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Polarisation Transfer and Breakup Effects
in
Deuteron Induced Nuclear Reactions

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A thesis submitted to the faculty of Science
for the degree of Doctor of Philosophy
at the University of Surrey

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ABSTRACT

Optical model studies of low energy (d,d) data show that deuteron-nucleus tensor potentials differ from theoretical model predictions. This suggests that measurement of additional observables, such as Polarization Transfer Coefficients (PTC), are needed to complement the existing data. Whether PTCs can clarify the experimental/theoretical ambiguity has been a matter of controversy. The first part of this thesis addresses this problem. We began by investigating whether the PTC, $K_{y}^{p}$, yields new information concerning the deuteron-nucleus tensor potential. We then examined the extent to which this coefficient can distinguish between the two types of tensor forces, $T_{r}$ and $T_{p}$. We showed that $K_{y}^{p}$ is strongly affected by tensor force effects, and that the origin of this sensitivity is the bi-linear combination of scattering amplitudes, $\text{Im}(Q_{00}Q_{21}^{*})$. We also found that, for realistic optical model parameters, $T_{21}$ and particularly $K_{y}^{p}$ discriminate between the effects of both tensor forces.

In the second part of this thesis we study the Weinberg State Expansion model (WSE) for (d,p) reactions. The weakly bound structure of the deuteron suggests the relevance of 3-body effects in the dynamics of deuteron stripping. At intermediate energies, the DWBA provides a much less reliable description of particle transfer reactions. Although the adiabatic theory (ADIA) has provided improvements over the conventional DWBA, recent experimental data suggest that it needs to be refined. The WSE method, in which the dominant contributions from the 3-body channels are explicitly included, is a way to systematically improve ADIA which appears as the lowest order solution in the WSE theory. In implementing the WSE model, we found that as the Weinberg basis size $N$ increases more c.m. n-p relative energies are simulated and readily included into the (d,p) calculations. We also showed that, when performing zero-range WSE calculations for $^{66}\text{Zn}(d,p)^{67}\text{Zn}$ (G.S.;$5/2^{-};l_{n} =3$) at 25 and 88.2 MeV, the results for $d\sigma/d\Omega$ and $iT_{11}^{*}$ converge for $N=35$. Although 35 Weinberg states were used in constructing the new basis, the reaction calculation reduced to a three coupled channels problem. Our calculations are therefore more efficient than the CDCC methods. The WSE results for $d\sigma/d\Omega$ and $iT_{11}^{*}$ were also compared against those of equivalent ADIA and Quasi-ADiabatic (QAD) methods. Our findings reveal that: a) the WSE model provides significant corrections to ADIA's predictions and as such constitutes an elegant mathematical justification of ADIA's ideas; b) the WSE results are overall in good agreement with those obtained using QAD.
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PART I

Polarisation Transfer in $(d, d)$ Reactions
CHAPTER I

- INTRODUCTION -

A standard approach to nuclear scattering is the optical potential model. In this model, the many-body interaction between the projectile and the individual nucleons of the target is replaced by a two-body potential. This potential is assumed to consist of two distinct parts, a spin-independent (or central) part and a spin-dependent component. This model has been essential in understanding and relating a vast range of nuclear data. However, despite the overall success of the optical model its spin-dependent part is problematic, in particular for deuteron induced reactions it remains poorly understood. Further consideration of the spin-dependent component of the optical potential is therefore necessary for a consistent description of spin-related properties. This is the object of our interest here in the context of polarized deuteron elastic scattering.

Previous calculations [Pe67, Ke73] using the Watanabe or Folding Model (FM) [Wa58] to evaluate the spin-independent (i.e. central) part of the optical model potential did not satisfactorily reproduce the differential cross-section data. The apparent failure of such calculations was understood to be caused by the absence of spin-dependent components in the optical potential used as well as the neglect of deuteron breakup, Pauli and other three-body effects. It is therefore vital that any analysis of the polarization of elastically scattered deuterons should, in the first instance, include the spin-dependent parts of the deuteron-nucleus optical potential.
In an early attempt to formally characterise the spin-dependent interaction, Satchler [Sa60] showed that if parity conservation and symmetry of the scattering matrix are satisfied, the spin-dependent parts of the interaction between a spin-1 projectile (e.g. deuterons) and a spinless target are of four types. These are a first-rank I,s (or vector spin-orbit) and three second-rank (or tensor) terms, normally denoted by \( T_r, T_1 \) and \( T_p \). The physical origin of the I,s term (first-rank) was given by Raynal [Ra63] who showed that the spin-orbit component of the nucleon-nucleus optical potential gives rise to a similar term for the deuteron-nucleus interaction. Regarding the physical origins of the tensor terms (second-rank), it was shown [Ra64, Te66] that the inclusion of the deuteron D-state in a FM calculation yields naturally the \( T_r \) tensor potential. It was also shown [St70] that a small \( T_1 \) tensor term is expected if deuteron breakup is considered, while the Pauli exclusion principle combined together with the deuteron D-state [Au76, Io76] generate the \( T_p \) tensor interaction.

Until recently, the only measured observables for polarized deuteron induced reactions were the differential cross-section, \( I(\theta) \), the vector analyzing power, \( iT_{11} \), and the three tensor analyzing powers, \( T_{20}, T_{21} \) and \( T_{22} \). Hooton and Johnson [Ho71] investigated the extent to which measurements of these five observables can be used to probe the deuteron-nucleus spin-dependent forces. They proved that if the spin-dependent potentials are small enough to be treated as perturbations, the I,s term produces, in first order, rank-one scattering amplitudes while the tensor components \( (T_r, T_1 \text{ and } T_p) \) generate rank-two scattering amplitudes. Their analysis of the expansions of the above observables in terms of the scattering amplitude elements reveals that, to first order, the spin-orbit interaction produces most of \( iT_{11} \), whereas
the tensor potentials generate most of the $T_{2q}$ ($q=0,1,2$).

Phenomenological studies of low energy deuteron elastic scattering data [Go79, Ma82] including $I(0),iT_{11}$ and the $T_{2q}$'s show results for the deuteron-nucleus spin-dependent interaction which are different from theoretical model predictions [To78]. In particular, the results differ in the expected ratio of the strengths of the real and imaginary parts of the second-rank tensor interactions. This suggests that measurements of the angular distributions of the cross-section and analyzing powers are not sufficient to clarify those ambiguities. Measurements of additional observables are therefore needed to complement the existing polarization data and provide an unambiguous description of the properties of the deuteron-nucleus tensor forces. Measurements of such observables are possible since polarization transfer experiments with polarized deuterons are now feasible. These additional observables are usually referred to as Polarization Transfer Coefficients or in short PTCs.

Whether the PTCs can clarify the just mentioned ambiguities has been a matter of controversy. The theoretical work of Lopes and Santos [Lo77] showed that polarization transfer experiments provide observables (i.e. PTCs) which show a selective sensitivity to the different components of the deuteron-nucleus interaction. They confirmed that measurements of the angular distributions of the PTCs determine quantities which have large tensor force effects. Furthermore, they concluded that the PTCs are not only sensitive to the deuteron-nucleus tensor force effects but are also likely to distinguish between them. On the other hand, Goddard [Go77a, Go78] suggested that at low energies (10-15 MeV) it would be very difficult to distinguish between the effects of a $T_p$ and a suitably chosen $T_2$ tensor interaction even, if PTCs and the usual five observables
are used. These two opposing claims therefore raise the necessity to examine the validity of either assertion.

The first part of this thesis addresses this problem. Our approach consists of two stages. Firstly, we investigate whether, as suggested by Lopes and Santos, the addition of PTCs yields new information concerning the tensor parts of the deuteron-nucleus potential. Secondly, we examine Goddard’s assertion by analysing the extent to which PTCs can distinguish between the deuteron-nucleus tensor forces, $T_r$ and $T_p$.

Regarding the first stage, the particular PTC we have chosen for this discussion is $K_y^{z'z'}$. The choice of this PTC is motivated by experimental considerations which are given in Appendix-A. We have carried out optical model calculations involving these observables to describe the elastic scattering of polarized deuterons from $^{90}$Zr at 15 MeV. Our results show that the PTC $K_y^{z'z'}$ is particularly sensitive to the presence of the $T_r$ type tensor interaction. To investigate this further we analysed this observed sensitivity within a spherical tensor decomposition [Ho71] of the elastic scattering amplitude.

As for the second stage, we performed additional calculations in order to assess the extent to which the PTC, $K_y^{z'z'}$, can distinguish between tensor interactions of $T_r$ and $T_p$ types. These calculations were carried out using realistic phenomenological optical potential form factors.

A close collaboration exists between our group and experimentalists at the Triangle Universities Nuclear Laboratory (TUNL) in the U.S.A. on this polarization transfer experiment. In fact, they [Ab86, Ab87] performed measurements of PTCs for the elastic scattering of polarized deuterons.
from $^9$Zr at 15 MeV. The preliminary testing phase at TUNL is finished and the first data gathering has taken place. We have recently received the first experimental data points for the combination, $\left(3K_{yz}^x + K_{yy}^x\right)$ at four c.m. angles [Ab88, Ab89]. These data points have now been analysed (at TUNL) for consistency and possible systematic errors.

The plan of this part of the thesis is as follows. In Chapter II we give the theoretical background and considerations that are necessary for a full description of the elastic scattering of polarized deuterons from a spin-zero target. We also derive the expressions of all the relevant reaction observables in terms of the scattering amplitude and appropriate vector and tensor spin operators. The investigation of the origin of sensitivity to tensor force effects that is displayed by the PTC $K_{yy}^x$ is carried out in section (3.2) of Chapter III. The question of discrimination between the effects of the $T_r$ and $T_p$ forces is examined in section (3.3). The summary of this work and our concluding remarks are given in section (3.4).
CHAPTER II

- THEORETICAL BACKGROUND AND GENERAL CONSIDERATIONS -

2.1) INTRODUCTION

For spin-1 particles, one distinguishes between vector and tensor polarization. The parameters describing the polarization states of a group of particles, are called spin tensor moments. Thus a beam is said to be polarized if any of these parameters are different from zero. In this work, we focus on the elastic scattering of low energy polarized deuterons from spinless medium weight targets and analyse the transfer of polarization from incoming to outgoing deuterons within an optical model framework.

To fully describe the polarization of the outgoing deuterons, the most common approach [Oh70, Oh72a, Oh72b] consists of a parameterization of the polarization of the outgoing particles in terms of the polarization observables and the polarization of the incoming particles. To achieve this parameterization we use the density matrix concept, illustrated in section (2.2). The application of this formalism to the spin-1 case is explained in section (2.3).

2.2) THE DENSITY MATRIX THEORY

A pure state is characterised by the coefficients $C_n$ of the expansion of its state vector $|\psi>$ in terms of the eigenvectors of some complete set of operators,
The mean value of an operator $A$ when the system is in the state $|\psi\rangle$ is given by

$$<A> = \sum_{n} A_{n} C_{n}^{*} C_{n},$$

where,

$$A_{n} = <U_{n}|A|U_{n}>.$$  

Non-pure states (or mixed states) are thought of as incoherent superposition of pure states, $|\psi_{i}\rangle$, with probabilities, $p_{i}$ ($\sum p_{i} = 1$). As shown above, to every pure state corresponds a mean value $<A_{i}>$. The mean value of an operator $A$ that corresponds to the mixed states system defined above, is therefore given by the grand average

$$<A> = \sum_{i} p_{i} <A_{i}> = \sum_{n} A_{n} C_{n}^{(1)} C_{n}^{(i)}.$$ 

The elements of the density matrix operator are defined [Fa57] as

$$\rho_{nn'} = \sum_{i} p_{i} C_{n'}^{(g)} C_{n}^{(i)},$$

which implies that the density matrix operator has the following form

$$\rho = \sum_{i} p_{i} |\psi_{i}\rangle <\psi_{i}|.$$ 

As a consequence of Eqn.(2.6) the density matrix operator of a pure state (i.e. a system where all the $p_{i}$ are equal to zero except one) is

$$\rho_{pnn} = |\psi\rangle <\psi|.$$
By substituting Eqn.(2.5) into Eqn.(2.4) one obtains the mixed state expectation value

\[< A > = \sum_{n,m} A_{nm} \rho_{mn} = \text{Trace} \left\{ A \rho \right\}, \tag{2.8}\]

where Trace refers to the sum of the diagonal elements of a matrix.

2.3)- SPIN-1 FORMALISM

The spin states of a particle taking part in a nuclear reaction are described as a statistical mixture of pure spin states in which the particles can be found. In general [Ha71] one needs \((2S+1)^2-1\) parameters to completely account for the polarization state of a spin-S particle. For spin-1 particles we require eight quantities.

Most of the earlier work with spin-1 particles (e.g. deuterons) was carried out using spherical tensor operators [La55, Sa60] as well as cartesian operators [Go58]. In our analysis, we make use of both types of operators (connections between the two types of operators can be found in Appendix-B). The spherical operators are chosen to form the components of irreducible tensors of rank-0 (identity matrix), rank-1 and rank-2. These components, denoted by \(\tau_{kq}\), transform under rotation like the spherical harmonics \(Y_{kq}\) and are defined [Br68] \(S = 1\)

\[\tau_{kq}(1) = \sum_{\sigma, \sigma'} \hat{\sigma} <1 \sigma \ kq \ | 1\sigma> \ |1\sigma> <1\sigma| \tag{2.9a}\]

where, \(\hat{\sigma} = (2k+1)^{1/2}\). The \(|1\sigma>\) are the simultaneous eigenstates of \(S^2\) and \(S_z\). The \(\tau_{kq}(1)\) satisfy the orthonormalization relation
\[
\text{Tr}\left\{ \tau_{kq} \tau_{k'q'}^+ \right\} = 3 \delta_{kk'}\delta_{qq'} \quad (2.9b)
\]

Any operator in spin space can be decomposed in terms of the \(\tau_{kq}\), including the density matrix operator. We therefore write

\[
\rho = (1/3) \sum_{k=0}^{2} \sum_{q=-k}^{q=k} (\rho_{kq}^{*} \tau_{kq}) \quad (2.10)
\]

where,

\[
\rho_{kq}^{*} = \text{Tr}\left\{ \rho_{kq} \tau_{kq}^+ \right\} = t_{kq}^{*} \quad (2.11)
\]

The definition of the \(\tau_{kq}\) means that \(\tau_{k,q} = (-1)^{q} \tau_{k,q}^+\). This in turn implies that the above defined tensor moments, \(t_{kq}^{*}\), have also the same symmetry property. Note that the \(t_{kq}\) are, according to Eqn.(2.8), nothing but the expectation values of the respective spin tensor operators \(\tau_{kq}\) in the incident beam.

In a nuclear reaction, the incoming and outgoing particles are either described by pure or mixed spin states. In the case of pure spin states we denote by \(|\chi_{i}\rangle\) and \(|\chi_{f}\rangle\) the spin states of incoming and outgoing particles respectively. The density operator associated with the incoming beam is

\[
\rho_{\text{pure}}^{i} = |\chi_{i}\rangle \langle \chi_{i}| \quad (2.12)
\]

and the spin states of the outgoing particles are given by

\[
|\chi_{f}\rangle = F(\theta,\Phi) \langle \chi_{i}| \quad (2.13)
\]

where \(F(\theta,\Phi)\) is the scattering amplitude matrix with elements \(F_{\sigma'\sigma}(\theta,\Phi)\). The density operator associated with the outgoing beam is thus
When the incoming and outgoing particles are described by mixed spin states, the density operator for the incoming beam is

$$\rho_{\text{mixed}}^{\text{i}} = \sum_i p_i \chi_i^\dagger \chi_i$$

while the density operator for the scattered particles takes this form

$$\rho_{\text{mixed}}^{\text{f}} = \sum_i p_i \left( F \chi_i^\dagger \chi_i F^+ \right) = F \rho_{\text{mixed}}^{\text{i}} F^+.$$  \hspace{1cm} (2.16)

Knowing $$\rho_{\text{mixed}}^{\text{f}}$$ means that it is possible to evaluate expectation values of any spin operator, $$Q$$ for example, associated with the outgoing particles. We write

$$\langle Q \rangle = \text{Tr}\{ \rho_{\text{mixed}}^{\text{f}} Q \} / \text{Tr}\{ \rho_{\text{mixed}}^{\text{f}} \}.$$  \hspace{1cm} (2.17)

The intensity of the scattered particles (differential cross-section) is [Da70]

$$I(\theta, \phi) = \text{Tr}\{ \rho_{\text{mixed}}^{\text{f}} \} = \text{Tr}\{ F(\theta, \phi) \rho_{\text{mixed}}^{\text{i}} F^+(\theta, \phi) \}.$$  \hspace{1cm} (2.18)

Using Eqns.(2.10-11) for $$\rho_{\text{mixed}}^{\text{i}}$$ in Eqn.(2.18), one thus obtains

$$I(\theta, \phi) = (1/3) \left[ \text{Tr}\{ F F^+ \} + \sum_{k \neq 0}^{2} \sum_{q=-k}^{q+k} t_{kq}^* \text{Tr}\{ F \tau_{kq} F^+ \} \right],$$

$$I(\theta, \phi) = I_0(\theta, \phi) \left[ 1 + \sum_{k, q} t_{kq}^* T_{kq}(\theta, \phi) \right],$$  \hspace{1cm} (2.19)

where, $$I_0(\theta, \phi) = (1/3) \text{Tr}\{ F F^+ \}$$ is the differential cross-section for an unpolarized incident beam, and the $$T_{kq}$$ are the analyzing powers of the
reaction, defined by

\[ T_{kq}(\theta, \phi) = \text{Tr} \left\{ F \tau_{kq} F^+ \right\} / \text{Tr} \left\{ F F^+ \right\}. \quad (2.20) \]

The \( \tau_{kq} \) symmetry relations combined with the fact that the elements of \( F \) satisfy parity conservation and rotational invariance [Ho71], impose certain constraints on the \( T_{kq} \)'s. Thus, provided the y-axis of a right handed coordinate system to which the \( T_{kq} \)'s are referred is along the normal to the reaction plane \( n \) and its z-axis is parallel to the incident beam direction (see Figs.(2.1)), then

\[ T_{kq} = (-1)^{k+q} T_{k,-q} \quad \text{and} \quad T_{kq}^* = (-1)^q T_{k,-q}. \quad (2.21) \]

The parameterization of the differential cross-section, when referred to the above coordinate system, in terms of the analyzing powers \( T_{kq} \) and the tensor moments \( t_{kq} \) is obtained using Eqns.(2.19) and (2.21)

\[ I = I_0 \left[ 1 + 2T_{11} Re(t_{11}) + T_{20} t_{20} + 2T_{21} Re(t_{21}) + 2T_{22} Re(t_{22}) \right]. \quad (2.22) \]

Usually, polarization transfer observables are expressed in the cartesian representation, so it is important at this stage to write the equivalent of Eqn.(2.22) in this representation. For this purpose we adhere to the Madison convention [Appendix-B], where it is recommended to use the helicity coordinate systems for the projectile and outgoing particle polarizations. For the projectile the frame has z-axis along its momentum, \( k_{in} \), y-axis normal to the scattering plane, \( n = k_{in} \times k_{out} \), and the x-axis such as to form a right handed coordinate system (see Fig.(2.1a)). As for the outgoing particles the \((x',y',z')\) coordinate system has \( z'\)-axis along the outgoing momentum, \( k_{out} \), the \( y' \) remains along \( n \), and \( x' \) chosen to form a right handed system (see Fig.(2.1b).
The cartesian or rectangular representation of spin operators was introduced by Goldfarb in 1958 [Go58], and is derived from the spin-1 angular momentum operators $S_x$, $S_y$ and $S_z$. These cartesian tensors are

$$
P_i = S_i; \quad P_{ij} = (3/2)[S_i S_j + S_j S_i] - 2S_{ij}; \quad (ij = x,y,z). \quad (2.23)
$$

The spherical tensor moments, $t_{kq}$, defined by Eqn.(2.10) have equivalent cartesian quantities denoted by $p_i$ and $p_{ij}$ ($ij = x,y,z$ in any order) and usually called vector and tensor polarizations, respectively. A list of the relationships between the $t_{kq}$ and the $(p_i, p_{ij})$ is given in Appendix-B. In order to establish the cartesian equivalent of the differential cross-section parameterization of Eqn.(2.22), one has to re-write the density matrix expansion of Eqn.(2.10) in terms of the cartesian operators, $(p_i, p_{ij})$ and $(P_i, P_{ij})$. The cartesian expression for the incoming density matrix reads
\[ \rho_{\text{inc}} = \frac{1}{3} \left[ 1 + \frac{3}{2} \left( p_x P_x + p_y P_y + p_z P_z \right) + \frac{2}{3} \left( p_{xy} P_{xy} + p_{yz} P_{yz} + p_{zx} P_{zx} \right) + \frac{1}{2} \left( p_{xx} - p_{yy} \right) \left( \begin{array}{ccc} P_{xx} - P_{yy} \end{array} \right) + \frac{1}{2} p_{zz} P_{zz} \right]. \] 

(2.24)

Given \( \rho_{\text{inc}} \) one can repeat the procedure starting from Eqn.(2.10) and ending at Eqn.(2.22), to derive the differential cross-section. Thus, in the cartesian representation Eqn.(2.22) becomes

\[ I = I_0 \left[ 1 + \frac{3}{2} p_y A_y + \frac{2}{3} p_x A_{xz} + \frac{1}{3} p_x A_{xx} + \frac{1}{3} p_{yy} A_{yy} + \frac{1}{3} p_{zz} A_{zz} \right], \] 

(2.25)

where we have defined the cartesian vector and tensor analyzing powers,

\[ \begin{bmatrix} A_i \\ A_{ij} \end{bmatrix} = \text{Tr.} \left\{ \mathbf{F} \left( \begin{bmatrix} P_i \\ P_{ij} \end{bmatrix} \right) \mathbf{F}^+ \right\} / \text{Tr.} \left\{ \mathbf{F} \mathbf{F}^+ \right\}; \ (ij = x,y,z), \] 

(2.26)

and the vector and tensor polarizations of the incident beam

\[ \begin{bmatrix} p_i \\ p_{ij} \end{bmatrix} = \text{Tr.} \left\{ \rho_{\text{inc}} \left( \begin{bmatrix} p_i \\ P_{ij} \end{bmatrix} \right) \right\}; \ (ij = x,y,z). \] 

(2.28)

The relationships between the \( T_{kj} \) and \( A_i, A_j \), when expressed in the Madison coordinate system are

\[ i T_{11} = (43/2) A_y, \] 

(2.27a)

\[ T_{20} = (41/2) A_{zz}, \] 

(2.27b)

\[ T_{21} = -(41/3) A_{xz}, \] 

(2.27c)

\[ T_{22} = (41/12) \left[ A_{xx} - A_{yy} \right]. \] 

(2.27d)
To obtain the tensor polarization of the outgoing particles, say $p_{z'z''}$, the general result given by Eqn.(2.17) can be used. Thus, the $z'z''$ polarization component of the outgoing particles is

$$p_{z'z''} = \frac{\text{Tr} \left\{ \rho^{\text{out}} \rho_{z'z''} \right\}}{\text{Tr} \left\{ \rho^{\text{out}} \right\}}, \quad (2.29)$$

or, expressing $\rho^{\text{out}}$ in terms of $F$, $F^+$, $\rho^{\text{inc}}$ and Eqn.(2.18) for $I(\theta,\Phi)$

$$p_{z'z''} I(\theta,\Phi) = \text{Tr} \left\{ F \rho^{\text{inc}} F^+ \rho_{z'z''} \right\}. \quad (2.30)$$

Inserting the expression of $\rho^{\text{inc}}$, Eqn.(2.24), into the RHS of Eqn.(2.30) one obtains in the general case

$$p_{z'z''} I = I_0 \left[ p_{z'z''} + \frac{3}{2} p_y K_{yy}^{z'z''} + \frac{2}{3} p_{xz} K_{xz}^{z'z''} + \frac{1}{3} p_{xx} K_{xx}^{z'z''} + \frac{1}{3} p_{yy} K_{yy}^{z'z''} + \frac{1}{3} p_{zz} K_{zz}^{z'z''} \right], \quad (2.31)$$

where we have defined the polarization function, $p_{z'z''}$, the $z'z''$ outgoing polarization produced by an unpolarized incident beam

$$p_{z'z''}(0,\Phi) = \frac{\text{Tr} \left\{ F F^+ \rho_{z'z''} \right\}}{\text{Tr} \left\{ F F^+ \right\}}, \quad (2.32)$$

and introduced the K functions, the Polarization Transfer Coefficients or PTCs. In this case, these are

$$\begin{pmatrix} K_{y}^{z'z''} \\ K_{ij}^{z'z''} \end{pmatrix} = \text{Tr} \left\{ F \left( \begin{pmatrix} p_i \\ p_j \end{pmatrix} \right) F^+ \rho_{z'z''} \right\} / \text{Tr} \left\{ F F^+ \right\}; \quad (i,j=x,y,z). \quad (2.33)$$

It should be noted that in this particular $(z'z'')$ example we have a vector-to-tensor ($K_{y}^{z'z''}$) and tensor-to-tensor ($K_{ij}^{z'z''}$) polarization.
transfer coefficients only. However, other coefficients can also be introduced when selecting different incoming vector or tensor polarizations. A PTC is a measure of the efficiency with which an incoming polarization component (be it vector or tensor) is transferred to an outgoing polarization component. It is, therefore, very important to study these coefficients in order to understand the mechanisms responsible for such polarization transfers, and eventually gain more insight into the underlying physics of the reaction.
3.1)- INTRODUCTION
As illustrated in Chapter-II, the polarization of elastically scattered deuterons can be parameterized in terms of measurable quantities such as, polarization functions, $p_{ij}$, polarization transfer coefficients or PTCs, $(K_{ij}^{kd}, K_{i}^{kq})$ and the incident beam polarizations, $(p_i, p_{ij})$. We choose the PTC, $K_{y}^{x'x'}$, as the polarization transfer observable that will be considered in our optical model analysis. We first begin by investigating whether or not this observable is sensitive to the deuteron-nucleus tensor forces.

