useful linear theories of elastic beams, it is convenient to introduce an extended definition of quadratic functionals, which symbolise a common property of energy equations occurring in these problems. A linear energy functional, say $\mathcal{U}$, can be generally defined by the expression

$$
\mathcal{U} = \int_{\Omega} \sum_{i,j,k} a_{ik} \psi_i \dot{\psi}_j \dot{\psi}_k \, d\Omega
$$

(1.1)
where \( i, k = 1, 2, \ldots, N_1 \), and \( F_j (\Psi_j) \) represents a linear partial differential operation \( F_j \) on the continuous argument function \( \Psi_j \), \( a_{ik} \) is a physical parameter which may be a function of the spatial variables defining the domain \( D_{\Omega} \), of the energy functional \( \mathcal{U} \). For some, or all, \( j = 1, 2, \ldots, N_1 \) the relation \( \Psi_j = \Psi_{1j} \), say, may hold. In this definition, the facts that \( \mathcal{U} \) is positive definite and \( \Psi_j \) possesses continuous derivatives so that the operation \( F_j (\Psi_j) \) is meaningful for all \( j \), are assumed on account of physical considerations.

In linear eigenvalue problems associated with beams, there exists two functionals of the above type, namely the strain energy and kinetic energy functionals. If in equation (1.1) the domain \( D_{\Omega} \) represents the one-dimensional beam domain \( 0 \leq z \leq L \), then this equation will represent the strain energy of the beam. The kinetic energy can also be written similarly:

\[
T = \sum_{i,k=1}^{N_2} b_{ik} E_i (\Psi_i) E_k (\Psi_k) dz \tag{1.2}
\]

where \( i, k = 1, 2, \ldots, N_2 \), and the quantities \( E_j \) and \( b_{ik} \) are defined similarly as \( F_j \) and \( a_{ik} \).

The argument functions \( \Psi_j \) represent the co-ordinates of the beam motion. For example, if a pre-twisted beam is vibrating flexurally, then \( \Psi_1 \) and \( \Psi_2 \) can be taken to be displacements in the two principal directions. If there are no elastic and dynamic interactions between the elements of an independent set formed from \( \Psi_j \), then both energy functionals reduce to "canonical" forms, i.e., \( a_{ik} = b_{ik} = 0 \) for \( i \neq k \), and the motion is called uncoupled. If such interactions exist, the motion is said to be coupled. Then the symmetry relations \( a_{ik} = a_{ki} \) and \( b_{ik} = b_{ki} \) will be valid.

By introducing matrix notation the functionals (1.1) and (1.2) can be written compactly as follows:

\[
\mathcal{U} = \frac{1}{2} \int_0^L (A \sigma_A)^T \mathbf{D} A \sigma_A \ dz \tag{1.3}
\]

\[
T = \frac{1}{2} \int_0^L (B \sigma_B)^T \mathbf{C} B \sigma_B \ dz \tag{1.4}
\]

where

\[
A = \begin{bmatrix} F_1 & F_2 & \cdots & F_N_1 \end{bmatrix} \\
B = \begin{bmatrix} E_1 & E_2 & \cdots & E_N_2 \end{bmatrix} \\
\sigma_A = \{ \phi_1, \phi_2, \ldots, \phi_N_1 \} \\
\sigma_B = \{ \phi_1, \phi_2, \ldots, \phi_N_2 \} \]
and $\mathbf{D}$ and $\mathbf{Q}$ are symmetric square matrices whose $(ik)$ th terms are
$2a_{ik}$ and $2b_{ik}$, respectively.

The vectors $\mathbf{C}_A$ and $\mathbf{C}_B$ can be expressed by simple transformations $\alpha$ and $\beta$, respectively, in terms of a linearly independent vector

$\mathbf{C} = \{ \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_N \}$

as

$\mathbf{C}_A = \alpha \mathbf{C}$

$\mathbf{C}_B = \beta \mathbf{C}$

Clearly, the vector $\mathbf{C}$ may coincide with $\mathbf{C}_A$ and/or $\mathbf{C}_B$. Substituting the foregoing transformations into the equations (1.3) and (1.4)

$\mathbf{L} = \frac{1}{2} \int \mathbf{M}^T \mathbf{D} \mathbf{M} \, dz$ (1.5)

$\mathbf{T} = \frac{1}{2} \int \mathbf{Q} \mathbf{B} \mathbf{Q}^T \, dz$ (1.6)

The energy equations for beams of interest in the present analysis are derived in Appendix 1, which also includes the forms of these equations in the notation of the foregoing equations. In the mean time it should be noted that, apart from the beam vibration problems considered in this thesis, it is possible to find other systems whose strain energy and kinetic energy can be represented in the foregoing forms. For example, the strain energy of bending of an isotropic and homogeneous rectangular plate lying in the $0 \leq x \leq a$, $0 \leq y \leq b$ plane can be written as

$\mathbf{U}_{\text{plate}} = \frac{3d}{2} \int_0^a \int_0^b \mathbf{M}^T \mathbf{D}_{\text{plate}} \mathbf{M} \, dx \, dy$

where

$\mathbf{A} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \end{bmatrix}$

$\mathbf{D}_{\text{plate}} = \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & 1-\mu \end{bmatrix}$

$\alpha = \{ 1, 1, 1 \}$

and $w$ is the deflection of the middle plane, $d$ the plate rigidity, $\mu$ the Poisson's Ratio.

1.2 Simple Approximation of the Field Functions: Displacement Method

The class of functions that are hoped to approximate the exact field functions will be called the "continuity functions". These can be written as

$\mathbf{C}^{(m_j)} = \sum_{i=1}^{m_j} g_{ji} c_{ji}$, $j = 1, 2, \ldots, N$ (1.7)
where \( \sigma_j^{(m_j)} \) denotes that the argument function \( \sigma_j \) is approximated by a linear combination of \( m_j \) number of "co-ordinate functions" \( g_{ji} \), and \( e_{ji} \) are undetermined time dependent constants (or, the "basis" of the linear combination).

The possibility of the foregoing representation is the subject of the theory of series expansion of arbitrary functions, and it is well documented in Ref. (7). It can be shown that any function continuous in the interval of its definition can be approximated arbitrarily closely by a linear combination of a system of "complete" functions in this interval. The definition of a complete system of functions is beyond the scope of the present analysis. However, it may suffice to note that, a system of functions can be said to be complete if any continuous function can be approximated arbitrarily closely by a linear combination of them.

Then, from Weierstrass's Theorem it is seen that the powers

\[ 1, z, z^2, \ldots \]

form a complete system of functions in every interval \( a \leq z \leq b \).

Throughout the analysis it will be assumed that the co-ordinates in the linear combination (1.7) i.e., \( g_{ji} \), are the simple powers and occur in their natural sequence.

The equation (1.7) in matrix notation becomes

\[ \sigma_j^{(m_j)} = G_j e_j \quad , \quad j = 1, 2, \ldots, N \tag{1.8} \]

where \( e_j = \{e_{j1}, e_{j2}, \ldots, e_{jm_j}\} \)

\[ G_j = \begin{bmatrix} g_{j1} & g_{j2} & \cdots & g_{jm_j} \end{bmatrix} \]

But

\[ \sigma = \left\{ \sigma_1^{(m_1)}, \sigma_2^{(m_2)}, \ldots, \sigma_N^{(m_N)} \right\} \]

Substituting the equation (1.8) into the foregoing equation

\[ \sigma = G e \tag{1.9} \]

where \( e = \{e_1, e_2, \ldots, e_N\} \)

\[ G = \begin{bmatrix} G_1 & G_2 & \cdots & G_N \end{bmatrix} \]

Hence, the approximated energy equations will be reduced to finite quadratic forms in the basis \( e_{ji} \), namely,

\[ \mathcal{L} = \frac{1}{2} e^T k e \tag{1.10} \]

\[ T = \frac{1}{2} e^T m e \tag{1.11} \]

where

\[ k = \int_0^L \left( A \alpha G^T \right) \cdot \mathcal{D} \cdot A \alpha G \cdot dz \tag{1.12} \]

\[ m = \int_0^L \left( B \beta G^T \right) \cdot \mathcal{Q} \cdot B \beta G \cdot dz \tag{1.13} \]
In the Rayleigh-Ritz method, the analysis essentially ends at this stage. For, all the essential boundary conditions are prescribed at the beginning of the problem by the selection of argument functions in the Ritz space and, therefore, the basis $e_{j_1}$, $j = 1, 2, \ldots, N$, are independent. Hence, the application of the Hamilton's Principle to equations (1.10) and (1.11) would be permissible.

In the present analysis, however, the Hamilton's Principle cannot be invoked at this stage. Because, by imposing constraints on the system, it is possible to find relations between $e_{j_1}$, and hence the latter cannot be subjected to arbitrary variations. This can be overcome by the transformation of the original basis to an independent set as follows.

Let

$$
\sigma^{r}_{j,k}, \quad \left\{ \begin{array}{c}
\sigma^{r}_{j,k} \\
\{ k = 0, 1, \ldots, q_{jr} \}
\end{array} \right.
$$

represent the $k$th derivative of the argument function $\sigma^{r}_{j}$ at a point $r$ (subsequently called a "node"; if a node belongs to the boundary of the beam it is termed a "terminal" node, otherwise it is called an "internal" node) on the beam (see Fig. 1.1). The number of nodes, $R_{j}$, and the number of successive derivatives, $q_{jr}$, at a node $r$, will be subjected to the following restrictions:

**Restriction 1.** The number of all $\sigma^{r}_{j,k}$ must be equal to the number of the undetermined constants $e_{j_1}$ in the linear combination (1.7), i.e.,

$$
m_{j} = \sum_{r=1}^{R_{j}} (1 + q_{jr}).
$$

**Restriction 2.** The set $\sigma^{r}_{j,k}$ must at least include those terms which are associated with the geometric boundary conditions imposable on the beam, for every argument function $\sigma^{r}_{j}$.

Let $\varphi^{r}_{j,k}$ denote, similarly, the values of $\sigma^{r}_{j,k}$ as obtained from the equation (1.8). Clearly

$$
\varphi^{r}_{j,k} = G^{r}_{j,k} e_{j}, \quad (1.14)
$$

Going through the values of $k$, for a fixed $r$ and $j$, let

$$
\varphi^{r}_{j} = \left\{ \begin{array}{c}
\varphi^{r}_{j,0} \\
\varphi^{r}_{j,1} \\
\vdots \\
\varphi^{r}_{j,q_{jr}}
\end{array} \right.
$$

and

$$
(C^{r}_{j})^{T} = \left[ \begin{array}{c}
(G^{r}_{j,0})^{T} \\
(G^{r}_{j,1})^{T} \\
\vdots \\
(G^{r}_{j,q_{jr}})^{T}
\end{array} \right]
$$

Going through the values of $r$ in the foregoing equations, for a fixed $j$, let
Similarly, for \( j = 1, 2, \ldots, N \), let
\[
\mathbf{\Phi} = \begin{bmatrix} \phi_1^1 & \phi_2^1 & \cdots & \phi_N^1 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^N & \phi_2^N & \cdots & \phi_N^N \end{bmatrix} \quad (1.17)
\]
\[
\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{bmatrix} \quad (1.18)
\]
Then the equations (1,14) are compactly written as
\[
\mathbf{\Phi} = \mathbf{C} \cdot \mathbf{e} \quad , \quad j = 1, 2, \ldots, N \quad (1.19)
\]
or,
\[
\mathbf{\Phi} = \mathbf{C} \cdot \mathbf{e} \quad (1.20)
\]
From the "restriction 1", it is clear that \( \mathbf{C} \) is a square matrix. Also, since the co-ordinate functions are complete and occur in their natural sequence, the matrix \( \mathbf{C} \) is non-singular. Hence from equation (1.20)
\[
\mathbf{e} = \mathbf{C}^{-1} \cdot \mathbf{\Phi} = \mathbf{D} \cdot \mathbf{\Phi} \quad (1.21)
\]
where \( \mathbf{D} = \mathbf{C}^{-1} \) is written for typographical simplicity. It should be noted that
\[
\mathbf{D} = \begin{bmatrix} D_1 & D_2 & \cdots & D_N \end{bmatrix} \quad (1.22)
\]
where \( D_j = C_j^{-1} \).
Substituting the equation (1.21) into the equation (1.9)
\[
\mathbf{\sigma} = \mathbf{G} \cdot \mathbf{D} \cdot \mathbf{\Phi} \quad (1.23)
\]
This result can be expressed, by a reverse process, by the following linear combination
\[
\mathbf{\sigma}'_j = \sum_{r=1}^{R_j} \sum_{k=0}^{q_{rj}} \bar{\mathbf{e}}_{j,k} \cdot \mathbf{\psi}_{j,k}^r \quad , \quad j = 1, 2, \ldots, N \quad (1.24)
\]
where, clearly, as the values \( \mathbf{\sigma}^r_{j,k} \) are assigned to the left hand side, the right hand side will go through the parameters \( \mathbf{\psi}^r_{j,k} \) (subsequently called generalised displacements). The functions \( \bar{\mathbf{e}}_{j,k} \) are called unit displacement functions. Hence, it is seen that the equation (1.24) can vanish only when all \( \mathbf{\psi}^r_{j,k} \) vanish, i.e., the combination (1.24) is linearly independent. Therefore, energy equations that will be obtained by the substitution of the equation (1.21) into the equations (1.10) and (1.11), namely,
\[
\mathcal{L} = \frac{1}{2} \mathbf{\psi}^T \mathbf{K} \mathbf{\psi} \quad (1.25)
\]
\[
\mathcal{T} = \frac{1}{2} \mathbf{\psi}^T \mathbf{M} \mathbf{\psi} \quad (1.26)
\]
where
\[
\mathbf{K} = \mathbf{D}^T \mathbf{k} \mathbf{D} \quad (1.27)
\]
\[
\mathbf{M} = \mathbf{D}^T \mathbf{m} \mathbf{D} \quad (1.28)
\]
can be used in the Hamilton's Principle. Thus, noting that the motion is harmonic with circular frequency \( \omega \), i.e., \( \dot{\Phi} = \dot{\dot{\Phi}} \cos \omega t \), and the matrices \( K \) and \( M \) are symmetric

\[
(K - \omega^2 M) \ddot{\Phi} = 0
\]

or,

\[
Z \ddot{\Phi} = 0
\]

The symmetric matrices \( K, M \) and \( Z \) are called the stiffness, inertia and dynamic-stiffness matrices of the beam, respectively.

The essential boundary conditions can now be applied directly to the equation (1.29), since by the restriction 2 the vector \( \dot{\Phi} \) contains all the necessary terms. Obviously, this will involve the deleting of the rows and columns of the dynamic-stiffness matrix, \( Z \), corresponding to such constraints.

The equation (1.29) can be considered as an approximation to the actual field equations. On the other hand, it is also the correct equation for a finite degrees of freedom system (subsequently called a model) which idealises the actual continuum. Implicit in the preceding analysis are the notions of the basic and improved models, which are discussed in the next articles.

1.2.1 The Basic Model

The basic beam model can be defined conveniently by the concept of "basic nodes". A node \( r \) is called a basic node if the number \( q_{jr} \) is such that there corresponds a force to every \( \dot{\sigma}_{jr} \). Then, the basic model is that which contains only basic terminal nodes.

The number \( q_{jr} \) of the basic model, say \( q_j \), obviously depends on a particular problem under consideration. For example, let the argument functions \( \dot{\sigma}_1 \) and \( \dot{\sigma}_2 \) represent the vibrational displacements in two perpendicular directions of a pre-twisted beam with symmetric cross-sections. Then there will be a unique correspondence only between the displacements (includes rotations) \( \dot{\sigma}_{1,0} \), \( \dot{\sigma}_{1,1} \), \( \dot{\sigma}_{2,0} \), \( \dot{\sigma}_{2,1} \), and the forces (includes moments) acting at the node \( r \). Therefore \( \bar{q}_1 = \bar{q}_2 = 2 \). In general, if \( \dot{\sigma}_j \) corresponds to the flexural (according to the Bernoulli-Euler theory) component of the motion then \( \bar{q}_j = 2 \), and if it corresponds to the torsional (according to the St. Venant theory) component then \( \bar{q}_j = 1 \).

The size of the basic dynamic-stiffness matrix, or the degrees of freedom given to the beam, will be \( 2(\bar{q}_1 + \bar{q}_2 + \ldots + \bar{q}_N) \), since the basic model has only two basic terminal nodes. It is interesting to note that this number equals the number of independent constants of integration in the exact field functions.

1.2.2 Improvements of the Basic Model

In order to improve the basic model properties it is necessary to
provide means which will enable the inclusion of an increased number of
terms in the linear combination (1.7) corresponding to the basic model.
Two such means are directly deduced from the analysis of article 1.2.
A third one, called the Internal Parameter method, is also given.

1.2.2.1 The Internal Node Method

In this method the degrees of freedom given to the basic model is
increased by introducing nodes in the region \( 0 < z < L \) of the beam in
such a way that to every \( \sigma_{jk}^x \) associated with such a node, there
 corresponds a force component. This limits the maximum number of degrees
of freedom that can be given to an internal node, to that of the basic
nodes. However, advantage can also be gained by using "degenerate"
internal nodes, i.e., nodes that are given less degrees of freedom than
that of basic nodes, especially in the vibration problems of beams where
flexural and torsional motions are coupled.

The positions of the internal nodes can be chosen arbitrarily. As
long as the number and nature of the internal nodes are kept constant, the
eigenvalues will be independent of any variations of their positions,
since such changes are equivalent to co-ordinate transformation and do not
alter the energies of the system.

1.2.2.2 The Higher Derivatives Method

This improved model contains only external nodes that are given more
number of degrees of freedom than that of basic nodes, by the inclusion
of the successive derivatives \( \sigma_{jk}^{x+1}, \sigma_{jk}^{x+2}, \ldots \) as the
additional generalised displacements. No forces, however, can be associated
with these extra co-ordinates.

This approach can be incorporated with the internal node method. The
procedure is self-explanatory, and the corresponding mathematical analysis
is already given in article 1.2.

1.2.2.3 The Internal Parameter Method

Refined models can also be derived by a linearly independent extension
of the unit displacement functions \( \bar{e}_r \) (see equation (1.24)). Let the
new approximating function thus obtained be

\[
\sigma_j = \sum_{r=1}^{2} \sum_{k=0}^{q_j} \bar{e}_{jrk} \cdot \psi_{jk}^x + f_j, \quad j=1,2,\ldots,N
\]
where, for simplicity it is assumed that the basic model is being improved, and

\[ f_j = \sum_{i=1}^{s_j} f_{ji} \cdot h_{ji} \quad , \quad s_j > 2q_j \]  \hspace{1cm} (1.30)

where \( h_{ji} \) are undetermined parameters, and \( f_{ji} \) are some functions of \( z \).

Evaluating the values of \( G_{j,k}^r \) (\( r=1,2; k=0,1,\ldots,q_j \)) it is seen that

\[ \psi_{j,k}^r = \sum_{r=1}^{2} \sum_{k=0}^{q_j} h_{j,k} \cdot \psi_{j,k} + f_{j,k}^r \]

Therefore, \( f_{j,k}^r = 0 \) , (\( r=1,2; k=0,1,\ldots,q_j \)) . \hspace{1cm} (1.31)

This result gives \( 2q_j \) conditions. The remaining \( s_j - 2q_j \) constants \( h_{ji} \) in equation (1.30) can be obtained by giving \( s_j - 2q_j \) number of values to the argument function (and/or its derivatives) at a number of points on the beam. Having thus determined \( f_j \), the dynamic-stiffness matrix can be derived as explained earlier. Evidently, this approach can be incorporated with the other two methods.

Bolotin \(^{(49)}\), Hurty \(^{(46,50)}\), and Argyris \(^{(51)}\) proposed, independently, similar methods for the refinement of basic elements. The first two authors define the functions which extend the unit displacement functions as satisfying the equation (1.31), and the remaining constants are obtained by considering displacements of internal points of the beam relative to the terminal nodes. Argyris, on the other hand, imposes the additional restraints that the extending functions also satisfy the zero force and moment conditions at the terminal nodes.

### 1.3 Piecewise Approximation of the Field Functions; Finite Element Method

The methods discussed in the preceding articles are based on the approximation over the whole region of a beam. Consequently, systematically improvable results, i.e., eigenvalues and eigenvectors, are obtained by the successive increase of the number of terms in the linear combination of the complete functions. A more powerful and versatile method of improvement of the results is based on the piecewise approximation of
the field functions. In effect this corresponds to segmenting the beam into a number of elements, whose motions are then synthesised through the conditions of compatibility of displacements and equilibrium of forces, to obtain the beam motion.

Consider a beam segmented into \( n \) number of elements. Let \( i = 0, 1, \ldots, n \) be a counter for all segmentation points, as shown in figure 1.2a, the points \( i = 0 \) and \( i = n \) referring to the ends of the beam. If the beam is "divided" into elements as shown in figure 1.2b, a typical element \( i \) having the terminal nodes \( i-1 \) and \( i \) (or, simply the element \( i \) ), can be considered as executing forced vibrations, the motion being excited by the adjacent portions of the beam. The dynamic properties of the element \( i \) will be determined by its potential and kinetic energies, and the work of the external harmonic forces, and can be idealised by the Displacement Method described in article 1.2.

The first step in the analysis is the derivation of all element dynamic-stiffness matrices. The synthesis of the beam motion can then be accomplished conveniently either by deriving the beam dynamic-stiffness matrix from component matrices, or by the Equivalent Transfer Matrix concept. Similarly, Equivalent Acceptance Matrices could be used, too. However, this does not exhibit any advantages over the other two techniques.

1.3.1 Assembly of Element Dynamic-Stiffness Matrices

For the present it will be convenient to denote the properties of an element \( i \) by a superscript \( i \). Then the same symbol without the superscript \( i \) will refer to the whole beam. This notation, however, will be dropped after this article.

From the equations (1.25) and (1.26) the potential and kinetic energies of the element \( i \) become

\[
\mathcal{U}^i = \frac{1}{2} (\psi^i)^T K^i \psi^i
\]

\[
\mathcal{T}^i = \frac{1}{2} (\dot{\psi}^i)^T M^i \dot{\psi}^i
\]

where

\[
K^i = (D^i)^T \int_0^{l_i} (A \alpha G^i)^T \cdot D^i \cdot A \alpha G^i \cdot dz \cdot D^i
\]

\[
M^i = (D^i)^T \int_0^{l_i} (B \beta G^i)^T \cdot G^i \cdot B \beta G^i \cdot dz \cdot D^i
\]

The matrices \( D \) and \( G \) are also incorporated by the superscript \( i \) because the continuity functions of the elements need not be the same for every element.
Let \( Q^i \) be the vector of all generalised forces acting on the element \( i \). It is assumed that \( Q^i \) and \( \mathbf{U}^i \) have a one-to-one correspondence, although the former may have a number of elements equal to zero. The work done by the external forces, \( W^i \), will be

\[ W^i = \left( Q^i \right)^T \cdot \mathbf{U}^i \]

Hence, the Hamiltonian of the element is

\[ H^i = T^i - \mathbf{U}^i + W^i \]

and the Hamiltonian of the whole beam will be

\[ H = \sum_{i=1}^{n} H^i = \sum_{i=1}^{n} \left( \frac{1}{2} \left( \mathbf{U}^i \right)^T M^i \cdot \mathbf{U}^i - \frac{1}{2} \left( \mathbf{U}^i \right)^T \mathbf{K}^i \cdot \mathbf{U}^i + \left( Q^i \right)^T \cdot \mathbf{U}^i \right) \]

This equation in matrix notation becomes

\[ H = \frac{1}{2} \left( \mathbf{U}^T \mathbf{M} \cdot \mathbf{U} \right) - \frac{1}{2} \left( \mathbf{U}^T \mathbf{K} \cdot \mathbf{U} \right) + \left( \mathbf{Q}^T \right) \cdot \mathbf{U} \]  

(1.32)

where

\[ \mathbf{U} = \begin{bmatrix} \mathbf{u}^1 & \mathbf{u}^2 & \ldots & \mathbf{u}^n \end{bmatrix} \]

\[ \mathbf{M} = \begin{bmatrix} M^1 & K^2 & \ldots & M^n \end{bmatrix} \]

\[ \mathbf{Q} = \begin{bmatrix} Q^1 & Q^2 & \ldots & Q^n \end{bmatrix} \]

However, the supervector \( \mathbf{U} \) is not independent, as the vectors \( \mathbf{u}^i \) and \( \mathbf{u}^{i+1} \), \( i=1,2,...,n-1 \), will have common terms on account of the conditions of analytical continuity at the joints. Let \( \mathbf{U} \) be a vector containing all the independent displacements of the beam. Then a transformation, say \( \Gamma \), having either zero or unit elements can be formed satisfying the following relation

\[ \mathbf{U} = \Gamma \cdot \mathbf{U} \]  

(1.33)

The corresponding transformation of the generalised forces can be obtained from the fact that the work done by the forces \( \mathbf{Q} \) on virtual displacements \( \delta \mathbf{U} \) is equal to the work done by the transformed force vector, \( \mathbf{Q} \), on virtual displacements \( \delta \mathbf{U} \), i.e.,

\[ (\mathbf{Q})^T \cdot \delta \mathbf{U} = (\mathbf{Q})^T \cdot \delta \mathbf{U} \]  

(1.34)

Substituting from equation (1.33) for

\[ (\mathbf{Q})^T \cdot \delta \mathbf{U} = (\mathbf{Q})^T \cdot \delta \mathbf{U} \]

Therefore

\[ \mathbf{Q} = \Gamma^T \cdot \mathbf{Q} \]  

(1.35)

Substituting the equation (1.33) into the equation (1.32) and noting the equation (1.34), the Hamiltonian of the beam is written in terms of the independent vector \( \mathbf{U} \), namely
where
\[ K = \Gamma^T \overline{K} \Gamma \]  \hspace{1cm} (1.37)
\[ M = \Gamma^T \overline{M} \Gamma \]  \hspace{1cm} (1.38)
are the assembled stiffness and inertia matrices, respectively.

On invoking Hamilton's Principle
\[ (K \ddot{\mathbf{q}} + M \dot{\mathbf{q}}) = 0 \]
For small vibrations of a conservative system vibrating with a circular frequency \( \omega \), the foregoing equation gives
\[ (K - \omega^2 M) \mathbf{\dot{q}} = 0 \]
or,\[ Z \mathbf{\ddot{q}} = 0 \]
where \( Z \) is the assembled beam dynamic-stiffness matrix.

The geometric boundary conditions can now be applied simply by deleting the rows and columns of the matrix \( Z \) corresponding to the constrained beam displacements. The resulting eigenvalue problem will be symmetric and can be solved efficiently by using a Similarity Method (59), e.g., the Jacobi Method. However, before the application of the latter it is necessary to transform an eigenvalue equation of the type \((A-\lambda B)\mathbf{X} = 0\) to the form \((C-\lambda I)\mathbf{Y} = 0\), where \(A, B, C\) are symmetric matrices, \(I\) is a unit matrix, and \(\mathbf{X}, \mathbf{Y}\) are eigenvectors. Two methods of doing this are shown in appendix 5.

In general, the methods described in article 1.2 can be used in the piecewise continuous approximation of the field functions with the following effects:

i) The use of basic elements is the minimum requirement for the solution of a vibration problem. Only the derivatives that occur in the essential boundary conditions will be made continuous at the junction points.

ii) The internal node and internal parameter methods retain the basic continuity conditions, but provide a means of improving the functional approximation for the elements.

iii) The method of higher derivatives will obviously enable a piecewise approximation where the derivatives higher than the basic ones can be considered in the smoothing of the approximations.

iv) A "mixed" formulation will achieve the above effects simultaneously.
In practice the assembly of the beam stiffness and inertia matrices as indicated by the equations (1.37) and (1.38) can be done by inspection.

1.3.2 Assembly of Elements by the Equivalent Transfer Matrices

The use of internal nodes, internal parameters, and higher derivatives increases the size of the element dynamic-stiffness matrix, and consequently the beam dynamic-stiffness matrix may become large enough to test the storage capacity of medium-sized digital computers. In such cases a full advantage of the symmetric eigenvalue problem given by the direct assembly of the element matrices cannot be taken, and a frequency iteration type solution has to be used. Then the following modifications on the already derived element dynamic-stiffness matrices may be advantageous.

Consider a beam element $i$, as shown in Fig. 1.3, whose displacement vector is assumed to contain, in addition to the basic displacements, a number of higher derivatives, displacements of internal nodes, and internal parameters. The generalised forces can be associated only with the basic displacements and there acts no external forces at the internal nodes. Hence, the element dynamic-stiffness matrix can be written in the following partitioned form

$$
\begin{bmatrix}
Z_{11} & Z_{21}^T & Z_{31}^T \\
Z_{21} & Z_{22} & Z_{23}^T \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix}
\begin{bmatrix}
\lambda_A^T \\
\lambda_C \\
\lambda_B^T
\end{bmatrix}
= \begin{bmatrix}
q_A \\
q_C \\
q_B
\end{bmatrix}
$$

(1.40)

where $\lambda_A^T$ and $\lambda_B^T$ denote the vectors of basic displacements at the node $i$ and $i+1$, $q_A$ and $q_B$ are the forces associated with these displacements, $\lambda_C$ is the vector of all displacements with which no forces can be associated. The foregoing equation can be represented by two equations, namely,

$$
\begin{bmatrix}
q_A \\
q_B
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{21}^T \\
Z_{31} & Z_{33}
\end{bmatrix}
\begin{bmatrix}
\lambda_A^T \\
\lambda_B^T
\end{bmatrix} +
\begin{bmatrix}
Z_{21} \\
Z_{32}
\end{bmatrix}
\lambda_C
$$

(1.41)

$$
Z_{22} \lambda_C =
-\begin{bmatrix}
Z_{21} & Z_{31}^T
\end{bmatrix}
\begin{bmatrix}
\lambda_A^T \\
\lambda_B^T
\end{bmatrix}
$$

(1.42)
The elimination of the vector $\mathbf{u}_C$ from equations (1.42) and (1.41) gives

$$
\begin{bmatrix}
Q_A \\
Q_B
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_A \\
\mathbf{u}_B
\end{bmatrix}
$$

(1.43)

Hence the size of the dynamic-stiffness matrix is reduced to that of the basic element. The penalty one has to pay for this is that the algebraic eigenvalue problem resulting from the assembly of the reduced dynamic-stiffness matrices will no longer be a simple one, i.e., its elements will be polynomials of the frequency.

Let the equation (1.43) be written in the following notation

$$
\begin{bmatrix}
Q_A \\
Q_B
\end{bmatrix} =
\begin{bmatrix}
Z_{AA} & Z_{AB} \\
Z_{BA} & Z_{BB}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_A \\
\mathbf{u}_B
\end{bmatrix}
$$

(1.44)

Using the sign convention of the figure 1.3b, i.e., putting $R_A = Q_A$ and $R_B = Q_B$, the foregoing equation can be put in the form

$$
\begin{bmatrix}
\mathbf{u}_B \\
\mathbf{u}_A
\end{bmatrix} =
\begin{bmatrix}
-Z_{AB} & Z_{AA} \\
Z_{BB} & -Z_{BA}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_A \\
\mathbf{u}_B
\end{bmatrix}
$$

(1.45)

or,

$$
\mathbf{u}_A = \sum \mathbf{u}_A
$$

where $\sum$ is the equivalent transfer matrix of the dynamic-stiffness matrix (1.43).

If the beam is divided into $n$ elements, denoting the ends of the beam by the subscripts 0 and $n$, it is easily shown that

$$
\begin{bmatrix}
\mathbf{u}_n \\
\mathbf{R}_n
\end{bmatrix} =
\begin{bmatrix}
1 \\
-\sum_i
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_0 \\
\mathbf{R}_0
\end{bmatrix}
$$

where $\sum_i$ is the equivalent transfer matrix of the $i$th element.

By inserting the end conditions in the foregoing equation a frequency determinant can be found. This is solved by iteration.

During the course of this research no occasion that forced the use of equivalent transfer matrices has arisen due to the strong convergence characteristics of the dynamic-stiffness matrices used. However, it is found invaluable in the analysis of article 1.3, in expanding infinite determinants.
1.4 Illustrative Examples

In deriving element dynamic-stiffness matrices, it is important that the potential and kinetic energies are expressed in dimensionless quantities. This, not only simplifies the numerical calculations, but also provides an additional generality in the analysis.

1.4.1 Torsional Vibrations According to the St.-Venant Theory

The potential and kinetic energies of this problem can be written as

\[ \mathcal{U} = \frac{1}{2} \frac{GJ}{\ell} \int_0^1 (\dot{\Theta}_G)^2 \, d\kappa \]  
(1.46)

\[ T = \frac{1}{2} \mu \Theta \frac{\kappa G^2}{\ell} \int_0^1 (\dot{\Theta}_G)^2 \, d\kappa \]  
(1.47)

where \( \kappa = z / \ell \), \( \Theta_G \) denotes the torsional twist \( \Theta \), and the beam is assumed to be uniform.

The continuity function is taken in the form

\[ \Theta = \sum_{j=1}^{m_1} \epsilon_{1j} \kappa^{j-1} \]  
(1.48)

where \( m_1 \) is the number of degrees of freedom given to the element, \( \epsilon_{1j} \) are the undetermined constants and, in the notation of equation 1.7,

\[ g_{1j} = \kappa^{j-1} \]. Thus

\[ G_1 = G = \begin{bmatrix} 1 & \kappa & \kappa^2 & \cdots & \kappa^{m_1-1} \end{bmatrix} \]  
(1.49)

From the equations (1.12) and (1.13), whose terms can be easily inspected from the equations (1.46) and (1.47), respectively, it can be shown that

\[ k = \frac{GJ}{\ell} \int_0^1 (G')^T G' \, d\kappa \]  
(1.50)

\[ m = \mu \Theta \frac{\kappa G^2}{\ell} \int_0^1 G^T G \, d\kappa \]  
(1.51)

The derivation of the stiffness and inertia, and hence the dynamic-stiffness matrices will depend on the geometric transformation matrix \( C_1 = C \), which in turn depends on the number of internal nodes and/or higher derivatives employed in the analysis. For simplicity, consider a basic element, i.e., \( m_1 = 2 \). Then only \( G_{1,0} \) and \( G_{1,1} \), where
i and i+1 refer to the ends of the element, may be considered in the element displacement vector. But, from equation (1.48)

\[ C_{i,0} = \mathbf{C}_i = e_{11}^1 + e_{12}^1 \]

\[ C_{i+1,0} = \mathbf{C}_{i+1} = e_{11}^1 + e_{12}^1 \]

or, in the notation of article 1.2, \( \psi = C \), where \( \psi = [\mathbf{C}_i \mathbf{C}_{i+1}] \),

\[ e = \left[ e_{11}^1 e_{12}^1 \right] , \text{ and } C = \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] . \] Therefore

\[ C^{-1} = D_1 = D = \left[ \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right] \quad (1.52) \]

Substituting equations (1.50), (1.51) and (1.52) in the equations (1.27) and (1.28), the following dynamic-stiffness matrix is obtained

\[ \begin{bmatrix} 1 - 2 \delta & -1 - \delta \\ -1 - \delta & 1 - 2 \delta \end{bmatrix} \] (1.53)

where \( \delta = \lambda \theta / 6 \).

