VIBRATION AND STABILITY ANALYSES OF
UNSTIFFENED AND STIFFENED COMPOSITE PLATES

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ABSTRACT

Vibration and stability studies have been undertaken on glass fibre reinforced polymer composite unstiffened and stiffened plates to optimise their dynamic properties. Boundary conditions, stiffeners and orthotropy of the material add to the complexity of a mathematical solution and to overcome this problem experimental and analytical studies were undertaken. The former method was carried out by impact hammer and an FFT digital signal analyser and the latter method was undertaken using finite element computer software.

The current research concentrated upon the procedures and possible techniques available to optimise the dynamic properties of the plate without introducing weight penalty with the object of achieving an efficient structural performance coupled with an economic design.

It has been shown that most of the increase in frequency and critical buckling load was directly related to the increase in stiffness of the stiffener and its position on the plate structure. The mode shapes have provided information regarding the most advantageous position for the setting of the stiffeners; they must be positioned away from nodal lines. The effect of the stiffener was significant for the fully clamped and clamped/free plates where only bending modes of vibration are present. However, for the completely free plates, where both bending and torsional modes of vibration could occur, the effect that the stiffeners have on the torsional modes was minimal. To locate precisely the position of the stiffener may be difficult when the plates are subjected to in-plane compressive loads, because higher order mode shapes may interchange.

The mass-saving advantage which has been obtained in this research has shown that the stiffened plates with top-hat stiffeners were seen to have higher natural frequencies, within a specific vibration mode, compared to stiffened plates with rectangular stiffeners (blade).
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IN MEMORY OF MY FATHER.
TO MY MOTHER.
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LIST OF SYMBOLS

a - width of plate
A_{11} - amplitude of plate deflection
A_s - cross-sectional area of stiffener
d - constant depending on the material density, beam geometrical dimensions and boundary conditions
b - length of plate
c_{ij} - coefficients
d - loss factor
d - depth of stiffener
D - flexural rigidity of isotropic plate
D_x, D_y, D_{xy} - bending rigidities of orthotropic plate
E - Young's modulus of isotropic plate
E_x, E_y - elastic modulue of orthotropic plate
f_{11} - fundamental natural frequency
f_{np} - p-th natural frequency
\tilde{f}_p - ratio of the p-th frequency (= f/f_{np})
G_{xy} - shear modulus
\nu_{sr} - complex function
I_s - second moment of area of stiffener
K - stiffness
k - buckling coefficient
k_p - p-th normal stiffness
k_p - p-th equivalent stiffness
l - length of the strain gauge
m, n - number of half-waves in X-X and Y-Y directions respectively
M - bending moment
M_c - mass of composite materials
M_f - mass of glass fibre
M_p - p-th normal mass
N - number of modes
P - in-plane compressive load
\[ P_{cr01} \] - critical buckling load of unstiffened plate
\[ P_{cr1} \] - critical buckling load of specific plate
\[ q \] - distributed lateral load per unit area
\[ Q \] - quality factor
\[ R \] - specific resistance of the strain gauge wire
\[ s \] - cross-sectional area of the strain gauge
\[ t \] - time
\[ t_o \] - thickness of unstiffened plate
\[ t_p \] - thickness of stiffened plate
\[ t_s \] - width of stiffener
\[ T \] - total potential energy
\[ u, v \] - displacement in \( X-X \) and \( Y-Y \) directions respectively
\[ U \] - strain energy
\[ V_f \] - fibre volume fraction
\[ w \] - deflection of plate function of \( x, y \) and \( t \)
\[ W \] - deflection of plate function of \( x \) and \( y \)
\[ W_n \] - amplitude of motion after \( n \) cycles
\[ W_{nr} \] - displacement responses at position \( "s" \) resulting from the excitation force at position \( "r" \)
\[ x, y, z \] - Cartesian coordinates
\[ X, Y, Z \] - coordinate axes of Cartesian coordinate system
\[ \alpha, \mu \] - parameters associated with the thickness of the stiffened plate
\[ \alpha_1 \] - phase angle of the \( i \)-th nodal point relative to the reference nodal point
\[ \beta \] - stiffening ratio \( (=A_s/bt_p) \)
\[ \beta_n \] - parameter accounts for the effect of fibres type
\[ \chi \] - stiffening ratio \( (=t_s/d) \)
\[ \delta \] - phase shift between stress and strain
\[ \Delta \] - uniform compressive load ratio \( (=P/P_{cr1}) \)
\[ \Delta_{cr1} \] - in-plane compressive load ratio \( (=P_{cr1}/P_{cr01}) \)
\[ \varepsilon \] - normal strain
\[ \phi \] - aspect ratio \( (=a/b) \)
\[ \phi_{i} \] - deflected mode shape in \( X \)-direction
\[ \phi_{sp}, \phi_{rp} \] - values of the \( p \)-th displacement mode at the
points "s" and "r", respectively

\( \gamma \) - shear stress

\( \Gamma_{mn} \) - frequency ratio \( (=f_{mn}/f_{11}) \)

\( \Gamma(t) \) - harmonic time function

\( \eta \) - lateral dimensionless length \( (=y/b) \)

\( \psi \) - specific damping capacity

\( \psi_j \) - plate deflected mode shape in \( \gamma \)-direction

\( \lambda_{mn} \) - dimensionless frequency parameter

\( \lambda^*_{mn} \) - longitudinal strain

\( \nu \) - Poisson's ratio

\( \theta \) - angle forming top-hat stiffener

\( \rho \) - plate density per unit area

\( \rho_c \) - density of the composite material

\( \rho_f \) - density of the glass fibre

\( \sigma_{cr} \) - elastic critical stress

\( \tau \) - shear stress

\( \tau \) - damping logarithmic decrement

\( \omega_{mn} \) - circular frequency

\( \xi \) - damping factor

\( \xi_p \) - ratio of the \( p \)-th modal damping

\( \zeta \) - lateral dimensionless length \( (=x/a) \)

\( \nu \) - Laplacian operator
1. INTRODUCTORY CONSIDERATIONS

1.1 USE OF PLATES IN ENGINEERING FIELDS

Structures consisting of rectangular plates have found an extensive use in many fields of engineering, from architectural structures, ship superstructures, bridge decks, hydraulic structures, pavements, machine parts, dams and reservoirs, to the high technology of the modern aerospace industries. These plates do not act in isolation but often as a part (structural component) of a more complex assembly where the boundary conditions must be specified at the plate edges, then the plate is investigated separately. Fig.1-1 shows the application of some plates, as a structural component, in different fields of engineering including those operating on water, in space and on earth. These plates must be built sufficiently strong to operate under the range of frequency interest, to be able to withstand severe conditions of excitation and vibration to which they will be exposed, and to survive their maximum resistance fatigue.

In order to ensure these requirements, a determination of the dynamic properties and a description of what the dynamic loading (excitation force) does to a plate must be obtained, where both stress prediction and fatigue life prediction are considered.
The fatigue life of a plate structure is dependent on many factors, such as [1]:

(a) material specification,
(b) load condition,
(c) structural geometry,
(d) excitation load frequency,
(e) environment conditions,
(f) plate history.

1.2 SOURCE OF EXCITATION AND ITS INFLUENCE

Generally, the plates under investigation are subjected to dynamic loading, where the source of excitation could be for example:

(a) wave impacts,
(b) gas flow under pressure,
(c) wind velocities,
(d) acoustic waves,
(e) seismic disturbances,
(f) traffic wheel load,
(g) engine mechanical vibration.

Such excitations can be classified as either steady state or transient. Steady state excitation is caused by continually running machines, such as engines or generators; these could be generated in the structure or externally to it. Transient excitation is caused by a short duration disturbance, such as
train or lorry passing over a road or bridge.

When the plate vibrates under its resonant frequency, large amplitudes will develop and, consequently, the plate can suffer a significant over-stress. The accumulation in maximum stress levels that is variable from cycle to cycle leads to failure through fatigue life [2]. The service life is defined as a function of the number of cycles producing vibration failure at a given stress level [3]. Such failure or damage is believed to be directly related to a reduction in stiffness and increase in damping. This reduction in stiffness will decrease the natural frequencies of the plate and shift them into the region where resonance could occur resulting in high stress levels. Therefore, fatigue damage in the form of cracks (usually invisible to the naked eye) could appear in a non-uniform distribution through the surface of the vibrating plate. The location of vibration cracks can differ from one mode to another depending on the high stress regions. The presence of vibration cracks and their location which may occur along the service life of a plate can be monitored by measuring its natural frequencies [4]. Fig.1-2 shows a sequence of failure associated with resonance conditions leading to a fatigue as a result of repeated stress.

Fig.1-3 illustrates a sketch showing the main structural details of a rectangular plate. Failure and damage may occur at the following points:
Fatigue cracks starting at the item (a) can spread rapidly throughout the plate.

1.3 EXCITATION TECHNIQUE AND RESPONSE SPECTRA

Studying the vibration data analysis of the input and the output in the time history (see Fig.1-4a and 1-4b) does not give much information for engineering purposes. A transformation to the frequency domain is necessary and gives the results of spectral analysis of the original time history in the form of an amplitude-frequency spectrum. The excitation spectrum represented by Fast Fourier Transform (FFT), shown in Fig.1-4c, is continuous and composed of large band of frequencies with maximum amplitude at zero hertz and decaying with increasing frequency. However, for the response spectrum (FFT) shown in Fig.1-4d, the frequencies are discrete and known from the sharp peaks where the energy is concentrated around them. The resonance occurs when a part of the bandwidth of frequencies of the excitation spectrum coincides with one or more of the natural frequencies of the plate.

1.4 RESONANCE AND ITS EFFECT

It is necessary to analyse the vibration of structures in
order to predict the natural frequencies and the response to
the expected excitation. This is because if the frequency
of the disturbing force coincides exactly with the frequency
of the free vibration of the structural system, resonance
occurs with high vibration amplitudes.

A very serious phenomenon caused by resonance, several
hundred cycles in some cases, frequently results in
dangerously large fatigue stress. Thus, in many engineering
situations it is not only necessary to consider the fatigue
strength under known static load, but it is also important to
determine the ability of such parts to withstand known
exciting forces at resonance.

The plate structure needs only to be analysed over the
expected frequency range of excitation where resonance should
be avoided. It is not usually necessary to predict all the
natural frequencies of the plate. This is because many of
these frequencies will not be excited and in any case they
may give small resonance amplitudes. The plate becomes
increasingly stiff for deformation in the higher mode shapes.
Consequently, the fundamental frequency is often of greater
interest than its higher natural frequencies because its
forced response in many cases is the largest.

1.5 CONTROL OF VIBRATION PROBLEMS

The objective considered to ensure a good structural design
of plates is usually that the frequencies of the free vibration system do not coincide with forcing frequencies; this can be achieved by removing the natural frequencies from the frequency region where excitation energy is available (see Figs.1-4c and 1-4d).

There is usually little that can be done to change the nature of the excitation forces. However, in the case of structures consisting of a series of plates, in which only a few natural resonant frequencies are present, these latter can be eliminated or they can be reduced. Alternatively, they can be removed from the range of the exciting frequencies by controlling the source of the excitation and its effect on the plate system. In addition, vibration of plates may be reduced by modifying its structure in accordance with one or more of the following methods:

(a) If the forcing function is of the discrete spectrum type, the natural frequencies of the plate may be shifted so that coincidence with forcing frequencies is avoided.

(b) If the forcing function is either of a discrete or a continuous spectrum type, the addition of damping will reduce the response at the resonant frequencies.

The first method may involve stiffening or making the plate structure lighter. For example, stiffening the system results in an increase in the natural frequencies of the structure. An investigation of the nodal patterns in the
plate often aids in determining the best location of stiffeners. These should be located to span the anti-nodal regions which results in maximum increase in the natural frequency.

The second method involves the introduction of damping. This is the only effective control technique where the forcing function has a relatively flat continuous spectrum. Methods of increasing damping include:

(a) replacement of the original material by another having higher internal damping,

(b) addition of vibration-damping material,

(c) alteration of the boundary conditions, if this does not affect the original design purpose,

(d) introduction of rubber joints between the surface of the plate and the clamped edges.

In this research, some methods will be undertaken in order to reduce the vibration effects on the plates. These methods are grouped into the following categories:

1.5.1 THE EFFECT OF NODAL REGIONS

In a structure which is being vibrated, points at which the displacement in a natural mode is relatively low, will occur at nodal regions. When the source of excitation is applied near a nodal point, its application minimises the vibration
response of the structure and such excitation force will then be incapable of giving more energy to the system. Hence, the plate will not suffer significant damage associated with fatigue failure phenomenon for that particular mode.

1.5.2 THE EFFECT OF THE BOUNDARY CONDITIONS

There exist several possible combinations of boundary conditions. The edges may have the classical boundaries such as clamped, simply supported and free edge conditions. The dynamic characteristics of the plate depend upon the boundary conditions, and for a complete understanding of structural dynamics it is necessary to investigate the influence of some boundary effects on the natural frequencies and mode shapes of plates. Consequently, three sets of boundary conditions will be studied; these are:

- (a) fully clamped (C-C-C-C),
- (b) completely free (F-F-F-F),
- (c) a combination of free and clamped (F-C-F-C).

1.5.3 ROLE OF COMPOSITES IN DYNAMICS

The general class of materials which are most suitable for dissipating the vibration in structures are polymer composites. These are a combination of matrix/fibre systems, which have characteristics not available within conventional
materials. Damping may be increased by the use of a high dissipative resin. However, highly dissipative resins will have a low stiffness.

This material is used structurally in buildings, and some research is going into the design of GRP bridges [5], however the aerospace industry continues to lead in the use of composites for high performance applications. In addition, such composite materials are also used in other engineering structures (submarines, high speed boats and hydrofoils, cranes and artificial limbs) for a variety of reasons. These are given in reference [6] as follows:

(a) composites can offer some economy in material, either from a cost or a weight saving consideration, in order to achieve a required natural frequency. This saving can be obtained by orientating the fibre reinforcements in the plane where they will be most effective.

(b) The damping value is variable from one component to another; this is not the case for conventional materials.

(c) Composites can often be molded with a variable thickness at no extra cost. This offers an additional freedom to economise in material.

Table 1-1 compares the mechanical properties of some composites with those of some conventional materials [7] and [8]. On a direct comparison of strength and stiffness there is little to be gained in using composites. However, when
densities are taken into consideration, the advantage of certain composites is mainly based on the achievement of strength/weight and stiffness/weight ratios which show a considerable importance in maximizing the natural frequencies with minimum weight for a specific dynamic stability requirement [9].

During vibration the constrained damping layers are subjected to shear effects which cause vibrational energy to be converted into heat and, hence, must dissipate through the material. The composite materials used must have the ability to withstand such energy without any undesirable effects. To satisfy these requirements, either glass reinforced polyester (GRP) or carbon fibre reinforced plastics (CFRP) can be used in the form of material composite plates or beams. However, it should be stressed that the source of high stiffness is the fibres, and that the damping is derived from the matrix. While the stiffness and damping are two quantities varying inversely, consideration must be given to the choice of the fibre volume fraction that offers a suitable stiffness-to-weight ratio.

In this investigation two categories of GRP materials will be tested. In the first category the fibres are randomly orientated in the matrix and defined as (R,R), while in the second category the fibres are orthogonally arranged and defined as (0°,90°).
1.6 OBJECTIVES OF THE RESEARCH

The aims of the investigations are as follows:

(a) to detect weaknesses of possible areas of material that would cause failure of the plate when used in actual service,

(b) to give the possible solutions to the vibration problems which will be achieved by investigating the natural frequencies, mode shapes and damping factor values of free vibration for a wide family of isotropic and orthotropic unstiffened and stiffened rectangular plates, made from Perspex and GRP composite materials,

(c) to find the corresponding optimal and economical solutions,

(d) to carry out a numerical analysis in order to obtain the eigenvalues in the form of natural frequencies and eigenvectors in the form of mode shapes, where exact mathematical solutions do not exist because of the complexity of the boundary conditions, difficulty of stiffeners and orthotropy of the material,

(e) to conduct several tests on unstiffened and stiffened plates in order to verify the accuracy of the numerical analysis.

To solve such cases, numerical solutions using the finite element method (FEM) and experimental investigations using a fast fourier transform (FFT) digital signal analyzer are the only approaches that can be employed.
The analyses presented in this thesis will first be applied to the isotropic unstiffened and stiffened plates made from Perspex and the numerical results will be compared with available closed results [10] (for the case of unstiffened plates only) to determine the accuracy of the computer program used. The present analysis will then be used to study the effect of the fibre orientations (random and bidirectional), thickness and mechanical properties on the dynamic response of unstiffened and stiffened GRP plates subjected to various boundary conditions. The Perspex plates are also utilised as a reference for the composite plates in accordance with their constant thicknesses and their correct dynamic mechanical properties; these were obtained from results given by the National Physical Laboratory (NPL) reference [11].
CHAPTER TWO

2. LITERATURE REVIEW

A discussion of the basic vibration methods and the theory relevant to structural vibration analyses were introduced by Thompson [12] and Beards [13]. They have described how structural parameters can be changed to achieve a desired dynamic performance, and the mechanisms and control of structural damping.

2.1 BACKGROUND OF RECTANGULAR PLATES

There have been a very large number of studies of eigenvalue and eigenvector problems of rectangular plates which have attracted the interest of researchers over many years. Leissa [14] and Szilard [15] have given an excellent review of the results obtained by analytical methods for free vibration of plates. However, it has been shown that the complexity of the modal analysis depends upon the boundary conditions. For the case of simply supported edges an exact solution can be readily obtained, but for the case where two opposite simply supported edges do not exist, the problem to be analyzed becomes more complex. Gorman [10] has analyzed theoretically the more difficult boundary conditions associated with isotropic rectangular plate as a free vibration problem, in which the plates had combinations of free, clamped and simply supported boundary conditions; these
were solved using the method of superposition and superimposing auxiliary cases for which exact solutions are available, and constraining them to satisfy the boundary conditions of interest. More importantly, he classifies the modal patterns into doubly anti-symmetric, doubly symmetric and symmetric/anti-symmetric groups and graphically presents the modal line patterns for the first six modes in each group. In addition, he presented the theoretical values of the dimensionless frequency parameter $\lambda_{mn}$ for all boundary condition combinations and different aspect ratios with Poisson's ratio equals to 0.3.

The dynamic behaviour of stiffened plates has been studied for the past few decades [16], but investigations of the stiffened plates with consideration of the eccentricity of the stiffeners have not been numerous [17], [18] and [19]. Troitsky [20] presented both theoretical development and pertinent solutions for stiffened plates including the problem of bending, stability and vibration. He presented an approximate analytical procedure for calculating the natural frequencies of stiffened plates using Rayleigh-Ritz method. The mode shape functions were assumed to be known and derived from Huber's non-homogeneous differential equation. This equation is valid only if the number of stiffeners is large (greater than three), that is, the distance between the stiffeners is very small as compared to the width of the plate, the stiffened plate then has two different flexural rigidities in the perpendicular directions and can be assumed
as an orthotropic and homogeneous flat plate with fictitious thickness.

The earlier models used to simplify the structure are the orthotropic plate [21] and [22] or the grillage system [23]. These models fail when the stiffeners are sparsely populated. Consideration of the plate and the stiffener separately and maintaining the compatibility of the two is a rational method.

The review of literature on the dynamics of rectangular plates revealed that, in most of the analyses, the plates with free boundary conditions had mostly been ignored. Considerable analyses have been undertaken on the completely free edge cases which have practical importance [24], [25] and [26].

2.2 VIBRATION OF UNSTIFFENED AND STIFFENED PLATES

Major parts of aerospace vehicles [27], naval vessels [28] and components of automobiles were originally manufactured in aluminium are now using composite materials in the form of plates or beams to enable weight savings to be made and high specific strength to be achieved [29] and [30]. Different problems related to the dynamic loads and vibrations of these composite plates have been treated in a few journals; these are discussed in the following sections. In addition, some experimental investigations and numerical methods have been
applied to the study of composite unstiffened and stiffened plates under dynamic conditions.

2.2.1 THEORETICAL ANALYSIS

Finite element method (FEM) has been widely used as an analytical tool in the field of structural vibration analysis. However, none of the proposed solutions gives exact results and it will be seen that the more general the system, the greater the degree of approximation.

(a) Classical Plates:

Kirk [31] has considered a study of the effect of the type of stiffeners based on the Ritz method. The procedure was limited to a simply supported plate along all edges and reinforced by a single integral stiffener placed along one of its centre lines. Two forms of stiffener cross-sections were studied, namely, rectangular sections and T-sections. In this reference it has been shown that a maximum increase in frequency for the symmetric mode of about three times that of the unstiffened plate occurs, whereas the frequency of the anti-symmetric mode was seen to be lower than that of the unstiffened plate. No attempt has been made to find a stiffener cross-section which will give the maximum frequency increases for the symmetric mode and the anti-symmetric mode.

An approximate method for the calculation of natural frequencies and mode shapes for rectangular panels which had stiffeners in one direction and one pair of sides simply
supported was developed by Long [32]. The greatest accuracy was achieved when membrane stresses were small compared with those of bending stresses (i.e., wide stiffener spacing and flexible stiffeners). Any loss of accuracy inherent in the proposed method was offset by the large reduction in computation time. It was shown that the largest discrepancy in the fundamental frequency calculated by the stiffness method was 1.3% higher than the composite beam-plate method developed by the author.

Using the grillage method Balendra and Shanmugen [33] analyzed the free vibration of plated structures such as plates, stiffened plates and cellular structures reinforced by:

(a) a single longitudinal stiffener,

(b) a single transverse stiffener with a distribution of longitudinal stiffeners,

(c) a series of transverse stiffeners and a distribution of stiffeners in the longitudinal direction.

The plates were either simply supported along all edges or clamped along all sides. They showed that, for a square plate, the addition of one and three stiffeners nearly doubles the first frequency. But, for a rectangular plate, the most effective way of increasing the fundamental frequency can be achieved by any one of the following:

(a) a single longitudinal stiffener;
(b) a single transverse stiffener with a distribution of longitudinal stiffeners;

(c) a distribution of stiffeners in the longitudinal direction.

Downs [34] has presented a theory with a computer subroutine which enables the properties of a thin rectangular plate element for both non-conforming and conforming elements, to be modelled in order to predict the vibration frequencies and modal shapes of a square plate subjected to four sets of boundary conditions. He showed that the frequencies determined from a particular numerical analysis do not necessarily occur in a monotonic sequence. Therefore, the magnitude of individual frequency errors was associated with similarity between mode shape and geometry of the finite element model.

Computer programs for the dynamic analysis of plates and shells using closed form, finite strip and finite element methods, have been developed recently by Hinton [35], where theory, software and applications were presented in details for six FORTRAN programs.

(b) Composite Plates:
Vibration of simply supported laminated plate, with two cross-ply glass fibre layers and infinite in length was analyzed by Jones [36]. The existence of plane strain conditions allow exact solution of the field equations. Curves of variation of exact natural frequencies versus
wavelength for several cases were calculated. The first two modes were plotted for various wavelengths and compared to classical plate theory. Flexural vibration is satisfactorily approximated by classical theory. Extensional vibration was approximated marginally for very long wavelengths.

Adams and Bacon [37] have predicted theoretically the effect of fibre orientation and laminate geometry on the flexural and torsional damping modulus values. A series of CFRP plate materials with fibre at +θ (off-axis), 0° (angle-ply), 0°/90° (cross-ply) and a general plate were investigated. It was indicated that the damping was associated with shear and direct stresses. Shear was seen to be the predominant factor in a lamination geometry that has high damping. Whereas, transverse direct stresses can sometimes give high energy dissipation whilst longitudinal direct stresses result in minimal damping. Very good agreement was obtained between theoretical predictions and experimental results in almost all cases.

The effect of the heterogeneous shear deformation over the thickness of the plate on the dynamical behaviour of laminated plates was investigated by Sun and Whitney [38]. Three sets of governing equations were derived according to different assumptions on the local transverse shear deformation and the interface conditions. These equations were evaluated by comparing these solutions with those of the exact one for harmonic wave propagations. A special formulation for laminates with in-plane symmetry was
presented. It was found that the effect of local shear deformation is highly dependent on the transverse shear rigidities of the constituent layers.

Bert [39] has presented a rationale for determining the optimal design for a thin simply supported plate consisting of four unidirectional filament reinforced composite materials. Four popular filamentary composite materials, having epoxy matrices and fibres of boron (BFRP), carbon (CFRP), glass (GFRP) and organic fibre (OFRP), were analysed. The application of the specific equations to these plates indicated that the optimal fundamental frequency parameter is a strong function of the plate aspect ratio and composite material.

Chao [40] has presented a design guide in dynamic analysis for the eccentrically stiffened laminated plates. Three typical cases were investigated for the frequency behaviour (natural frequencies and mode shapes) as related to the stiffener depth or eccentricity and spacing between stiffeners for a specially orthotropic, antisymmetric cross ply and multiple isotropic lay-ups. The results obtained were consistent with the classical lamination theory in which the laminate is unstiffened.

Mathematical predictions of the dynamic properties of carbon glass fibre sandwich hybrid laminated composites have been developed by Ni et al. [41]. The energy method was used for predicting the dynamic properties of hybrid beams, while a
finite element analysis technique was used for predicting the mode shape, resonant frequency and damping of the hybrid plates. It has been shown that by adding a small amount of CFRP to the surface of GRP, can lead to a higher flexural modulus and lower density than the original GRP. The theoretical analysis indicated that the effect of core material on the modulus and damping in flexure is generally not large. Therefore, it allows some freedom in choosing the orientation of GRP core and even different core materials. It was shown that, choice of the optimum proportions depends on the ratio of the cost of carbon to glass composite. At current UK prices, a composite containing about 60% GRP gives the best specific stiffness per unit cost.