3.2)- SENSITIVITY OF $K_{y}^{x'x'}$ TO TENSOR FORCES
We focus on the elastic scattering of polarized deuterons from medium weight nuclei (40 $\leq A \leq$ 90) at low energies, $E_d^{1ab} = 13-15$ MeV. In our analysis we consider the PTC $K_{y}^{x'x'}$ in addition to the cross-section and the four analyzing powers.

3.2.1- Evidence of the Dependence of $K_{y}^{x'x'}$ on the Tensor Forces
The expression of $K_{y}^{x'x'}$, obtained from Eqn.(2.33), is

$$K_{y}^{x'x'} = \frac{\text{Tr.}\left\{ F P_y F^{+} P_{x'x'}\right\}}{\text{Tr.}\left\{ F F^{+}\right\}} , \quad (3.1)$$

with, $F$ the scattering amplitude operator and $P_y, P_{x'x'}$ the cartesian tensor operators defined in Eqn.(2.23). To numerically evaluate the
angular distribution of $K_{Y}^{2,2}$, one needs to calculate the scattering amplitude matrix $F$.

The scattering of two particles (at energy $E$) can be described in terms of the matrix elements between plane waves states of the transition operator $M(E)$ [Me62]:

$$M(E) = V(r) + V(r) \left[ E - H_0 - V(r) + i\epsilon \right]^{-1} V(r), \quad (3.2)$$

where $H_0$ is the kinetic energy operator, and $V(r)$ is the deuteron-nucleus interaction. The transition operator, $M(E)$, has a simple relation with the scattering amplitude operator, $F$, namely

$$F_{\sigma_1 \sigma_2}(k_2, k_1) = - (\mu / 2 \pi \hbar^2) \langle k_2 \sigma | M(E) | k_1 \sigma \rangle, \quad (3.3)$$

where $k_2$ and $k_1$ in this notation refer to the final and initial relative momenta.

The numerical evaluation of the elements $F_{\sigma_1 \sigma_2}$ requires the knowledge of the potential $V(r)$. If invariance under rotations of the coordinate axes, parity conservation and time reversal invariance are imposed on the spherically symmetric interaction between a spin-1 particle and a spin-0 nucleus, then the most general form of $V(r)$ is given by the local optical model potential [Sa60]

$$V(r) = V_c(r) + V_l(r) \mathbf{l} \cdot \mathbf{s} + V_t(r) T_r + V_d(r) T_1 +$$

$$\frac{1}{2} \left[ V_p(r) T_p + T_p V_t(p') \right], \quad (3.4)$$

where $T_{tp}$ are spin tensor operators defined by
\[ T_r = (s \hat{\sigma})^2 - \frac{2}{3}; \quad (\hat{\sigma} = \hat{r}/|\hat{r}|) \, , \quad (3.5a) \]
\[ T_1 = (l \cdot s)^2 + \frac{1}{2}(l \cdot s) - \frac{2}{3} l^2; \quad (l = |l|) \, , \quad (3.5b) \]
\[ T_p = (s \cdot p)^2 - \frac{2}{3} p^2; \quad (p = |p|) \, . \quad (3.5c) \]

s, r, p and l are the deuteron spin, position, momentum and orbital angular momentum operators, respectively. The five form factors \( V_i(r) \) are complex functions of the deuteron-nucleus separation.

We first investigate whether or not the addition of a tensor force term, say \( T_r \), to the central and l.s interactions affects the polarization observables (i.e. \( I, T_{k_3} \) and as an example of PTCs, \( K_{y}^{x z} \)). To achieve this, we carried out numerical calculations for the reaction \(^{90}\text{Zr}(d,d)\) at 15 MeV using DDTP (a computer program [Go77b] based on optical model potentials that include \( T_r \) and/or \( T_p \) tensor interactions) where the \( T_r \) tensor interaction is considered, and where only the central and l.s terms are present. The optical model parameters used in this calculation are taken from [Go78] and are reproduced here as Table-1. These potentials give the best fit to all four elastic analyzing powers \((iT_{11}, T_{20}, T_{21}, T_{22})\) and the cross-section.

Figure (3.1) shows the calculated \( K_{y}^{x z} \) and \( T_{21} \). The cross-section, \( I \), and the other analyzing powers, \( iT_{11}, T_{20} \) and \( T_{22} \), are not shown since they do not display as significant effects as those observed in \( K_{y}^{x z} \) and \( T_{21} \). The inclusion of a complex \( T_r \) tensor interaction generates large effects in \( K_{y}^{x z} \) over a wide angular range \((50-140^\circ)\). The effects of the \( T_r \) potential on \( T_{21} \) are much smaller than those observed in \( K_{y}^{x z} \) but are appreciable when compared with the rest of the observables.
Table-1

Optical Model Potentials

<table>
<thead>
<tr>
<th>TARGET</th>
<th>(^{46}\text{Ti})</th>
<th>(^{90}\text{Zr})</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENERGY</td>
<td>13.0</td>
<td>15.0</td>
</tr>
<tr>
<td>\text{REAL CENT. V (MeV)}</td>
<td>101.8</td>
<td>79.7</td>
</tr>
<tr>
<td>\text{R (fm)}</td>
<td>1.07</td>
<td>1.34</td>
</tr>
<tr>
<td>\text{A (fm)}</td>
<td>0.81</td>
<td>0.63</td>
</tr>
<tr>
<td>\text{IMAG. CENT. V}</td>
<td>14.0</td>
<td>28.5</td>
</tr>
<tr>
<td>\text{R}</td>
<td>1.38</td>
<td>1.29</td>
</tr>
<tr>
<td>\text{A}</td>
<td>0.70</td>
<td>0.48</td>
</tr>
<tr>
<td>\text{REAL LS V}</td>
<td>5.10</td>
<td>5.20</td>
</tr>
<tr>
<td>\text{R}</td>
<td>0.64</td>
<td>0.83</td>
</tr>
<tr>
<td>\text{A}</td>
<td>0.41</td>
<td>0.34</td>
</tr>
<tr>
<td>\text{REAL Tr} \text{(D2-FLD)}</td>
<td>0.30</td>
<td>0.60</td>
</tr>
<tr>
<td>\text{R}</td>
<td>1.70</td>
<td>1.65</td>
</tr>
<tr>
<td>\text{A}</td>
<td>0.90</td>
<td>0.47</td>
</tr>
<tr>
<td>\text{IMAG. Tr} \text{(D3-FLD)}</td>
<td>0.70</td>
<td>3.60</td>
</tr>
<tr>
<td>\text{R}</td>
<td>1.45</td>
<td>1.24</td>
</tr>
<tr>
<td>\text{A}</td>
<td>0.59</td>
<td>0.65</td>
</tr>
</tbody>
</table>

The potential shapes (WOOSAX, THOMAS, ...etc) are taken from Goddard's thesis [Go78] and are shown at the end of Appendix-B.
Fig. (3.1) - Angular distribution of the polarization transfer coefficient $K_{yz}^2$ and the tensor analyzing power $T_{21}$ for $^{90}$Zr(d,d) at 15 MeV. The solid lines indicate calculations with a $T_r$ interaction and the dashed curves show calculations without $T_r$. 
The origin of this sensitivity to the tensor force effect in $K_y^{x'z'}$ is examined in the next sub-section.

### 3.2.2 Analysis of the Source of Sensitivity in $K_y^{x'z'}$ to Tensor Forces

In order to understand why $K_y^{x'z'}$ is affected by the presence of the $T_r$ tensor interaction, one needs to know the relationships between the various terms in the potential of Eqn.(3.4) and the matrix elements of the scattering amplitude.

We showed in section (2.2) that any spin operator can be written as a linear combination of the irreducible spin-tensor operators, $\tau_{kq}$. In particular, we can apply this to the scattering amplitude matrix elements, $F_{\sigma'\sigma}(k_2, k_1)$. We adopt the following notation

$$F(E) = \sum_{k=0}^{\infty} \sum_{q=-k}^{+k} Q_{kq}(E) \tau_{kq}^+, \quad (3.6)$$

and since

$$F_{\sigma'\sigma}(k_2, k_1) = <k_2 \sigma' | F(E) | k_1 \sigma>, \quad (3.7)$$

we obtain

$$F_{\sigma'\sigma}(k_2, k_1) = \sum_{k', q} Q_{kq}(k_2, k_1) <k_1 \sigma | \tau_{kq}^+ | k_2 \sigma>. \quad (3.8)$$

Using the properties of the $\tau_{kq}$'s in any coordinate system that has its $y$-axis along the normal to the scattering plane, $k_1 \times k_2$, and assuming rotational invariance and parity conservation of the $Q_{kq}(k_2, k_1)$, then [Ho71]

$$Q_{kq} = (-1)^{k+q} Q_{k,-q}. \quad (3.9)$$

Eqns (3.6-9) imply that $F(E)$ takes on the following form when written in the $[|1 \sigma>\}$ basis (z-axis along the incident beam direction)
\[
\mathbf{F} = \begin{bmatrix}
  a & b & c \\
  d & e & -d \\
  c & -b & a
\end{bmatrix},
\] (3.10)

where,

\[
\begin{align*}
\alpha &= Q_{00} + (41/2) Q_{20}, \\
\beta &= (43/2) (Q_{11} - Q_{21}), \\
\gamma &= (43) Q_{22}, \\
\delta &= -(43/2) (Q_{11} + Q_{21}), \\
\epsilon &= Q_{00} - (42) Q_{20}.
\end{align*}
\] (3.11)

Only four of the five amplitudes in Eqn.(3.10) are independent since they satisfy the relation [Ho71]

\[
c = (\alpha - \epsilon) - (42) (\beta + \delta) \cot(\theta),
\] (3.12)

where \(\theta\) is the scattering angle.

Hooton and Johnson [Ho71] demonstrated that if the spin-dependent forces in the deuteron-nucleus potential are assumed to be weak in comparison to the central interaction, then it is possible to treat the former using a perturbative prescription. They showed that, to first order, \(Q_{00}\) is produced by the central component of the deuteron-nucleus potential, the \(Q_{1q}\) by the spin-orbit term and the \(Q_{2q}\) amplitudes by the tensor components. These findings will be used later to interpret the expansion of \(K_y^{x,z}\).

Now we have enough information which can be used to investigate the origin of sensitivity in \(K_y^{x,z}\) to tensor force effects. As given by Eqn.(3.1), the PTC \(K_y^{x,z}\) is expressed in both the projectile \((x,y,z)\) and
the outgoing \((x',y',z')\) particle helicity frames. It is straightforward to switch from one coordinate system to the other by a rotation \(\theta\) around the \(y\)-axis. Therefore, one can re-express the tensor operator \(P\) with respect to the \((x,y,z)\) frame as

\[
P_{x'y'} = \sin^2(\theta) P_{xx} + \cos^2(\theta) P_{zz} + \sin(2\theta) P_{zx}.
\]  

(3.13)

Inserting Eqn.(3.13) into (3.1) yields

\[
K_{y}^{z'z'} = \sin^2(\theta) K_{y}^{xx} + \cos^2(\theta) K_{y}^{zz} + \sin(2\theta) K_{y}^{zx}.
\]  

(3.14)

It is clear from Eqn.(3.14) that the expansions of \(K_{y}^{xx}\), \(K_{y}^{zz}\) and \(K_{y}^{zx}\) will yield that of \(K_{y}^{z'z'}\). As an example, we only consider the expansion of one of the coefficients, say \(K_{y}^{xx}\). By definition this PTC is given by

\[
K_{y}^{xx} = \text{Tr.} \left\{ F P_{y} F^{+} P_{xx}\right\} / \text{Tr.} \left\{ F F^{+}\right\}.
\]  

(3.15)

In expanding \(K_{y}^{xx}\) we must expand \(F\), \(F^{+}\), \(P_{y}\), \(P_{xx}\). Thus,

\[
F = \sum_{k=0}^{2} \sum_{q=-k}^{+k} Q_{qk} \tau_{qk}^{+},
\]  

(3.16a)

\[
F^{+} = \sum_{m_{n}} Q_{mn}^{*} \tau_{mn},
\]  

(3.16b)

\[
P_{y} = \sum_{i_{j}} B_{ij} \tau_{ij}^{+},
\]  

(3.16c)

\[
P_{xx} = \sum_{b_{d}} D_{bd}^{*} \tau_{bd},
\]  

(3.16d)

Substituting Eqns.(3.16) into (3.15) and using the orthonormalization of the \(\tau_{qk}\)'s , we obtain

\[
K_{y}^{xx} = \left( \sum_{i_{j}} \sum_{b_{d}} B_{ij} D_{bd}^{*} \tau_{ij} \right) / \left( \sum_{k,q} |Q_{qk}|^2 \right),
\]  

(3.17)
where we have introduced the new quantity, $\chi_{ij}^{bd}$, defined as

$$\chi_{ij}^{bd} = \text{Tr}\left\{ F \tau^{+}_{ij} F^{+} \tau_{bd} \right\}. \quad (3.18)$$

The complex numbers $\theta$ and $\phi$, can be obtained by inverting Eqns.(3.16c,16d) (the $P_y$, $P_{xx}$, ...etc. and the $\tau_{k_3}$ matrices are given in Appendix-B). The structure of expressions (3.17-18) remains the same for all other PTCs apart from minor changes of the complex constants $\theta$ and $\phi$ appearing in the numerator. We have now obtained a general prescription which provides a working model for the expansion of PTCs in terms of the scattering amplitude matrix elements, $Q_{k_q}(k_2, k_i)$.

In order to achieve the expansion of $K_{y}^{xx'}$ we replace $K_{y}^{xx}$, $K_{y}^{zz}$ and $K_{y}^{xx}$ by their respective expanded expressions into Eqn.(3.14) bearing in mind the existence of the relationship between the $Q_{2q}$'s (see Eqn.(3.12)). The details of the derivations and simplifications can be found in Appendix-C. The final result of the $K_{y}^{xx'}$ expansion is given by

$$K_{y}^{xx'} = \left\{ K1 + K2 + K3 + K4 + K5 \right\}, \quad (3.19)$$

where

$$K1 = \left[ \frac{6 \text{Im.} (Q_{0_0} Q_{21}^*)}{3 I_0} \right], \quad (3.20a)$$

$$K2 = \left[ \frac{-2 \text{Im.} (Q_{0_0} Q_{11}^*)}{3 I_0} \right], \quad (3.20b)$$

$$K3 = \left[ \frac{6 \text{Im.} (Q_{11} Q_{22}^*)}{3 I_0} \right], \quad (3.20c)$$

$$K4 = \left[ \frac{54 \text{Im.} (Q_{20} Q_{11}^*)}{3 I_0} \right], \quad (3.20d)$$

$$K5 = \left[ \frac{-6 \text{Im.} (Q_{20} Q_{21}^*)}{3 I_0} \right]. \quad (3.20e)$$
and $I_0$ is the cross-section due to an unpolarized incident deuteron beam

$$I_0 = \left[ |Q_{00}|^2 + |Q_{20}|^2 + 2 |Q_{11}|^2 + 2 |Q_{21}|^2 + 2 |Q_{22}|^2 \right]. \quad (3.21)$$

To identify the magnitudes of the contributions to $K_y^{x'z'}$ from the $K_i$ ($i=1-5$), we carried out optical model calculations including the $T^r$ type tensor interaction for the reaction $^{90}\text{Zr}(d,d)$ at 15 MeV. The computer code DDTP was used as well as a number of additional routines to, firstly, extract the complex amplitudes, $Q_{00}(\theta)$, $Q_{11}(\theta)$, $Q_{20}(\theta)$, $Q_{21}(\theta)$ and $Q_{22}(\theta)$ from DDTP’s scattering amplitude matrix and, secondly, to substitute them into Eqns.(3.19-21) to evaluate the angular distribution of $K_y^{x'z'}(\theta)$.

The results of the individual contributions to $K_y^{x'z'}$, $K_i$ ($i=1-5$), and $K_y^{x'z'}$ itself are shown in Figs.(3.2-4), respectively. The largest and second largest contributions to $K_y^{x'z'}$ arise from the quantities $K_1$ and $K_2$ respectively. The much smaller terms $K_3$ and $K_5$ have little impact on $K_y^{x'z'}$. $K_4$ is appreciable only within a short backward angular range ($100-160^\circ$). Although the term $K_2 = \text{Im}(Q_{00} Q_{11}^*)$ is the second largest contributor to $K_y^{x'z'}$, it does not bring about any new information because we already know [Ho71] that it is the leading term in $t_{11}$ and that it is mainly due to the spin-orbit interaction. Therefore, the quantity, $K_1 = \text{Im}(Q_{00} Q_{21}^*)$, is very important in the sense that it is the driving term in $K_y^{x'z'}$ and that it contains the element, $Q_{21}$, which as we mentioned earlier in this section is generated by the tensor interaction parts of the deuteron-nucleus potential. Notice that the large magnitude of $K_1$ emanates from the presence of the amplitude $Q_{00}$ which is due to the central part of the deuteron-nucleus interaction.
Fig. (3.2): Angular distributions of the K1 and K2 components of $K_y^{z',z}$ as defined by Eqs. (3.20a-b). These are obtained for the reaction $^9\text{Be}(d,d)$ at 15 MeV.
Fig. (3.3) - Angular distributions of the K3 and K4 components of \( K_y^{2+3} \) as defined by Eqs. (3.20c-d). These are obtained for the reaction \(^{90}Zr(d,p)\) at 15 MeV.
Fig. (3.4) - Angular distributions of the K5 component of $K^y_{x'y'}$ as given by Eqn. (3.20e) and that of $K^y_x$ itself. These are for the reaction $^{40}$Zr($^3$He,$^4$He) at 15 MeV.
Thus in $K_y^{z'z}$ we have here an observable which not only singles out the effects of the tensor forces but also contains large scattering amplitude combinations that can easily be used to extract accurate information concerning these forces. For instance, if one considers the PTC $K_y^{z'z'}$ in conjunction with the tensor analyzing power $T_{21}$ (dominated by Re.$(Q_{00}^*, Q_{21}^*)$) then it is in principle possible to obtain additional information that can be utilized to clarify the differences between theoretical model predictions [To87] of the deuteron-nucleus spin-dependent interaction and phenomenological observations of other studies [Go79, Ma82].

It is clear from our results that the origin of sensitivity to tensor force effects displayed by $K_y^{z'z'}$ is due to the presence of a large bi-linear combination, Im.$(Q_{00}^*, Q_{21}^*)$, of two scattering amplitude elements one of which (i.e. $Q_{21}^*$) is understood to be generated by the tensor part of the deuteron-nucleus potential. In the next section we investigate whether $K_y^{z'z'}$ is capable of distinguishing between the two tensor forces $T_r$ and $T_p$.

3.3) **DISCRIMINATION BETWEEN $T_r$ AND $T_p$**

Deuteron optical model calculations that include a $T_p$ tensor interaction were first performed by Goddard [Go77a, Go78]. The essence of Goddard’s study consisted of a numerical comparison between the effects of the $T_p$ tensor force and that of an appropriate $T_r$ tensor potential. The forms of the $T_r$ and $T_p$ potentials and their associated operators are
where $s, p$ and $r$ are the deuteron's spin, momentum and position operators, respectively ($\hat{r} = \frac{r}{|r|}$ and $p = |p|$). Goddard started his comparison using the following classical argument: if $p$ is nearly parallel to $r$, then the operator expressions on the right hand sides of Eqns.(3.22-23) are related to each other through

$$T_r = \frac{2}{3} \left( s \cdot \hat{r} \right)^2 - \frac{2}{3} = \frac{1}{p^2} \left[ \left( s \cdot p \right)^2 - \frac{2}{3} p^2 \right] = \frac{1}{p^2} T_p.$$  \hspace{1cm} (3.24)

Goddard further assumes that the $T_p$ potential can be transformed from the expression on the left hand side of Eqn.(3.23) to just $V_p(r) T_p$. This clearly means that $V_p(r)$ commutes with the operator $T_p$. Therefore, as a consequence of the classical argument and the assumption just mentioned, Goddard concludes that, at low energies, a potential of the form $V_t(r) T_p / p^2$ exerts the same force on a particle of spin-1 as a potential of the form $V_t(r) T_r$. In other words, for a given form factor $V_t(r) = V_u(r)$, it is possible to construct a $V_p(r)$ such that $V_p(r) = V_u(r) / p^2$.

To test these conclusions Goddard presented optical model calculations in which the effects of a $T_r$ potential with a given shape are compared against the effects of a $T_p$ potential whose form factor is given by $V_p(r) = V_u(r) / p^2$. This comparison was carried out for two reactions, $^{46}$Ti$(d,d)'$ at 13 MeV and $^{56}$Fe$(d,d)'$ at 30 MeV. The optical potential
parameters (real Woods-Saxon shapes) used for that purpose were chosen so as to give good fits to the cross-section, I\textsubscript{0}, and the vector analyzing power, iT\textsubscript{11}, only. The results of Goddard’s optical model calculations suggest that the elastic S-matrix calculated with a T\textsubscript{r} potential is very similar to that of a suitably chosen T\textsubscript{r} potential (i.e. when T\textsubscript{r} and T\textsubscript{p} potentials are assumed to have similar radial shapes) and that measurements of more observables, such as PTCs, will therefore not help to differentiate between the two potentials.

An obvious criticism of Goddard’s calculations lies in the fact that optical model analyses of experimental data suggest that the best fits to all polarization observables, I, iT\textsubscript{11}, T\textsubscript{20}, T\textsubscript{21} and T\textsubscript{22}, are obtained [Ke73, Go79] when different geometries are used to simulate the strengths of the T\textsubscript{r} tensor potentials. These, in general, do not satisfy the commutation assumption adopted by Goddard. Moreover, these analyses show that the T\textsubscript{r} potential is a complex function of r and that the imaginary part is rather large when compared with the folding model predictions.