A number of more complicated models found for the St.-Venant beams are presented in article 1.5.

The assembly procedure will be demonstrated by a beam divided into three basic elements, as shown below

\[ 0 \ 1 \ 1 \ 2 \ 2 \ 3 \ 3 \]

The supervector of all element displacements (see equation 1.34) is

\[ \vec{\mathbf{\theta}} = \left\{ \theta_0 \ \theta_1 \ \theta_1 \ \theta_2 \ \theta_2 \ \theta_3 \right\} \]

while the independent beam vector is

\[ \mathbf{\theta} = \left\{ \theta_0 \ \theta_1 \ \theta_2 \ \theta_3 \right\} \]

Clearly, \( \vec{\mathbf{\theta}} = \mathbf{\Gamma} \theta \), or

\[ \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \]

Hence, the rectangular matrix is the required assembly matrix \( \Gamma \) occurring in the equation (1.34). The extension of schematism to the general case is straightforward.

1.4.2 *Flexural Vibrations According to the Timoshenko Theory*

The starting point is again the potential and kinetic energy
equations, which have the following non-dimensional forms

\[ \lambda = \frac{1}{2} \int_0^1 \frac{E I \pi^2}{\ell} \left\{ (\sigma'_1)^2 + s \sigma'_2 (\sigma'_2 - \sigma'_1)^2 \right\} \, d\kappa \] (1.54)

\[ T = \frac{1}{2} \int_0^1 \mu \rho (2 \pi \dot{\sigma}_1^2 + \dot{\sigma}_2^2) \, d\kappa \] (1.55)

where \( \sigma'_1 \) and \( \sigma'_2 \) denote the average rotations and average displacements. In the conventional theory these quantities represent the bending slope, and total deflection (deflection due to bending + deflection due to shearing). Let

\[ \lambda = \begin{bmatrix} \frac{d}{d\kappa} & \frac{d}{d\kappa} \\ s & 0 & -s \\ 0 & 1 & 0 \\ -s & 0 & s \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]

\[ \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} \quad \dot{\sigma} = \begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{bmatrix} \]

\[ \kappa = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 1 \end{bmatrix} \]

Then the equations (1.54) and (1.55) can be written in the forms of equations (1.5) and (1.6), respectively.

The derivation of an element dynamic-stiffness matrix will depend on the forms of continuity functions. Any form of the latter can be selected with appropriate choice of internal nodes and/or higher derivatives. For example, it is assumed that continuity functions are required in the following forms

\[ \sigma_1 = \sum_{j=1}^4 \epsilon_{1j} \kappa^{j-1} = G_1 \epsilon_1 \] (1.56)

\[ \sigma_2 = \sum_{j=1}^4 \epsilon_{2j} \kappa^{j-1} = G_2 \epsilon_2 \] (1.57)

Since the total number of undetermined coefficients is eight, the vector of nodal displacements must contain eight elements. This can be done by taking two basic internal nodes, i.e.,

\[ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sigma_1^1 & \sigma_1^\kappa & \sigma_1^{i+1} & \sigma_2^1 & \sigma_2^\kappa & \sigma_2^{i+1} \end{bmatrix} \] (1.58)

Clearly, a number of combinations of internal nodes and higher derivatives are also possible. In all cases, however, the matrices \( k \) and \( m \) of
equations (1.12) and (1.13) will be the same for the functions (1.56) and (1.57). By substituting the latter equations into the former equations, it can be shown that the matrices $k$ and $m$ are as follows

$$k = \int_0^1 \begin{bmatrix} S_1 G_1^T G_1 + (G_1^T)^T G_1 & -S_1 G_1^T G_2 \\ -S_1 (G_2^T)^T G_1 & S_1 (G_2^T)^T G_2 \end{bmatrix} d\alpha \cdot \frac{EI\sigma_k}{t}$$

$$m = \int_0^1 \begin{bmatrix} R_1 G_1^T G_1 & 0 \\ 0 & R_2 G_2^T G_2 \end{bmatrix} d\alpha \cdot \mu$$

Now, consider the case of two internal nodes. The undetermined coefficients are transformed into the elements of the vector (1.58) in the usual way, i.e.,

$$\begin{bmatrix} \sigma_1^{i_1} \\ \sigma_1^{i_2} \\ \sigma_1^{i_{i+1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \kappa_1 & \kappa_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = c_1 e_1$$

where $0 < \kappa_1 < \kappa_2 < 1$ denote the positions of internal nodes. Similarly,

$$\begin{bmatrix} \sigma_2^{i_1} \\ \sigma_2^{i_2} \\ \sigma_2^{i_{i+1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \kappa_1 & \kappa_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = c_2 e_2$$

Thus

$$\begin{bmatrix} \begin{bmatrix} \sigma_1^{i_1} \\ \sigma_1^{i_2} \end{bmatrix} \\ \begin{bmatrix} \sigma_2^{i_1} \\ \sigma_2^{i_2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

and, therefore,

$$D = \begin{bmatrix} c_1^{-1} & c_2^{-1} \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$ Then from the equations (1.27) and (1.28) the dynamic stiffness matrix becomes

$$Z = D^T (k - \lambda m) D$$

where $\lambda$ is the frequency parameter.

With the internal nodes dividing the beam into equal portions, i.e., $\kappa_1 = \frac{1}{3}$ and $\kappa_2 = \frac{2}{3}$, the dynamic-stiffness matrix for a uniform beam will be as shown in Table 1.1. Some results obtained using this matrix are given in Table 1.2. It is seen that the speed of convergence of the eigenvalues is very good. This method has also been applied, by the author, to doubly tapered Timoshenko beams with noteworthy success.
Table 1.1
Non-Dimensional Dynamic-Stiffness Matrix for Timoshenko Beams

\[
Z = \begin{bmatrix}
Z_{11} & \frac{Z_{11}^T}{2} \\
\frac{Z_{21}}{2} & Z_{22}
\end{bmatrix}
\]

\[
Z_{11} = \begin{bmatrix}
6216 + 128.8 & -128.8 \lambda \\
-7938 + 99.8 & 18144 + 648.8 \\
-2268 + 36.8 & -12474 - 81.8 & 18144 + 648.8 \\
-546 + 19.8 & 2268 - 36.8 & -7938 + 99.8 & 6216 + 128.8
\end{bmatrix}
\]

\[
Z_{21} = \begin{bmatrix}
840 & \text{Symmetric} \\
-1197 & 0 \\
504 & -1701 & 0 \\
-1197 & 504 & -1197 & 840
\end{bmatrix}
\]

\[
Z_{22} = \begin{bmatrix}
6216.8 - 128.8 \lambda \\
-7938.8 - 99.8 \lambda & 18144.8 - 648.8 \lambda \\
-2268.8 + 36.8 \lambda & -12474.8 - 81.8 \lambda & 18144.8 - 648.8 \lambda \\
-546.8 - 19.8 \lambda & 2268.8 + 36.8 \lambda & -7938.8 - 99.8 \lambda & 6216.8 - 128.8 \lambda
\end{bmatrix}
\]

Table 2.1
Eigenvalues of Clamped-Free Uniform Timoshenko Beam with
R = 0.0025
S = 100.00

<table>
<thead>
<tr>
<th>( \omega^2 )</th>
<th>\begin{tabular}{l} \text{FINITE ELEMENT METHOD} \ \text{Number of Elements} \end{tabular}</th>
<th>\begin{tabular}{l} \text{Exact} \ \text{Ref. (33)} \end{tabular}</th>
<th>% \text{Error} \text{at 4} \text{Elements}</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mode</strong></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>11.73</td>
<td>11.69</td>
<td>11.69</td>
</tr>
<tr>
<td>2</td>
<td>483.61</td>
<td>347.91</td>
<td>346.97</td>
</tr>
<tr>
<td>3</td>
<td>3817.23</td>
<td>2146.50</td>
<td>2099.59</td>
</tr>
<tr>
<td>4</td>
<td>43196.54</td>
<td>7009.29</td>
<td>5762.75</td>
</tr>
<tr>
<td>5</td>
<td>54436.58</td>
<td>15797.79</td>
<td>12349.41</td>
</tr>
</tbody>
</table>
1.5 Convergence Properties

The accuracy which may be obtained by the Finite Element method depends directly on the extend to which the continuity functions are able to approximate the actual field functions. The analytical continuity conditions at junction points are prescribed, and, therefore, as the number of elements is increased the closer will the eigenvalues and eigenvectors get to their exact values. Also, since the approximation is of the energy type, the convergence of the eigenvalues will start from an upper bound. This is usually explained by the fact that the continuity functions cannot represent the actual field function exactly, and any variation from the exact shape can be thought of as resulting from the application of internal constraints, which have the effect of stiffening the model thus producing higher eigenvalues.

In the preceding analysis a number of techniques, which are in effect devices for improving the functional approximation of the field functions, have been presented for the development of finite beam models. The "best" model will in general differ for each individual problem. However, in coupled vibration problems of beams an estimation of the "best" model can be made by studying the convergence characteristics of the component motions, i.e., bending and torsion vibrations, independently.

1.5.1 Torsional Vibrations

In this case closed solutions of the eigenvalues and eigenvectors at n elements can be found. The method is given in Appendix 4, and consists of finding an equivalent transfer matrix as shown in the article 1.3.2, whereby reducing the assembly procedure to consecutive multiplication of element matrices. It is not, however necessary to derive the transfer matrix, for the frequency equation can be obtained from the elements of the reduced element dynamic-stiffness matrix. The procedure is explained for basic elements, and results obtained for various higher order models are then summarised.

1. Basic elements (model T1): Consider the dynamic-stiffness matrix (1.53), namely,

\[
\begin{bmatrix}
1 - 2\delta & -1 - \delta \\
-1 - \delta & 1 - 2\delta
\end{bmatrix}, \quad \delta = \lambda \theta / 6
\]

From the application of equations (A4.3) and (A4.13)

\[
(1 - 2\delta) + (-1 - \delta) \cos \alpha = 0
\]

where \( \alpha = (2r+1)\pi / 2n, \quad r = 0,1,\ldots, n-1 \) for clamped-free beams, and
\[ \alpha = \frac{r \pi}{n}, \quad r = 1, 2, \ldots, n \] for free-free and clamped-clamped beams, and \( n \) denotes the number of elements. Hence

\[ \lambda_0 = 6 \cdot \frac{1 - \cos \alpha}{2 + \cos \alpha} \hspace{1cm} (1.59) \]

By writing

\[ \cos \alpha = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{2j!} \alpha^{2j} \]

it can be shown that the series expansion of a function of the type

\[ \lambda_\Theta = c \cdot \frac{1 - \cos \alpha}{a + b \cos \alpha} \]

is given by

\[ \lambda_\Theta = \frac{c}{2(a + b)} \alpha^2 \left( 1 + A_1 \alpha^2 + A_2 \alpha^4 + \ldots \right) \hspace{1cm} (1.60) \]

where \( A_j = \frac{x_{j+1}}{x_j}, \quad j = 1, 2, \ldots \), and \( x_j \) is the coefficient of \( \alpha^{2j} \) in the binomial expansion of

\[ \left( 1 - \frac{b}{a+b} \frac{\alpha^2}{2!} \left( 1 - \frac{1}{4!} \frac{\alpha^2}{6!} \alpha^4 - \ldots \right) \right)^{-1} \]

For example, \( A_1 = \frac{(5b-a)}{12(a+b)} \)
\[ A_2 = \frac{(61b^2 - 28ab + a^2)}{360(a+b)^2} \]

Hence the equation (1.59) becomes

\[ \lambda_\Theta = \alpha^2 \left( 1 + \frac{\alpha^2}{12} + \frac{\alpha^4}{360} + \ldots \right) \hspace{1cm} (1.61) \]

The exact values of \( \Omega_\Theta^2 = n^2 \lambda_\Theta \) are given by

\[ \left( r + \frac{1}{r} \right)^2 \pi^2 \alpha^2 n^2 \] for clamped-free beams, and
\[ r^2 \cdot \pi^2 \alpha^2 n^2 \] for clamped-clamped (or, free-free) beams. Thus, from the equation (1.61) it is seen that as \( n \to \infty \), the finite element solutions converge to the exact values uniformly.

ii. Elements with one internal node (model T2)

Element continuity function

\[ e_1 + \kappa e_2 + \kappa^2 e_3 \]

Element displacement vector

\[ \begin{bmatrix} \Theta_i \\ \Theta_{\kappa i} \\ \Theta_{i+1} \end{bmatrix} \]

where \( 0 < \kappa_i < 1 \) is the position of the internal node. Taking \( \kappa_i = \frac{1}{2} \), the following dynamic-stiffness matrix is obtained

\[ \begin{bmatrix} 7 - 4 \delta & -3 - 2 \delta & 1 + \delta \\ -3 - 2 \delta & 16 - 16 \delta & -8 - 2 \delta \\ 1 + \delta & -3 - 2 \delta & 7 - 4 \delta \end{bmatrix} \]

\[ \delta = \lambda_\Theta / 10 \hspace{1cm} (1.62) \]
The corresponding reduced dynamic-stiffness matrix is

\[
\frac{1}{4(1-\delta)} \begin{bmatrix}
15\delta^2 - 52\delta + 12 & -5\delta^2 - 8\delta - 12 \\
-5\delta^2 - 8\delta - 12 & 15\delta^2 - 52\delta + 12
\end{bmatrix}
\]

Hence, the frequency equation becomes

\[
(3 - \cos \alpha) \lambda^2_{\theta} - 8(13 + 2\cos \alpha)\lambda_{\theta} + 240(1 - \cos \alpha) = 0
\]

\[(1.63)\]

It can be shown that the smaller root of this equation converges to the exact values, whilst the other diverges to infinity.

iii. Elements with one internal parameter (model T3) : The function \( \kappa(1-\kappa) \) satisfies the zero displacement conditions at the nodes, and can be used in extending the unit displacement functions of the basic element, namely, \( 1 - \kappa \) and \( \kappa \) (these are obtained from the relation \( \Theta = G D \Theta \), where \( G \) and \( D \) are given by the equations (1.49) and (1.52), respectively). Then the same frequency equation as that for one internal node element will be obtained. Argyris's type internal parameter (see article 1.2.2.3) can be introduced by the function \( \kappa^2(1-\kappa)^2 \) which satisfies the zero displacement and torque conditions at the nodes of the basic element. The resulting dynamic stiffness matrix and the frequency equation will be

\[
\begin{bmatrix}
105(1 - 4\delta) & -21\delta & -105(1 + 2\delta) \\
-21\delta & 2(1 - \delta) & -21\delta \\
-105(1 + 2\delta) & -21\delta & 105(1 - 4\delta)
\end{bmatrix}
\]

\[(1.64)\]

where \( \delta = \lambda_{\theta} / 12 \).

\[(19 - \cos \alpha) \lambda^2_{\theta} - 120(5 + \cos \alpha)\lambda_{\theta} + 1440(1 - \cos \alpha) = 0
\]

\[(1.65)\]

which again has one convergent and one divergent roots.

iv. Elements with two internal nodes (model T4) :

Element displacement vector \( \mathbf{v} = \left\{ \Theta_i, \Theta_i, \Theta_{i+1}, \Theta_{i+1} \right\} \)

Element continuity function \( \Theta = e_i + \kappa e_2 + \kappa^2 e_3 + \kappa^3 e_4 \)

where \( 0 < \kappa, \kappa_i < 1 \) are the positions of the internal nodes. Taking \( \kappa_i = \frac{1}{3} \) and \( \kappa_i = \frac{2}{3} \) the following dynamic-stiffness matrix and frequency equation are obtained

\[
\begin{bmatrix}
148 - 128\delta & -189 - 99\delta & 54 + 36\delta & -13 - 19\delta \\
-189 - 99\delta & 452 - 64\delta & -297 + 81\delta & 54 + 36\delta \\
54 + 36\delta & -297 + 81\delta & 452 - 64\delta & -189 - 99\delta \\
-13 - 19\delta & 54 + 36\delta & -189 - 99\delta & 148 - 128\delta
\end{bmatrix}
\]

\[(1.66)\]

where \( \delta = \lambda_{\theta} / 42 \).

\[(4 + \cos \alpha) \lambda^3_{\theta} - 30(18 - \cos \alpha) \lambda^2_{\theta} + 360(32 + 3\cos \alpha)\lambda_{\theta} - 25200(1 - \cos \alpha) = 0
\]

\[(1.67)\]
It should be noted that this equation and the equation (1.63) hold for any positions of internal nodes. The foregoing equation can be solved by Cardan’s formulae. Then it can be shown that the smallest root converges to the exact eigenvalues, whilst the other two roots diverge to infinity as the number of elements is increased.

v. Elements with higher derivatives (model T5):

Element displacement vector \( \mathbf{u} = \{ \Theta_1, \Theta_1', \Theta_1', 1+1' \} \)

Element continuity function \( \Theta = e_1 + \kappa e_2 + \kappa^2 e_3 + \kappa^3 e_4 \)

The corresponding dynamic stiffness matrix will be

\[
\begin{bmatrix}
-36-156\delta & 3-22\delta & -36-54\delta & 3+13\delta \\
3-22\delta & 4-4\delta & -3-13\delta & -1+3\delta \\
-36-54\delta & 3-13\delta & -36-156\delta & -3+22\delta \\
3+13\delta & -1+3\delta & -3+22\delta & 4-4\delta
\end{bmatrix}
\]  \tag{1.68}

where \( \delta = \lambda / e \). The reduction of this matrix to the 2x2 basic size by the elimination of the displacements \( \Theta_1 \) and \( \Theta_1' \), which are not associated with nodal forces, leads to the frequency equation (1.67).

However, as far as the displacement formulation is concerned, such solutions will be fallacious for this model. The correct values must be obtained by the dynamic-stiffness matrix assembly procedure.

The accuracy of the five models have been studied in detail. Table 1.3 shows some typical results obtained for clamped-free beams. The eigenvalues obtained by various models are compared against the size of the beam dynamic-stiffness matrix (or, the degrees of freedom given to the beam), which need to be solved. It is seen that all the higher models except the model T5.3 are superior to the basic model. As it would be expected, two internal node elements give much better eigenvalues than the one internal node elements at the same number of degrees of freedom, especially for the higher modes of vibration. Two internal node elements and the higher derivative elements compete very closely. In general, the latter yields slightly lower results.

The eigenvectors, on the other hand, are exact at the nodes considered, for all models. This can be seen from the equation (A4.14) of Appendix 4.

It is interesting to note that in the foregoing equations (1.63), (1.65) and (1.67) the terms occurring with the higher powers of \( \lambda e \) are small compared to the other terms. Thus, as an approximation, one can write the following frequency equations
Model T1: \[ \lambda_0 = 6 \left(1 - \frac{\cos \alpha}{2 + \cos \alpha}\right) = \alpha^2 \left(1 + \frac{\alpha^2}{12} + \frac{\alpha^4}{360} + \ldots \right) \]
Model T2: \[ \lambda_0 = 30 \left(1 - \frac{\cos \alpha}{13 + \cos \alpha}\right) = \alpha^2 \left(1 - \frac{\alpha^2}{60} - \frac{\alpha^4}{1080} + \ldots \right) \]
Model T3: \[ \lambda_0 = 12 \left(1 - \frac{\cos \alpha}{5 + \cos \alpha}\right) = \alpha^2 \left(1 - \frac{\alpha^2}{360} - \ldots \right) \]
Model T4: \[ \lambda_0 = 70 \left(1 - \frac{\cos \alpha}{32 + 3 \cos \alpha}\right) = \alpha^2 \left(1 - \frac{\alpha^2}{420} - \frac{\alpha^4}{33260} + \ldots \right) \]

where the series on the right hand sides are obtained from the formula (1.60). It is seen that the approximated forms of the higher upper bound models give lower bounds to the exact eigenvalues, and better approximations than the basic model (i.e., model T1). For the well known lower bound model, namely Duncan's model, it can be shown that
\[ \lambda_0 = 2 \left(1 - \cos \alpha\right) = \alpha^2 \left(1 - \frac{\alpha^2}{12} + \frac{\alpha^4}{360} - \ldots \right) \]
i.e., this is a "dual" of the present basic model.

1.5.2 Bending Vibrations

Similar analysis can be carried out for the bending vibrations of the Bernoulli-Euler beams. The closed forms of the frequency equations are, however, practically impossible to determine, and one is forced to use digital computers.

The derivation of the element matrices follows the same pattern as that for torsional vibrations, but now a basic node is given two degrees of freedom (deflections and rotations). The dynamic-stiffness matrices for the basic, one-internal-node and one-higher-derivative uniform Bernoulli-Euler beam elements are shown in Table 1.6.

The basic element for uniform beams has been studied by a number of authors. Archer(43) and Leckie and Lindberg(44) derived, independently, the basic element dynamic-stiffness matrix, and shown that it gives better results, i.e., frequencies and mode shapes, than other methods such as Finite Differences method, Stodola method, Myklestadt's method and other methods where continuous beam parameters are physically lumped. Uhrig(52) demonstrated that the basic element dynamic-stiffness matrix can be obtained by the expansion of the terms of the exact dynamic-stiffness matrix in powers of the frequency parameter \( \lambda \). Leckie and Lindberg(44) have also shown that the error of the eigenvalues in this
idealisation is proportional to $1/n^4$, where $n$ is the number of elements, for all boundary conditions. Clearly, this error law would be valid when a uniform convergence level has been reached. In general this is attained by the first four eigenvalues, after 4 or 5 elements.

Tables 1.4a to 1.4d are prepared to indicate the typical improvements that can be expected when using one-internal-node and one-higher-derivative elements in vibration problems of uniform Bernoulli-Euler beams. It is seen that the improvements vary with boundary conditions and modes of vibration, and are substantial especially for the higher modes.

The internal node and higher derivative elements compete closely. The choice between the two models would depend on the type of data required from the eigenvectors. In the present analysis the internal node method is preferred, for it provides more points for plotting the beam deflections and slopes. The higher derivative method will be more advantageous when moments and shear forces, as well as deflections and slopes are required.

An important feature of the higher derivative approximation is the possibility of imposing the natural boundary conditions. In Tables 1.4a to 1.4c eigenvalues obtained from the use of one-higher-derivative elements with moment imposition at the appropriate ends of clamped-free, clamped-pinned and pinned-pinned beams are shown. A particular case, namely the free-free Bernoulli-Euler beam, where the application of the higher derivative approximation with imposition of the natural boundary conditions can be effective, is given in Table 1.7. The results obtained by the use of the two-higher derivative elements are also included in this Table in order to demonstrate the effect of imposing all the natural conditions at the ends of the beam.

The basic, one-internal-node and one-higher-derivative finite element solutions obtained for clamped-free, sliding-free, pinned-free and free-free wedges are compared in Tables 1.5a, 1.5b, 1.5c and 1.5d, respectively. The explicit forms of these matrices are not included here, but can be inspected from the catalogue given in Appendix 5. For example, in the notation of the latter, the one-internal-node element dynamic-stiffness matrix for a beam tapered both in depth and width will be

\[
\begin{align*}
\left\{ \sum_{j=0}^{4} b_j \lambda_Y^{(1)} - \lambda_i \sum_{j=0}^{2} a_j \lambda_Y^{(1)} \right\}
\end{align*}
\]

It is interesting to note that, at 8 degrees of freedom, the one-internal node element solutions improve the third eigenvalue of the clamped--
free wedge obtained by the basic elements by 12.2%, whilst the corresponding improvement in the case of the uniform beam is 1.3%.

The comparison of Tables 1.3 and 1.4a shows that the basic element approximation can be applied with higher accuracy for the case of bending vibrations than for that of torsional vibrations. For example, at 8 degrees of freedom the percentage error of the fourth mode eigenvalue of clamped-free beam for torsion is about 18%, whilst for bending it is about 3%. This fact is valuable in the analysis of vibrations of beams where the torsional and bending vibrations are coupled due to the non-collinearity of the shear centre and mass centre axes.
Table 1.3. Torsional Vibrations
Eigenvalues of Clamped-Free St.-Venant Beams by Various Finite Elements Models.
Exact eigenvalues are 2.46740
22.2066
61.6850
120.903

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Exact eigenvalues are (53)
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485.520
3806.75
14661.3
### Table 1.4

Bending Vibrations

b. Eigenvalues of Uniform CLAMPED-PINNED Beams by Various Finite Element Models

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Table 1.4
Bending Vibrations

c. Eigenvalues of Uniform PINNED-PINNED Beams
by Various Finite Element Models

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Exact eigenvalues are: 97.4091
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7890.66
24972.4
Table 1.4
Bending Vibrations

d. Eigenvalues* of Uniform Clamped-Clamped Beams

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*Exact eigenvalues are \(^{(53)} 500.564
3803.58
14654.7
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**Expressions in brackets indicate the relationship between
the degrees of freedom and number of elements, \(n\).
Table 1.5
Bending Vibrations
a. Frequency Values of the CLAMPED-FREE Wedge by Various Finite Element Models

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Exact values are (64): 2.306, 3.900, 5.484
Table 1.5
Bending Vibrations

b. Frequency Values of the SLIDING-FREE Wedge
by Various Finite Element Models

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Exact eigenvalues are (54): 2.568, 4.209, 5.810
# Table 1.5

**Bending Vibrations**

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Exact values are (64): 3.302, 4.965, 6.573.
Table 1.5
Bending Vibrations

d. Frequency Values of the FREE-FREE Wedge
by Various Finite Element Models

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Exact values are (64): 3.572, 5.269, 6.898.
Table 1.6
Dynamic-Stiffness Matrices
for Various Flexural Finite Element Models

a. Basic Element $\delta = \lambda / 840$

\[
\begin{bmatrix}
6 - 156 \delta & \text{Symmetric} \\
-3 - 22 \delta & 2 - 4 \delta \\
-6 - 34 \delta & -5 - 15 \delta \\
5 + 15 \delta & 1 + 3 \delta & -5 + 22 \delta & 2 - 4 \delta
\end{bmatrix}
\]

b. One-Internal-Node Element $\delta = \lambda / 792$

\[
\begin{bmatrix}
2546 - 2092 \delta & \text{Symmetric} \\
569 - 114 \delta & 166 - 8 \delta \\
-1792 - 880 \delta & -448 - 88 \delta & 3584 - 5632 \delta \\
960 + 160 \delta & 160 + 12 \delta & 0 & 640 - 128 \delta \\
-754 - 262 \delta & -121 - 29 \delta & -1792 - 880 \delta & -960 - 160 \delta & 2546 - 2092 \delta \\
121 + 29 \delta & 19 + 3 \delta & 448 + 88 \delta & 160 + 12 \delta & -569 + 114 \delta & 166 - 8 \delta
\end{bmatrix}
\]

c. One-Higher-Derivative Element $\delta = \lambda / 792$

\[
\begin{bmatrix}
1200 - 21720 \delta & \text{Symmetric} \\
600 - 3732 \delta & 384 - 832 \delta \\
30 - 281 \delta & 22 - 69 \delta & 6 - 6 \delta \\
-1200 - 6000 \delta & -600 - 1812 \delta & -50 - 181 \delta & 1200 - 21720 \delta \\
600 + 1812 \delta & 216 + 532 \delta & 8 + 52 \delta & -600 + 3732 \delta & 384 - 832 \delta \\
-30 - 181 \delta & -8 - 52 \delta & 1 - 5 \delta & 30 - 281 \delta & -22 + 69 \delta & 6 - 6 \delta
\end{bmatrix}
\]
Table 1.7
Bending Vibrations
Comparison of Eigenvalues Obtained by Higher Derivative Elements for a FREE-FREE Beam

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</table>

For the exact eigenvalues see Table 1.4d.
Fig.1.1. Node Counting System

Fig.1.2. Segmentation of a Beam

Fig.1.3. Terminal Nodal Forces
(a). Displacement Method
(b). Transfer Matrix Method
SECTION 2

2. APPLICATIONS TO PRE-TWISTED BEAMS

In this Section the general non-dimensional forms of the stiffness, inertia and dynamic-stiffness matrices for pre-twisted beams are presented. The analysis includes the effects of shear and rotatory inertia, the coupling of bending and torsion deformations due to the non-collinearity of the shear centre and centroidal axes, and the double taper. Explicit forms of the matrices are given for basic elements, with polynomial approximation of argument functions which are taken to be displacements in the principal directions and torsional twist. The use of the principal co-ordinates greatly simplifies the calculations and the subsequent computer programming. However, a method for transformation to a fixed co-ordinate system is shown.

According to the formalism of Article 1.1, the potential energy of a beam is completely defined by the matrices \( A, \alpha, \sigma, \psi \), and the kinetic energy is similarly defined by the matrices \( B, \beta, \phi, \Omega \). Therefore, in the following analysis the energies will be identified simply by specifying these quantities. It should, however, be noted that this notation is not introduced only for reference purposes. Their direct use in the derivation of the element stiffness, etc., matrices substantially simplifies the calculations.

The general expressions presented in this Section can be used when internal node displacements and higher derivatives are included in the element displacement vectors.

2.1 Pre-Twisted Uniform Beams

The first part of this article investigates the bending-bending-torsion vibrations of uniform beams where the effects of shear and rotatory inertia are negligible. In the second part these effects are taken into account.

For the co-ordinate system used the reader is referred to the figure A1.2.

2.1.1 The Non-Dimensional Dynamic-Stiffness Matrix for a Slender Beam with Non-Symmetric Cross-Sections

The strain energy of the beam is assumed to be given by the
equation (A1.25). Using the equations (A1.28) and (A1.29), the non-
dimensionalised form of this equation can be identified with the
following matrices

\[
A = \begin{bmatrix}
\frac{\partial^2}{\partial \chi^2} & \frac{\partial}{\partial \chi} & \frac{1}{2} \\
\frac{\partial^2}{\partial \chi^2} & \frac{\partial}{\partial \chi} & \frac{1}{2} \\
\frac{\partial}{\partial \chi} & \frac{1}{2} & 1
\end{bmatrix}
\]

\[
\Sigma = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
\begin{bmatrix}
1 & 0 & -\Phi V^* \\
0 & 1 & -\Phi V^* \\
-\Phi V^* & -\Phi V^* & -\Phi^2 V^* + \frac{1}{2} \\
\end{bmatrix}
\end{bmatrix}
\]

\[
\alpha = \begin{bmatrix}
1 & 1 & 1
\end{bmatrix}
\]

The non-dimensionalisation is affected by

\[
\kappa = \frac{z}{\ell}, \quad \eta = \frac{v}{\ell}, \quad \xi = \frac{u}{\ell}
\]

and the resulting non-dimensional physical beam parameters occurring in
the foregoing equations are defined as follows

\[
\gamma = \frac{I_X}{I_G}, \quad \gamma^* = \frac{I_U}{\ell I_X}
\]

\[
\gamma^*_V = \frac{I_V}{\ell I_X}, \quad \gamma^*_T = \frac{1}{\ell^2 I_X}
\]

\[
\gamma = E \frac{I_G}{J_G}
\]

Similarly, the kinetic energy given by the equation (A1.30) can be
made non-dimensional by the use of the quantities (2.2). The result
will be as follows

\[
\mathcal{B} = \beta = \begin{bmatrix}
1 & 1 & 1
\end{bmatrix}
\]

\[
\mathcal{J} = \begin{bmatrix}
1 & 0 & -\varepsilon_X \\
0 & 1 & \varepsilon_Y \\
-\varepsilon_X & \varepsilon_Y & \varepsilon_Y^2
\end{bmatrix}
\]

\[
\mathcal{O} = \{ \eta, \dot{\varepsilon}, \dot{\theta} \}
\]

where \( \varepsilon_X = e_X / \ell \), \( \varepsilon_Y = e_Y / \ell \), \( \gamma_o = r_o / \ell \), \( \gamma = r_G / \ell \),
and the following relationship between \( \gamma_o \) and \( \gamma_G \) is valid, namely,

\[
\gamma_o^2 = \gamma_G^2 + \varepsilon_X^2 + \varepsilon_Y^2
\]
The derivation of the stiffness and inertia matrices proceeds as described in Article 1.2. In this case the argument functions are,

\[ \sigma_1 = \eta, \quad \sigma_2 = \xi, \quad \sigma_3 = \vartheta \]

i.e., \( N = 3 \). It will be convenient to write the matrix, \( G \), of the continuity functions (see equation 1.9) as

\[ G_{YXO} = \begin{bmatrix} G_Y & G_X & G_\Theta \end{bmatrix} \] (2.4)

where \( G_1 = G_Y, \ G_2 = G_X \) and \( G_3 = G_\Theta \). By using the conditions (1.16) the transformation matrix, \( \Gamma \), will be found as

\[ D_{YXO} = \begin{bmatrix} D_Y & D_X & D_\Theta \end{bmatrix} \] (2.5)

Hence, the stiffness and inertia matrices for this problem can be determined from the formulae (1.27) and (1.28), i.e.,

\[ K_{YXO} = D_{YXO}^T \int_0^1 (A\alpha G_{YXO})^T A\alpha G_{YXO} \, dx \cdot D_{YXO} \] (2.6)

\[ H_{YXO} = D_{YXO}^T \int_0^1 (B\beta G_{YXO})^T B\beta G_{YXO} \, dx \cdot D_{YXO} \] (2.7)

The stiffness matrix is now obtained by substituting the equations (2.1), (2.4) and (2.5) into the equation (2.6), and performing the simple integration and matrix multiplications. The inertia matrix is obtained in the similar manner. The results of these operations are listed below

i. The stiffness matrix

\[ K_{YXO} = \begin{bmatrix} K_{11}^T & K_{12}^T & K_{13}^T \\ K_{21} & K_{22} & K_{23}^T \\ K_{31} & K_{32}^T & K_{33} \end{bmatrix} \cdot \frac{E I G_X}{\iota} \] (2.8)

where

\[ K_{11} = K_{Y} + \bar{\Phi}^2(Y R_{X0} - X R_{Y0} + \bar{\Phi}^2 N_{X0}) \] (2.8a)

\[ K_{21} = 2 \bar{\Phi} \left\{ (N_{10} - N_{20}) + \bar{\Phi}^2(N_{X0} - N_{Y0}) \right\} \] (2.8b)

\[ K_{31} = -\bar{\Phi} \left\{ (N_{10} - \bar{\Phi}^2 N_{Y0}) \nu_U^* - 2\bar{\Phi} \nu_V^* \right\} \] (2.8c)

\[ K_{22} = \nu_X^2 X_0 + \bar{\Phi}^2(\nu X_0 - \nu X_0 + \bar{\Phi}^2 \nu X_0) \] (2.8d)

\[ K_{32} = -\bar{\Phi} \left\{ (N_{10} - \bar{\Phi}^2 N_{X0}) \nu_V^* - 2\bar{\Phi} \nu_U^* \right\} \] (2.8e)
The matrices \( K_{\cdot \cdot} \), \( K_{\cdot \cdot}, P_{\cdot \cdot}, R_{\cdot \cdot}, N_{\cdot \cdot}, N_{\cdot \cdot}, M_{\cdot \cdot}, M_{\cdot \cdot} \) are defined in Article 2.2.2 by equations (2.30a) to (2.30h). The rest of the matrices in the foregoing equations have the following forms:

\[
K_{T_{Y1}} = \mathcal{D}_T \left[ \left( \mathcal{G}_{\Theta} \right)^T \mathcal{G}_{Y1} \right] dx \, dY, \quad (Y \rightarrow X) \tag{2.9a}
\]

\[
K_{T_{Y2}} = \mathcal{D}_T \left[ \left( \mathcal{G}_{\Theta} \right)^T \mathcal{G}_{Y2} \right] dx \, dY, \quad (Y \rightarrow X) \tag{2.9b}
\]

\[
K_{T_{Y3}} = \mathcal{D}_T \left[ \left( \mathcal{G}_{\Theta} \right)^T \mathcal{G}_{Y3} \right] dx \, dY, \quad (Y \rightarrow X) \tag{2.9c}
\]

\[
K_{\Theta} = \mathcal{D}_T \left[ \left( \mathcal{G}_{\Theta} \right)^T \mathcal{G}_{\Theta} \right] dx \, d\Theta \tag{2.9d}
\]

The notation \((Y \rightarrow X)\) after an equation implies that the same equation exists with all the \(Y\)s in it replaced by \(X\). This notation will be used throughout this section.

ii. The inertia matrix

\[
M_{XX\Theta} = \begin{bmatrix}
M_{Y0} & 0 & -\epsilon_X M_{\Theta Y} \\
0 & M_{X0} & \epsilon_Y M_{\Theta Y} \\
-\epsilon_X M_{\Theta Y} & \epsilon_Y M_{\Theta Y} & \epsilon^2 \Theta 
\end{bmatrix} \tag{2.10}
\]

where

\[
M_{\Theta Y} = \mathcal{D}_T \left[ \left( \mathcal{G}_{\Theta} \right)^T \mathcal{G}_{Y} \right] dx \, dY, \quad (Y \rightarrow X) \tag{2.10a}
\]

\[
M_{\Theta} = \mathcal{D}_T \left[ \left( \mathcal{G}_{\Theta} \right)^T \mathcal{G}_{\Theta} \right] dx \, d\Theta \tag{2.10b}
\]

iii. The dynamic-stiffness matrix

\[
Z_{YY\Theta} = \begin{bmatrix}
K_{11} - \lambda M_{Y0} & \text{Symmetric} \\
K_{21} & K_{22} - \lambda M_{X0}
\end{bmatrix}
\]

\[
K_{31} + \lambda \epsilon_X M_{\Theta Y} \\
K_{32} - \lambda \epsilon_Y M_{\Theta X} \\
K_{33} - \lambda M_{\Theta}
\]

where
The explicit form of the dynamic-stiffness matrix (2.11) will obviously depend on the continuity functions, i.e., $G_Y$, $G_X$, $G_\Theta$. Although there is a wide variety of complete sets which can be used in the approximations, the simple polynomials provide the simplest matrices with satisfactory accuracy, and when non-uniform beams are considered no additional difficulties arise in the integration process. On the other hand, any polynomial set, for example the Legendre polynomials, which can be reduced to the simple power series, will give the same dynamic-stiffness matrix as that obtained from the latter. The component matrices occurring in the equations (2.8a) to (2.8f), (2.10), and (2.11a) to (2.11c) will be termed the "submatrices" of the approximation.