Analysis of vibration and damping of multilayered cylindrical shells consisting of an arbitrary number of orthotropic material layers was developed by Alam and Asnani [42] and [43]. This analysis has been applied by the same investigators [44] to study the damping for axi-symmetric vibrations of fibre reinforced composite laminated shells. The analysis considers bending, extension and shear deformations in each of the layers and also included rotary, longitudinal translatory and transverse inertia.

An orthotropic plate in which finite elements in the shape of parallelograms were used by Malhotra et al. [45] to study the effect of fibre orientation and boundary conditions on the frequency of four rhombic plates made of glass/epoxy,
kevlar/epoxy, boron/epoxy and graphite/epoxy. The properties of the four plates were based on a 70% fibre volume fraction in which eight layered symmetric lay-up laminae \((0, -\theta, 0, -\theta/0, 0, -\theta, 0, 0)\) were used. Five boundary conditions (CCFC, CCCC, CSSC, CCSC, SSFS) and four skew angles \((\pi/12, \pi/6, \pi/4, \pi/3)\) were considered. It was observed that:

(a) for a given boundary condition and skew angle, the variation of the fundamental frequency parameter, \(\sqrt{\lambda_1}\), with fibre orientation, \(\theta\), follows the same pattern for all materials considered,

(b) for a given boundary condition and skew angle, the nature of the variable \(\sqrt{\lambda_1}\) with \(\theta\) is dependent on the skew angle of the plate,

(c) for a given boundary condition, the value of \(\theta\) at which maximum \(\sqrt{\lambda_1}\) occurs is different for different skew angles,

(d) in general, the value of the frequency parameter increases with increasing skew angles for all boundary conditions.

A Rayleigh-Ritz formulation was used by the previous investigators [46] to study the effect of fibre orientation and boundary conditions on the fundamental frequency of a square graphite/epoxy plate of variable thickness (linear variation) along the x-direction and the y-direction. The study was conducted using eight boundary conditions (e.g., CCCC, CCSC, CCFC, CCSS, CCFS, CCFF, SSFS and FCFF). It was observed that:
(a) for a linear variation in thickness along the $x$-direction or the $y$-direction, $\sqrt{\lambda}$ increases as the degree of thickness of the taper, $\eta$, is increased from 0.0 to 1.0 for almost all boundary conditions and fibre orientation,

(b) the fibre angle, at which $\sqrt{\lambda}$ is maximum for a given boundary condition, is also dependent on $\eta$,

(c) it would be possible to choose the fibre angle that gives maximum $\sqrt{\lambda}$ for a given boundary condition and degree of taper,

(d) patterns of variation of $\sqrt{\lambda}$ with fibre angle is dependent on $\eta$ for most of the boundary conditions.

A Rayleigh-Ritz method which includes the transverse shear effect in calculating the fundamental natural frequencies of thin orthotropic shells with simply supported boundaries has been developed by Lee [47] using the energy method. It has been shown that the transverse shear effect should be included in calculating the natural frequencies of advanced composite materials because of their high ratio of elastic modulus to shear modulus.

Exact solutions for the dynamic response of simply supported symmetrically laminated cross-ply rectangular plates were presented by Khdeir and Reddy [48] using a classical plate theory and a higher order shear deformation theory. The two procedures were used to investigate the dynamic response of three layers cross-ply $(0^\circ/90^\circ/0^\circ)$ square $(a=b=5t)$ laminated plates under various loads. The numerical results were
compared with those obtained by Reissner-Mindlin shear deformation plate theory (FSDT) as well as the classical laminate plate theory (CPT). It was shown that the stresses obtained by the classical plate theory were significantly different from the first order and higher order theories. The stresses predicted by the classical plate theory are closer to the first order theory than to those of the higher order theory, which is the closest to the elasticity solution.

2.2.2 EXPERIMENTAL INVESTIGATIONS

In the current experimental work, the stiffeners were attached to the plate by the use of epoxy resin which results in a continuous line attachment. However, in some structures consisting of plates, the connection of the plate-stiffener could be achieved by using point attachments such as rivets or continuous line attachment such as welds. For this subject matter, some experimental work has been undertaken by Fahy and Wee [49] to obtain an idea of the effect of different methods of plate-stiffener connection on acoustically induced strains in a series of mild steel plates. It has been found that the point attachments can be preferable to line attachments. The magnitude of the benefits gained from attaching the stiffener at discrete points is not explained by existing theoretical vibration analysis.
(a) Classical Plates:

Hazell [50] described an experimental study of flat square cantilevered plate by holographic interferometry. The disadvantages of this technique over the usual method of using transducers to measure the response were:

(a) it is not possible to measure the phase angle, but its sign can be measured,

(b) the dynamic range is not as great as with piezoelectric transducers,

(c) the technique is tedious and time consuming.

However, he has showed some definite advantages in using holographic interferometry; these are:

(a) the ability to observe the full field transfer function of the plate without resorting to mode plotting,

(b) mode development, mode transition and compound mode behaviour can all be observed directly in real time as the plate is sine swept,

(c) the overall effects of additive damping, adjustment in boundary conditions, mass changes, etc., can be immediately observed and recorded photographically.

Yi et al. [51] presented a modal analysis of vibrational plates by means of impulse excitation and strain gauges. It has been shown that this technique is convenient, reliable, does not cause additional stiffness and mass, and can obtain higher precision for the situation of multi-point measurement
in which the strain gauge is suitable, especially, for the small and light structural parts where accelerometers are inappropriate.

Hazell and Michell [52] have measured experimentally eigenvalues and eigenvectors of both square and rectangular clamped plates using digital spectrum analysis and holographic interferometry. It has been shown that:

(a) the experimental mode shapes for both the square and rectangular plates were in good agreement with the beam mode predictions given by Warburton [53] and Bolotin’s mode shapes as given in reference [54],

(b) the experimental eigenvalues for the square plate were in good agreement with the theoretical results given in reference [55],

(c) the experimental eigenvalues for the rectangular plate \( \phi = 2/3 \) agree with the limited theoretical predictions available in references [56] and [57].

(b) Composite Plates:
Vibration characteristics of cantilevered triangular plates 958.1mm x 1092mm (with straight angle and clamped along 1092mm) having a solid core of 2024-T3 aluminum with six boron/epoxy composite material layers bonded symmetrically on both sides were investigated experimentally and analytically by Clay and Cooper [58]. Each of the three plates had different ply orientation; these are:

(a) where the fibres were parallel to the clamped edge
(i.e. \( \theta=0^\circ \)),

(b) where the fibres were orientated at \(+45^\circ\) to the clamped edge,

(c) where the fibres were normal to the clamped edge (i.e. \( \theta=90^\circ \)).

Experimental results showed that the fundamental frequency of aluminium plate can be increased 77% by adjusting the fibre orientation with no change in the plate mass. Results also showed that the damping coefficients could be controlled with filament orientation and that the maximum damping occurred for modes having maximum straining of the matrix and minimum straining of the filaments. Regardless of the filament orientation, all reinforced plates had a greater damping component than that for the aluminium plates. The NASA structural analysis (NASTRAN) computer program (level 12) was used to calculate the natural frequencies and mode shapes for each of the above plates. Analytical results were in qualitative agreement with experimental ones for the mode shape variation with filament orientation. However, in some cases, qualitative frequency correlation was poor. To obtain better frequency results, a more sophisticated finite element mesh, including effects of interlaminar shear, matrix viscoelasticity and metal/epoxy interface properties was suggested.

White [59] has undertaken an experimental study which investigates the strain response in two types of plates to
acoustic excitation in the frequency range 80Hz to 800Hz at levels up to 155dB likely to be encountered by aircraft components in service. One composite plate was constructed of four layers (0°/90°/90°/0°) of commercially available pre-impregnated sheets of CFRP composed of DX-210 resin with HT-S carbon fibre reinforcement. The final thickness was 1mm and 60% fibre volume fraction. A metal plate used for comparison was 1.2mm thick and made of aluminium alloy BSSL71. Both plates measured 250mm square. The spectral density, probability distribution, peak and zero count properties of the induced strains were described. Comparison between the behaviour of the two types of plates were made and non-linear dynamic response of the CFRP plate was observed. The strain gauges used to measure the surface strains induced in the test structures were attached near the four clamped edges of the square plate, whereas the microphone used to measure the sound pressure level was mounted near the centre of the plate. Difficulties can occur in predicting some resonant frequency modes because of the presence of nodal lines. Alteration of the above locations (strain gauges and microphone) may give different results.

Cawley and Adams [60] have undertaken an experimental investigation into the natural modes of various free/free plates (square aluminium plate, square CFRP plate and trapezoidal CFRP plate). The near free/free condition was achieved by hanging the plates from flexible strips, excitation was provided by loudspeaker which was
non-contacting except via the acoustic medium. The response was detected by small piezoelectric strain gauges bonded to the plate. The mass of the strain gauges and the stiffness of the connecting wires were sufficiently small and, therefore, their influence was neglected. A comparison with 8-node, 40 d.o.f shell finite element analysis was made and the worst error between both techniques was 6.3%.

Crawley [61] has studied experimentally the effect of aspect ratio on the natural frequencies and mode shapes on graphite/epoxy plates of aspect ratio 1 and 2, measuring 76mm x 76mm and 152mm x 76mm, respectively. Additionally, a set of 152mm x 76mm symmetric sandwich plates consisting of graphite/epoxy plies bonded onto aluminium core were built and tested. Each of the three plates was composed of eight graphite/epoxy layers orientated in three different lay-up schemes that included: \((0^\circ/0^\circ/\pm 30^\circ)_s\), \((0^\circ/\pm 45^\circ/90^\circ)_s\) and \((\pm 45^\circ/\pm 45^\circ)_s\). Natural frequency and mode shape results were compared with those calculated by a finite element analysis and it was noticed that the dynamic flexural moduli appeared to be less than the static in-plane moduli, even after transverse shear effects have been taken into account. This difference was probably due to uncertainty in the in-plane material properties. The agreement between observed and calculated mode shapes was excellent.

A rational method for obtaining the necessary modulus and damping parameters of laminated plates and beams, using the
best theoretical and experimental results available was provided by Ni and Adams [62]. The specimens, which were made from glass fibre in DX-20 epoxy resin (GRP), were moulded using two methods, hot compression moulding and vacuum bag. It was shown that problems can arise for specimens manufactured by the vacuum bag technique because of the rough surface. It is, therefore, most important to be able to measure the thickness and fibre volume fraction of the specimens accurately in order to have available laminate parameters at the same fibre volume fraction as those which it is intended to use in practice.

Another experimental method for predicting the dynamic mechanical properties of laminated composite beams was developed by the previous investigators [63]. In order to make the experimental data more reliable a non-contacting optical transducer was used for measuring the central amplitude of specimens. These specimens were conditioned to minimise the effect of the environmental history such as internal stresses caused by machining and the relaxation of stresses induced during manufacture on the modulus and damping measurements. It was shown that such technique can be used to predict satisfactorily the dynamic modulus and damping of symmetric laminated composites in the form of a beam. This was justified by comparing the theoretical predictions and experimental results for four types of laminates, unidirectional, cross-plied, generally plied \((0^\circ,-60^\circ,60^\circ)\), and \((0^\circ,90^\circ,45^\circ,-45^\circ)\), both in GLASS/DX-210

Experimental and theoretical investigations into the effects of fibre length, matrix type and fibre orientation were undertaken by Willaway and White [64] with a view to optimizing the dynamic properties of laminated CFRP plates using a combination of short (continuous and conventional) highly dissipation resin matrices. It was shown that the loss factors of CFRP laminated plates may be significantly increased by using flexible highly dissipative resin matrices whilst maintaining high specific stiffness properties. A similar but lesser effect can be achieved using short fibres. The authors have indicated that it may be possible to optimise the dynamic properties of CFRP, with respect to high specific modulus and damping, by using short aligned fibres in a flexible, highly dissipative matrix; in practice it is impossible to achieve the required volume fraction. Consequently, the use of combination of short aligned/random and highly dissipative resin matrices in a CFRP structural configuration may provide the most suitable compromise.

2.3 VIBRATION-BUCKLING OF STIFFENED AND UNSTIFFENED PLATES WITH IN-PLANE FORCES

If the in-plane loads are compressive, the free vibration frequencies will be expected to decrease as the loads are increased. If the loads are increased sufficiently so that the lowest frequency approaches zero, the resulting loads
will have reached their critical values which corresponds to the onset of buckling. Thus, the buckling problem may be considered as a special case of the free vibration problem.

A number of researchers have attempted to solve the buckling problem for axially loaded rectangular plates in which all four sides simply supported. Allen and Bulson [65], Brush and Almroth [66] have undertaken considerable research effort into the buckling behaviour of stiffened and unstiffened plates with various boundary conditions, and subjected to a different compressive in-plane loads. The authors analysed these engineering plate components in order to withstand the initial elastic buckling loads and the ultimate or collapse strengths.

A study of the influence of constant axial compressive loads on natural frequencies of uniform single-span beams with different end conditions were presented by Bokaian [67]. His analytical results indicate that the variation of the normalized natural frequency with the normalized axial force is exactly the same for pinned/pinned, pinned/sliding and sliding/sliding end conditions (where pinned and sliding are defined as: ─── and ─── , respectively) and this variation can be expressed in a closed form solution. In addition, this solution describes roughly the above variation for sliding/free, clamped/clamped and clamped/sliding beams under the fundamental mode of vibration. Further, apart from a region close to buckling, the above variation is almost the
same for clamped/pinned, pinned/free and free/free end conditions. The author has claimed that the closed form expression can also be applied for any beam boundary conditions when such beam vibrates in a high mode. No investigations were undertaken to explain the points at which the intersections between the curves (normalized natural frequency versus normalized axial force) occur and no mode shapes were presented.

2.3.1 THEORETICAL ANALYSIS

(a) Classical Plates:
An approximate formula for natural frequencies and flexural vibration of isotropic plates, originally developed by Warburton [68] using characteristic beam functions in Rayleigh's method, was modified by the same investigator to enable it to be applied to orthotropic plates with an extension to include the effect of uniform, direct in-plane forces. The approach permitted the determination of the reasonably accurate natural frequencies and buckling loads for a given plate involving any combination of free, simply supported or clamped edges. The applicability and accuracy of the approach was illustrated by comparing these results with numerical ones from a number of specific plate problems.

Rectangular finite elements based on Reissner type variational statement for plate bending were applied to stability and free vibration of isotropic rectangular plates.
by Reddy and Tsay [69] for various edge conditions. These types of mixed rectangular elements (linear and quadratic) were algebraically simple and, it was claimed, yielded more accurate solutions for the critical loads and natural frequencies when compared to conventional finite elements.

Bassily and Dickinson [70] have presented a Ritz solution for the problem of buckling and lateral vibration of rectangular isotropic plates having arbitrary combinations of simply supported and free edges and subjected to various in-plane loads. Numerical examples for square plate and rectangular plate ($\phi$=2) with CCCC and CFCF boundaries under either direct in-plane load or under edge shear were presented, in order to illustrate the applicability of the approach and to indicate the order of errors that may result in the determination of the out-of-plane characteristics, when using simplified assumptions for the in-plane stress field. It was shown that, under uniform stress distribution, the buckling load was found to be 5% higher than that obtained by using the more realistic modified distribution, a discrepancy which is undesirable in practical situations.

Diez et al. [71] investigated square plates with edges elastically restrained against rotation and subjected to uniformly distributed in-plane normal and shear stresses; this investigation was for aeronautical and naval engineering applications. The plate displacement function was first expressed in terms of a polynominal coordinate function which
identically satisfy the boundary conditions. A frequency equation was then generated by using a variational formulation. Numerical results for both the fundamental mode and the first fully symmetrical mode for three boundary conditions (simply supported, elastically supported and rigidly clamped) were tabulated. These results were more accurate than that available in reference [72] (0.03% difference). No explanation has been mentioned regarding the change in the mode shapes affected by the non-linearity of the frequency coefficient Ω as a function of the shear stress parameter λ when the in-plane stress (tension) is varied.

The strip method, which is a variation of the finite element method has been used successfully by Chan and Foo [73] in problems of vibration analysis of rectangular plates subjected to in-plane forces and under pure shear. They demonstrated that, in the case of uniaxial compression, a linear relationship between the frequency parameter λ and the buckling coefficient k can exist when the buckling modes and the vibration modes are similar. However, in the case of pure shear the buckling modes are quite different from the vibration modes and the relationship between λ and k are non-linear. No study was made to explain the intersections which occur between λ versus k curves for different modes.

Kielb and Han [74] have presented numerical solutions for the fundamental natural frequency and mode shape of a rectangular plate, loaded by in-plane hydrostatic force for a wide
variety of aspect ratios, boundary conditions, and load magnitude. Design curves and approximate formulae were presented which provide a simple means of determining the fundamental frequency parameter. However, no mode shapes were presented to explain the non-linearity deviation of the frequency parameter $\lambda$ versus buckling coefficient $k$.

A Rayleigh-Ritz approach was presented by Kalas and Dickinson [75] which permits:

(a) the prediction of the natural frequencies and mode shapes of rectangular welded plates subjected to any practical in-plane stress field,

(b) the study of the related elastic buckling problem.

However, no mode shapes or nodal patterns were presented.

Katzer and Murray [76] have shown that simply supported panels with high aspect ratios can buckle either locally or globally. The buckling mode may change from local to global with interactive buckling in the transition region, as the number of stiffeners increases. Graphs of the theoretical buckling stress against depth of stiffener show that as $d$ increases from zero, where the panels behave as an isotropic flat plate, up to an intermediate value the panels behave as orthotropic plates. Beyond certain intermediate values the buckling stress decreases and the local buckling mode is largely governed by torsional effects in the deep narrow stiffeners. Experiments were carried out to verify the
application of the finite strip method to stiffened plates, and it was demonstrated that this change can be brought either by changes in the number or depth of stiffeners.

Vibration frequencies and buckling loads were presented by Leissa and Ayoub [77] for simply supported rectangular plates subjected to a pair of oppositely directed concentrated forces each acting normal to an edge. The forces were considered to be either tensile or compressive. The analysis was long and difficult. The problem was solved by the Ritz method. The results were presented for three plate aspect ratios ($\phi = 0.5, 1$ and $2$) and it was noticed that the in-plane forces caused by concentrated forces acting along the boundaries have the same influence in the natural frequencies as those caused by uniform normal forces.

Mukhopadhyay [78] and [79] developed a semi-analytical finite difference method for vibration and stability analysis of plates having concentric (symmetric) and eccentric stiffeners, possessing different boundary conditions, mass and stiffness properties and varying number of stiffeners. The effect of stiffeners was suitably incorporated in the free vibration and stability equations. The natural frequencies and the critical stresses of the above stiffened plates were compared with available published data [80], [81] and [82] given in part-I and [83], [84] and [85] given in part-II. Excellent agreement has been obtained with these references. The difference in the fundamental frequency of
the stiffened plate between neglected and restraint of the
in-plane displacements along all edges was less than 2% as
was shown in reference [86]. Although natural frequencies of
stiffened plates where torsional stiffness is neglected have
shown a small reduction in the natural frequencies when
torsional stiffness is included in the analysis.

(b) Composite Plates:
The effect of boundary conditions on the bending under
uniform transverse load, fundamental frequencies of
transverse vibration and buckling under uniform compression
of unsymmetrically laminated rectangular plates was
investigated by Whitney [87]. Five sets of boundary
conditions corresponding to various clamped and simply
supported edges were treated. It was found that for certain
orientation of anti-symmetric angle-ply plates, membrane
boundary conditions can significantly influence the plate
response. The results showed that the reduced bending
stiffness (RBS) approximation did not agree well with coupled
laminate solutions for such orientations. Further studies of
the effect of membrane boundary conditions on the response of
anti-symmetric angle-ply laminates would be useful.

An 18 d.o.f triangular plate finite element in bending with
anisotropic symmetrically laminated composite material was
formulated by Alex et al. [88], in order to develop a
minicomputer program for the static, free vibration and
buckling analysis. The program is able to give a
three-dimensional plot for the static deflection and mode
shapes of the plate. Its efficiency, simplicity and practicality was evaluated through a series of examples.

Pad and Peterson [89] have proposed contour plotting method to display mode shapes of free vibration and buckling of laminate plates. Only square laminate plates symmetric about their respective mid-planes and fully clamped were investigated. The study was conducted on ($\theta=0^\circ, \theta=15^\circ, \theta=30^\circ, \theta=45^\circ$) single-layer laminate plates and ($0^\circ/90^\circ/0^\circ$, $45^\circ/-45^\circ/45^\circ$) tri-layer laminate plates. The mode shapes displayed by the use of contour method demonstrate the peaks and valleys of the mode shapes and also reveal the fibre direction of the single layer laminate plate. The effect of changing fibre orientation relative to clamped boundaries on the mode shapes was fully explained by this technique.

2.3.2 EXPERIMENTAL INVESTIGATIONS

(a) Classical Plates:
Some experimental work on the natural frequencies of a rectangular square plate subjected to in-plane compressive load has been undertaken by Lurie [90]. He showed considerable discrepancy between the theoretical predictions and experimental results and he suggested that the difference was due to the effects of initial geometrical imperfections in the test of plate specimens. He based his agreement on some theoretical work on clamped circular plates undertaken by Massonnet [91] who showed a similar deviation when imperfections were taken into account.
White and Teh [92] have undertaken an experimental work on the dynamic response of an aircraft-type aluminium alloy plate, with fully clamped boundaries, subjected to combined acoustic excitation and uniaxial in-plane compression. It has been shown that under slow sweep sinusoidal excitation, the percentage change in resonance frequency with excitation sound pressure level increased with increasing levels of applied compression. However, under broad band excitation, no shift in the resonance frequency was observed with increasing sound pressure level for a particular loading condition, and the rate of decrease of the second resonance was found to be larger than that of the fundamental resonance. The experimental procedure undertaken was similar to that described in reference [59].

The stress distribution in some practical in-plane loaded plates has been obtained either directly, via strain gauges, or indirectly, from the measurement of transverse deflections or initial imperfection profiles by Ilanko and Tillmav [93]. This stress distribution has been incorporated into a finite difference computer program set up to evaluate the natural frequencies of the plates. A comparison has been made between both techniques and, it was noted that the difference increased with increasing in-plane stress. This difference was due to the distribution of in-plane stress that occurred in practical plates due to the growth of initial geometrical imperfections with increasing applied load.
Abramovich et al. [94] have tested the vibration of a series of stringer-stiffened circular-cylindrical shells subjected to a combined axial compression and external pressure state loading. Some shells were tested with clamped boundary conditions and some on nominal simple supports. Their study also included an assessment of the influence of the order of loading on the behaviour of the shells before and at buckling. They represented the buckling loads corresponding to different combinations of loading in the form of the so-called buckling interaction curves. It was noticed that the sequence of loading was initially an axial compressive force which was followed by an increase in the external pressure until buckling occurred. However, if the sequence of loading was reversed the magnitudes of the buckling loads were not affected.

(b) Composite Plates:
Rao [95] has presented a theoretical work dealing with the prediction of elastic buckling loads (axial compression) of composite sandwich rectangular panels with a grid core. Each of the panel edges was considered to be clamped or simply supported, and the results were obtained using conventional orthotropic plate theory. A large number of $0^\circ/90^\circ/45^\circ/-45^\circ$ lamination schemes leading to quadridirectional, tridirectional and bidirectional composites was examined under buckling loads. The results indicated that by proper choice of lamination scheme, significant increases in buckling loads compared to quasi-isotropic case can be
obtained when $a/b < 1$ (the lower the value of $a/b$, the higher the gain) and for $a/b > 2$, the corresponding gains were smaller. For $a/b > 3$, having a greater number of $45^\circ$ layers was found to be more advantageous. This was also the case for $a/b < 0.75$ having a greater number of $0^\circ$ layers.
CHAPTER THREE

3. BASIC ELEMENTS OF THE THEORY OF VIBRATIONS

3.1 PART A: MATHEMATICAL ANALYSIS

3.1.1 INTRODUCTION

Structures made of advanced composites such as glass reinforced plastics often consist of a number of layers. The laminate can be manufactured from either continuous or discontinuous fibres which can be unidirectionally aligned, randomly orientated or in specified orientations according to a specific dynamic requirement. These layers, therefore, can be quasi-isotropic, quasi-orthotropic or anisotropic. When orthotropic laminae are stacked to form a laminate, the resulting structure is generally anisotropic. The laminate will be orthotropic only for certain stacking sequences.

The derived mathematical equations in this chapter are for orthotropic and isotropic materials only. No attempt has been made to consider the anisotropic case.

3.1.2 BASIC ASSUMPTIONS OF ORTHOTROPIC PLATE

The plate, shown in Fig.3-1a, is assumed to satisfy the thin plate theory; that is:

(a) the thickness of the plate is small compared to the
dimensions $a$ and $b$ shown in Fig.3-1a. The smallest lateral dimension of the plate is at least ten times larger than its thickness,

(b) the deflections are small compared to the plate. A maximum deflection from one tenth to one fifth of the thickness is considered as the limit for small deflection theory. This limitation can also be stated in terms of length; i.e., the maximum deflection is less than one fiftieth of the smaller span length,

(c) the relationship between stress and strain is linear,

(d) the material of the plate is elastic, homogeneous and possesses different elastic properties in the two orthogonal directions,

(e) the thickness of the plate is uniform and small in comparison with its other dimensions,

(f) the middle plane of the plate, according to Kirchoff hypothesis, does not stretch during bending and remains a neutral surface.