In order to examine the validity of Goddard’s conclusions, we performed optical model comparison between the T\textsubscript{r} and T\textsubscript{p} potentials using realistic phenomenological V\textsubscript{u}(r) form factors. In our comparison we considered similar radial shapes for the two tensor interactions, T\textsubscript{r} and T\textsubscript{p}. The calculations were carried out for two reactions, 46\textsuperscript{Ti}(d,d) at 13 MeV and 90\textsuperscript{Zr}(d,d) at 15 MeV. The optical potential parameters, taken from [Go79], are reproduced in Table-1.

The results in Figs.(3.5-8) show T\textsubscript{20}(0), T\textsubscript{21}(0), T\textsubscript{22}(0) and K\textsubscript{y}\textsuperscript{z}z' for both reactions. We note that the effects of the T\textsubscript{r} and T\textsubscript{p} tensor forces on the tensor analyzing powers, T\textsubscript{20} and T\textsubscript{22}, are similar for both
systems. The cross-section and $iT_{11}$ are not affected by the tensor forces. The tensor analyzing power $T_{21}$ and the PTC $K_{z}^z$ however show completely different sensitivities to the two tensor forces in the case of the $^{50}$Zr$(d,d)$ reaction. In the $^{46}$Ti$(d,d)$ case, the effects of the tensor forces are much more apparent in $T_{21}$ than in $K_{y}^z$ where we observe differences only over a small angular range.

It is clear from these results that the effects of the $T_{r}$ and $T_{p}$ tensor potentials (even if they are of the same shape) on $T_{21}$ are not similar. This is true for both reactions. Moreover, with respect to the $^{90}$Zr reaction, and contrary to Goddard’s suggestion, the PTC $K_{y}^z$ shows significant differences between the effects of the two tensor potentials. We therefore conclude that Goddard’s main assumptions, viz,

a) the choice of a smoothly varying function of $r$ (Woods-Saxon shape) as the $V_{r}(r)$ form factor so that the corresponding $V_{p}(r)$ commutes with the operator $T_{p}$,

b) the use of optical potential parameters that give good fits to only the cross-section and the vector analyzing power,

c) and the fact that only real parts of the $T_{r}$ and $T_{p}$ potentials were considered,

do not hold in general.
Fig. (3.5) - Shows the angular distributions of the tensor analyzing powers $T_{20}$ and $T_{21}$ for the $^{46}$Ti(d,d) reaction at 13 MeV. The solid lines represent calculations with a $T_r$ tensor force and the dashed curves indicate calculations with a $T_p$ tensor potential.
Fig. (3.6) - Shows the angular distributions of the tensor analyzing power $T_{22}$ and the PTC $K_{y}^{x'z'}$ for the $^{46}$Ti$(d,p)$ reaction at 13 MeV. The solid lines represent calculations with a $T_T$ tensor force and the dashed curves indicate calculations with a $T_T$ tensor potential.
Fig. (3.7) - Shows the angular distributions of the tensor analyzing powers $T_{20}$ and $T_{21}$ for the $^{90}$Zr$(d,d)$ reaction at 15 MeV. The solid lines represent calculations with a $T_r$ tensor force and the dashed curves indicate calculations with a $T_p$ tensor potential.
Fig. (3.8). Shows the angular distributions of the tensor analyzing power $T_{22}$ and the PTC $K_{y}^{2'2'}$ for the $^{90}$Zr($d,d$) reaction at 15 MeV. The solid lines represent calculations with a $T_r$ tensor force and the dashed curves indicate calculations with a $T_p$ tensor potential.
Our results also indicate that it is misleading to use optical potential parameters (shapes and geometries), that give acceptable fits to only the cross-section and the vector analyzing power, in calculations that are aimed at probing tensor force effects. This is, because tensor force effects show up predominantly in the second rank types of spin observables such as $T_{21}$ and from this work the PTC, $K^{x'z'}_y$.

Having established that $K^{x'z'}_y$ and $T_{21}$ are sensitive to $T_r$ and $T_p$, one would like to make a comparative study of the different sensitivities in order to investigate which amongst these two observable is the best probe of a particular tensor force. To achieve this, we perform calculations for the $^{90}$Zr reactions using $T_r$ and $(T_r + \delta T_r)$ as well as $T_p$ and $(T_p + \delta T_p)$ interactions. The notations $\delta T_{r,p}$ refer to a small variation of the depths of the $T_r$ and $T_p$ forces, respectively. In fact, we take $\delta T_{r,p}$ to be equal to one sixth of the respective original depths.

The results of these calculations for $K^{x'z'}_y$ and $T_{21}$ are shown in Fig.(3.9). Adding a small $T_r$ potential induces changes in $K^{x'z'}_y$ at all angles, whereas in $T_{21}$ only the peaks are slightly affected. The effects of a small $T_p$ force are only localized at the peaks for both observables. We conclude:

i) none of the observables is particularly sensitive to small variations of the $T_p$ tensor potential,

ii) only the PTC, $K^{x'z'}_y$, is particularly sensitive to small variations of the $T_r$ tensor potential.
Fig. (3.9) - Represents the results of the angular distributions of $K_{y}^{+}$ and $T_{21}$ when using a $T_r$, $(T_r + \text{small } T_r)$, $T_p$ and $(T_p + \text{small } T_p)$ for the reaction $^{90}\text{Zr} (d, d)$ at 15 MeV. Small $T_r, T_p$ refer to small variations in the depths of their respective potentials, and are taken to be equal to one sixth of those depths.
The above calculations suggest that $K_{yz}^{z'}$ shows a relatively stronger sensitivity to small variations of the depth of the $T_r$ potential than to that of $T_p$. To clarify this point we analyse the behaviour of the observable as a function of the depths of both tensor forces. The absolute differences for $K_{yz}^{z'}$ as a function of the depth $V$ at the c.m. angles, 50, 65, 110 and 165° (i.e. $\Delta K_{yz}^{z'}(V) = |K_{yz}^{z'}(V) - K_{yz}^{z'}(V)|$ for a fixed $\theta_{c.m.}$) are shown in Fig.(3.10) for the $^{90}\text{Zr}(d,d)$ case. At all four c.m. angles, the slopes of the $\Delta K_{yz}^{z'}(V)$ curves are much steeper in the $T_r$ than in the $T_p$ case indicating that $K_{yz}^{z'}$ is more sensitive to variations of the $T_r$ tensor force than to those of $T_p$. We conclude that $K_{yz}^{z'}$ is a sensitive observable to the $T_r$ tensor interaction, and that its experimental determination is highly recommended as far as future analyses of the $T_r$ tensor force are concerned.
Fig. (3.10) - Shows the absolute differences for $K_y^{Z^*}$ as a function of the depth of the $T_p$ potentials. These are for the $^{90}_{\text{Zr}}(d,p)$ reaction at 15 MeV. Each graph represents the results for a particular c.m. angle. We chose $\Theta_{\text{cm}} = 50, 65, 110$ and 165.
3.4)- CONCLUSIONS

We have shown that the Polarization Transfer Coefficient $K_{y}^{z'}$ which corresponds to the transfer of the incoming vector polarization along the \( y \)-axis (\( p_{y} \)) to the outgoing tensor polarization (\( p_{x'y'} \)), is strongly affected by tensor force effects. The origin of this sensitivity is the bi-linear combination, \( \text{Im}(Q_{00}Q_{*}^{*}) \), where \( Q_{21} \) is one of the scattering amplitude matrix elements that is mainly due to the tensor force parts of the deuteron-nucleus potential.

We also investigate whether it is possible to distinguish between the effects of $T_{r}$ and $T_{p}$ tensor forces at low energies. For this, we carried out optical model calculations with $T_{p}$ and $T_{r}$ tensor interactions. Contrary to Goddard’s suggestion [Go77a, Go78], we have found that for realistic optical model parameters, $T_{21}$ and $K_{y}^{z'}$ discriminate between the effects of $T_{r}$ and $T_{p}$ potentials. The results of our $T_{r}$ - $T_{p}$ comparison suggest that:

i) The statements and semi-classical arguments made by Goddard [Go77a, Go78] are not valid in general. For instance, he assumes that the tensor potentials form factors are slowly varying functions of \( r \), which automatically allow the commutator, \( [V_{r}(r), T_{p}] \), to be small. This is not always the case and in particular the potentials required to produce good fits to the data show much more radial structure.

ii) Assuming similar radial shapes (but different from Goddard’s) to simulate the tensor potentials, $T_{r}$ and $T_{p}$, the effects on some polarization observables, namely, $T_{21}$ and $K_{y}^{z'}$, are different. Therefore, it is possible to distinguish, even at low energies, between the two types of tensor forces.
The pair \( (\mathbf{T}^{x'y'}, K'^x_y) \) when used in conjunction with other observables (tensor analyzing powers and PTCs) may prove very useful in disentangling and clarifying ambiguities relating to the ratio of the strengths of the real and imaginary parts of the tensor interaction. Furthermore, we show that \( K'^x_y \) is particularly sensitive to the \( \mathbf{T} \) tensor interaction.

Preliminary data for the \( ^{90}\text{Zr}(d,d) \) reaction at 15 MeV, have been obtained at TUNL [Ab86, Ab87]. The recently published [Ab88, Ab89] experimental points (four data points) are for the combination (3K'^x' y' + K'^x' y'') and are shown in Fig.(3.11). Optical model calculations carried out by [Ab89] show that the trend of the obtained theoretical predictions is not that far from the few experimental points.
Fig. (3.11) - Shows the angular distribution of \( (3K_y^z z' + K_y^z z') \) for the \(^{90}\text{Zr}(d,d)\) reaction at 15 MeV. The solid curve represents optical model predictions when using a complex \( T_r \) tensor force, whereas the dashed line indicates the result without \( T_r \). The parameters of the \( T_r \) potential are:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Real</th>
<th>Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V ) (MeV)</td>
<td>0.50</td>
<td>3.50</td>
</tr>
<tr>
<td>( r ) (fm)</td>
<td>1.65</td>
<td>1.24</td>
</tr>
<tr>
<td>( a ) (fm)</td>
<td>0.47</td>
<td>0.65</td>
</tr>
</tbody>
</table>

This graph and the potential parameters are reproduced from [Ab89].
PART II

Deuteron Breakup Effects

in

The Weinberg States Expansion Model
CHAPTER IV

- INTRODUCTION -

The analysis of direct nuclear reaction processes has always been at the forefront of nuclear physics studies and is to a large extent responsible for our current knowledge of nuclear structure and nuclear interactions. Being the simplest composite nucleus, the deuteron is often considered as an appropriate probe for experimental reaction investigations. In fact an ever increasing data base of deuteron induced reactions, at various bombarding energies, has and still is the subject of ongoing detailed theoretical examinations.

As a part of these efforts, the central aim of this work is to study breakup effects in deuteron induced nuclear reactions. These reactions are considered as the simplest theoretical context in which such studies can be made. The long term thinking behind this and other related theoretical analysis is to eventually use what we learn from deuteron induced reactions to investigate analogous effects in reactions induced by other loosely bound projectiles such as $^6$Li, $^7$Li, $^9$Be.

In this part of the thesis we consider deuteron stripping reactions A(d,p)B, and specifically reactions at intermediate incident energies, $E_{d}^{lab} \sim 100$ MeV. At this energy regime, one is sure that any breakup effects associated with the Coulomb force are negligible [Ci65, Gi66, Jo70]. In order to extract nuclear structure information from such experimental data, one requires an accurate comparison of data with the predictions of a realistic direct reaction theory. One such theory is the Distorted Wave Born Approximation or DWBA [Ho53, To61, Au70, Le73, Gi75].
The DWBA method has enjoyed some success in describing (d,p) and (p,d) cross-sections [Ph68] and polarization data [Yu68, Bj69]. By construction however, the DWBA does not explicitly account for breakup effects and assumes that the reaction takes place between deuteron-target and proton-residual nucleus elastic scattering states. The accuracy and validity of the DWBA has been the subject of a number of studies using three-body models [e.g. Bo72] and it is now established that this method needs to be extended to include explicitly these breakup effects.

Considerable theoretical efforts have also been devoted to developing other theories which treat the three-body aspects of stripping reactions in terms of the exact three-particle scattering techniques [Mi65, Re67, Bo72, Ki73]. However, the application of these three-body techniques to the (A+2)-body problem of stripping is not clear since they involve a large number of partial waves and complicated interactions.

One attempt to improve the DWBA, by Johnson and Soper [Jo70], included approximately the deuteron breakup channels within the Adiabatic approximation; denoted ADIA in the following. In this method, the deuteron-target system is treated as a 3-body structure (n + p + target), but the centre of mass (c.m.) energy of the n-p pair in the breakup states is assumed degenerate with that of the deuteron elastic channel. Systematic improvements in the description of transfer reaction angular distributions were obtained using the ADIA model [Ha71b, Sa71, Oh74, Wa77].

Although the Adiabatic model has in general provided improvement over the DWBA, a number of stripping and pick-up transitions were not well reproduced by either theory [Ha71b, Sa71, Oh74]. Moreover, recent
experimental data [St86, St87] on large l-transfer (d,p) transitions at 100 MeV are in no way described by ADIA. This has required a reassessment of ADIA.

Two techniques have been developed to extend the adiabatic method which, while not requiring an exact three-body treatment, do contain the dominant contributions from the three-body channels. In the first of these models, the so-called Quasi-Adiabatic approximation or QAD [Am84, St90], the c.m. energy of the broken up n-p pair is no longer assumed to be degenerate with the deuteron elastic channel. Having said this, there is a certain flexibility as to the choice of the most appropriate c.m. energy.

The second technique which can extend systematically the Adiabatic formalism, is the Weinberg States Expansion method denoted WSE in this work. This method, proposed by Johnson and Tandy [Jo74], has never been investigated quantitatively. This part of the thesis is concerned with the analysis of the WSE formalism, its numerical feasibility and its application to deuteron stripping reactions. An assessment is made of the quality of improvements that are expected from this method when compared against the ADIA theory.

A different approach, called the Coupled Discretized Continuum Channels method denoted CDCC, was proposed by Rawitscher [Ra74] and the Pittsburgh group [Fa76, Au78]. This technique, subsequently refined by the Kyushu collaboration [Ka86, Au87] has been used to analyse transfer reactions. In this method the n-p breakup continuum is discretized, resulting in a truncated set of coupled channel equations. A major shortcoming of the CDCC calculations is that they are complex to carry out and are
computationally very expensive. This is one reason why the extension, of the numerically efficient ADIA method is a desirable option. However, one should remark that CDCC calculations provide a reliable reference against which approximate breakup treatments can be checked.

The plan of this part of the thesis is as follows. In the next Chapter the formal theory and theoretical models appropriate to stripping reactions, as well as a brief derivation of the basic points of the DWBA, ADIA, QAD and to a lesser degree CDCC methods are given. The apparent failure of ADIA to account for certain stripping transitions and the QAD recommendations regarding the refinement of ADIA’s main approximation have led to the introduction of the WSE method. The formal theory of the WSE model is the subject of Chapter VI. Having established most of the theoretical ingredients of the WSE method, we discuss the numerical implementation of the formalism. This numerical implementation and the comparison of the WSE predictions against ADIA and QAD form the subject matter of Chapter VII. Our summary, concluding remarks and suggestions for future work are presented in Chapter VIII.
CHAPTER V

THEORETICAL MODELS FOR DIRECT NUCLEAR REACTIONS

5.1) - INTRODUCTION

Nuclear reactions are in general divided into two groups: reactions that proceed by a direct-reaction mechanism and those that proceed by a compound-nucleus reaction scheme. The latter class of reactions are assumed to involve complicated excitations of many degrees of freedom of the many-body system. However, there exist cases where only a few degrees of freedom are excited, the other degrees of freedom remaining effectively passive. Such simpler reactions fall into the direct-reaction category.

In this work we study the simplest direct nuclear reaction, namely deuteron stripping, or (d,p), reactions. In the previous Chapter we cited most of the theories developed to analyse such reactions. In this Chapter we discuss the formal theory common to all of these theories.

5.2) - FORMAL THEORY OF (d,p) REACTIONS

Deuteron stripping reactions are represented schematically by

$$d(p+n) + A \rightarrow p + B(n+A),$$

where A and B are the initial (target) and final (residual) nuclei, d and p stand for the incident deuteron and the outgoing proton, respectively. It is assumed that the residual nucleus is a bound state of the (n+A) system. The incident deuteron and outgoing proton wave vectors are denoted by $K_d$ and $k_p$ respectively, and their c.m. energies are given by
\[ E_d^{\text{cm}} = \left( \frac{\hbar^2 K_d^2}{2 \mu_i} \right), \quad (5.1) \]

\[ E_p^{\text{cm}} = \left( \frac{\hbar^2 k_p^2}{2 \mu_f} \right) = E_d^{\text{cm}} + Q. \quad (5.2) \]

Here \( \mu_i \) and \( \mu_f \) are the reduced masses in the initial and final channel, respectively, and \( Q \) represents the Q-value of the reaction,

\[ Q = E_p^{\text{cm}} - E_d^{\text{cm}} = \varepsilon_n - \varepsilon_d, \quad (5.3) \]

with \( \varepsilon_{n,d} \), defined positive, the separation energy of the neutron from the residual nucleus and the deuteron binding energy respectively.

Strictly speaking, the \( ^A(d,p)^B \) reaction is a many-body problem, therefore the total Hamiltonian \( \mathcal{H} \) of the system should represent all degrees of freedom of the target (residual) nuclei as well as those of the deuteron (outgoing proton). Thus, in the incident channel we have

\[ \mathcal{H} = h_A + h_{np} + \mathcal{F}_{\text{inc}} + V_{dA}, \quad (5.4a) \]

and in the outgoing channel

\[ \mathcal{H} = h_B + \mathcal{F}_{\text{out}} + V_{np} + V_{pA}, \quad (5.4b) \]

where \( h_A \), \( h_B \) and \( h_{np} \) are the internal Hamiltonians of the target, residual nucleus and deuteron. The \( \mathcal{F}_{\text{inc, out}} \) are the appropriate channel kinetic energy operators, \( V_{np} \) is the free neutron-proton interaction and \( V_{dA} \) describes the deuteron-target interaction and is usually considered as the sum of the many-body neutron-target \( V_{nA} \) and the proton-target \( V_{pA} \) interactions. The Coulomb interaction \( V_c \) is omitted for simplicity.
The total wave function for a deuteron with incident momentum \( \mathbf{K}_d \) on a target \( A \), obeying outgoing wave boundary conditions in all open channels, satisfies the \((A+2)\)-body Schrödinger equation

\[
\mathcal{H} \Psi_{K_d}^{(+)}(\xi, p, n) = E \Psi_{K_d}^{(+)}(\xi, p, n), \quad (5.5)
\]

where \( E \) is the total energy of the d-A system, \( E = E_d^\text{cm} - \varepsilon_A - \varepsilon_d \), and \( \xi \) represents all the internal degrees of freedom of the target \( A \). The eigenstates of the internal Hamiltonian of the target are assumed to form a complete set of states denoted by \( \{ \Sigma^A_{0}(\xi) \} \) such that the target ground state \( \Sigma^A_{0}(\alpha=0) \) satisfies (target binding energy)

\[
h_A \Sigma^A_{0}(\xi) = -\varepsilon_A \Sigma^A_{0}(\xi). \quad (5.6)
\]

Thus, outside the range of the neutron and proton-target interactions (i.e., \( V_{dA} = 0 \)), the incident waves part of the solution to Eqn.(5.5) is

\[
\Psi_{K_d}^{(+)}(\xi, p, n)^\text{inc} = \exp(i \mathbf{K}_d \cdot \mathbf{R}) \phi_d(r, p, n) \Sigma^A_{0}(\xi), \quad (5.7)
\]

where \( \phi_d \) is the deuteron ground state and \( \Sigma^A_{0} \) is the initial state of the target. As the deuteron beam enters the nuclear field of the target all sorts of processes take place. In particular, the nuclear potentials \( V_{nA} \) and \( V_{pA} \) exert forces capable of distorting and breaking up the loosely bound deuteron, with a probability of capture of the neutron (proton) to the target nucleus. Provided one or more stripping channels are open, then asymptotically \( \Psi_{K_d}^{(+)} \) will contain outgoing proton waves. Thus

\[
\Psi_{K_d}^{(+)}(\xi, p, n) \xrightarrow{r_p \to \infty} -\frac{\mu_f}{2\pi\hbar^2} T_{dF} \Sigma_{0}^B(\xi, n) \frac{e^{ik_p r_p}}{r_p}, \quad (5.8a)
\]
where $T_{dp}$ denotes the matrix elements of the transition operator $T$,

$$T_{dp} = \langle k_p | T | K_d > ,$$  \hspace{1cm} (5.8b)

with $\Sigma^B_{f}(\xi,n)$, an eigenstate of $\hbar_B$, the final state of the residual nucleus $(n+A)$. For clarity, we have omitted spin and spin-projection indices. All $(d,p)$ reaction information concerning a transition to a particular final state of $B$ is calculable from the matrix elements of $T$.

The various methods to be discussed in the next sections all have as their aim an estimate of $T_{dp}$ as it is necessary for the determination of the reaction observables. We start by reviewing the DWBA method.

5.3) - THE DISTORTED WAVE BORN APPROXIMATION

The transition matrix elements appearing in Eqn.(5.8b) may be written as [Au70]

$$T_{dp} = \langle e^{ik_p \cdot r_p} \Sigma_B f(\xi,n) | V_{np} + V_{pA} | \Psi^{(+)}_{K_d}(\xi_{p,n}) > , \hspace{1cm} (5.9)$$

where $\Psi^{(+)}$ is the exact many-body wave function of Eqn.(5.5). The many-body wave function $\Psi^{(+)}$ can always be expanded in terms of the target states $\{ \Sigma^A_{A}(\xi) \}$. In deriving the DWBA, an alternative form of $T_{dp}$ is useful. This is obtained by including the effects of an arbitrary two-body distorting potential, $U_{pA}$, on the final state. It is shown [Au70, Bo72] that Eqn.(5.9) is equivalent to

$$T_{dp} = \langle \chi^{(-)}_{K_d} \Sigma_f^B | V_{pA} U_{pA} | \Psi^{(+)} > + \langle \chi^{(-)}_{K_d} \Sigma_f^B | V_{np} | \Psi^{(+)} > , \hspace{1cm} (5.10)$$

where $\chi^{(-)}$ is the proton-residual nucleus distorted wave generated by $U_{pA}$. A suitably chosen $U_{pA}$ can cancel a large part of $V_{pA}$, thus removing
the first term in Eqn.(5.10). The many-body wavefunction $\Psi^{(+)}_{K_d}$ contains, asymptotically, waves describing both deuteron elastic scattering and broken up n-p pairs. In the DWBA, transfer is assumed to take place between two-body states and $\Psi^{(+)}_{K_d}$ is replaced by the product of a deuteron internal wave function and a c.m. optical model wave function $\Xi^{(+)}_{K_d}$ which describes deuteron elastic scattering. Taking all these approximations together leads to the DWBA transition matrix elements

$$T^{\text{DWBA}}_{d,p} = \langle \chi_k^{(-)} \Sigma_{B} | V_{np} | \Sigma_{A} \phi_{d} \Xi^{(+)}_{K_d} \rangle.$$  \hspace{1cm} (5.11)

As the deuteron approaches the target its internal state must be distorted away from $\phi_{d}$ and may break up into an unbound n-p pair, thus generating flux in channels not included in the DWBA. We now discuss three-body models which attempt to include such channels explicitly.

5.4) - THE ADIABATIC APPROXIMATION

In this method the deuteron-target system is treated as a three-body structure. The adiabatic approximation (or ADIA) consists of considering the c.m. energy of the n-p pair in the breakup states as degenerate with the deuteron elastic channel. In their work, Johnson and Soper [Jo70] have in fact explained how this idea is implemented in the context of a three-body model of the deuteron-target system.