A catalogue of the submatrices of polynomial approximation is given in Appendix 5, from which a variety of stiffness, inertia and dynamic-stiffness matrices can be formed for basic and one-internal node flexural elements, basic and one, and two, internal-node torsional elements, and for any combination of these.

The explicit form of the basic dynamic-stiffness matrix resulting from the approximations:
\[
G = \begin{bmatrix} 1 & \kappa \\ \end{bmatrix}, \\
G_Y = \begin{bmatrix} 1 & \kappa & \kappa^2 & \kappa^3 \\ \end{bmatrix}, \\
G_X = \begin{bmatrix} 1 & \kappa & \kappa^2 & \kappa^3 \\ \end{bmatrix}
\]
and including the effects of rotatory inertia and shear displacements, is given in Table 2.1. From this, the basic form of the matrix (2.11) can easily be observed.
The Non-Dimensional Dynamic-Stiffness Matrix for a Non-Symmetric Beam, Including the Effects of Rotatory Inertia and Shear Displacements

The kinetic energy due to the rotatory motion of cross-sections, expressed in principal co-ordinate system, is given by the equation (A1.38), which when non-dimensionalised by the quantities (2.2), becomes

\[
\begin{bmatrix}
-\Phi & -\Phi \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial x}
\end{bmatrix}
\]

\[
\hat{\Omega} = \{\hat{\gamma}, \hat{\xi}\}
\]

\[
\Omega = \begin{bmatrix}
\gamma & 0 & 0 & \gamma \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
\gamma & 0 & 0 & \gamma
\end{bmatrix}
\]

\[
\hat{\Phi}^T = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]

The effect of rotatory inertia alone will affect the translational inertia matrix, given by the equation (2.10), only. It is noticed that in the foregoing equations only the co-ordinates \(\gamma\) and \(\xi\) are present. This is a result of the simplifying assumptions made in the theory, which is presented in Appendix 1. The substitution of the continuity functions \(G_X\) and \(G_Y\) into the general inertia matrix equation (1.28) will result in the following rotatory inertia matrix

\[
M_R = \begin{bmatrix}
M_{R11} & M_{R12}^T & 0 \\
M_{R21} & M_{R22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\mu I_0 G_X^l
\]

(2.13)

where

\[
M_{R11} = \Phi \gamma M_X + R_Y 
\]

(2.13a)

\[
M_{R21} = \Phi (N_0 - \gamma N_0) 
\]

(2.13b)

\[
M_{R22} = \Phi^2 M_X + \gamma R_X 
\]

(2.13c)

* See the equations (2.30) in the next Article for the definition of the matrices \(M_X, M_Y, R_Y, R_X, N_0,\) and \(N_{0}^\lambda\).
and the zeros are introduced to accommodate the torsional co-ordinate $\Theta^i$ (and also, $\zeta$ and $\psi$ when the shear displacement is considered). With this notation the complete inertia matrix becomes

$$I'_{yx} = I_{yx}^t + M_R \quad (2.14)$$

ii. The effect of shear displacement is such that both the translational inertia matrix (2.10) and the stiffness matrix (2.8) will be modified. Two methods of the finite treatment of beam vibration problems where the shear displacement effect is considered had been successfully applied by the author in relation to non-uniform Timoshenko beams. In the first method the static equations of equilibrium are used in forming the continuity functions, and the resulting basic element matrix is of the order $4 \times 4$. In the second approach (see Article 1.4.2) the use of two internal nodes is made for permitting the approximation of bending slopes and total deflections by two separate cubic polynomials. The size of the element matrices in this case is $8 \times 8$. Both these methods can be extended to the case of coupled vibrations. However, the derivation of the element matrices becomes complicated and the convergence characteristics is rather poor if the first method is used, and, on the other hand, the second method will give rise to matrices of large order which will not permit an effective application on an average sized digital computer.

The third method presented here is particularly suitable to coupled vibration problems. It differs from the above mentioned approaches in that the shear displacements are explicitly introduced into the analysis by assigning two extra degrees of freedom to a basic node, corresponding to shear displacements in two perpendicular directions.

The displacement vector of the problem will be taken as

$$\begin{bmatrix} V_b \\ U_b \\ \Theta \\ V_s \\ U_s \end{bmatrix}$$

in the principal frame, subscripts $b$ and $s$ referring to bending and shear, respectively. If the corresponding non-dimensional vector is taken as

$$\begin{bmatrix} \eta \\ \xi \\ \Theta \\ \zeta \\ \psi \end{bmatrix} \quad (2.15)$$

where

$$\eta = \frac{V_b}{\ell} , \quad \xi = \frac{U_b}{\ell} , \quad \zeta = \frac{V_s}{\ell} , \quad \psi = \frac{U_s}{\ell} \quad (2.16)$$

the kinetic energy of translation, as obtained from equations (A1.35), (A1.39) and (A1.29), can be identified with the following forms
Let the matrix of continuity functions, \( G \), be taken as

\[
\begin{bmatrix}
G_Y & G_X & G_{\Theta} & G_{S_Y} & G_{S_X}
\end{bmatrix}
\]  

which has one-to-one correspondence with the vector (2.15). By using the conditions (1.16), the transformation matrix, \( D \), will be found as

\[
\begin{bmatrix}
D_Y & D_X & D_{\Theta} & D_{S_Y} & D_{S_X}
\end{bmatrix}
\]  

Hence, it can be shown that the new inertia matrix of translation is

\[
\mathbf{I}^{XY\Theta} = \begin{bmatrix}
\mu a l^3
\end{bmatrix}
\]  

where

\[
M_{\Theta SY} = D_{SY}^T \int_0^1 G_T \mathbf{G} \cdot dX \cdot D_{\Theta} \quad (Y \rightarrow X)
\]

and the remaining submatrices \( M_{S_Y} \), \( M_{S_X} \), \( M_{S_{YY}} \), and \( M_{S_{XX}} \) are defined by the equations (2.37a) and (2.37b) of the Article 2.2.3.

The stiffness matrix due to shear displacements is obtained from the non-dimensional forms of the strain energy equation (A1.41), namely,
\[ A = \begin{bmatrix} -\Phi & -\Phi & \frac{\partial}{\partial \kappa} & \frac{\partial}{\partial \kappa} \\ & & & \\ \sigma = \{ \zeta \ , \ \psi \} \\ & & & \\ \mathcal{D} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \end{bmatrix} \cdot k\mathcal{G}\mathcal{G}' \\
\end{bmatrix} \]

\[ (2.21) \]

\[ \mathcal{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \end{bmatrix} \]

Hence, the stiffness matrix of shear can be written as

\[ K_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{44} & \mathcal{D}^T \cdot k\mathcal{G}\mathcal{G}' \\ 0 & K_{54} & K_{55} \end{bmatrix} \]

where

\[ K_{44} = K_{S_Y} + \Phi^2 M_{SYY} \]

\[ K_{54} = \Phi N_S \]

\[ K_{55} = K_{S_Y} + \Phi^2 M_{SXX} \]

\[ (2.22a) \]

\[ (2.22b) \]

\[ (2.22c) \]

and the zeros are introduced for the accommodation of the co-ordinates \( \eta, \xi \) and \( \Theta \). Again, the definitions of the submatrices \( K_{S_Y} \), \( K_{S_Y} \) and \( N_S \) are given by the equations (2.37c) and (2.37d) of Article 2.2.3.

The combined stiffness matrix of the element is obtained by the addition of the equations (2.22) and (2.8), namely,

\[ S_{KYY}^k = K_{YY}^k + K_S \]

\[ (2.23) \]

Similarly, the inertia matrix where shear and rotatory inertia effects are included, is obtained from the equations (2.20) and (2.13), namely,

\[ S_{MYY}^k = S_{MYY}^k + M_{\kappa} \]

\[ (2.24) \]

iii. The dynamic-stiffness matrix

\[ S_{ZYY}^{\kappa \kappa} = \]
The explicit form of this matrix is given in Table 2.1 for the basic polynomial approximation, i.e.,

\[ G_X = G_X = \begin{bmatrix} 1 & \kappa & \kappa^2 & \kappa^3 \end{bmatrix} \]

\[ G = G_{Sy} = G_{Sx} = \begin{bmatrix} 1 & \kappa \end{bmatrix} \]

A variety of improved models can be obtained from the submatrices listed in Appendix 5.

2.2 Pre-Twisted Non-Uniform Beams

The matrices derived in the preceding part of this Section can be employed in the analysis of any non-uniform beam by considering the beam as made up of stepped uniform segments. While such analysis can be effective for beams where the non-uniformities are gradual and small, in the case of taper of the rather high order the convergence properties of the results will be slow, resulting in an increase in computer time and space.

In cases where the non-uniformity of sections is analytically defined, the dynamic-stiffness matrix can be derived without any difficulty, with the aid of polynomial approximation. As a result, computer time and space will be saved, and there will be more scope to get more accurate results.

In this Article the element matrices for a pre-twisted slender beam which is linearly tapered in both the principal directions are
presented. The analysis is restricted to bending-bonding vibrations. However, it can be extended to non-symmetric beams, too, if the analytical expressions for the cross-sectional moduli of such beams are available. Later, the modifications necessary for taking into account the effects of shear and rotatory inertia are briefly indicated.

General expressions are given for the stiffness and inertia matrices. The formulae for the derivations of submatrices are also presented. The matrices listed in Appendix 5 can be utilised in determining the explicit forms of the stiffness and inertia matrices with polynomial approximation, for basic and one-internal-node elements.

2.2.1 Cross-Sectional Moduli for a Linearly Tapered Beam

The cross-sectional shapes of the beam are taken so that the areas and second moments of area conform with the following relations

\[ A \propto h_x h_y \]

\[ I_{Gx} \propto h_x^3 h_y \]

\[ I_{Gy} \propto h_x^3 h_y \]

where \( h_x \) and \( h_y \) are some dimensions of the sections in the principal directions \( X \) and \( Y \), respectively. For example, for rectangular, circular, triangular and elliptical sections the foregoing relations will be true.

Consider a beam element whose terminal nodes are denoted by \( i \) and \( i+1 \). The cross-sectional properties of this beam at the nodes \( i \) and \( i+1 \) will be referred to by subscripts \( i \) and \( i+1 \), respectively.

In the case of linear taper, it can be shown that \( A \), \( I_{Gx} \) and \( I_{Gy} \) at any point on the beam can be written as

\[ A = A_i A^* \]

\[ I_{Gx} = I_{Gx_i} I_{Gx}^* \]

\[ I_{Gy} = I_{Gy_i} I_{Gy}^* \]

where

\[ A^* = a_0 + a_1 \kappa + a_2 \kappa^2 \]

\[ I_{Gx}^* = b_0 + b_1 \kappa + b_2 \kappa^2 + b_3 \kappa^3 + b_4 \kappa^4 \]

\[ I_{Gy}^* = c_0 + c_1 \kappa + c_2 \kappa^2 + c_3 \kappa^3 + c_4 \kappa^4 \]
where, \( a_0 = 1 \) ; \( b_0 = 1 \) ; \( c_0 = 1 \)

\[
a_1 = K_{Y_1} + \frac{H_{X_1}}{a_0} \quad ; \quad b_1 = 3K_{Y_1} + \frac{H_{X_1}}{a_1} \quad ; \quad c_1 = 3H_{X_1} + \frac{H_{Y_1}}{a_1}
\]

\[
a_2 = H_{Y_1} \quad ; \quad b_2 = 3H_{Y_1} \left( H_{Y_1} + H_{X_1} \right) \quad ; \quad c_2 = 3H_{X_1} \left( H_{Y_1} + H_{X_1} \right)
\]

\[
b_3 = H_{X_1}^{2} \left( H_{X_1} + \frac{3H_{X_1}}{a_2} \right) \quad ; \quad c_3 = H_{X_1}^{2} \left( H_{X_1} + \frac{3H_{X_1}}{a_2} \right)
\]

\[
b_4 = H_{X_1}^{3} \quad ; \quad c_4 = H_{X_1}^{3} \frac{H_{Y_1}}{a_4}
\]

where,

\[
H_{Y_i} = \left( \frac{h_{Y_{i+1}}}{h_{Y_i}} \right) - 1 \quad \text{(2.26a)}
\]

\[
H_{X_i} = \left( \frac{h_{X_{i+1}}}{h_{X_i}} \right) - 1 \quad \text{(2.20b)}
\]

The quantities \( H_{Y_i} \) and \( H_{X_i} \) will subsequently be called element depth and width taper parameters, respectively.

If the beam depth and width taper parameters are denoted by

\[
H_{Y_n} = \left( \frac{h_{Y_n}}{h_{Y_0}} \right) - 1
\]

\[
H_{X_n} = \left( \frac{h_{X_n}}{h_{X_0}} \right) - 1
\]

then the following relations can be shown

\[
H_{Y_i} = H_{Y_n} / \left( n + i.H_{Y_n} \right)
\]

\[
H_{X_i} = H_{X_n} / \left( n + i.H_{X_n} \right)
\]

\[
I_{G_{Y_i}} = I_{G_{Y_n}} \left( 1 + i.H_{Y_n} / n \right) \left( 1 + i.H_{X_n} / n \right)
\]

\[
I_{G_{X_i}} = I_{G_{X_n}} \left( 1 + i.H_{X_n} / n \right) \left( 1 + i.H_{X_n} / n \right)
\]

where \( i = 0, 1, 2, \ldots, n-1 \), and \( n \) is the number of elements.

\[\text{2.2.2 The Non-Dimensional Dynamic-Stiffness Matrix for Linearly Tapered Slender Beams}\]

The non-dimensional form of the strain energy, in the principal co-ordinate system, for the present case can be shown to be identifiable
with the following forms (see equation A1.30)

\[
A = \begin{bmatrix}
-\Phi^2 & -\Phi \beta & -2\Phi \frac{\beta}{\delta x} & -2\Phi \frac{\beta}{\delta x^2} & -\Phi \frac{\beta}{\delta x^2} \\
0 & \frac{\beta}{\delta x^2} & 2\frac{\beta}{\delta x^3} & 0 & 0 \\
0 & 0 & \frac{\beta}{\delta x^2} & 0 & 0 \\
0 & 0 & 0 & \frac{\beta}{\delta x^2} & 0 \\
0 & 0 & 0 & 0 & \frac{\beta}{\delta x^2}
\end{bmatrix}
\]

\[
\sigma = \begin{bmatrix}
\eta \\
\xi
\end{bmatrix}
\]

\[
\mathbf{D} = \begin{bmatrix}
I_{1G_X}^* & 0 & 0 & -I_{1G_X}^* & I_{1G_X}^* & 0 \\
0 & \gamma_1 \lambda_{1G_Y}^* & \gamma_1 \lambda_{1G_Y}^* & 0 & 0 & \gamma_1 \lambda_{1G_Y}^* \\
0 & \gamma_1 \lambda_{1G_Y}^* & \gamma_1 \lambda_{1G_Y}^* & 0 & 0 & \gamma_1 \lambda_{1G_Y}^* \\
-I_{1G_X}^* & 0 & 0 & I_{1G_X}^* & -I_{1G_X}^* & 0 \\
I_{1G_X}^* & 0 & 0 & -I_{1G_X}^* & I_{1G_X}^* & 0 \\
0 & \gamma_1 \lambda_{1G_Y}^* & \gamma_1 \lambda_{1G_Y}^* & 0 & 0 & \gamma_1 \lambda_{1G_Y}^*
\end{bmatrix}
\]

\[
\alpha^T = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

where \( \gamma_1 = \frac{I_{1G_Y}}{I_{1G_X}} \), and the non-dimensionalisation is affected by the quantities (2.2).

Taking \( G = \begin{bmatrix} G_Y & G_X \end{bmatrix} \), the stiffness matrix can be written as

\[
\bar{K}_{YX} = \begin{bmatrix}
\bar{K}_{11} & \bar{K}_{12}^T \\
\bar{K}_{21} & \bar{K}_{22}
\end{bmatrix}
\]

where \( (\sim) \) denotes that taper is considered, and the submatrices have the following forms

\[
\bar{K}_{11} = \sum_{j=0}^{h_1} \left[ b_j \left( K_{Y,j} + \Phi^2 \left( \Phi^2 N_{X,j} - H_{Y,j} \right) \right) + a_j \gamma_1 \Phi^2 R_{Y,j} \right]
\]

\[
\bar{K}_{21} = \sum_{j=0}^{h_1} 2 \Phi \left( b_j \left( N_{1,j} - \Phi^2 N_{Y,j} \right) - c_j \gamma_1 \left( K_{2,j} - \Phi^2 N_{Z,j} \right) \right)
\]

\[
\bar{K}_{22} = \sum_{j=0}^{h_1} \left[ c_j \gamma_1 \left( K_{X,j} + \Phi^2 \left( \Phi^2 M_{X,j} - H_{X,j} \right) \right) + b_j \Phi^2 R_{X,j} \right]
\]
where,
\[
\begin{align*}
K_{Yj} &= D_Y^T \left\{ \int_0^1 \chi_j \cdot (G_{Yj}''')^T G_{Yj} d\chi \right\} D_Y, \quad (Y \leftrightarrow X) \quad (2.30a) \\
R_{Yj} &= D_Y^T \left\{ \int_0^1 \chi_j \cdot (G_{Yj}')^T G_{Yj} d\chi \right\} D_Y, \quad (Y \leftrightarrow X) \quad (2.30b) \\
M_{Yj} &= D_Y^T \left\{ \int_0^1 \chi_j \cdot (G_{Yj})^T G_{Yj} d\chi \right\} D_Y, \quad (Y \leftrightarrow X) \quad (2.30c) \\
\Pi_{Yj} &= D_Y^T \left\{ \int_0^1 \chi_j \cdot (G_{Yj}''')^T G_{Yj} d\chi \right\} D_Y, \quad (Y \leftrightarrow X) \quad (2.30d) \\
N_1 &= D_X^T \left\{ \int_0^1 \chi_j \cdot (G_{Xj})^T G_{Xj} d\chi \right\} D_Y \\
N_2 &= D_X^T \left\{ \int_0^1 \chi_j \cdot (G_{Xj}')^T G_{Xj} d\chi \right\} D_Y \\
N_3 &= D_X^T \left\{ \int_0^1 \chi_j \cdot (G_{Xj})^T G_{Xj} d\chi \right\} D_Y \\
N_4 &= D_X^T \left\{ \int_0^1 \chi_j \cdot (G_{Xj})^T G_{Xj} d\chi \right\} D_Y \\
N_5 &= D_X^T \left\{ \int_0^1 \chi_j \cdot (G_{Xj})^T G_{Xj} d\chi \right\} D_Y \\
\end{align*}
\]

ii. The kinetic energy of translation of linearly tapered beam can be determined from the following non-dimensional matrix forms
\[
\begin{align*}
\dot{b} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \beta \\
\dot{c} &= \begin{bmatrix} \eta & \xi \end{bmatrix} \\
c &= \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{c}_1 & \bar{c}_2 \end{bmatrix} \begin{bmatrix} 4 \end{bmatrix} \\
\end{align*}
\]
Hence, the inertia matrix becomes
\[
\begin{align*}
\tilde{M}_{XX} &= \begin{bmatrix} M_{Ya} & \bar{M}_{Ya} \end{bmatrix} \cdot \begin{bmatrix} \bar{c}_1 & \bar{c}_2 \end{bmatrix} \begin{bmatrix} 4 \end{bmatrix} \\
\end{align*}
\]
where
\[
\begin{align*}
\tilde{M}_{Ya} &= \sum_{j=0}^{2} a_j M_{Yj} \quad (Y \leftrightarrow X) \quad (2.22a) \\
\end{align*}
\]

iii. The dynamic-stiffness matrix associated with the equations (2.29) and (2.31) will then be
\[
\bar{K}_{XX} = \begin{bmatrix} K_{11} - \lambda_i \bar{M}_{Ya} & \bar{K}_{21} \\
\bar{K}_{21} & K_{22} - \lambda_i \bar{M}_{Ya} \end{bmatrix} \\
\begin{align*}
\bar{K}_{11} &= \tilde{K}_{11} - \lambda_i \tilde{M}_{Ya} \\
\bar{K}_{21} &= \tilde{K}_{21} \\
\bar{K}_{22} &= \tilde{K}_{22} - \lambda_i \tilde{M}_{Ya} \\
\end{align*}
\]
\[
\begin{align*}
\bar{K}_{XX} &= \begin{bmatrix} K_{11} - \lambda_i \bar{M}_{Ya} & \bar{K}_{21} \\
\bar{K}_{21} & K_{22} - \lambda_i \bar{M}_{Ya} \end{bmatrix} \\
\end{align*}
\]
2.2.3 Modifications of the Stiffness and Inertia Matrices of Linearly Tapered Slender Beams Due to the Effects of Rotatory Inertia and Shear Deflection

The inclusion of the effects of rotatory inertia and shear deflection exactly follows the same pattern of the analysis presented in Article 1.1.2. Therefore, here only the results are given.

\[
\mathbf{\bar{K}}_{\text{XX}} = \mu I_{g_{x_1}} \ell \sum_{j=0}^{b} \left[ b_j R_y^2 + c_j \Phi^2 \gamma_j M_y^2 \right] \left( c_j \Phi^2 N_x^2 \right) \\
\mathbf{\bar{M}}_{\text{XX}} = \mu I_{g_{x_1}} \ell \sum_{j=0}^{b} \left[ b_j R_y^2 + c_j \Phi^2 \gamma_j M_y^2 \right] \left( c_j \Phi^2 N_x^2 \right) \\
\mathbf{\bar{S}}_{\text{XX}} = \mu I_{g_{x_1}} \ell \sum_{j=0}^{b} \left[ b_j R_y^2 + c_j \Phi^2 \gamma_j M_y^2 \right] \left( c_j \Phi^2 N_x^2 \right)
\]

\[
\mathbf{\bar{C}}_{\text{YX}} = \mu I_{g_{x_1}} \ell \sum_{j=0}^{b} \left[ b_j R_y^2 + c_j \Phi^2 \gamma_j M_y^2 \right] \left( c_j \Phi^2 N_x^2 \right)
\]

where,

\[
\mu_{S_{yj}} = D_{SY}^T \left\{ \int_0^1 \kappa^j \cdot G_{SY}^T \cdot G_{SY} \, d\kappa \right\} D_S
\]

\[
M_{S_{yj}} = D_{SY}^T \left\{ \int_0^1 \kappa^j \cdot G_{SY}^T \cdot G_{SY} \, d\kappa \right\} D_S
\]

\[
K_{S_{yj}} = D_{SY}^T \left\{ \int_0^1 \kappa^j \cdot (G_{SY}^T \cdot G_{SY}) \, d\kappa \right\} D_S
\]

\[
N_{S_{yj}} = D_{SY}^T \left\{ \int_0^1 \kappa^j \cdot G_{SY}^T \cdot (G_{SY}^T \cdot G_{SY}) \, d\kappa \right\} D_S
\]
Table 2.1
Non-Dimensional Basic Dynamic-Stiffness Matrix for an Uniform Asymmetric Beam, Including the Effects of Rotatory Inertia and Shear Displacements.

\[
S_{\text{sym}} = \begin{bmatrix}
Z_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
Z_{55} & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

\[
Z_{11} = \begin{vmatrix}
12 + \Phi^2 \left( \frac{12}{5} + \frac{24}{35} \phi^2 \right) + \frac{13}{35} \Phi^2 \\
-\lambda \left( \frac{13}{35} + R \left( \frac{6}{5} + \frac{13}{35} \phi^2 \right) \right) \\
6 + \Phi^2 \left( \frac{6}{5} + \frac{24}{35} \phi^2 \right) + \frac{11}{210} \Phi^2 \\
-\lambda \left( \frac{11}{210} + R \left( \frac{6}{5} + \frac{11}{210} \phi^2 \right) \right) \\
-12 - \Phi^2 \left( \frac{12}{5} + \frac{24}{35} \phi^2 \right) - \frac{13}{35} \Phi^2 \\
-\lambda \left( \frac{9}{70} + R \left( \frac{6}{5} + \frac{9}{70} \phi^2 \right) \right) \\
6 + \Phi^2 \left( \frac{1}{5} + \frac{2}{5} \phi^2 \right) - \frac{12}{5} \Phi^2 \\
-\lambda \left( \frac{13}{420} + R \left( \frac{1}{10} + \frac{13}{420} \phi^2 \right) \right) \\
2 - \Phi^2 \left( \frac{1}{15} + \frac{2}{5} \phi^2 \right) + \frac{11}{210} \Phi^2 \\
\lambda \left( \frac{1}{140} + R \left( \frac{1}{30} + \frac{11}{210} \phi^2 \right) \right) \\
\end{vmatrix}
\]
\[ z_{22} = \begin{bmatrix} 12 \nu \Phi^2 \left( \frac{12}{5} \nu + \frac{24}{5} + \frac{13}{35} \nu \Phi^2 \right) \\ -\lambda \left( \frac{13}{35} + \nu \left( \frac{6}{5} + \frac{13}{35} \Phi^2 \right) \right) \\ 6 \nu + \Phi^2 \left( \frac{6}{5} \nu + \frac{2}{5} + \frac{11}{210} \nu \Phi^2 \right) \\ -\lambda \left( \frac{11}{210} + \nu \left( \frac{11}{210} \nu + \frac{13}{35} \Phi^2 \right) \right) \\ -12 \nu + \Phi^2 \left( -\frac{12}{5} \nu - \frac{24}{5} + \frac{9}{70} \nu \Phi^2 \right) \\ -\lambda \left( \frac{9}{70} + \nu \left( -\frac{6}{5} \nu + \frac{9}{70} \Phi^2 \right) \right) \\ 6 \nu + \Phi^2 \left( \frac{1}{5} \nu + \frac{2}{5} + \frac{13}{420} \nu \Phi^2 \right) \\ -\lambda \left( -\frac{15}{420} + \nu \left( \frac{1}{10} \nu + \frac{15}{420} \Phi^2 \right) \right) \\ 2 \nu + \Phi^2 \left( -\frac{1}{5} \nu - \frac{2}{5} - \frac{11}{210} \nu \Phi^2 \right) \\ -6 \nu - \Phi^2 \left( \frac{6}{5} \nu + \frac{2}{5} + \frac{11}{210} \nu \Phi^2 \right) \\ 4 \nu + \Phi^2 \left( \frac{4}{15} \nu + \frac{2}{15} + \frac{11}{210} \nu \Phi^2 \right) \\ -\lambda \left( \frac{11}{210} + \nu \left( \frac{1}{10} \nu + \frac{11}{210} \Phi^2 \right) \right) \end{bmatrix} \]

**SYMMETRIC**

\[ \begin{bmatrix} \nu \Phi^2 \left( \frac{12}{5} \nu + \frac{24}{5} + \frac{13}{35} \nu \Phi^2 \right) \\ \nu \Phi^2 \left( \frac{6}{5} \nu + \frac{2}{5} + \frac{11}{210} \nu \Phi^2 \right) \\ \nu \Phi^2 \left( \frac{1}{5} \nu + \frac{2}{5} + \frac{13}{420} \nu \Phi^2 \right) \\ \nu \Phi^2 \left( -\frac{1}{5} \nu - \frac{2}{5} - \frac{11}{210} \nu \Phi^2 \right) \\ \nu \Phi^2 \left( \frac{4}{15} \nu + \frac{2}{15} + \frac{11}{210} \nu \Phi^2 \right) \\ \nu \Phi^2 \left( \frac{4}{15} \nu + \frac{2}{15} + \frac{11}{210} \nu \Phi^2 \right) \end{bmatrix} \]

\[ \lambda \left( \frac{13}{35} + \nu \left( \frac{6}{5} + \frac{13}{35} \Phi^2 \right) \right) \]

\[ \lambda \left( \frac{15}{420} + \nu \left( \frac{15}{420} \nu + \frac{13}{35} \Phi^2 \right) \right) \]

\[ \lambda \left( \frac{11}{210} + \nu \left( \frac{11}{210} \nu + \frac{13}{35} \Phi^2 \right) \right) \]

\[ \lambda \left( \frac{1}{10} \nu + \frac{15}{420} \Phi^2 \right) \]

\[ \lambda \left( \frac{11}{210} + \nu \left( \frac{1}{10} \nu + \frac{11}{210} \Phi^2 \right) \right) \]

\[ \lambda \left( \frac{1}{10} \nu + \frac{11}{210} \Phi^2 \right) \]

\[ \lambda \left( \frac{1}{10} \nu + \frac{11}{210} \Phi^2 \right) \]
Table 2.1 (cont.)

\[
\begin{align*}
Z_{21} &= \begin{pmatrix}
-\left(\Phi^2 + \frac{1}{3} \lambda \right) \left(\frac{\nu - 1}{\nu + 1}\right) \\
-(2 + \frac{1}{3} \Phi^2 - \frac{1}{10} \lambda) \\
-\left(\Phi^2 - \frac{1}{3} \lambda\right) \\
(2 + \frac{1}{3} \Phi^2 - \frac{1}{10} \lambda)
\end{pmatrix} \\
Screw & \quad Symmetric \\
& \quad (\nu + 1) \Phi
\end{align*}
\]

\[
Z_{31} = \begin{pmatrix}
\frac{1}{8} \Phi^2 (\Phi \gamma_v - \Phi \gamma_u) \\
+ \frac{7}{20} c_x \lambda \\
-\Phi (1 + \frac{1}{12} \Phi^2) \gamma_u \\
+ \frac{3}{20} c_x \lambda
\end{pmatrix} \\
+ \frac{1}{20} c_x \lambda \\
\Phi (1 + \frac{1}{12} \Phi^2) \gamma_u \\
+ \frac{7}{20} c_x \lambda
\end{pmatrix} \\
-\Phi (1 + \frac{1}{12} \Phi^2) \gamma_u \\
-\frac{1}{20} c_x \lambda
\end{pmatrix} \\
\Phi (1 + \frac{1}{12} \Phi^2) \gamma_u \\
-\frac{1}{20} c_x \lambda
\end{pmatrix}
\]

\[
Z_{41} = Z_{52} = -\lambda \left[\begin{array}{c}
\frac{21}{60} \\
\frac{1}{9} \\
\frac{1}{21} \\
\frac{1}{21}
\end{array}\right] \\
Z_{43} = \frac{1}{6} \lambda c_x \left[\begin{array}{c}
2 \\
1 \\
2 \\
1
\end{array}\right] \\
Z_{53} = -\frac{1}{6} \lambda c_y \left[\begin{array}{c}
2 \\
1 \\
1 \\
2
\end{array}\right]
\]

Table 2.1 (contd.,)

\[ z_{32} = \begin{bmatrix}
\frac{1}{2} \Phi^2 (4 \nu_U - \Phi \nu_V) & -\Phi (1 + \frac{1}{12} \Phi^2) \nu_V & -\frac{1}{2} \Phi^2 (4 \nu_U + \Phi \nu_V) & \Phi (1 + \frac{1}{12} \Phi^2) \nu_V \\
-\frac{7}{20} c_Y \lambda & -\frac{1}{20} c_Y \lambda & -\frac{3}{20} c_Y \lambda & +\frac{1}{30} c_Y \lambda \\
-\frac{3}{20} c_Y \lambda & -\frac{1}{30} c_Y \lambda & -\frac{7}{20} c_Y \lambda & +\frac{1}{30} c_Y \lambda
\end{bmatrix} \]

\[ z_{33} = \begin{bmatrix}
(\Phi^2 \nu_T + \frac{1}{\nu'_{\Phi}}) - \frac{1}{6} \lambda & -(\Phi^2 \nu_T + \frac{1}{\nu'_{\Phi}}) - \frac{1}{6} \lambda \\
-(\Phi^2 \nu_T + \frac{1}{\nu'_{\Phi}}) - \frac{1}{6} \lambda & (\Phi^2 \nu_T + \frac{1}{\nu'_{\Phi}}) - \frac{1}{6} \lambda
\end{bmatrix} \]

\[ z_{44} = z_{55} = \begin{bmatrix}
(1 + \frac{1}{6} \Phi^2) s - \frac{1}{6} \lambda & -(1 + \frac{1}{6} \Phi^2) s - \frac{1}{6} \lambda \\
-(1 + \frac{1}{6} \Phi^2) s - \frac{1}{6} \lambda & (1 + \frac{1}{6} \Phi^2) s - \frac{1}{6} \lambda
\end{bmatrix} \]

\[ z_{54} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} s \Phi \]
2.3 Transformation to a Fixed Co-ordinate System

In the determination of the dynamic-stiffness matrices presented in the preceding analysis the "principal co-ordinate system" was used. However, a local fixed frame, generally taken in the direction of the principal axes at the origin, can be used, too. If the continuity functions are chosen with their meaning implied in the preceding analysis, the resulting dynamic-stiffness matrix will yield the same eigenvalues, but, evidently, the eigenvectors will be those associated with the fixed directions.