3.1.3 GEOMETRY OF RECTANGULAR PLATE

A sketch of essential elements of a plate is represented in Fig.3-1a; it is a rectangular thin plate of width $a$, length $b$ and thickness $t$. The plate’s rectangular cartesian coordinates are $X$, $Y$ and $Z$ where $X$ and $Y$ lie in the middle plane and $Z$ is measured from the middle plane. While the body is undergoing deformation, caused by the external load,
q_z, acting normally to the surface of the plate, the
displacement components of a typical point within the plate
are u, v and w in the X, Y and Z directions, respectively.
Thus, u and v are in-plane displacements and w is the
transverse displacement with respect to the mid-plane.

Consider now, an element cut out from an orthotropic plate by
planes parallel to the Z-X, and Z-Y planes. Then, if this
element is subjected to the action of normal stress, \( \sigma_x \),
uniformly distributed over two opposite sides in the
X-direction, as shown in Fig.3-1b. This results in:

\[
\begin{align*}
- \text{extension in } X-X \text{ axis} & \quad \varepsilon_x' = \frac{\sigma_x}{E_x} \\
- \text{contraction in } Y-Y \text{ axis} & \quad \varepsilon_y'' = -\nu_x \frac{\sigma_x}{E_x}
\end{align*}
\] .... (3.1.1)

where, \( E_x \) is the elastic modulus in X-direction,
\( \nu_x \) is Poisson's ratio in the X-direction.

If the same element is subjected to the action of normal
stress, \( \sigma_y \), uniformly distributed over two opposite sides in
the Y-direction, this results in:

\[
\begin{align*}
- \text{contraction in } X-X \text{ axis} & \quad \varepsilon_y' = -\nu_y \frac{\sigma_y}{E_y} \\
- \text{extension in } Y-Y \text{ axis} & \quad \varepsilon_y'' = \frac{\sigma_y}{E_y}
\end{align*}
\] .... (3.1.2)

Superposing (3.1.1) and (3.1.2), this results in the
following expressions:
If the considered element is under pure shearing stress, then the values of shear modulus, \( G_{xy} \), and shear strain, \( \gamma_{xy} \), may be expressed as:

\[
G_{xy} = \frac{E_x E_y}{E_x + (1 + 2\nu_{xy})E_y} \quad \text{..... (3.1.4)}
\]

\[
\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \quad \text{..... (3.1.5)}
\]

where \( \tau_{xy} \) is the tangential stress.

### 3.1.4 GOVERNING FUNDAMENTAL EQUATIONS

#### 3.1.4.1 UNSTIFFENED PLATES

(a) **Orthotropic Plates:**

The stress-strain relationships for a thin orthotropic rectangular plate lying in the \( X-Y \) plane derived from equations (3.1.3) and (3.1.5) can result in the following matrix form expression:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} \quad \text{..... (3.1.6)}
\]

The coefficients, \( c_{ij} \), are related to the plate material
properties by the expressions:

\[ c_{11} = \frac{E_x}{(1 - \nu \nu_y)}, \quad c_{22} = \frac{E_y}{(1 - \nu \nu_y)}, \quad c_{33} = G_{xy} \]
\[ c_{12} = \nu \frac{E_x}{(1 - \nu \nu_y)}, \quad c_{21} = \nu \frac{E_y}{(1 - \nu \nu_y)}. \]

The stress components \( \sigma_x \) and \( \sigma_y \), shown in Fig.3-1c, produced by bending moments in the plate element. On another hand, the stress component \( \tau = \tau_{xy} = \tau_{yx} \) produce twisting moments. Thus, by integration of normal stress and shear stress; the bending and twisting moments can be evaluated as:

\[
M_x = \int_{-t/2}^{+t/2} \sigma_x zdz
\]

\[ \ldots \quad (3.1.7a) \]

\[
M_y = \int_{-t/2}^{+t/2} \sigma_y zdz
\]

\[ \ldots \quad (3.1.7b) \]

\[
M_{xy} = \int_{-t/2}^{+t/2} \tau_{xy} zdz
\]

\[ \ldots \quad (3.1.7c) \]

By considering the equilibrium of the infinitesimal plate element, shown in Fig.3-1c, this leads to the three fundamental equations of static; these are:

\[
\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma P_z = 0 \quad \ldots \quad (3.1.8)
\]

The strains, stresses and moments in an element of the plate
have been developed in Appendix A.

Consequently, the governing differential equation of a thin orthotropic rectangular plate subjected to lateral loading can be obtained in the following form:

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2(D_1 + 2D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q_z
\]

in which the constants \( D_x, \ D_1, \ D_{xy} \) and \( D_y \), are given by the expressions:

\[
D_x = c_{11} t_p^3 / 12, \quad D_1 = c_{12} t_p^3 / 12
\]

\[
D_y = c_{22} t_p^3 / 12, \quad D_{xy} = c_{33} t_p^3 / 12
\]

introducing the following notation,

\[
2H = 2(D_1 + 2D_{xy})
\]

where, \( 2H \) represents the effective torsional rigidity of the orthotropic plate.

This results in:

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q_z
\]

This is the governing differential equation of an orthotropic plate, and known in the technical literature as: "Huber's
equation'.

(b) **Isotropic Plates:**

In the case of isotropic plates the mechanical properties are the same in all directions and can be expressed as:

\[ E_x = E_y = E \]
\[ \nu_x = \nu_y = \nu \]

and,

\[ E = E = E \]

which leads equation (3.1.9) to be written in the following form:

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_x}{D} \]

where, \( D = \frac{Et^3}{12(1-\nu^2)} \) is the flexural rigidity of the isotropic plate.

By using the two-dimensional Laplacian operator, the above equation can be written as:

\[ D \nabla^4 w = q_z \]

where, \( \nabla^4 = \nabla^2 \nabla^2 \) is the biharmonic differential operator.

(i) **Free Undamped Vibration Analysis:**

In the free vibration of plates there is no lateral loading \( q_z \). However, there will exist an inertial body force which must be taken into consideration. This inertial force is due to the oscillatory nature of the plate motion. Fig.3-1d shows the quantity of the inertial body force of surface \( dA = dx \cdot dy \) and a magnitude \( \rho \cdot dA \cdot \frac{\partial^2 w}{\partial t^2} \) representing the
acceleration.

Replacing $q_1$ of equation (3.1.13) by the inertial force, $-\rho \ddot{w}$, results in:

$$D \nabla^4 w = -\rho \ddot{w} \quad \ldots \quad (3.1.14)$$

where, $\rho$ is the mass of the plate per unit area.

$w$ can be expressed as:

$$w(x, y, t) = W(x, y) f(t) \quad \ldots \quad (3.1.15)$$

Since wave type motions are of interest let the solution be harmonic in time, i.e.,

$$f(t) = e^{i\omega t} \quad \ldots \quad (3.1.16)$$

where, $\omega$ is the circular frequency of oscillation, $i = \sqrt{-1}$.

Hence, the differential equation for the spatial variation becomes:

$$D \nabla^4 W - \omega^2 \rho W = 0 \quad \ldots \quad (3.1.17a)$$

If the following lateral dimensionless parameters ($\zeta = x/a$, $\eta = y/b$) are introduced, then after some rearrangements, equation (3.1.17a) can be evaluated in a convenient form:
\[
\frac{\partial^4 W}{\partial \eta^4} + 2\phi^2 \frac{\partial^2 W}{\partial \eta^2 \partial \zeta^2} + \phi^4 \frac{\partial^4 W}{\partial \zeta^4} - \phi^2 \lambda^2 W = 0 
\]

where, \( \lambda \) is the frequency parameter given by:

\[
\lambda = \lambda_{m_n} = \omega_{m_n} b^2 \sqrt{\rho/D}
\]

where, \( m=1,2,.. \) and \( n=1,2,.. \) indicate the number of half waves in mode shape in \( X \) and \( Y \) directions respectively,

\( \phi \) is the plate aspect ratio: \( \phi=a/b \)

For lightly damped plates, it is possible to define a constant based on the plate geometry and material properties as follows:

\[
(\lambda/\omega)_{m_n} = b^2 \sqrt{\rho/D}
\]

(ii) **Stability Analysis (Buckling):**

Consider an isotropic flat plate with specified boundary conditions and subjected to an in-plane compressive load, \( P \), per unit length uniformly distributed along one pair of opposite edges, and applied at the boundaries of the middle surface.

In the current buckling analysis the lateral force \( q_x \) in the governing differential equation of the plate is replaced by \( P\phi^2 W/\partial x^2 \) and the differential equation of equilibrium can be written as:
In the stability analysis of plates usually the critical load is of practical importance (i.e., the minimum buckling load); this corresponds to the first mode of the buckled plate shape. In order to obtain approximate analytical expressions for the critical value of edge load, $P_{cr}$, the energy method can be used advantageously (see Appendix B).

(iii) **Vibration with In-Plane Load Analysis:**
Assuming that the plate material is linearly elastic, and that the shear deformation and rotatory inertia are negligible, the differential equation of motion for small deflections of a plate loaded in X-direction is:

$$ Dv^4W + \frac{P}{D} \, \frac{\partial^2 W}{\partial x^2} = 0 \quad \ldots \quad (3.1.20) $$

Using equation (3.1.16) the differential equation for the spatial variation becomes:

$$ Dv^6W + \omega^2 \rho W + P \, \frac{\partial^2 W}{\partial x^2} = 0 \quad \ldots \quad (3.1.21) $$

By introducing the integers $m$ and $n$ corresponding to the number of half waves in $X$ and $Y$ directions, respectively. Equation (3.1.22) results in:

$$ f_{mn}^2 = \left( Dv^6W + P \, \frac{\partial^2 W}{\partial x^2} \right) / 4\pi^2 \rho W \quad \ldots \quad (3.1.23) $$
where, \( f_{mn} = \omega_{mn} / 2\pi \)

Calculations show that, for all values of \( \phi \), the minimum loads occur for \( m=1 \). This corresponds to the smallest critical buckling load eigenvalue. Accordingly,

\[
f_{1n}^2 = \left( \frac{Dv^4W + P \delta^2W/\delta x^2}{4\pi^2 \rho W} \right) \quad (3.1.24)
\]

It can be seen, from equation (3.1.24), that the frequency of vibration will be either increased or decreased, depending on the direction of the in-plane load. It is clear, then, that for the compressive loads the frequency of vibration will be reduced. Thus, when the frequency is zero, \( P \) will be the critical buckling load.

By putting \( f_{mn}^2 \) equal to zero in equation (3.1.23) the equation can be reduced to the following form:

\[
Dv^4W + P \frac{\delta^2W}{\delta x^2} = 0 \quad (3.1.25)
\]

This corresponds to equation (3.1.20) shown in section (ii) in the stability analysis.

It can be noted that, in some literature the critical buckling load may be expressed in the form:

\[
P_{cr} = k D(\pi/b)^2 \quad (3.1.26)
\]
The coefficient \( k \) is found to be a function of the aspect ratio \( \phi \) and the wavelength parameter \( n \).

For a given \( \phi \), the value of \( n \) may be chosen to yield the smallest buckling eigenvalue, at which the plate can lose its stability.

In order to have better understanding of the influence of the in-plane compressive loads on the natural frequencies of the plate, it is convenient to produce this relationship in the form of graphs with normalized axis. Therefore, it is assumed that there is a given ratio between the in-plane uniform compressive load \( P \) and the critical buckling load \( P_{cr} \).

Then,

\[ \Delta = \left( \frac{P}{P_{cr}} \right) \]  \( \ldots (3.1.27) \)

Similarly, a given ratio between the frequency \( f_{mn} \) and the fundamental frequency \( f_{11} \) of the unloaded plate can be defined as:

\[ \Gamma_{mn} = \left( \frac{f_{mn}}{f_{11}} \right) \]  \( \ldots (3.1.28) \)

3.1.4.2 STIFFENED PLATES

If the number of stiffeners is very large, greater than three for instance, the stiffened plate may be considered as orthotropic and homogeneous with a fictitious thickness.
However, if the plate is reinforced by only a few stiffeners, say one or two, it cannot be considered orthotropic and homogeneous and the effect of stiffeners must be introduced in the governing differential equation. Even for the case of a simply supported plate along its four edges, the solution tends to be complex.

Appendix C illustrates the mathematical differential equations of rectangular plates reinforced either by the addition of stiffeners which are symmetric relative to the middle plane of the plate, or are eccentric. The mathematical solutions are either very complex, or are impossible to solve.

3.1.5 GENERAL FORM OF THE TRANSVERSE DEFLECTION \( W \)

In general the form of \( W \) is not known because of the difficulties of boundary conditions, stiffeners effect and orthotropy of material. However, an assumed form of \( W \) can be represented by a series of approximations of the form:

\[
W(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \phi_i(x) \psi_j(y) \quad \ldots \quad (3.1.29)
\]

where, \( \phi_i \), \( \psi_j \) are the beam mode shapes in \( X \) and \( Y \) directions respectively,

\( A_{ij} \) are the amplitude coefficients defined by the boundary conditions.
It has been shown in the literature [14] and [15] that for any given plate problem the boundary conditions, stiffeners and orthotropy of the material may create numerous mathematical difficulties. Closed-form solutions exist only for relatively few cases, such as all edges simply supported or two opposing sides simply supported. However, various approximate techniques can be used, notably numerical techniques, such as finite element method (FEM).
3.2 PART B: NUMERICAL ANALYSIS

3.2.1 GENERALITY

It is not sufficiently realistic to approximate structures by single degree of freedom models. Every real structure can be defined by a series of lumped masses each having several restraints and, therefore, several degrees of freedom to describe its motion. Computational and computer techniques are extensively used in the analysis of such multi-degree of freedom structures. However, it is essential for the analytical and numerical bases of any program used to be understood, to ensure its relevance to the problem considered, and that the program does not introduce unacceptable approximations and calculation errors. For this reason the basic theory and governing equations for multi-degree of freedom structures were derived, and numerical methods may have to be used to solve the differential equations by transforming them into a set of linear algebraic equations, which can be solved by matrix method.

In order to save time and considerable economics in writing programs, computer packages are widely used and available for routine analysis associated with the sophisticated structural models. One computer program 'ABAQUS' [96] was utilized to solve the governing differential equations using finite element approach that is presented in references [97], [98] and [99].
3.2.2 DISCRETIZATION OF PLATES

In order to yield good approximations of the system behaviour, the physical model is divided into a large number of finite elements. Consequently, the stiffened and unstiffened plates are divided longitudinally and transversally into a number of individual elements, "meshes". Thus, in the current investigation, the flat plate, plates with one and three stiffeners (rectangular and top-hat) are divided into 100, 110, 150, 160 and 300 elements respectively, as shown in Figs.3-2a, 3-2b, 3-2c, 3.2d and 3.2e.

3.2.3 CHOICE OF ELEMENT TYPE

Throughout the development of the finite element method, much attention has been devoted to the investigation of the appropriate element. Some elements tend to be complicated to formulate, expensive and difficult to use, because of the large number of nodal variables that result from the interpolations choice, and the difficulty to deal with such variables.

The shell elements used for the numerical investigations are of type (S4R), from the ABAQUS package, and are defined as a 4-node reduced integration, doubly curved shell. The doubly curved shell elements use five active degrees of freedom per node. These are three displacement components and two
in-surface rotation components. However, six degrees of freedom are used in practice at each node (δx, δy, δz, θx, θy, and θz).

Several numerical examples are presented in chapter 4 and are compared with the available exact solution. Reference [10] clearly indicates the accuracy and efficiency of the present technique.

3.2.4 EIGENVALUE EXTRACTION

The matrix equations of motion for linear multi-degree of freedom systems usually appear as:

\[
[M]\{\ddot{f}(t)\} + [C]\{\dot{f}(t)\} + [K]\{f(t)\} = \{0\}
\]

where, \([K]\) is the stiffness matrix,

\([M]\) is the mass matrix,

\([C]\) is the damping matrix.

The solution is in the following form:

\[
\{f(t)\} = \{\phi\}e^{it}\omega
\]

where, \([\phi]\) is the column matrix of eigenvectors,

\(\omega\) is the circular frequency of oscillation.

Substituting the assumed solution into equation (3.2.1), it gives:
\[ (-\omega^2[M] + \omega[C] + K)\{\phi\} = \{0\} \quad \ldots \quad (3.2.3) \]

The effect of damping has been neglected ([C]=[0]).

The matrices [M] and [K] are positive and definite, so that the eigensystem (equation 3.2.3) has real, positive eigenvalues only.

The program has the ability to extract eigenvalues and, hence, the natural frequencies of vibration associated with the corresponding displacement shapes.

The available methods for the structural eigenvalue problem are:

(a) Lanczos method,
(b) subspace iteration method.

The latter method was implemented in this program, in which the Householder and Q-R algorithms are used to extract eigenvalues and eigenvectors of vibration of the various plates.

3.2.5 SUBSPACE ITERATION - THE BASIC ALGORITHM

In this program the subspace iteration method represented in references [100], [101] and [102] is the basic algorithm used; the procedure is a simultaneous inverse power iteration in which a small set of base vectors is created.
A 'subspace' is defined and is transformed by iteration into the space containing the lowest few eigenvectors of the overall system at which point these lowest eigenvectors and, hence, natural frequencies of vibration are naturally available.

The natural frequencies of a plate structure are obtained either unloaded or during loading which includes the effects of the preload. Subspace iteration provides the basic eigenvalue technique, and the program allows a shift to be introduced so that, for example, rigid body modes may be included.

The various parts of this algorithm are fully discussed and presented in Appendix D. This algorithm is represented in reference [103].

3.2.6 RIGID BODY MODES

Looking at an infinitesimal electrical strain gauge of length l, cross-sectional area s and density ρ. The specific resistance of the wire material is given as:

\[ R = \frac{\rho l}{s} \] .... (3.2.4)

After differentiation with respect to R, l and s, with a further slight rearrangement, the longitudinal strain \( \lambda^* \) of this gauge length is:

\[ \lambda^* = \frac{dl}{l} = k \cdot \frac{dR}{R} \] .... (3.2.5)
where $k$ is the gauge factor ($a^2$)

if $A^* = 0$ there is no strain, the gauge length has undergone rigid body motion only.

The rigid body modes are, therefore, those displacement modes that the structure must be able to undergo as a rigid body without deformations (strains) being developed in it.

An example of free/free stiffened plate is illustrated in Fig.3-3, the plate is able to translate uniformly in $X$, $Y$ and $Z$ directions and rotate about $\theta_x$, $\theta_y$ and $\theta_z$ without developing strains. The eigenvalues corresponding to these modes are negative. The number of rigid body modes that a structure must be able to undergo can usually be identified without difficulty by inspection [104].

3.2.7 EIGENVALUE BUCKLING PREDICTION

The program contains a capability for estimating elastic buckling by eigenvalue extraction. This estimation is useful for stiff structures, where the pre-buckling response is almost linear. The buckling load estimate is obtained as a multiplier of the pattern of "live" loads, which are added to a set of "dead" loads. The "dead" state of the structure could have resulted from any type of response history, i.e., plasticity, creep, etc..
The program calculates the change in stiffness associated with the addition of the live loads, and estimates these loads in order to cause collapse of the system.

Assuming that the change in stiffness is proportional to the change in live load magnitude then, the eigenproblem,

\[(K_p + \lambda \Delta K) f_i = 0 \quad \ldots \quad (3.2.5)\]

is solved.

where, \(\Delta K_{pq}\) is the change in stiffness going from load \(P_n\) to load \(P_n + Q_n\),

\(f_i\) is the eigenvector corresponding to the associated mode shape,

\(\lambda_i\) are multipliers which provide the estimated buckling load.

This estimated the collapse load which is given as:

\[P_n + \lambda_i Q_n \quad \ldots \quad (3.2.7)\]

Since the program uses subspace iteration for eigenvalue extraction, several modes can be extracted simultaneously, although in most geometries the lowest mode is the only one of interest. However, it is sometimes useful to see if the collapse loads are similar in several modes.
4. EXPERIMENTAL PROGRAMMES

4.1 PART A: MEASUREMENT OF DYNAMIC AND STATIC MECHANICAL PROPERTIES OF SOME GRP COMPOSITE MATERIALS.

4.1.1 GENERALITY

If the plates are required to operate under a small range of dynamic loadings (excitation forces), which excite several natural frequencies of the plate, then it might be possible to shift these frequencies outside of the particular range of interest by stiffening the plate structure or making it lighter. However, these natural frequencies cannot be totally shifted outside the frequency range of interest when this range is wide. These vibration levels or vibrational energy can be significantly reduced by increasing the damping in the dominant modes throughout the application of viscoelastic damping technology which primarily limits the amplitude of response at resonance [105].

In order to achieve this requirement, one, two, three or more layers of viscoelastic material should be introduced to have the properties of frequency and temperature sensitivities. It is, therefore, essential that resin materials are selected after consideration of operational temperatures and frequencies [105].

In this investigation two types of GRP materials have been
tested; in both cases the fibre weight and its surface area were kept constant for each type of reinforcement. However, the resulting thickness may be variable in the hand lay up fabrication method which was the one used in this investigation. Consequently, this variation in thickness will affect the fibre volume fraction and, therefore, will alter slightly the mechanical properties of these composite materials [107] and [108].

The purpose of this investigation is:

(a) to obtain data on the effect of the variation of thickness in a composite plate for instance the effect of change in fibre volume fraction on the dynamic properties of GRP composite beams, which are cut from different plates,

(b) to use these data for a modal analysis of the plates where the elastic modulus and Poisson’s ratio are required,

(c) to use these data to optimise the dynamic properties of laminated GRP rectangular plates, when the frequency range of application is wide.

However, it should be noted that the determination of the elastic properties of fibre composites is important for optimum design, quality control and damage detection.

4.1.2 MATERIALS

4.1.2.1 Polymer Matrix Material:
The matrix material used for the manufacturing of specimens
consisted of a liquid polyester resin. Chemical catalyst and accelerator were added to polymerise the matrix resin; the accelerator controlled the initial hardening of the resin. Therefore, the proportion by the weight of catalyst and accelerator to polyester was controlled and corrected to achieve the suitable optimum properties of the cured laminate [109].

4.1.2.2 Glass Fibre:
The most widely used glass in GRP work is known as E-glass. This glass is formed from very thin fibres which are assembled together in two different types:

(a) Chopped Strand Mat (CSM), which consists of chopped glass fibres about 50mm long, of weight 450gm/m² and bonded together in a random pattern to form a mat with resinous binder, and the resulting laminate is assumed to have the same mechanical properties in all directions and, therefore, is considered to be quasi-isotropic for the purpose of the current analytical studies. The lay-up sequence was composed of 3-ply: (R,R) (see Fig.4-1a).

(b) Woven Glass Cloth (WGC), which consists of glass fibre fabric plain weave 428 (330 gm/m²), orthogonally arranged and the proportion of glass in each direction is 50%. The lay-up sequence was composed of 6-ply: (0°,90°) (see Fig.4-1b).

4.1.2.3 Composite Materials:
By combining an elastic phase (filaments, fibres or fibrous
material) with viscoelastic phase (matrix material) the composite materials are achieved, and are likely to be used in many applications where high specific stiffness and damping are required. The source of high stiffness is the fibres, while that for the high damping is the matrix.

A series of beams were manufactured in order that the dynamic properties of various types of fibre-resin combinations could be measured. In the current experimental work, the GRP composite materials are fabricated in the form of thin nearly uniform plates produced by traditional hand lamination method. Accurate control of thickness is not easy in this case, because of the hand lay-up fabrication method. Consequently, variation in thicknesses can alter the fibre volume fraction and, hence, this may influence the dynamic properties of such composites particularly the flexural modulus (stiffness) and damping factor. The actual thickness of the specimens manufactured by hand lay-up method are approximately 2.0mm and 2.4mm for the (0°, 90°)_s and the (R,R)_s, respectively.

4.1.3 PREDICTION OF LAMINATE PROPERTIES

Particular attention is given to the influence of the load intensity in the material and to its frequency of application. Consequently, the static or dynamic elastic property of composite materials is dependent upon the
frequency of the applied load.

4.1.3.1 Static Case:
Under static conditions, the modulus of elasticity is simply defined as:

\[ E = \sigma / \varepsilon \quad \ldots \quad (4.1.1) \]

In order to determine these static components, several methods can be employed. The simplest by considering the law of mixtures which require a knowledge of the fibre volume fraction and this can be determined by heating the composite for temperature below the molten temperature of glass. The test procedure consists of putting a specimen of mass \( M_c \) in an oven (furnace) at 450°C for about three hours, then the polyester resin is burned off to leave the glass fibre, which is then weighted.

The law of mixture relationship between the elastic modulus of composite, \( E_c' \), for specific direction and a specific fibre volume fraction, \( V_f \), is expressed as:

\[ E_c' = \beta_c E_f V_f + E_m V_m \quad \ldots \quad (4.1.2) \]

Since \( V_f + V_m = 1 \), equation (4.1.2) may be written as:

\[ E_c' = \beta_c E_f V_f + (1 - V_f) E_m \quad \ldots \quad (4.1.3) \]

where, \( \beta_c \) is a parameter accounts for the effect of fibres type,
c, f, m refer to composite, fibre and matrix respectively.

4.1.3.2 Dynamic Case:
Providing that the beam is vibrating at a low amplitude, then there must exist a linear relationship between stress and strain at any point on the beam. Realizing that the stress and strain are varying sinusoidally with time, then it should be noted that, the strain at any point is given by:

\[ \varepsilon = \varepsilon_0 \sin(\omega t - \delta) \quad \ldots \quad (4.1.4) \]

where, \( \delta \) is the phase shift between stress and strain, and depends on the frequency of the force application, \( \omega \) is the circular frequency.