Here we will assume that the couplings between the ground and excited states of both target and residual nuclei are weak. It is then possible to treat the target as a structureless core and the residual nucleus as a neutron-core bound state thus suppressing the internal coordinate $\xi$ of the target. The coordinate system appropriate to such a three-body structure is shown in Fig.(5.1), with $r$, $R$ as the relative and centre of
mass coordinates of the neutron and proton respectively. Explicitly,

\[ r = r_p - r_n \quad \text{and} \quad R = \frac{1}{2} \left( r_p + r_n \right), \] (5.12)

where the origin of the coordinate system is the c.m. of the target nucleus which we assume is infinitely heavy.

![Diagram of coordinate system](image)

*FIG. (5.1) - Shows the coordinate system adopted throughout this work to represent the (d,p) reaction channel variables.*

In the incident channel the total Hamiltonian \( \mathcal{H} \) for the three-body model takes the form

\[ \mathcal{H} = \mathcal{H}_{np} + \mathcal{V}_{cm} + \left( V_{nA}(R - r/2) + V_{pA}(R + r/2) \right). \] (5.13)

where (\( m \) is the nucleon mass)

\[ \mathcal{H}_{np} = \mathcal{V}_{np} + V_{np}, \] (5.14a)

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and
\[ \mathcal{H}_{np} = -\left(\frac{\hbar^2}{2m}\right) \nabla^2_r \quad \text{and} \quad \mathcal{H}_{cm} = -\left(\frac{\hbar^2}{4m}\right) \nabla^2_R . \] (5.14b)

The three-body wave function, initiated by an incident deuteron beam of kinetic energy \( E_d^{cm} \), satisfies \( (E = E_d^{cm} - E_d) \)
\[ \mathcal{H}_{K_d} \Psi^{(n)}(r,R) = E \Psi^{(n)}(r,R) . \] (5.15)

In this model \((n + p + \text{core}) T_{dp} \) reduces to [e.g. Sa65, Au70]
\[ T_{dp} = \langle \chi^{(p)}(r_p) , \Phi_n(r_n) | V_{np} | \Psi^{(n)}_{K_d} > , \] (5.16)
where \( \Phi_n \) is the neutron-core bound state wave function and \( \chi^{(p)} \) is the proton distorted wave generated by \( V_{pa} \), describing proton-core elastic scattering at the proton energy \( E_p^{cm} \).

As can be seen from Eqn.(5.16), the numerical evaluation of \( T_{dp} \) requires accurate knowledge of \( \Psi^{(n)} \) only in a very limited region of space, that is for \( r \leq \) the range of \( V_{np} \) and for \( R \) within the range of \( \Phi_n \).
This implies that, in the case where \( V_{np} \) is assumed to be a zero-ranged interaction, then only the components of \( \Psi^{(n)} \) with \( n \) and \( p \) in a relative \( S \) state contribute to the stripping matrix element. It is therefore legitimate to expand \( \Psi^{(n)} \) in terms of the eigenstates of \( \Psi_{np} \). These states form a complete set, viz,
\[ \hbar_{np} \phi_d = -\varepsilon_d \phi_d \quad \text{and} \quad \hbar_{np} \phi_k = \varepsilon_k \phi_k , \quad (\varepsilon_k \geq 0) \] (5.17a)
with
\[ \langle \phi_d | \phi_k > = 0 \quad \text{and} \quad \langle \phi_k | \phi_{k'} > = \delta(k - k') , \] (5.17b)
\[ | \phi_d > < \phi_d | + \int dk | \phi_k > < \phi_k | = 1 , \] (5.17c)
where $|\phi_d\rangle$ is the deuteron ground state and the $|\phi_k\rangle$ represent the scattering states of the n-p system. Thus one can write \[\text{[Jo70]}\] (with no subscript in $\Psi^{(r)}$)

$$\Psi^{(r)}(r,R) = \phi_d(r) \chi_0(R) + \int dk \phi_k(e_k, r) \chi_k(e_k, R).$$ \hspace{1cm} (5.18)

Substituting (5.18) into (5.15) yields at $r = 0$

$$\left[ E + \epsilon_d - \mathcal{F}_{\text{em}} - V_{nA}(R) - V_{pA}(R) \right] \phi_d(r=0) \chi_0(R) + \int dk \left[ E - \epsilon_k - \mathcal{F}_{\text{em}} - V_{nA}(R) - V_{pA}(R) \right] \phi_k(e_k, r=0) \chi_k(e_k, R) = 0. \hspace{1cm} (5.19)$$

In ADIA the continuum relative energies $\epsilon_k$ in Eqn.(5.19) are replaced by the deuteron binding energy $-\epsilon_d$. An immediate consequence of such an approximation is that (5.19) becomes a differential equation for $\Psi^{(r)}(0,R)$ and that in the zero-range limit for $V_{np}$ one can write

$$\Psi^{(r)}(0,R) = \phi_d(0) \tilde{\chi}(R), \hspace{1cm} (5.20a)$$

where $\tilde{\chi}(R)$ satisfies

$$\left[ E + \epsilon_d - \mathcal{F}_{\text{em}} - V_{nA}(R) - V_{pA}(R) \right] \tilde{\chi}(R) = 0. \hspace{1cm} (5.20b)$$

Having obtained $\Psi^{(r)}$, one can now evaluate $T_{dp}$ which, according to Eqn.(5.16),

$$T_{dp}^{\text{ADIA}} |_{Z,R} = G_0 \int dR \left[ \chi^{(r)}_{\phi_p}(R) \right]^* \left[ \Phi_n(R) \right]^* \tilde{\chi}(R), \hspace{1cm} (5.21a)$$

where

$$G_0 = \int dr V_{np}(r) \phi_d(r), \hspace{1cm} (5.21b)$$

is the so-called zero-range constant.
The ADIA method is easily generalized to include finite-range corrections [Jo70]. In ADIA, the replacement of \( e_k \) by \( -e_d \) in Eqn.(5.19) means that the contribution from the n-p continuum states to the total wave function \( \Psi^{(r)} \) is incomplete in the sense that all of these states have the same c.m. energy. Nevertheless, this in itself is an improvement over the DWBA method where such effects are not considered at all.

In the DWBA, \( \Psi^{(r)} \) is substituted with an elastic wave function implying that the behaviour of \( \Psi^{(r)} \) well inside the nucleus (strong interaction zone) is replaced by a function that only has the correct asymptotic form. This is not the case in ADIA, since the geometry of the new ADIA potential (Eqn.(20b) of [Jo70]) generates deuteron c.m. wave functions that contain explicitly outgoing waves associated with deuteron breakup into low-energy n-p relative S states as well as elastic scattering.

5.5) - THE QUASI-ADIABATIC APPROXIMATION

In order to improve upon ADIA, Amakawa and co-workers [Am84] extended the adiabatic theory to include positive relative n-p energies when they introduced the Quasi-Adiabatic approximation or QAD.

Denoting by \( U(r,R) \) the sum of the nucleon-target optical potentials \( V_{nA} \) and \( V_{pA} \), both containing spin-orbit forces, that appear in the three-body Hamiltonian of Eqn.(5.13), then the angular average of Eqn.(5.15) (when considering only S-wave relative n-p motion) is

\[
\left[ E - \hbar_{np} - T_{cm} - U(r,R) \right] \tilde{\Psi}^{(r)}(r,R) = 0 ,
\]

(5.22)

where \( U(r,R) \) is the angular average of \( U(r,R) \). By replacing \( \hbar_{np} \) by \( -e_d \) in
Eqn.(5.22), one obtains the ADIA equation

\[
\begin{bmatrix}
E_d^\text{cm} - \mathcal{T}_{\text{cm}} - \hat{U}(r,R)
\end{bmatrix} \psi^{\text{AD}}(r,R) = 0.
\] (5.23)

The elastic and breakup parts of $\psi^{\text{AD}}$ can be extracted by projection onto the deuteron ground state [Co89, St90]. Thus it is possible to write the breakup component as

$$
\psi^{\text{AD,BU}}(r,R) = \psi^{\text{AD}}(r,R) - \psi^{\text{AD,EL}}(r,R).
$$ (5.24)

The QAD equation is obtained by decomposing the solution of Eqn.(5.22) into elastic and breakup components to produce the inhomogenous equation for $\psi^{\text{BU}}(r,R)$

\[
\begin{bmatrix}
E - \hbar^2 \nabla^2 - \mathcal{T}_{\text{cm}} - \hat{U}(r,R)
\end{bmatrix} \psi^{\text{BU}}(r,R) = \begin{bmatrix}
\hat{U}(r,R) - U^\text{opt}(R)
\end{bmatrix} \psi^{\text{EL}}(r,R),
\] (5.25)

where $U^\text{opt} = \left(E_d^\text{cm} - \mathcal{T}_{\text{cm}} \right) \psi^{\text{EL}}$ is an optical potential that generates the elastic part $\psi^{\text{EL}}$ of the total wave function [Co89]. The breakup piece of the wave function is thus the solution of an equation in which the elastic component appears in a source term. The QAD equation is obtained using the following approximations:

a) as in ADIA, the Hamiltonian $\hbar^2 \nabla^2$ in Eqn.(5.25) is replaced by either a number or a function independent of $r$. For the moment we denote this quantity by $\hat{\epsilon}$ and we will discuss specific choices later on,

b) the ADIA has proved to be very successful in describing deuteron elastic scattering data. Thus, it is plausible to consider the elastic part of the adiabatic calculation to be a good approximation to the exact wave function, namely $\psi^{\text{EL}}(r,R) = \psi^{\text{AD,EL}}(r,R)$. 

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The QAD equation (5.25) for the breakup component of the three-body wave function becomes [Co89, St90]

\[ [E - \tilde{e} - \mathcal{U}_{cm}(r,R)] \Psi^{QA,BU}(r,R) = \left[ \mathcal{U}(r,R) - \mathcal{U}^{opt}(R) \right] \Psi^{AD,EL}(r,R). \] (5.26)

Amakawa et al. [Am84] suggested a simple two parameter prescription for \( \tilde{e} \) given by (L is the c.m. deuteron angular momentum)

\[ \tilde{e}_L = E_0 \exp \left(-L/L_0\right), \] (5.27)

where \( E_0 \) (energy) and \( L_0 \) (angular momentum) are two parameters that can be adjusted to either fit the data or other reliable calculations. A more physically consistent prescription for the mean breakup energy is to assume that the breakup part of the ADIA wave function \( \Psi^{AD,BU} \) provides a reasonable first guess of the breakup wave function and the n-p relative motion components in it [Co89]. An estimate of \( \tilde{e} \), the expectation value of \( n_{np} \) in \( \Psi^{AD,BU} \), is

\[ \tilde{e}_{JL}(R) = -\varepsilon_d + \frac{\hbar^2}{2 \mu_{np}} \phi_d(r) \frac{\partial \Sigma^{AD,BU}_{JL}(r,R)}{\partial r} \frac{\partial \Sigma^{AD,BU}_{JL}(r,R)}{\partial r} , \] (5.28)

where the \( \Sigma^{AD,BU}_{JL}(r,R) \) are the J,L components of \( \Psi^{AD,BU}(r,R) \). Clearly, the second term in the RHS of Eqn.(5.28) represents the correction which the QAD model generates to complement the ADIA prediction. These corrections are seen to arise from a consideration of the \( r \)-dependence of \( \Sigma^{AD,BU}_{JL} \).

The expression of Eqn.(5.28) is one of a number of plausible theoretical prescription for \( \tilde{e} \), but is without a rigorous justification. However, recent QAD calculations [St90] for the \(^{66}\text{Zn}(d,p)^{67}\text{Zn} \) reaction at \( E^\text{lab} = 88.2 \) MeV, show significant improvements when compared against equivalent ADIA results.

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It appears, following the apparent success of the above QAD calculations, that it is worthwhile to devote more theoretical effort to account for and systematically include the n-p centre of mass energy within the (d,p) formalism. In the next Chapters, we will discuss one such effort, namely the WSE theory. As mentioned earlier in this Chapter, there is another method which is also widely used to analyse (d,p) reactions, namely, the Coupled Discretised Continuum Channels method (CDCC). In the next section, we briefly introduce the CDCC method. The complete derivations and discussions of this method can be found in two recent and excellent review articles [Ka86] and [Au87] as well as in the work of Iseri et al. [Is85, Is86].

5.6 - THE COUPLED DISCRETISED CONTINUUM CHANNELS

In the CDCC method, the treatment of the relative n-p centre of mass is circumvented by discretizing the n-p continuum. Thus, using the eigenstates of the n-p Hamiltonian $\h_{np}$, one can expand $\Psi^{(+)}$ as follows

$$\Psi^{(+)}(r,R) = \chi^{(+)}(r,R) \phi_0(r) + \int dk \, \chi^{(+)}(e_k, r, R) \phi_k(e_k, R), \quad (5.29)$$

where the total breakup channel contributions are given by the results of the integral over the continuum of $k$ values. However, the $k$ continuum renders any coupled channel calculations impracticable. To remedy this, Rawitscher [Ra74] and the Pittsburgh group [Fa76, Au78] introduced the concept of continuum discretization within a coupled channel formalism, namely, the CDCC method. In this method, a maximum value $k_{max}$ is assigned to $k$ and the momentum continuum is discretized into bins with a common width $\Delta k$. As a result, the integration in Eqn.(5.29) is replaced by a
summation over the k bins to generate the CDCC wave function $\Psi_{\text{CDCC}}$, with the n-p wave function in each bin (i) being described by a representative function $\hat{\phi}_i$ [e.g. Is83].

It was shown [Ya86], that convergence of the elastic transition matrix elements can be obtained for acceptable values of $k_{\text{max}}$ and $\Delta k$. Increasing attention has been payed [e.g. Ka77, Ka78, Ka86] to the CDCC model to the point where routine calculations are now feasible, though computationally very expensive. As such, the CDCC method provides benchmark calculations by which more computationally efficient but approximate breakup treatment methods can be rated. However, one should emphasize that for breakup studies the ADIA method can include deuteron D-state and full spin-orbit distortion effects, whereas the CDCC method has yet to be applied to transfer reactions in anything but the zero-range approximation [Is83, Is85].

As an alternative to the CDCC and as a systematic extension of the Adiabatic ideas, we study in the next Chapters the WSE formalism and investigate the quality of improvements it provides over the ADIA model.
CHAPTER VI

- FORMAL THEORY OF THE WEINBERG STATES EXPANSION MODEL -

6.1) - INTRODUCTION

We mentioned in section (5.4) that the adiabatic approximation (ADIA) needs to be refined. Two methods, based on the ideas of ADIA, the Quasi-Adiabatic approximation (QAD) and the Weinberg States Expansion (WSE) were proposed. An outline of the QAD approximation is given in section (5.5) of Chapter V. In this Chapter, we lay down the theoretical basis and the main approximations of the WSE formalism.

In the next sections we describe the three-body model of the (d,p) reaction, recognize the three-body effects of interest, introduce the basic approximation of the theory and finally establish the set of coupled channel equations that needs to be solved.

6.2) - FORMAL THEORY OF THE APPROXIMATE THREE-BODY MODEL

The three-body wave function describing an incident deuteron beam of c.m. kinetic energy $E^\text{cm}_d$ and internal binding energy $\varepsilon_d$ satisfies the Schrödinger equation ($E = E^\text{cm}_d - \varepsilon_d$)

$$ H \Psi^{(i)}(r,R) = E \Psi^{(i)}(r,R), \quad (6.1) $$

where $H$ is the three-body Hamiltonian of Eqn.(5.13). If we explicitly specify the incident boundary condition of a deuteron with internal wave function $\phi_d$ and incident momentum $K_d$, then
where the physical total wave function $\Psi(\tau,R)$ is to be calculated in the limit $\epsilon \to 0^+$. The interaction $U(\tau,R)$ to be used in Eqn.(6.2) has been discussed by several authors [Ju67, An68, Mu68]. Their work shows that the three-body model can in principle be equivalent to the (A+2)-body system, if appropriate effective interactions are adopted. Moreover, they showed that such effective interactions can be, to a good approximation, given by the sum of local phenomenological neutron- and proton-target optical potentials evaluated at half the incident deuteron energy. Throughout this work we assume that the effective interaction is of the following form (omitting the Coulomb interaction $V_c(R)$)

$$U(\tau,R) = V_{nA} \left( \frac{1}{2} E_d , R - r/2 \right) + V_{pA} \left( \frac{1}{2} E_d , R + r/2 \right), \quad (6.3)$$

where $V_{nA}$ and $V_{pA}$ are the nucleon-target optical potentials.

An obvious criticism of Eqn.(6.3) is the total absence of terms from Pauli induced breakup. These effects are due to the possibility of having nucleon momenta in the projectile wave function overlapping with momenta already occupied in the target. This problem and the validity of the other assumptions contained in Eqn.(6.3), including the energy dependence of the nucleon-target optical potentials, have been studied in detail by a number of authors [e.g. Wa76, Ko84, Is86, To86]. As for the Pauli effects, it has been proved [To86] that they populate higher breakup energies than the deuteron ground state energy, and that only small corrections are obtained when compared against the original ADIA calculations.
The (d,p) stripping amplitude in the three-body model is [Au70]

\[ T_{dp} = < \chi^{(s)}_p (r_p), \Phi_n (r_n) | V_{np} | \Psi^{(s)} >, \]

(6.4)

where \( \Phi_n \) and \( \chi^{(s)}_p \) are as described in section (5.4).

If we consider a sufficiently short ranged \( V_{np} \) potential, then we only require \( \Psi^{(s)}(r=0,R) \) to evaluate accurately the stripping amplitude of Eqn.(6.4) [Jo70, Jo74]. It is therefore expected that in the limit of the zero-range approximation for \( V_{np} \), the wave function \( \Psi^{(s)}(0,R) \) which obeys much simpler boundary conditions than the total three-body wave function, can be reduced to an effective two-body analysis. This in fact constitutes an important step of the WSE formalism in the sense that for a given \( V_{np} \) interaction it is possible to develop an effective two-body method for the projected wave function \( V_{np} | \Psi^{(s)} > \). Clearly, this prescription is by no means an accurate determination of \( \Psi^{(s)} \) in the region of configuration space where the neutron and proton are widely separated, however these parts of \( \Psi^{(s)} \) are not necessary for a precise computation of the transition matrix element \( T_{dp} \) of Eqn.(6.4). It should be noted that, since the projected wave function \( V_{np} | \Psi^{(s)} > \) appears in Eqn.(6.4), the WSE technique can be thought of as being well adapted to the accurate determination of the stripping amplitude \( T_{dp} \).

The exact three-body wave function \( \Psi^{(s)}(r,R) \) of Eqn.(6.2) satisfies the integral equation [Jo74]

\[ |\Psi^{(s)}> = |\phi_d>|K_d> + G_{np} \left[ V_{nA} + V_{pA} \right] |\Psi^{(s)}>, \]

(6.5a)

where

\[ G_{np} = \left( E + i\epsilon - \hbar_{np} - K_{cm} \right)^{-1}, \]

(6.5b)
is the three-body Green’s function. The total wave function $|\Psi^{(e)}\rangle$ contains elastic and breakup components. To explicitly see the breakup part, we use the spectral decomposition of the three-body Green function in terms of the eigenstates of $\hat{h}_{np}$. Thus ($\epsilon_\kappa = \hbar^2 \kappa^2 / m$)

$$G_{np} = \frac{|\phi_d > < \phi_d|}{E + i\epsilon + \epsilon_d - \mathcal{E}_{cm}} + \int \frac{dk}{E + i\epsilon - \epsilon_\kappa - \mathcal{E}_{cm}} |\phi_\kappa > < \phi_\kappa| . \quad (6.6)$$

Substituting (6.6) into (6.5a) yields

$$|\Psi^{(e)}\rangle = |\phi_d > |\chi_{d}^{(e)} > + \int dk |\phi_\kappa^{(e)} > |\chi_\kappa^{(e)} > , \quad (6.7a)$$

where $\chi_{d}^{(e)}$ is the elastic deuteron c.m. scattering wave function given by

$$|\chi_{d}^{(e)} > = |K_d > + \frac{1}{E + \epsilon_d + i\epsilon - \mathcal{E}_{cm}} |\phi_d^+ |V_{nu} + V_{PD}|\Psi^{(e)} > \quad , (6.7b)$$

and the continuum components $\chi_\kappa^{(e)}(R)$, which describe the motion of the c.m. of an n-p pair in a state with relative energy $\epsilon_\kappa$, are

$$|\chi_\kappa^{(e)} > = \frac{1}{E - \epsilon_\kappa + i\epsilon - \mathcal{E}_{cm}} |\phi_\kappa^{(e)} |V_{nu} + V_{PD}|\Psi^{(e)} > . \quad (6.7c)$$

In the DWBA, where breakup effects are ignored, the second term in the RHS of Eqn.(6.7a) is neglected and $\chi_d^{(e)}(R)$ is assumed to be equivalent to an optical wave function.

In the WSE method, the two terms of Eqn.(6.7a) are taken into account by expanding the total wave function in terms of an appropriate set of relative motion states. However, as mentioned in the previous Chapter, the success of the zero-range approximation in describing stripping
reactions implies that the restriction to only $^3S$ states of the n-p pair system is an acceptable approximation. In the case where one includes the spin-orbit components of the nucleon optical potentials contributions to the projected wave function $V_{np} | \Psi^{(+)} >$ from singlet spin states can arise. These contributions are found [Ha74] to be small, and are neglected in this work.

6.3) - THE WEINBERG STATES EXPANSION OF THE WAVE FUNCTION

Since for the evaluation of the the stripping amplitude $T_{dp}$, we require the three-body wave function $\Psi^{(+)}$ only for $r \leq$ range of $V_{np}$, any approximate wave function, say $\tilde{\Psi}^{(+)}$, that obeys

$$V_{np} | \tilde{\Psi}^{(+)} > = V_{np} | \Psi^{(+)} > ,$$

(6.8)

will generate the correct transition amplitude $T_{dp}$. Thus it is not crucial in this case to calculate the true form of the wave function for large n-p separations. From Eqn.(6.7a) the projected wave function is

$$V_{np} | \Psi^{(+)} > = V_{np} | \phi_d > | \chi_d^{(+)} > + \int dk V_{np} | \phi_k^{(+)} > | \chi_k^{(+)} > .$$

(6.9)

The first term on the RHS of Eqn.(6.9) represents the elastic part while the second term contains all the breakup pieces. It is this second term which other theories either completely neglect, e.g. DWBA, or partially include, e.g. ADIA and QAD.

Since only the projected wave function $V_{np} | \Psi^{(+)} >$ is needed to accurately calculate the transition amplitude, this enables one to expand the wave function in terms of discrete set of states, orthogonal within the range of $V_{np}(r)$. A convenient set of states are the Weinberg [We63]
eigenstates \( \{ \phi_i(r) \} \), the solutions of

\[
\left[ -\varepsilon_d - \varepsilon_{np} - \alpha_i V_{np}(r) \right] \phi_i(r) = 0, \quad i = 1, 2, \ldots
\]  

(6.10a)

with the orthonormalization relation

\[
< \phi_i | V_{np} | \phi_j > = -\delta_{ij}.
\]  

(6.10b)

The only physical state among the \( \{ \phi_i(r) \} \) is the first one and, when \( i = 1 \), \( \alpha_i = 1 \), and \( \phi_1 \) is proportional to the deuteron ground state \( \phi_d \). The \( \alpha_i \) are real numbers that increase monotonically with \( i \) and are obtained by solving the eigenvalue equation (6.10a). The Weinberg states are particularly well suited to our purpose since they possess the same weight function in the orthogonality expression (6.10b) as the weighting of \( |\Psi^{(s)}> \) in the transition amplitude \( T_{dp} \).