For simplicity, consider a basic beam element of the type investigated in Article 2.1.2. The strain energy of this element in the fixed co-ordinate system can be written as

\[ \mathcal{W} = \frac{1}{2} \mathbf{e}^T \mathbf{k}_{yx\Theta} \mathbf{e} \]

where \( \mathbf{e} = \{ \mathbf{e}_Y, \mathbf{e}_X, \mathbf{e}_\Theta \} \) are the undetermined constants occurring in the equation (1.9), i.e.,

\[ \mathbf{\eta} = \mathbf{G}_Y \mathbf{e}_Y, \quad \mathbf{\xi} = \mathbf{G}_X \mathbf{e}_X, \quad \mathbf{\Theta} = \mathbf{G}_\Theta \mathbf{e}_\Theta \]

and the square matrix \( \mathbf{k}_{yx\Theta} \) is given by the equation (1.12). The undetermined constants are then obtained, from the geometry of the element, in terms of the element nodal displacements, which are now in the fixed directions. Thus, denoting the displacements in the fixed frame by a bar on the corresponding principal displacements

\[ \mathbf{\bar{\mathbf{e}}} = \mathbf{\Gamma} \mathbf{e} \]

where

\[ \mathbf{\bar{\mathbf{e}}} = \{ \mathbf{\bar{\eta}}_i, \mathbf{\bar{\eta}}_i', \mathbf{\bar{\xi}}_i, \mathbf{\bar{\xi}}_i', \mathbf{\bar{\Theta}}_i, \mathbf{\bar{\eta}}_{i+1}, \mathbf{\bar{\eta}}_{i+1}', \mathbf{\bar{\xi}}_{i+1}, \mathbf{\bar{\xi}}_{i+1}', \mathbf{\bar{\Theta}}_{i+1} \} \]

say. Since a fixed local non-dimensional co-ordinate system is used, \( \mathbf{\bar{\eta}}_i, \mathbf{\bar{\xi}}_i \) and \( \mathbf{\bar{\eta}}_{i+1}, \mathbf{\bar{\xi}}_{i+1} \) are obtained from the equation

\[
\begin{bmatrix}
\mathbf{\eta} \\
\mathbf{\xi}
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & \sin \Theta \\
-\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
\mathbf{G}_Y & 0 \\
0 & \mathbf{G}_X
\end{bmatrix}
\begin{bmatrix}
\mathbf{e}_Y \\
\mathbf{e}_X
\end{bmatrix}
\]

by putting \( \kappa = 0 \) and \( \kappa = 1 \), respectively. The resulting transformation matrix can then be written as

\[ \mathbf{\Gamma} = \mathbf{C}_{yx\Theta} \]

Therefore

\[ \mathbf{\bar{\mathbf{e}}} = \mathbf{\Gamma} \mathbf{C}_{yx\Theta} \mathbf{e} \]

Therefore

\[ \mathbf{e} = \mathbf{D}_{yx\Theta} \mathbf{\Gamma}^{-1} \mathbf{\bar{\mathbf{e}}} \]

where the supermatrices \( \mathbf{C}_{yx\Theta} \) and \( \mathbf{D}_{yx\Theta} \) are defined before, and the matrix
\( \Gamma^{-1} \) has the form shown in Table 2.2 for this particular case. Hence, the strain energy becomes

\[
2\lambda = \frac{1}{2} \Phi^T (\Gamma^{-1})^T D_{11}^T \Phi \kappa_{11} \Phi \Gamma^{-1} \Phi^T
= \frac{1}{2} \Phi^T (\Gamma^{-1})^T \kappa_{11} \Gamma^{-1} \Phi
\]

Similarly, it can be shown that

\[
T = \frac{1}{2} \Phi^T (\Gamma^{-1})^T \kappa_{11} \Gamma^{-1} \Phi
\]

Therefore, it is seen that the matrix \( \Gamma^{-1} \) is the required matrix for the transformation of the dynamic-stiffness matrix (2.11) to a fixed frame, in the case of basic elements. Similar matrices can be found for other types of elements.

### Table 2.2
Transformation Matrix for Basic Elements

```
[ cos\theta_1 0 - sin\theta_1 0 0 0 0 0 ]
[ -sin\theta_1 cos\theta_1 - \phi sin\theta_1 0 0 0 0 0 ]
[ \phi sin\theta_1 cos\theta_1 0 0 0 0 0 0 ]
[ 0 0 0 0 cos\theta_{i+1} 0 - sin\theta_{i+1} 0 ]
[ 0 0 0 - \phi sin\theta_{i+1} cos\theta_{i+1} - \phi cos\theta_{i+1} sin\theta_{i+1} 0 0 ]
[ \phi cos\theta_{i+1} sin\theta_{i+1} - \phi sin\theta_{i+1} cos\theta_{i+1} 0 0 0 0 0 0 ]
[ 0 0 0 0 sin\theta_{i+1} 0 cos\theta_{i+1} 0 0 ]
[ 0 0 0 0 0 0 cos\theta_{i+1} 0 0 ]
[ 0 0 0 0 0 0 0 0 0 ]
```

2.4 **Numerical Results**

During the course of this research a number of programmes have been written, in Algol, towards solution of various vibration problems of beams. Among these, one included the effects of shear and rotatory inertia on the bending—bending—torsion vibrations of uniform beams, the other, on the other hand, included the effect of linear double taper on the bending—bending vibrations. Each programme was incorporated with a simple procedure which enabled the treatment of any boundary conditions.
without re-calculation of the beam dynamic-stiffness matrix.

In this Article a few results obtained for the eigenvalues of various pre-twisted beams are shown in order to indicate the degree of accuracy that can be expected in the applications of the Finite Element Method, as formulated here, to such problems. For the present the discussion of eigenvectors is omitted. However, it should be stated that they were at least as accurate as the corresponding eigenvalues.

The Tables 2.3 to 2.9 are prepared to show the eigenvalues obtained by the use of various number of elements, in each case. These results are then compared with those obtained by other authors where available. Better values of frequencies can, evidently, be obtained by considering greater number of elements. On the other hand, an extrapolation from an assumed error law may also improve the results substantially. The Tables include frequencies obtained using the formula

\[ \Omega_n^4 = \Omega^4 + \sum_{s=1}^{n} \phi_s / n^{2s} \quad (2.38) \]

where \( \Omega_n^4 \) is the frequency parameter from \( n \) elements, \( \Omega^4 \) is the improved frequency parameter, and \( \phi_s \) is a constant function. This formula is based on the error law obtained in the case of classical beams, and should be used only when a uniform convergence level has been reached.

The Tables 2.3 and 2.4 show the convergence characteristics of the eigenvalues, \( \Omega \), in the case of bending-bending vibrations of clamped-free uniform beams. Comparison with the existing solutions given by other authors shows that the basic elements yield satisfactory results for the first four modes at five elements. The matrix size of the eigenvalue problem for five elements is 50x50 in this case, and the corresponding computer time is about two minutes on Elliott 503. This favourably compares with the Finite Difference results, which are extrapolated from 8,10 and 16 elements corresponding to matrix sizes 16x16, 20x20 and 32x32, respectively. Extrapolation of the Finite Element solutions by the formula (2.38) from 3 and 4 elements gives, consistently, slightly better frequency values than those obtained from 5 elements.

Table 2.5 shows the improvements that can be expected when one

* The extrapolation was done by the formula (32)

\[ \Omega_n^4 = \Omega^4 + \sum_{s=1}^{n} \phi_s / n^{2s} \]
internal node twisted beam elements are used in the case of uniform clamped-free beams. Comparison with respect to the degrees of freedom given to the beam shows only a slight improvement, which becomes greater at the higher modes. This can be seen by comparing the fourth column of Table 2.3 with the third column of Table 2.5 (see also Article 1.5.2). In order to show the power of the internal node technique in the determination of the characteristics of the higher modes of vibration, the frequency values of the fifth mode obtained by the 8 basic element approximation, which gives the beam 32 degrees of freedom, are also included in Table 2.5. These are only slightly (about 0.05%) better than the corresponding values obtained from 2 one-internal-node elements approximation, which gives the beam only 16 degrees of freedom.

In Table 2.6 the first five frequency ratios*, obtained by the basic element approximation, of a linearly tapered cantilever beam of 30, 60, and 90 degrees pre-twist angles are shown. It is seen that the frequency ratios converge rapidly as the number of elements increases. Comparison with the Finite Difference solutions, which are extrapolated from the frequencies obtained by the use of 8, 10 and 16 elements, indicates that satisfactory results are obtained from 5 basic elements. The Finite Difference results are slightly lower. This is to be expected, since in this method the eigenvalues are approached from a lower bound.

A detailed investigation of the vibration characteristics of uniform and tapered pre-twisted beams with various end conditions is given in the second part of the thesis.

The Table 2.7 demonstrates the accuracy that can be expected in applications where bending and torsion vibrations are coupled due to the non-collinearity of the shear centre and centroidal axes, the beam being uniform and the cross-sections having one axis of symmetry. The solutions obtained by the basic elements, elements having one torsional internal node, and elements having one basic internal node are compared, for a clamped-free case, with the exact values calculated by using the method described in Appendix 3. In Tables 2.7a, b and c

* The term frequency ratio denotes the ratio of any \( \Omega^2 \) value of a beam to the first \( \Omega^2 \) value of the uniform classical beam having the same cross-sectional properties at the root section.
the matrix sizes are given by 3n, 4n and 6n, respectively, n being the number of elements. The percentage errors in frequency parameters $\Omega$, at a matrix size of only 12x12 are: 4.3%, 3.2%, 1.2% for the fourth mode, 1.4%, 0.20%, 0.13% for the third mode, 0.6%, 0.6%, 0.05% for the second mode, for the three models considered. The columns that correspond to 12 degrees of freedom are marked by an (*).

The effects of rotatory inertia, and rotatory inertia and shear displacements are presented in Table 2.8, in relation to a pre-twisted uniform beam, with pre-twist angles 30 and 90 degrees, and ratio of flexural rigidity 16. Only solutions obtained by basic elements are shown. The first five frequency ratios with the both effects included are available. The present results agree closely with the latter values at even a few number of elements.

An advantage of the present method in problems where the effects of shear and rotatory inertia are involved is the possibility of investigating the two effects separately with the greatest ease.

"Degenerate" internal nodes can also be used in the problems where the effects of shear displacement and rotatory inertia have to be taken into account. However, as the following table shows, the inclusion of internal nodes for the improvement of the shear displacements is inferior to the basic element approximation.

Frequency Ratios
of a 90° pre-twisted uniform cantilever beam of ratio of flexural rigidities 16,
including the effects of shear and rotatory inertia, by

<table>
<thead>
<tr>
<th>Mode</th>
<th>4 Basic elements (24 degrees of freedom)</th>
<th>3 elements with one internal node for shear displacements (24 degrees of freedom)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0500</td>
<td>1.0422</td>
</tr>
<tr>
<td>2</td>
<td>2.8526</td>
<td>2.8283</td>
</tr>
<tr>
<td>3</td>
<td>3.7567</td>
<td>3.8447</td>
</tr>
<tr>
<td>4</td>
<td>15.7216</td>
<td>14.7261</td>
</tr>
<tr>
<td>5</td>
<td>28.4503</td>
<td>28.8376</td>
</tr>
</tbody>
</table>
The Table 2.9 contains the Finite Element solutions for a clamped-free beam of the Aerofoil section and 30 degrees of pre-twist. These results compare favourably with the Runge-Kutta solutions, at a few number of elements. Further analysis, which is not included here, has shown that in the bending-bending-torsion vibrations the consideration of torsional internal nodes has a similar effect to that discussed in relation to the bending-torsion vibrations.

In Table 2.9 the effects of rotatory inertia and rotatory inertia and shear displacements, which are small due to the rather slender nature of this beam, are also shown.

The foregoing examples demonstrate the accuracy that can be expected in various coupled vibration problems of clamped-free beams. In general, the finite element solutions converge to the exact values. The speed of convergence, however, varies with boundary conditions, as well as the pre-twist angle and ratio of flexural rigidities. In order to present an idea of the convergence properties in the extreme case of the beam parameters considered in this thesis, the frequency values, \( \Omega \), of pre-twisted uniform beams having various boundary conditions, the Tables 2.10a to 2.10f are prepared. Results obtained by the basic, one-internal-node and one-higher-derivative elements are presented for a 90 degrees pre-twisted beam having ratio of flexural rigidities 256, and compared with those given by Anliker and Trosch(29) where available.
Table 2.3
Bending–Bending Vibrations
Frequency Values of Uniform Cantilever Beams by Basic Elements

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \phi n = 30^\circ, \nu = \frac{1}{4} )</th>
<th>( \phi n = 60^\circ, \nu = \frac{1}{4} )</th>
<th>( \phi n = 90^\circ, \nu = \frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FINITE ELEMENT METHOD</td>
<td>Extrapolated from 3 and 4 elements</td>
<td>Finite Difference Method (32) extrapolated from 8, 10 and 16 elements</td>
</tr>
<tr>
<td></td>
<td>FINITE ELEMENT METHOD</td>
<td>Extrapolated from 3 and 4 elements</td>
<td>Finite Difference Method (32) extrapolated from 8, 10 and 16 elements</td>
</tr>
<tr>
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<td>( \Omega ) Number of Elements</td>
<td>( \Omega ) Number of Elements</td>
<td>( \Omega ) Number of Elements</td>
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<tr>
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<td>2.6400 2.6385 2.6381 2.6379 2.6379</td>
<td>2.6379 2.6379 2.6379 2.6379 2.6379</td>
<td>2.6379 2.6379 2.6379 2.6379 2.6379</td>
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<td>2.5469 2.5469 2.5469 2.5469 2.5469</td>
<td>2.5469 2.5469 2.5469 2.5469 2.5469</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Phi n = 30^\circ, \nu = \frac{1}{4} & \quad \Phi n = 60^\circ, \nu = \frac{1}{4} & \quad \Phi n = 90^\circ, \nu = \frac{1}{2}
\end{align*}
\]
Table 2.4
Bending-Bending Vibrations
Frequency Values of Uniform Cantilever Beams by Basic Elements

<table>
<thead>
<tr>
<th>Mode</th>
<th>Number of Elements</th>
<th>Extrapolated from 3 and 4 elements</th>
<th>Finite Element Method</th>
<th>Anliker and Tresca Method Ref. (29)</th>
<th>Slyper Stodola Method Ref. (27)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
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<td></td>
<td>Ωn = 30°, ν = 256</td>
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<tr>
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<td>8.2287</td>
<td>8.1580</td>
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### Table 2.5

**Bending-Vibrating Vibrations**

**Frequency Values of Uniform Cantilever Beams by One-Internal-Mode Elements**

These values are obtained from 8 basic elements.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Number of Elements</th>
<th>Extrapolated Method</th>
<th>Finite Difference</th>
<th>Anliker and Trossel Ref. (29)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Phi_n = 30^\circ$, $\nu = \frac{1}{4}$</td>
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<tr>
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<td>$1.8773$</td>
<td>$1.8773$</td>
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<th>Finite Difference</th>
<th>Anliker and Trossel Ref. (29)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5</td>
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</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>Number of Elements</th>
<th>Extrapolated Method</th>
<th>Finite Difference</th>
<th>Anliker and Trossel Ref. (29)</th>
</tr>
</thead>
<tbody>
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* These values are obtained from 8 basic elements.
Table 2.6
Bending-Bending Vibrations
Frequency Ratios of Linearly Tapered Clamped-Free Beam by Basic Elements.

\( \psi = 1 \), \( H_n = -0.25 \), \( K_n = 0.75 \)

<table>
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<tr>
<th>Mode</th>
<th>Frequency Ratio</th>
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<th>Finite Element Method</th>
<th>Extrapolated from 4 and 5 elements</th>
<th>Finite Difference Method (32) extrapolated from 8, 10 and 16 elements</th>
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</thead>
<tbody>
<tr>
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<td>0.8766</td>
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<td>0.8310</td>
<td>0.8798</td>
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</table>

\( \Phi_n = 30^\circ \)

\( \Phi_n = 60^\circ \)

\( \Phi_n = 90^\circ \)
Table 2.7
Bending-Torsion Vibrations
Frequency Parameters of Uniform Cantilever Beams by Various Finite Element Models

\[ \frac{a_X}{b} = 0.8 \quad , \quad \frac{1}{\rho} \beta^2 = 324 \]

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \Omega^2 )</th>
<th>Finite Element Method</th>
<th>Exact (App. 3)</th>
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</thead>
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<td>Number of Elements</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>a. Basic elements</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1175.43</td>
<td>464.887</td>
<td>458.870</td>
</tr>
<tr>
<td>3</td>
<td>5648.39</td>
<td>5530.42</td>
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<td>2593.09</td>
<td>2321.82</td>
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<td>b. Elements with one torsional internal node</td>
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<td></td>
<td></td>
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<td>2</td>
<td>1015.38</td>
<td>461.515</td>
<td>455.800</td>
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<td>4692.43</td>
<td>3450.39</td>
<td>3395.82</td>
</tr>
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<td>c. Elements with one basic internal node</td>
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<td></td>
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* These columns correspond to the matrix size of 12x12.
Table 2.8
Bending-Bending Vibrations
Frequency Ratios of Uniform Cantilever Beams by Basic Elements, Including the Effects of Rotatory Inertia and Shear.

\( \gamma = 16 \), \( k = 5/6 \), \( L = 6 \) in., \( h = 1 \) in.

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<tr>
<th>Freq. Ratio</th>
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<th>Extrapolated from 4 and 5 elements</th>
<th>Range-Kutta Method Ref. (34)</th>
<th>Rayleigh Ritz Method Ref. (22)</th>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Phi_n = 30^\circ ), Effect of rotatory inertia only</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Phi_n = 60^\circ ), Effect of shear and rotatory inertia</td>
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<td><strong>Endurance Experiment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Phi_n = 90^\circ ), Effect of rotatory inertia only</td>
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<td></td>
<td></td>
<td></td>
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<td><strong>Endurance Experiment</strong></td>
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<tr>
<td>( \Phi_n = 90^\circ ), Effect of shear and rotatory inertia</td>
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<td>28.4503</td>
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</table>
Table 2.9  
Bending-Bending-Torsion Vibrations  
Frequency Ratios of Uniform Aerofoil Section  
Cantilever Beams by Basic Elements, Effects  
of Rotatory Inertia and Shear Included.

\[
\begin{align*}
&l=6 \text{ in.}, \quad \Phi_{\alpha}=30^\circ, \quad I_{C_x}=8.4\times10^{-4} \text{ in}^4, \quad I_{g_x}=6.7\times10^{-3} \text{ in}^4, \\
&\epsilon_x=7.6\times10^{-5} \text{ in.}, \quad \epsilon_y=4.7\times10^{-2} \text{ in.}, \quad C_l=9.14\times10^{-2} \text{ in}^2, \quad E=31\times10^6 \text{ lb/in}^2.
\end{align*}
\]

<table>
<thead>
<tr>
<th>Mode</th>
<th>Number of Elements</th>
<th>Extrapolated from 4 and 5 elements</th>
<th>Range-Kutta Method</th>
<th>Ref. (32)</th>
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Table 2.10
Bending-Bending Vibrations

a. Frequency Values of Pre-twisted CLAMPED-FREE Beams

by Various Finite Element Models

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Anliker and Troesch\(^{22}\): 1.9, 3.52, 6.9, 9.7.
Table 2.10
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b. Frequency Values of Pre-twisted PINNED-FREE Beams
by Various Finite Element Models
\( \gamma = 256, \quad \phi_n = 90^\circ \)

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Anliker and Troesch\(^{(29)}\): 4.8, 7.4, 10.6, 13.8
Table 2.10
Bending-Bending Vibrations
d. Frequency Values of Pre-twisted CLAMPED-PINNED Beams
by Various Finite Element Models
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Anliker and Troesch(29): 5.9, 8.2, 10.3, 12.65
Table 2.10
Bending-Bending Vibrations
e. Frequency Values of Pre-twisted PINNED-PINNED Beams
by Various Finite Element Models
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Anliker and Troesch\(^{(29)}\): 3.2, 5.2, 8.8, 11.9
Table 2.10
Bending-Bending Vibrations
f. Frequency Values of Pre-twisted CLAMPED-CLAMPED Beams
by Various Finite Element Models.
ψ = 256, θN = 90°

<table>
<thead>
<tr>
<th>Degrees of Freedom</th>
<th>Mode</th>
<th>Basic Elements</th>
<th>Internal Node Elements</th>
<th>Higher Derivative Elements</th>
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</tbody>
</table>

Anliker and Troesch\(^{(29)}\): 7.0, 9.46, 13.07, 13.37
SECTION 3.

3. APPLICATIONS TO CONSTRAINED BEAMS

In this section the application of the finite element method to vibration problems of uniform (or linearly tapered) beams carrying concentrated masses, and being supported on elastic or rigid supports is presented. The analysis can be useful in the approximate examination of lacing on turbine blade vibration. The inherent characteristics of the method enable the most versatile treatment of the subject; beams containing any number of concentrated masses, and rigid and/or elastic supports are treated with almost the same ease as if they did not exist. Such a generality cannot be approached by the more conventional methods.

Analytical solutions consist of breaking the beam into a number of spans and solving the system of differential equations connected with each span under approximate boundary conditions. This is, however, a very tedious method. The exact analysis can be simplified, as shown by Yu Chen (54), by the use of the Laplace Transform method. Such an analysis is presented in Appendix 3 for single concentrated mass and single spring support. A Rayleigh-Ritz type procedure was developed by Lee and Saibel (55), where knowledge of the frequencies and mode shapes of the beam under no intermediate constraint but with the same end conditions is required.

The present method can be extended to vibration problems of beams carrying sprung mass systems, and beams supported on elastic continuous supports.

Fig. 3.1 Beam element with concentrated mass and spring support.
3.1 Non-Dimensional Dynamic-Stiffness Matrix for Flexural Motion

The beam element that will be considered contains a concentrated mass $m$, and a linear spring support of modulus $s$, as shown in Fig. 3.1. The kinetic energy of this system is equal to the sum of the kinetic energies of the beam and the concentrated mass due to their translational and rotational motions and the kinetic energy of the spring. The potential energy of the system is equal to the sum of the strain energies of the beam and spring. The former may include the effect of shear displacements.

The quadratic forms associated with the kinetic and potential energies of the beam are obtained as described in the previous Section. In order to simplify the presentation of the material only linearly tapered Bernoulli-Euler beams will be considered. However, the analysis can be extended to the general case if required.

The approximated strain and kinetic energies of the beam element will be given by the following quadratic forms (these are directly deduced from the equations (2.29), (2.32) and (2.34))

$$\mathcal{U}_1 = \frac{1}{2} l^2 EI \cdot \delta^T \left( \sum_{j=0}^{k} b_j K_{Y_j} \right) \delta$$

$$T_1 = \frac{1}{2} \mu \Omega l \cdot \dot{\delta}^T \left( \sum_{j=0}^{2} a_j M_{Y_j} \right) \dot{\delta}$$

where the explicit forms of the matrices $K_{Y_j}$ and $M_{Y_j}$ under polynomial approximation of $G_Y$ are given in Appendix 5 for basic and one-internal-node elements. The use of internal nodes provides great amount of flexibility into the analysis of beams which are rigidly supported at many intermediate points. The analytical work, however, is considerably increased. Nevertheless, the application of the theory of alternate and confluent alternate matrices simplifies the calculations greatly. The theory of such matrices is presented in Ref. (57).

3.1.1 Concentrated-Mass Inertia Matrix

The kinetic energy of the concentrated mass $m$ is given by

$$T_2 = \frac{1}{2} \mu \Omega \cdot \dot{\delta}^T \left( \bar{m} \left| \dot{\delta}^T \right|^2 \right) \left( \bar{m} \cdot \frac{k^2}{\ell^2} \left| \dot{\delta}^T \right|^2 \right)$$

where $\bar{m} = m/\mu \Omega l$, and $k$-radius of gyration of the concentrated mass.
In accordance with the equation (1.23) the energy equation (3.1) is approximated by the continuity function

$$\eta = G_Y \Phi_Y$$

(3.2)

Hence

$$T_2 = \frac{1}{2} D_Y \left[ 3.\Phi_Y^T \left( \bar{M}_{Ye} \Phi_Y + \bar{M}_{Ye} \Phi_Y \right) + \bar{Y}_e \right] \Phi_Y$$

where

$$\bar{M}_{Ye} = D_Y \left[ G^T \Phi_Y G \right]_{k=e} = D_Y$$

$$\bar{R}_{Ye} = D_Y \left[ G^T \Phi_Y G \right]_{k=e} = D_Y$$

For basic element approximation $G_Y = \left[ 1, \kappa, \kappa^2, \kappa^3 \right]$, the foregoing matrices will have the following forms

$$M_{Ye} = \begin{bmatrix}
(1-3\epsilon^2+2\epsilon^3)^2 & \epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 \\
\epsilon^2(1-\epsilon)^2 & (1-3\epsilon^2+2\epsilon^3)(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 \\
\epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & (3-2\epsilon)(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 \\
\epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & (1-3\epsilon)^2
\end{bmatrix}$$

Symmetric

$$R_{Ye} = \begin{bmatrix}
36\epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 & \epsilon^2(1-\epsilon)^2 \\
-6(1-\epsilon)^2 & (1-3\epsilon)^2 & (1-\epsilon)^2 & (1-3\epsilon)^2 \\
-6\epsilon^2(1-\epsilon)^2 & 6\epsilon^2(1-\epsilon)^2 & (1-3\epsilon)^2 & (1-3\epsilon)^2 \\
6\epsilon^2(1-\epsilon)(2-3\epsilon) & -6\epsilon^2(1-\epsilon)(2-3\epsilon) & -6\epsilon^2(1-\epsilon)(2-3\epsilon) & 2(2-3\epsilon)^2
\end{bmatrix}$$

Symmetric

The mass of the spring can be taken into account as a first approximation, by a concentrated mass equal to $\frac{1}{3}$ of the mass of the spring situated at the spring support.

3.1.2 Spring Stiffness Matrix

The potential energy of the spring in non-dimensional form is

$$\mathcal{U}_2 = \frac{1}{2} \ell^{-1} E_r G' \chi \bar{\eta}^2 | \kappa = \pi$$

(3.3)

where

$$\bar{\eta} = \eta \ell^3 / E_r G' \chi$$

Substituting the equation (3.2) into (3.3)

$$\mathcal{U}_2 = \frac{1}{2} \ell^{-1} E_r G' \chi \bar{\eta}^2 \Phi^T M_{Ye} \Phi$$

where
Clearly, this matrix has the same form as the matrix \( M_{\epsilon} \), with \( \epsilon \) replaced by \( \pi \) in the latter.

The dynamic-stiffness matrix corresponding to the element shown in Fig. 3.1 can now be obtained from the total potential energy \( U = U_1 + U_2 \), and the total kinetic energy \( T = T_1 + T_2 \). Thus

\[
Z_Y = \left[ \sum_{j=0}^{h} b_j M_{X_j}^T + \bar{\sigma} M_{X_{\pi}} \right] - \\
\lambda_i \left[ \sum_{j=0}^{2} a_j M_{X_j}^T + \bar{\rho}_1 \sum_{j=0}^{4} b_j M_{X_j}^T + \bar{\rho} \left( M_{X_{\epsilon}} + \frac{k^2}{\ell^2} M_{X_{\epsilon}} \right) \right]
\]

where \( \bar{\rho} = \frac{T}{G_{X_{\pi}}^2 \ell^2} \).

3.2 Non-Dimensional Dynamic-Stiffness Matrix for Torsional Motion

The dynamic-stiffness matrix for the St-Venant torsional vibrations of a uniform beam element carrying a mass and containing a torsional spring can be obtained by inspection of the equation (3.4), and appropriate modifications of the symbols used. Thus, let

- \( s \) = torsional spring stiffness
- \( m \) = mass of the "flywheel"
- \( k \) = polar radius of gyration of the flywheel.

Defining the following non-dimensional quantities

\[
\bar{s} = s / G J \\
\bar{m} = \left( m / \mu \right) \left( k / h_o \right)^2 \\
\lambda_\theta = \omega^2 \mu \left( \ell^2 / h_o \right) / G J
\]

and neglecting the inertia of the spring the following dynamic-stiffness matrix is obtained

\[
Z_\theta = \left( K_\theta + \bar{s} M_{\theta \pi} \right) - \lambda_\theta \left( M_\theta + \bar{m} M_\epsilon \right) \quad (3.5)
\]

where the matrices \( K_\theta \) and \( M_\theta \) are defined as in the equations (2.9d) and (2.10b), respectively, and

\[
M_{\theta \pi} = D_{\theta}^T \left| G_{\theta}^T \ G_{\theta} \right| \kappa = \pi \cdot D_{\theta}, \ \ (\pi \rightarrow \epsilon)
\]

For basic elements \( G_\theta = \left[ \begin{array}{c} 1 \ \kappa \end{array} \right] \), and the equation (3.5) becomes
\[ Z = \left[ \frac{(1+(1-\pi^2)^2) - \lambda_\theta \left( \frac{1}{2} + (1-\epsilon)^2 \frac{m}{I} \right)}{(-1+\pi(1-\pi)\pi - \lambda_\theta \left( \frac{1}{12} + (1-\epsilon)^2 \frac{m}{I} \right)} \right] \]

3.3 Numerical Results

The accuracy of the dynamic-stiffness matrices derived in this section for beams carrying concentrated masses and supported on elastic springs has been tested against the exact solutions obtained by the method described in Appendix 3. Tables 3.1 and 3.2 show two simple cases, namely, a beam carrying a central mass, and a beam supported at its mid-span by a spring support. Close agreement exists between the exact and basic element solutions.

Fig. 3.2 shows the effect of the variation of the position of one concentrated mass on the frequency values of uniform clamped-free Bernoulli-Euler beams, for a mass ratio (i.e., concentrated mass / mass of the beam) of 0.1. From the energy considerations it is immediately deduced that all the eigenvalues will be less than, or equal to, the eigenvalues of the unconstrained beam, and the maxima and minima of the frequency curves will occur at the nodes and anti-nodes of the modal patterns of the unconstrained beam. Fig. 3.2 confirms this theoretically and experimentally. Dual conclusions apply to the effect of spring supports.
TABLE 3.1. Frequency Parameter Ratio of a Cantilever Beam Carrying one Central Concentrated Mass (Concentrated Mass / Mass of the Beam = 0.1)

<table>
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<th>(Frequency Ratio)²</th>
<th>Number of Elements</th>
<th>Exact (App.3)</th>
<th>Rayleigh-Ritz Ref.(55)</th>
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<tr>
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<td>0.3315</td>
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<td>0.3311</td>
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<td>256</td>
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<tr>
<td>5</td>
<td>558.29</td>
<td>465.93</td>
<td>459.74</td>
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</table>

TABLE 3.2. Frequency Parameters of a Cantilever Beam Supported by a Linear Spring at its Mid-Span

\[ \frac{s L^3}{E I} = 1.0 \]

<table>
<thead>
<tr>
<th>( N^4 )</th>
<th>FINITE ELEMENT METHOD</th>
<th>EXACT (App.3)</th>
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</thead>
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<tr>
<td>MODE</td>
<td>Number of Elements</td>
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<td>3</td>
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<tr>
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</table>
Fig. 3.2. Frequencies of a Uniform Clamped-Free Beam with a Concentrated Mass.
SECTION 4

4. CONCLUSIONS

The theoretical formulation and practical applications of the Displacement analysis of coupled vibrations problems of beams by the Finite Element idealisation have been presented in this part of the thesis. The theoretical formulation of the subject has been unified and systematised by the introduction of a matrix notation representation of the quadratic energy functionals. Practical applications covering a large class of problems have shown that the method is ideally suited to vibration problems of beams; simplicity, versatility and accuracy are three properties comparing in its favour.

The generalised displacement finitisation of the energy functionals can be achieved by two techniques. The first is concerned with the approximation of the argument functions in the whole domain occupied by the beam. In this context two methods need to be clearly distinguished, namely the Rayleigh-Ritz method and the Displacement method. The former is characterised by its variational formulation, and as a result, the essential boundary conditions have to be imposed on the approximating sequence of complete functions before the reduction of the energy functionals into finite quadratic forms; the latter, on the other hand, is based on the approximation of the field functions, and therefore the boundary conditions are imposed after the energy finitisation. The second technique is a generalisation of the displacement method, where the field functions are approximated piecewise continuously, leading to a Finite Element approximation.

In the Displacement method of Finite Element approximation (or simply the Finite Element method), systematically improvable results can be obtained either by decreasing the size of elements, or by increasing the degree of functional approximation in each element. The latter can be done in a straightforward manner simply by extending, linearly independently, the unit displacement functions, as in the Bolotin-Hurty and Argyris methods of internal parameters. However, the internal node and higher derivative methods are preferred not only for the facts that they are more suitable for practical applications and yield, if not better, about the same
accuracy but also they provide additional directly useful data concerning the modal patterns of vibrations.

Examples considered demonstrate the applicability of the method to various vibration problems of beams of some complexity and practical importance. It is shown that by using a few numbers of elements satisfactory results can be obtained. The relative simplicity and versatility of the analysis and its ideal suitability for digital computation makes it desirable in the applications to coupled beam vibration problems; a field where the application of the Finite Element method is somehow neglected in the past.

Based on the theoretical analysis and data presented in this part of the thesis, the following additional conclusions are drawn:

1. Polynomial approximation converges uniformly to the exact results from an upper bound.

2. For the same degree of functional approximations, the internal node and higher derivative models are superior to the other refined models considered, the latter providing slightly better results, for the same degrees of freedom.

3. In torsional vibrations the accuracy of the basic elements is lower than that of the basic elements in bending.

4. From the foregoing conclusion it is deduced that in coupled vibration, where torsional and bending motions take place simultaneously, satisfactorily refined models can be obtained by the consideration of only torsional internal nodes or torsional higher derivatives. Table 2.7 confirms this fact and shows that the use of such degenerate nodes also improves the non-dominantly torsional higher modes.

5. For a fixed polynomial approximation and degrees of freedom the internal node and higher derivative models yield very close results. The choice between these two techniques will depend mainly on the required nodal pattern data. For example, in bending vibrations both methods will give displacements and slopes at the terminal nodes (i.e., junction points). If internal nodes are used, the beam displacement vector will contain such displacements at additional points, thus enabling a more accurate determination of the displacement modal patterns. On the other hand, the higher derivative method will give values for second, third, ..., etc., derivatives at the terminal nodes, thus enabling an accurate determination of the bending moments and shear forces. Evidently a compromise can be obtained by combining the two techniques.
6. If a beam element is incorporated only with internal nodes, for a fixed element number, the eigenvalues will be invariant under the positional changes of the internal nodes. Because such changes are equivalent to coordinate transformations which do not alter the energies of the system.

7. The above conclusion can be generalised as follows: If two finite models are obtained from the same polynomial approximation and contain, in equal or un-equal numbers, internal nodes, internal parameters and higher derivatives, the eigenvalue problems obtained from the assembly of the same number of elements will give the same eigenvalues provided the vectors of the terminal node displacements are identical for the two models at all nodes.