The input stress for the viscoelastic material is given by:

\[ \sigma = \sigma_0 \sin \omega t \quad \ldots \quad (4.1.5) \]

For each quarter cycle, the energy \( Z \) absorbed by the composite beam from time \( t_0 = 0 \) to \( t_1 = \pi/2\omega \) during vibration is [110]:

\[ Z = \int_{t_0}^{t_1} \sigma \, d\varepsilon \quad \ldots \quad (4.1.6) \]

After integration, the following expression can be obtained:
The first term represents the maximum elastic energy stored in the beam during quarter cycle, and is denoted by:

\[ Z_s = \left( \frac{\sigma \varepsilon_o}{2\omega} \right) \cos \delta \] \hspace{1cm} (4.1.8)

\( Z_s \) vanishes when integrated over a complete cycle.

The second term, denoted by:

\[ \Delta Z = \left( \pi \frac{\sigma \varepsilon_o}{4\omega} \right) \sin \delta \] \hspace{1cm} (4.1.9)

represents the energy dissipated in the form of heat during a quarter cycle.

The quantity,

\[ \psi = \frac{\Delta Z}{Z_s} \] \hspace{1cm} (4.1.10)

is known as specific loss or specific damping capacity.

Equation (4.1.10) can also be written in the following form,

\[ \psi = \left( \frac{\pi}{2} \right) \tan \delta \] \hspace{1cm} (4.1.11)

\( \tan \delta \) is known as the dissipative or loss factor,

\[ \tan \delta = d \] \hspace{1cm} (4.1.12)

Considering now \( E^* \) of equation (4.1.2) as a complex modulus [111]:

\[ E^* = \left( \frac{\sigma \varepsilon_o}{2\omega} \right) \cos \delta \] \hspace{1cm} (4.1.7)


\[ E^*_c = E'_c + iE''_c \] .... (4.1.13)

where, \( i = \sqrt{-1} \)

If the strain and stress in equation (4.1.4) and (4.1.5) are written in complex form:

\[ \varepsilon^* = \varepsilon_0 e^{i\omega t} e^{-i\delta} \] .... (4.1.14)

\[ \sigma^* = \sigma_0 e^{i\omega t} \] .... (4.1.15)

then,

\[ E^*_c = \frac{\sigma^*_0}{\varepsilon^*_0} e^{i\delta} = \frac{\sigma_0}{\varepsilon_0} (\cos\delta - i\sin\delta) \] .... (4.1.16)

which gives:

\[ E'_c = E_0 \cos\delta \quad \text{and} \quad E''_c = E_0 \sin\delta \] .... (4.1.17)

where,

\[ E_0 = \frac{\sigma_0}{\varepsilon_0} \]

Similarly for \( \mathcal{E}_c \) and \( \Delta \mathcal{E} \), \( E'_c \) and \( E''_c \) are known as the storage modulus and loss modulus, respectively.

In the experimental work, it has been found that the dynamic flexural modulus is a function of the resonance frequency and the mechanical properties of the beam, and can be calculated from Euler's equation [112]:

\[ E' = 48\pi^2 \rho \left( \frac{1}{h} \right)^2 \left( \frac{f_n}{k_n} \right)^2 \] .... (4.1.18)
where,

- \( k_n \): constant depending on the boundary conditions of the beam, both ends clamped or free: \(- k_1 = 22.27,\)
- \( f_n \): fundamental natural frequency,
- \( \rho \): density of composite material,
- \( l \): active length of the beam,
- \( h \): thickness of the beam.

4.1.4 DETERMINATION OF DAMPING

Generally, the mathematical description of the damping factor is complicated; thus, experimental models have been developed to evaluate the damping factor present in the oscillatory system.

4.1.4.1 Rule of Mixtures:

It is assumed that the composite damping ratio is the sum of the products of the volume fractions and damping ratios of the matrix and fibre [113], that is,

\[
\xi_c = V_{f} \xi_{f} + V_{m} \xi_{m} \quad \ldots \quad (4.1.19)
\]

A second method is the rule of mixtures weighted by the bending stiffness, namely:

\[
\xi_c = \xi_f \left( \frac{E_I}{E_{c c}} \right) + \xi_m \left( \frac{E_I}{E_{m m}} \right) \quad \ldots \quad (4.1.20)
\]

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$E_c I_c$ is the bending stiffness of the composite which is given by:

$$E_c I_c = E_f I_f + E_m I_m \quad \ldots \quad (4.1.21)$$

and,

$$I_c = I_f + I_m \quad \ldots \quad (4.1.22)$$

A full complex stiffness method assumes that the composite damping ratio can be presented as:

$$
\xi = \frac{\xi + \xi_f (E_f V / E_m V)}{1 + (E_f V / E_m V)} \quad \ldots \quad (4.1.23)
$$

Once the ply damping characteristics are known, either from calculation or experiment, there is still a problem in predicting the damping of a composite whose plies have general orientations.

4.1.4.2 Rate of Decay Method:

A convenient way to determine the amount of damping factor present in a system is to measure the rate of decay of free oscillations.

By introducing a term $\gamma$ called logarithmic decrement which is defined as:

$$\gamma = \ln \left( \frac{\text{Amplitude of motion}}{\text{Amplitude of motion one cycle later}} \right) \quad \ldots \quad (4.1.24)$$
After n cycles have elapsed, equation (4.1.24) can be written as:

\[ T = \frac{1}{n} \ln \left( \frac{W_1}{W_n} \right) \quad \ldots \quad (4.1.25) \]

Note that,

\[ T = \frac{2\pi \xi_c}{(1 - \xi_c^2)^{1/2}} \quad \ldots \quad (4.1.26) \]

which is an exact equation, and for low damping (\( \xi_c \ll 1 \)) an approximate equation can be obtained in the following form:

\[ T = \frac{2\pi \xi_c}{\sqrt{1 - \xi_c^2}} \quad \ldots \quad (4.1.27) \]

Hence,

\[ \xi_c = \frac{1}{2\pi n} \ln \left( \frac{W_1}{W_n} \right) = d/2 = \tan(\delta/2) = 1/2Q \quad \ldots \quad (4.1.28) \]

where \( \xi, d, \delta \) and \( Q \) are damping factor, loss factor, phase shift between stress and strain and quality factor, respectively. All these quantities represent the amount of damping present in a system with different ways of nomination.

4.1.4.3 Half Power Bandwidth Technique:

The damping factor can be obtained by applying the half power bandwidth method to individual resonant peaks, as shown in Fig.4-2. The half power bandwidth technique is based on a lightly damped single degree of freedom system. The structural natural frequency values are known from the discrete sharp peaks in the frequency spectrum. The half
power points of the resonant peak are determined by tracing straight lines between the discrete values, and then the damping factor is obtained using the following expression:

\[
\xi_i = \frac{(f_a - f_b)}{2f_i} \quad \ldots \quad (4.1.29)
\]

The positions of \(f_a\), \(f_b\) and \(f_i\) are shown in Fig.4-2.

4.1.5 EXPERIMENTAL PROCEDURE

A series of beams were manufactured in order that their static and dynamic properties, with differing fibre volume fraction, could be measured.

4.1.5.1 Density:
The density was determined by measuring the difference between the weight of specimen in air and in distilled water. The density, \(\rho_c\), is then determined from the relationship,

\[
\rho_c = \frac{W_{(air)}}{W_{(air)} - W_{(water)}} \quad \ldots \quad (4.1.30)
\]

4.1.5.2 Fibre Volume Fraction:
The method used here was the resin combustion method which has been described in section 4.1.3.1. The fibre volume fraction, \(V_f\), of the specimen is then determined from the relationship:
4.1.5.3 Experimental Tests:

(a) Static Test:
Specimens of the same material as the plate were cut in the form of rectangular strips (300x25.5)mm², where the thickness variation does not exceed ± 2% within each specimen (t = 2.0 mm). In order to establish the elastic modulus and Poisson's ratio, one axially and one laterally positioned strain gauges were bonded to each face of the specimen, as shown in Fig.4-3. Both axially positioned gauges were mounted longitudinally to the load direction to enable detection of strain resulting from the longitudinal motion. Likewise the laterally positioned gauges were mounted transversely to the load direction to detect lateral contraction caused by longitudinal motion. The gauge of the electrical resistance strain gauges was 8mm.

An Instron machine model TT-CM-LM4, shown in Fig.4-4, was used to input the static tensile load through a cross head speed of 0.005cm/min. A data logger which was manufactured in the department of electrical and electronic engineering of the University of Surrey was used to record the outputs of the four strain gauges.

The axial load versus the axial strain curves were plotted on
a scale drawing by an HP86B computer using a digital program. Typical test results of load-strain curves with a specific fibre volume fraction, for both categories of GRP composites, are shown in Figs.4-5a and 4-5b. The average of the load and strain readings for the axially and laterally positioned gauges have only been reproduced on the graphs.

Some results of the experimental static test are presented in Table 4-1.

(b) Dynamic Test:
One test beam was manufactured with different thickness, for the determination of \( V_G \), \( E_s \) and \( \xi \). Material specifications for all tested beams are shown in Table 4-2. Each composite beam was clamped from both sides in the vertical plane and tested using the arrangement shown in Fig.4-6a. Fig.4-6b shows a photograph of the B & K flexural resonance apparatus and a typical output signal.

The beam is ordinarily excited by a non-contacting electromagnetic shaker activated by an oscillator which is tuned to the natural frequency of the beam. Under these circumstances, only one mode is excited and the decay is immediately logarithmic once the exciter is de-energized. The vibration level of the beam is detected by an electromagnetic pick-up placed a few millimetres from the centre of the beam. The decaying signal is amplified and displayed, as shown in Fig.4-7, where the damping factor can
be calculated using equation (4.1.28).

The dynamic flexural data are obtained by testing the beams under the fundamental mode only. No attempt has been made to determine the frequency effect on the modulus of elasticity and damping of GRP materials. The reason for this decision is that the test equipment was not sufficiently accurate to yield satisfactory results for the higher modes. However, for a specific range of frequency, the modulus of elasticity and damping of some composite materials are not highly dependent upon frequency [114].
4.2 PART B: EXPERIMENTAL MODAL ANALYSIS

4.2.1 INTRODUCTION

The object of this section is to investigate the dynamic effects of stiffeners on quasi-isotropic and orthotropic rectangular plates. The modal analysis was performed on six plates (three were made from GRP randomly orientated fibre and three from bidirectional fibre composites). In addition to the composite plates, a further three plates were made from Perspex and were used as guides and references to check the accuracy of the finite element modelling of the structural system. The total distributed mass of rectangular stiffeners was kept constant but the number was varied between zero and three per plate. The positions of these stiffeners are shown in Fig.4-8. The height and the sum of the widths of rectangular stiffeners were also kept constant. In addition to the rectangular beam stiffeners, top-hat stiffeners [115] and [116] were investigated analytically only, due to the difficulty of fabricating small top-hat stiffeners with the required dimensions. The flexural rigidity of the stiffener was kept the same for the case of the plate with one top-hat stiffener and plate with three top-hat stiffeners. However, the cross-section, shape and dimensions of the stiffeners were modified which leads to an extra weight. The geometrical configuration and dimensions of these stiffeners are illustrated in Fig.4-9, and were made
from randomly orientated chopped strand mat GRP composites. Green et al. [117] showed experimentally that hull panels reinforced by this type of stiffeners give better results than woven roving cloth GRP composites.

4.2.2 CHARACTERISTIC VIBRATION PARAMETERS

Because of the complex nature of the response phenomenon of plates to excitation forces, a considerable amount of testing is required. In particular, the problem of simulating a force excitation and response by means of continuous and discrete frequency testing are discussed to enable a good understanding of the experimental testing techniques. In addition, a section which describes the special facilities and techniques used in testing the various plates under dynamic loading conditions are also included.

4.2.2.1 Frequency Response Function (FRF):
If the structure has many important resonant frequencies or complicated damping factor values, the only way to determine its dynamic behaviour is by measurement methods, which form an important part of vibration analysis used as a tool to determine the characteristics of structures, which are, indeed, complicated to analyse theoretically.

In order to determine these dynamic characteristic, the frequency response function must be measured. This is considered to be the most convenient means of describing the
dynamic characteristics of the structure, and can be obtained by expressing the relationship between the input and the output, that is achieved by dividing the Fourier Transform (FT) of the response by the FT of the force or vice versa [118].

In fact, there are several frequency response functions depending on whether the relationship between the response and excitation is taken under the form (output/input) or (input/output), i.e.,

- Accelerance, Inertance ............... (acceleration/force)
- Mobility .................................. (velocity/force)
- Receptance, Compliance, Admittance .... (displacement/force)
- Apparent Mass, Effective Mass ........... (force/acceleration)
- Impedance ................................. (force/velocity)
- Dynamic Stiffness, Apparent Stiffness .. (force/displacement)

All these functions contain basically the same information about the dynamic behavior of the structure, the choice can be determined by the available equipment.

The transformation from the time domain to the frequency domain requires the use of FT, which represents any real waveform of any structure as an addition of a series of sines and cosines which are harmonically related, (see Fig.4-10). The magnitude of the vibration may be represented in terms of its spectral characteristics and its variation with time by the surface shown in Fig.4-10. The height of a point on the surface raised over the time-frequency planes defines the magnitude of the vibration at a particular time and
frequency.
At a specific instant of time, the vibration is described by its variation of magnitude with frequency in the plane normal to the time axis, i.e., in the plane defined by "Spectral Analysis" shown in the two dimensional plot in Fig.4-11a.

The variation of the magnitude of the vibration at a particular frequency as a function of time is shown by passing a plane normal to the frequency axis, i.e., in the plane defined by "Magnitude time history Analysis" and shown in the two dimensional plot in Fig.4-11b.

4.2.2.2 Fourier Transform (FT):
The Fourier integral is defined by the equation [119] and [120]:

\[
W(t) = \int_{-\infty}^{+\infty} W(\omega) e^{i\omega t} dt \quad \text{..... (4.2.1)}
\]

and its inverse is defined as:

\[
W(\omega) = \int_{-\infty}^{+\infty} w(t) e^{-i\omega t} dt \quad \text{..... (4.2.2)}
\]

where, \(w(t)\) represents the signal in the time domain,
\(W(\omega)\) represents the signal in the frequency domain.

4.2.2.3 Discrete Fourier Transform (DFT):
If the period \(T\) is extended to infinity the discrete frequency spectrum becomes continuous,
\[ W(\omega) = \lim_{n \to \infty} \Delta t \sum_{n=0}^{N-1} W(n, \omega), \quad \ldots \quad (4.2.3) \]

\( \Delta t \) is the time interval between samples.

For \( N \) (degree of freedom) very large number, the discrete function can be written as:

\[ W(\omega) = \Delta t \sum_{n=0}^{N-1} W(n, \Delta t) e^{-i\omega n\Delta t}, \quad \ldots \quad (4.2.4) \]

4.2.2.4 Fast Fourier Transform (FFT):

The fast Fourier transform algorithm is only an approximation of the desired FT.

4.2.2.5 Transfer Function (TF):

Strictly speaking, the term transfer function is the ratio of the Laplace transform (LT) of the output to the Laplace transform (LT) of the input. In the frequency domain, however, the real part of \( \omega = \omega^r + i\omega \) is zero, and the LT becomes the FT. Thus, the TF can be defined as:

\[ H(\omega) = \frac{V(\omega)}{W(\omega)} \frac{\text{FT of the output}}{\text{FT of the input}} \quad \ldots \quad (4.2.5) \]

where, the \( W(\omega) \) and \( V(\omega) \) are the FT of \( w(t) \) and \( v(t) \), respectively.
4.2.2.6 N-Degrees of Freedom System:
Considering that the plate is vibrating at a low amplitude, there must, therefore, exist a direct linear relationship between the input and the output. If a force $p_r$, as shown in Fig. 4-12, is applied at point "r", an acceleration $\ddot{w}_r$ is yielded on the surface at point "s" in the $z$-direction.

Experience has shown that for an analytical study, it is most convenient to express the output in terms of displacement instead of acceleration. Thus, in harmonic motion the displacement $w$ can be written as:

$$w(t) = W \sin \omega t \quad \ldots \quad (4.2.6)$$

The velocity and acceleration can be simply determined by differentiating the above equation,

$$\dot{w}(t) = \omega W \sin \omega t = \omega W \sin (\omega t + \pi/2) \quad \ldots \quad (4.2.7)$$

$$\ddot{w}(t) = -\omega^2 W \sin \omega t = \omega^2 W \sin (\omega t + \pi) \quad \ldots \quad (4.2.8)$$

so that in harmonic motion the acceleration is proportional to the displacement and is directed towards the origin, this is done by introducing a factor of $\omega^2$ in the equation of displacement,

$$\ddot{w}(t) = -(\omega)^2 w(t) \quad \ldots \quad (4.2.9)$$

The general way to describe the elastic behaviour of the
plate system is by discretizing it. In doing so, the equation for a linear dynamic system with \( N \) degrees of freedom read:

\[
[M]\{\ddot{w}\} + [C]\{\dot{w}\} + [K]\{w\} = \{p\}
\]

\[\ldots \quad (4.2.10)\]

The displacement responses at position "s" (see Fig.4-12), resulting from the excitation force at position "r" under a frequency \( f \), can be written as:

\[
W(f) = \sum_{p=1}^{N} \frac{1}{k_p} \frac{\phi_{sp} \phi_{rp} P_r(f)}{1 - \bar{f}_p^2 + 2 \xi_p \bar{f}_p}
\]

\[\ldots \quad (4.2.11)\]

where, \( N \) is the number of modes,
- \( \phi_{sp}, \phi_{rp} \) are the values of the \( p \)-th displacement mode at the points "s" and "r",
- \( \xi_p \) is the ratio of the \( p \)-th modal damping,
- \( \bar{f}_p \) is the ratio of the \( p \)-th frequency, \( \bar{f}_p = f / f_{np} \), \( f_{np} \) is the \( p \)-th natural frequency,

\[
2\pi f_{np} = \sqrt{\frac{K_p}{M_p}}
\]

\[\ldots \quad (4.2.12)\]

where, \( K_p \) and \( M_p \) are \( p \)-th normal stiffness and mass,
- \( W(f) \) and \( P(f) \) are the FT of the displacement response \( w(t) \) and excitation force \( p(t) \), respectively.

By letting the input \( P_r(f) \) equal to unity the FT of the impulse \( h(t) \) for \( N \) degrees of freedom can be obtained in the
following form:

\[ H^e(f) = \sum_{p=1}^{N} \frac{1}{k_p} \frac{\phi_{sp} \phi_{rp}}{1 - \frac{T_p^2}{(1 - \xi_{rp}^2)} + 2\xi_{rp} T_p} \]

\[ (4.2.13) \]

\( H^e(f) \) is a complex function and can be separated into a real part and imaginary part, i.e.;

\[ H^e(f) = R_H^e(f) + i I_H^e(f) \]

\[ (4.2.14) \]

where,

\[ R_H^e(f) = \sum_{p=1}^{N} \frac{1}{k_p} \frac{1 - \frac{T_p}{(1 - \xi_{rp}^2) + (2\xi_{rp} T_p)^2}}{2 \xi_{rp} T_p} \]

\[ (4.2.15) \]

\[ I_H^e(f) = \sum_{p=1}^{N} \frac{1}{k_p} \frac{2 \xi_{rp} T_p}{(1 - \xi_{rp}^2) + (2\xi_{rp} T_p)^2} \]

\[ (4.2.16) \]

where,

\[ k_p = k_p / (\phi_{sp} \phi_{rp}) \]

is called the equivalent stiffness.

4.2.3 INSTRUMENTATION FOR VIBRATION MEASUREMENT

The transformation from the time domain to the frequency
domain is obtained using the basic experiment set-up shown in Fig.4-13a and a photograph in Fig.4-13b. Both a hammer (Bruel & Kjaer Hammer type 8202) and shaker were used to excite the plates through their resonant frequencies. The hammer had a force transducer mounted on its head. A piezoelectric accelerometer which monitored the response of the plate was attached to the point where the response was to be measured; the former through a frequency wave form recorder (DL1200) to Hewlett-Packard mini computer in which an Acquire software package enabled the computations of the frequency response function (FRF) to be made, the latter was monitored through an oscillator and double beam oscilloscope to an X-Y plotter.

In the current investigation the shaker was used to demonstrate the mode shapes of vibration of the various plates and the impact hammer was utilised to obtain the frequency response functions and, hence, the first four natural frequency modes.

4.2.3.1 Shaker Testing:
As can be seen, from Fig.4-14a, that the shaker was suspended and then mounted to the plate via the force transducer. The load cell may add appreciable mass to the plate structure which causes the force measured by the load cell to be greater than the force actually applied to the plate structure. The shaker was connected to the load cell through a slender rod called stringer (see Fig.4-14b), to allow the
plate to move freely in the other directions. This slender rod, shown in Fig.4-14c, has a strong axial stiffness, but weak bending and shear stiffness. Consequently, it carries only axial loads but negligible moments or shear loads.

In order to demonstrate the mode shapes of vibration of these plates, it is necessary to measure two parameters at each resonant frequency of the plate; these are:

(a) the displacement amplitude at a sufficient number of points,
(b) the corresponding phase relationship between these points.

The number of accelerometers and their location on the plate will depend on the mode shape to be measured. As an example, Fig.4-15 shows the suitable location of the mounting points for demonstrating the first and second mode shapes of the F-C-F-C plate. A vibration pick-up for several points on the plate were made and the motion of each accelerometer was recorded (see Fig.4-16). Another pick-up (shaker), held at fixed point on the plate, provides a signal used as a phase reference, to determine whether the plate motion at various points is in phase or out of phase with the input.

By connecting the points of different amplitudes at the same resonant frequency, the mode shape corresponding to this resonant frequency can be traced out. For the higher modes this method tends to be too complex to distinguish the mode
shapes. Consequently, a large number of accelerometers are required.

It should be noted that the natural frequencies measured are those of the plate plus the mass loading from accelerometers. The effect of this mass is an important consideration and this may alter the natural frequency of the plate itself as a result of the mass of the accelerometers being a significant fraction of the effective mass of a particular mode. The exact natural frequency of the plate can be determined from the following experimental procedure:

(a) measure a typical frequency response function of the plate using the desired accelerometer,

(b) mount another similar accelerometer (in addition to the first) at the same point and repeat the measurement,

(c) compare the two measurements and investigate the frequency shifts and amplitude changes.

If the two measurements differ significantly, the mass loading is a problem and this should be taken into account. An accelerometer with less mass may be a good solution to eliminate this significant difference. However, this latter solution still inappropriate for the small structures (beams investigated in section 4.1, for instance) because the extra mass loading originated from the accelerometer was significant in comparison with the effective mass of the beam. A non-contacting electromagnetic transducer was used
to measure the response at the desired point without extra mass loading.

4.2.3.2 Impact Testing:
The transformation from the time domain to the frequency domain is obtained using the basic experiment set-up shown in Fig.4-13a and a photograph shown in Fig.4-13b. An FFT was used to conveniently convert a time function into its frequency components. The time history can, for example, be represented by the following expression, as given in reference [121]:

\[ P(t) = \sum_{n=1}^{N} P_n e^{i\omega_n t} \quad \ldots \quad (4.2.17) \]

where, \( P_n \) are the FFT components at the discrete frequencies \( \omega_n \).

The time function is sampled every \( \Delta T \) of a total time window of \( (N \Delta T) \). The data can be sampled between 2\( \mu \)-s-20ms for a total of 4096 samples per channel, the final choice of time depends upon the frequency range of interest. Generally, however, only 1024 points of the data are used so that reflections and noise during impact are removed from the signals. In addition, to enable the impact to be reduced but more energy to be produced to excite the whole plate without damage or without introducing non-linear responses, a
resilient material (rubber or plastic) can be introduced between the hammer and plate.

During the impact hammer operation, the input and the output signals have been recorded from a Hewlett-Packard mini computer in which an Acquire software package was implemented. A typical time records history for the completely free plate have been presented in Fig.1-4 (see chapter 1). The waveform produced by the impact hammer excitation, represented in Fig.1-4a, is transient (short duration) energy transfer event. The typical accelerometer output response by the plate system, at fixed point and direction is illustrated in Fig.1-4b. The Fast Fourier Transform (FFT) in logarithmic magnitude of the force-time input is illustrated in terms of magnitude-frequency in Fig.4-17, which shows that -10DB corresponds to 222Hz; this is an acceptable reduction as the frequency range of interest, in this case, is only 100 Hz.

A typical magnitude against frequency relationship for the transfer function curve for the fully clamped (0°,90°) composite plate with no stiffener is shown in Fig.4-18. Each resonant frequency has a peak value corresponding to the natural frequency of that mode. As can be seen from Fig.4-18, strong frequency components are noted at 85 Hz, 149 Hz, 199 Hz and 246 Hz which are the fundamental and the three harmonic modes, respectively.

In addition to obtaining a natural frequency from the
frequency response curve a structural damping factor can be derived from the formula given in equation (4.1.29).

4.2.4 EXPERIMENTAL PROCEDURE

Three plates made from Perspex of aspect ratio 1.25 and plate thickness 2.0mm were first tested; the purpose of this test was to act as a reference for the composite plates concerning the variation of the thickness resulting from the hand lay-up lamination method.

Six composite plates (three quasi-isotropic and three orthotropic) of aspect ratio 1.25 were made. The three quasi-isotropic plates were manufactured from three chopped strand mats (each of weight/surface 450gm/m$^2$), in a polyester resin matrix; the final thickness was $(2.4 \pm 0.1)$mm. The three orthotropic plates were fabricated from bidirectional glass fibre fabric plain weave 428 (330gm/m$^2$) in a polyester resin matrix; the final thickness of the composite plate was $(2.0 \pm 0.1)$mm.