Having identified a suitable complete set of states, one can expand the three-body wave function as

\[
\Psi^{(s)}(r, R) = \sum_{i=1}^{\infty} \phi_i(r) \chi_i^{(s)}(R),
\]  

(6.11a)

where, using Eqn.(6.10b)

\[
\chi_i^{(s)}(R) = -< \phi_i | V_{np} | \Psi^{(s)} >.
\]  

(6.11b)

If one considers that there is some upper limit on the n-p continuum energies which will contribute significantly to \( V_{np} |\Psi^{(s)}> \), then it is plausible to truncate the summation in Eqn.(6.11a) to a certain number \( N \) of terms. Of course, this truncation must be checked and the required number \( N \) determined by performing accurate numerical calculations.
Inserting the truncated version of Eqn.(6.11a) into (6.2), multiplying both sides by $\phi^*_i(r) V_{np}(r)$, and integrating over $r$, yields ($i = 1, \ldots, n$)

$$
\left[ E + i\epsilon + \epsilon_d - \sigma_{em} \right] |\chi_i^{(+)}> = i\epsilon \delta_{ii} N_d |K_d> - <\phi_i | V_{np} (V_{nA} + V_{pA}) |\Psi^{(+)}> - <\phi_i | V_{np} (\lambda_{np} + \epsilon_d) |\Psi^{(t)}> , (6.12a)
$$

where

$$
N_d = - <\phi_i | V_{np} | \phi_d > , \text{ thus } |\phi_d> = N_d |\phi_i>. \quad (6.12b)
$$

The truncated expansion of $|\Psi^{(t)}>\text{, when substituted into the second and third terms on the RHS of Eqn.(6.12a)}\text{ results in a set of N coupled-channel two-body Schrödinger equations,}$

$$
\left[ E + i\epsilon - \sigma_{em} - u_i(R) \right] |\chi_i^{(+)}> = i\epsilon \delta_{ii} N_d |K_d> + \sum_{j \neq i}^{N} u_{ij}(R) |\chi_j^{(+)}> , (6.13a)
$$

where the following definitions have been used

$$
u_{ij}(R) = V_{ij}(R) + \beta_{ij} (\alpha_j - 1) , \quad (6.13b)$$

$$V_{ij}(R) = - <\phi_i | V_{np} (V_{nA} + V_{pA}) | \phi_j> , \quad (6.13c)$$

$$\beta_{ij} = <\phi_i | V_{np}^2 | \phi_j> , \quad (6.13d)$$

Expression (6.13c) represents the coupling potentials $V_{ij}(R)$ which describe the interaction of an n-p pair with the target core as it evolves from the relative motion state $\phi_j$ to $\phi_i$. The modified coupling potentials $u_{ij}(R)$ of Eqn.(6.13b) contain the real coupling constant terms $\beta_{ij}(\alpha_j - 1)$ which are due to the fact that the Weinberg states are not eigenfunctions of $\lambda_{np}$. The n-p Hamiltonian $\lambda_{np}$ can generate transitions
between the Weinberg states themselves, and it is evident that the representation of \( h_{np} \) in the \( \{ \phi_i(r) \} \) basis is not diagonal. This can be easily seen when using Eqns.(6.10) to obtain the matrix elements of the n-p Hamiltonian \( h_{np} \ (= \mathcal{F}_{np} + V_{np}) \), i.e.

\[
< \phi_i | V_{np} h_{np} | \phi_j > = - \left( \beta_{ij} (\alpha_j - 1) - e_d \delta_{ij} \right).
\]  

As they stand, the N coupled-channel equations of Eqn.(6.13a) are not expressed in standard form. This is due to the presence of the real constant coupling terms in the potentials \( U_{ij} \) of Eqn.(6.13b). It will be shown in the next Chapter that the constant coupling can be removed by an appropriate diagonalisation of a constant coupling matrix.

A characteristic of these coupled channel equations (6.13a) is that only the first (i=1) channel solution \( |\chi_{i}^{(+)}> \) contains the elastic component of \( |\psi^{(+)}> \) and that every channel \( i=1,2,...,N \) wave function explicitly includes breakup effects. This can be illustrated using Eqns.(6.11b) and (6.7a), i.e.

\[
|\chi_{i}^{(+)}> = \delta_{i1} N_d |\chi_{d}^{(+)}> - \int dk <\phi_i | V_{np} | \phi_k^{(+)}> |\chi_k^{(+)}>.
\]  

Furthermore, if one retains only the first term of the expansion (6.11a) as the lowest order approximation for \( |\psi^{(2)}> \), then coupled equations reduce to the Adiabatic (ADIA) equation [Jo70]

\[
\left[ E_d + ie - \mathcal{F}_{cm} - V_{11}(R) \right] |\chi_{i}^{(+)}> = ie N_d |K_d>.
\]  

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For the same $V_{nA}$ and $V_{pA}$, the coupling potential $V_{11}(R)$ (defined in Eqn.(6.13c)) is numerically identical to the ADIA potential

$$V_{\text{ADIA}}(R) = \frac{<\phi_d | V_{np} (V_{nA} + V_{pA}) | \phi_d >}{<\phi_d | V_{np} | \phi_d >} \quad \text{(6.16b)}$$

This shows that, in the WSE method, the ADIA approximation appears as the solution of lowest order. In this sense an expansion that consists of several Weinberg states is expected to provide an improved description of higher n-p relative energy components which are not included accurately in ADIA. Thus, solving the N coupled-channel equations (6.13a) should, in principle, provide improvements over ADIA.
7.1) INTRODUCTION

In this Chapter we implement numerically the WSE formalism developed in the previous Chapter for the analysis of deuteron stripping reactions at intermediate energies. The WSE method proceeds by expanding the total wave function $|\Psi^{(4)}\rangle$ in terms of a complete set of Weinberg eigenstates.

The first step is to construct the Weinberg states basis defined in Eqns.(6.10). For this purpose, one has to choose an n-p interaction model $V_{np}(r)$. We take the n-p interaction to be the central Hulthén potential. This potential is convenient since the corresponding Weinberg states can be expressed analytically.

To solve the N coupled channel equations for the c.m. wave functions $|\chi_i^{(4)}\rangle$, one needs to remove the constant coupling terms. The second step in this work consists of rearranging the coupled equations such that no constant coupling terms are present.

The solutions can be obtained once all the coupling potentials $V_{ij}(R)$ are determined. The third step of the calculations is concerned with the numerical evaluation of these potentials and the solutions of the coupled equations.

The obtained expansion of the projected wave function is used to determine numerically the stripping transition amplitude $T_{dp}$. In this
final phase of the calculations we study the convergence of calculated observables as a function of the size $N$ of the Weinberg basis used. We also apply the WSE method to analyse a $(d,p)$ reaction. The obtained results are then compared against equivalent ADIA and QAD calculations.

7.2) - CONSTRUCTION OF THE WEINBERG STATES BASIS

In this section we determine the Weinberg eigenstates for an $n$-$p$ interaction of the Hulthén form, denoted here by $V_{np}^H(r)$. As noted earlier the first solution, $\phi_1(r)$, of the Weinberg equation (6.10a) is proportional to the deuteron ground state $\phi_d(r)$ whose expression, for this potential, is given by (where $S_d(r) = r \phi_d(r)$ and $N$ is a normalization constant) [e.g. Ya54]

$$S_d(r) = N ( e^{\gamma r} - e^{-\gamma r} ), \quad (7.1a)$$

and satisfies the radial equation (6.10a) ($\mu_1 = 1$, $\alpha_1 = 1$ and $\phi_1 = N_d \phi_d$)

$$S_d''(r) - \frac{2 \mu_{np}}{\hbar^2} V_{np}^H(r) S_d(r) = k^2 S_d(r), \quad (7.1b)$$

where

$$k^2 = \frac{2 \mu_{np} E_d}{\hbar^2}, \quad \text{with} \quad E_d = +2.226 \text{ MeV}. \quad (7.1c)$$

The constant $\gamma$ is determined by reproducing the deuteron binding energy. It is found [Ya54] that the value of this constant is given by

$$\gamma = 6.255 \text{ k}. \quad (7.1d)$$

Substituting (7.1a) into (7.1b) yields the parameters of $V_{np}^H(r)$

$$V_{np}^H(r) = \frac{V_0}{\left( e^{\beta r} - 1 \right)}, \quad (7.2a)$$
where

\[
\beta = \gamma - k , \quad (7.2b)
\]

\[
V_0 = - \frac{n^2}{2U_{np}} \beta^2 \left[ 1 + \frac{2k}{\beta} \right] . \quad (7.2c)
\]

Having completely determined the n-p interaction, one can now proceed to solve the Weinberg equation (6.10a). The radial equation for the \( S_i(r) \) (with \( \phi_i(r) = (S_i(r)/r) Y^0_0(r) \)) is

\[
\left[ - k^2 + \frac{d}{dr} - \frac{2\mu_{np} \alpha_i}{\hbar} V_{np}^H(r) \right] S_i(r) = 0 , \quad (7.3)
\]

and inserting the expression of \( V_{np}^H(r) \) into (7.3) yields

\[
S''_i(r) + \left[ \alpha_i \sqrt{V_0 / (e^{Br} - 1)} - k^2 \right] S_i(r) = 0 , \quad (7.4a)
\]

where

\[
V_0 = - \frac{2\mu_{np} V_0}{n^2} . \quad (7.4b)
\]

The structure of Eqn.(7.4a) shows that asymptotically \( (r \to \infty) \) the \( S_i(r) \) satisfy

\[
S''_i(r) - k^2 S_i(r) = 0 , \quad (7.5)
\]

therefore it is appropriate to assume a trial solution of the form

\[
S_i(r) = e^{kr} f_i(r) . \quad (7.6)
\]

In this case the \( f_i(r) \) satisfy

\[
f''_i(r) - 2k f'_i(r) + \left[ \alpha_i \sqrt{V_0 / (e^{Br} - 1)} \right] f_i(r) = 0 . \quad (7.7)
\]

and making a change of variable to
yields
\[ z = e^{-\beta r}, \quad (7.8) \]
yields
\[ z(1-z) f''_i + \omega_i (1-z) f'_i + \omega_i f_i = 0. \quad (7.9a) \]
where
\[ \omega_i = \left\{ \alpha_i V_0 / \beta^2 \right\}, \quad i = 1, 2, \ldots, N. \quad (7.9b) \]
If we assume the \( f_i \) to be represented by the power series expansion
\[ f_i = \sum_{p=0}^{\infty} a_p^i z^p, \quad (7.10) \]
yielding the indicial equation (assuming \( a_0^i \neq 0 \))
\[ a_0^i \left[ s(s-1) + \omega_i s \right] = 0, \quad (7.11b) \]
the roots of which are \( s_1 = 0 \) and \( s_2 = (1 - \omega_i) \). Taking \( s = s_1 \), the general coefficient (i.e. coefficient of \( z^j \)) has the form
\[ a_j^i \left[ j(j+1) + \omega_i (j+1) \right] + a_j^i \left[ \omega_i - \omega_i + j(j-1) \right] = 0, \quad (7.12a) \]
giving rise to a one-term recurrence relation for the power series coefficients, i.e.
\[ a_{j+1}^i = a_j^i \left\{ \frac{\omega_i - \omega_i + j(j-1)}{j(j+1) + \omega_i (j+1)} \right\}, \quad (7.12b) \]
where the \( \omega_i \) (\( i = 1, 2, \ldots, N \)) are as defined in Eqn.(7.9b).

It was shown [Ne66] using other methods that such S-wave differential equation can be solved analytically and that the following relationship is always true for an attractive Hulthén potential
\[ k = \left( \omega_i - i^2 \right) / \left( 2i / \beta \right). \quad (7.13) \]
As a check of Eqn.(7.13), one can readily obtain, when \( i = 1 \) and \( \alpha_1 = 1 \), the same expression of \( V_0 \) as the one given earlier by Eqn.(7.2c), namely

\[
\frac{2k}{\beta} = \omega_1 - 1.
\]  

(7.14a)

Note that when multiplying both sides of Eqn.(7.14a) by \( i \) we obtain

\[
\frac{2ik}{\beta} = i \omega_1 - i.
\]  

(7.14b)

The expression of \( \frac{2ik}{\beta} \) can also be obtained from Eqn.(7.13). We thus have

\[
i \omega_1 - \omega_1 + i(i-1) = 0.
\]  

(7.14c)

Note that the numerator of the recurrence relation (7.12b) is identical to the expression (7.14c). In fact, this indicates that the numerator of (7.12b) vanishes when \( j = i \), thus terminating the power series expansion.

We now use the recurrence relation (7.12b) to evaluate the coefficients \( a_p^i \) in the expansion (7.10) of the function \( f_i \) which in turn is substituted into Eqn.(7.6) to finally produce the Weinberg states \( S_i(r) \):

\[
S_i(r) = e^{kr} \sum_{p=0}^{i} a_p^i e^{-p\beta r},
\]

\[
= \sum_{p=0}^{i} a_p^i \exp(-p\rho_p r),
\]  

(7.15a)

where

\[\rho_p = k + p\beta.\]

(7.15b)

We have, thus far, derived the analytical form of the Weinberg states \( \phi_i(r) \) for the Hulthén potential. In the next stages of the calculations, we will need to have available the numerical values of \( N \) Weinberg states. To achieve this, we wrote the computer program WEINBERG [La88] which is
based on most of the equations derived above as well as the orthonormality relation of Eqn.(6.10b). The parameters of the Hulthén potential used to construct the Weinberg states are

\[
\begin{align*}
    k &= 0.232 \text{ fm}^{-1}, \\
    \beta &= 5.255 \text{ fm} \text{ k} = 1.220 \text{ fm}^{-1}, \\
    V_0 &= -84.86 \text{ MeV}.
\end{align*}
\] (7.16)

The orthonormalization integrals in our computer code were evaluated using accurate integration routines. However, one ought to point out that these integrals can be determined analytically, and the details of these derivations are given in Appendix-D. An example of the first three Weinberg states, \( S_i(r), i=1,2,3 \), and the Hulthén potential \( V^H_{np}(r) \) are shown in Fig.(7.1). Three important features of these states are as follows:

1) all the oscillations occur within the range of \( V^H_{np}(r) \),
2) the exponential fall off at large r values,
3) the state \( S_i(r) \) has \( i \) nodes within the range of \( V^H_{np}(r) \).

Having constructed a set of Weinberg states, our next step consists of solving the \( N \) coupled channel equations. This can be done, once the problem concerning the presence of constant coupling terms in the coupling potentials \( U_{ij}(R) \) is solved. In the next section we discuss the details of a technique that can be used to circumvent such a problem.
FIG. (7.1) - The first three radial Weinberg S-states for the Hulthén potential, and the Hulthén potential itself.
7.3) - DIAGONALISATION OF THE REAL CONSTANT COUPLING MATRIX

We stated in the introduction section of this Chapter that the $N$ coupled channel equations (6.13) for the c.m. wave functions $|\chi^{(+)}_i>$ cannot be solved while remaining in that form. This situation is conveniently illustrated using an appropriate matrix notation [Jo74]. Thus, if we define the column of solutions

$$|\chi^{(+)}_{K_1}| = \text{Column } \{ |\chi^{(+)}_1>, |\chi^{(+)}_2>, \ldots, |\chi^{(+)}_N> \} ,$$

(7.17a)

and the elements of the real constant coupling matrix $c$ as

$$c_{ij} = \beta_j (\alpha_j - 1) ,$$

(7.17b)

then the $N$ coupled equations (6.13a) take the following form

$$\left[ E_d + ic - g_{cm} \right] |\chi^{(+)}^{(t)}> = c |\chi^{(+)}^{(t)}> + \psi |\chi^{(+)}^{(o)}> ,$$

(7.17c)

where $\psi$ is the coupling potential matrix, whose elements are given by Eqn.(6.13c). Note that, for clarity, the incoming deuteron boundary condition for the first channel has not been included. The solutions of Eqn.(7.17c) can be adequately defined by specifying the outgoing boundary conditions in each channel. To do so, one has to prove that the matrix $c$ can always be diagonalised.

It is convenient to re-write the matrix $c$ as

$$c = \alpha^{-1} \circ \alpha ,$$

(7.18a)

where

$$\alpha = \text{Diagonal } \{ 1, (\alpha_2 - 1)^{1/2}, \ldots, (\alpha_N - 1)^{1/2} \} ,$$

(7.18b)

and

$$b = \begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix} ,$$

(7.18c)
with
\[ x = \text{Row} \left\{ \beta_{12}(\alpha_2-1)^{1/2}, \ldots, \beta_{1N}(\alpha_N-1)^{1/2} \right\}, \quad (7.18d) \]

\[ b_{ij} = (\alpha_i-1)^{1/2} \beta_{ij}(\alpha_j-1)^{1/2}, \quad i,j = 2,3,\ldots,N. \quad (7.18e) \]

The \( \beta_{ij} \) are real numbers defined in Eqn.(6.13d), and the \( \alpha_i \) are real constants whose expressions are obtained from Eqn.(7.13), i.e.

\[ \alpha_i = i \left\{ \frac{1 + i (\beta / 2k)}{1 + (\beta / 2k)} \right\}. \quad (7.19) \]

The sub-matrix \( b \) is by construction real and symmetric, and as such can always be diagonalised by a similarity transformation. In other words it is possible to determine a non-singular matrix \( \mathbf{u} \), so that the matrix \( b \) reduces to a diagonal matrix \( \mathbf{s} \), whose diagonal elements are known as the eigenvalues of \( b \). Therefore, as a consequence of the diagonalization of \( b \), the real but non-symmetric matrix \( \mathbf{B} \) becomes similar to a triangular matrix. The diagonal elements of such a matrix (i.e. \( \mathbf{B} \)) are therefore given by \( \{ 0, \lambda_2, \ldots, \lambda_N \} \), where the \( \lambda_i \) represent the real eigenvalues of the sub-matrix \( b \). Since the diagonal elements of a triangular matrix are its eigenvalues and that provided these eigenvalues are distinct, then such a matrix is similar to a diagonal one. In fact, the condition on the eigenvalues of the triangular matrix \( \mathbf{B} \) is always satisfied since the matrix \( b \) is positive definite.

We have, so far, shown that \( \mathbf{B} \) is similar to a diagonal matrix. This result together with the expression (7.18a), prove that \( \mathbf{c} \) is diagonalisable and as such we can write

\[ \mathbf{c} = \mathbf{A}^{-1} \lambda \mathbf{A}, \quad (7.20a) \]
where $A$ is a non-singular but not unitary matrix to be determined, and $\lambda$ is a diagonal of the form

$$\lambda = \text{Diagonal} \left\{ \lambda_1 = 0, \lambda_2, \ldots, \lambda_N \right\}. \quad (7.20b)$$

The usefulness of Eqn.(7.20a) and the question concerning the determination of the matrix $A$ will be discussed later. First, one should concentrate on the diagonalisation of the sub-matrix $b$. The elements $b_{ij}$ of $b$ are defined in Eqn.(7.18d), where the real constants $\beta_{ij}$, $\alpha_{ij}$ are given by Eqns.(6.13d) and (7.19) respectively. The $\beta_{ij}$ represent the matrix elements, in the Weinberg basis, of the $V_{np}$ operator squared.

The determination of the $\beta_{ij}$ involves integrals with Weinberg states similar to those encountered during the orthonormalization phase. These integrals can be obtained analytically using the Weinberg states expressions (7.15) and the Hulthén potential (7.2). The analytical result for the $\beta_{ij}$ element is given by Eqn.(D.15) of Appendix-D. Despite the apparent simplicity, expression (D.15) turned out to be very tricky to handle numerically. This situation is due to the summation of alternating (in signs) but comparable coefficients products $a_p^i a_l^j$, resulting in numerical cancellations as $i$ or $j$ reach the value of 20.

As an alternative to the above analytical approach, we opted for a numerical evaluation of $\beta_{ij}$. The $\beta_{ij}$ results combined with the values of the $\alpha_{ij}$ are then used to produce the matrix $b$. Having done this, we now need to diagonalise $b$. We do not intend to elaborate on diagonalisation techniques since a great deal of specialised literature is available. Nonetheless, we recommend as a good reading of this subject the eleventh Chapter of this ref. [Pr86]. Diagonalisation routines can be called from
known numerical libraries, e.g. N.A.G. or HARWELL. It turned out that the diagonalisation of large real symmetric matrices produces more precise results than those obtained when diagonalising non-symmetric matrices. Therefore, diagonalising \( b \) should not be a problem, because by definition the sub-matrix \( b \) is real and symmetric. We have used two different routines to diagonalise \( b \), one from the N.A.G. library - F02ABF - and the other from HARWELL - EA06C. This way, we provide an independent check of either routines. We have so far obtained the eigenvalues \( \lambda_i \) \((i=2,N)\) and eigenvectors of the sub-matrix \( b \). However, the diagonalisation of \( c \) has yet to be completed since the matrix \( A \) is still undefined. In what follows we discuss a technique that we used to determine the matrix \( A \).

The diagonalisation of a large non-symmetric matrix (e.g. \( B \) or \( C \)) can produce misleading results. This is due to the fact that the eigenvalues of this kind of matrices are very sensitive to small variations of the matrix elements. Since, by construction the matrices \( B \) and \( C \) are not balanced - i.e. corresponding rows and columns of \( B \) or \( C \) do not have comparable norms, then it is not advisable to pursue a straightforward diagonalisation of either matrix. However, the diagonalisation of a real and symmetric matrix (e.g. \( b \)) - by definition a well balanced matrix - can always be performed to a very good precision. Thus, combining this with the fact that \( B \) is similar to a triangular matrix whose diagonal elements \( [0, \lambda_2, \ldots, \lambda_N] \) are actually its eigenvalues, means that it is possible to extract the eigenvectors of \( B \) form those of \( b \). An immediate consequence of the diagonalisation of \( B \) is that we can now determine accurately the matrix \( A \), since \( B \) and \( C \) are related by Eqn.(7.18a).

This technique is best illustrated by a simple example, the case where \( N = 3 \). From Eqn.(7.18c) we have
and as a result of the diagonalisation of the 2x2 sub-matrix $b$, we can write ($\lambda_2$ and $v^{(2)}$ are known)

$$b v^{(2)} = \lambda_2 v^{(2)},$$

(7.21b)

where

$$v^{(2)} = \begin{bmatrix} g \\ h \end{bmatrix},$$

(7.21c)

is the eigenvector of $b$ corresponding to the eigenvalue $\lambda_2$. Since $\lambda_2$ is also an eigenvalue of $B$, then the following vector ("$f"$ unknown)

$$Y^{(2)} = \begin{bmatrix} f \\ g \\ h \end{bmatrix},$$

(7.21d)

is an eigenvector of $B$ if and only if

$$B Y^{(2)} = \lambda_2 Y^{(2)},$$

(7.21e)

which implies that

$$f = \left( x_{12} g + x_{13} h \right) / \lambda_2.$$  

(7.21f)

The eigenvector $Y^{(2)}$ has to be normalised to unity. This condition is always satisfied when it is possible to find a real constant $\eta$ such that

$$\eta \begin{bmatrix} f \\ g \\ h \end{bmatrix} \eta \begin{bmatrix} f \\ g \\ h \end{bmatrix} = 1,$$

(7.21g)

resulting in (since $(g^2 + h^2) = 1$)

$$\eta = \frac{1}{(f^2 + 1)^{1/2}}.$$  

(7.21h)
The same procedure is repeated to determine $Y^{(3)}$. The fact that the *first column* of $B$ is made up of zeros, and that the eigenvalue $\lambda_1$ is equal to zero means that the eigenvector $Y^{(1)}$ can only take the following form

$$Y^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (7.21i)$$

We have therefore proved that it is possible, in this special case, to accurately diagonalise the unbalanced non-symmetric matrix $B$ using the eigenvalues and eigenvectors of the sub-matrix $b$. Of course, such technique can be generalised to any dimension $N$, thus providing an alternative diagonalisation of $B$.