8. Frequency extrapolation based on the error laws of Finite Element idealisation for the uncoupled problems can be used with advantage in the cases of coupled vibrations.

9. The use of internal nodes and higher derivatives improves the convergence characteristics, in terms of degrees of freedom, of the basic element results for all boundary conditions. The determination of the "best" model requires further research.

The refined models are particularly advantageous in cases where the modal patterns are complex and physical parameters are non-uniform. However, in the flexural vibrations of clamped-free beams the basic elements are considered to be adequate.

10. The vibrations of beams that are constrained by rigid supports at many intermediate points, carrying concentrated masses, and being supported on elastic supports are best analysed by the use of internal nodes.

11. The accuracy of the "eigensolve" procedure given in Appendix 6 was tested against the solutions obtained from the closed frequency equations, presented in Article 1.5.1 for the torsional vibrations. It was found that the two sets of results were identical for all matrix sizes considered. This study is not included in the thesis.
VIBRATION CHARACTERISTICS
OF
SLENDER BEAMS
WITH
VARIOUS END CONDITIONS
In this part of the thesis the vibration characteristics of un-twisted and pre-twisted slender beams of uniform and tapered symmetric sections, and with various end conditions are investigated through the results obtained by the Finite Element method. The application of the latter to these problems has already been discussed in detail in the previous Part of the thesis.

The following notation and terminology are used throughout the discussion.

i. Consider a linearly pre-twisted and tapered slender beam whose ends are denoted by A and B. Omitting the case of sliding ends, nine types of beams are constructed by imposing the following conditions at A and B

<table>
<thead>
<tr>
<th>End A</th>
<th>End B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free (F)</td>
<td>Free</td>
</tr>
<tr>
<td>Pinned (P)</td>
<td>Pinned</td>
</tr>
<tr>
<td>Clamped (C)</td>
<td>Clamped</td>
</tr>
</tbody>
</table>

where the letters in brackets indicate the abbreviations that will be used subsequently; boundary conditions of a beam will be indicated by the notation \( a - b \), where a and b represent the abbreviations for the conditions at the left end (or, root) and right end (or, tip) of the beam, respectively.

ii. The term eigenvalue (or, frequency parameter) will denote the quantity

\[
\Omega_0^2 = \omega^2 \mu \Omega_0 \frac{L^4}{EI_G x_0}
\]

iii. The square root of the ratio of non-zero eigenvalues of a beam to the smallest non-zero eigenvalue of the uniform classical beam with the same boundary conditions will be referred by the term frequency ratio.

iv. The square root of the ratio of first (second, third,...) non-zero eigenvalue of a beam to the first (second, third,...) non-zero eigenvalue of a uniform classical beam with the same boundary conditions will be referred by the term F-ratio.

v. If the end conditions of a beam can be interchanged without altering the frequencies of the system, it will be said that the reversibility condition applies. This fact will be denoted by the symbol " \( \leftrightarrow \) ", e.g., F \( \leftrightarrow \) P, C \( \leftrightarrow \) P, etc.

vi. The ratios of dynamic displacements of a beam to the displacement at a particular section (subsequently called reference section) of the beam are uniquely determined, and will be referred to as the displacements (or, amplitudes) relative to the reference section.
SECTION 5.

5. BEAMS WITHOUT PER-TWIST

The theoretical results presented in this section were obtained by using five one-internal-node elements. The greatest matrix size in this representation is 22x22, and occurs in the case of free-free (or, F - F) beams. The corresponding computer time, on Elliott 503, is about two minutes.

The comparison of eigenvalues thus obtained with the exact solutions of uniform Bernoulli-Euler beams showed that the accuracy was relatively the lowest for C - C beams. However, even then the percentage errors of the first five eigenvalues were 0.004, 0.00147, 0.00318 and 0.0891. In general, the results presented are considered exact to two figures after the decimal point.

5.1 Uniform Beams

In this case there are only four sets of end conditions which have distinct frequency spectra. Firstly, from the physical considerations it is obvious that the reversibility condition will apply. Secondly, from the comparison of the complementary energy and Hamiltonian formulation of the problem it can be shown that the non-zero eigenvalues of F => P and F => F are the same as those of C => F and C => C beams. This, in fact, implies that the free and clamped conditions can be interchanged without altering the frequencies of the beam. Further analysis shows that the eigenmodes of F => P and F => F beams are the second derivatives of the eigenmodes of the C => P and C => C beams.

The first four frequency ratios have been listed in Table 5.1 for reference purposes.

5.2 Linearly Tapered Beams

Consider a linearly tapered beam with a particular set of end conditions, say P-C, and depth and width taper $\frac{H_y}{n}$ and $\frac{H_x}{n}$, as shown in Fig.5.1a.

If the end conditions are interchanged, as shown in Fig.5.1b, the beam can still be considered P-C, but with the modified depth and
Table 5.1
Frequency Ratios of Uniform Classical Beams

<table>
<thead>
<tr>
<th>Mode</th>
<th>Boundary Condition</th>
<th>$F \leftrightarrow F$</th>
<th>$C \leftrightarrow C$</th>
<th>$P \leftrightarrow F$</th>
<th>$P \leftrightarrow C$</th>
<th>$G \leftrightarrow F$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>2</td>
<td></td>
<td>2.7565</td>
<td>3.2406</td>
<td>4</td>
<td>6.2669</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>5.4039</td>
<td>6.7613</td>
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<td>17.5475</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>8.9330</td>
<td>11.3623</td>
<td>16</td>
<td>34.3861</td>
<td></td>
</tr>
</tbody>
</table>

* Numbers in brackets are the $\Omega^2$ values for the first mode.

Fig. 5.1
Interchange of End Conditions of a Linearly Tapered Beam.
width taper \( \Pi_n^t \) and \( \Pi_n^w \) which are connected to taper values \( \Pi_n^t \) and \( \Pi_n^w \) by the following relationships (see Fig. 5.1)

\[
\Pi_n^t = -\Pi_n^w / (1 + \Pi_n^w) \quad (5.1)
\]

\[
\Pi_n^w = -\Pi_n^t / (1 + \Pi_n^t) \quad (5.2)
\]

Thus, if the eigenvalues of a P-C beam with taper values \( \Pi_n^t \) and \( \Pi_n^w \) are known, the eigenvalues of the C-P beam with taper values \( \Pi_n^t \) and \( \Pi_n^w \) can be determined from the equation

\[
(\text{eigenvalues of C-P beam with taper } \Pi_n^t, \Pi_n^w) = (1 + \Pi_n^w)^2 (\text{eigenvalues of P-C beam with taper } \Pi_n^t, \Pi_n^w) \quad (5.3)
\]

since the eigenvalues are referred to the root section.

Clearly, this equation holds for all boundary conditions, and can be considered as a modified reversibility condition for tapered beams.

The relationships (5.1) and (5.2) are shown graphically in Fig. 5.2, and can be used for determination of taper values when end conditions are interchanged.

The equation (5.3) indicates that the nine boundary conditions need be investigated only in the negative range of taper parameters*.

The frequencies for positive taper parameters, in a wider range, can then be obtained from these results.

5.2.1 Single Taper

5.2.1.1 Depth Taper

The first four frequency ratios of depth tapered P-F, C-F, C-P, F-F, P-P and C-C beams are given in Figs. 5.3 to 5.8. It is seen that the frequency ratio for the first mode of C-F beams decreases gradually as the depth taper parameter increases. The frequency ratio for the subsequent higher modes shows a marked increase with the depth taper. In the cases of other boundary conditions, however, the frequency ratios of all modes show marked increases with the increase in depth taper.

This general trend also applies to F-C, P-P and P-C beams, as it can be

* ie., \( \Pi_n^t \) and \( \Pi_n^w \)
shown easily from the application of the equation (5.3)

Figs. 5, 9a, b and c show the variation of F-ratio of first, second and third modes, respectively, with depth taper for various boundary conditions. F-C, F-P and P-P beams are not included in these figures: F-ratios of the first two cases can be obtained from equation (5.3), and F-ratios of P-P beams closely follow those of the C-C beam. It is seen that, the relative effects of the changes in end conditions are marked for the first mode. For the subsequent higher modes the curves tend to approach each other. This indicates that at higher modes of vibration the effect of depth taper on the F-ratios is almost the same for all end conditions considered.

For a value of depth taper parameter, the F-ratios of all modes decrease as the boundary conditions are taken in the order of Table 5.2. Hence it may be stated that:

a. In the positive range of the depth taper parameter, F-ratios of a beam with a fixed depth taper and boundary conditions at its root (tip) increase (decrease) as the tip (root) is subjected to successively stiffer conditions,

b. In the negative range of the depth taper parameter, on the other hand, the introduction of successively stiffer conditions at the tip (root), for a fixed boundary condition at the root (tip), results in the decrease (increase) of F-ratios.

Table 5.2
Decrease of F-ratios with Boundary Conditions

<table>
<thead>
<tr>
<th>Positive Taper</th>
<th>Negative Taper</th>
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</thead>
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<tr>
<td>F - C</td>
<td>C - F</td>
</tr>
<tr>
<td>F - P</td>
<td>P - F</td>
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<tr>
<td>P - C</td>
<td>C - P</td>
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<tr>
<td>F - F</td>
<td>F - P</td>
</tr>
<tr>
<td>P - P</td>
<td>P - P</td>
</tr>
<tr>
<td>C - C</td>
<td>C - C</td>
</tr>
<tr>
<td>P - F</td>
<td>F - P</td>
</tr>
<tr>
<td>C - P</td>
<td>F - C</td>
</tr>
<tr>
<td>C - F</td>
<td>F - P</td>
</tr>
</tbody>
</table>
P-ratios of F-F and C-C, F-P and C-P beams follow each other closely; the variations get increasingly marked as the depth taper parameter increases, both in the positive and negative range. It should be noted that these pairs of beams have equal non-zero frequencies when the depth taper parameter is zero.

The P-ratios of first seven modes are plotted in Fig. 5.11 for the extreme values of depth taper parameter considered. It is seen that as the mode number increases the P-ratios of beams with various end conditions tend to the same value, i.e., the end conditions have negligible effect on the P-ratios of higher modes of beams tapered in depth. For boundary conditions lying in the centre section of Table 5.2, the P-ratios of all modes are effected nearly to the same degree. For cases which lie at the top and bottom sections of Table 5.2, however, gradual variations of the P-ratios with the mode number occurs. These changes are essentially confined to the first three modes, and greatest in the C-F (or, F-C) beams.

The frequencies are slightly more sensitive to changes in depth taper for the negative range of depth taper parameter than for the positive range.

The typical properties displayed by the mode shapes as a result of variations in depth taper are illustrated in Figs. 5.15a to 5.16a, in relation to the first four modes of P-F beams. These curves are plotted relative to the displacements of the free end. Similar plots can be made for the C-F and F-F beams. In the C-F, P-P and C-C cases, however, the displacements have to be plotted relative to an internal reference section. If the latter is taken sufficiently near to the tip of the beam, the following general characteristics of the mode shapes are observed,

a. The positions of nodes and anti-nodes are affected by the depth taper; anti-nodes are shifted towards the root as the depth taper increases. The nodes show the same tendency, except in the first mode of P-F beams, where the movement is slightly towards the tip.

b. The anti-node displacements increase as the depth taper increases. This is particularly marked at anti-node positions nearer to the root section.

c. The anti-node displacements increase more rapidly with depth taper as the mode number increases, and are slightly more sensitive to changes in the depth taper for the negative range of depth taper parameter than for the positive range.
d. The effect of increasing depth taper is in general, to increase relative amplitudes. There are, however, regions of the beam, near the vicinity of the nodes, where the generalisation does not apply. This could be explained by the notion of natural reference sections.

The natural reference sections, in relation to a particular mode of two beams with different taper and the same boundary conditions and reference section, are the points of intersection of modal curves. The existence of such points is ensured from the properties a and b.

Now, consider the flexural rigidity of a depth tapered beam at one of its reference sections (ordinary or natural). From geometrical considerations, it can be seen that, the ratio of flexural rigidity at a section to the left (right) of the reference section to that of the reference section will be decreased (increased) as the depth taper increases. Consequently, the relative amplitudes of the sections lying on the left (right) of the reference section will increase (decrease) as the depth taper increases.

The effect of increasing depth taper on mode shapes is, in general, to give rise to regions bounded by reference sections containing nodes and anti-nodes, in this order. Thus from above, it is seen that, as the depth taper increases the relative amplitudes in such a region will, at first, decrease and then increase as viewed from left to right. This change will clearly occur in the region bounded by nodes (see Figs. 5.14a to 5.16a)

5.2.1.2 Width Taper

The effects of width taper on the frequency ratios of the first four modes of P-P, C-F and C-P beam are shown in Figs. 5.3, 5.4 and 5.5 respectively. It is seen for the three cases that

a. The frequency ratios decrease gradually for all modes of vibration as the width taper parameter increases.

b. The change of the frequency ratio with the width taper is marked in the first mode, but gradually decreases as the mode number increases.

c. The frequency ratios are more sensitive to changes in width taper for the negative range of the width taper parameter than for the positive range.

The equation (5.3) in the present case becomes

$$\left\{ \begin{array}{l} \text{frequency ratio of a} \\
\text{beam with taper } H_{X_n} \end{array} \right\} = \left\{ \begin{array}{l} \text{frequency ratio of the beam} \\
\text{with interchanged end conditions and taper } H_{X_n} \end{array} \right\}$$  \hspace{1cm} (5.4)
Thus it is seen that, the frequency ratios of the above beams with interchanged end conditions, namely F-P, F-C and P-C beams, will increase gradually as the width taper increases.

The effects of negative width taper parameter on the frequency ratios of the first four modes of F-F, P-P and C-C beams are shown in Figs. 5.6, 5.7 and 5.8, respectively. For the three cases, the changes in frequency ratios over the range of width taper parameter considered is negligibly small for all modes. From equation (5.4) it is further deduced that the changes in the frequency ratios for the positive range of width taper parameter will be even smaller.

Figs. 5.10a, b and c show the variation of F-ratios of the first, second and third modes, respectively, with the width taper for various boundary conditions. The relative effects of the changes in end conditions are marked for the first mode, but as the mode number increases the curves approach each other rapidly.

The negligible effect of the width taper on the F-ratios of higher modes of vibration is clearly illustrated in Fig. 5.12, where the first seven F-ratios are plotted for various boundary conditions and extreme values of the width taper considered. It is seen that the variations are essentially confined to the first two modes, and are greatest in the C-F (or, F-C) beams.

The relative effects of width taper on the F-ratios of beams with various boundary conditions follow the properties displayed by the depth taper, but in a reduced scale.

a. In the positive range of width taper parameter, F-ratios of a beam of a fixed width taper and root (tip) conditions increase (decrease) as the tip (root) is subjected to successively stiffer conditions.

b. In the negative range of width taper parameter, F-ratios of a beam of a fixed width taper and root (tip) conditions decrease (increase) as the tip (root) is subjected to successively stiffer conditions.

c. For a value of width taper, F-ratios of all modes decrease as the boundary conditions are taken in the order of Table 5.2, the relative changes being directly proportional to the absolute value of the width taper parameter.

Figs. 5.13b to 5.16b show the mode shapes of the first four modes of vibration of F-F beams, where the amplitudes are plotted relative to the tip section. The typical characteristics of these curves, which also apply to beams with other end conditions, provided the reference...
sections are taken sufficiently near the tip section, are
a. The positions of nodes and anti-nodes are affected by the width
taper; the nodes and anti-nodes are shifted slightly towards the tip
section.
b. The anti-node displacements increase as the width taper increases.
This is particularly marked at anti-node positions nearer to the root.
c. The anti-node displacements increase more rapidly with the width
taper as the mode number increases, although the corresponding changes
in the F-ratios are not appreciable, and are more sensitive to
variations in the width taper for the negative range of the width
taper parameter than for the positive range.
d. The effect of increasing width taper is in general, to increase
relative amplitudes. The regions where this does not apply could be
explained, as in the case of depth taper, by considering the natural
reference sections.

Experimental confirmation concerning the mode shapes of beams
tapered in depth and width is given in Figs. 5.17 and 5.18, in relation
to third and fourth modes of C-F beams.

5.2.2 Double Taper

In Figs. 5.3, 5.4 and 5.5 the first four frequency ratios of
P-F, C-F and C-P beams tapered in depth and width are plotted, for
constant values of depth (width) taper values, against the frequency
ratios of beams tapered only in width (depth). In the range of the
taper parameters considered, a straight line relationship is clearly
indicated. Similar properties were also found to be valid for beams
with other end conditions.

Further study of these lines shows that they are very nearly
coincident with lines passing through the origin and having slopes
equal to the F-ratio of beams of single taper, the taper parameter
being equal to the constant values of taper parameters considered
for beams having double taper. Thus it can be written that

\[
\begin{align*}
& \text{frequency ratios of beams of double taper } \left\{ H_n^Y, H_n^X \right\} = \\
& \text{F-ratio of a frequency ratios of beams } \left\{ \text{beam of single } H_n^Y, (H_n^X) \right\} + \xi \\
& \text{taper } H_n^Y, (H_n^X) \\
& \text{and taper values } H_n^Y, H_n^X
\end{align*}
\]

which can be written, more simply, as

\[
\begin{align*}
& \text{F-ratios of beams of double taper } \left\{ H_n^Y, H_n^X \right\} = \\
& \text{F-ratios of beams of single taper } H_n^Y \left\{ \text{of single taper } H_n^X \right\} + \xi \\
& \text{double taper } H_n^Y, H_n^X \\
& \text{and taper values } H_n^Y, H_n^X
\end{align*}
\]

... (5.5)
where $\zeta$ is a function of depth and width taper and is negligible in the range of taper parameters considered in the thesis.

This empirical relationship was suggested, in connection with C-F beams, by Thomas (63), who has also shown that a similar relationship holds for the deflections of beams having double taper, namely,

$$\left\{ \text{deflection of beam of double taper } H_y, H_x \right\} = \frac{\left( \text{deflection of beam of single taper } H_y \right) \left( \text{deflection of beam of single taper } H_x \right)}{\text{(deflection of uniform beam)}} + \epsilon$$

(5.6)

at points sufficiently remote from the nodes of the uniform beam.

The present analysis has shown that the equation (5.6) also applies to beams with other end conditions. This is illustrated in Tables 5.3 and 5.4 for C-F and C-C beams, respectively.

The accuracy of Thomas's formulae (i.e., equations 5.5 and 5.6) depends, as it would be expected from the perturbation theory, on the closeness of the taper parameters to the zero value. In the range of parameters considered in this thesis, namely $-0.5 \leq H_y \leq 0.5$, the frequencies and mode shapes obtained by the Finite Element method agree closely with those obtained by the application of the above empirical relations to the results obtained for beams having single taper. Therefore, these relations can be employed with confidence in the discussion of vibration characteristics of beams of double taper. It would, however, be superfluous to dwell on these, since the characteristics of beams of single taper have already been discussed.
Table 5.3
Comparison of Mode Shapes of Clamped-Pinned Beams of Double Taper

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( H_X = -0.5 )</th>
<th>( H_Y = 0 )</th>
<th>( H_X = 0 )</th>
<th>( H_Y = 0 )</th>
<th>( H_X = -0.5 )</th>
<th>( H_Y = 0 )</th>
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<tbody>
<tr>
<td>First Mode</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0</td>
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<td>0</td>
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<tr>
<td>0.1</td>
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<td>0.140</td>
<td>0.159</td>
<td>0.505</td>
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<td>0.530</td>
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<td>1.523</td>
<td>1.016</td>
<td>1.074</td>
<td>2.119</td>
</tr>
<tr>
<td>0.4</td>
<td>1.441</td>
<td>2.299</td>
<td>2.166</td>
<td>1.550</td>
<td>1.530</td>
<td>2.168</td>
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<td>2.593</td>
<td>1.997</td>
<td>1.974</td>
<td>2.719</td>
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<td>2.219</td>
<td>2.719</td>
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<td>2.450</td>
<td>2.182</td>
<td>2.168</td>
<td>2.719</td>
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<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.719</td>
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<td>Second Mode</td>
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<td>-1</td>
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<td>Third Mode</td>
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<td>0.282</td>
<td>0.670</td>
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<td>-0.214</td>
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<td>0.769</td>
<td>0.767</td>
<td>0.767</td>
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Table 5.4
Comparison of Mode Shapes of Clamped-Clamped Beams of Double Taper

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( \text{FINITE ELEMENT METHOD} )</th>
<th>( \text{Eq. (5.6)} )</th>
</tr>
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Fig. 5.2. Relation between Taper Parameters for Interchanged end Conditions.
Fig. 5.3. Frequency Ratios of Tapered Pinned-Free Beams
Fig. 5.3. Frequency Ratios of Tapered Pinned-Free Beams.
Fig. 5.4. Frequency Ratios of Tapered Clamped-Free Beams.
Fig. 5.4. Frequency Ratios of Tapered Clamped-Free Beams.
Fig. 5.5. Frequency ratios of tapered clamped-pinned beams.
Fig. 5.5. Frequency Ratios of Tapered Clamped-Pinned Beams.
Fig. 5.6
Frequency Ratios of Tapered Free-Free Beams

- - - - Constant Width ($H_{x}=0$)
- - - - Constant Depth ($H_{y}=0$)

Fig. 5.7
Frequency Ratios of Tapered Pinned-Pinned Beams.
Fig. 5.8

Frequency Ratios of Tapered Clamped-Clamped Beams.
Fig. 5.9
Variation of P-ratios with Depth Taper
(b) SECOND MODE

- pinned-clamped
- free-free (also clamped-clamped)
- pinned-free (also clamped-pinned)
- clamped-free

---

(e) THIRD MODE

- clamped-free
- pinned-clamped

Fig. 5.9
Variation of F-ratios with Depth Taper
Fig. 5.10
Variation of F-ratios with Width Taper
Fig. 5.11
Variation of F-ratios with Mode Number

- pinned-clamped
- free-free (also clamped-clamped)
- pinned-free
- clamped-pinned
- clamped-free
Fig. 5.12

Variation of P-ratios with Mode Number
Fig. 5.14 Second Mode Shapes of Tapered Pin-Fixed Beams
Fig. 5.36 Fourth mode shapes of tapered pinned-free beams.

(a) Constant Width

(b) Constant Depth
Fig. 5.17 Third mode shapes of tapered clamped-free beams.
Fig. 5.18 Fourth Mode Shapes of Tapered Clamped-Free Beams
Pre-twisted beams of symmetric sections execute simultaneous
flexural motions in the two principal directions giving coupled bending-
bending vibrations. A pre-twisted beam element and the co-ordinate
system to which its motion is referred are shown in Figs. 6.1 and 6.2,
respectively. The analysis concerning the generation of dynamic-
stiffness matrices for such elements is presented in Article 2.2.2.

The theoretical results presented in this Section are obtained by
using eight elements of cubic polynomial approximation. The accuracy
that can be expected when using this idealization has already been
demonstrated in the previous Part of the thesis.

6.1 Uniform Beams

The frequency ratios of pre-twisted uniform beams with various-
end conditions were determined for pre-twist angles in the range
$0^\circ \leq \Phi_n \leq 90^\circ$, and for ratio of flexural rigidities in the range
$1 \leq \psi \leq 256$.

The effect of coupling on the frequency ratios of $P \leftrightarrow P$, $C \leftrightarrow F$,
$C \leftrightarrow P$, $F \leftrightarrow F$, $P \leftrightarrow F$ and $C \leftrightarrow C$ beams is shown in Figs. 6.3 to 6.8, where
the frequency ratios are plotted against the square root of the ratio
of flexural rigidities (subsequently called "$\psi^{1/2}$-ratio"). The
latter may be interpreted as the ratio of the characteristic dimension
of a beam cross-section in the $X$ direction to that in the $Y$ direction,
ic., $h_x/h_y$.

At $\psi^{1/2} = 1$, pre-twisted uniform beams with any boundary conditions
have double frequencies, which are not affected by the pre-twist angle.
This is to be expected, since for such cross-sectional shapes the
second moment of area about any centroidal axis is invariant of its
orientation, and the product of inertia and hence the coupling term

* The frequency ratios of pre-twisted beams are taken relative to the
fundamental frequency of the untwisted beam in $Y$ direction. This is
consistent with the definition given on page 123.
in the strain energy expression; in equation (A1.25), vanishes

The following properties of the curves of frequency ratio against \( \gamma^2 \)-ratio are deduced from Figs. 6.3 to 6.8.

a. At zero pre-twist angle the frequency ratios belonging to vibrations in the two principal directions are clearly distinguished. The frequency ratios corresponding to vibrations in the Y direction are independent of \( \gamma^2 \)-ratios, whilst those corresponding to vibrations in the X direction vary linearly with the \( \gamma^2 \)-ratios. The two frequency ratios are equal for those values of \( \gamma^2 \)-ratios (subsequently called critical \( \gamma^2 \) ratios) equal to the ratio of any two frequency ratios of uniform Bernoulli-Euler beam. Some values of the critical \( \gamma^2 \)-ratios can be inspected from Table 5.1, for various boundary conditions.

b. For values of \( \gamma^2 \)-ratios sufficiently near, and greater than, unity, the coupling term may be neglected. Then, the increase of the pre-twist angle, in the range 0 to 90 degrees, increases the equivalent stiffness of the beam in the y direction, and decreases that in the x direction, consequently, in the vicinity of \( \gamma^2 = 1 \) (\( \gamma^2 \leq 1 \)), the frequency ratios corresponding to vibration in the y direction will increase, and those corresponding to vibration in the x direction will decrease, as the pre-twist angle is increased.

c. As the \( \gamma^2 \)-ratio increases, frequency ratios continuously increase (at least do not decrease), for all modes of vibration.

d. As the pre-twist angle increases, the common frequency at a critical point is separated into two frequencies; one being greater and the other smaller than the common frequency. The amount of separation increases with the pre-twist angle.

The foregoing properties, which apply to all boundary conditions considered, will be referred to as the properties of the frequency ratio against \( \gamma^2 \)-ratio curves.

The Fig. 6.9 is included to demonstrate that the effect of coupling is similarly reflected on the frequency values of bending-torsion vibrations. In this case, the role of the pre-twist angle is taken by the eccentricity of the shear centre and centroidal axes, and the ratio of rigidities now represents the ratio of the torsional rigidity to flexural rigidity.

The correlation between the frequency ratios and pre-twist angle are shown in Figs. 6.10 to 6.15, for \( P \leftrightarrow P \), \( C \leftrightarrow P \), \( C \leftrightarrow P \), \( P \leftrightarrow P \), and \( C \leftrightarrow C \) beams, respectively. It is seen that the effect of pre-twist angle on frequency ratios are dependent on the value of the ratio of flexural rigidities, mode of vibration and boundary conditions. These
changes may be inspected by considering the "trapezoidal" and "open trapezoidal" sections, which are formed by the frequency ratio lines of pre-twisted beams of zero pre-twist angle, in Figs. 6.3 to 6.8.

The domain of a trapezoidal section contains the frequency ratio curves corresponding to two successive modes of vibration. It can be stated, as a generalisation, that as the pre-twist angle increases, the frequency corresponding to the higher mode decreases, and that corresponding to the lower mode increases. The relative variations of the frequency ratios with pre-twist angle in the trapezoidal sections become increasingly marked as the boundary conditions are taken in the order $F \leftrightarrow F$, $P \leftrightarrow F$, $P \leftrightarrow P$ and $C \leftrightarrow F$. It is interesting that for this order the critical $\beta$-ratios also get larger. In the cases of $C \leftrightarrow C$ and $C \leftrightarrow P$ beams, the changes in frequency ratios are more marked than those for $F \leftrightarrow P$ and $P \leftrightarrow P$ beams, respectively.

In the open trapezoidal sections the frequency ratios for the second and successively higher modes of $P \leftrightarrow F$, $C \leftrightarrow F$, $F \leftrightarrow F$ and $P \leftrightarrow P$ beams decreases as the pre-twist angle increases. As the ratio of flexural rigidities increases, the frequency ratios tend to approach constant values. The frequency ratio for the first mode, on the other hand, increases slightly with the pre-twist angle and the ratio of flexural rigidities. These changes are small, and shown separately in Fig. 6.16. The frequency ratios of $C \leftrightarrow P$ and $C \leftrightarrow C$ beams show similar characteristics for pre-twist angles up to about thirty degrees. The fundamental frequencies of these beams, however, increase rapidly with the pre-twist angle and ratio of flexural rigidities. For larger pre-twist angles the effect of coupling on frequencies becomes more complex.

Anliker and Troesch (29) calculated the analytical frequency values, $\Omega$, for ratio of flexural rigidities 4, 25 and 256. These values are included in Figs. 6.11 to 6.15, and agree closely with the present results. Additional confirmation is given in the case of $C \leftrightarrow P$ beams using frequency ratios obtained by Slyper (27), and experimental results. Again the agreement is good (see Fig. 6.4). Some typical experimental frequency ratios are also compared in Table 6.1 with the Finite Element solutions.

The deviations of the frequencies of pre-twisted beams from those of the corresponding beams with zero pre-twist angle are shown in

* See page 160
Figs. 6.17, 18 and 19, for pre-twist angles 30, 60 and 90 degrees, respectively. In plotting those graphs, the endwise and chordwise frequencies of untwisted beams are taken in ascending order, i.e., it is considered as a pre-twisted beam of zero pre-twist angle. It is seen that, discontinuities occur, as it would be expected, at the critical $\psi^\frac{2}{3}$-ratios. For a particular mode of vibration deviations increase with the pre-twist angle.

The mode shapes of pre-twisted beams are actually space curves, and depend on the mode of vibration and boundary conditions, as well as the pre-twist angle and ratio of flexural rigidities. A convenient method of presentation of mode shapes of pre-twisted beams is to plot components of displacements in two directions parallel to the principal axes at the root section. These are referred to as the $y$ and $x$ components of the mode shapes. It is also convenient to extend this convention to the case of untwisted beams by taking the frequency ratios in ascending order, and classifying the corresponding mode shapes of the independent vibrations in the two principal directions as the first, second, etc. mode shapes of a pre-twisted beam of zero pre-twist angle. The usefulness of this extension lies in the fact that it enables an estimation of the type of mode shape that may be expected in the case of non-zero pre-twist angle, from the inspection of the frequency ratio against $\psi^\frac{2}{3}$-ratio plots.

For small pre-twist angles, it is sensible to expect the dominant component of the mode shape of a mode of vibration to be of the type which is associated with the same mode of the pre-twisted beam of zero pre-twist angle and of the same ratio of flexural rigidities. Then, by observation of the curves of frequency ratio against $\psi^\frac{2}{3}$-ratio, it can be seen that as the ratio of flexural rigidities increases, the weaker component of the mode shape will tend to become the dominant component as the next critical $\psi^\frac{2}{3}$-ratio is approached.

For larger values of pre-twist angle, as the pre-twist angle increases, the weaker component of the mode shape in general, increases and the stronger component decreases.

In order to demonstrate the foregoing points in isolation from the additional complexities caused by the end constraints, the $x$ and $y$ components of the first four mode shapes of $F \leftrightarrow F$ beams are plotted in Figs. 6.20 to 6.23, for various pre-twist angles and ratios of flexural rigidities. It is interesting to note that, at 90 degrees pre-
twist angle, the x component of mode shapes of any mode of vibration is the mirror image, or inverted mirror image, of the y component. This is due to the symmetry of the boundary conditions, and also applies to P → P and C → C beams.

For values of ratio of flexural rigidities greater than the highest critical value associated with a mode of vibration, the y component of the mode shapes is the dominant (at least not weaker) component, and mode shapes are not substantially affected by further increases in the ratio of flexural rigidities.

The study of the mode shapes of beams with other end conditions showed that in general, the above properties are also valid in these cases.

In the cases of C ↔ P and C → C beams, it was noted that the effect of coupling on the frequencies was severe in the open trapezoidal sections, especially for larger pre-twist angles. The corresponding changes in the mode x and y components of the first two mode shapes are illustrated in Fig. 6.24 for C → C beams of 60 degrees pre-twist and various ratios of flexural rigidities.

Theoretical and experimental verification of the mode shape results are given in Fig. 6.25, in connection with a C ↔ F beam of 90 degrees pre-twist and ratio of flexural rigidities 4, 64 and 256. It is seen that, there is a close agreement between the Finite Element solutions and those obtained by Finite Differences (32) (γ = 4), and Runge-Kutta (22) (γ = 64, 256) methods, and the experimental values. The first three mode shapes given by the Stodola method (27) (γ = 64) also show satisfactory agreement.

6.2 Linearly Tapered Beams

In linearly pre-twisted and tapered beams the equation (5.3), which expresses the relationship between the eigenvalues of a tapered beam and those of the same beam with interchanged end conditions, has to be modified, since the eigenvalues are also dependent on the pre-twist angle and the ratio of flexural rigidities at the root section, γ_0. Thus it can be written that

\[
\text{(eigenvalues of pre-twisted beam with parameters } H_{y_n}, H_{x_n}, \bar{\Phi}_n, \gamma') = (1 + H_{y_n})^2 \text{(eigenvalues of the beam with interchanged end conditions,)}
\]

\[
\text{(and parameters } H_{y_n}, H_{x_n}, \bar{\Phi}_n, \gamma_0') \quad \ldots (6.1)
\]
where \( \Pi_X^n \) and \( \Pi_X^o \) are defined by equations (5.1) and (5.2), respectively, and \( \gamma_0' \) is given by the equation

\[
\gamma_0' = \gamma_0 \frac{(1 + \Pi_X^o)^2}{(1 + \Pi_X^o/n)^2}
\]  

(6.2)

In the following discussion the frequency ratios of pre-twisted tapered beams are mainly analysed through the plots of frequency ratio against \( \gamma_0^3 \)-ratio. In order to clarify the difference between the uniform and tapered beams, a "skeleton" form of the frequency ratio against \( \gamma_0^3 \)-ratio plot, displaying the frequency ratio lines for zero pre-twist angle, is shown below, for a pre-twisted beam having depth taper \( a \), and width taper \( b \). A horizontal line \( Y_i \) represents the \( i \) th frequency ratio of an untwisted beam having depth taper \( a \) and width taper \( b \), and an inclined line \( X_i \) has a slope equal to the \( i \) th frequency ratio of an untwisted beam having depth taper \( b \) and width taper \( a \). Clearly, if the frequency ratios are taken in ascending order, for a particular value of \( \gamma_0' \), the \( Y_i \) and \( X_i \) lines will be associated with the vibrations of a pre-twisted beam of zero pre-twist angle in the \( Y \) and \( X \) directions, respectively. For \( a=b=0 \) this representation reduces to that used for uniform beams in the previous article.
The critical \( \gamma_{b}^{\hat{o}} \)-ratios are now defined as follows

\[
\begin{align*}
\text{Critical } \gamma_{b}^{\hat{o}}-\text{ratios of} \\
\text{pre-twisted beam of} \\
taper parameters \\
H_y = n, H_x = b \\
\text{untwisted beam of taper} \\
parameters H_y = n, H_x = b
\end{align*}
\]

\[
\begin{align*}
\text{i th frequency ratio of} \\
\text{untwisted beam of taper} \\
parameters H_y = n, H_x = b \\
\text{j th frequency ratio of} \\
\text{untwisted beam of taper} \\
parameters H_y = b, H_x = a
\end{align*}
\]

\[i, j = 1, 2, 3, \ldots\]

since the depth and width taper for vibrations in Y direction become width and depth taper, respectively, for vibrations in X direction. It should be noted that for zero taper, ie., \( a = b = 0 \), the foregoing relationship reduces to the definition of critical ratios for uniform beams.