The strips of composite representing the stiffeners were manufactured from randomly orientated glass fibre in a polyester resin matrix; the strip had a total thickness of 6mm and a depth of 6.8mm. The strips were bonded onto the plates with epoxy resin. Likewise strips of bidirectional glass reinforcement were manufactured having thickness and depth of 6mm and 7mm, respectively.
Three sets of boundary conditions, for each plate material (Perspex, \((R,R)\) and \((0^\circ,90^\circ)\)) were simulated in the experimental work; these are:

(a) a near free/free \((F-F-F-F)\) boundary condition was obtained by suspending the plates vertically from a frame by soft elastics attached to the mid-points of its four corners. The suspension gives rigid body modes of vibration of the plate at very low frequencies (in the range of 1–2 Hz for the Perspex unstiffened plate) compared with the lowest plate natural frequency,

(b) the condition of fully clamped \((C-C-C-C)\) boundaries was achieved by clamping the plate between two steel frames with four bolts per side as shown in Fig.4-19; the whole system was then fixed onto a strong and stiff timber frame.

(c) the free/clamped \((F-C-F-C)\) plate boundary conditions were fully clamped along the two longitudinal edges and were free on the other two edges; the system was then fixed in the above described frame.
CHAPTER FIVE

5. THEORETICAL ANALYSES

5.1 PART A: VIBRATIONAL ANALYSIS OF COMPOSITE PLATE REINFORCED BY SINGLE STIFFENER HAVING MINIMUM MASS AND SUITABLE CROSS-SECTION

5.1.1 INTRODUCTION

The composites which are to be discussed in this section are rectangular plates which are exposed to vibrations with a possible induced fatigue failure of the plate through the development of resin cracking, fibre debonding and delamination between plies [122]. These vibrations could be the results of an impact from a foreign body, wave impacts, high wind velocities, engine vibrations etc. Consequently, the determination of the dynamic characteristics of these plates becomes important and a decision must be made as to whether the structural system is able to support such vibrations; unless the ratios $a/t_p$ and $b/t_p$ are small, where $a$, $b$ and $t_p$ are the width, breadth and thickness of the plate respectively, the unstiffened plates are unlikely to be suitable for such applications.

In addition to the dynamic problems of plates it is often necessary to minimise the maximum plate deflections without introducing weight penalty and this would be achieved by incorporating stiffeners within the plate. Two geometrical shapes of stiffeners are often used in GRP composite plates;
these are rectangular cross-section (blade) and top-hat stiffeners.

Minguez [123] has undertaken experimental work on the different behaviour of buckling of aluminium-alloy panels reinforced by stiffeners having either open or closed cross section, namely Z and 90° hat shaped stiffeners. The experimental results showed that, the shape of the stiffener having a closed cross-section may alter the buckling mode shape of the panel and this eventually resulted in a considerable increase of the critical buckling load.

Several experimental and analytical studies were conducted on a composite top-hat and a blade stiffened plate under uniaxial compression [124], [125], [126] and [127]. However, no attempt has been made to investigate the effects of the frequency and mode shape of these plates resulting from the alteration of the cross-sectional configurations of these stiffeners.

In the current investigation top-hat and rectangular cross-section stiffeners are incorporated into the plate and the whole vibrated at its four lowest modes. The mass and the material properties of both types of stiffeners were kept constant throughout. An analytical study of the effect of stiffener geometry on the vibrations of the stiffened plate was made. The reliability of the analytical procedure was verified by experimental analysis on rectangular stiffened plates using an impact hammer and roving accelerometer. The
comparisons have been discussed in section 6.4

5.1.2 THE STIFFENED AND UNSTIFFENED PLATES.

The initial lateral dimensions of the unstiffened plate and its thickness were first chosen on the basis of the small deflection thin plate theory; these are given in section 3.1.1 as (a) and (b).

In order to minimise the vibration amplitude of a plate with given mass and thickness, it is necessary to maximise the natural frequencies. This can be achieved by removing a uniform thickness of material and forming it into one or several stiffeners. The mode shape of the unstiffened plate will indicate the area of high amplitude where the newly formed stiffener should be positioned to reduce this amplitude.

The thickness $t_p$ of the stiffened plate of mass $m$ can be obtained by equating the volumes of stiffened and unstiffened plates shown in Figs.5-1a and 5-1b:

$$V = a(b.t_p + A_s) = a.b.t_0$$

Hence,

$$t_p = t_0 - A_s/b$$

where, $A_s$ is the area of the stiffener cross-section,

$$A_s = t_s.d,$$
\( t_0 \) is the thickness of the unstiffened plate,
\( b \) is the length of the plate perpendicular to the stiffener.

Equation (5.1.1b) can be written in the following form:

\[
\frac{t_0}{t_p} = 1 + \beta \quad \ldots \quad (5.1.1c)
\]

where, \( \beta = \frac{A_s}{b \cdot t_p} \)

The aims of the current work are to investigate the resonant frequency characteristics for stiffened and unstiffened composite plates whilst retaining the mass of the whole plate stiffener system constant. An optimization method was undertaken in order to find the thinnest possible plate with the most suitable stiffener. The latter should have an advantageous cross-section and minimum mass which would result in an increase in the dynamic structural efficiency of the plate. Top-hat and rectangular stiffeners with different cross-sectional configurations have been studied. The variables of the cross-sections were width and thickness of the stiffener while the mass of this latter was kept constant.

A fully clamped fibre/matrix composite plate was used to investigate the stiffener dimensions to enable it to remain undeformed during vibration. A parameter study of the quantity \( \beta = \frac{A_s}{b \cdot t_p} \) obtained from equation (5.1.1c) was made by comparing the variation of this quantity with the fundamental frequency. A parameter study on the plate was
undertaken to determine the most appropriate size of single stiffener; the aspect ratio of the plate was 1.25 where the largest length was 375mm. The plate was manufactured from three plies of chopped strand glass fibre mat each of weight 450gm/m² impregnated with polyester resin; the final thickness was (2.4 ± 0.1)mm.

5.1.3 NUMERICAL ANALYSIS

As the boundary conditions and stiffener effects on the composite plate are complex, the form of the deflection \( w \) is not known. However, an assumed form satisfying the boundary conditions and stiffener effect can be chosen as:

\[
w(x,y,t) = W(x,y) \cdot \Gamma(t) \quad \ldots \quad (5.1.2)
\]

where,

\( \Gamma(t) \) is assumed to be an harmonic time function,

\( W(x,y) \) represents the shape of the deflected middle surface of the vibrating plate; the latter satisfies the boundary conditions, stiffener size and its material mechanical properties.

The maximum deflection in an unstiffened plate at say, the fundamental mode of vibration, can be reduced to zero by the introduction of a rigid stiffener at that position and the new position of the maximum amplitudes of vibration will be displaced to locations on either side of the stiffener. The plate will then effectively be divided into two half plates,
the dimensions of each will be $bxa/2$ with zero deflection along the length of the stiffener. For such deflections the fundamental mode of vibration can be written in the form:

$$W(x, y) = \psi(x) [A_1 \phi_1(y) + A_2 \phi_2(y)] \quad \ldots \quad (5.1.3)$$

where,

$\psi_1$ is the deflected mode shape of the plate in X-direction and consists of one half-wave,

$\phi_1, \phi_2$ are the deflected mode shapes in Y-direction,

$A_1, A_2$ are the amplitude coefficients satisfying the boundary conditions, stiffener size and its material mechanical properties.

The fundamental frequencies of vibration of rectangular stiffened plates can be determined by applying the strain energy method and using the Rayleigh-Ritz procedure [20]:

$$\theta (V - \omega K) A_i \quad \ldots \quad (5.1.4)$$

where, $i=1,2$ in this case.

The amplitude coefficients $A_i$ can readily be obtained for the simply supported case where the mode shapes are known. However, the values of $A_i$ are difficult to determine in the case of the fully clamped stiffened plate.

To circumvent this problem a finite element technique was
used to analyse the plates and to perform eigenvalue extractions. The finite element package selected was the ABAQUS program. The procedure was discussed in section (3.2.5).

By using the finite element procedure the stiffened and unstiffened plates were divided longitudinally and transversally into a large number of elements (as was shown in Fig.3-2). A relationship between the ratio $\beta$ and the fundamental frequency of the stiffened plate was obtained and plotted in graphical form.

By considering the stiffener shown in Fig.5-1c, the moment of inertia about the middle plane of the plate is:

$$I_s = t_s \cdot d^3/12 + t_s \cdot d \cdot (d/2 + t_p/2)^2 \quad \ldots \quad (5.1.5a)$$

which becomes after some rearrangement,

$$I_s = t_s \left( d^3 + 1.5 t_p \cdot d^2 + 0.75 t_p^2 \cdot d \right) / 3 \quad \ldots \quad (5.1.5b)$$

It can be seen from equation (5.1.5b) that the moment of inertia $I_s$ is dependent on the plate thickness $t_p$, stiffener depth $d$ and stiffener width $t_s$.

In the current investigation, the plate thickness $t_p$ remains constant. Therefore, any increase or decrease in frequency value is believed to be due to the stiffener dimensions $t_s$ and $d$. 

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By introducing the following constant parameters:

\[ \psi = 1.5t_p \]
\[ \mu = 0.75t_p^2 \]

Equation (4.2.5b) can be written as:

\[ I_s(t_s, d) = t_s(d^3 + \psi \cdot d^2 + \mu \cdot d)/3 \ldots \quad (5.1.5c) \]

It can be seen from the above equation that the depth \( d \) has a much greater effect on \( I_s \) than the width \( t_s \) because of the high power degree.

The ABAQUS finite element program was again used to undertake a parameter study of the effect of cross-stiffening \( (\beta = A_s/b \cdot t_p) \) on the natural frequency of the plate. Fig. 5-2 shows the different sectioned stiffeners having the same mass, used in this study.

5.1.4 THE GEOMETRICAL CONFIGURATION OF THE STIFFENER

Having obtained the maximum cross-sectional area (rectangular cross-section) of the stiffener with respect to that of the plate geometry to give the maximum frequency of the stiffened plate (see Fig. 6-11), it will now be necessary to determine the most suitable and weight efficient geometric shape to gain the highest possible fundamental frequency of the stiffened plate. Clearly the increase in frequency by stiffening is directly related to the moment of inertia of
the stiffener about the centroidal axis of the plate. In the following calculations it has been shown that the centroidal axis is very close to the middle surface of the plate and, consequently, the moment of inertia will be taken about this middle surface. The error in this assumption is 3% for a rectangular stiffener of area 3x6mm$^2$ and plate thickness of 2.4mm (plate dimensions of 375mm by 300mm).

To reduce the computation, because of the large number of elements, the stiffeners were investigated independently of the plate with fixed boundary conditions in the X and Y directions as shown in Fig.5-3; this defines approximately the boundary conditions when the stiffener is attached to the plate. This figure also shows typical stiffeners with the imposed boundary conditions, vibrating in their fundamental mode. The results are given in Table 6-8.
5.2 PART B: VIBRATIONAL ANALYSES OF PLATES SUBJECTED TO IN-PLANE LOADS

5.2.1 INTRODUCTION

The following investigations concentrate upon the dynamic response of rectangular stiffened and unstiffened composite plates when subjected to static and zero in-plane compressive loads applied at the same time as the dynamic forces; the plates are assumed to be completely free along two parallel edges and fully fixed along the other edges (F-C-F-C). Fig.5-4 shows a plate example in hull girder which is subjected to in-plane load. Considerable vibration within the plate components of the superstructure can be set-up under working conditions and these may have a severe effect upon the plate stability.

Both a theoretical and an experimental analyses have been undertaken to calculate the critical buckling load and the natural frequencies of the systems. The former solution used a finite element method and the computation package employed was the ABAQUS program (discussed in section 3.2); this latter was based upon a small deflection elastic analysis. The experimental technique was performed using the experimental set-up described in section 4.2.3.

The aims of the current work are to investigate the resonant
frequency characteristics for stiffened and unstiffened composite plates. The unstiffened plates will be discussed first and the objects here are:

(a) to investigate the resonant frequency shift when the plate is under in-plane loading of various magnitudes up to first buckling load of the plate,

(b) to define the range of compressive loads by constructing curves of natural frequencies against different loading increments.

Thin plates which are incorporated into certain types of structures have to be stiffened when the breadth/thickness ratio of the plate exceeds certain limits. These limits are given in section 3.1.1, with respect to the basis of the small deflection thin plate theory. In some structures, the addition of stiffeners to the plate would introduce a weight penalty. Consequently, it would be advantageous to keep the total mass of the plate constant by taking a thin uniform layer of constant thickness from the plate and forming it into one or more stiffeners; this could lead to an increase in the natural frequencies and higher buckling loads whilst retaining the same mass of the system.

In the following sections the analytical and experimental techniques will be used to investigate (F-C-F-C) plates of GRP composite material having lateral dimensions of 300mm by 375mm. Similar dimensioned plates, with one central stiffener spanning between the edges, are also considered.
In addition, the effect of in-plane loadings on the stiffened and unstiffened plates will be included.
Each complete quasi-isotropic flat plate of aspect ratio 1.25 was manufactured from three chopped strand mats as discussed in section 4.2.4.

5.2.2 THE TESTING PROCEDURE

Fig.5-5a shows the test arrangement for each panel and Fig.5-5b shows a photograph of the set-up. It consists of a mechanism which provides a clamping system formed by a pair of metallic strips at the top and bottom edges of the GRP plates; top clamps were free to move vertically between two metal columns which had PTFE strips bonded to them to provide near frictionless movement and to prevent eccentricities from developing. A load distribution bar was provided along the top edge of the plate to enable a uniformly distributed load to be applied to it. The in-plane compressive load was applied through a lever system and the load was applied to the plate at one third points along the distribution bar; dead loads were applied at one end of the lever system.

The plates were subjected to a series of base load excitation spectra by means of an instrumented impact hammer. The lower modes of resonance were detected using an accelerometer strategically placed on the plates. The vibration responses were measured for six load increments. It should be noted that the compressive load values on the plate were only a
fraction of those predicted for critical buckling, so that
damage did not occur to the plate and non-linearity and
plastic deformation were not introduced into the problem.
The time domain force and acceleration signals from the
models were digitally sampled and stored in block sizes of
4096 points in the DL 1200 multi-channel digital recorder and
controlled by a Datalab ACQUIRE digital signal processing
software which was mounted on a HP-200 computer. The
sampling intervals used were selected to prevent signal
aliasing after capture, the signals were transferred to the
Acquire software for subsequent conditioning and fast fourier
transform (FFT) processing into the frequency domain, as
described in section 4.2.3.

The numerical analysis computed the natural frequencies of
the multi-degree of freedom of the plates. The procedure
followed was similar to the free vibration but with a load
in-plane at each node.
6. RESULTS AND DISCUSSIONS

6.1 DYNAMIC AND STATIC PROPERTIES.

Typical amplitude-time curve for the randomly orientated fibre composites with fibre volume fraction $V_f = 14\%$ is shown in Fig. 4-7. The amplitude is seen to be maximum at $t=0$ seconds and decaying with increasing time. This decay illustrates the damping of the considered composite beam at a specific frequency and a fixed fibre volume fraction.

Graphs of $V_f$-dependence on the dynamic storage modulus $E'$ for both categories of GRP composites are shown in Figs. 6-1a and 6-2a on linear scales. From the results of $E'$, it is seen that the dynamic modulus of elasticity is increasing when the fibre volume fraction increases. This increase varies in a linear form in the region where the content of glass fibre is kept constant while the fibre volume fraction is varied by changing the quantity of polyester resin used, which results in an increase of the thickness of the composite beam.

The variations of the damping ratio $\xi$ with fibre content are shown in Figs. 6-1b and 6-2b for the $(0^\circ,90^\circ)$ and $(R,R)$ categories, respectively. The results indicate that the relationship is non-linear with a noticeable fall in the region of high fibre
content and it is clearly observed that at low fibre volume fractions of the composite exhibits high damping factors.

By comparing the difference in the two curves at the same value of fibre volume fraction, \( V_f' \), it is noted that the dynamic storage modulus, \( E' \), for the fibre \((0^\circ,90^\circ)\) is higher than that with the fibre \((R,R)\). Similarly, the fibre \((0^\circ,90^\circ)\) gives slightly higher damping than the fibre \((R,R)\), this is more noticeable when the results are compared in the region around \( V_f'=15\% \) to \( V_f'=45\% \), where there is a high \( E' \) corresponding to the \((0^\circ,90^\circ)\) fibre. The first phenomenon, concerning \( E' \), appears to be due to the good reinforcement properties of the fibre in \((0^\circ,90^\circ)\) which gives high \( E' \). However, the second phenomenon, concerning the damping, is believed to be related to the polyester resin used to fabricate the beams; it is observed that this polymer has a significant damping effect when utilising the orthotropic \((0^\circ,90^\circ)\) composite than with the quasi-isotropic \((R,R)\) fibre composite. Consequently, the dynamic mechanical properties are highly depend upon the polyester resin used, fibre orientation, disposition of fibre and fibre length.

The critical parameters for the accurate measurement of \( E' \) are the sample dimensions, density and resonant frequency. The magnetic attraction exercised by the transducers on the sample (as described in section 4.1.5.3b) was taken into consideration and this was reduced to a minimum by either decreasing the size of the ferromagnetic masses mounted on
the sample or by increasing their distance from the transducers.

Since the damping is a dimensionless quantity, its measurement was not affected by errors in the sample dimensions. However, appreciable errors can originate from measurement of amplitude and decay time.

The error due to energy losses in the clamps was neglected in comparison with the internal losses in the material itself because $\xi>0.5\%$. More details about the energy losses in the clamps are given in reference [112].

6.2 EFFECT OF THE BOUNDARY CONDITIONS

The boundary conditions of a plate can vary depending upon the purpose of its use. The alteration of these boundaries, under a constant plate mass, has a large effect on the natural frequencies and the mode shapes.

Table 6-1 shows the effects on the first four natural frequencies of 2.4mm (R,R), composite plate. The fully clamped plate is seen to have the highest natural frequencies; it is three times higher than that for the completely free case. The combination of boundaries free and clamped is also seen to be about two times higher than the completely free one. It should be also pointed out that the natural frequencies of a plate having its four edges simply
supported are lower than if the same plate is fully clamped. Therefore, the greater the degree of fixity the higher will be the natural frequencies, which results in high stiffness of the system.

6.3 PLATES WITH RECTANGULAR STIFFENERS.

Before investigating the stiffened composite plates experimentally and theoretically a decision must be made on whether the stiffeners will be welded onto the plate utilising one of the two possible lay-up schemes mentioned below, to enable the stiffened plate to gain the highest possible natural frequency.

(a) the stiffener laminae are parallel to the plate middle plane, as shown in Fig.6-3a,

(b) the stiffener laminae are perpendicular to the plate middle plane, as shown in Fig.6-3b.

In order to develop an understanding of the stiffness and damping of these stacking arrangements, a square cross-section beam (6.7x6.7)mm$^2$ made of 12-ply ($0^\circ,90^\circ$) glass fibre, bonded together in a polyester resin was tested under vibration using the test arrangements described in section 4.1.5.4. This beam was tested in the X-X axis (laminae parallel to the X-X axis), and in the Z-Z axis (laminae perpendicular to the Z-Z axis).
Referring to equation (4.1.18) described in section 4.1.3.2, it can be seen that the stiffness of the square beam representing the stiffener is a function of the frequency only, whether this beam is under vibration along 1-1 or 2-2 axis, as shown in Fig.6-3c. Hence, in the present case, the beam stiffness can be written as:

\[ EI = \mu f_n^2 \]

where, \( \mu \) is a constant which depends on the material density, beam geometrical dimensions and boundary conditions;

\[ \mu = 4\pi^2 \rho l_1 h^2 / \kappa_n^2 \]

Experimental results show that the fundamental frequency of the beam, vibrating along the 1-1 axis is higher than if the same beam is vibrating along the 2-2 axis (see Fig.6-3c).

The damping factor can be obtained from the curves illustrated in Figs.6-3d and 6-3e. It is seen that in Fig.6-3d, the decay is logarithmic and the damping ratio can readily be obtained, if equation (4.1.25) is used. However, the damping curve shown in Fig.6-3e seems to be very complicated and decaying in a sinusoidal form.

For a given frequency the displacement of any nodal point on the plate from its mean position is equal to the product of the maximum displacement and the sine of the phase angle.
\[ w_i(\alpha) = W_i \sin \alpha \]

where,

- \( w_i(\alpha) \) is the displacement of the \( i \)-th nodal point,
- \( W_i \) is the maximum displacement of the \( i \)-th nodal point,
- \( \alpha \) is the phase angle of the \( i \)-th nodal point relative to the reference nodal point.

The lines joining the computed displacement \( w_i(\alpha) \) are plotted on a scale drawing of the plate to produce the mode shape when, for a specific resonant frequency, the displacement of each nodal point is maximum. The resultants of this product at each nodal point on the surface of the unstiffened and stiffened composite plates with various boundary conditions are presented in Figs. 6-4a & 6-4b, 6-5a & 6-5b, 6-6a & 6-6b, where "a" refers to the randomly orientated composite unstiffened and stiffened plates and "b" to the bidirectional ones.

It is shown that the natural frequencies of GRP rectangular plates can be modified by incorporating stiffeners. The mode shapes of the unstiffened plates will indicate the areas of high amplitude where stiffeners could be positioned to reduce the amplitude but they must be positioned away from nodal lines. The relationship between the stiffener and the plate geometrical dimensions and the stiffener mass value must be such as to minimise the maximum deflection (this will be discussed in section 6.4). The effect of the stiffener is
significant in increasing the frequency for fully clamped and clamped/free plates where only bending modes of vibration are present. However, for the plates in free/free conditions where both bending and torsional modes of vibration are present, the effect that the stiffeners have on the torsional modes is minimal.

The maximum deflection in unstiffened plates can be reduced to zero at that position by the introduction of stiff stiffeners and the maximum amplitude of vibration will be distributed on either side of the stiffener. The plate will then have the appearance of being divided into two half plates; these will have a common clamped edge along the length of the stiffener.

The mode shapes of the unstiffened plates are not dependent upon the plate mechanical properties and its thickness, but do depend upon the lateral dimensions of the plate and upon its boundary conditions. However, for stiffened plates the mode shape is dependent upon the mechanical and geometrical properties of the stiffener and upon the equivalent properties for the plate. This has been demonstrated in Figs.6-4a & 6-4b, 6-5a & 6-5b and 6-6a & 6-6b where it will be seen that, for fully clamped conditions and a composite plate thickness of 2.4 mm the frequency of the first mode of vibration of the plate with one stiffener has a value of 148 Hz and this is greater than the frequency of the first mode of the plate with three stiffeners which has a value of only
142 Hz. For the plate of 2.0 mm thickness the corresponding frequencies are 169.77 Hz and 193.12 Hz, respectively.

If three stiffeners are positioned in a rectangular plate it does not necessary imply that the fundamental frequency of this plate will be greater than that for the plate with a single stiffener; the boundary conditions must also be considered (see section 6.2). It can be seen that in Fig.6-5a, for the case of free/clamped rectangular composite plate of thickness 2.4 mm, in mode-1, there is an increase in frequency, but for fully clamped condition with three stiffeners there is a lower fundamental frequency than for the single stiffened plate. Also it can be seen that for the case of a rectangular plate with free/free boundary conditions, the frequencies for the three stiffeners for both thicknesses of plates are less than those for the single stiffener case.

Plate with high stiffness-to-weight ratio will have higher fundamental frequencies than those with lower ratios; this has been demonstrated with the 2.0mm bidirectional and 2.4mm randomly orientated fibre composites.

The solutions to the fundamental and the first three harmonics, and the percentage difference between analytical and experimental results for perspex plates are given in Tables 6-2, 6-3 and 6-4, for fully clamped, free/clamped and free/free boundary conditions, respectively. For the first
three modes the exact solution is lower than that obtained from the analytical one; in addition, the experimental solutions are lower than those from the exact ones. The shaker in the experimental technique adds an additional mass to the plate system; this was explained in section 4.2.3. This mass has not been considered in the theoretical or analytical solutions, consequently, this source of inaccuracy will reduce slightly the frequency value of the experimental result. Fully fixed conditions are difficult to realise in vibration and impact work and the difference between the analytical and experimental are no doubt from this.

The finite element and Gorman's frequency predictions [10] for the isotropic perspex unstiffened plates are compared in Tables 6-2, 6-3 and 6-4. It can be observed that there is a non-monotonic frequency sequence between both techniques; for some modes the natural frequencies are higher and for other modes are lower than the finite element values. The source of these variations is believed to be associated with the similarity between mode shape and the geometry of the finite element model. Downs [34] has studied this behaviour in some detail and has shown similar results.

The composite plates were investigated experimentally and theoretically using the same procedure as the perspex plates. The results for the various boundary conditions are tabulated in Tables 6-5, and 6-6. It can be seen from Tables 6-2 and 6.6a that the natural frequency variation of the different
techniques for the isotropic perspex plates were generally less than those of the composite GRP plates. It should be noted that the mechanical properties of the Perspex plates do not vary greatly. These mechanical properties were deduced from Graph E-1 and Graph E-2 (see Appendix E), given by the national physical laboratory (NPL) [11].

Figs. 6-7a and 6-7b show the variation of natural frequencies as \( E \) and \( v \) vary; consequently it will be seen that the variation in thickness of the composite plates, due to the method of manufacture, will lead to either a small increase or decrease in frequency.

6.4 RECTANGULAR AND TOP-HAT STIFFENERS.

The theoretical mode shape results, shown in Figs. 6-8, 6-9 and 6-10, for the 2.4mm \((R,R)\) plates with varying boundary conditions and reinforced by the addition of either one or three top-hat stiffeners of equal stiffness and variable mass were investigated theoretically only. These mode shapes are similar to those obtained for the plates reinforced by the addition of rectangular stiffeners (shown in Figs. 6-4a, 6-5a and 6-6a), except for the following cases:

(a) interchange between mode-3 and mode-4 for the fully clamped plate with one central stiffener,

(b) different mode shapes regarding mode-4 for the free/clamped plate with one central stiffener,
(c) interchange between mode-1 and mode-3 for the completely free plate with three stiffeners.