As a consequence of the diagonalisation of $B$, one can always write

$$B = D^{-1} \lambda D, \quad (7.22)$$

where $D$ is a matrix whose *columns* are given by the eigenvectors $Y^{(1)}$ in this *order* ($i=1,2,...,n$), and $\lambda$ is the diagonal of Eqn.(7.20b). Since the matrix $c$, as defined in Eqn.(7.18a), can be expressed in terms of $B$, then using Eqn.(7.22) we obtain

$$c = \alpha^{-1} \left( D^{-1} \lambda D \right) \alpha, \quad (7.23)$$

thus, after comparison with Eqn.(7.20a) we deduce the following

$$A = D \alpha. \quad (7.24)$$

The computer code MDIAG [La89a], developed to perform these calculations, has built in it several consistency checks as well as being capable of generating the same results when using different library routines. We carried out these calculations for various values of $N$, and
the results obtained, i.e. the $\lambda_i$, $\alpha$ and $\alpha^\dagger$, are stored for further applications.

We have demonstrated that the matrix $c$ is diagonalisable. This result implies that it is now possible to remove the constant coupling matrix $c$ from the coupled equations (7.17c). This is done by replacing $c$ by the diagonalised expression (7.20a), thus defining a new basis in which the coupled equations have no constant coupling terms, namely,

$$\left[ E_d + i e - T_{cm} \right] |F^{(\alpha)}> = \lambda |F^{(\alpha)}> + w |F^{(\alpha)}> ,$$

(7.25a)

where we introduced the new column solutions

$$|F^{(\alpha)}> = \alpha |\chi^{(\alpha)}> ,$$

(7.25b)

and the coupling potentials matrix

$$w = \alpha \varpi \alpha^{-1} .$$

(7.25c)

The elements of the matrix $w$ are given by (using Eqn.(6.13c) for $V_{kq}$)

$$W_{ij}(R) = - \sum_{k,q} N_{ik} < \phi_k | V_{np} (V_{nA} + V_{pA}) | \phi_q > A_{qi}^{-1} ,$$

$$W_{ij}(R) = - < \Omega_i^N | V_{np} (V_{nA} + V_{pA}) | \Delta_j^N > ,$$

(7.25d)

where the new basis states are defined by

$$< \Omega_i^N | = \sum_{k=1}^N A_{ik} < \phi_k | \neq < \Delta_j^N | ,$$

(7.25e)

$$| \Delta_j^N > = \sum_{q=1}^N | \phi_q > A_{qi}^{-1} ,$$

(7.25f)

with orthonormality relation
We have thus far showed that it is possible to construct a discrete set of n-p eigenstates to represent the three-body continuum effects. These states are also used to expand the three-body wave function \( \Psi^{(q)}(r, R) \). Moreover, we showed that the resulting coupled equations can be reduced to a convenient form. However, little has been mentioned about the way the continuum spectrum of the n-p interaction is simulated. The best way to illustrate how this continuum is mapped is to analyse the main WSE approximation from the n-p Hamiltonian standpoint.

We have explained in Chapter VI that in the Weinberg states basis the matrix representation of the n-p Hamiltonian \( h_{np} \) is not diagonal (see Eqn.(6.14)). In fact, the selection of N Weinberg states to describe the n-p spectrum is effectively equivalent to the replacement of \( h_{np} \) by an approximate Hamiltonian [Jo74]

\[
\begin{align*}
\tilde{h}_{np}^N &= \sum_{ij}^N | \phi_i > \left[ < \phi_i | V_{np} h_{np} | \phi_j > \right] < \phi_j | V_{np} .
\end{align*}
\]

where the expression between square brackets is given by Eqn.(6.14). In the new basis, \( \tilde{h}_{np}^N \) has this representation

\[
\begin{align*}
\tilde{h}_{np}^N &= \sum_{ij}^N | \Delta_i^N > \left[ < \Omega_i^N | V_{np} h_{np} | \Delta_j^N > \right] < \Omega_j^N | V_{np} ,
\end{align*}
\]

and since we have shown that in the new basis the matrix \( \tilde{c} \), whose elements are \( \tilde{c}_{ij}(\alpha_j - 1) \), is diagonal (i.e. \( \tilde{c} = \alpha^{-1} \lambda \alpha \)) then the square brackets expression in Eqn.(7.26b) will have non-zero elements only when \( i = j \). Thus, in the new basis we obtain for \( \tilde{h}_{np}^N (\lambda_1 = 0) \)

\[
\tilde{h}_{np}^N = - \sum_i^N | \Delta_i^N > \left[ \lambda_i - \epsilon_d \right] < \Omega_i^N | V_{np} .
\]
where the $\lambda_i$ (i=1,...,N) are the eigenvalues of the sub-matrix $b$. Thus, as a result of the diagonalisation of $b$, we have replaced the continuum spectrum of $\mathcal{H}_{np}$ by a discretised version whereby the c.m. energy of the n-p pair is now represented by the eigenvalues $\lambda_i$. Of course, one should indicate that since we use a finite number of Weinberg states, our simulation of the continuum eigenstates of $\mathcal{H}_{np}$ is approximate. However, we expect these discrete eigenvalues to draw closer when $N$ is increased.

The results of a series of diagonalisations ($N = 1$-$35$) are shown in Fig.(7.2). The three curves represent, where possible, the three lowest values of the $\lambda_i$'s obtained for a fixed $N$. As expected, we see from the trend of the curves that as $N$ increases the $\lambda_i$'s become closer. Moreover, the results of Fig.(7.2) show that, if we consider a (d,p) reaction with $E_{d}^{1st}$ near 100 MeV, then for a basis dimension $N=30$, we have available - in addition to the ADIA channel (i.e. i = 1) - two more open channels. Such extra channels simulate the effects due to deuteron breakup into high n-p relative momentum components and represent the WSE's corrections to the ADIA model. By having these additional channels available within the coupled channel procedure, we are effectively including explicitly their contributions to the total wave function $|\Psi(+)\rangle$.

The $N$ coupled channel equations (Eqn.(7.25a)), when expressed in the new basis, do not contain any constant coupling terms. In the next section we examine the new basis states, evaluate the coupling potentials $W_{ij}(R)$ and discuss the numerical method used to solve the $N$ coupled equations (7.25a). The obtained solutions are then analysed in conjunction with the evaluation of the amplitude $T_{dp}$. 

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FIG.(7.2)- Represents the spectrum of the Hamiltonian $\lambda_n$ as a function of the dimension $N$ of the Weinberg states basis. These results are for the n-p Hulthen interaction of Eqns.(7.2).
7.4) SOLUTIONS OF THE N COUPLED-CHANNEL EQUATIONS

We have shown, in the preceding section, that the complicated coupled equations (6.13a) of Chapter VI can be transformed into a traditional coupled channel problem when expressed in a new basis. The expansion (6.11a) combined with equations (7.25b) and (7.25f), give rise to the new representation of the total wave function,

\[ \Psi^{(+)} = \sum_{i=1}^{N} \Delta_i^N | F_i^{(+)} > , \]  

(7.27a)

where

\[ | F_i^{(+)} > = - \langle \Omega_i^N | V_{np} | \Psi^{(+)} > . \]  

(7.27b)

When the incoming deuteron boundary condition for the \( i = 1 \) channel is shown, then explicitly Eqn.(7.25a) is \( (E_1 = E_d - \lambda_1) \)

\[ \left[ E_i + ie - g_m - W_{1i}(R) \right] F_i^{(+)}(R) = i e \delta_{i1} N_d e^{iK_d.R} + \sum_{j \neq 1}^{N} W_{i,j}(R) F_j^{(+)}(R) . \]  

(7.28)

To solve these equations we require the coupling potentials \( W_{ij}(R) \), which are matrix elements in the new basis states (see Eqn.(7.25d)). To do so, we first evaluate these states.

The Weinberg states calculated in Section (7.2) and the matrices \( A \) and \( A^{-1} \) computed in Section (7.3) are used, according to Eqns.(7.25e) and (7.25f), to construct the new basis states. The results are shown in Figs.(7.3) and (7.4) where we present the first few states \( | \Delta_i^N >, | \Omega_i^N > \) when \( N = 35 \). It is interesting to note the behaviour of the states \( | \Omega_i^N > \) at small (within the range of \( V_{np} \)) n-p separations.
To interpret this, we consider as an example the state $|\Omega^N_3^5\rangle$. According to Eqn.(7.28c), this state is a linear combination of 35 Weinberg states where the amplitude of each state is given by the elements of the third row of the matrix A. A close examination of that row indicates that its elements, $A_{3k}^k (k=1-35)$, change signs and magnitudes thus picking up, as k increases, proportions of even more oscillatory Weinberg states giving rise to the products $A_{3k}^k |\phi_k\rangle$, which when added together produce cancellations at small r values. These cancellations guarantee that the orthonormality relation (7.25g) is satisfied. As for the $|\Delta^N_j\rangle$ of Fig.(7.3), we notice that although these n-p continuum states are constructed out of $N = 35$ Weinberg states, their magnitudes are smaller than the $|\Omega^N_i\rangle$ states. An estimate of the relative order of magnitude of the $|\Delta\rangle$ and $|\Omega\rangle$ is given in the caption of Fig.(7.4). Moreover, a comparison of the $|\Delta^N_j\rangle (k=1-3)$ states of Fig.(7.3) with the first 3 Weinberg states of Fig.(7.1) shows that the $|\Delta^N_j\rangle$ wave functions are smaller in magnitude and that their exponential fall off occurs at relatively higher r values.

In order to complete the expansion (7.27a) of $|\Psi^{(c)}\rangle$, we now need to solve the coupled equations (7.28) for the c.m. wave functions $|\Gamma^{(c)}_i\rangle$. We require the coupling potentials $W_{ij}(R)$ defined by Eqn.(7.25d). The potentials $V_{nA}$ and $V_{pA}$ appearing in the expression of $W_{ij}$ were defined in Section (6.2) as the respective nucleon-nucleus optical potentials evaluated at half the incident deuteron lab. energy. Explicitly

$$W_{ij}(R) = -2\pi \int_0^\infty r^2 \Omega^N_i(r) V_{np}(r) \Delta^N_j(r) \left[ N(r,R) + P(r,R) \right] dr, \quad (7.29a)$$
FIG.(7.3)- Shows the first three $\Delta >$ states of the new basis. These states are obtained from a Weinberg basis dimension of $N = 35$. 

$\Delta_{i}^{35} (i=1,2,3)$

$r \ [\text{fm}]$
FIG. (7.4)- Shows the first three $|\Omega\rangle$ states of the new basis. These states are obtained from a Weinberg basis dimension of $N = 35$. Note the substantial change in the vertical scale from the $|\Delta\rangle$ graphs. An estimate of the relative order of magnitude of the $|\Omega\rangle$ and $|\Delta\rangle$ states is obtained from Eqn. (7.25g). Thus, inside the range of $V_{np}$ we have: $\Omega(r), \Delta(r) \sim \frac{1}{q}$, where $q = \int dr \ V_{np}(r)$. 

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where (with \( \cos \theta = \xi \))

\[
N(r,R) = \int \left( \sum_{nA}^{+1} \left\{ \left[ \frac{r^2}{4} + R^2 - rR\xi \right] \right\} \right) d\xi, \quad (7.29b)
\]

\[
P(r,R) = \int \left( \sum_{pA}^{+1} \left\{ \left[ \frac{r^2}{4} + R^2 + rR\xi \right] \right\} \right) d\xi. \quad (7.29c)
\]

We assume that the nucleon-nucleus optical potentials are purely central. In fact, in all the subsequent calculations, the spin-orbit components are neglected in both entrance and exit channels.

In calculating the coupling potentials \( W_{ij} \), we used for \( V_{nA} \) and \( V_{pA} \) the global optical model parameters of Becchetti and Greenlees [Be69] with \( E = E_{d}^{\text{lab}} / 2 \). The target chosen is \(^{66}\text{Zn}\) and the incident deuteron lab. energy is \( E_{d}^{\text{lab}} = 88.2 \text{ MeV} \). The results obtained are shown in Figs.(7.5-7) where the real and imaginary parts of all elements of the \( 3 \times 3 \) coupling potential matrix \( W \) are presented for \( N = 35 \). These results show that the off-diagonal potentials are surface peaked and that their ranges are approximately of the size of the target nucleus plus the range of \( V_{np} \). Note that the off-diagonal potential, e.g. \( W_{21}(R) \), which represents the coupling of the ADIA channel to the n-p continuum state \( |\Delta_{2}^{35} \rangle \), reaches a value of about 9 MeV at \( R \approx 4.5 \text{ fm} \). Such a potential strength indicates a strong coupling that would eventually manifest itself at the level of the c.m. wave function \( F_{2}^{(n)}(R) \). It is important to point out that the \( W_{ij} \) coupling potentials are not symmetric in the channel indices, thus rendering the coupled channel calculations unusual. As will be seen later, this does not cause any problems since the coupled channel code we use accepts \( W_{ij} \neq W_{ji} \).
FIG. (7.5) - Displays the real and imaginary parts of the first row's elements of the 3x3 coupling potential matrix $W$. These are for the $^{66}\text{Zn(d,p)}^{67}\text{Zn}$ reaction at $E_{\text{lab.}} = 88.2$ MeV and a Weinberg basis dimension of $N = 35$. 
FIG. (7.6) - Displays the real and imaginary parts of the second row's elements of the 3x3 coupling potential matrix $W$. These are for the $^{66}\text{Zn}(d,p)^{67}\text{Zn}$ reaction at $E_{\text{lab}} = 88.2$ MeV and a Weinberg basis dimension of $N = 35$. 

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FIG. (7.7) - Displays the real and imaginary parts of the third row's elements of the $3 \times 3$ coupling potential matrix $W$. These are for the $^{66}_{\text{Zn}}(d,p)^{67}_{\text{Zn}}$ reaction at $E_{\text{lab}} = 88.2$ MeV and a Weinberg basis dimension of $N = 35$. 
During the calculation of the coupling potential $W_{ij}(R)$ we made sure that, in our double folding potential code DFOLD [La89b], the following volume integrals are always satisfied

$$\int dR W_{ij}(R) = \delta_{ij} \int dR \left[ V_{nA}(R) + V_{pA}(R) \right]. \quad (7.30)$$

Having established the numerical procedure for the calculation of the $W_{ij}(R)$, our next step is to solve the coupled equations (7.28). The channel energy as defined in Eqn.(7.28) implies that the channel wave numbers are given by

$$K_i = \left\{ \frac{4 m (E_d - \lambda_i)}{\hbar^2} \right\}^{1/2}, \quad (7.31a)$$

therefore, the first channel has the elastic deuteron wave number since $\lambda_1 = 0$. However, all the channels for which $E_i = E_d - \lambda_i$ is negative, decay exponentially outside the range of the potentials $W_{ij}$, and the $K_i$ become complex, i.e.

$$K_i \rightarrow i\kappa_i, \quad \kappa_i = \left\{ \frac{4 m (\lambda_i - E_d)}{\hbar^2} \right\}^{1/2}. \quad (7.31b)$$

In any practical application the number of closed ($E_i < 0$) or open ($E_i \geq 0$) channels is determined by the incident deuteron energy $E_d$ and the eigenvalues $\lambda_i$. However, we have shown in the previous section that as $N$ is increased the eigenvalues $\lambda_i$ cover a larger range of positive energies. Therefore, for any constant value of $E_d$ we deduce from Fig.(7.2) the number of open channels available for a fixed $N$. For example, if $N = 15$ and $E_d = 88$ MeV then only two open channels are available, namely, those corresponding to $E_1 = 88$ MeV ($\lambda_1 = 0$ MeV) and $E_2$.
\( \approx 62 \text{ MeV (since } \lambda_2 = 26 \text{ MeV). In this case we examine the effects due to the second channel } |F_2^{(+)}> \text{ on the total wave function } |\Psi^{(t)}> \).

Similar structures to that of the coupled equations (7.28) is widely encountered in nuclear physics, hence the existence of several dedicated computer codes. A recent such code is the general purpose Coupled Reaction Channels Program called FRESCO by I.J. Thompson [Th88]. We have chosen to use FRESCO to generate the solutions \( |F_i^{(+)}> \text{ since it can handle non-symmetric coupling potentials. However, for checking purposes we also wrote TWOCEQ [La89c] a two coupled channel Fortran code to solve (7.28) in the case where we have two open channels.}

Returning to the example mentioned above, \( N = 15 \) and \( E_a = 88 \text{ MeV}, \) where we have two open channels. Equations (7.28) become (\( e \rightarrow 0 \))

\[
\begin{pmatrix}
E_1 - \tilde{\epsilon}_{cm} & 0 \\
0 & E_2 - \tilde{\epsilon}_{cm}
\end{pmatrix}
\begin{pmatrix}
F_1^{(+)} \\
F_2^{(+)}
\end{pmatrix} =
\begin{pmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{pmatrix}
\begin{pmatrix}
F_1^{(+)} \\
F_2^{(+)}
\end{pmatrix}.
\tag{7.32}
\]

We have chosen the \(^{66}\text{Zn(d,p)}^{67}\text{Zn} \) reaction at \( E_a^{1x_b} = 88.2 \text{ MeV}, \) because a QAD calculation has been reported [St90] for the same reaction. This is useful in the sense that we can now perform a qualitative comparison between the predictions of both theories, the WSE and QAD. The numerical method used to solve Eqn.(7.32) is discussed in Appendix-E.

The results, shown in Fig.(7.8), represent the moduli of the c.m. wave functions \( F_1^{(+)}(R) \) and \( F_2^{(+)}(R) \) for the surface partial wave, \( L = 13. \) Figure (7.8) shows that the effects due to the strong and surface peaked coupling potentials \( W_{12}(R) \) and \( W_{21}(R) \) have been translated into relatively large amplitudes of \( |F_2^{(+)}(R)| \) in the nuclear surface region.
FIG. (7.8) - Shows the moduli of the c.m. motion wave functions $F_1^{(+)}(R)$ and $F_2^{(+)}(R)$ for the surface partial wave $L = 13$. These calculations are for the reaction $^{66}$Zn$(d,p)^{67}$Zn at $E_{l_{ab}} = 88.2$ MeV. The Weinberg basis dimension was taken equal to $N = 15$. 

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It is interesting to examine the impact of the second channel c.m. function on the stripping transition amplitude $T_{dp}$. To do so, we first need to express the total wave function $|\Psi^{(s)}\rangle$ in terms of the $|\Delta_i^{N}\rangle$ and the $|F_1^{(s)}\rangle$ according to Eqn.(7.27a). Thus, if we assume that the two open channel wave functions $|F_1^{(s)}\rangle$ and $|F_2^{(s)}\rangle$ calculated above, constitute an accurate representation of $|\Psi^{(s)}\rangle$, then we can write

$$|\Psi^{(s)}\rangle = |\Delta_1^{15}\rangle |F_1^{(s)}\rangle + |\Delta_2^{15}\rangle |F_2^{(s)}\rangle ,$$

(7.33)

which in turn is substituted into the stripping amplitude $T_{dp}$ of Eqn.(6.4) to produce

$$T_{dp} = <\chi_p^{(s)}(r_p), \Phi_n(r_n) | \sum_{i=1}^{2} V_{np}(r_i) |\Delta_1^{15}\rangle |F_i^{(s)}\rangle > .$$

(7.34)

We mentioned in Chapter VI that the zero-range approximation for $V_{np}(r)$ is quite successful in describing stripping calculations. Such a situation is reproduced here if we replace $V_{np} |\Delta_1^{15}\rangle$ of Eqn.(7.34) by

$$V_{np}(r) \Delta_1^{15}(r) \equiv G_i \delta (r) ,$$

(7.35)

where the $G_i$ are real constants representing the strengths of the $\Delta_1^{15}(r)$ wave functions, and are given by

$$G_i = \int dr V_{np}(r) \Delta_1^{15}(r) .$$

(7.36)

As a result, the six dimensional $T_{dp}$ integral expression (7.34) becomes, in the zero-range approximation, three dimensional, viz

$$T^{ZR}_{dp} = \int dR \left(\chi_p^{(-)}(R)\right)^* \Phi_n(R) \left\{ \sum_{i=1}^{2} G_i F_i^{(s)}(R) \right\} .$$

(7.37)
One can see from Eqn.(7.37) that the second channel c.m. function \( F_2^{(+)}(R) \) contributes to \( T_{\text{dp}}^{ZR} \) through the quantity \( G_2 F_2^{(+)}(R) \). It is therefore informative to compare the moduli of \( G_1 F_1^{(+)}(R) \) and \( G_2 F_2^{(+)}(R) \) in order to assess the second channel's contribution. Using the calculated functions \( F_1^{(+)}(R) \), \( F_2^{(+)}(R) \) and the appropriate \( G_i \) factors as given in Table-II, we evaluated the moduli of the above mentioned combinations.

The results, shown in Fig.(7.9), prove that when we add the second channel wave function to the first one (each times the appropriate \( G_i \) factor) we obtain large modifications of \( \Psi^{(+)}(r,R) \) in the nuclear surface region. We therefore conclude that breakup effects, associated with an n-p pair of c.m. relative energy simulated by \( \lambda_2 \) (\( N = 15 \)), are important in computing the stripping amplitude \( T_{\text{dp}} \). From this result, we understand that as the basis size \( N \) increases more discrete positive energies simulating the n-p continuum are introduced through the \( \lambda_i \) into the coupled equations, thus allowing contributions to \( T_{\text{dp}} \) from additional open channels.

However, it must be stressed that in addition to open channels, the contributions from physically closed (\( E_1 < 0 \)) channels should also be considered. To find out whether open and closed channels are needed and subsequently determine the size of the Weinberg basis, one has to perform stripping calculations for various values of \( N \) until convergence of the predicted observables is reached. The convergence question and the comparison of the WSE calculations with other models form the subject matter of the next section.
FIG. (7.9) - Shows the moduli of $G_1 F_1^{(+)}(R)$ and $[G_1 F_1^{(+)}(R) + G_2 F_2^{(+)}(R)]$ with the individual moduli of the functions $F_1^{(+)}$ and $F_2^{(+)}$ as shown in Fig. (7.8). These calculations are for the reaction $^{66}Zn(d,p)^{67}Zn$ at $E^{\text{lab.}}_d = 88.2$ MeV, and for the $L = 13$ partial wave. The Weinberg basis dimension was taken equal to $N = 15$. The zero-range constants $G_1$ and $G_2$ are given in Table - II.
TABLE-I

Relevant Constants

$$\phi_d(r) = N e^{-kr} (1 - e^{-\beta r})$$,

with

$$k = 0.232 \text{ fm}^{-1}$$,

$$\beta = 1.220 \text{ fm}^{-1}$$,

$$N = \left[ \frac{2k(k + \beta)(2k + \beta)}{\beta^2} \right]^{1/2} = 0.873 \text{ fm}^{-1/2}$$.

The constant \(N_d\) defined as \(- \langle \phi_i | V_{ap} | \phi_d \rangle\) is equal to

$$N_d = 4.020 \text{ MeV}^{1/2}$$.

TABLE-II

The \(G_i\) \((i \text{ open channels})\) zero-range constants as a function of the Weinberg states dimension \(N\)

(G \(i\) units : \text{MeV}^{1/2} \text{ fm}^{3/2})

<table>
<thead>
<tr>
<th>(N \rightarrow)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_2)</td>
<td>+9.21</td>
<td>+12.86</td>
<td>+14.41</td>
<td>+15.31</td>
<td>+15.90</td>
<td>+16.32</td>
<td>+16.64</td>
</tr>
<tr>
<td>(G_3)</td>
<td></td>
<td></td>
<td>-7.01</td>
<td>-7.53</td>
<td>-7.95</td>
<td>-8.30</td>
<td></td>
</tr>
</tbody>
</table>
7.5) CONVERGENCE OF THE CALCULATED WSE OBSERVABLES
AND COMPARISON WITH OTHER METHODS

The work presented in the previous sections is mainly concerned with
the preparation phase of the WSE model for eventual practical
applications. During that phase, we identified and solved a number of
numerical problems, then constructed most of the building blocks
necessary to perform WSE stripping calculations for different Weinberg
basis sizes (N).