Figs. 6.26 and 6.27 show the variation of frequency ratio with \( \gamma_{b}^{\hat{o}} \)-ratio for tapered P-F beams of depth taper parameters -0.5 and 0.5, respectively. Frequency ratios are plotted for 0, 30, 60 and 90 degrees pre-twist angles.

At zero pre-twist angle, the frequency ratios belonging to independent vibrations in the X and Y directions, which are clearly distinguished, correspond to vibrations of width and depth tapered untwisted beams, respectively. From Article 5.2, it is known that, in the case of untwisted P-F beams the increase of taper parameter increases the frequency ratios for depth taper, and decreases the frequency ratios for width taper. Therefore, as the depth taper of the pre-twisted beam increases, the ordinates of the critical points will also increase. This implies that the heights of trapezoidal zones of the frequency ratio against \( \gamma_{b}^{\hat{o}} \)-ratio plots increase with depth taper. Similarly, the critical ratios also increase with depth taper, since the increase of the heights of trapezoidal zones is accompanied by the decrease of slopes of the non-parallel sides. Hence, as a generalisation, it may be stated that the increase of depth taper increases the frequency ratios of pre-twisted P-F beams. This is especially true for small pre-twist angles, and for ratios of flexural rigities, \( \gamma_{b}^{\hat{o}} \), lying in the intervals corresponding to open trapezoidal sections, ie., for relatively thinner beams.

Figs. 6.28 and 6.29 show the variation of frequency ratios with \( \gamma_{b}^{\hat{o}} \)-ratio for tapered P-F beams of width taper parameters -0.5 and 0.5, respectively. Frequency ratios are plotted for 0, 30, 60 and 90 degrees pre-twist angles.

The frequency ratios for zero pre-twist angle in this case, correspond
to those of depth tapered beam executing vibrations in the X direction, and width tapered beam executing vibrations in the Y direction. Following a similar reasoning as in the case of depth taper, it can be seen that, as the width taper increases the heights of trapezoidal sections and the critical \( \gamma^2_0 \)-ratios will decrease. Consequently, the increase of width taper will in general, reduce the frequency ratios of pre-twisted P-F beams. Again, this is especially true for small pre-twist angles, and relatively thinner beams.

From Figs. 6.26 to 6.29, it is seen that the effect of coupling on the frequency ratios of depth tapered P-F beams display similar properties as those for uniform beam. It should, however, be noted that depth, or width, tapered pre-twisted beams do not have double frequencies.

The study of the curves of frequency ratio against \( \gamma^2_0 \)-ratio for pre-twisted tapered beams with other end conditions leads to similar deductions. Verification is given in Figs. 6.30 to 6.37 for C-F and C-C beams tapered in depth, or width, and having pre-twist angles 0, 30, 60 and 90 degrees.

In the cases of P-F and C-F (also, P-P and C-P) beams the frequency ratio for the first mode is not appreciably affected by variations in pre-twist angle and ratio of flexural rigidities at the root section. For C-C (also, C-P) beams the effect of coupling on the fundamental frequency is more marked, as in the case of uniform beams.

From the foregoing discussion it is seen that the frequency ratios of pre-twisted tapered beams possess the combined properties of the frequency ratios of untwisted tapered and pre-twisted uniform beams. The coupling phenomenon, which characterised by the presence of pre-twist, causes added complications in assessing the effect of taper.

From the consideration of the frequency characteristics of untwisted tapered beams discussed in Article 5.2, a rough estimate of the effect of depth and width tapers on frequency ratios of pre-twisted beams can be made, as it is done above in connection with P-F beams, by inspecting the changes that occur in the "skeleton" structure of the frequency against \( \gamma^2_0 \)-ratio plots.

The mode shapes of pre-twisted tapered beams also possess the combined properties of mode shapes of untwisted tapered and pre-twisted uniform beams. In particular, the correspondence noted in the cases of uniform beams between the dominant component of mode shape and the frequency ratio against \( \gamma^2_0 \)-ratio curves also applies to tapered beams. This correspondence, which also provides a more precise description of
the effects of depth and width taper, may be stated as follows: The effect of increasing depth (width) taper is such that the frequency ratios of pre-twisted beams give, approximately, similar properties as those of untwisted depth (width) tapered beams if the coupling is dominant for the y direction, and those of the untwisted width (depth) tapered beams if the coupling is dominant for the x direction. This is supported by Figs. 6.38 and 6.39, which show the variation of the second mode frequency ratios of C-P beams with depth and width taper, respectively, for various pre-twist angles and ratios of flexural rigidities at the root section. Fig. 6.38 can be explained with reference to Figs. 6.30 and 6.31. From the latter it is seen that for large values of ratio of rigidities (\( \gamma > 16 \)), the dominant component of the mode shape will be of the second mode type of a depth tapered untwisted beam, for which the second mode frequency ratio increases as the depth taper increases (Fig. 5.4b). Hence, as the foregoing statement asserts, the pre-twisted beam displays similar changes with the increase of depth taper. For smaller values of ratio of rigidities, on the other hand, the dominant component of the mode shape will be of the first mode type of a width tapered untwisted beam, for which the first mode frequency ratio decreases (Fig. 5.4a) as the width taper increases. Hence, the pre-twisted beam again shows similar changes as the depth taper increases*. Fig. 6.39 can be explained similarly.

A general study of the mode shapes of tapered pre-twisted beams, which is not included in the present analysis, confirmed the above facts, and showed that similar deductions can in general, be made for the other modes of vibration, and for beams with other end conditions.

The properties of the curves of frequency ratio against \( \gamma_0^{\frac{1}{2}} \)-ratio are not altered by variations in depth, or width, taper, and the boundary conditions. However, the quantitative effect of coupling varies with taper and boundary conditions. A rough estimate of this can be made

* It should be noted that this does not violate the statement made previously, namely, that the effect of depth taper is, in general, to increase the frequency ratios of pre-twisted beams tapered in depth. Because, the interval of ratio of rigidities corresponding to dominant vibrations in the x direction is relatively small compared to that corresponding to dominant vibrations in the y direction.
by inspecting the amount of frequency "separation", which occurs as a result of variations in the pre-twist angle, at the critical points. From Figs.6.26 to 6.37, it can be observed that the amount of separation increases (decreases) as the depth (width) taper increases. Since for the same variations of taper, the co-ordinates of the critical points undergo similar changes, the relative effect of coupling may be related to the relative changes that occur in the "skeleton" configuration of the frequency ratio against \( \nu^2 \)-ratio plot. This idea can also be used in assessing the relative effects of boundary conditions.

A large number of beams of double taper were also considered for various boundary conditions. The vibration characteristics of these may be summarised by noting that the Thomas’s equations (i.e., equations 5.5 and 5.6) may be used, as a first approximation, in estimating the frequencies and mode shapes of pre-twisted beams having double taper. Table 6.2 is prepared to show the accuracy that can be expected in the extreme case of the variables considered.

In the case of pre-twisted beams of double taper, degenerate solutions exist for equal depth and width taper and ratio of flexural rigidities at the root section equal to unity.

The vibration characteristics of C-F beams having double taper and ratio of flexural rigidities at the root section equal to unity were calculated by Carnegie, Dawson and Thomas (32) using a Finite Differences approximation. In Table 2.6 a comparison of the frequency ratios obtained by the Finite Element method with those obtained in Ref. (32) is given.
Table 6.1
Comparison of Experimental and Theoretical Frequency Ratios of Clamped-Free Beams.

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<td>5.27</td>
</tr>
<tr>
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<td>17.55</td>
</tr>
<tr>
<td>4</td>
<td>34.40</td>
<td>34.39</td>
</tr>
<tr>
<td></td>
<td>$\Theta_1 = 15^\circ, \gamma = 64$</td>
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</tr>
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</tr>
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</tr>
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* Beams tested were of rectangular sections.
Table 6.2
Comparison of Frequency Ratios of Pre-twisted Beams of Double Taper

\[ \Phi_n = 90^\circ \quad \gamma_0 = 256 \]

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<th>Taper Parameter</th>
<th>Finite Element Method</th>
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<tr>
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<td>1st mode</td>
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<td>( H_N )</td>
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<td>( H_N )</td>
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Clamped-Free

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<thead>
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<td>(3.8272)</td>
<td>(11.4471)</td>
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Pinned-Free

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Clamped-Clamped

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<td>-0.5</td>
<td>(1.6767)</td>
<td>(2.9519)</td>
<td>(5.5342)</td>
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*Numbers in brackets are obtained from equation (5.5)
Fig. 6.1
A Typical Beam Element

Fig. 6.2
Relations Between Co-ordinate Axes
Fig. 6.3
Correlation between Frequency and $\nu^{\frac{1}{2}}$ Ratios of Pinned-Free Uniform Beams
Correlation between Frequency and $\frac{\psi}{l}$ Ratios of Clamped-Free Uniform Beams
Correlation between Frequency and $\sqrt{\nu}$ Ratios of Clamped-Pinned Uniform Beams
Correlation between Frequency and $\psi^{1/2}$ Ratios of Free-Free Uniform Beams
Fig. 6.7
Correlation between Frequency and $\sqrt{\nu}$ Ratio of Pinned-Pinned Uniform Beams

- $0^\circ$ pre-twist
- $30^\circ$ "
- $60^\circ$ "
- $90^\circ$ "

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Fig. 6.8
Correlation between Frequency and $\nu^3$ Ratios of Clamped-Clamped Uniform Beams
Fig. 6.9
Effect of Coupling on the Frequency Values
in Bending-Torsion Vibrations
Fig. 6.10
Variation of Frequency Ratios of Pinned-Free Uniform Beams with Pre-twist Angle
Variation of Frequency Ratios of Clamped-Free Beams with Pre-twist Angle
Fig. 6.12 Variation of Frequency Ratios of Clamped-Pinned Uniform Beams With Pre-twist Angle
Fig. 6.14
Variation of Frequency Ratios of Pinned-Pinned Beams with Free-Twist Angle
Fig. 6, 15
Variation of Frequency Ratios
of Clamped-Clamped Uniform
Beams with Pre-twist Angle

--- Third Mode
--- Fourth Mode
--- First Mode
--- Second Mode

--- Another and Tresch (29)
Frequency Ratios of the First Mode of Vibration of Pre-twisted Uniform Beams
Fig. 6.17

F-ratios of 30° Pre-twisted uniform Beams
First Mode Shapes of Pre-twisted Prestressed Beams
Fig. 6-21. Second Mode Shapes of Pre-twisted Free-Free Beams
a. 50 degrees Pre-twist
Fig. 6.21  Second mode shapes of pre-twisted free-free beams
  b.  60 degrees pre-twist
Fig. 6.21
Second Mode Shapes of Pre-twisted Free-Free Beams
- 90 degrees Pre-twist
Fig. 6.22
Third Mode Shapes of Pre-twisted Free-Free Beams

b. 60 degrees Pre-twist
Fig. 6.24
Mode Shapes of 69 Fre. Twisted Clamped-Clamped Beams, Second Mode

- y component
- x component

$y = 100$
$y = 16$
$y = 64$
$y = 196$
Mode Shapes of 90° Pre-twisted Clamped-Free Beams
Fig. 6.25  Mode Shapes of 90° Pre-twisted Glimmer-Tire Beams

- (a) FOURTH MODE
- (b) THIRD MODE

- Experimental (22)
- Range-Kutta (27)
- Stodola (52)
- Finite Difference

Relative Amplitude
Fig. 6.26

Correlation between Frequency and $\psi_0^{1/2}$ Ratios of Depth Tapered Pinned-Free Beams

$H_y = -0.5; H_x = 0$
Fig. 6.27
Correlation between Frequency and $\sqrt{\frac{I_y}{I_n}}$ Ratios of Depth Tapered Pinned-Free Beams

$\Pi_y = +0.5; \Pi_x = 0$
Correlation between Frequency and $\psi^\frac{1}{2}$ Ratios of Width Tapered Pinned-Free Beams
Correlation between Frequency and $\eta_0^{1/2}$ Ratios of Width Tapered Pinned-Free Beams
Correlation Between Frequency and $\gamma_0^{1/2}$ Ratios of
Depth Tapered Clamped-Free Beams
Fig. 6.31

Correlation between Frequency and $\frac{\Psi^2}{\Theta^2}$ Ratios of Depth Tapered Clamped-Free Beams
Fig. 6.32

Correlation between Frequency and $\sqrt[1/2]{\gamma_o}$ Ratios of Width Tapered Clamped-Free Beams
Correlation between Frequency and $\nu^\frac{1}{2}$ Ratios of Width Tapered Clamped-Free Beams
Correlation between Frequency and $\gamma^1_o$ Ratios of Depth Tapered Clamped-Clamped Beams
Fig. 6.35

Correlation between Frequency and $\Psi_0^{1/2}$ Ratios of Depth Tapered Clamped-Clamped Beams

$\Psi_0^{1/2}$
Fig. 6.36

Correlation between Frequency and $\sqrt{\frac{\nu}{\rho \omega}}$ Ratios of Width Tapered Clamped-Clamped Beams
Fig. 6.37

Correlation between Frequency and $\theta_{0.5}^{0.5}$ Ratios of Width Tapered Clamped-Clamped Beams
Variation of Second Mode Frequency Ratios of Pre-twisted Beams Tapered in Depth.
Variation of Second Mode Frequency Ratios of Pre-twisted Beams Tapered in Width
Experimental work was carried out with the object of confirming the practical applicability of the theory. It is needless to say that, the success of experimental analysis would depend on the extend to which the hypothesis underlying the theory could be satisfied by physical representation of the "theoretical" beam, as well as the instrumentation systems used for the excitation of natural vibration conditions and the measurement of frequencies and mode shapes.

The beams, which were taken from the stock available in the Vibrations Laboratory of Surrey University, were of rectangular section and had 10 in. effective length; an additional 2 in. was used for clamping purposes. Cross-sectional dimensions of beams were chosen so as the effects of shear deflection and rotatory inertia could be neglected. The pre-twisted beams had been pre-twisted in intervals of 15 degrees, up to 90 degrees.

The beams were clamped at one end in a rigid steel block, which was in turn bolted rigidly to a large vibration table. An electromagnet, placed near the free end, was used to vibrate the beam. In some cases the excitation was also done by a lead zirconate strain gauge cemented near the clamped end of the beam. This method was particularly satisfactory in yielding the higher modes of vibrations.

Power was supplied to the magnet from an oscillator through an amplifier having a large frequency range. A lead zirconate crystal, cemented near the clamped end of the beam and connected to the Y plates of a cathode ray oscilloscope, detected the amplitudes of vibration. In some cases, the latter was also picked up by a capacitance type probe mounted near the tip. The frequency of vibration of the beam was measured by an accurately calibrated frequency counter.

During a test, the frequency supply from the oscillator was varied until the amplitude of the trace on the oscilloscope screen indicated a maximum, i.e., resonance. The corresponding natural frequency was read off.
from a decade oscillator, by forming a steady Lissajous figure on the oscilloscope screen; the decade oscillator was connected to the X plates of the oscilloscope and the frequency output was varied until the trace on the oscilloscope screen was stationary circle, the oscilloscope being set to the external time base.

A schematic arrangement of the apparatus is shown in Fig. 7.1. For a detailed study of the instrumentation the reader is referred to Ref. (22).
In this Part of the thesis vibration characteristics of slender beams of symmetric sections and having linear pre-twist and/or taper were investigated for various end conditions. The latter included all possible combinations that can be formed by imposing free, pinned and clamped conditions at either ends of a beam. Based on the data and discussion presented, the following conclusions are drawn.

1. Beams without Pre-twist
1.1. The "free" condition can be replaced by the "clamped" condition without altering the frequencies of uniform beams.

1.2. Frequency ratios of tapered beams exhibit the following features:
   i. As the depth taper increases the frequency ratios for all end conditions considered also increase, except for the first frequency ratio of clamped-free beams, which decreases gradually with increasing depth taper.
   ii. As the width taper increases the frequency ratios gradually decrease for pinned-free, clamped-free and clamped-pinned beams, gradually increase for free-pinned, free-clamped and pinned-clamped beams, and do not show any appreciable changes for free-free, pinned-pinned and clamped-clamped beams. As the mode number increases the effect of width taper on the frequency ratios rapidly decreases.
   iii. When the end conditions of a tapered beam are interchanged, the frequency ratios can be determined from those of the original beam.

1.3. The mode shapes of beams executing free vibrations are uniquely determined relative to a reference section. When the latter is taken sufficiently near the tip, the mode shapes of tapered beams exhibit the following features:
   i. The increase of depth taper increases anti-node displacements and, in general, shifts the positions of nodes and anti-nodes towards the root section.
   ii. The increase of width taper also increases anti-node displacements, but the positions of nodes are now shifted slightly towards the tip section, and the positions of anti-nodes are not appreciably altered.
1.4. The frequency ratios and mode shapes of beams tapered both in depth and width can be determined approximately, from those of uniform beams, beams tapered only in depth and beams tapered only in width.

1.5. The effect of depth taper is, in general, greater than that of width taper. The exception is the first mode of clamped-free beams, where the frequency ratios decrease more rapidly for increasing width taper than depth taper.

1.6. The deviations of frequencies of tapered beams from those of uniform beams of the same length and material and having cross-sectional dimensions equal to those of the tapered beams at the root section, increase for all modes of vibration, as the boundary conditions are taken in a certain order. As the mode number increases, however, these deviations tend to be the same for all end conditions considered.

2. Beams with Linear Pre-twist

2.1. An untwisted beam can be considered as a pre-twisted beam of zero pre-twist angle by taking its frequencies belonging to vibrations in the two principal directions in ascending order. Then the variation of the frequency ratios with the ratio of rigidities (at the root section) is represented by straight lines which intersect at "critical" points.

2.2. In the cases of pinned-free, clamped-free, free-free and pinned-pinned beams the frequency ratios of the first mode of vibration increases as the pre-twist angle or ratio of rigidities increases. These changes are, however, small; being about $3\%$ in the extreme cases considered. The frequency ratios of second and successively higher modes decrease as the pre-twist angle increases for values of ratio of flexural rigidities greater than the greatest critical-ratio corresponding to that mode of vibration.

2.3. In the cases of clamped-pinned and clamped-clamped beams the foregoing conclusion applies for pre-twist angles up to about thirty degrees. The changes, however, now occur more rapidly. For higher pre-twist angles the effect of coupling becomes more complex and requires a more detailed examination.

2.4. The mode shapes of pre-twisted beams have components in the directions parallel those of principal axes at the root section. For a particular mode of vibration, the dominant component is of the type associated with the same mode of a pre-twisted beam of zero pre-twist angle, for relatively small pre-twist angles. As the pre-twist angle increases, the dominant component, in general, decreases and the
weaker component increases.

2.5. The frequency ratios and mode shapes of pre-twisted beams having double taper can be approximately estimated from those of pre-twisted uniform beams and pre-twisted beams having single depth and width tapers.

2.6. The curves of frequency ratio against the ratio of flexural rigidities are useful in analysing the nature and effect of coupling caused due to the presence of pre-twist. These curves exhibit the following general properties:

i. Pre-twisted beams with equal depth and width taper (including uniform beams) have double frequencies for the ratio of flexural rigidities equal to unity, for all pre-twist angles.

ii. The frequency ratios increase as the ratio of flexural rigidities increases.

iii. At critical points the critical frequencies are separated into two frequencies, one being greater and the other smaller than the critical frequency. This separation increases with pre-twist angle.

2.7. The foregoing properties yield an adequate picture of the effect of coupling on frequency ratios, and are not altered by the presence of linear taper, and variations in the end conditions.

2.8. The amount of frequency "separation" at the critical points in general increases if the changes in taper, or boundary conditions, result in the increase of the co-ordinates of the critical points. From this, the relative effects of taper and boundary conditions on the vibration characteristics of pre-twisted beams can be roughly estimated by considering the vibration characteristics of untwisted beams.
APPENDIX 1

THE DERIVATION OF ENERGY EQUATIONS

1. The Frenet-Serret Equations

Consider a point Q on an analytic curve s, as shown in Fig. A1.1b. The position vector \( \vec{r} \), where a bar under a symbol is used to indicate the vector character of a quantity, being measured from an arbitrary origin. Then the unit vector \( \frac{\partial \vec{r}}{\partial s} \) is along the tangent to the space curve at point Q and is directed along the increasing s. Thus

\[
\frac{\partial \vec{r}}{\partial s} = \vec{t} \tag{A1.1}
\]

Since the vector \( \vec{t} \) is of unit length, from \( \vec{t} \cdot \vec{t} = 1 \)

\[
\vec{t} \cdot \frac{\partial \vec{t}}{\partial s} = 0
\]

Hence the vector \( \vec{t} \) is perpendicular to \( \frac{\partial \vec{t}}{\partial s} \). Therefore one may write

\[
\frac{\partial \vec{t}}{\partial s} = \kappa \vec{n} \tag{A1.2}
\]

where \( \vec{n} \) is a unit vector perpendicular to \( \vec{t} \) and is called the principal normal vector, and the scalar \( \kappa \) is called the curvature of the curve at point Q. The direction of \( \vec{n} \) is chosen so that \( \kappa \) is non-negative.

The binormal vector \( \vec{b} \) is defined by the relation

\[
\vec{b} = \vec{t} \times \vec{n} \tag{A1.3}
\]

Since

\[
\vec{b} = (\vec{t} \times \vec{n}) = (\vec{t} \times \vec{n}) = (\vec{t} \cdot \vec{n} - \vec{n} \cdot \vec{t}) = 1
\]

the binormal vector is of the unit length. Thus

\[
\vec{b} \cdot \frac{\partial \vec{b}}{\partial s} = 0 \tag{A1.4}
\]

i.e., the vector \( \frac{\partial \vec{b}}{\partial s} \) is perpendicular to \( \vec{b} \). On the other hand, from equations (A1.2) and (A1.3)
\[ t \cdot b = n \cdot b = 0 \]

Therefore
\[ \frac{d}{ds}(t \cdot b) = -b \cdot \frac{d}{ds}t = -\kappa b \cdot n \]

Thus \( t \cdot \frac{db}{ds} = 0 \), i.e., the vector \( \frac{db}{ds} \) is perpendicular to \( t \).

Therefore
\[ \frac{d}{ds}b = -\kappa n \]

where \( \kappa \) is called the torsion of the curve at point \( Q \). With respect to the choice of signs, the torsion is positive if the binormal performs a twisted motion in the direction from \( n \) to \( b \) around the tangent while it moves along the curve in direction of increasing arc length.

The vectors \( t \), \( n \), and \( b \) in this order form a right-handed system of orthogonal unit vectors, which is called a triad. Then
\[ n = b \times t \]
\[ \frac{dn}{ds} = b \times \frac{dt}{ds} + \frac{db}{ds} \times t = b \times \kappa n - \tau n \times t = -\kappa t + \tau b \] (A1.6)

2. The Strain Energy in Twisted Asymmetric Beams (the effect of shear displacement neglected)

The positions of a beam cross-section before and after deformation is shown in Fig. A1.1a. The unit vectors \( \xi \) and \( \eta \) are parallel to the principal axes whose orientation relative to an arbitrary frame will be denoted by \( \Theta(z) \), where \( z \) is the axis of the beam through its centroids. Let \( \xi \) and \( \eta \) be the same unit vectors after deformation. The relative position of the cross-section is now given by
\[ \Theta_s(s) = \Theta(z) + \Theta(z) \] (A1.7)

where \( \Theta \) denotes torsional displacement about the shear centre and \( s \) is the curve length of the locus of shear centres.

Consider a triad \( t \), \( n \), \( b \) originating from \( 0 \). Assuming that the tangent to the locus of shear centres is normal to the
section before and after deformation, the following relations hold
\[ \xi_4 = \eta \cos \phi + b \sin \phi \]
\[ \eta_4 = -\eta \sin \phi + b \cos \phi \]  
\( (A1.8) \)

Consider a point \( P = (\xi, \eta) \). The position vector of this point after deformation is given by
\[ \gamma_P = \gamma + \xi \xi_4 + \eta \eta_4 \]  
\( (A1.9) \)

The vectorial equation of an infinitesimal beam fibre associated with the point \( P \) will then be
\[ d\gamma_P = ds \left( \xi + \xi \frac{\partial \xi}{\partial s} + \eta \frac{\partial \eta}{\partial s} \right) \]  
\( (A1.10) \)

From equation (A1.8)
\[ \frac{\partial \xi_4}{\partial s} = \frac{\partial \phi}{\partial s} \frac{\partial \xi}{\partial s} + \frac{\partial \eta}{\partial s} \cos \phi + \frac{\partial b}{\partial s} \sin \phi \]
\[ \frac{\partial \eta_4}{\partial s} = -\frac{\partial \phi}{\partial s} \xi + \frac{\partial \eta}{\partial s} \sin \phi + \frac{\partial b}{\partial s} \cos \phi \]  
\( (A1.11) \)

Substituting equations (A1.5) and (A1.6) in the foregoing equations
\[ \frac{\partial \xi}{\partial s} = \frac{\partial \phi}{\partial s} \frac{\partial \xi}{\partial s} - \kappa \frac{\partial \xi}{\partial s} \cos \phi + \gamma \left( b \cos \phi - \eta \sin \phi \right) \]
\[ \frac{\partial \eta}{\partial s} = -\frac{\partial \phi}{\partial s} \xi + \kappa \frac{\partial \eta}{\partial s} \sin \phi - \gamma \left( b \sin \phi + \eta \cos \phi \right) \]  
\( (A1.12) \)

Further, using the inverse relations corresponding to equations (A1.8), namely,
\[ n = \xi_4 \cos \phi - \eta_4 \sin \phi \]
\[ b = \xi_4 \sin \phi + \eta_4 \cos \phi \]

equations (A1.12) become
\[ \frac{\partial \xi}{\partial s} = -\frac{\partial \phi}{\partial s} \xi + \eta \left( \gamma + \frac{\partial \phi}{\partial s} \right) \cos \phi \]
\[ \frac{\partial \eta}{\partial s} = \frac{\partial \phi}{\partial s} \xi - \frac{\partial \xi}{\partial s} \left( \gamma + \frac{\partial \phi}{\partial s} \right) \sin \phi \]  
\( (A1.13) \)

By the definition of scalar product
\[ \cos \phi = n \cdot \xi_4 \]
\[ \sin \phi = -n \cdot \eta_4 \]

Also, from equations (A1.1) and (A2.2)
\[ n = \frac{1}{\kappa} \left( \partial^2 \gamma_P / \partial s^2 \right) \]

Hence, equations (A1.13) become
The torsion of the curve at point \( P_1 \) is clearly given by
\[
\gamma = \left( \frac{\partial \Theta}{\partial s} + \frac{\partial \phi}{\partial s} \right)
\]
Therefore
\[
\frac{\partial \xi}{\partial s} = -\frac{\partial^2 \rho}{\partial s^2} \cdot \xi + \frac{\partial \Theta}{\partial s} \eta
\]
\[
\frac{\partial \eta}{\partial s} = -\frac{\partial^2 \rho}{\partial s^2} \cdot \eta + \frac{\partial \Theta}{\partial s} \xi
\]
Substituting these in the equation (A1.10)
\[
dr_2 = ds \left[ \left( 1 - \frac{\partial^2 \rho}{\partial z^2} \cdot \bar{O}_1 \vec{P}_2 \right) \xi + \frac{\partial \Theta}{\partial s} (\eta \bar{O}_1 - \xi \eta) \right]
\]
\[
dr_2 = ds \left[ \left( 1 - \frac{\partial^2 (r+w)}{\partial z^2} \cdot \bar{O}_1 \vec{P}_2 \right) \xi + \left( \frac{\partial \Theta}{\partial s} + \frac{\partial \phi}{\partial s} \right) (\eta \bar{O}_1 - \xi \eta) \right]
\]
where \( \bar{\phi} = \frac{\partial \phi}{\partial s} \)

The vector \( \vec{O}_1 \vec{P}_2 \) may be written as
\[
\vec{O}_1 \vec{P}_2 = \bar{O} \vec{P} + \Theta \vec{t} \times \bar{O} \vec{P}
\]
Further, assuming that the radius of curvature is very large compared to the beam length, the equation (A1.17) becomes
\[
dr_2 = ds \left[ \left( 1 - \frac{\partial^2 (r+w)}{\partial z^2} \cdot \bar{O} \vec{P} \right) \xi + \left( \bar{\phi} + \frac{\partial \phi}{\partial s} \right) (\eta \bar{O}_1 - \xi \eta) \right]
\]
Hence
\[
|dr_2| = ds \left[ 1 - \frac{\partial^2 (r+w)}{\partial z^2} \cdot \bar{O} \vec{P} + \left( \frac{1}{2} \bar{\phi}^2 + \bar{\phi} \frac{\partial \phi}{\partial s} \right) \bar{O} \vec{P}^2 \right]
\]
where non-linear terms are neglected.

The length of the fibre before deformation can be obtained by the inspection of the foregoing equation. Denoting the length of the locus of shear centres before deformation by \( \xi \)
\[
|dr_2| = dc \left( 1 - \frac{\partial^2 \rho}{\partial z^2} \cdot \bar{O} \vec{P} + \frac{1}{2} \bar{\phi}^2 \right)
\]
Let \( \xi_0 \) be the strain in the fibre joining the shear centres of
two infinitesimally close sections, i.e., \( ds = (1 + \varepsilon_o) \, dc \). Putting this into the equation (A1.19) and noting that \( \varepsilon_o \) is small

\[
|dr_p| = dc \left[ 1 + \varepsilon_o - \frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} + \frac{1}{2} \frac{\partial r^2}{\partial \theta} \frac{\partial}{\partial \theta} \right] \quad (A1.21)
\]

Then the strain in the fibre associated with point \( P \) is given by

\[
\varepsilon_p = \frac{|dr_p| - |dr_p|}{|dr_p|} = \frac{\varepsilon_o - \frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} + \frac{1}{2} \frac{\partial r^2}{\partial \theta} \frac{\partial}{\partial \theta} \vec{r}}{1 - \frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} + \frac{1}{2} \frac{\partial r^2}{\partial \theta} \frac{\partial}{\partial \theta} \vec{r}} = \varepsilon_o - \frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} + \frac{1}{2} \frac{\partial r^2}{\partial \theta} \frac{\partial}{\partial \theta} \vec{r} \quad (A1.22)
\]

where it is assumed that the radius of curvature is very large and \( \ddot{\theta}, \dot{r}^2 \ll 1 \) for all points of cross-sections.

The strain \( \varepsilon_o \) can be determined from the criterion that the axial force acting on the area \( d\sigma_1 \) must vanish when integrated over the whole section, namely,

\[
\int_{\sigma_1} (\varepsilon_o - \frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} + \frac{1}{2} \frac{\partial r^2}{\partial \theta} \frac{\partial}{\partial \theta} \vec{r}) E \, d\sigma_1 = 0
\]

Thus

\[
\varepsilon_o = \frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} - \frac{1}{2} \frac{\partial r^2}{\partial \theta} \frac{\partial}{\partial \theta} \vec{r} \quad (A1.23)
\]

where \( I_{P_0} \) is the polar moment of area about the shear centre.

From equations (A1.22) and (A1.23)

\[
\varepsilon_p = -\frac{\dot{\theta}(w \cdot \vec{w} + \Theta \cdot \vec{r})}{\dot{\theta}^2} \vec{F} + \Theta \frac{\partial}{\partial \theta} \left( \vec{r}^2 - \frac{I_{P_0}}{\sigma_1} \right)
\]

Now the strain energy can be expressed as

\[
\mathcal{U} = \frac{1}{2} \int \left\{ \int_{\sigma_1} \varepsilon_p^2 \, d\sigma_1 + GJ \left( \frac{\partial \Theta}{\partial \theta} \right)^2 \right\} \, dz \quad (A1.24)
\]

where \( GJ \) is the torsional rigidity, the strain energy of torsion being assumed to be given by the St. Venant theory. Choosing the co-ordinate system as shown in Fig. A1.2

\[
\frac{w}{GJ^p} = i \mu + \frac{i}{2} \nu
\]

Thus the strain energy becomes
$$
abla = \frac{1}{2} \int_0^l \left\{ E \left[ I_{GXX} \left( \frac{\partial^2 \nu}{\partial x^2} \right)^2 + 2 I_{GXY} \left( \frac{\partial^2 \nu}{\partial x \partial y} \right) \left( \frac{\partial^2 \mu}{\partial x \partial z} \right) + I_{GYY} \left( \frac{\partial^2 \mu}{\partial y^2} \right)^2 \right] + \\
\left[ E \Phi^2 \left( I_T - \frac{I_{P_0}^2}{I_{P_0}} \right) + GJ \left( \frac{\partial \theta}{\partial z} \right)^2 \right] - \\
2 E \Phi \left[ \left( e_X I_{P_0} + I_{T_0} \right) \frac{\partial^2 \nu}{\partial x^2} + \left( e_Y I_{P_0} + I_{T_0} \right) \frac{\partial^2 \nu}{\partial x \partial y} \right) \frac{\partial \theta}{\partial z} \right\} dz \\
$$

where

$$I_{GXX} = \int_{\Omega} y^2 \, d\Omega$$

$$I_{GXY} = \int_{\Omega} xy \, d\Omega$$

$$I_{GYY} = \int_{\Omega} x^2 \, d\Omega$$

$$I_{P_0} = \int_{\Omega} (u^2 + v^2) \, d\Omega$$

$$I_T = \int_{\Omega} (u^2 + v^2)^2 \, d\Omega$$

$$I_{T_0} = \int_{\Omega} u(u^2 + v^2) \, d\Omega$$

$$I_{T_0} = \int_{\Omega} v(u^2 + v^2) \, d\Omega$$

and $e_X$ and $e_Y$ are measured from the centroidal frame.