A rectangular stiffener is able to rotate with the amplitude of deflection of the plate and may have zero displacement along its length which results in becoming a nodal line during vibration. However, a top-hat is able to restrain the plate and does not suffer any twisting. It seems reasonable to assume, therefore, that the two stiffeners behave in different ways to prevent the plate undergoing vibration deflections, which depend on whether the cross-sectional area of the stiffener is rectangular or top-hat.

Taking the natural frequencies of the unstiffened plate as a reference point for the comparison between both types of stiffeners (see Tables 6-5a, 6-5b and 6-5c for the (R,R), rectangular stiffeners and Tables 6-7a, 6-7b and 6-7c for the top-hat stiffeners). It can be seen that the application of stiffeners having a closed cross-section (top-hat) on the plate skin can give a noticeable increase in the natural frequencies. This increase is believed to be due to the closed cross-section of stiffeners; this cross-section offers a torsional rigidity which is higher than the rectangular one.

The analytical and experimental analyses of the plates given in this section reinforce the fact that vibration behaviour of complicated systems must not be dependent upon one analysis procedure only. Comprehensive modelling of finite
element solutions from an experimental procedure should be undertaken before parameter studies are made.

6.5 TOP-HAT STIFFENER OF MINIMUM MASS AND MAXIMUM STIFFNESS

An economical solution to increasing the natural frequency of the plate by removing an uniform layer of material from the plate and forming it into a central stiffener has been discussed in section 4.2.2. This result could be achieved without introducing material degradation throughout the life of the structure whilst still maintaining a constant mass. Fig.6-11 illustrates, in graphical form, the relationship between the stiffening ratio $\beta \left(=\frac{A_s}{bt_p}\right)$ and the fundamental frequency $f_{11}$ obtained for a specific aspect ratio and a specific boundary condition (C-C-C-C in the present case).

It is seen from Fig.6-11 that for a value of $0<\beta<0.03686$, corresponding to plates (1-4), shown in Fig.6-12, the fundamental frequency is varying linearly and this indicates that the fundamental mode shape consists of a single half-wave in each direction. Beyond a ratio of $\beta>0.03686$, the relationship is non-linear and the stiffener will start to act as a nodal line, because of its high stiffness in comparison with the stiffness of the plate. By gradually increasing $\beta$ the condition for which the plate will be divided into two half plates, with a new boundary conditions along the length of the stiffener, can be achieved. This condition corresponds to the stiffener becoming a nodal line.
and remaining straight during vibration, as can be seen in Fig.6-12 (plates 6-10). It has been shown in Fig.6-11 that for a certain definite value of $\beta=0.1$ the fundamental frequency reaches its maximum, about 150 Hz, then decreases. This decrease in frequency is due to the plate cross-section $(b.t_s)$ being less than 10% of the cross-sectional area of the stiffener which makes it completely rigid. For $\beta>0.25$ the above described relationship is decreasing linearly with negative slope.

Referring to Fig.6-13, it is shown that in most cases the fundamental natural frequency with respect to the stiffener ratio $x=t_s/d$ does not follow a linear relationship. In the case of Hat-1 corresponding to $\theta=30^\circ$ ($\theta$ is the angle forming top-hat stiffener as indicated in Fig.5-2), it is easy to discern a significant maximum at about $x=0.78$ which gradually dies out by increasing $\theta$ to $90^\circ$. On the other hand, for all values of $30^\circ<\theta<90^\circ$, it is seen that the minimum and the maximum frequency occurs at the same value of $x$ corresponding to $x=1.05$ and $x=1.26$ respectively, but with a different frequency magnitude which increases when $\theta$ increases. Beyond certain value of $x>1.8$, the frequency function $f_{11}$ takes the form of an exponential and converges to the same frequency value when $x$ tends to 3.

For the stiffener type Hat-4 corresponding to $\theta=90^\circ$ with constant width $t_s$ and variable depth $d$, it can be seen that the fundamental frequency falls rapidly with increasing $x$ and
takes the form of an exponential curve. This decrease in frequency appears to be related to the stiffener depth \( (d) \), while the stiffener width was kept constant, and this was noticed in equation (4.2.5c) where the only variable that affects the frequency calculation was the depth \( d \).

For the stiffener type Hat-5, the value of the fundamental frequency remains almost constant, or perhaps even increases slightly in a small region of \( \chi \) with a positive slope and then remains constant beyond this region with a frequency value of 1113 Hz. This phenomenon is believed to be related to the variation of the stiffener width \( (t) \), while the stiffener depth \( (d) \) was kept constant and this does not affect the fundamental frequency greatly because the stiffness of the whole assembly was not increased significantly.

In the case of the stiffener with a rectangular cross-section, the variation of the fundamental frequency versus \( \chi \) shows a slight decrease for \( 0.5<\chi<1.22 \). No apparent variation was noticed for \( \chi>1.22 \) where the curve takes a constant frequency value of about 1056 Hz.

Table 6-8 illustrates a comparison in fundamental frequency between the individual stiffeners and the stiffened plates when these stiffeners are attached to them. For each type of stiffener with the same mass but variable \( \chi \) the frequency variation shows a similar trend. Therefore, the procedure of investigating the stiffener separately from the plate is
shown to be an acceptable analytical approach. In addition, by comparing the behaviour of the stiffener attached to the plate, vibrating in its first fundamental mode (shown in Fig.6-14) with that of the separate stiffener (shown in Fig.5-3) further demonstrates that the modes are similar.

An attempt has been made to find a stiffener cross-section that offers a maximum natural frequency of the stiffened plate without introducing a weight penalty. It has been shown that if the variation of the frequency, for a specific interval of \( \beta \) (0<\( \beta <0.03 \)) is linear, the fundamental mode shapes are symmetric, identical and always consisting of one half wave in both directions. If, however, \( \beta \) lies between 0.03 and 0.28 the relationship between it and the fundamental frequency is non-linear and the frequency function takes the form of a concave curve with a maximum at \( \beta =0.1 \). Beyond this maximum, the fundamental frequency function \( f_{11} \) falls and becomes at the value of \( \beta >0.28 \) linear with negative slope.

6.6 PLATES SUBJECTED TO IN-PLANE LOADS

6.6.1 UNSTIFFENED PLATES

The first seven natural frequencies derived from the theoretical and experimental analysis for the composite plate of aspect ratio of 1.25 are given in Table 6-9; the applied in-plane load varies between zero and 1500N/m, uniformly distributed. It can be seen that there is good agreement between the two methods of analysis with a 2% variation
between the two techniques. The introduction of the in-plane compressive loads have two main effects on the frequencies of the plate compared with those for the unloaded plates; these are:

(a) a reduction of natural frequencies,

(b) an interchange of higher mode shapes.

The former effect is well known; the latter effect is related to the number of nodal lines or half waves in the direction of loading. Figs. 6-15, 6-16 and 6-17 show the relationship between the square of the ratio \( \Gamma_{mn} = \frac{f_{mn}}{f_{11}} \) (harmonic frequency/fundamental frequency of unloaded plate) against in-plane load as a ratio \( \Delta = P/P_{cr} \) (uniform compressive load/critical buckling load) of the loaded plate for aspect ratios of 0.2, 1.25 and 3.75, respectively. Table 6-10 has been presented to demonstrate the interchange of mode shapes within specific limits of \( \Delta \); six limits have been defined in column 1 and the dominant mode shapes are given in the main body of this table. Fig. 6-18 shows the corresponding mode shapes and the change in the nodal lines. Fig. 6-19 to 6-24 illustrate the various mode shapes for the limiting values of \( \Delta \). These values are known to be the points at which the intersections between the different sets of curves, shown in Fig. 6-15, occur. Figs. 6-25 and 6-26 show the mode shapes for vibration (when \( P = 0 \)) and buckling (when \( f = 0 \)), respectively.

The relevant practical part of Fig. 6-15 lies between the values of \( \Delta \) of 0 to 1; this limit gives the fundamental and
harmonic frequencies and mode shapes of the plate which is loaded in the plane of the plate up to first buckling. The objective of investigating the region in which $\Delta$ is greater than unity was to provide a better understanding of the behaviour of the system within the practical range; clearly the values of mode shapes for $\Delta$ greater than unity have no practical value. However, if the plate was constrained from buckling in its $(m=1, n=1)$ mode by adding a knife edge support across its centre line, $\Delta$ could be increased beyond unity and the plate would eventually buckle in its $(m=1, n=2)$ mode as $f_{12}$ reduces to zero.

The natural frequencies and the critical buckling loads have been plotted against aspect ratios of the composite plates and are shown in Fig. 6-27 for the fundamental and first six modes. It can be seen that the relationship between fundamental frequency and aspect ratio up to 4 is approximately linear. The harmonic frequencies are not linear but decreases exponentially with increase in the aspect ratio $\phi$ ($=b/a$); they tend to become asymptotic to the fundamental frequency as $\phi$ tends to infinity. In the case of the buckling load against $\phi$ it can be seen that for low values of $\phi$ all critical buckling modes converge to approximately the same value.

6.6.2 STIFFENED PLATES
The vibrational characteristic and mode shapes of the
unstiffened plates can provide information regarding the most advantageous position for the location of stiffeners in the stiffened plates. To enable these plates to retain the constant mass of the unstiffened one a thin uniform layer of constant thickness was taken from the latter plate and formed into one or more stiffeners, (as described in section 5.1.2).

An analytical investigation was initially undertaken to find the stiffener cross-section which gave the maximum frequency and the maximum critical buckling load for the first mode. The parameters investigated were the ratio of the (fundamental frequency of the stiffened plate/fundamental frequency of unstiffened plate) \( \Gamma_1 \) and the ratio of (first critical buckling load of stiffened plate/first critical buckling load of unstiffened plate) \( \Delta_{crl} \) for a wide range of stiffening ratios \( \beta = A_x/b.t \); cross-section of rectangular stiffener/cross-section of plate perpendicular to stiffener).

Figs. 6-28 and 6-29 show the relationships between \( \Gamma_1 \) versus \( \beta \) and \( \Delta_{crl} \) versus \( \beta \), respectively for a series of 'one stiffener' plates with the same mass but with variation in plate thickness and stiffener sizes; the aspect ratio of the plate was equal to 1.25 and the aspect ratio of the rectangular cross-section of the stiffener \( (d/t_s) \) was equal to 2.0. For the values of \( \beta \) equal to zero, which corresponds to an unstiffened plate, it has been shown that both the shape of the fundamental vibration mode and that of the buckling mode are identical and that \( m \) equals 1 and \( n \) equals
1 in both lateral dimensions. As $\beta$ increases a situation develops where the central stiffener acts as a support thus dividing the plate into two halves with lateral dimensions \((a \times b/2)\) and imposing a zero deflection along the stiffener. With further increase in stiffener size and reduction in plate thickness the relationship between $\beta$ and frequency ratio $\Gamma_1$ and between buckling load ratio $\Delta_{cr1}$ approach a maximum and then reduce again with further increase in $\beta$. As $\beta$ increases in value to reach the maximum of the graph the influence of the stiffness of the stiffener to that of the plate is the criterion on which the natural frequency depends but as $\beta$ increases in value beyond these maxima, the critical component which is influencing the values of the buckling load and frequency is the reduction in the stiffness of the plate. It will be seen, in Fig. 6-28, that the maximum value of $\Gamma_1$ corresponds to a value of $\beta$ equal to 0.01; from Fig. 6-29 the maximum value of $\Delta_{cr1}$ is when $\beta$ is equal to 0.02.

Fig. 6-30 shows the relationship between $\Gamma_1$ and $\beta$ for a series of plates with varying thicknesses and different stiffener sizes; they are subjected to different static in-plane compressive load increments. The relationship is similar to that of Figs. 6-28 and 6-29. It is seen that considerable increase in the fundamental frequency mode, gained by stiffening the plate, reaches a maximum at the same value of $\beta$ obtained from Fig. 6-28, consequently the in-plane compressive load has no effect on the positions of these
maxima values.

The stiffener size can be computed from the curve $A_{cr1}$ against $\beta$, shown in Fig.6-29, where the maximum value of $A_{cr1}$ gave a value of $\beta$ equal to 0.02 which corresponds to a plate thickness of $t_p$ equal to 2.352mm and a cross sectional area of the stiffener equal to $3 \times 6 \text{mm}^2$, (the actual thickness of the unstiffened plate $t_p$ is equal to 2.4mm).

Fig.6-31 illustrates the dynamic behaviour of the above stiffened plate under different static in-plane load increments. It is seen that, generally, the behaviour is the same as the unstiffened one (c.f. Fig.6-15) and the intersection between the curves interchange the sequence of some of the mode shapes as a result of the increase in the in-plane compressive load.

Table 6-11 summarises the results of this latter study and shows the percentage shift in frequencies and critical buckling loads for the fundamental and the first six harmonics. The fundamental mode is increased significantly with higher modes showing a small shift with the exception of the vibration modes 3 and 7 and the buckling modes 5 and 6 which corresponds to shape C and shape G, respectively, of the unstiffened plate. A possible explanation for this result is that both mode shapes are similar (see Figs.6-25 and 6-26) and the stiffener is placed in the area of high stress, thus reducing the stress levels to a low value. In
some other modes the percentage shift is small, this result can also be directly related to the stiffener position, the latter could be a nodal line where the stiffener would have little effect.

The negative percentage shift in the natural frequencies, shown in Table 6-11a, could be due to the relative positioning of the stiffener with respect to the nodal line position or due to the buckling analysis curve ($\Lambda_{\text{cr}} \sim \beta$ shown in Fig.6-29) from which the stiffener size was computed.
7. CONCLUSIONS AND FUTURE RESEARCH

7.1 INTRODUCTION

Vibration and stability studies of fibre reinforced polymer stiffened and unstiffened plate structures were undertaken by experimental and analytical techniques. The former method was carried out by impact hammer and spectrum analysis and the latter method was undertaken using a standard finite element computer package for vibration and stability analyses. Attention was focused on the procedures that optimise the dynamic properties to provide reliable and efficient structural design, taking into account the mass-saving advantage. The maximum structural efficiency for these composite materials was achieved through an understanding of their structural behaviour as well as through a variety of factors that affect their performance. This understanding will enable the selection and design of new materials to be made which are capable of withstanding severe conditions of vibration and stability to which they may be exposed in engineering practice.

A systematic experimental investigation in conjunction with the latest theoretical and numerical procedures was undertaken to obtain a relationship between changes in
property and in dynamic factors in order to ensure that efficient composite plate structures meet demands for reduced costs, required weight targets and performance goals.

The main conclusions reached were that most of the potential increase in frequency was directly related to the increase in stiffness of the stiffener and its position on the plate structure. This was accomplished by reinforcing particular parts of the plate (areas of high amplitude) with one or more stiffeners. Therefore, predictions of excessive amplitudes within the plate structure were made initially to determine exactly in which regions the stiffeners should be introduced. To enable the stiffened plate to retain the constant mass of the unstiffened one, a thin uniform layer of constant thickness was taken from the plate and formed into one or more stiffeners.

7.2 FINITE ELEMENT CORRELATION

The correlation was based on comparing the results from the theoretical predictions and the measured data from a model test. This correlation has involved two major investigations; these were:

(a) the analysis of two of the modal parameters (that is the frequencies and the mode shapes) both of which were compared and the degree of variation was studied.
(b) the modifications to the structural configurations of the plate structure.

The difference between the analytical and experimental natural frequencies may be attributed to either one or a combination of the following causes:

(a) a misrepresentation of the true boundary conditions,

(b) incorrect material properties, due to thickness variation in the manufacturing of the composite materials technique,

(c) secondary mass loadings from the experimental technique originating from the mass of the accelerometers, the shaker attachment, etc.,

(d) the frequency dependence on the temperature variation,

(e) the modulus of elasticity and Poisson's ratio dependence on the range of frequency.

The natural frequency variations derived from the analytical and experimental techniques for the isotropic perspex plates were generally less than those of the GRP composite plates. The unstiffened plates manufactured from these two materials produced natural frequencies with less variation than those with one or three stiffeners.

In a composite material the fibres provide the stiffness and
hence the greater frequency whereas the matrix material provides a high degree of material damping.

7.3 BOUNDARY CONDITIONS EFFECT

The alteration of the boundary conditions from fully clamped, clamped/free to completely free were shown to have a large effect on the natural frequencies and mode shapes within a specific plate (the plates having similar geometrical and mechanical properties). The fundamental natural frequency of the fully clamped plate was seen to be approximately three times greater than that of the completely free and two times greater than that of the free/clamped ones.

7.4 STIFFENER AND ITS POSITION EFFECTS

The natural frequencies of rectangular plates can be modified by incorporating stiffeners. The vibrational characteristic and mode shapes of the unstiffened plates can provide information regarding the most advantageous position for the setting of the stiffeners in the stiffened plates, but they must be positioned away from nodal lines. The relationship between the stiffener and the plate geometrical dimensions and the stiffener mass value must be such as to minimise the maximum deflection. The effect of the stiffener is significant in increasing the frequency for fully clamped and clamped/free plates where only bending modes of vibration are
present. However, for the completely free plates where both bending and torsional modes of vibration are present, the effect that the stiffeners have on the torsional modes is minimal.

In the current investigation it has been shown that the lower mode shapes of the unstiffened plates (perspex, randomly orientated fibre and bidirectional composites) do not depend upon the plate mechanical properties and its thickness, but do depend upon the lateral dimensions of the plate and upon its boundary conditions. However, for stiffened plates the mode shape is dependent upon the mechanical and geometric properties of the stiffener and of the plate.

Design curves of fundamental frequency versus stiffener geometrical configuration have been presented for various types of top-hat and blade stiffeners, where the source for increasing frequency and reducing the vibration amplitude levels, for a specific frequency region, without introducing weight penalty was fully investigated. It has been shown that composite plates with open-section stiffeners such as blade stiffeners require a higher mass to reduce the amplitude of a specified vibration mode compared with the stiffened plates with top-hat stiffeners.

If three stiffeners are employed to stiffen a rectangular plate it does not necessarily imply that the fundamental
frequency of this plate will be greater than that for the single stiffener plate; the boundary conditions must also be considered. However, reinforcement of the GRP plates by stiffeners having high stiffness-to-weight ratio (i.e., CFRP) is an efficient solution for the vibration and stability analyses of these systems.

7.5 IN-PLANE FORCE EFFECT

Relationships between the first seven vibration modes and the in-plane compressive loads for rectangular plates under a combination of free and clamped edges (F-C-F-C) have been presented. It has been shown that the increase in magnitude of the in-plane load reduces the natural frequencies. This becomes zero when the in-plane load is equal to the critical buckling load of the plate. An interchange between some mode shapes occurs when the aspect ratio lies between 0.2 and 1.25; it is related to the change in the number of half-waves which are parallel to the load direction. Different characteristics were obtained for panels of aspect ratio 3.75 compared with those above and it was shown that the relationship between the square of the frequency ratio and the in-plane compressive load ratio are practically linear with no interchange between modes. Unstiffened and stiffened plates of the same aspect ratio show similar interchange characteristics between modes for the above relationship.
It has been shown that the natural frequencies and critical loads are proportional to the flexural rigidity of the plate and its thickness. The necessity of designing an efficient plate structure can be improved either by increasing the flexural rigidity or by increasing the thickness. If the latter alternative is adopted, the weight of the plate system is increased, and this is undesirable for the aerospace industry and to a lesser extent for the shipbuilding industry.

An economical solution for the former alternative is to introduce stiffeners in the longitudinal direction. The stiffener which is added, also increase the weight of the plate structure, but it has been shown that such increase in weight is generally much smaller than that introduced by increase in the thickness of the plate. The stiffener can also be placed transversally, but its effect on the buckling strength of the plate is negligible unless the number of stiffeners is large and these are so positioned that they are very close to each other.

7.6 PERSPEX AND COMPOSITES AS A MODELLING MATERIAL

In general, it has been found that the ideal candidate material for modelling plate structures should, for modal survey applications, possess both high specific stiffness and high strength characteristics. Such properties are well
pronounced in the fibre reinforced polymer materials and this has resulted in it being very suitable for use in the analyses of plate behaviour. Accordingly, significant economies can be achieved by using isotropic perspex in the fabrication of plates models.

Unfortunately, although both materials (GRP composite and perspex) possessed the required linear response characteristics, the dynamic behaviour (mode shapes) of the composite and perspex unstiffened plates were quite similar. However, the mode shapes for the stiffened plates were dissimilar in some modes.

The high correlation achieved between finite element and experimental analyses of both perspex and composite plate structures confirms that the perspex is indeed a suitable material for modelling composite plate structures.

7.7 SUMMARY

Possible methods and different techniques were involved to raise the natural frequencies of the various composite plates to a value above the frequency range where the spectrum of the excitation forces is minimum. It was seen that in the case of some plates in which only a small number of resonant frequencies are present within the range of interest, they can be rejected or eliminated from the range of excitation forces.
frequencies. Alternatively, they can be reduced if one or a combinations of the following are involved:

(a) minimization of the maximum structural deflection by reinforcement of transversal stiffeners,

(b) alteration of the boundary conditions (if this does not affect the design purpose),

(c) location of the excitation source, if possible, on a nodal line (a line of zero displacement),

(d) maximization of the first few fundamental frequencies by the choice of suitable dynamic stiffness-to-weight ratio of materials,

(e) increase of damping in the dominant modes throughout the application of viscoelastic damping technology.

No attempt has been made to investigate the temperature dependence on the dynamic modulus and loss factor of composite materials presented in this thesis. However, it should be pointed out that such dependence is far from negligible and could, during operation, displace the natural frequencies of the plate system into the range of those frequencies causing the resonance. This could produce considerable deformation modes to the plate which will, consequently, be unable to withstand the vibrational effects.

The analytical and experimental analyses of the plates reinforce the fact that vibration and stability behaviour of
composite plates must not depend upon one analysis procedure only. Comprehensive modelling of a finite element solution from an experimental procedure should be undertaken before parameter studies are made. Several theoretical solutions and experimental techniques have been undertaken in order to select the composite with the required properties and to define the desirable conditions for their static and dynamic processing and to ensure a safe plate which would be able to operate under the required range of forcing frequencies.

7.8 SUBJECTS FOR FUTURE WORK

In addition to the vibration problems, many other problems must be considered and examined. Among them are those of changes in material properties with elevated temperatures. This can lead to failure of the plate through other factors related to the variation of temperature. Therefore, investigations of the temperature as a function of space and time are required for the entire plate structure.

The principal undesirable effect of temperature on plate structures could be a result of the deterioration of structural stiffness or reduction of the value of the vibration frequencies. It should be noted that the presence of thermal stresses in a structural element, such as a plate, can produce thermal buckling and creep. The creep may be able to be ignored in small structures, but it must be taken
into account when designing a large plate structure where the composite materials may be more highly stressed. It should be pointed out that a simple reduction in stiffness due to temperature heating may again be cited as the principal effect on the natural frequencies. This effect may be large in dynamic response. Therefore, a great deal of attention must be focused on temperature and its effects on plates.

An extensive study must, therefore, be made of the properties and behaviour at high and low temperatures with a view to their eventual application as structural materials operating at the required range of temperature and frequency.

The plate components involved depend on the particular application and vary from one to another. Some of these plates are generally exposed to highly complex combinations of different factors, such as: transverse loads, in-plane loads, shear loads, impact stresses and temperature environment. It would be desirable to analyse and discuss separately and in combination each of the resulting factors as follow:

(a) investigate their effects and interaction on fatigue failures caused by near resonant vibrations, in attempt to select the optimum of the range of operation

(b) provide criteria for judging resonant fatigue strength which may play a very significant role in reducing the stress levels.
Studies of these phenomenon response and the rate of change of the physical properties of these composite materials must be conducted and will continue to be done to ensure safe working process.
REFERENCES


[47] Lee D.G., "Calculation of Natural Frequencies of Vibra-


[81] Olson, M.D., and Hazell, C.R., "Vibration Studies on Some Integral Rib Stiffened Panels" Journal of Sound and


[96] ABAQUS, Computer Program Developed by Hibbit, Karlsson and Solensen Inc. Province, Rhode Island, U.S.A.


TABLES
<table>
<thead>
<tr>
<th>Fibre Type</th>
<th>Density (g/cm³)</th>
<th>Flexural modulus (GN/m²)</th>
<th>Flexural strength (GN/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-glass Random mats</td>
<td>1.4-1.8</td>
<td>3.5-10</td>
<td>0.04-0.1</td>
</tr>
<tr>
<td>E-glass Woven</td>
<td>1.6-1.9</td>
<td>1.5-28</td>
<td>0.3-0.6</td>
</tr>
<tr>
<td>E-glass Unidir</td>
<td>1.8-2.2</td>
<td>14-31</td>
<td>1.0-1.5</td>
</tr>
<tr>
<td>Steels Monolithic</td>
<td>7.9</td>
<td>180-200</td>
<td>0.35-2.1</td>
</tr>
<tr>
<td>Alumin. Monolithic</td>
<td>2.7</td>
<td>60-80</td>
<td>0.1-0.43</td>
</tr>
</tbody>
</table>

Table 1-1 Mechanical properties of some typical engineering composite materials.
Table 4-1 Mechanical properties of GRP laminate beams under static conditions,
(a) chopped strand mat,
(b) woven glass cloth.
### Table 4-2 Full details of the specimens investigated under dynamic conditions.