We have shown that as N increases, more open channels are explicitly
introduced into the calculations of the stripping transition amplitude.
When N = 15, the effects of such additional channels on $T_{dp}$ are found to
be particularly substantial in the nuclear surface region. This is an
encouraging result which provided the basis for investigating further
whether such effects persist as N increases or become negligible beyond a
particular value of N. This investigation has been implemented at both
low and intermediate energies.

Moreover, since the ADIA prescription appears as the solution of
lowest order (N=1) in the WSE approach, then one expects the N $\gg$ 1 WSE
calculations to improve on the predictions of ADIA. In this context, it
is interesting to examine the nature of the obtained improvements by
comparing the ADIA and WSE results for both observables, the
cross-section and the vector analyzing power.

The WSE predictions for the same observables are also compared
against those obtained using an equivalent QAD calculation.
The objectives we have outlined above can be adequately re-phrased by addressing the following outstanding questions:

a) what is the number $N$ of Weinberg states that is required to achieve numerical convergence of the calculated observables at medium and intermediate deuteron energies?

b) what are the corrections to the ADIA method that we obtain when using this new formalism?

c) how do the WSE predictions of the cross-section and the vector analyzing power compare with the equivalent QAD results?

The answers to the questions posed above are sequentially discussed in what follows.

We first of all begin with the convergence issue. The approach adopted here consists of examining the predicted stripping observables for ascending $N$ values. For purposes of our analysis, we performed zero-range WSE calculations. The procedure for a typical zero-range WSE stripping calculation is explained below since it is repeated several times during the convergence investigation.

Once the n-p interaction model is identified, a Weinberg states basis of a given size $N$ can be constructed and then used to expand the three-body wave function. The substitution of the expanded wave function into the three-body Schrödinger equation produces a set of $N$ coupled equations that contain real constant coupling terms (i.e. matrix $c$). The diagonalisation of the matrix $c$ allows the creation of a new basis
constructed from N Weinberg states) in which the N coupled equations transform into a regular coupled channel problem. The resulting coupling potentials are determined using the new basis states and a given set of nucleon-nucleus optical potentials. The coupling potentials are in turn fed to a coupled channel code (FRESCO) to generate the c.m. motion wave functions $F_i^{(c)}(R)$ ($i=1,2,...,N$). The latter are then used as part of an input data deck for the transfer reactions code (TWOFNR) to evaluate relevant zero-range observables. The other part of TWOFNR's data deck consists of the reaction details and the exit channel optical parameters that are necessary to produce the outgoing proton distorted wave $\chi_p^{(3)}(R)$ and the bound neutron wave function $\Phi_n(R)$.

This one N zero-range WSE stripping calculation is best illustrated by the Chart-D1 of Appendix-D.

For the convergence analysis, we have chosen the $^{66}$Zn(d,p)$^{67}$Zn reaction at $E_{d}^{lab} = 88.2$ MeV. As mentioned earlier, this choice is motivated by the fact that this reaction has already been studied by a collaboration between our group and experimentalists at Indiana university [St90]. Moreover, since they used QAD to describe the experimental angular distributions of the cross-section $(d\sigma/d\Omega)$ and the vector analyzing power $(T_{11})$, then it is possible to compare our WSE calculations with their equivalent QAD results. We return to this point in the course of answering question c) above.

In implementing our analysis, the c.m. motion wave functions $F_i^{(c)}$ were calculated from the underlying nucleon-nucleus optical potentials of Becchetti and Greenlees [Be69] and the Hulthén interaction of Eqns.(7.2). The final-state proton optical parameters used in our calculations were
derived [St90] from fitting the cross-section and vector analyzing power data of $^{67}\text{Zn}(p,p)^{67}\text{Zn}$ at $E_{p}^{1\text{lab}} = 91.8$ MeV. The bound neutron wave function was evaluated in a Woods-Saxon well with shape parameters $r = 1.25$ fm and $a = 0.65$ fm, and a spin-orbit potential depth $V_{\text{SOR}} = 6.0$ MeV. The real well depth was adjusted to reproduce the neutron separation energy $\varepsilon_{n}$. The proton optical parameters are shown in Table-III.

One should mention at this stage that following the breakup of the deuteron into a free n-p pair, a neutron of spin $s_{n}$ is captured into one of the allowed quantum mechanical states of the target nucleus $^{66}\text{Zn}$. These states ($\Phi_{n}(r_{n})$ in our notation) are determined by the n-$^{66}\text{Zn}$ interaction denoted here by $V_{n}^{A}(r_{n})$, and each of them has a specific orbital, $l_{n}$, and total angular momentum $j_{n}$ such that $j_{n} = l_{n} + s_{n}$. Such states are assumed in this work to be neutron-target bound states whose absolute energy is given by the separation energy of a neutron from the residual nucleus $^{67}\text{Zn}$. In fact, in the reaction under study we are interested in a transition whereby the the residual nucleus is left in its Ground State (G.S.) with $l_{n} = 3$ and $j^{\pi} = 5/2^{-}$. The results of zero-range WSE stripping calculations shown in Figs.(7.10-11), represent the convergence of the calculated cross-section and vector analyzing power as a function of the Weinberg basis dimension N. We clearly see that the results have effectively converged for $N = 35$. Furthermore, it turned out that for (d,p) reactions below 100 MeV only very few of the $|\Delta_{i}^{N}\rangle$ represent physically open channels. Thus, although a large number of Weinberg states was used in constructing the $\{\Omega_{i}^{N}, \Delta_{j}^{N}\}$ basis, the stripping calculation reduced to only a very small coupled channels analysis (i.e. three coupled equations at the most).
TABLE-III

Exit Channel Distorting Potential parameters and Forms

(Units are in MeV and fm)

<table>
<thead>
<tr>
<th>Energy</th>
<th>91.80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Central</td>
<td>V</td>
</tr>
<tr>
<td></td>
<td>r</td>
</tr>
<tr>
<td></td>
<td>a</td>
</tr>
<tr>
<td>Imaginary Central-Volume</td>
<td>V</td>
</tr>
<tr>
<td></td>
<td>r</td>
</tr>
<tr>
<td></td>
<td>a</td>
</tr>
<tr>
<td>Coulomb radius</td>
<td>r_c</td>
</tr>
</tbody>
</table>

Optical Potential

\[ p + ^{67}\text{Zn} \]

*Proton optical potential are of the following forms

\[ V_{\text{pB}}(r) = -V_{\text{CR}} f_{\text{CR}}(r) - iV_{\text{CI}} f_{\text{CI}}(r), \]

where \( f_i \) is the conventional Woods-Saxon form factor

\[ f_i(r_p) = \left[ 1 + \exp \left( \frac{r_p - r_0^i}{a_i} \frac{A^{1/3}}{A^{1/3}} \right) \right]^{-1} \]

1 These parameters are reproduced from [St90].
FIG. (7.10) - Represents the convergence of the angular distribution of the differential cross-section for the $^{66}\text{Zn}(d,p)^{67}\text{Zn}(G.S.;5/2^+;I_n = 3)$ at $E_d = 88.2$ MeV. These results are obtained by performing zero-range WSE stripping calculations for various ascending values of $N$ (i.e. $N = 15, 20, 25, 30$ and $35$).
FIG. (7.11) - Represents the convergence of the angular distribution of the vector analyzing power $iT_{11}$ for the $^{66}$Zn(d,p)$^{67}$Zn(G.S.;5/2$^{-};l_1=3$) reaction at $E_{lab} = 68.2$ MeV. These results are obtained by performing zero-range WSE stripping calculations for various ascending values of $N$ (i.e. $N = 15, 20, 25, 30$ and $35$).
A close examination of the contributions of each channel to the WSE calculated observables \( \frac{d\sigma}{d\Omega} \) and \( iT_{11} \) shows that, e.g. in the N=35 case, the third channel's input is relatively small when compared against the first two. To illustrate this, we carried out N=35 zero-range WSE calculations to evaluate the individual contributions of the three open channels for the \(^{66}\text{Zn}(d,p)^{67}\text{Zn}\) reaction at \( E_{\text{lab}}^d = 88.2 \) MeV.

The results presented in Fig.(7.12) show, for both observables, the contribution of each channel as well as the resulting prediction when all channels are added together. A clearer picture concerning the third channel's input is obtained when we estimate the impacts of the first channel, then the first plus the second and finally all three channels separately. From Fig.(7.13) we see that, for both observables, the contribution of the third channel is rather small. In fact these results suggest that only two out of the three available channels need to be considered. Moreover, our results indicate that the effects of a deuteron breaking up into an n-p pair whose c.m. continuum energy is simulated by \( \lambda_2 = 20 \) MeV (see Fig.(7.2)) are certainly very important in the study of deuteron stripping reactions at intermediate energies.
FIG. (7.12) - The angular distribution of the cross-section $\frac{d\sigma}{d\Omega}$ and the vector analyzing power $T_{11}$ for the $^{66}$Zn($d,p$)$^{67}$Zn(G.S.;$5/2^+;I_n=3$) at $E_{\text{lab}} = 88.2$ MeV. These results represent N=35 zero-range WSE calculations when considering the contributions of each channel separately, and when all three channels are present.
FIG. (7.13) - The angular distribution of the cross-section $d\sigma/d\Omega$ and the vector analyzing power $I_{11}$ for the $^{66}\text{Zn}(d,p)^{67}\text{Zn}$(G.S.;5/2$^+_1$; $J^e$=3) at $E_{\text{lab}}$ = 88.2 MeV. These results represent $N=35$ zero-range WSR calculations when considering the contributions of the first channel, then the first plus second channel and finally all three channels together. The results indicate that the effects simulated by the third channel are relatively small.
The above convergence investigation was carried out at an intermediate incident energy, namely, \( E^{\text{lab}}_d = 88.2 \text{ MeV} \). In order to complete the convergence analysis of the WSE method, we require the same investigation to be performed at a lower incident energy. For this purpose we use the same stripping reaction as above, but this time the incident deuteron lab. energy is chosen, for example, to be \( E^{\text{lab}}_d = 25 \text{ MeV} \).

At this energy regime, one immediately infers from Fig.(7.2) that if \( 5 \leq N \leq 35 \), then up to two open channels can be considered in the coupled equations. In this calculation, the final-state proton potential parameters are evaluated using the standard proton-nucleus optical potential of Becchetti and Greenlees evaluated at the proton lab. energy \( E^{\text{lab}}_p = 29.5 \text{ MeV} \).

We repeated the same one N zero-range WSE numerical procedure, defined above, for several values of N. However, due to the low value of \( E^{\text{lab}}_d \) we performed one- and two-open channels zero-range WSE calculations. The results, shown in Figs.(7.14-15), indicate that the calculated cross-section and vector analyzing power for the same \(^{66}\text{Zn} \) reaction at \( E^{\text{lab}}_d = 25 \text{ MeV} \) have converged for \( N = 35 \). One should point out, that the quality of this convergence is slightly inferior to the one obtained at the intermediate energy of 88.2 MeV. Nonetheless, it is clear from the graphs of the cross-section and the analyzing power that the \( N = 30 \) and \( N = 35 \) calculations have the same structure and only differ by a small amount in magnitude. These results prove that even at low energies the WSE method converges for a manageable Weinberg states dimension, and that the number of coupled channel equations remains small (at the most two).
FIG. (7.14) - Represents the convergence of the angular distribution of the differential cross-section for the $^{66}_{\text{Zn}}(d,p)^{67}_{\text{Zn}}(G.S.;5/2^+;l = 3)_{\text{lab}}$ at $E_{\text{d}} = 25$ MeV. These results are obtained by performing zero-range WSE stripping calculations for various ascending values of $N$ (i.e., $N = 20, 25, 30$ and 35).
FIG. (7.15) - Represents the convergence of the angular distribution of the vector analyzing power $I_{11}^V$ for the $^{66}\text{Zn}(d,p)^{67}\text{Zn}(\text{G.S.}; \frac{5}{2}^+; l = 3)$ at $E_{\text{lab}} = 25$ MeV. These results are obtained by performing zero-range WSE stripping calculations for various ascending values of $N$ (i.e., $N = 20, 25, 30$ and 35).
Secondly we attempt, in what follows, to answer the question relative to the corrections to ADIA that we obtain when using the WSE approach. In order to achieve this, we carried out zero-range WSE stripping calculations for the $^{66}$Zn(d,p)$^{67}$Zn (G.S., $5/2^-$, $l_n = 3$) at 88.2 MeV in the cases where $N=1$ and $N=35$. In both cases we considered open channels since we have already shown that the convergence of the calculated reaction observables can be achieved without the closed channels' contributions.

Figure (7.16) displays the calculated moduli of $G_{1}^{\uparrow}(R)$ and $G_{2}^{\downarrow}(R) + G_{3}^{\uparrow}(R)$ for the partial waves $L = 0, 7, 13$ and 20. The additional channels that the WSE method introduces into the stripping calculation, when compared against the ADIA channel, tend to modify the contribution of the projected wave function $\psi^{\uparrow}(\mathbf{r})$ to the zero-range stripping amplitude $T_{dp}^{ZR}$. These modifications occur at the nuclear surface region and are, for this reaction, important for the surface partial wave $L=13$. 

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FIG. (7.16)- Shows the moduli of $G_{P_1}^{(+)}$ for the ADIA channel $N=1$, and $[G_{P_1}^{(+)} + G_{P_2}^{(+)} + G_{P_3}^{(+)}]$ for the three open WSE channels with $N=35$, for the $L = 0, 7, 13$ and $20$ partial waves. These results are for the $^{66}$Zn(d,p)$^{67}$Zn reaction at $E_{\text{lab}}^d = 88.2$ MeV.
In order to establish whether these modifications are improvements or not, we have to evaluate the reaction observables. The trends of the obtained observables are then qualitatively compared against the available experimental data [St90]. By carrying out this comparison we do not attempt to fit the data for the reaction under investigation, but simply to use such data as a reference against which the theoretical predictions of both theories ADIA and WSE are qualitatively compared. The results, shown in Figs.(7.17-18), respectively represent the ADIA (N=1) and the N=35 WSE predictions of the cross-section and vector analyzing power for the \(^{66}\text{Zn}(d,p)^{67}\text{Zn}\) (G.S.; \(5/2^+\); \(I_n=3\)) reaction at \(E_{\text{lab}}^d = 88.2\) MeV.

When qualitatively compared with the data of [St90] (not shown in our graphs) the N=35 zero-range WSE observables tend to be closer (in structure not in magnitude) to the data than those predicted by ADIA. In particular the vector analyzing power \((T_{11})\) where a sort of forward angles structure is observed, and especially the peak at about \(\theta_{\text{em}} = 25^\circ\). This proves that the modifications brought about by the additional WSE channels are in fact improvements over ADIA and that the explicit inclusion of the correct n-p continuum c.m. energies is crucial in stripping calculations.
FIG. (7.17) - Shows the angular distribution of the cross-section $d\sigma/d\Omega$ as predicted by ADIA ($N=1$) and WSE ($N=35$). These results are for the same $^{66}Zn(d,p)^{67}Zn$ (G.S.: $5/2^-$; $l_n=3$) reaction at $E_{d}^{lab}=88.2$ MeV.
FIG. (7.18) - Shows the angular distribution of the vector analyzing power $i T_{11}$ as predicted by ADIA ($N=1$) and WSE ($N=35$). These results are for the same $^{66}$Zn($d,p$)$^{67}$Zn (G.S.; $5/2^+ ; I_n=3$) reaction at $E_{\text{lab}} = 85.2$ MeV.
We finally answer the third question using a comparative approach. It should be noted that it is possible to carry out QAD calculations since all the relevant computer codes are locally available. Thus, to generate the equivalent to our zero-range N=35 WSE calculation, we switched off the spin-orbit forces from both entrance and exit channels and selected the zero-range mode when performing the QAD calculations.

The calculations were carried out for the $^{66}\text{Zn}(d,p)^{67}\text{Zn}$ (G.S.; $5/2^-$; $l_n = 3$) at $E_{\text{lab}}^{\text{d}} = 88.2$ MeV. The obtained results for the cross-section and the vector analyzing power, including those of ADIA, are presented together in Figs. (7.19-20) respectively.

The first important comment to make about these results is that the predictions of our zero-range N=35 WSE calculations are in general similar to those obtained using the QAD method. This result is encouraging since it was shown [St90] that the full (i.e. with spin-orbit forces, and transfer from both S and D states of the deuteron) QAD calculations satisfactorily reproduce the data. The second remark that can be made is that the predictions of both theories, WSE and QAD, are for both observables consistently different from those of ADIA. This result provides a strong evidence supporting the importance of breakup effects in (d,p) reactions and gives a quantitative indication concerning the corrections to the ADIA model. In fact, these two methods (WSE and QAD) were purposely developed in order to improve on ADIA's treatment of these particular effects. The results of this relatively simple zero-range WSE calculations have exactly done that and it is hoped that, when the full implementation of this model is attained, attempts to rigorously reproduce (d,p) experimental data can be made.
FIG.(7.19)- Shows the comparison of the angular distribution of the cross-section $d\sigma/d\Omega$ as predicted by the ADIA ($N=1$), QAD and WSE ($N=35$) methods. These results are for the same $^{66}\text{Zn}(d,p)^{67}\text{Zn}$ (G.S.; $S/2^+; I=3$) reaction at $E_{\text{lab}} = 85.2$ MeV.
FIG. (7.20) - Shows the comparison of the angular distribution of the vector analyzing power \( \tilde{T}_{11} \) as predicted by the ADIA (N=1), QAD and WSE (N=35) methods. These results are for the same \( ^{66}\text{Zn}(d,p)^{67}\text{Zn} \) (G.S.: \( 5/2^-; l=3 \)) reaction at \( E_{d}^{\text{lab}} = 88.2 \text{ MeV} \).
CHAPTER VIII

- CONCLUSIONS -

8.1) SUMMARY AND CONCLUSIONS

We mentioned in Chapter V that a number of direct nuclear reactions theories can be applied to analyse (d,p) reactions. We have indicated that the DWBA, in which the n-p relative motion is assumed to be that of a free deuteron, provides a much less reliable description of particle transfer reactions. In an attempt to improve on the DWBA, Johnson and Soper introduced the ADIAbatic theory where the effects of continuum n-p states are partially taken into account. Although ADIA has provided systematic improvements over the conventional DWBA, recent experimental data [St86, St87] suggest that it needs to be refined. Two techniques have thus been developed to extend ADIA, namely the Quasi-ADIabatic (QAD) and Weinberg States Expansion (WSE) methods. In the QAD method the c.m. energy of the n-p pair is no longer assumed to be degenerate with the deuteron elastic channel (as in ADIA), but more efforts were devoted to obtaining realistic prescriptions of that c.m. energy.

Although the WSE was developed in 1974, it has never been investigated quantitatively. Our task was therefore to analyse the WSE formalism and to numerically implement it. Thus, in Chapter VI we discussed in detail the fundamental theoretical points of the WSE method, and in Chapter VII we constructed and numerically tested the main building blocks of the formalism and eventually applied it to analyse a stripping reaction.
In implementing the WSE theory we carried out several preparatory calculations. These are as follows

1) the construction of a Weinberg states basis for the n-p system. The n-p interaction model chosen for this purpose is the central Hulthén potential. We only considered S-waves n-p breakup states in our truncated Weinberg states expansion of the three-body wave function,

2) the resulting N coupled channel equations for the c.m. motion wave functions contain real constant coupling terms. We showed that this problem can be overcome by diagonalising an appropriate matrix and re-writing the N coupled in a new basis. We also found that the eigenvalues of such a matrix simulate the continuum spectrum of the n-p Hamiltonian,

3) the solutions to the new set of N coupled channel equations required the construction of the new basis states and the channel coupling potentials that describe the interaction of a continuum n-p pair with the nuclear target. An attractive feature of this method is that the new basis states depend only on the n-p system information and contain no reference to the reaction under consideration.

4) To solve the N coupled equations we wrote, for checking purposes, a two open channel code TWOCEQ and used the general purpose coupled channels code FRESCO when more than two channels were considered. The obtained c.m. motion wave functions were either analysed independently or passed on to a reaction transfer code TWOFNR to evaluate all the relevant deuteron stripping observables.
By performing these calculations and obtaining the encouraging results of Section (7.5), we have effectively proved that the Weinberg eigenstates of the n-p system do provide an adequate basis for the representation of the three-body wave function. Particularly when seeking the accurate determination of the stripping amplitude \( T_{dp} \). Moreover, by construction the WSE coupled channel equations for the c.m. motion wave functions do explicitly include, in an elegant mathematical way, deuteron breakup effects. We have also seen that the WSE formalism provides the means to systematically investigate the leading corrections to ADIA, since it is readily shown that the latter appears as a suitable lowest (i.e. \( N = 1 \) result) order solution to the present theory.

It must be stressed at this point that the aim of the present work is not the rigorous reproduction of experimental data, but is to test the concepts of the WSE method and prepare it for an eventual full implementation in the future. To achieve our present aims, we carried out zero-range WSE stripping calculations using a simple n-p interaction model and neglecting the spin-orbit forces from both entrance and exit channels.

The main body of our calculations was devoted to the answer of a number of important questions (see Section (7.5)). The approach adopted in answering these questions was to examine the results of zero-range WSE calculations for ascending numbers of Weinberg states starting from \( N = 1 \). As \( N \) increases, more of the new basis states \( | \Delta_1^N \rangle \) become open. In fact, it turned out that for \((d,p)\) reactions below 100 MeV, only very few of the \( | \Delta_1^N \rangle \) represent open channels. The result is that the number of physically important \( | \Delta_1^N \rangle \) states is very small. Thus, although a large
number (N=35) of underlying Weinberg states may be used in constructing the $| \Delta_i^N >$ and $| \Omega_j^N >$, the reaction calculations reduce to only a very small coupled channel calculations. At low energies (e.g. 25 MeV) the number of physically important $| \Delta_i^N >$ states is found to be even smaller.

The WSE method has been applied to the $^{66}$Zn(d,p)$^{67}$Zn reaction at 88.2 and 25 MeV. We have seen that the cross-section and vector analysing power results have effectively well converged for N = 35 at $E^{1a b}_d = 88.2$ MeV and have, to a lesser degree, converged for the same N value when $E^{1a b}_d = 25$ MeV. Furthermore, we have shown that the WSE model provides substantial improvements over ADIA's predictions. Thus, following these positive results and bearing in mind that the WSE formalism was designed to extend and improve ADIA, we indirectly proved that this model justifies mathematically the ADIAbatic ideas.

It is worth pointing out that the dimension of the coupled channel equations used in obtaining these results was at most three, thus making our calculations computationally more efficient than the equivalent CDCC methods. However, it is important to note that the WSE is not well adapted for elastic scattering, whereas the CDCC technique can be applied to elastic scattering as well as stripping reactions. The comparison of our N = 35 zero-range WSE results with those of the equivalent QAD indicates that the cross-section and the vector analyzing power predictions of both theories are overall in good agreement.
8.2) SUGGESTIONS FOR FUTURE WORK

It has been shown [St90] that if the transfer from both the S and D states of the deuteron is included in finite range stripping calculations then a much better reproduction of the experimental data is obtained, particularly when polarization observables are involved. Therefore, it would be interesting to consider within the framework of the WSE method, a more realistic n-p interaction model in order to account for deuteron D-state contribution to the stripping amplitude. Other improvements to the present WSE calculations could be brought about by including the spin-orbit forces in the entrance and exit channels.

The work of Rawitscher [Ra74] in the related problem of deuteron elastic scattering indicated that the coupling of continuum D-states of the n-p system has a significant effect on the calculations. It is therefore worth investigating, using the WSE formalism, whether the effects of such states on the (d,p) transition amplitude are as important as in the elastic scattering case.