The strain energy given by the foregoing equation may be written as

$$\nabla = \frac{1}{2} \int_0^l (A \sigma \sigma)^T \mathcal{D} A \sigma \sigma \, dz \quad (A1.26)$$

where

$$A = \left[ \begin{array}{ccc} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial}{\partial x} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 1 \end{array} \right]$$

$$\sigma = \{ \nu, u, \theta \}$$

$$\mathcal{D} = E \left[ \begin{array}{ccc} I_{GXX} & I_{GXY} & -\Phi I_u \\ I_{GXY} & I_{GYY} & -\Phi I_v \\ -\Phi I_u & -\Phi I_v & \Phi^2 I + GJ/E \end{array} \right]$$

$$\mathcal{A} = \left[ \begin{array}{ccc} 1 & 1 & 1 \end{array} \right] \quad (A1.27)$$

Clearly, the following notation is used in the strain coupling matrix
\[
I_u = e_y I_{p_o} + I_{Tu}
\]
\[
I_v = e_x I_{p_o} + I_{Tv}
\]
\[
I = I_T - I_{p_o}^2 / \Theta
\]

Since
\[
I_{Gx} = I_{Gx} \cos^2 \Theta + I_{GY} \sin^2 \Theta
\]
\[
I_{Gy} = I_{Gx} \sin^2 \Theta + I_{GY} \cos^2 \Theta
\]
\[
I_{Gxy} = (I_{Gy} - I_{Gx}) \sin \Theta \cos \Theta
\]
\[
I_u = I_v \sin \Theta + I_u \cos \Theta
\]
\[
I_v = I_v \cos \Theta - I_u \sin \Theta
\]

where
\[
I_v = e_x I_{p_o} + I_{Tv}
\]
\[
I_u = e_y I_{p_o} + I_{Tu}
\]

the coupling matrix \( \mathbf{J} \) can also be expressed as follows
\[
\mathbf{J} = \begin{bmatrix}
\cos \Theta & \sin \Theta & 0 \\
-\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
E_{I_{Gx}} & 0 & -\Phi E_{I_{U}} \\
0 & E_{I_{GY}} & -\Phi E_{I_{V}} \\
-\Phi E_{I_{U}} & -\Phi E_{I_{V}} & \phi^2 E_I + GJ
\end{bmatrix}
\begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 \\
\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Alternatively, the strain energy can be expressed in the principal co-ordinate system. This is easily done by noting that
\[
\begin{bmatrix}
v \\ u \\ x
\end{bmatrix}
= \begin{bmatrix}
\cos \Theta & \sin \Theta \\
-\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
y \\ x
\end{bmatrix}
\begin{bmatrix}
v \\ u
\end{bmatrix}
\]

For example, when the high order cross-sectional moduli are neglected in the equation (A1.25), the resulting expression can be identified with the following forms
\[
A = \begin{bmatrix}
-\Phi^2 & -\Phi^2 & -2 \phi \frac{\partial}{\partial z} & -2 \phi \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z}
\end{bmatrix}
\]
\[
\sigma = \{ V \cdot U \cdot \Theta \}
\]
\[
\mathbf{\chi}^T = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
3. The Kinetic Energy in Twisted Asymmetric Beams (rotary inertia neglected)

The kinetic energy of the beam is given by

\[ T = \frac{1}{2} \int_0^L \left\{ \mu \mathbf{A} \left( \frac{3}{2} \mathbf{G}_{x} \cdot \mathbf{G}_{x} + \frac{2}{3} \mathbf{G}_{y} \cdot \mathbf{G}_{y} + \lambda_{G}^2 \dot{\theta}^2 \right) \right\} \, dz \quad (A1.31) \]

where \( \lambda_G \) = polar radius of gyration about centroid 
\( \mu \) = mass per unit volume

But from Fig.A1.1a
\[ \mathbf{G}_{z} = \mathbf{w} - \theta \mathbf{k} \times \mathbf{G}_{O} \]

where \( \mathbf{k} \) is a unit vector in the direction of z axis. Also, from Fig.A1.2
\[ \mathbf{G}_{O} = ie_x + j e_y \]

Therefore
\[ \mathbf{G}_{z} = (i \mu + j v) - \theta \mathbf{k} \times (i e_x + j e_y) \]

or,
\[ \mathbf{G}_{z} = i \mu + \frac{1}{2} y \]

Hence energy equation (A1.31) becomes

\[ T = \frac{1}{2} \int_0^L \mu \mathbf{A} \left\{ (v - e_x \dot{\theta})^2 + (\mu + e_y \dot{\theta})^2 + \lambda_{G}^2 \dot{\theta}^2 \right\} \, dz \quad (A1.33) \]

The foregoing equation has the form

\[ T = \frac{1}{2} \int_0^L (B \beta \dot{\sigma})^T O \mathbf{B} \beta \dot{\sigma} \, dz. \quad (A1.34) \]

with
The Effect of Rotatory Inertia

In the previous Article the kinetic energy due to the out-of-plane rotations of cross-sections is ignored. This effect can be taken into account by considering the rotational motion of sections relative to their centres of gravity. Denoting the components of these in the $xy$ frame by $\Lambda_x$ and $\Lambda_y$, the additional kinetic energy becomes (see Fig. A1.3)

$$\mathcal{T}_R = \frac{1}{2} \mu \int_0^l \left( I_{Gx} \dot{\Lambda}_x^2 + 2 I_{Gxy} \dot{\Lambda}_x \dot{\Lambda}_y + I_{Gy} \dot{\Lambda}_y^2 \right) d\sigma_1 \, dz$$  \hspace{1cm} (A1.37)

The rotations $\Lambda_x$ and $\Lambda_y$ are assumed to be given by

$$\frac{\partial w}{\partial z} = \frac{\partial u}{\partial z} + i \frac{\partial v}{\partial z} = i \Lambda_y + i \Lambda_x$$

Hence

$$\mathcal{T}_R = \frac{1}{2} \mu \int_0^l \left\{ I_{Gx} (\dot{w}')^2 + 2 I_{Gxy} \dot{v}' \dot{u}' + I_{Gy} (\dot{u}')^2 \right\} d\sigma_1 \, dz$$

This expression has the form of the equation (A1.34) with

$$\mathbf{B} = \left[ \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right]$$
\[ \sigma = \{ \ddot{v}, \ddot{u} \} , \beta = \begin{bmatrix} 1 & 1 \end{bmatrix} \]
\[ \mathbf{C} = \begin{bmatrix} I_{Gx} & I_{Gxy} \\ I_{Gxy} & I_{Gy} \end{bmatrix} \]

From Fig. A1.2, and the relationship
\[
\begin{bmatrix} I_{Gx} & I_{Gxy} \\ I_{Gxy} & I_{Gy} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} I_{Gx} & 0 \\ 0 & I_{Gy} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}
\]
the foregoing forms can be written in the principal frame as
\[ \mathbf{B} = \begin{bmatrix} -\Phi & -\Phi & \frac{3}{2} & \frac{3}{2} \end{bmatrix} \]
\[ \mathbf{\dot{\sigma}} = \begin{bmatrix} \dot{v} \\ \dot{u} \end{bmatrix} \]
\[ \mathbf{C} = \begin{bmatrix} I_{Gy} & 0 & 0 & I_{Gy} \\ 0 & I_{Gx} & -I_{Gx} & 0 \\ 0 & -I_{Gx} & I_{Gx} & 0 \\ I_{Gy} & 0 & 0 & I_{Gy} \end{bmatrix} \]
\[ \beta^T = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \]

5. The Effect of Shear Displacements

As with problems involving the effects of shear displacement it is convenient here to consider the total deflection of the beam as made up of bending and shear deflections\(^{(17)}\), i.e.,
\[ \mathbf{\gamma} = \mathbf{\gamma}_b + \mathbf{\gamma}_s \]
or
\[ i \mathbf{\mu} + \frac{j}{2} \mathbf{\nu} = \frac{i}{2} (\mathbf{\mu}_b + \mathbf{\mu}_s) + \frac{j}{2} (\mathbf{\nu}_b + \mathbf{\nu}_s) \]
where subscripts \(b\) and \(s\) refer to bending and shear, respectively.

With this notation the strain energy due to shear is
\[ \mathcal{U}_s = \frac{1}{2} \int \kappa \mathcal{G} \{(\mathbf{\mu}' - \Lambda_y)^2 + (\mathbf{\nu}' - \Lambda_x)^2\} \, dz \]
where the meaning of the rotations \(\Lambda_y\) and \(\Lambda_x\) must be modified as
\[ i \Lambda_y + \frac{j}{2} \Lambda_x = \frac{i}{2} \mathbf{\mu}_b + \frac{j}{2} \mathbf{\nu}_b \]
since the shear displacements have no rotational components, and \(k\) is the Timoshenko shear coefficient. It should be noted that
when the effect of shear displacement is considered the strain energy given by the equation (A1.25) and the rotatory energy given by the equation (A1.37) should be written in terms of the rotations $\Lambda_x$ and $\Lambda_y$, whilst the equation (A1.33) remains the same but the meaning of equation (A1.39) is implied.

The equation (A1.40) can be written in the form (A1.26) with

$$A = \left[ \begin{array}{ccc} \frac{\partial}{\partial z} & 1 & \frac{\partial}{\partial z} \\ \end{array} \right]$$

$$\sigma = \left\{ \begin{array}{c} u \\ \Lambda_y \\ \Lambda_x \end{array} \right\}$$

$$D = \left[ \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] k \alpha \Gamma$$

$$\alpha = \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right]$$

Alternatively, these forms can be expressed in terms of the displacements in principal directions as follows

$$A = \left[ \begin{array}{ccc} -\Phi & -\Phi & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \end{array} \right]$$

$$\sigma = \left\{ \begin{array}{c} V_s \\ U_s \end{array} \right\}$$

$$D = k \alpha \Gamma \cdot \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\alpha^T = \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right]$$
Deformation of Beam Sections
A Moving Triad
Fig. Al.2 REFERENCE FRAMES

Fig. Al.3 OUT-OF-PLANE ROTATIONS
APPENDIX 2

NOTE ON THE DERIVATION OF EQUATIONS OF MOTION

The free motion of the beam at time $t$, as determined by its strain energy $U$ and kinetic energy $T$, is given by the Hamilton's Principle, i.e.,

$$\delta \int_{t_1}^{t_2} H \, dt = 0$$

where $t_1 - t_2$ is an arbitrary time interval, and the quantity $H = T - U$ is called the Hamiltonian of the system.

Substituting the equations (1.5) and (1.6) into the foregoing equation and allowing the variations of the displacement vector

$$\delta \int_{t_1}^{t_2} H \, dt = \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \left\{ \left[ (B\beta \delta \sigma)^T \mathcal{I} B\beta \delta \sigma + (B\beta \delta \sigma)^T \mathcal{I} B\beta \delta \sigma \right] - \left[ (A\alpha \sigma)^T \mathcal{D} A\alpha \delta \sigma + (A\alpha \delta \sigma)^T \mathcal{D} A\alpha \sigma \right] \right\} \, dz \, dt$$

where it is assumed that the operator matrices $A$ and $B$ are diagonal. From the symmetry of the matrices $\mathcal{D}$ and $\mathcal{C}$

$$\left( (A\alpha \sigma)^T \mathcal{D} A\alpha \delta \sigma \right)^T = (A\alpha \delta \sigma)^T \mathcal{D} A\alpha \sigma \quad (A2.2)$$

$$\left( (B\beta \delta \sigma)^T \mathcal{C} B\beta \delta \sigma \right)^T = (B\beta \delta \sigma)^T \mathcal{C} B\beta \delta \sigma \quad (A2.3)$$

By substituting equations (A2.2) and (A2.3) in the equation (A2.1), and noting that the former can be considered as identities between 1x1 matrices

$$\delta \int_{t_1}^{t_2} H \, dt = \int_{t_1}^{t_2} \int_0^L \left\{ (B\beta \delta \sigma)^T \mathcal{C} B\beta \delta \sigma - (A\alpha \sigma)^T \mathcal{D} A\alpha \delta \sigma \right\} \, dz \, dt$$

Integrating the first term in the above equation partially with respect to time

$$\delta \int_{t_1}^{t_2} H \, dt = - \int_{t_1}^{t_2} \int_0^L \left\{ (B\beta \delta \sigma)^T \mathcal{C} B\beta \delta \sigma + (A\alpha \sigma)^T \mathcal{D} A\alpha \delta \sigma \right\} \, dz \, dt$$
since $\delta=0$ when $t = t_1$ and $t = t_2$. By making use of the symmetry properties of coupling matrices $D$ and $Q$

$$\delta\int_{t_i}^{t_2} H dt = -\sum_{i=1}^{q} \int_{t_i}^{t_f} \left[ (Q_{ij} B) \delta t + (D_{ij} A) \delta t \right] dt$$

The equations of motion are obtained, as it is well known, from partial integration of this equation. This can be done easily if the equation (A2.4) is written in expanded form. However, a different method, which produces the same effect, will be employed here so that the resulting equations of motion can be written in a compact form. Consider a series of diagonal operator matrices $\Delta_{A_1}, \Delta_{A_2}, \ldots, \Delta_{A_r}$ and $\Delta_{B_1}, \Delta_{B_2}, \ldots, \Delta_{B_q}$ defined as follows:

$$B = \Delta_{B_1} \cdot \Delta_{B_2} \cdot \ldots \cdot \Delta_{B_q} = \prod_{s=1}^{q} \Delta_{B_s}$$

$$A = \Delta_{A_1} \cdot \Delta_{A_2} \cdot \ldots \cdot \Delta_{A_r} = \prod_{s=1}^{r} \Delta_{A_s}$$

where $\Delta_{A_j}$ and $\Delta_{B_j}$ involve the simple operator $\delta/\delta x$ and/or constants as their elements. It will be convenient to extend these operators by the following convention

$$\Delta_{A_0} = \Delta_{B_0} = \mathcal{E}$$

where $\mathcal{E}$ is a conforming unit matrix. The integration operators $\phi_{A_j}, \phi_{B_j},$ and $\Phi_{A_j}, \Phi_{B_j}$ are now defined as

$$\phi_{A_j} \cdot \Delta_{A_j} = \mathcal{E}_{A_j}$$

$$\Phi_{A_j} \cdot \Delta_{A_j} = \mathcal{E}_{A_j}$$

$$\phi_{B_j} \cdot \Delta_{B_j} = \mathcal{E}_{B_j}$$

$$\Phi_{B_j} \cdot \Delta_{B_j} = \mathcal{E}_{B_j}$$

$$\mathcal{E}_{A_0} = \mathcal{E}_{B_0} = \mathcal{E}$$

where the diagonal matrices $\mathcal{E}_{A_j}$ and $\mathcal{E}_{B_j}$ have the following one-to-one correspondence with the operator $\Delta_{A_j}$

i) a diagonal term of $\mathcal{E}_{A_j}$ is equal to -1 if the corresponding term in $\Delta_{A_j}$ involves the operator $\delta/\delta x$ or else it is equal to 1.

ii) a diagonal term of $\mathcal{E}_{B_j}$ is equal to 1 if the corresponding
term in $\Delta_{A_i}$ is $\delta/\delta z$ else it is equal to zero.
The correspondance between $E_{B_i}$, $\beta_{B_i}$ and $\Delta_{B_i}$ is the same. From equations (A2.7) and (A2.10) it is seen that $\phi_{A_0}$ and $\phi_{B_0}$ are unit matrices.

With this notation the first partial integration of the equation (A2.11) becomes

$$\int_{t_1}^{t_2} J(t) dt = - \int_{t_1}^{t_2} \left[ \left( \begin{array}{c} (\Delta_{B_i} \beta_{B_i})^T \phi_{B_i} \beta_{B_i} \delta \sigma + (\Delta_{A_i} \alpha \sigma)^T \phi_{A_i} \alpha \delta \sigma \end{array} \right) \right]_0^L +$$

$$+ \int_0^L \left[ (\Delta_{B_i} \beta_{B_i})^T \phi_{B_i} \beta_{B_i} \delta \sigma +$$

$$+ (\Delta_{A_i} \alpha \sigma)^T \phi_{A_i} \alpha \delta \sigma \right] dt$$

Thus, by successive application of the integration procedure indicated in the foregoing equation, it is not difficult to show that the result of the complete integration can be written as follows

$$\int_{t_1}^{t_2} J(t) dt = - \int_{t_1}^{t_2} \left[ \sum_{i=1}^{q} \left( \sum_{s=1}^{1} \Delta_{B_i} \beta_{B_i})^T \phi_{B_i} \beta_{B_i} \delta \sigma +$$

$$+ \sum_{s=1}^{q} \left( \sum_{i=1}^{1} \Delta_{A_i} \alpha \sigma)^T \phi_{A_i} \alpha \delta \sigma \right) \right]_0^L +$$

$$+ \int_0^L \left[ \left( \sum_{s=1}^{q} \Delta_{B_i} \beta_{B_i})^T \phi_{B_i} \beta_{B_i} +$$

$$+ \left( \sum_{s=1}^{q} \Delta_{A_i} \alpha \sigma)^T \phi_{A_i} \alpha \delta \sigma \right) \right] dz \right] dt$$

By making use of the equations (A2.5), (A2.6) and (A2.8), (A2.9), and noting that the multiplication of diagonal matrices is commutative, the equation (A2.11) becomes
On invoking the Hamilton's Principle the time and space integrals in the foregoing equation may be dropped. Hence the equations of motion are

\[
\sum_{j=1}^{q} \left( \prod_{s=1}^{i-1} \Delta_{B_{s-1}} \delta \sigma \right)^T \prod_{s=1}^{i-1} E_{B_{s}} \cdot B_{j} \cdot \prod_{s=i+1}^{q} \Delta_{B_{s}} \cdot B_{j} \cdot \delta \sigma + \\
\sum_{j=1}^{q} \left( \prod_{s=1}^{j-1} \Delta_{A_{s-1}} \delta \sigma \right)^T \prod_{s=1}^{j-1} E_{A_{j}} \cdot A_{j} \cdot \prod_{s=j+1}^{r} \Delta_{A_{s}} \cdot \alpha \delta \sigma \Big|^{r}_{0} = 0
\]

\[
\sum_{j=1}^{q} \left( \prod_{s=1}^{i-1} \Delta_{B_{s-1}} \delta \sigma \right)^T \prod_{s=1}^{i-1} E_{B_{s}} \cdot B_{j} \cdot \prod_{s=i+1}^{q} \Delta_{B_{s}} \cdot \beta \delta \sigma + \\
\sum_{j=1}^{q} \left( \prod_{s=1}^{j-1} \Delta_{A_{s-1}} \delta \sigma \right)^T \prod_{s=1}^{j-1} E_{A_{j}} \cdot A_{j} \cdot \prod_{s=j+1}^{r} \Delta_{A_{s}} \cdot \alpha \delta \sigma \Big|^{r}_{0} = 0
\]

The equations (A2.13) and (A2.14) are called the field and boundary equations, respectively.
APPENDIX 3

EXACT ANALYSIS OF EQUATIONS OF MOTION

1. Bending-Torsion Vibrations

The equations of motion, which can be obtained from the application of the Hamilton's Principle to the energy equations derived in Appendix 1, can be written as follows

\[ \dddot{\eta} - \Omega^2 (\eta + \epsilon_x \Theta) = 0 \quad \text{(A3.1)} \]

\[ \dddot{\Theta} + \Omega^2 \rho [\epsilon_x \eta + (\frac{\epsilon_x}{\rho} + \frac{\epsilon_G}{\rho}) \Theta] = 0 \quad \text{(A3.2)} \]

where the beam is assumed to be uniform and symmetric about the X axis, and the symbols are defined in the nomenclature.

Eliminating \( \eta \) (or, \( \Theta \)) from the foregoing equations

\[ [D^6 + \Omega^2 \rho (\epsilon_x^2 + \frac{\epsilon_G^2}{\rho}) D^4 - \Omega^2 \rho \frac{\epsilon_x \Theta}{\rho} + \Omega^2 \rho \frac{\epsilon_G^2}{\rho}] \eta, \Theta = 0 \]

where \( D = \frac{d}{d\xi} \). This equation has solutions of the form \( \exp(m\xi) \)

where \( m \) is given by the roots of the equation

\[ (m^2)^3 + \Omega^4 \rho (\epsilon_x^2 + \frac{\epsilon_G^2}{\rho}) (m^2)^2 - \Omega^2 \rho \frac{\epsilon_x \Theta}{\rho} = 0 \quad \text{(A3.3)} \]

Using the notation

\[ a = \Omega^2 \rho (\epsilon_x^2 + \frac{\epsilon_G^2}{\rho}) \quad \text{(A3.4)} \]

\[ b = -\Omega^4 \quad \text{(A3.5)} \]

\[ c = -\Omega^2 \rho \frac{\epsilon_G^2}{\rho} \quad \text{(A3.6)} \]

and defining a new variable, say \( \lambda \), by

\[ \lambda = m^2 + a/3 \quad \text{(A3.7)} \]

the characteristic equation (A3.3) becomes

\[ \lambda^3 + p \lambda + q = 0 \quad \text{(A3.8)} \]

where

\[ p = b - a^2/3 \quad \text{(A3.9)} \]

\[ q = c - ab/3 + 2a^3/27 \quad \text{(A3.10)} \]

It can be shown that when \( p \) and \( q \) are real the roots of the equation (A3.8) depends on the discriminant

\[ \Delta = 4p^3 + 27q^2 \quad \text{(A3.11)} \]
in the following fashion

1) $\Delta > 0$, one real root and two imaginary roots

2) $\Delta \leq 0$, three real roots

Substituting the equations (A3.4) to (A3.6) in (A3.9) and (A3.10), and the results into the equation (A3.11), it can be shown that

$\Delta < 0$. Thus the equation (A3.8) has three real roots, which

will be given by the following formulas (30)

$$
\lambda_1 = 2\sqrt{-\frac{p}{3}} \cos \left( \frac{\phi}{3} \right)
$$

$$
\lambda_2 = 2\sqrt{-\frac{p}{3}} \cos \left( \frac{\phi}{3} + \frac{2\pi}{3} \right)
$$

$$
\lambda_3 = 2\sqrt{-\frac{p}{3}} \cos \left( \frac{\phi}{3} + \frac{4\pi}{3} \right)
$$

where $\cos \phi = \frac{3q}{2p \sqrt{-p/3}}$.

From the transformation (A3.7) it is seen that the roots of the equation (A3.3) will have the following forms

$$
\pm \left( \lambda_1 - a/3 \right)^{1/2}
$$

(A3.12)

$$
\pm \left( \lambda_2 - a/3 \right)^{1/2}
$$

$$
\pm \left( \lambda_3 - a/3 \right)^{1/2}
$$

By Descartes' Rule of Signs (30), however, it can be concluded that the equation (A3.3) has only one positive and one negative real roots, and the remaining four roots are imaginary. The real roots will be given by the expression (A3.12), since

$$
\left( \lambda_1 - a/3 \right)^{1/2} = \left\{ 2\sqrt{-\frac{1}{3}} \left( \frac{b-a^2/3}{c} \right) \cos \left( \frac{\phi}{3} - \frac{a}{3} \right) \right\}^{1/2}
$$

$$
= \left\{ 2\sqrt{(a/3)^2 + \Omega^2/3} \cos \left( \frac{\phi}{3} - \frac{a}{3} \right) \right\}^{1/2} > 0
$$

and $0 < \phi \leq 180^\circ$

Using the notation

$$
s_1 = \left( \left| \lambda_1 - a/3 \right| \right)^{1/2}
$$

$$
s_2 = \left( \left| \lambda_2 - a/3 \right| \right)^{1/2}
$$

$$
s_3 = \left( \left| \lambda_3 - a/3 \right| \right)^{1/2}
$$

the roots of the characteristic equation (A3.3) will, therefore, be as follows

$$
s_1, -s_1, js_2, -js_2, js_3, -js_3
$$

where $j = \sqrt{-1}$. Hence, the general solutions of the differential equation are obtained as
\[ \eta = A_1 \cosh s_1 \kappa + A_2 \sinh s_1 \kappa + A_3 \cos s_2 \kappa + A_4 \sin s_2 \kappa + A_5 \cos s_3 \kappa + A_6 \sin s_3 \kappa \]
\[ \Theta = B_1 \cosh s_1 \kappa + B_2 \sinh s_1 \kappa + B_3 \cos s_2 \kappa + B_4 \sin s_2 \kappa + B_5 \cos s_3 \kappa + B_6 \sin s_3 \kappa \]
where \( A_j \) and \( B_j \) are constants which are connected by the equation (A3.1) or (A3.2).

For a clamped-free beam, for example, the boundary conditions are \( \eta = \eta' = \Theta = 0 \) at \( \kappa = 0 \), and \( \eta'' = \eta''' = \Theta' = 0 \) at \( \kappa = 1 \). Application of these and the elimination of the six integration constants through the equation (A3.1) results in the following frequency determinant

\[
\begin{vmatrix}
P_1 & P_2 & P_3 \\
Q_1 & Q_2 & Q_3 \\
R_1 & R_2 & R_3
\end{vmatrix} = 0 
\] (A3.13)

where

\[ P_j = \begin{bmatrix} 1 & 0 \\ 0 & s_j \end{bmatrix}, \quad j = 1, 2, 3, \]

\[ Q_j = s_j^2 \begin{bmatrix} \cosh s_1 & \sinh s_1 \\ \sinh s_1 & \cosh s_1 \end{bmatrix}, \quad j = 1, 2, 3. \]

\[ Q_j = -s_j^2 \begin{bmatrix} \cos s_j & \sin s_j \\ -s_j \sin s_j & s_j \cos s_j \end{bmatrix}, \quad j = 2, 3. \]

\[ R_1 = (s_1^4 - \Omega_1^4) \begin{bmatrix} 1 & 0 \\ s_1 \sinh s_1 & s_1 \cosh s_1 \end{bmatrix}, \quad j = 2, 3. \]

2. Vibrations of Constrained Beams

The equation of motion of the beam shown in Fig. A3.1 can be written in non-dimensional form as follows

\[
\eta'' + 3 \eta' \delta (\kappa - \eta) - \Omega^2 [1 + \bar{m} \delta (\kappa - \epsilon)] \eta = 0 \] (A3.14)
where harmonic motion is assumed, and
\[ \bar{f} = \frac{sL^3}{EI_0}, \quad \bar{m} = m/\mu_0 l^2, \quad \omega^2 = \omega_0^2/\mu_0 L^4/EI_0 \]
and the Dirac delta function \( \delta(\kappa - \epsilon) \) is defined as
\[ \delta(\kappa - \epsilon) = 0 \quad \text{if} \quad \kappa \neq \epsilon \]
\[ \delta(\kappa - \epsilon) = \infty \quad \text{if} \quad \kappa = \epsilon \]
\[ \int_{-\infty}^{\infty} \delta(\kappa - \epsilon) \, d\kappa = 1 \]
The function \( \delta(\kappa - \eta) \) is defined similarly.

Taking the Laplace Transform of the equation (A3.14)
\[ t^4 \eta^*(t) - t^3 \eta(0) - t^2 \eta'(0) - t \eta''(0) - \eta'''(0) + \]
\[ \bar{s} \eta(\eta) e^{-t\eta} - \bar{m} \omega^4 \eta(\eta) e^{-t\eta} - \bar{m} \omega^4 \eta^*(t) = 0 \quad (A3.15) \]
where \( \eta^*(t) \) denotes the transform of the deflection \( \eta(\kappa) \) with respect to the variable \( t \), and \( e \) denotes the exponential function.

![Fig.A3.1 A constrained beam.](image)

For a cantilever beam the boundary conditions are
\[ \eta(0) = \eta'(0) = 0 \quad (A3.16) \]
\[ \eta''(0) = \eta'''(0) = 0 \quad (A3.17) \]
Substituting the former conditions into the equation (A3.15) and solving for \( \eta^*(t) \)
\[ \eta^*(t) = \frac{t \cdot \eta''(0) + \eta'''(0)}{t^4 - \omega^4} + \bar{m} \omega^4 \frac{\eta(\eta) e^{-t\eta}}{t^4 - \omega^4} - \bar{s} \eta(\eta) e^{-t\eta} \]
Hence the inverse transformation gives
\[ \eta(\kappa) = \frac{1}{2} \frac{\eta''(0)}{\Omega^2} \left[ \cosh \Omega \kappa - \cos \Omega \kappa \right] + \frac{1}{2} \frac{\eta'''(0)}{\Omega^3} \left[ \sinh \Omega \kappa - \sin \Omega \kappa \right] \\
+ \frac{1}{2} \frac{\eta''(0)}{\Omega^2} \mathcal{M}(\kappa - \varepsilon) \left[ \sinh \Omega (\kappa - \varepsilon) - \sin \Omega (\kappa - \varepsilon) \right] \cdot \eta(\varepsilon) \\
- \frac{1}{2} \frac{\eta''(0)}{\Omega^2} \mathcal{M}(\kappa - \pi) \left[ \sinh \Omega (\kappa - \pi) - \sin \Omega (\kappa - \pi) \right] \cdot \eta(\pi) \]

where \( \mathcal{M}(\kappa - \varepsilon) \) is a unit-step-function placed at \( \kappa = \varepsilon \).

The function \( \mathcal{M}(\kappa - \pi) \) is defined similarly.

The four unknowns \( \eta''(0), \eta'''(0), \eta(\varepsilon), \) and \( \eta(\pi) \) are obtained by imposing the boundary conditions (A3.17) and the values of the function \( \eta(\kappa) \) at \( \kappa = \varepsilon \) and \( \kappa = \pi \) on the equation (A3.18). The result is written in matrix notation

\[
\begin{pmatrix}
\cosh \Omega & \frac{1}{2} \sinh \Omega & -\frac{\mathcal{M}}{\Omega^2} \sinh(1-\varepsilon) & \mathcal{M} \sinh(1-\varepsilon) \\
+ \cos \Omega & + \sin \Omega & + \sin(1-\varepsilon) & + \sin(1-\varepsilon) \\
\sinh \Omega & (\cosh \Omega & - \frac{3}{2} \cosh(1-\varepsilon) & \mathcal{M} \cosh(1-\varepsilon) \\
- \sin \Omega & + \cos \Omega & + \cos(1-\varepsilon) & + \cos(1-\varepsilon) \\
\frac{1}{\Omega^4} \left( \cosh \Omega \pi \right) & \frac{1}{\Omega^2} \left( \sinh \Omega \pi \right) & -2 & \mathcal{M} \mathcal{M} \mathcal{M}(\pi - \varepsilon) \left( \begin{array}{c}
sinh(\pi - \varepsilon) \\
- \sin(\pi - \varepsilon) \\
\end{array} \right) \\
\frac{1}{\Omega^4} \left( \cosh \Omega \pi \right) & \frac{1}{\Omega^2} \left( \sinh \Omega \pi \right) & -2 & \mathcal{M} \mathcal{M} \mathcal{M}(\pi - \varepsilon) \left( \begin{array}{c}
sinh(\pi - \varepsilon) \\
- \sin(\pi - \varepsilon) \\
\end{array} \right) \\
- \cos \Omega \varepsilon & - \sin \Omega \varepsilon & - \sin \Omega(\pi - \varepsilon) & -2 \\
\end{pmatrix}
\]

The corresponding eigenvector is

\[ \eta''(0), \eta'''(0), \eta(\pi), \eta(\varepsilon) \]
APPENDIX 4

POWERS OF SQUARE MATRICES WITH
DISTINCT ROOTS

Astin (58) recently proved two theorems which provide a
generalization to the Cayley-Hamilton theorem. The first gives
the explicit expression for \( A^N \) as a polynomial in \( A \), where \( A \) is
square matrix. The second expands the first result to the case
of matrix series. Here the main interest is in the first one
which is quoted below:

If \( A \) is a square matrix of order \( P \), with distinct latent roots \( t_1, t_2, \ldots, t_P \) then (writing \( A^0 = \text{unit matrix} \))

\[
D A^N = D_{P-1} A^{P-1} = D_{P-2} A^{P-2} - \ldots - D_0 A^0
\]  

(A4.1)

where 

\[
D = \begin{bmatrix}
1 & t_1 & \cdots & t_1^{P-1} \\
1 & t_2 & \cdots & t_2^{P-1} \\
& \ddots & \ddots & \cdots \\
1 & t_P & \cdots & t_P^{P-1}
\end{bmatrix}
\]

and where \( D_j \) is obtained from \( D \) by replacing the \((j-1)\)th column
of \( D \) by

\[
\begin{bmatrix}
t_1^N \\
t_2^N \\
\vdots \\
t_P^N
\end{bmatrix}
\]

It is interesting to note that \( D \) and \( D_j \) are alternants.
Consequently \( D_j/D \) can be obtained by the procedure described in
Ref. (57) for determination of bialternants.

Consider a simple dynamic-stiffness matrix of the following
special form.
where the elements may be some functions of the frequency parameter. The corresponding transfer matrix will be given by

\[
A = \frac{1}{P_2} \begin{bmatrix} -P_1 & -\Delta \\ \frac{1}{x} & -P_1 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} a & d \\ c & a \end{bmatrix}
\]  

(Al.3)

If the corresponding system is assembled from n such elements the overall transfer relation will be given by the n th power of the above matrix. Denoting the roots of the matrix (Al.3) by \( t_1 \) and \( t_2 \) and making use of the Astin's Theorem, the i th power of this matrix becomes

\[
A^i = \frac{1}{\Delta^i(t_1-t_2)} \begin{bmatrix} a(t_1^{i}-t_2^{i})+t_1^{i}t_2^{i}-t_1^{i}t_2^{i} & d(t_1^{i}-t_2^{i}) \\ c(t_1^{i}-t_2^{i}) & a(t_1^{i}-t_2^{i})+t_1^{i}t_2^{i}-t_1^{i}t_2^{i} \end{bmatrix} 
\]

(Al.4)

where

\[
t_1 = a-\sqrt{cd} = a-jb, \quad (b = \sqrt{-cd}) \]

(Al.5)

\[
t_2 = a+\sqrt{cd} = a+jb, \quad (j = \sqrt{-1})
\]

(Al.6)

In the case of St. Venant beams the foregoing matrix will express the relation of the torque and displacements at the origin to the respective quantities at the i th station. Thus, denoting torque by \( T \) and twist by \( \Theta \)

\[
\begin{bmatrix} \Theta_i \\ T_i \end{bmatrix} = A^i \begin{bmatrix} \Theta_o \\ T_o \end{bmatrix}
\]

(Al.7)

For a clamped-free beam it is seen that the frequency equation will be

\[
a(t_1^n-t_2^n)+t_1^n t_2^n-t_1^n t_2^n = 0
\]

(Al.8)

where the beam is assumed to consist of n elements. On the other hand, since \( \Theta_o = 0 \), from equation (Al.7) one obtains

\[
\Theta_1 = \frac{T_0 d}{\Delta^i} \frac{t_1^{i}-t_2^{i}}{t_1-t_2}
\]

(Al.9)
But by De Moivre's Theorem
\[(a+jb)^i = \Delta^i (\cos i\alpha + j\sin i\alpha) \quad (A4.10)\]
\[(a-jb)^i = \Delta^i (\cos i\alpha - j\sin i\alpha) \quad (A4.11)\]
where \(\tan \alpha = \frac{b}{a}\), and the relation \(\Delta^2 = a^2 + b^2\) is consistent with the meaning employed in the matrix (A4.3), namely \(\Delta = P_2\), i.e., \(P_2\) is the square root of the determinant of the transfer matrix.

Thus \(\cos \alpha = \frac{a}{\Delta}\).

From the substitution of the equations (A4.10) and (A4.11) in the equation (A4.9)
\[\Theta_i = -j \cdot \frac{To}{2b} \cdot \frac{(a+jb)^i - (a-jb)^i}{\Delta^i} \]
\[= \frac{To}{b} \sin i\alpha \quad (A4.12)\]
Similarly the equation (A4.8) becomes
\[\left(\frac{a-jb}{a+jb}\right)^n = -1\]
or, \(\cos n\alpha = 0\)
Therefore \(\alpha = (2r+1)\pi/2n\), \(r = 0, 1, 2, \ldots, n-1\)
Hence, the solutions for frequencies and mode shapes of a clamped-free St.-Venant beam segmented into \(n\) elements can be obtained from the relations
\[b = a \cdot \tan\left(\frac{2r+1}{2n}\right)\pi\quad (A4.13)\]
\[\Theta_i = \text{(constant)} \cdot \sin\left(\frac{2r+1}{2n}\right)i\pi\quad (A4.14)\]
respectively.