<table>
<thead>
<tr>
<th>Specimen Code</th>
<th>Lay-Up</th>
<th>Number of Layers</th>
<th>Fibre Volume Fraction, Vf%</th>
<th>Fibre Type</th>
<th>Method of Manufacture</th>
</tr>
</thead>
<tbody>
<tr>
<td>RES-C045</td>
<td>Resin</td>
<td>0</td>
<td>0</td>
<td>—————</td>
<td>Hand Lay-Up</td>
</tr>
<tr>
<td>ROF3L1</td>
<td>(R,R)s</td>
<td>3</td>
<td>14</td>
<td>CSM-E</td>
<td>Hand Lay-Up</td>
</tr>
<tr>
<td>ROF3L2</td>
<td>(R,R)s</td>
<td>3</td>
<td>20</td>
<td>CSM-E</td>
<td>Hand Lay-Up</td>
</tr>
<tr>
<td>ROF3L3</td>
<td>(R,R)s</td>
<td>3</td>
<td>40</td>
<td>CSM-E</td>
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</tr>
<tr>
<td>OAF6L1</td>
<td>(0° 90°)</td>
<td>6</td>
<td>38</td>
<td>WSC-E</td>
<td>Hand Lay-Up</td>
</tr>
<tr>
<td>OAF6L2</td>
<td>(0° 90°)</td>
<td>6</td>
<td>58</td>
<td>WSC-E</td>
<td>Hand Lay-Up</td>
</tr>
<tr>
<td>OAF6L3</td>
<td>(0° 90°)</td>
<td>6</td>
<td>66</td>
<td>WSC-E</td>
<td>Hand Lay-Up</td>
</tr>
</tbody>
</table>
### Table 6-1 Effect of boundary conditions on the natural frequencies of unstiffened (R,R) composite plate.

<table>
<thead>
<tr>
<th>BOUNDARY CONDITIONS</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C-C-C-C</td>
<td>84.22 (85)</td>
<td>148.60 (151)</td>
<td>197.04 (178)</td>
<td>246.45 (236)</td>
</tr>
<tr>
<td>F-C-F-C</td>
<td>63.57 (65)</td>
<td>69.77 (67)</td>
<td>95.56 (92)</td>
<td>152.44 (147)</td>
</tr>
<tr>
<td>F-F-F-F</td>
<td>28.89 (31)</td>
<td>36.55 (37)</td>
<td>61.56 (64)</td>
<td>68.25 (73)</td>
</tr>
</tbody>
</table>

(••) Parenthesized values denote experimental natural frequencies
Table 6-2 Theoretical and experimental comparison of natural frequencies of various C-C-C-C Perspex plates.

<table>
<thead>
<tr>
<th>STIFFENER NUMBER</th>
<th>NODE NUMBER</th>
<th>METHODS</th>
<th>VARIATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>ZER0 STIFFENER</td>
<td>1</td>
<td>48.0</td>
<td>48.1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>84.4</td>
<td>84.8</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>110.1</td>
<td>112.5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>143.3</td>
<td>140.7</td>
</tr>
<tr>
<td>ONE STIFFENER</td>
<td>1</td>
<td>-</td>
<td>122.0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-</td>
<td>130.2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>172.5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-</td>
<td>174.6</td>
</tr>
<tr>
<td>THREE STIFFENERS</td>
<td>1</td>
<td>-</td>
<td>259.2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-</td>
<td>265.7</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-</td>
<td>276.8</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-</td>
<td>292.3</td>
</tr>
</tbody>
</table>

a: Exact Solution (Ref.[10]),
b: Finite Element Solution (Abaqus),
c: Experimental Result (FFT).
Table 6-3 Theoretical and experimental comparison of natural frequencies of various F-C-F-C Perspex plates.

<table>
<thead>
<tr>
<th>STIFFENER NUMBER</th>
<th>MODE NUMBER</th>
<th>METHODS</th>
<th>VARIATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exact</td>
<td>F.E</td>
</tr>
<tr>
<td>ZERO STIFFENER</td>
<td>1</td>
<td>35.6</td>
<td>36.3</td>
</tr>
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<td></td>
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<td>39.9</td>
<td>39.8</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>57.2</td>
<td>54.4</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>93.2</td>
<td>87.0</td>
</tr>
<tr>
<td>ONE STIFFENER</td>
<td>1</td>
<td>-</td>
<td>42.5</td>
</tr>
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<td></td>
<td>2</td>
<td>-</td>
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<td></td>
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<tr>
<td>THREE STIFFENERS</td>
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<td>-</td>
<td>69.9</td>
</tr>
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<td></td>
<td>2</td>
<td>-</td>
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<td></td>
<td>4</td>
<td>-</td>
<td>138.9</td>
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</table>

- a: Exact Solution (Ref.[10]),
- b: Finite Element Solution (Abaqus),
- c: Experimental Result (FFT).
Table 6-4 Theoretical and experimental comparison of natural frequencies of various F-F-F-F Perspex plates.

<table>
<thead>
<tr>
<th>STIFFENER NUMBER</th>
<th>MODE NUMBER</th>
<th>METHODS</th>
<th>VARIATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>a: Exact</td>
<td>b: F.E</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Solution</td>
<td>Solution</td>
</tr>
<tr>
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<td></td>
<td>(Ref.[10])</td>
<td>(Abaqus)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a: Exact</td>
<td>b: F.E</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Solution</td>
<td>Solution</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Ref.[10])</td>
<td>(Abaqus)</td>
</tr>
<tr>
<td></td>
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<td>a: Exact</td>
<td>b: F.E</td>
</tr>
<tr>
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<td>Solution</td>
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<td>(Ref.[10])</td>
<td>(Abaqus)</td>
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<td>a: Exact</td>
<td>b: F.E</td>
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<td>Solution</td>
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<tr>
<td></td>
<td></td>
<td>(Ref.[10])</td>
<td>(Abaqus)</td>
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</table>

a: Exact Solution (Ref.[10]),
b: Finite Element Solution (Abaqus),
c: Experimental Result (FFT).
<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st Freq.</td>
<td>2nd Freq.</td>
<td>3rd Freq.</td>
<td>4th Freq.</td>
<td></td>
</tr>
<tr>
<td>Zero stif.</td>
<td>0.00</td>
<td>84.22</td>
<td>148.60</td>
<td>197.04</td>
<td>246.45</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(85)</td>
<td>(151)</td>
<td>(178)</td>
<td>(236)</td>
<td></td>
</tr>
<tr>
<td>One stif.</td>
<td>4.33</td>
<td>148.00</td>
<td>155.57</td>
<td>257.75</td>
<td>268.82</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(133)</td>
<td>(139)</td>
<td>(260)</td>
<td>(270)</td>
<td></td>
</tr>
<tr>
<td>Three stif.</td>
<td>4.33</td>
<td>142.08</td>
<td>185.04</td>
<td>279.9</td>
<td>361.75</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(132)</td>
<td>(180)</td>
<td>(259)</td>
<td>(363)</td>
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</table>

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st Freq.</td>
<td>2nd Freq.</td>
<td>3rd Freq.</td>
<td>4th Freq.</td>
<td></td>
</tr>
<tr>
<td>Zero stif.</td>
<td>0.00</td>
<td>63.57</td>
<td>69.77</td>
<td>95.56</td>
<td>152.44</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(65)</td>
<td>(67)</td>
<td>(92)</td>
<td>(147)</td>
<td></td>
</tr>
<tr>
<td>One stif.</td>
<td>4.33</td>
<td>70.97</td>
<td>74.10</td>
<td>148.48</td>
<td>158.91</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(78)</td>
<td>(82)</td>
<td>(139)</td>
<td>(162)</td>
<td></td>
</tr>
<tr>
<td>Three stif.</td>
<td>4.33</td>
<td>90.49</td>
<td>90.81</td>
<td>140.48</td>
<td>182.32</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(67)</td>
<td>(94)</td>
<td>(135)</td>
<td>(193)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st Freq.</td>
<td>2nd Freq.</td>
<td>3rd Freq.</td>
<td>4th Freq.</td>
<td></td>
</tr>
<tr>
<td>Zero stif.</td>
<td>0.00</td>
<td>28.89</td>
<td>36.55</td>
<td>61.56</td>
<td>68.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(31)</td>
<td>(37)</td>
<td>(64)</td>
<td>(73)</td>
<td></td>
</tr>
<tr>
<td>One stif.</td>
<td>4.33</td>
<td>32.19</td>
<td>37.13</td>
<td>66.16</td>
<td>83.56</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(31)</td>
<td>(39)</td>
<td>(67)</td>
<td>(84)</td>
<td></td>
</tr>
<tr>
<td>Three stif.</td>
<td>4.33</td>
<td>29.10</td>
<td>38.43</td>
<td>69.15</td>
<td>101.45</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>(28)</td>
<td>(38)</td>
<td>(68)</td>
<td>(99)</td>
<td></td>
</tr>
</tbody>
</table>

( ) Parenthesized values denote experimental natural frequencies

Table 6-5 Theoretical and experimental comparison of natural frequencies of various (R,R) composite plates,
(a) fully clamped (C-C-C-C),
(b) free/clamped (F-C-F-C),
(c) completely free (F-F-F-F),
rectangular stiffeners.
<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td>0.00</td>
<td>88.68 (85)</td>
<td>155.83 (149)</td>
<td>203.31 (199)</td>
<td>264.79 (246)</td>
<td></td>
</tr>
<tr>
<td>One</td>
<td>4.33</td>
<td>169.77 (153)</td>
<td>179.23 (158)</td>
<td>277.77 (252)</td>
<td>288.21 (268)</td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>4.33</td>
<td>193.12 (189)</td>
<td>229.07 (235)</td>
<td>326.07 (310)</td>
<td>443.75 (463)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td>0.00</td>
<td>65.73 (63)</td>
<td>73.68 (78)</td>
<td>105.57 (103)</td>
<td>172.10 (155)</td>
<td></td>
</tr>
<tr>
<td>One</td>
<td>4.33</td>
<td>78.11 (77)</td>
<td>81.83 (82)</td>
<td>174.78 (182)</td>
<td>180.93 (183)</td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>4.33</td>
<td>107.48 (97)</td>
<td>108.11 (120)</td>
<td>191.24 (200)</td>
<td>224.97 (217)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
<th>1st Freq.</th>
<th>2nd Freq.</th>
<th>3rd Freq.</th>
<th>4th Freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td>0.00</td>
<td>31.20 (28)</td>
<td>39.82 (40)</td>
<td>66.41 (69)</td>
<td>75.51 (73)</td>
<td></td>
</tr>
<tr>
<td>One</td>
<td>4.33</td>
<td>39.60 (36)</td>
<td>39.63 (37)</td>
<td>75.54 (72)</td>
<td>96.72 (99)</td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>4.33</td>
<td>34.77 (32)</td>
<td>40.98 (44)</td>
<td>79.82 (80)</td>
<td>110.98 (113)</td>
<td></td>
</tr>
</tbody>
</table>

(..) Parenthesized values denote experimental natural frequencies.

Table 6-6: Theoretical and experimental comparison of natural frequencies of various (0°,90°)₅ composite plates,
(a) fully clamped (C-C-C-C),
(b) free/clamped (F-C-F-C),
(c) completely free (F-F-F-F),
rectangular stiffeners.
Table 6-7 First four theoretical natural frequencies of (R,R)<sub>2</sub> plate with one and three top-hat stiffeners,
(a) fully clamped (C-C-C-C),
(b) free/clamped (F-C-F-C),
(d) completely free (F-F-F-F).

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st Freq.</td>
</tr>
<tr>
<td>Zero stif.</td>
<td>0.00</td>
<td>84.22</td>
</tr>
<tr>
<td>One stif.</td>
<td>6.57</td>
<td>192.77</td>
</tr>
<tr>
<td>Three stif.</td>
<td>11.78</td>
<td>197.32</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st Freq.</td>
</tr>
<tr>
<td>Zero stif.</td>
<td>0.00</td>
<td>63.57</td>
</tr>
<tr>
<td>One stif.</td>
<td>6.57</td>
<td>79.60</td>
</tr>
<tr>
<td>Three stif.</td>
<td>11.78</td>
<td>133.17</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stif. No.</th>
<th>% Increase in material</th>
<th>NATURAL FREQUENCIES (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st Freq.</td>
</tr>
<tr>
<td>Zero stif.</td>
<td>0.00</td>
<td>28.89</td>
</tr>
<tr>
<td>One stif.</td>
<td>6.57</td>
<td>46.14</td>
</tr>
<tr>
<td>Three stif.</td>
<td>11.78</td>
<td>46.08</td>
</tr>
</tbody>
</table>
### Table 6-8 Frequency comparison between stiffener alone and attached to the plate.

<table>
<thead>
<tr>
<th>STIFFENER TYPE</th>
<th>FUNDAMENTAL FREQUENCY (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stiffener alone</td>
</tr>
<tr>
<td>Unstiffened plate</td>
<td>—</td>
</tr>
<tr>
<td>HAT-2</td>
<td>HAT2-1</td>
</tr>
<tr>
<td>$\phi=50$</td>
<td>HAT2-5</td>
</tr>
<tr>
<td>HAT-3</td>
<td>HAT3-1</td>
</tr>
<tr>
<td>$\phi=80$</td>
<td>HAT3-5</td>
</tr>
<tr>
<td>HAT-5</td>
<td>HAT5-1</td>
</tr>
<tr>
<td>$\phi=90$</td>
<td>HAT5-4</td>
</tr>
<tr>
<td>REC</td>
<td>REC 2.0</td>
</tr>
<tr>
<td>REC 5.0</td>
<td>1056.00</td>
</tr>
</tbody>
</table>
### Table 6-9 Comparison of theoretical and experimental natural frequencies of F-C-F-C composite plate subjected to in-plane compressive load.

<table>
<thead>
<tr>
<th>Applied load (N/m)</th>
<th>Natural frequencies (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>0</td>
<td>63.57</td>
</tr>
<tr>
<td>100</td>
<td>61.58</td>
</tr>
<tr>
<td>200</td>
<td>59.53</td>
</tr>
<tr>
<td>500</td>
<td>52.94</td>
</tr>
<tr>
<td>1000</td>
<td>38.98</td>
</tr>
<tr>
<td>1500</td>
<td>14.83</td>
</tr>
</tbody>
</table>

Note: Parenthesized values denote experimental natural frequencies.
<table>
<thead>
<tr>
<th>Dominant shape number</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
<th>5th mode</th>
<th>6th mode</th>
<th>7th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta &lt; 0.83125$</td>
<td>Shape A</td>
<td>Shape B</td>
<td>Shape C</td>
<td>Shape D</td>
<td>Shape E</td>
<td>Shape F</td>
<td>Shape G</td>
</tr>
<tr>
<td>$0.83125 &lt; \Delta &lt; 1.15625$</td>
<td>Shape A</td>
<td>Shape B</td>
<td>Shape C</td>
<td>Shape E</td>
<td>Shape D</td>
<td>Shape F</td>
<td>Shape G</td>
</tr>
<tr>
<td>$1.15625 &lt; \Delta &lt; 2.1625$</td>
<td>Shape A</td>
<td>Shape B</td>
<td>Shape C</td>
<td>Shape E</td>
<td>Shape F</td>
<td>Shape D</td>
<td>Shape G</td>
</tr>
<tr>
<td>$2.1625 &lt; \Delta &lt; 2.30$</td>
<td>Shape A</td>
<td>Shape B</td>
<td>Shape E</td>
<td>Shape C</td>
<td>Shape F</td>
<td>Shape D</td>
<td>Shape G</td>
</tr>
<tr>
<td>$2.30 &lt; \Delta &lt; 2.40$</td>
<td>Shape A</td>
<td>Shape B</td>
<td>Shape E</td>
<td>Shape C</td>
<td>Shape F</td>
<td>Shape G</td>
<td>Shape D</td>
</tr>
<tr>
<td>$2.40 &lt; \Delta &lt; 3.0$</td>
<td>Shape A</td>
<td>Shape B</td>
<td>Shape E</td>
<td>Shape F</td>
<td>Shape C</td>
<td>Shape G</td>
<td>Shape D</td>
</tr>
</tbody>
</table>

$\Delta$—in-plane compressive load ratio.

Table 6-10 Interchange phenomenon between the higher modes of F-C-F-C polyester/glass composite plate, subjected to in-plane compressive load, with aspect ratio $\phi=1.25$.  

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Table 6-11 Percentage shift due to stiffener effect for the F-C-F-C composite plate:
(a) natural frequencies,
(b) critical buckling.

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Natural frequencies (Hz)</th>
<th>% shift due to stiffener</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>before stiffening</td>
<td>after stiffening</td>
</tr>
<tr>
<td>1</td>
<td>63.57</td>
<td>68.61</td>
</tr>
<tr>
<td>2</td>
<td>69.77</td>
<td>71.20</td>
</tr>
<tr>
<td>3</td>
<td>95.56</td>
<td>124.7</td>
</tr>
<tr>
<td>4</td>
<td>152.4</td>
<td>150.4</td>
</tr>
<tr>
<td>5</td>
<td>179.9</td>
<td>183.2</td>
</tr>
<tr>
<td>6</td>
<td>186.6</td>
<td>184.8</td>
</tr>
<tr>
<td>7</td>
<td>212.2</td>
<td>251.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Critical buckling load (N/m)</th>
<th>% shift due to stiffener</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>before stiffening</td>
<td>after stiffening</td>
</tr>
<tr>
<td>1</td>
<td>1583.2</td>
<td>1971.2</td>
</tr>
<tr>
<td>2</td>
<td>2047.2</td>
<td>2130.9</td>
</tr>
<tr>
<td>3</td>
<td>3506.9</td>
<td>3903.4</td>
</tr>
<tr>
<td>4</td>
<td>3942.5</td>
<td>4001.9</td>
</tr>
<tr>
<td>5</td>
<td>4063.0</td>
<td>6514.7</td>
</tr>
<tr>
<td>6</td>
<td>5497.1</td>
<td>8338.6</td>
</tr>
<tr>
<td>7</td>
<td>7914.7</td>
<td>8338.6</td>
</tr>
</tbody>
</table>
Fig. 1-1 Application of plates in different fields of engineering.
Fig.1-2 Vibration-failure associated with resonance conditions.

Figure 1-3 Possible locations of vibration-cracks in a fully clamped rectangular plate.
Figure 1-4 Vibration data of input and output in the time and frequency domains.
Figure 3-1 Elements of a rectangular plate.
Figure 3-2 Division of unstiffened and stiffened plates into a large number of finite elements.
Figure 3-3 Rigid body modes of free/free stiffened plate.
(a) chopped strand mat,

(b) woven glass cloth.

Figure 4-1 Use of E-glass in GRP work.
Figure 4-2 Positions of $f_a$, $f_b$ and $f_1$ on peak.

Figure 4-3 Position of strain gauges.
Figure 4-4 Instron machine used for static test.
The Composite Elastic Modulus is $7.5$ GPa
The Standard Error of Estimate of Load is $1.75$ Kg
The Linear Correlation Coefficient is $0.99976$
The Poisson's Ratio is $2.64$

Figure 4-5a Typical load-strain curve for $(R,R)_x$ composite beam.

The Composite Elastic Modulus is $12.9$ GPa
The Standard Error of Estimate of Load is $6.45$ Kg
The Linear Correlation Coefficient is $0.99996$
The Poisson's Ratio is $0.16427$

Figure 4-5b Typical load-strain curve for $(0^0,90^0)_x$ composite beam.
Figure 4-6a Block diagram of typical measuring arrangement for clamped/clamped composite beam test.
Figure 4-6b The B&K flexural resonance apparatus and a typical output signal, recorded on a double beam oscilloscope.
Figure 4-7 Typical signal recorded during flexural tests in glass fibre composites.
Figure 4-8 Plates with different number of stiffeners.

- $t_s = 6.0$ mm and $d_s = 6.8$ mm for $(R,R)$ composites.
- $t_s = 6.0$ mm and $d_s = 7.0$ mm for $(0^\circ,90^\circ)$ composites.

Figure 4-9 Geometrical configuration and dimension of top-hat stiffeners reinforcing the plate.

(a) one central stiffener,
(b) three stiffeners.
Figure 4-10 Real signal generated by adding up sine waves.
Figure 4-11a Magnitude-time history analysis.

Figure 4-11b Spectral analysis.

Figure 4-12 Plate vibration system.
Figure 4-13a  Experiment apparatus of FFT frequency response.
Figure 4-13b Fast Fourier Transform equipment.
Figure 4-14a Shaker attachment (free/free condition).

Figure 4-14b The load cell mass which affects the input force.

Figure 4-14c Slender rod (stringer).
Figure 4-15 Three dimensional graph showing the relationship between frequency domain and modal domain.
Figure 4-17 FFT in logarithmic magnitude of the force-time input.
Figure 4-18 FFT of fully clamped (0°, 90°), GFRP composite plate.
Figure 4-19 Exploded view of plate-clamping apparatus.
Figure 5-1 Unstiffened and stiffened plates of equal masses.
In all cases the cross-sectional area, \( A_5 = 14.0 \text{mm}^2 \).

Figure 5-2 Cross-sectional areas of various stiffeners having same mass and same material properties with variable configurations.
Figure 5-3 Division of stiffeners into finite elements associated with the first mode shapes of vibration.
Figure 5-4 Plate example in hull girder.
Figure 5-5a Plate loading device.
Figure 6-1a Variation of dynamic modulus, $E'$, with fibre volume fraction, $V_f$, of (R,R) composite beams in flexure.

Figure 6-1b Variation of damping factor, $\xi$, with fibre volume fraction, $V_f$, of (R,R) composite beams in flexure.
Figure 6-2a Variation of dynamic modulus, $E'$, with fibre volume fraction, $V_f$, of $(0^0, 90^0)_s$ composite beams in flexure.

Figure 6-2b Variation of damping factor, $\xi$, with fibre volume fraction, $V_f$, of $(0^0, 90^0)_s$ composite beams in flexure.
Figure 6-3 Possible lay-up schemes for the stiffener and the difference in damping decay.
Figure 6-4a First four theoretical mode shapes of fully clamped rectangular composite plates with different numbers of transverse stiffeners. Lay-up 3 plies ($R,R$), corresponding to 2.4 mm thickness of the plate.
<table>
<thead>
<tr>
<th></th>
<th>1-st mode</th>
<th>2-nd mode</th>
<th>3-rd mode</th>
<th>4-th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0) Stiffener</td>
<td>f=68.68 Hz</td>
<td>f=155.83 Hz</td>
<td>f=203.31 Hz</td>
<td>f=264.79 Hz</td>
</tr>
<tr>
<td>(1) Stiffener</td>
<td>f=49.77 Hz</td>
<td>f=179.77 Hz</td>
<td>f=277.77 Hz</td>
<td>f=280.21 Hz</td>
</tr>
<tr>
<td>(3) Stiffeners</td>
<td>f=123.12 Hz</td>
<td>f=229.69 Hz</td>
<td>f=326.07 Hz</td>
<td>f=443.75 Hz</td>
</tr>
</tbody>
</table>

Dashed lines: Original mesh  
Solid lines: Displaced mesh

Figure 6-4b First four theoretical mode shapes of fully clamped rectangular composite plates with different numbers of transverse stiffeners. Lay-up 6 plies $(0^\circ, 90^\circ)_s$ corresponding to 2.0 mm thickness of the plate.
Figure 6-5a First four theoretical mode shapes of free/clamped rectangular composite plates with different numbers of transverse stiffeners. Lay-up 3 plies $(R,R)$ corresponding to 2.4 mm thickness of the plate.
<table>
<thead>
<tr>
<th>Stiffener</th>
<th>1-st mode</th>
<th>2-nd mode</th>
<th>3-rd mode</th>
<th>4-th mode</th>
</tr>
</thead>
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<td>f=73.68 Hz</td>
<td>f=105.57 Hz</td>
<td>f=172.09 Hz</td>
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<tr>
<td>1</td>
<td>f=78.11 Hz</td>
<td>f=91.83 Hz</td>
<td>f=174.78 Hz</td>
<td>f=189.93 Hz</td>
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<tr>
<td>2</td>
<td>f=107.48 Hz</td>
<td>f=109.11 Hz</td>
<td>f=151.24 Hz</td>
<td>f=224.97 Hz</td>
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</tbody>
</table>

Dashed lines: Original mesh
Solid lines: Displaced mesh

Figure 6-5b First four theoretical mode shapes of free/clamped rectangular composite plates with different numbers of transverse stiffeners. Lay-up 6 plies (0°,90°) corresponding to 2.0 mm thickness.
Figure 6-6a First four theoretical mode shapes of completely free rectangular composite plates with different numbers of transverse stiffeners. Lay-up 3 plies (R,R), corresponding to 2.4 mm thickness.
<table>
<thead>
<tr>
<th>Stiffener</th>
<th>1-st mode</th>
<th>2-nd mode</th>
<th>3-rd mode</th>
<th>4-th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>f=31.20 Hz</td>
<td>f=39.62 Hz</td>
<td>f=68.41 Hz</td>
<td>f=75.31 Hz</td>
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<tr>
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<td>f=39.60 Hz</td>
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<td>f=75.54 Hz</td>
<td>f=98.72 Hz</td>
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<tr>
<td>2</td>
<td>f=34.77 Hz</td>
<td>f=40.98 Hz</td>
<td>f=79.82 Hz</td>
<td>f=110.98 Hz</td>
</tr>
</tbody>
</table>

Dashed lines: Original mesh  
Solid lines: Displaced mesh

Figure 6.6b First four theoretical mode shapes of completely free rectangular composite plates with different numbers of transverse stiffeners. Lay-up 6 plies $(0^0, 90^0)$ corresponding to 2.0 mm thickness.
Figure 6-7a Variation of natural frequencies with elastic modulus, $E$.