The foregoing equations also apply to free-free beams if \(\alpha\) is taken as
\[\alpha = r\pi/n\quad , \quad r = 1, 2, \ldots, n\]
APPENDIX 5

A CATALOGUE OF SUBMATRICES FOR
KINETIC BEAM PROBLEMS IN FINITE ELEMENT APPROXIMATION
BY POLYNOMIALS

In referring to the submatrices the notation of Section 2 is
modified by incorporating the symbols of Section 2 with superscripts
in brackets which indicate the number of internal nodes used. All
internal nodes are taken in positions that divide the beam element
into equal portions.

The reciprocals of various transformation matrices, namely the
matrices $D_x$, $D_y$, and $D_\theta$ are also included at the end of the catalogue.
\[
\begin{align*}
K_{Y_0}^{(0)} &= K_{X_0}^{(0)} = \begin{bmatrix}
1260 & 650 & -1260 & 650 \\
650 & 420 & -650 & 210 \\
-1260 & -650 & 1260 & -650 \\
650 & 210 & -650 & 420 \\
\end{bmatrix}, &
R_{Y_0}^{(0)} &= R_{X_0}^{(0)} = \begin{bmatrix}
1008 & 84 & -1008 & 84 \\
84 & 112 & -84 & 28 \\
-1008 & -84 & 1008 & -84 \\
84 & -28 & -84 & 112 \\
\end{bmatrix} \\
K_{Y_1}^{(0)} &= K_{X_1}^{(0)} = \begin{bmatrix}
650 & 210 & -650 & 420 \\
210 & 105 & -210 & 105 \\
-650 & -210 & 650 & -420 \\
420 & 105 & -420 & 315 \\
\end{bmatrix}, &
R_{Y_1}^{(0)} &= R_{X_1}^{(0)} = \begin{bmatrix}
504 & 84 & -504 & 0 \\
84 & 28 & -84 & -14 \\
-504 & -84 & 504 & 0 \\
0 & -14 & 0 & 84 \\
\end{bmatrix} \\
K_{Y_2}^{(0)} &= K_{X_2}^{(0)} = \begin{bmatrix}
504 & 147 & -504 & 357 \\
147 & 56 & -147 & 91 \\
-504 & -147 & 504 & -357 \\
357 & 91 & -357 & 266 \\
\end{bmatrix}, &
R_{Y_2}^{(0)} &= R_{X_2}^{(0)} = \begin{bmatrix}
288 & 60 & -288 & 24 \\
60 & 16 & -60 & -12 \\
-288 & -60 & 288 & 24 \\
-24 & -12 & 24 & 72 \\
\end{bmatrix} \\
K_{Y_3}^{(0)} &= K_{X_3}^{(0)} = \begin{bmatrix}
441 & 126 & -441 & 315 \\
126 & 42 & -126 & 84 \\
-441 & -126 & 441 & -315 \\
315 & 84 & -315 & 231 \\
\end{bmatrix}, &
R_{Y_3}^{(0)} &= R_{X_3}^{(0)} = \begin{bmatrix}
180 & 42 & -180 & -30 \\
42 & 11 & -42 & -11 \\
-180 & -42 & 180 & 30 \\
-50 & -11 & 30 & 65 \\
\end{bmatrix} \\
K_{Y_4}^{(0)} &= K_{X_4}^{(0)} = \begin{bmatrix}
396 & 114 & -396 & 282 \\
114 & 36 & -114 & 78 \\
-396 & -114 & 396 & -282 \\
282 & 78 & -282 & 204 \\
\end{bmatrix}, &
R_{Y_4}^{(0)} &= R_{X_4}^{(0)} = \begin{bmatrix}
120 & 30 & -120 & -30 \\
50 & 8 & -50 & -10 \\
-120 & -30 & 120 & 30 \\
-50 & -10 & 30 & 60 \\
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
H_y^{(0)} &= H_x^{(0)} = \begin{bmatrix} -504 & -252 & 504 & -42 \\ -252 & -56 & 42 & 14 \\ 504 & 42 & -504 & 252 \\ -42 & 14 & 252 & -56 \end{bmatrix}, \\
H_y^{(1)} &= H_x^{(1)} = \begin{bmatrix} -42 & -42 & 252 & 0 \\ -42 & -14 & 42 & 7 \\ 252 & 42 & -462 & 210 \\ 0 & 7 & 210 & -42 \end{bmatrix}, \\
H_y^{(2)} &= H_x^{(2)} = \begin{bmatrix} 12 & -8 & 198 & -1 \\ -8 & -4 & 43 & 3 \\ 198 & 43 & -408 & 176 \\ -1 & 3 & 176 & -52 \end{bmatrix}, \\
H_y^{(3)} &= H_x^{(3)} = \begin{bmatrix} 18 & 0 & 171 & -3 \\ 0 & -1 & -42 & 1 \\ 171 & 42 & -360 & 150 \\ -3 & 1 & 150 & -95 \end{bmatrix}, \\
H_y^{(4)} &= H_x^{(4)} = \begin{bmatrix} 16 & 2 & 152 & -4 \\ 2 & 0 & 40 & 0 \\ 152 & 40 & -520 & 330 \\ -4 & 0 & 130 & -20 \end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
N_{20}^{(0)} &= (N_{10}^{(0)})^T = \\
&= \begin{bmatrix}
0 & -420 & 0 & 420 \\
420 & -210 & -420 & 210 \\
0 & 420 & 0 & -420 \\
-420 & -210 & 420 & 210
\end{bmatrix} \\
N_{40}^{(0)} &= (N_{20}^{(0)})^T = \\
&= \begin{bmatrix}
-1260 & -252 & -1260 & 252 \\
252 & 0 & -252 & 42 \\
1260 & 252 & 1260 & -252 \\
-252 & -42 & 252 & 0
\end{bmatrix} \frac{1}{2520}
\end{align*}
\]

\[
\begin{align*}
N_{21}^{(0)} &= (N_{11}^{(0)})^T = \\
&= \begin{bmatrix}
-252 & -126 & 252 & 294 \\
84 & -28 & -84 & 112 \\
252 & 126 & -252 & -294 \\
-556 & -98 & 336 & 182
\end{bmatrix} \frac{1}{420}
\end{align*}
\]

\[
\begin{align*}
N_{41}^{(0)} &= (N_{21}^{(0)})^T = \\
&= \begin{bmatrix}
-468 & -168 & -792 & 144 \\
-24 & -12 & -166 & 30 \\
468 & 168 & 792 & -144 \\
-66 & -12 & 276 & -12
\end{bmatrix} \frac{1}{2520}
\end{align*}
\]

\[
\begin{align*}
N_{22}^{(0)} &= (N_{12}^{(0)})^T = \\
&= \begin{bmatrix}
-252 & -84 & 252 & 252 \\
0 & -14 & 0 & 84 \\
252 & 84 & -252 & -252 \\
-252 & -70 & 252 & 168
\end{bmatrix} \frac{1}{420}
\end{align*}
\]

\[
\begin{align*}
N_{42}^{(0)} &= (N_{22}^{(0)})^T = \\
&= \begin{bmatrix}
-216 & -54 & -540 & 90 \\
-30 & -9 & -138 & 21 \\
216 & 54 & 540 & -90 \\
-18 & 3 & 270 & -15
\end{bmatrix} \frac{1}{2520}
\end{align*}
\]

\[
\begin{align*}
N_{23}^{(0)} &= (N_{13}^{(0)})^T = \\
&= \begin{bmatrix}
-216 & -66 & 216 & 228 \\
-24 & -12 & 24 & 72 \\
216 & 66 & -216 & -228 \\
-192 & -54 & 192 & 156
\end{bmatrix} \frac{1}{420}
\end{align*}
\]

\[
\begin{align*}
N_{43}^{(0)} &= (N_{23}^{(0)})^T = \\
&= \begin{bmatrix}
-114 & -30 & -390 & 60 \\
-21 & -6 & -105 & 15 \\
114 & 50 & 390 & -60 \\
-3 & 0 & 255 & -15
\end{bmatrix} \frac{1}{2520}
\end{align*}
\]

\[
\begin{align*}
N_{24}^{(0)} &= (N_{14}^{(0)})^T = \\
&= \begin{bmatrix}
-180 & -54 & 180 & 210 \\
-30 & -11 & 30 & 65 \\
180 & 54 & -180 & -210 \\
-150 & -43 & 150 & 145
\end{bmatrix} \frac{1}{420}
\end{align*}
\]

\[
\begin{align*}
N_{44}^{(0)} &= (N_{24}^{(0)})^T = \\
&= \begin{bmatrix}
-66 & -18 & -294 & 42 \\
-14 & -4 & -82 & 11 \\
66 & 18 & 294 & -42 \\
2 & 1 & 238 & -14
\end{bmatrix} \frac{1}{2520}
\end{align*}
\]
\[
\begin{align*}
M_{x_0}^{(1)} = M_{x_0}^{(1)} &= \begin{bmatrix}
54392 & 2964 & 22880 & -1160 & 6812 & -754 \\
2964 & 208 & 2288 & -312 & 754 & -78 \\
22880 & 2288 & 146432 & 0 & 22880 & -2288 \\
-4160 & -312 & 0 & 3328 & 4160 & -312 \\
6812 & 754 & 22880 & 4160 & 54392 & -2964 \\
-754 & 78 & 2288 & -312 & -2964 & 208
\end{bmatrix} \cdot \frac{1}{360360}
\end{align*}
\]

\[
M_{x_1}^{(1)} = M_{x_1}^{(1)} = \begin{bmatrix}
5616 & \text{Symmetric} \\
442 & 39 \\
7280 & 832 & 75216 \\
-312 & 0 & 3823 & 1664 \\
3406 & 442 & 15600 & 3818 & 48776 \\
-312 & -39 & 1456 & -312 & -2522 & 169
\end{bmatrix} \cdot \frac{1}{360360}
\]

\[
M_{x_2}^{(1)} = M_{x_2}^{(1)} = \begin{bmatrix}
1108 & \text{Symmetric} \\
111 & 12 \\
3528 & 416 & 39936 \\
288 & 48 & 3328 & 1024 \\
2428 & 331 & 11648 & 3248 & 44268 \\
-201 & -27 & -1040 & 264 & -2191 & 142
\end{bmatrix} \cdot \frac{1}{360360}
\]

\[
M_{x_3}^{(1)} = M_{x_3}^{(1)} = \begin{bmatrix}
399 & \text{Symmetric} \\
48 & 6 \\
1952 & 256 & 25296 \\
336 & 48 & 2688 & 704 \\
1939 & 268 & 9072 & 2596 & 40469 \\
-153 & -21 & -784 & -216 & -1923 & 121
\end{bmatrix} \cdot \frac{1}{360360}
\]

\[
M_{x_4}^{(1)} = M_{x_4}^{(1)} = \begin{bmatrix}
228 & \text{Symmetric} \\
30 & 4 \\
1312 & 176 & 14336 \\
288 & 40 & 2048 & 512 \\
1606 & 223 & 7232 & 2240 & 37208 \\
-123 & -17 & -608 & -176 & -1700 & 104
\end{bmatrix} \cdot \frac{1}{360360}
\]
\[
\begin{align*}
R_{Y_0}^{(1)} &= R_{X_0}^{(1)} = \begin{bmatrix} 954096 \\ 22308 & 16016 \\ -878592 & -27456 & 1757184 \\ 137280 & -4576 & 0 & 146432 \\ -75504 & 5148 & -878592 & -137280 & 954096 \\ -5148 & -2860 & 27456 & -4576 & -22308 & 16016 \\ \end{bmatrix} \cdot \frac{1}{360360} \\
R_{Y_1}^{(1)} &= R_{X_1}^{(1)} = \begin{bmatrix} 202488 \\ 13156 & 1716 \\ -164736 & -9152 & 878592 \\ 52032 & 2288 & 73216 & 73216 \\ -37752 & -4004 & 713856 & -105248 & 751608 \\ -9152 & 1430 & 18304 & -6864 & 9152 & 14300 \\ \end{bmatrix} \cdot \frac{1}{360360} \\
R_{Y_2}^{(1)} &= R_{X_2}^{(1)} = \begin{bmatrix} 59072 \\ 5460 & 624 \\ -18304 & 0 & 589728 \\ 19532 & 2496 & 73216 & 46592 \\ -40768 & -5460 & 567424 & -92768 & 608192 \\ -7696 & -1092 & 9152 & -6556 & 9152 & 15208 \\ \end{bmatrix} \cdot \frac{1}{360360} \\
R_{Y_3}^{(1)} &= R_{X_3}^{(1)} = \begin{bmatrix} 24804 \\ 2834 & 551 \\ 17472 & 2912 & 439296 \\ 16640 & 2288 & 66360 & 33280 \\ -42276 & -5746 & 456768 & -85200 & 499044 \\ -6526 & -923 & 2912 & -6240 & 3614 & 12389 \\ \end{bmatrix} \cdot \frac{1}{360360} \\
R_{Y_4}^{(1)} &= R_{X_4}^{(1)} = \begin{bmatrix} 14648 \\ 1878 & 248 \\ 26624 & 3744 & 346112 \\ 14688 & 2032 & 59904 & 25600 \\ -41272 & -5622 & 372736 & -74592 & 414008 \\ -5766 & -818 & 1248 & -5372 & 7014 & 11740 \\ \end{bmatrix} \cdot \frac{1}{360360}
\end{align*}
\]
\[
K_0^{(1)} = K_0^{(1)} = \begin{bmatrix}
4032863 & 901296 & 262944 & \text{Symmetric} \\
-2838528 & -709632 & 5677096 \\
1520640 & 253440 & 0 & 1013760 \\
-1194336 & -191664 & -2838528 & -1520640 & 4032863 \\
191664 & 30096 & 709632 & 253440 & -901296 & 262944
\end{bmatrix} \cdot \frac{1}{27720}
\]

\[
K_1^{(1)} = K_1^{(1)} = \begin{bmatrix}
749232 & 99792 & 21384 & \text{Symmetric} \\
-152064 & -25344 & 2838528 \\
456192 & 50688 & 608256 & 506880 \\
-597168 & -74448 & -266464 & -1056448 & 3283632 \\
117216 & 15048 & 684288 & 202752 & -801504 & 241560
\end{bmatrix} \cdot \frac{1}{27720}
\]

\[
K_2^{(1)} = K_2^{(1)} = \begin{bmatrix}
382272 & 42756 & 7040 & \text{Symmetric} \\
185856 & 29568 & -2162388 \\
308552 & 38720 & 608256 & 362976 \\
-568128 & -76824 & -1228864 & -4166608 & 2916672 \\
119592 & 16208 & 629576 & 190784 & -748968 & 227216
\end{bmatrix} \cdot \frac{1}{27720}
\]

\[
K_3^{(1)} = K_3^{(1)} = \begin{bmatrix}
262152 & 34320 & 4752 & \text{Symmetric} \\
291456 & 42240 & 1824788 \\
251528 & 33792 & 576464 & 321924 \\
-553600 & -76560 & -2116224 & -825792 & 266932 \\
122232 & 16896 & 587136 & 183744 & -709368 & 215160
\end{bmatrix} \cdot \frac{1}{27720}
\]

\[
K_4^{(1)} = K_4^{(1)} = \begin{bmatrix}
208224 & 20568 & 3936 & \text{Symmetric} \\
329472 & 46464 & 1622016 \\
221884 & 30528 & 543672 & 282624 \\
-537696 & -74832 & -1951448 & -761856 & 2489184 \\
123408 & 17136 & 552344 & 178568 & -767652 & 204576
\end{bmatrix} \cdot \frac{1}{27720}
\]
\[ H_{x0}^{(1)} = H_{x0}^{(1)} = \begin{bmatrix}
-1908192 & -322032 & \text{Symmetric} \\
404976 & 51912 & -3514368 \\
1757184 & 9152 & 0 & -292864 \\
-274560 & -10296 & 1757184 & 274560 & -1908192 \\
151008 & 7520 & -54912 & 9152 & 404976 & -322032
\end{bmatrix} \frac{1}{560360} \]

\[ H_{x1}^{(1)} = H_{x1}^{(1)} = \begin{bmatrix}
-44616 & -3452 & \text{Symmetric} \\
26312 & 4576 & -3452 & -146432 \\
329472 & 18304 & -1757184 \\
64604 & -4576 & 146432 & -146432 \\
75504 & 8008 & 1427712 & 210496 & -1865376 \\
18304 & 2860 & -36608 & 15728 & 378664 & -28600
\end{bmatrix} \frac{1}{560360} \]

\[ H_{x2}^{(1)} = H_{x2}^{(1)} = \begin{bmatrix}
-9560 & -832 & \text{Symmetric} \\
4992 & 4576 & -832 & -878592 \\
82368 & 4576 & -3452 & -86528 \\
47424 & -5616 & 146432 & -86528 \\
95160 & 12428 & 1180608 & 193956 & -18832320 \\
13884 & 2028 & -22880 & 12688 & 357344 & -26000
\end{bmatrix} \frac{1}{560360} \]

\[ H_{x3}^{(1)} = H_{x3}^{(1)} = \begin{bmatrix}
-15912 & -468 & \text{Symmetric} \\
5016 & 4576 & -468 & -439296 \\
8756 & -832 & 4576 & -113152 & -56576 \\
35152 & -4576 & 146432 & -113152 & -56576 \\
104988 & 14144 & 1007136 & 189488 & -1786512 \\
11180 & 1612 & -14560 & 10608 & 338000 & -23764
\end{bmatrix} \frac{1}{560360} \]

\[ H_{x4}^{(1)} = H_{x4}^{(1)} = \begin{bmatrix}
-16000 & -352 & \text{Symmetric} \\
2424 & 2496 & -352 & -212992 \\
15312 & 2496 & -212992 \\
25920 & -352 & 78972 & -352 & 38912 \\
111680 & 15216 & 885248 & 188160 & -1738240 \\
9120 & 1512 & -9984 & 8576 & 320040 & -21776
\end{bmatrix} \frac{1}{560360} \]
\[
\begin{align*}
N_{20}^{(1)} &= (N_{10}^{(1)})^T = \\
&= \begin{bmatrix}
0 & -62568 & 126720 & 101376 & -126720 & 24552 \\
62568 & -13860 & -38016 & 25344 & -24552 & 4356 \\
-126720 & 38016 & 0 & -202752 & 126720 & 38016 \\
-101376 & -25344 & 202752 & 0 & -101376 & 25344 \\
126720 & 24552 & -126720 & 101376 & 0 & -62568 \\
-24552 & -4356 & -38016 & -25344 & 62568 & 13860
\end{bmatrix} \frac{1}{27720} \\
N_{21}^{(1)} &= (N_{11}^{(1)})^T = \\
&= \begin{bmatrix}
-36696 & -6204 & 97152 & 29568 & -60456 & 16236 \\
4488 & -616 & 4224 & 4928 & -8712 & 2288 \\
-29568 & -2112 & -67584 & -101376 & 97152 & 40128 \\
-40128 & -4576 & -101376 & -5632 & -61248 & 20768 \\
66264 & 8316 & -29568 & 71808 & -36696 & -56364 \\
-15840 & -2068 & -42240 & -20416 & 58060 & 15244
\end{bmatrix} \frac{1}{27720} \\
N_{22}^{(1)} &= (N_{12}^{(1)})^T = \\
&= \begin{bmatrix}
-15576 & -1548 & 54912 & 12672 & -39336 & 13392 \\
-176 & -132 & 5632 & 1760 & -5456 & 1936 \\
-29568 & -4224 & -67584 & -67584 & 97152 & 38016 \\
-17600 & -2112 & -56320 & -5632 & -58720 & 18304 \\
45144 & 6072 & 12672 & 54912 & -57816 & -52008 \\
-12584 & -1716 & -4032 & -17248 & 55416 & 12760
\end{bmatrix} \frac{1}{27720} \\
N_{23}^{(1)} &= (N_{13}^{(1)})^T = \\
&= \begin{bmatrix}
-6816 & -828 & 33792 & 5664 & -26976 & 12768 \\
432 & -72 & 4224 & 768 & -3792 & 1776 \\
29568 & -4224 & -67584 & 50688 & 97152 & 55904 \\
10176 & -1344 & -53792 & -5376 & -25616 & 16608 \\
36584 & 5052 & 33792 & 45024 & -70176 & -48672 \\
-10992 & -1524 & -38016 & -15072 & 49008 & 12336
\end{bmatrix} \frac{1}{27720} \\
N_{24}^{(1)} &= (N_{14}^{(1)})^T = \\
&= \begin{bmatrix}
-3816 & -508 & 22772 & 2240 & -18456 & 11924 \\
364 & -54 & 3006 & 320 & -2644 & 1666 \\
27648 & -3904 & -67584 & -40960 & 95232 & 34112 \\
7360 & -1024 & -20480 & -5120 & -13120 & 15360 \\
31464 & 4412 & 4532 & 38720 & -76776 & -46036 \\
-9916 & -1382 & -35008 & -13440 & 44924 & 11954
\end{bmatrix} \frac{1}{27720}
\end{align*}
\]
\[ \begin{align*}
  N_{k_0}^{(1)} &= \left( N_{3_0}^{(1)} \right)^t \begin{bmatrix}
  -180180 & -13156 & -137280 & 36608 & -42900 & 4004 \\
  13156 & 0 & 9152 & 2283 & -4004 & 286 \\
  137280 & 9152 & 0 & -73216 & -137280 & 9152 \\
  -36608 & -2288 & 73216 & 0 & -36608 & 2288 \\
  42900 & 4004 & 137280 & 36608 & 180180 & -13156 \\
  -4004 & -286 & 9152 & -2288 & 13156 & 0
  \end{bmatrix} \begin{bmatrix}
  1664 \\
  182 \\
  9152 \\
  1872 \\
  10816 \\
  104
  \end{bmatrix}
\end{align*} \]

\[ \begin{align*}
  N_{k_1}^{(1)} &= \left( N_{3_1}^{(1)} \right)^t \begin{bmatrix}
  -27196 & -2340 & -32032 & 8944 & -24856 & 1664 \\
  624 & 104 & 2288 & 728 & -3094 & 182 \\
  9152 & 0 & 73216 & 36608 & -128128 & 9152 \\
  4784 & 416 & 36608 & -1664 & -31824 & 1872 \\
  18044 & 2340 & 105268 & 27664 & 152984 & -10816 \\
  -910 & 104 & 6864 & -1560 & 13760 & 104
  \end{bmatrix} \begin{bmatrix}
  1664 \\
  182 \\
  9152 \\
  1872 \\
  10816 \\
  104
  \end{bmatrix}
\end{align*} \]

\[ \begin{align*}
  N_{k_2}^{(1)} &= \left( N_{3_2}^{(1)} \right)^t \begin{bmatrix}
  -5616 & -546 & -7072 & 2288 & -19916 & 1118 \\
  338 & -39 & 416 & 208 & 2678 & 143 \\
  7488 & -1248 & 73216 & -23296 & -111488 & 7904 \\
  1664 & 204 & 16640 & -1664 & 28704 & 1664 \\
  13104 & 1794 & 80288 & 21008 & 151404 & -9022 \\
  -494 & 65 & 4992 & -1040 & 14066 & 169
  \end{bmatrix} \begin{bmatrix}
  1664 \\
  182 \\
  9152 \\
  1872 \\
  10816 \\
  104
  \end{bmatrix}
\end{align*} \]

\[ \begin{align*}
  N_{k_3}^{(1)} &= \left( N_{3_3}^{(1)} \right)^t \begin{bmatrix}
  -1662 & -186 & -1248 & 400 & -17682 & 932 \\
  147 & 18 & 0 & 32 & 2427 & 125 \\
  8736 & -1248 & 59904 & -16640 & -96096 & 6656 \\
  1264 & 176 & 6656 & -1536 & -25984 & 1488 \\
  10398 & 1454 & 61152 & 16240 & 113778 & -7588 \\
  -329 & 44 & 3536 & -696 & 14161 & -213
  \end{bmatrix} \begin{bmatrix}
  1664 \\
  182 \\
  9152 \\
  1872 \\
  10816 \\
  104
  \end{bmatrix}
\end{align*} \]

\[ \begin{align*}
  N_{k_4}^{(1)} &= \left( N_{3_4}^{(1)} \right)^t \begin{bmatrix}
  -798 & -102 & -2248 & -208 & -16136 & 830 \\
  90 & 12 & 32 & 32 & 2230 & 115 \\
  7584 & -1056 & 46592 & -12544 & -83104 & 5600 \\
  1136 & -160 & 1792 & -1408 & -23536 & 1328 \\
  8382 & 1158 & 46816 & 12752 & 99242 & -6430 \\
  -218 & -29 & -2464 & -464 & 14122 & -242
  \end{bmatrix} \begin{bmatrix}
  1664 \\
  182 \\
  9152 \\
  1872 \\
  10816 \\
  104
  \end{bmatrix}
\end{align*} \]
\[
M_{S_{X_0}}^{(0,0)} = M_{S_{X_1}}^{(0,0)} = \begin{bmatrix} 147 & 21 & 63 & -14 \\ 63 & 14 & 147 & -21 \end{bmatrix} \cdot \frac{1}{420}
\]

\[
M_{S_{Y_0}}^{(0,0)} = M_{S_{X_2}}^{(0,0)} = \begin{bmatrix} 13 & 3 & 22 & -4 \\ 15 & 4 & 90 & -10 \end{bmatrix} \cdot \frac{1}{420}
\]

\[
M_{S_{Y_0}}^{(0)} = M_{S_{X_0}}^{(0)} = \begin{bmatrix} 20 & 10 \\ 10 & 20 \end{bmatrix} \cdot \frac{1}{60},
\]

\[
M_{S_{Y_1}}^{(0)} = M_{S_{X_1}}^{(0)} = \begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} \cdot \frac{1}{60}
\]

\[
M_{S_{Y_2}}^{(0)} = M_{S_{X_2}}^{(0)} = \begin{bmatrix} 2 & 3 \\ 3 & 12 \end{bmatrix} \cdot \frac{1}{60},
\]

\[
M_{S_{Y_0}}^{(1)} = M_{S_{X_0}}^{(1)} = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \cdot \frac{1}{6},
\]

\[
M_{S_{Y_1}}^{(1)} = M_{S_{X_1}}^{(1)} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \cdot \frac{1}{6}
\]

\[
N_{S_0}^{(0)} = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} \cdot \frac{1}{6},
\]

\[
N_{S_1}^{(0)} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \cdot \frac{1}{6},
\]

\[
N_{S_2}^{(0)} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \cdot \frac{1}{6},
\]

\[
N_{S_0}^{(1)} = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix} \cdot \frac{1}{3}
\]

\[
K_{S_0}^{(0)} = \begin{bmatrix} 148 & -189 & 54 & -13 \\ -189 & 432 & -297 & 54 \\ 54 & -297 & 432 & -189 \\ -13 & 54 & -189 & 148 \end{bmatrix} \cdot \frac{1}{450}
\]
\[
\begin{align*}
M_{\Theta}^{(0,0)} &= M_{\Theta}^{(0,0)} = \mu(0) \\
M_{\Theta}^{(0,1)} &= M_{\Theta}^{(1,0)} \cdot T \\
M_{\Theta}^{(1,1)} &= M_{\Theta}^{(1,1)} = \mu(1) \\
M_{\Theta}^{(2,1)} &= M_{\Theta}^{(2,1)} = \frac{1}{1680} \\
M_{\Theta}^{(0,0)} &= M_{\Theta}^{(0,0)} = \frac{1}{60} \\
M_{\Theta}^{(1,0)} &= M_{\Theta}^{(1,0)} = \frac{1}{60} \\
M_{\Theta}^{(2,0)} &= M_{\Theta}^{(2,0)} = \frac{1}{1680} \\
M_{\Theta}^{(0)} &= \frac{2}{16} \\
M_{\Theta}^{(1)} &= \frac{4}{2} \\
M_{\Theta}^{(2)} &= \frac{128}{128}
\end{align*}
\]
\[
\begin{align*}
M^{(0,1)}_{\Theta Y} &= M^{(0,1)}_{\Theta X} = \begin{bmatrix} 79 & 5 & 112 & -8 & 19 & -2 \\ 19 & 2 & 112 & 8 & 79 & -5 \end{bmatrix} \\
M^{(1,1)}_{\Theta Y} &= M^{(1,1)}_{\Theta X} = \begin{bmatrix} 57 & 3 & 16 & -8 & -3 & 0 \\ 44 & 4 & 192 & 0 & 44 & -14 \\ -3 & 0 & 16 & 8 & 57 & -3 \end{bmatrix} \\
K^{(0,0)}_{T Y_1} &= \begin{bmatrix} 0 & 60 & 0 & -60 \\ 0 & -60 & 0 & 60 \end{bmatrix} \cdot \frac{1}{60} = K^{(0,0)}_{T X_1} \\
K^{(0,0)}_{T Y_2} &= \begin{bmatrix} 60 & 0 & -60 & 0 \\ -60 & 0 & 60 & 0 \end{bmatrix} \cdot \frac{1}{60} = K^{(0,0)}_{T X_2} \\
K^{(0,0)}_{T Y_3} &= \begin{bmatrix} -30 & -5 & -30 & 5 \\ 30 & 5 & 30 & -5 \end{bmatrix} \cdot \frac{1}{60} = K^{(0,0)}_{T X_3} \\
K^{(1,0)}_{T Y_1} &= \begin{bmatrix} 240 & 180 & -240 & 60 \\ -480 & -240 & 480 & -240 \\ 240 & 60 & -240 & 180 \end{bmatrix} \cdot \frac{1}{60} = K^{(1,0)}_{T X_1} \\
K^{(1,0)}_{T Y_2} &= \begin{bmatrix} 120 & -10 & 120 & 10 \\ -120 & 20 & 120 & -20 \\ 0 & -10 & 0 & 10 \end{bmatrix} \cdot \frac{1}{60} = K^{(1,0)}_{T X_2} \\
K^{(1,0)}_{T Y_3} &= \begin{bmatrix} -72 & -11 & -48 & 9 \\ 84 & 12 & 36 & -8 \\ -12 & -1 & 12 & -1 \end{bmatrix} \cdot \frac{1}{60} = K^{(1,0)}_{T X_3}
\end{align*}
\]
\[ \mathbf{D}^{(0)} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \mathbf{D}_{X}^{(0)} = \mathbf{D}_{Y}^{(0)} \]

\[ \mathbf{D}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix} = \mathbf{D}_{X}^{(1)} = \mathbf{D}_{Y}^{(1)} \]

where internal node is taken at \( \mathcal{K} = \frac{1}{2} \)

\[ \mathbf{D}^{(2)} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 18 & -45 & 36 & -9 \\ -9 & 27 & -27 & 9 \end{bmatrix} = \mathbf{D}_{X}^{(2)} = \mathbf{D}_{Y}^{(2)} \]

where internal nodes are taken at \( \mathcal{K} = \frac{1}{2}, \frac{2}{3} \)

\[ \mathbf{D}_{X}^{(0)} = \mathbf{D}_{X}^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \]

\[ \mathbf{D}_{X}^{(1)} = \mathbf{D}_{X}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 66 & 13 & -32 & 32 & -54 & 5 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 24 & 4 & 0 & 16 & -24 & 4 \end{bmatrix} \]

where internal node is taken at \( \mathcal{K} = \frac{1}{2} \)
APPENDIX 6

NOTE ON THE NUMERICAL EIGENVALUE PROBLEM

The determination of the beam eigenvalues and eigenvectors by the Finite Element Method requires the solution of the eigenvalue problem

$$ B \mathbf{v} = \lambda A \mathbf{v} $$  \hspace{1cm} (A6.1)

where the matrices $A, B$ are symmetric. This equation can be reduced to the simpler form

$$ C \mathbf{v} = \lambda E \mathbf{v} $$  \hspace{1cm} (A6.2)

where the matrix $C$ is symmetric, and $E$ is a unit matrix, $\mathbf{v}$ is a new set of eigenvectors. For the solution of eigenvalue problems of the foregoing type there exists a number of established direct methods (e.g., see Ref. (39)). Among these, the Jacobi method appears to be the simplest one. A discussion of the Jacobi method and its Algol procedure is presented in Ref. (60).

The most commonly used method of reducing the equation (A6.1) to (A6.2) is as follows:

By making use of the positive definiteness property, the matrix $A$ is decomposed (61) into the form

$$ A = NN^T $$

where $N$ is a lower triangular matrix. Thus, the equation (A6.1) becomes

$$ B \mathbf{v} = \lambda NN^T \mathbf{v} $$

Putting $\mathbf{v} = N^T \mathbf{v}$ the foregoing equation reduces to the form (A6.2) with

$$ C = N^{-1} A (N^T)^{-1} $$

which is symmetric since $N$ is triangular.

In the present thesis a different method, which produces the same effect with less number of operations, is used. It is based on finding a lower (or, upper) triangular matrix, say $D$, so that $DA$ is upper (lower) triangular. Then $DAD^T$ will be diagonal. Let $F = DAD^T$. Therefore

$$ A = D^{-1} F (D^T)^{-1} $$

Substituting this into the equation (A6.1)

$$ B \mathbf{v} = \lambda D^{-1} F (D^T)^{-1} \mathbf{v} $$
Therefore

\[ F^{-\frac{1}{2}} D B D^T F^{-\frac{1}{2}} \cdot F^\frac{1}{2} (D^T)^{-1} V = \lambda F^\frac{1}{2} (D^T)^{-1} V \]

which will reduce to the form \((A,G,B)\) on putting \( V = F^\frac{1}{2} (D^T)^{-1} V \). The matrices \(DA, D,\) and \(DB\) can be obtained by performing Gaussian elimination on Matrix \(A\), while simultaneously performing the same operations on a unit matrix and the matrix \(B\), respectively.

The Algol procedure "eigensolve", written by the author, which performs the foregoing operations is shown in the next page. The procedure parameters are

- \(n\) the size of the matrix to be solved
- \(A\) the inertia matrix
- \(B\) the stiffness matrix

The eigenvalues and eigenvectors will be put in \(D\) and \(A\), respectively. "rot" will contain the number of Jacobi rotations required to achieve an accuracy which the computer finds satisfactory. (see Ref. (60)).
procedure eigensolve(n, A, D, B, rot);
value n;
integer n, rot;
array A, D, B;
begin integer i, j, k;
real r;
array C[1:n, 1:n];
for i:=1 step 1 until n do
for j:=1 step 1 until n do
C[i, j] := if i=j then 1.0 else 0.0;
for k:=1 step 1 until n-1 do
for i:=k+1 step 1 until n do
if A[i, k] # 0 then
begin r:=A[i, k]/A[k, k];
for j:=1 step 1 until n do
B[i, j] := B[i, j] - r*B[k, j];
C[i, j] := C[i, j] - r*C[k, j];
end;
for i:=1 step 1 until n-1 do
for j:=i+1 step 1 until n do
begin C[i, j] := C[j, i];
C[j, i] := 0.0;
end;
for i:=1 step 1 until n do
D[i] := 1.0/sqrt(A[i, i]*C[i, i]);
for i:=1 step 1 until n do
for j:=1 step 1 until n do
begin r:=0;
for k:=1 step 1 until n do
r := r + B[i, k]*C[k, j];
A[i, j] := r*D[i]*D[j];
end;
for i:=1 step 1 until n do
for j:=1 step 1 until n do
C[i, j] := C[i, j]*D[j];
jacobi(n, true, A, D, B, rot);
for i:=1 step 1 until n do
for j:=1 step 1 until n do
begin r:=0;
for k:=1 step 1 until n do
r := r + C[i, k]*B[k, j];
A[i, j] := r;
end
end eigensolve;
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12. See Ref. (11)
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