Figure 6-7b Variation of natural frequencies with Poisson's ratio, $v$. 
Figure 6-8 First four theoretical mode shapes of fully clamped rectangular composite plates with one and three top-hat stiffeners. Lay-up 3 plies (R,R), corresponding to 2.4 mm thickness of the plate.
<table>
<thead>
<tr>
<th></th>
<th>1-st mode</th>
<th>2-nd mode</th>
<th>3-rd mode</th>
<th>4-th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0 Stiffeners</strong></td>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
<td><img src="image4" alt="Image" /></td>
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<tr>
<td></td>
<td>f= 63.57 Hz</td>
<td>f= 69.77 Hz</td>
<td>f= 95.56 Hz</td>
<td>f= 152.44 Hz</td>
</tr>
<tr>
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<tr>
<td></td>
<td>f= 79.60 Hz</td>
<td>f= 81.32 Hz</td>
<td>f= 193.13 Hz</td>
<td>f= 195.40 Hz</td>
</tr>
<tr>
<td><strong>3 Stiffeners</strong></td>
<td><img src="image9" alt="Image" /></td>
<td><img src="image10" alt="Image" /></td>
<td><img src="image11" alt="Image" /></td>
<td><img src="image12" alt="Image" /></td>
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<tr>
<td></td>
<td>f= 133.17 Hz</td>
<td>f= 133.55 Hz</td>
<td>f= 198.71 Hz</td>
<td>f= 248.58 Hz</td>
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</tbody>
</table>

Figure 6-9 First four theoretical mode shapes of free/clamped rectangular composite plates with one and three top-hat stiffeners. Lay-up 3 plies (R,R), corresponding to 2.4 mm thickness of the plate.

Dashed lines: Original mesh
Solid lines: Displaced mesh
Figure 6-10 First four theoretical mode shapes of completely free rectangular composite plates with one and three top-hat stiffeners. Lay-up 3 plies (R,R), corresponding to 2.4 mm thickness of the plate.
Figure 6-11 Relationship between stiffening ratio and fundamental frequency for a fully clamped glass/polyester composite plate reinforced by one central stiffener.
<table>
<thead>
<tr>
<th>Plate number</th>
<th>Plate thickness ( t_p ) ((\text{mm}))</th>
<th>Stiffener cross-section ( A_s ) ( (\text{mm}^2))</th>
<th>Fundamental frequency ( (\text{Hz}))</th>
<th>First fundamental mode shape</th>
</tr>
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<td>0 x 0</td>
<td>84.22</td>
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</tr>
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<td>2</td>
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<td>1 x 2</td>
<td>87.15</td>
<td></td>
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<td>2.37867</td>
<td>2 x 4</td>
<td>97.62</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.352</td>
<td>3 x 6</td>
<td>115.54</td>
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<td>5</td>
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<td>6</td>
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<td>147.74</td>
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</tr>
<tr>
<td>7</td>
<td>2.14</td>
<td>7 x 14</td>
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<td>8</td>
<td>2.0587</td>
<td>8 x 16</td>
<td>146.78</td>
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<td>9 x 18</td>
<td>143.59</td>
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</tr>
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<td>10</td>
<td>1.8667</td>
<td>10 x 20</td>
<td>138.54</td>
<td></td>
</tr>
</tbody>
</table>

Figure 6-12 Effect of stiffener size on the fundamental mode shape of fully clamped glass polyester composite plate.
Figure 6-13 Relationship between stiffener geometrical configuration and fundamental frequency.
Figure 6-14 Fundamental mode shape of fully clamped plate with one central top-hat stiffener.
Figure 6-15 Square of frequency ratio $\Gamma = (f/f_{01})$, versus in-plane compressive load ratio $\Delta = (P/P_{cr1})$, for F-C-F-C polyester/glass (R, R) composite plate, with aspect ratio $\phi = 1.25$. 
Figure 6-16 Square of frequency ratio $\Gamma = (\bar{f} / \bar{f}_{01})$, versus in-plane compressive load ratio $\Delta = (P / P_{cr1})$, for F-C-F-C polyester/glass (R,R) composite plate, with aspect ratio $\phi = 0.2$. 
Figure 6-17 Square of frequency ratio $\Gamma = (\tilde{f}/f_{c1})^2$, versus in-plane compressive load ratio $\Delta = (P/P_{cr1})$, for F-C-F-C polyester/glass (R,R) composite plate, with aspect ratio $\phi = 3.75$. 
Figure 6-18 Nodal lines for F-C-F-C polyester/glass unstiffened (R,R) composite plate.
Figure 6-19 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R), subjected to in-plane compressive load, with $0<\Delta<0.83$ and $\phi=1.25$. 
Figure 6-20 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R)$_s$, subjected to in-plane compressive load, with 0.83$\leq$\(\Delta\)\leq$1.156 and $\phi$=1.25.
Figure 6-21 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R), subjected to in-plane compressive load, with $1.156 < \Delta < 2.16$ and $\phi = 1.25$. 
Figure 6-22 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R), subjected to in-plane compressive load, with 2.16<Δ<2.30 and φ=1.25.
Figure 6-23 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R), subjected to in-plane compressive load, with $2.30<\Delta<2.40$ and $\phi=1.25$. 
Figure 6-24 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R)
subjected to in-plane compressive load, with
2.40<Δ<3.0 and φ=1.25.
Figure 6-25 First seven theoretical mode shapes of vibration of F-C-F-C polyester/glass composite plate type (R,R).
Figure 6-26 First seven theoretical mode shapes of buckling of F-C-F-C polyester/glass composite plate type (R,R).
Figure 6-27 Variation of the first seven theoretical: (a) natural frequencies, (b) critical buckling loads, versus aspect ratio $\phi$ for F-C-F-C polyester/glass composite plate, when $P=0$ and $f=0$ respectively.
Figure 6-28 Relationship between stiffening ratio, \( \beta = (A_s / b.t_p) \), and fundamental frequency ratio \( \Gamma_1 = (f_1 / f_{01}) \), for F-C-F-C polyester/glass (R,R) composite plate when \( P = 0 \).
Figure 6-29 Relationship between stiffening ratio, $\beta = (A_s / b.t_p)$, and in-plane compressive critical load ratio, $\Delta_{cr1} = (P_{cr1} / P_{cr01})$, for $F$-$C$-$F$-$C$ polyester/glass $(R,R)_s$ composite plate when $f=0$. 

\[ \Delta_{cr1} = \left( \frac{P_{cr1}}{P_{cr01}} \right) \]
Figure 6-30 Relationship between stiffening ratio, $\beta=(A_s/b.t_p)$, and fundamental frequency ratio $\Gamma_1=(f_1/f_{o1})$, for F-C-F-C polyester/glass (R,R) composite plate, under different in-plane compressive load increment.
Figure 6-31 Square of frequency ratio $\Gamma = \left( f / f_{g1} \right)^2$, versus in-plane compressive load ratio, $\Delta = (P / P_{cr1})$, for F-C-F-C polyester/glass $(R, R)_z$ composite plate, reinforced by the addition of central stiffener, $\phi = 1.25$. 
APPENDIX A

STRAIN, STRESS AND MOMENT IN AN ELEMENT OF THE PLATE

A.1 Strains in an element of the plate

The strains in a thin layer of element can be obtained from a simple geometrical consideration and will be represented by the following equations:

\[ e_x = \frac{\partial u}{\partial x} = -z \sigma_x^2 w / \partial x^2 \quad \ldots \quad \text{A.1} \]

\[ e_y = \frac{\partial v}{\partial y} = -z \sigma_y^2 w / \partial y^2 \quad \ldots \quad \text{A.2} \]

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2 z \sigma_x^2 w / \partial x \partial y \quad \ldots \quad \text{A.3} \]

where,

\[ u = -z \partial w / \partial x \]

\[ v = -z \partial w / \partial y \]

A.2 Stresses in an element of the plate

By elastic theory, the stresses corresponding to these strains are determined by the known relationships,

\[ \sigma_x = -z E_x (\sigma_x^2 w / \partial x^2 + \nu_x \sigma_y^2 w / \partial y^2) / (1 - \nu_x \nu_y) \quad \ldots \quad \text{A.4} \]

\[ \sigma_y = -z E_y (\sigma_y^2 w / \partial y^2 + \nu_y \sigma_x^2 w / \partial x^2) / (1 - \nu_x \nu_y) \quad \ldots \quad \text{A.5} \]

\[ \tau_{xy} = -2 z G_{xy} \partial^2 w / \partial x \partial y \quad \ldots \quad \text{A.6} \]
A.3 Moments in an element of the plate

The stress components $\tau_x$ and $\tau_y$ produce bending moments in the plane element. Thus by integration of normal stress, the bending moments by unit of the length are obtained:

$$M_x = \int_{-t_p/2}^{t_p/2} z \sigma_x \, dz = -E \frac{t^3_p}{x_p} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) / 12 (1 - \nu \nu) \ldots \quad A.7$$

$$M_y = \int_{-t_p/2}^{t_p/2} z \sigma_y \, dz = -E \frac{t^3_p}{y_p} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) / 12 (1 - \nu \nu) \ldots \quad A.8$$

Similarly, the twisting moments produced by shear stress $\tau = \tau_{xy} = \tau_{yx}$ can be calculated by integration,

$$M = M_{xy} = M_{yx} = \int_{-t_p/2}^{t_p/2} z \tau \, dz = -G \frac{t^3}{x_p} \frac{\partial^2 w}{\partial x \partial y} \ldots \quad A.9$$

denoting

$$D_x = E \frac{t^3}{x_p} / 12 (1 - \nu x \nu), \quad D_y = E \frac{t^3}{y_p} / 12 (1 - \nu y \nu)$$

$$D_{xy} = G \frac{t^3}{x_p y_p} / 12 \ldots \quad A.10$$

and,

$$D_{yx} = G \frac{t^3}{x_p y_p} / 12$$

the following expressions can be obtained,

$$M_x = -D_x \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \ldots \quad A.11$$

$$M_y = -D_y \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \ldots \quad A.12$$

$$M_{xy} = -2D_x D_y \frac{\partial^2 w}{\partial x \partial y} \ldots \quad A.13$$

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APPENDIX B

ENERGY METHOD IN STABILITY ANALYSIS

The potential energy accumulated in the thin layer of the element during the deformation is given by the well known expression:

\[ dU = \left( \varepsilon_x \sigma_x / 2 + \varepsilon_y \sigma_y / 2 + \gamma_{xy} \tau_{xy} / 2 \right) dx dy dz \quad \ldots \quad B.1 \]

or by using the equations (A.1), (A.2), (A.3), (A.4), (A.5) and (A.6) for the case of an isotropic plate yields,

\[ dU = \frac{E}{2(1-\nu^2)} \left[ \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + \left( \frac{\partial^2 W}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + 2(1-\nu) \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 \right] dx dy dz \quad \ldots \quad B.2 \]

From which by integration, the potential energy can be obtained,

\[ U = \int_0^a \int_0^b \int_0^t \, dU \quad \ldots \quad B.3 \]

\[ U = \frac{D}{2} \int_0^a \int_0^b \left[ \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + \left( \frac{\partial^2 W}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + 2(1-\nu) \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 \right] dx dy \quad \ldots \quad B.4 \]

The kinetic energy \( T \) of the buckled plate is:

\[ T = \frac{P}{2} \int_0^a \int_0^b \left( \frac{\partial W}{\partial x} \right)^2 dx dy \quad \ldots \quad B.5 \]

where, \( P \, dy \) is the load on an element.
From here, it is known that the critical value of edge load, \( P_{cr} \), occurring when \( U = T \). Consequently, by equating (B.4) and (B.5), \( P_{cr} \) can be obtained in a difficult form.

If the following important parameters are now introduced

\[
K = \frac{P}{D} \left( \frac{b}{\pi} \right)^2 \quad \text{.... B.6}
\]

\[
\phi = \frac{b}{a} \quad \text{.... B.7}
\]

Therefore, there must exist a relationship between \( k \) and \( \phi \) which could be directly written as:

\[
K = f(\phi) \quad \text{.... B.8}
\]

\[
dk/d\phi = 0 \quad \text{.... B.9}
\]

Equation (b.4) yields \( k_{\text{min}} \), which corresponds to the first critical buckling load.
APPENDIX C

GOVERNING DIFFERENTIAL EQUATIONS FOR STIFFENED PLATES.

C.1 STIFFENERS SYMMETRIC RELATIVE TO THE MIDDLE PLANE

C.1.1 Free undamped vibration

The stiffeners are assumed to be symmetrical relative to the plate middle plane, as shown in Fig. ; i.e., the influence of stiffener eccentricity is not considered. The differential equation of isotropic stiffened plate is given by:

\[ D v^4 w + \sum_{k=1}^{n_x} \delta( y - b_k) \left[ EI_{x_k} \frac{\partial^4 w}{\partial x^4} \right] + \sum_{k=1}^{n_y} \delta( x - a_k) \left[ EI_{y_k} \frac{\partial^4 w}{\partial y^4} \right] + \\
+ \left[ \bar{\rho} + \sum_{k=1}^{n_x} \delta( y - b_k) \bar{\rho}_{x_k} + \sum_{k=1}^{n_y} \delta( x - a_k) \bar{\rho}_{y_k} \right] w = 0 \tag{C.1} \]

C.1.2 Stability analysis

The differential equation of the buckled stiffened plate loaded in X and Y directions is:

\[ D v^4 w + \sum_{k=1}^{n_x} \delta( y - b_k) EI_{x_k} \frac{\partial^4 w}{\partial x^4} + \sum_{k=1}^{n_y} \delta( x - a_k) EI_{y_k} \frac{\partial^4 w}{\partial y^4} + \\
+ P x \frac{\partial^2 w}{\partial x^2} + P y \frac{\partial^2 w}{\partial y^2} + \sum_{k=1}^{n_x} \delta( y - b_k) P_{x_k} \frac{\partial^2 w}{\partial x^2} + \\
+ \sum_{k=1}^{n_y} \delta( x - a_k) P_{y_k} \frac{\partial^2 w}{\partial y^2} = 0 \tag{C.2} \]
C.1.3 Vibration with in-plane load

The differential equation is given by:

\[ D V^4 w + \sum_{k=1}^{n_x} \delta(y - b_k) \frac{\partial^4 w}{\partial x^4} + \sum_{k=1}^{n_y} \delta(x - a_k) \frac{\partial^4 w}{\partial y^4} + \\
+ w \left[ \bar{\rho} + \sum_{k=1}^{n_x} \delta(y - b_k) \bar{\rho}_{x_k} + \sum_{k=1}^{n_y} \delta(x - a_k) \rho_{y_k} \right] + p_x \frac{\partial^2 w}{\partial x^2} + \\
+ p_y \frac{\partial^2 w}{\partial y^2} + \sum_{k=1}^{n_x} \delta(y - b_k) p_{x_k} \frac{\partial^2 w}{\partial x^2} + \sum_{k=1}^{n_y} \delta(x - a_k) p_{y_k} \frac{\partial^2 w}{\partial y^2} = 0 \]

.... C.3

C.2 ECCENTRIC STIFFENERS

The governing differential equations for the eccentrically stiffened rectangular isotropic plate shown in Fig... can be obtained as:

\[ D V^4 w + \sum_{k=1}^{n_x} \delta(y - b_k) \left[ -ES_{x_k} \frac{\partial^3 u}{\partial x^3} + EI_{x_k} \frac{\partial^4 w}{\partial x^4} + GJ_{x_k} \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] + \\
+ \sum_{k=1}^{n_x} \delta(x - a_k) \left[ -ES_{y_k} \frac{\partial^3 v}{\partial y^3} + EI_{y_k} \frac{\partial^4 w}{\partial y^4} + GJ_{y_k} \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] + \\
+ w \left\{ \bar{\rho} + \sum_{k=1}^{n_x} \delta(y - b_k) \left[ \bar{\rho}_{x_k} + \rho J_{x_k} \frac{\partial^2 w}{\partial y^2} \right] + \\
+ \sum_{k=1}^{n_y} \delta(x - a_k) \left[ \bar{\rho}_{y_k} + \rho J_{x_k} \frac{\partial^2 w}{\partial x^2} \right] \right\} \]

.... C.4

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Assuming harmonic vibration,

\[ u(x,y,t) = U(x,y)e^{i\omega t} \quad \text{and} \quad v(x,y,t) = V(x,y)e^{i\omega t} \]

and using equation (2.1.13) yields:

\[
\begin{align*}
D^2 W &+ \sum_{k=1}^{n} \delta(y - b_k) \left( -ES_{x_k} \frac{\partial^3 U}{\partial x^3} + EI_{x_k} \frac{\partial^4 W}{\partial x^4} + GJ_{x_k} \frac{\partial^4 W}{\partial x^2 \partial y^2} - \frac{1}{\omega^2} \left( \rho W + \sum_{k=1}^{n} \delta(y - b_k) \left[ \frac{\partial W}{\partial x} - \rho J_{x_k} \frac{\partial^2 W}{\partial x^2} \right] \right) \right) \\
&+ \sum_{k=1}^{n} \delta(x - a_k) \left( -ES_{y_k} \frac{\partial^3 V}{\partial y^3} + EI_{y_k} \frac{\partial^4 W}{\partial y^4} + GJ_{y_k} \frac{\partial^4 W}{\partial x^2 \partial y^2} - \frac{1}{\omega^2} \left( \rho W + \sum_{k=1}^{n} \delta(x - a_k) \left[ \frac{\partial W}{\partial y} - \rho J_{y_k} \frac{\partial^2 W}{\partial y^2} \right] \right) \right)
\end{align*}
\]

\[ \ldots \quad \cdots \quad \text{C.5} \]
APPENDIX D

D.1 SUBSPACE ITERATION - THE BASIC ALGORITHM

It is assumed that, at the i-th iteration, a set of m base vectors, \( \{V\}_{(i)}^{n} \) \( i=1,2,...,m \) exist. These vectors can be arranged as a set of column vectors in the matrix \([V]_{(i)}\). The number of rows of \([V]_{(i)}\) is the size of the complete set of equation, and the number of columns is the dimension of the subspace, m.

The first step in the algorithm is to define a new set of base vectors by solving the following equations:

\[
[K][V]_{(i+1)} = [M][V]_{(i)} \quad .... \quad D.1
\]

The stiffness and mass matrices of the plate structure are then projected onto the subspace by:

\[
[K^*] = [V]_{(i+1)}^T[K][V]_{(i+1)} \quad .... \quad D.2
\]

and

\[
[M^*] = [V]_{(i+1)}^T[M][V]_{(i+1)} \quad .... \quad D.3
\]

so that the reduced mass and stiffness, \([M^*]\) and \([K^*]\) are available. These are of dimension m by n which corresponds to the chosen dimension of the subspace.

The eigenproblem:

\[
(-\omega^2[M^*] + [K^*])(\phi) = 0
\]

\[ .... \quad D.4 \]
is solved completely in the subspace, using Householder and Q-R methods.

Once the eigenvectors are defined in the reduced space, they can be transformed back to the full space of the structural problem, to define all the \([V]_n\) for the next iteration,

\[
[V]_{i+1} = [\hat{V}]_{i+1} [\phi] 
\]

where, \([\phi] = ([\phi]_1 \phi_2 \ldots \phi_m)\)

This complete an iteration.

D.2 HOUSEHOLDER'S METHOD

This method begins by determining a matrix \(P^{(1)}\) with \(P^{(1)} = [P^{(1)}]^{-1}\) and with the additional property that the matrix \(A^{(2)} = P^{(1)} A P^{(1)}\) has entries \(a_{n,j}^{(2)} = a_{j,n}^{(2)} = 0\) for each \(i=1,2,\ldots,(n-2)\). The tridiagonal matrix is produced by repeating this procedure \((n-3)\) times, where the \(k\)-th step is described by:

\[
A^{(k+1)} = P^{(k)} A^{(k)} P^{(k)} 
\]

for a matrix \(P^{(k)}\) chosen to have the properties:

\[
[P^{(k)}]^{-1} = P^{(k)} 
\]

and

\[
a_{n-k+1,j}^{(k+1)} = a_{j,n-k+1}^{(k+1)} = 0
\]

for each \(j=1,2,\ldots,(n-k-1)\).
The matrix \( A^{(k)} \) will be tridiagonal in its last \((k-1)\) rows and columns and has the following form:

\[
A^{(k)} =
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
0 & \ddots & \ddots & \ddots & \cdots \\
0 & 0 & \ddots & \ddots & \cdots \\
0 & 0 & 0 & \ddots & \ddots \\
\end{pmatrix}
\]

where, \( i = (n-k+2) \) for convenience of notation.

To illustrate the construction of \( P^{(k)} \), suppose that \( A^{(k)} \) has been formed as shown in equation (D.9).

Set,

\[
P^{(k)} = I - 2ww^T \quad .... \quad D.10
\]

where \( w \) is a vector with the property that \( w^Tw = 1 \). Then \( P^{(k)} \) has the property that,

\[
[P^{(k)}]^2 = (I - 2ww^T)^2 = I \quad .... \quad D.11
\]

and hence \( [P^{(k)}]^{-1} = P^{(k)} \). This ensures that \( P^{(k)} \) satisfies equation (D.8).

In order to guarantee that equation D.3 also holds, it is chosen that \( w_j = 0 \) for each \( j = i-1, \ldots, n \) and \( w_j \) for \( j = 1, 2, \ldots, i-2 \).

So that,
for each \( j = 1, 2, \ldots, (i-2) \).

The precise procedure for the choice of \( w_1', w_2', \ldots, w_{i-2} \) is detailed in the following algorithm.

Computational accuracy requires that the larger diagonal elements be initially placed toward the upper left corner of the matrix. This is accomplished by simply interchanging appropriate rows and columns of the matrix.

D.3 HOUSEHOLDER ALGORITHM

This algorithm is used to reduce the general matrix to a symmetric tridiagonal form; that is, the only non-zero entries in the matrix lie either on the diagonal or on the subdiagonals directly above or below the diagonal. To obtain a symmetric tridiagonal matrix \( A_{n-1} \) similar to the symmetric matrix \( A = A' \), construct the following matrices \( A_1', A_2', \ldots, A_{n-1} \) where, \( A_k = a_{i,j}^{(k)} \) for each \( k = 1, 2, \ldots, (n-1) \).

------------

INPUT dimension \( n \); matrix \( A \).

OUTPUT \( A_{n-1} \).

Step 1 For \( k = 1, 2, \ldots, (n-2) \) do Steps 2-10.

Step 2 Set \( i = n-k+2 \).
Step 3 Set $q = (\sum [a_{i-1,j}^{(k)}]^2)_{j=1,\ldots,(i-2)}$

Step 4 If $a_{i-1,i-2}^{(k)} = 0$ then set $s=q$

else set $s = qa_{i-1,i-2}^{(k)}/|a_{i-1,i-2}^{(k)}|$.

Step 5 Set $RSQ = (q^2 + sa_{i-1,i-2}^{(k)})$.

(Note: $RSQ=2r^2$)

Step 6 Set $v_{1-1} = 0$.

(Note: $v_1 = \ldots = v_n = 0$, but are not needed)

Set $v_{i-2} = a_{i-1,i-2}^{(k)} + s$

If $i>3$ then for $j=i,\ldots,i-3$ set $v_j = a_{i-1,j}^{(k)}$

(Note: $w = v/(2RSQ)^{1/2} = v/2r$)

Step 7 For $j=i,\ldots,i-1$ set $u_j = (\sum a_{j,1}^{(k)} v_1)/RSQ$

where $1 = 1,\ldots,(i-2)$.

(Note: $A_k v/RSQ = A_k v/2r^2$)

Step 8 Set $\text{PROD} = \sum u_j$ where $j=i,\ldots,(i-2)$

(Note: $v^t u = v^t A_k v/2r^2$)

Step 9 For $j=i,\ldots,i-1$ set $z_j = u_j - (\text{PROD} v_j)/2RSQ$

(Note: $z = u - v^t u v/2RSQ = u - v^t u v/4r^2 = u - w w^t u$

$= A_k w/2 - w w^t A_k w/2r$)

Step 10 For $1,\ldots,i-2$ (Note: Compute $A_{k+1} = A_k - vz^t - vz^t$

$= (I - 2ww^t) A_k (I - 2ww^t)$.)

for $j=1,\ldots,i-1$

set $a_{1j}^{(k+1)} = a_{1j}^{(k)} - v_j z_j - vz_{j1}$;

$a_{1j}^{(k+1)} = a_{j1}^{(k+1)}$

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set $a_{11}^{(k+1)} = a_{11}^{(k)} - 2v_1 z_j$;

set $a_{11}^{(k+1)} = a_{11}^{(k)} - 2v_1 z_i$;

for $j=1,...,i-3$

set $a_{i-1,i}^{(k+1)} = 0; a_{j,i-1}^{(k+1)} = 0;$

set $a_{i-1,i-2}^{(k+1)} = a_{i-1,i-2}^{(k)} - v_{i-2} z_{i-1}$;

$a_{i-2,i-1}^{(k+1)} = a_{i-2,i-1}^{(k+1)}$

(Note: The other elements of $A_{k+1}$ are the same as $A_k$)

Step II OUTPUT ($A_{n-1}^{(k+1)}$)

The process is complete. $A_{n-1}^{(k+1)}$ is symmetric, tridiagonal and similar to $A_k$)

STOP.
APPENDIX E

Graph E-1: Variation of $E'$ and tan$\delta$ with frequency for PMMA at 21°C. Point symbols as follows; $\bullet$, $\Delta$: Tensile forced non-resonance data; $\bigcirc$, $\triangle$: Flexural resonance; $\bigcirc$, $\triangle$: Longitudinal resonance; $\bigcirc$, $\triangle$: Ultrasonic pulse propagation.

Graph E-2: $\nu'$ and tan$\delta$ plotted against frequency for PMMA at 21°C. Point symbols as in Graph E-1.

PMMA 21°C