Finally, since we have concentrated our efforts only on physically open channels, it is in principle feasible to investigate whether the extension to closed channels generates additional contributions to the stripping transition amplitude. If they exist, these contributions must be associated with some unknown underlying pieces of physics. This may be of particular relevance to WSE stripping calculations at low energies.
APPENDIX-A

INCIDENT BEAM POLARIZATION FOR A PRECISE

PTC MEASUREMENT

Measurements of PTCs for spin-1 polarized incident particles involve what is called a double scattering experiment. These experiments are very difficult to perform, and obtaining good statistics out of them is not always easy to achieve [Sa73]. Nonetheless, it is possible to measure with good precision certain PTCs in some particular cases. Identifying one of these cases is the subject matter of this appendix.

The experimental set-up described by [Oh73], and reproduced here as Fig.(A.1), is used to perform experiments aimed at measuring polarization observables for the scattering of polarized deuterons from several nuclei and at different energies. This represents a double scattering experiment whereby the first scattering center is the target of interest and the second scattering center is the polarimeter (second target - $^3$He cell). The scattered deuteron beam from the first target is directed towards the polarimeter which analyzes the polarization of the scattered deuterons from the first scattering center.

The most general form of the polarization after the first scattering (described in the x'y'z' frame) is given by a set of equations representing every possible outgoing polarization component and the differential cross-section, in terms of the initial polarization components as well as other spin functions. This set of equations can be found in the references [Oh70, Oh72a, Oh72b, Oh73], and is for the general case where the symmetry axis of the incident polarized beam is arbitrary. It is shown [Oh73] that if one considers the case where the
symmetry axis is chosen to be along the y-axis, thus allowing \( p_{xx} = p_{zz} = -\frac{1}{2} p_{yy} \), then Eqns.(2.25-31) of the main text simplify into

\[
I = I_0 \left[ 1 + \frac{3}{2} p_y A_y + \frac{1}{2} p_{yy} A_{yy} \right]. \tag{A.1}
\]

\[
p_{zx} I = I_0 \left[ q_{z,x} + \frac{3}{2} p_y K_y z + \frac{1}{2} p_{yy} K_{yy} z' \right]. \tag{A.2}
\]

Fig. (A.1) - Shows the experimental lay out of [Oh73], where \( \hat{S} \) is a unit vector indicating the symmetry axis of the incident beam which is, in this case, chosen to be parallel to the y-axis. The positions of the four detectors (\( \theta_2 \) and \( \phi_2 \)) are as indicated in the text.

Note that Eqns.(A.1-2) relate to the first scattering process. The analysis of the outgoing deuterons polarization, when using the particular geometry of the experiment mentioned above can be carried out by considering the azimuthal dependence of the cross-section for the analyzing reaction (polarimeter). We make use of a number of equations from [Oh73] that are related to the experimental set up of Fig.(A.1). The azimuthal dependence of the cross-section can be written as
\[ I(\theta_2, \phi_2) = I_0(\theta_2) \left[ 1 + A \cos(\phi_2) + B \sin(\phi_2) + C \sin(2\phi_2) + D \cos(2\phi_2) + E \right] \quad \text{(A.3)} \]

The experimental set up of Fig.(A.1) contains four detectors labelled by Left, Right, Up, Down (L, R, U, D). The four detectors are positioned at \( \theta_2 = 54.7^\circ \) and \( \phi_2 = 0, 180, 270 \) and \( 90^\circ \), respectively. The value of \( 54.7^\circ \) for the angle \( \theta_2 \) is a characteristic of the polarimeter used. The coefficients in Eqn.(A.3) are

\[
A = \frac{3}{2} p_{y'} A_{y'}(\theta_2) + \frac{2}{3} p_{x'z'} A_{x'z'}(\theta_2), \\
B = \frac{3}{2} p_{x'} A_{y'}(\theta_2) - \frac{2}{3} p_{y'z'} A_{x'z'}(\theta_2), \\
C = -\frac{1}{3} p_{x'z'} \left[ A_{x'z'}(\theta_2) - A_{y'z'}(\theta_2) \right], \\
D = \frac{1}{6} \left( p_{x'z'} - p_{y'z'} \right) \left[ A_{x'z'}(\theta_2) - A_{y'z'}(\theta_2) \right], \\
E = \frac{1}{2} p_{z'x'} A_{z'x'}(\theta_2), 
\]

where the analyzing powers with double-primed indices refer to those of the analyzing reaction and the polarization components with primed indices represent the polarization of the scattered deuterons from the first target. The observed yields in each detector (details in [Oh73]) is of the form (e.g. for the down detector)

\[ Y_D = n \Delta \Omega_D I_D(\theta_2, \phi_2 = 90^\circ), \quad \text{(A.5)} \]

where \( I_D \) is given by Eqn.(A.3) with \( \phi_2 = 90^\circ \), \( n \) is the number of incident particles on the analyzer and \( \Delta \Omega_D \) is the effective solid angle subtended by the Down detector. If we assume that the four solid angles are equal \( (\Delta \Omega_L = \Delta \Omega_R = \Delta \Omega_U = \Delta \Omega_D) \) then the sum of the four detectors' yields gives the following total yield.
Eqn. (A.6) shows that by adding the yields observed in each detector, thus improving the statistics and hence the accuracy of the measurements, one obtains a straightforward means of measuring the quantity $E$. By virtue of Eqn. (A.4), the accurate measurement of the quantity $E$ means that $p_{z'z'}$ can also be measured with good precision since $A_{z'z'}$ is a known characteristic of the polarimeter used. This is a very important point in our analysis. The reason being the linkage of $p_{z'z'}$ to PTCs in Eqn. (A.2). In other words, we have here a very good case for a possible accurate measurements of PTCs. Due to the flexibility, offered by most ion sources, regarding the choice of the the incoming deuterons' polarization it is now feasible to select certain types of polarizations. Therefore, if we consider $p_y = 2/3$ and $p_{yy} = 0$ then Eqn. (A.2) transforms into

$$p_{z'z'} I = I_0 \left[ p_{z'z'} + K_{y'} z'z' \right].$$  \hspace{1cm} (A.7)

In deuteron elastic scattering, polarization functions (e.g. $p_{z'z'}$) are in general simply related, via time reversal invariance, to analyzing powers, and are therefore easily measured. We conclude from Eqn. (A.7) that provided, one prepares a purely vector polarized incident beam (i.e $p_y = 2/3$; $p_{yy} = 0$) with its symmetry axis along the normal to the scattering plane, then it is possible to measure with good accuracy the PTC $K_{y'} z'z'$. 

\hspace{1cm} 131
A) THE MADISON CONVENTION

The main text of the so-called Madison-convention can be found in the Proceedings of the Third International Symposium on Polarization Phenomena in Nuclear Reactions, University of Wisconsin Press, Madison, 1970. The main four points of this convention are given below:

THE MADISON CONVENTION

I) Polarization effects involving spin-1 particles should be described either by spherical tensor operators $\tau_{kq}$, with normalization given by $\text{Tr}(\tau_{kq} \tau_{k'q'}^+) = 3 \delta_{kk'}\delta_{qq'}$, or by cartesian operators $S_i$, $(3/2)$

$$(S_i S_j + S_j S_i) - 2 \delta_{ij}$ (i or j = x,y,z). $S_i$ denote the usual spin-1 angular momentum operators.

II) The state of spin orientation of an assembly of particles, referred to as polarization, should be denoted by the symbols $t_{kq}$ (spherical) or $p_i$, $p_{ij}$ (cartesian). These quantities should be referred to a right-handed coordinate system in which the positive z-axis is along the direction of momentum of particles, and the y-axis is along $k_i \times k_o$ for the nuclear reaction which the polarized particles initiate, or from which they emerge.

III) Terms used to describe the effects of initial polarization of a beam or target on the differential cross-section for a nuclear reaction
should include the modifiers analyzing or efficiency, and should be
denoted by $T_{k_{in}}$ (spherical) or $A_{ij}$ (cartesian). These quantities
should be referred to a right-handed coordinate system in which the
positive $z$-axis is along the beam direction of the incident particles and
the $y$-axis is along $k_{in} \times k_{out}$ for the reaction in question.

IV) - In the expression for a nuclear reaction $A(b,c)D$ an arrow placed
over a symbol denotes a particle which is initially in a polarized state
or whose state of polarization is measured.

We omitted in Chapter-II the specific expressions of the spherical and
cartesian operators in terms of the usual spin-1 angular momentum
operators, $S_i (i=x,y,z)$, and the $3 \times 3$ unit matrix, $\mathbb{1}$. These are given by

$$
\mathbb{1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix};
S_x = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix};
S_y = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & -i \\
0 & i & 0
\end{pmatrix};
S_z = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
$$

The spherical and cartesian tensors are as follows:

Table-A1

<table>
<thead>
<tr>
<th>spherical</th>
<th>cartesian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{00} = 1$</td>
<td>$P_x = S_x$</td>
</tr>
<tr>
<td>$\tau_{10} = (3/2) S_z$</td>
<td>$P_y = S_y$</td>
</tr>
<tr>
<td>$\tau_{1\pm1} = \mp (1/213) (S_x \pm i S_y)$</td>
<td>$P_z = S_z$</td>
</tr>
<tr>
<td>$\tau_{20} = (11/2) (3S_z^2 - 2)$</td>
<td>$P_{xx} = 3S_z^2 - 2$</td>
</tr>
<tr>
<td>$\tau_{2\pm1} = \mp (1/213) [(S_x \pm i S_y)S_z$</td>
<td>$P_{yy} = 3S_z^2 - 2$</td>
</tr>
<tr>
<td>$\tau_{2\pm2} = (1/213) (S_z \pm i S_y)^2$</td>
<td>$P_{zz} = 3S_z^2 - 2$</td>
</tr>
<tr>
<td></td>
<td>$P_{xy} = (3/2) (S_x S_y + S_y S_z)$</td>
</tr>
<tr>
<td></td>
<td>$P_{xz} = (3/2) (S_x S_z + S_z S_x)$</td>
</tr>
<tr>
<td></td>
<td>$P_{yz} = (3/2) (S_y S_z + S_z S_y)$</td>
</tr>
</tbody>
</table>
The spherical and cartesian tensor moments are related through the following expressions:

**Table-A2**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{10} = (43/2) \ p_z )</td>
<td>( p_x = (-1/3) \ (t_{11} - t_{1-1}) )</td>
</tr>
<tr>
<td>( t_{1\pm1} = \mp 43/2 \ (p_x \pm i \ p_y) )</td>
<td>( p_y = (i/3) \ (t_{11} + t_{1-1}) )</td>
</tr>
<tr>
<td>( t_{20} = (1/42) \ p_{zz} )</td>
<td>( p_z = (42/3) \ t_{10} )</td>
</tr>
<tr>
<td>( t_{2\pm1} = \mp (1/13) \ (p_{xz} \pm i \ p_{yz}) )</td>
<td>( p_{xx} = (43/2)(t_{22} + t_{2-2}) - t_{20}/42 )</td>
</tr>
<tr>
<td>( t_{2\pm2} = (1/243) \ (p_{xx} - p_{yy} \pm 2i \ p_{xy}) )</td>
<td>( p_{yy} = (43/2)(t_{22} + t_{2-2}) - t_{20}/42 )</td>
</tr>
<tr>
<td></td>
<td>( p_{xy} = \pm (i/43)(t_{22} - t_{2-2}) )</td>
</tr>
<tr>
<td></td>
<td>( p_{xz} = (43/2)(t_{21} + t_{2-1}) )</td>
</tr>
<tr>
<td></td>
<td>( p_{yz} = (43/2)(t_{21} - t_{2-1}) )</td>
</tr>
</tbody>
</table>

Note that the same relations (those given in Table-A2) hold between the \( T_{kq} \) and the \( A_k, A_q \). It is straightforward to extract the relationships between cartesian and spherical tensor operators as defined in Table-A1. These are given by

**Table-A3**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_x = - (41/3) \ (\tau_{11} - \tau_{1-1}) )</td>
<td></td>
</tr>
<tr>
<td>( p_y = + (i/3) \ (\tau_{11} + \tau_{1-1}) )</td>
<td></td>
</tr>
<tr>
<td>( p_z = + (42/3) \ \tau_{10} )</td>
<td></td>
</tr>
<tr>
<td>( p_{xy} = - (i43/2)(\tau_{22} - \tau_{2-2}) )</td>
<td></td>
</tr>
<tr>
<td>( p_{xz} = - (i43/2)(\tau_{21} - \tau_{2-1}) )</td>
<td></td>
</tr>
<tr>
<td>( p_{yz} = + (i43/2)(\tau_{21} + \tau_{2-1}) )</td>
<td></td>
</tr>
<tr>
<td>( p_{xx} = + (i43/2)(\tau_{22} + \tau_{2-2}) - (41/2) \ \tau_{20} )</td>
<td></td>
</tr>
<tr>
<td>( p_{yy} = - (i43/2)(\tau_{22} + \tau_{2-2}) - (41/2) \ \tau_{20} )</td>
<td></td>
</tr>
<tr>
<td>( p_{zz} = + 42 \ \tau_{20} )</td>
<td></td>
</tr>
</tbody>
</table>
Using the $S_i$ ($i=x,y,z$) and the equations in Table-A1 we have expressed all the necessary matrices (i.e. $\tau_{1m}'s$ and the $P_i, P_j$) which are needed in Appendix-C. These matrices are shown below:

**Table-A4**

\[
\begin{align*}
\tau_{00} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
\tau_{11} &= -(43/2) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \\
\tau_{1-1} &= +(43/2) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
\tau_{10} &= +(43/2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
\tau_{22} &= +(43) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
\tau_{2-2} &= +(43) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\tau_{20} &= +(41/2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
\tau_{21} &= -(43/2) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \\
\tau_{2-1} &= +(43/2) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{align*}
\]

We have not given the matrices, $P_x, P_y$ and $P_z$, because they are identical to the $S_i$ ($i=x,y,z$) which have already been displayed above.

**Table-A5**

\[
\begin{align*}
P_{xy} &= (3i/2) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\
P_{xz} &= (3/24i) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \\
P_{yz} &= (3i/24i) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
P_{xx} &= (1/2) \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}; \\
P_{yy} &= (1/2) \begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix}; \\
P_{zz} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]
B)- DEFINITIONS OF THE POTENTIAL FORM FACTORS OF TABLE-1.

The potential form factors used in our calculations and referred to in Table-1 of the main text are defined for a target of mass AT as (with \( x = (r-R_{AT}^{1/3})/A \))

**WOOSAX** : Woods-Saxon

\[ V(r) = - V \left( 1 + e^x \right)^{1} . \]

**WS-DER** : Woods-Saxon Derivative (normalized to unity)

\[ V(r) = - 4 V e^x / \left( 1 + e^x \right)^2 . \]

**THOMAS** : Thomas formfactor

\[ V(r) = - \left( 2 V / A r \right) \left( e^x / \left( 1 + e^x \right)^2 \right) . \]

**D2-FLD** This potential is similar to a Woods-Saxon second derivative. It was designed to approximate the real part of the \( T_r \) potential predicted by the Folding Model. It was proposed by Keaton and Armstrong [Ke73],

\[ V(r) = -\left(2V/A^2\right) \left( e^x/(1+e^x)^2 \right) \left[ (A/r) - (1-e^x)/(1+e^x) \right] . \]

**D3-FLD** : This potential, which resembles a Woods-Saxon third derivative, also proposed by [Ke73] to approximate the imaginary \( T_r \) Folding Model potential,

\[ V(r) = -\left(8V/RA\right) \left( e^x/(1+e^x)^3 \right) \left[ (r/A)(1-4e^x+e^{2x}) - (1-e^{2x}) \right] . \]
In this appendix we give more details of the derivations that led to the expansion of $K_y^{x'z'}$ in terms of the $Q_{1m}$'s. The result of this derivation is given by Eqns.(3.19-21) in section (3.2).

For clarity, one needs to re-write some of the key equations that are necessary for the algebra. These equations are as follows:

$$K_y^\alpha = \text{Tr.} \left\{ F P_y F^+ P_\alpha \right\} / \text{Tr.} \left\{ F F^+ \right\}, \quad (C.1)$$

where $\alpha = xx$, $yy$ and $xz$. The $P$'s are defined in section (3.2) their matrix representations are given in Appendix-B. The $F$ matrix is given by

$$F = \begin{bmatrix} a & c \\ d & -e \\ c & a \end{bmatrix}, \quad (C.2)$$

where,

$$a = Q_{00} + (41/2) Q_{20},$$

$$b = (43/2) ( Q_{11} - Q_{21} ),$$

$$c = 43 Q_{22},$$

$$d = - (43/2) ( Q_{11} + Q_{21} ),$$

$$e = Q_{00} - (42) Q_{20}. \quad (C.3)$$

Eqn.(3.12) of section (3.2) yields

$$Q_{22} = (43/2) Q_{20} + 2 Q_{21} \cot(\theta), \quad (C.4)$$

where $\theta$ refers to the scattering angle. The expansion of $K_y^{x'z'}$ starts from Eqn.(3.14)
\[ K_{y}^{x-z} = \sin^{2}(\theta) K_{y}^{xx} + \cos^{2}(\theta) K_{y}^{zz} + \sin(2\theta) K_{y}^{xz}. \]  

(C.5)

We now expand \( K_{y}^{xx}, K_{y}^{zz} \) and \( K_{y}^{xz} \) in terms of the \( Q_{lm} \)'s and then substitute them into Eqn.(C.5) in order to obtain the equivalent expansion for \( K_{y}^{x-z} \). Eqns.(C.1-4) will be used for this purpose. The first step in this derivation is to obtain the expansions of \( K_{y}^{xx}, K_{y}^{zz} \) and \( K_{y}^{xz} \). In what follows we give a detailed derivation for the PTC, \( K_{y}^{xx} \). The same method applies to the two others, \( K_{y}^{zz} \) and \( K_{y}^{xz} \), therefore only their expanded expressions are given. Eqn.(C.1) implies that \( K_{y}^{xx} \) has the form

\[ K_{y}^{xx} = \text{Tr.} \left\{ F P_{y} F^{+} P_{xx} \right\} / \text{Tr.} \left\{ F F^{+} \right\}, \]  

(C.6)

We have showed in section (3.2) that \( K_{y}^{xx} \) can be written in terms of the \( Q_{lm} \)'s as follows

\[ K_{y}^{xx} = \left( \sum_{i} \sum_{j} \sum_{b} \sum_{c} B_{ij} D_{bd}^{*} \chi_{ij}^{bd} \right) / \left( 3 \sum_{k} \sum_{q} | Q_{kq} |^{2} \right), \]  

(C.7)

where,

\[ E_{ij} = (1/3) \text{Tr.} \left\{ P_{y} \tau_{ij} \right\}, \]  

(C.8)

\[ D_{bd}^{*} = (1/3) \text{Tr.} \left\{ P_{xx} \tau_{bd}^{+} \right\}, \]  

(C.9)

\[ \chi_{ij}^{bd} = \text{Tr.} \left\{ F \tau_{ij}^{+} F^{+} \tau_{bd} \right\}. \]  

(C.10)

Notice that in Eqn.(C.7) the index \( i \) takes the values \( 0 \) and \( 1 \) (\( P_{y} \) is a rank-1 tensor operator) whereas the index \( \nu \) goes from \( 0 \) to \( 2 \) (\( P_{xx} \) is a rank-2 tensor operator). Using the matrix expression of the \( P \)'s and the \( \tau_{kq} \)'s (given in Appendix-B) one determines the complex constants \( E_{ij} \) and \( D_{bd}^{*} \). We obtain for \( K_{y}^{xx} \) the following

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3 I_0 K_y = (-i/2) \left[ \chi_{11}^{20} + \chi_{11}^{22} + \chi_{11}^{22} + \chi_{11}^{22} \right] - i(6)^{1/2} \left[ \chi_{11}^{20} + \chi_{11}^{20} \right], \quad \text{(C.11)}

where I_0 refers to the unpolarized cross-section and has the form

\[ I_0 = \left[ |Q_{00}|^2 + |Q_{20}|^2 + 2|Q_{11}|^2 + 2|Q_{21}|^2 + 2|Q_{22}|^2 \right]. \quad \text{(C.12)}

The determination of the \( \chi \)'s is easily obtained by substituting Eqns.(C.2-3) for \( F \) and hence \( F^+ \) and the \( \tau_k \)'s matrices in Eqn.(C.10). We obtain the following expressions (using the notations of Eqn.(C.2))

\[
\chi_{11}^{2-2} = \chi_{11}^{22} = \left[ -i(2)^{1/2}/3 \right] \text{Im.}(\& \ c^*), \quad \text{(C.13)}
\]

\[
\chi_{11}^{22} = \chi_{11}^{22} = \left[ -i(2)^{1/2}/3 \right] \text{Im.}(\alpha \ & \ c^*), \quad \text{(C.14)}
\]

\[
\chi_{11}^{20} = \chi_{11}^{20} = \left[ (i)(3)^{-1/2}/3 \right] \left[ \text{Im.}(\alpha \ & \ c^*) + \text{Im.}(\& \ c^*) + 2\text{Im.}(e \ d^*) \right], \quad \text{(C.15)}
\]

where we have used the result: \( b \ a^* - a \ b^* = 2i \ \text{Im.}(b \ a^*) \). Thus,

\[
3 I_0 K_y^{\chi x} = \left( -4(2)^{1/2}/9 \right) \left[ \text{Im.}(\alpha \ & \ c^*) + \text{Im.}(\& \ c^*) + (1/2) \ \text{Im.}(e \ d^*) \right]. \quad \text{(C.16)}
\]

Finally, using Eqns.(C.3) one writes \( K_y^{\chi x} \) in terms of the \( Q_{im} \)'s

\[
3 I_0 K_y^{\chi x} = -2(3)^{1/2} \text{Im.}(Q_0 Q_{11}^*) + 6(3)^{1/2} \text{Im.}(Q_0 Q_{21}^*) - 4(6)^{1/2} \text{Im.}(Q_{20} Q_{11}^*) - 12 \text{Im.}(Q_{11} Q_{22}^*) + 12 \text{Im.}(Q_{21} Q_{22}^*). \quad \text{(C.17)}
\]

Similarly, we obtain the expansions of \( K_y^{\chi x} \) and \( K_y^{\chi x} \). These are given by

\[
3 I_0 K_y^{\chi x} = -2(3)^{1/2} \text{Im.}(Q_0 Q_{11}^*) - 6(3)^{1/2} \text{Im.}(Q_0 Q_{21}^*) + 5(6)^{1/2} \text{Im.}(Q_{20} Q_{11}^*) + 3(6)^{1/2} \text{Im.}(Q_{20} Q_{21}^*) + 6 \text{Im.}(Q_{11} Q_{22}^*) - 6 \text{Im.}(Q_{21} Q_{22}^*), \quad \text{(C.18)}
\]
\[ 3 I_0 K_{xy}^{xx} = -9(2)\cdot 1/2 \text{Im.} (Q_{00} Q_{20}^*) + 3(3)^{1/2} \text{Im.} (Q_{00} Q_{22}^*) + 3(6)^{1/2} \text{Im.} (Q_{22} Q_{20}^*) \\
+ 18 \text{Im.} (Q_{11} Q_{21}^*) . \tag{C.19} \]

Now as the expansions of the three PTCs appearing in Eqn.(C.5) are completed, one can go on to finally determine the expansion of \( K_{xy}^{xx} \). To this end, we substitute Eqns.(C.16-18) into Eqn.(C.5) to obtain

\[ 3 I_0 K_{xy}^{xx} = -243 \text{Im.} (Q_{00} Q_{11}^*) + 346 \left \{ \cos^2(\theta) \text{Im.} (Q_{22} Q_{20}^*) + \sin(2\theta) \text{Im.} (Q_{22} Q_{20}^*) \right \} \\
+ \left \{ \sin(2\theta) \left \{ 343 \text{Im.} (Q_{00} Q_{22}^*) - 9/12 \text{Im.} (Q_{00} Q_{20}^*) \right \} - 6(3\cos(2\theta) \text{Im.} (Q_{00} Q_{21}^*) \right \} \\
+ \left \{ 18\sin(2\theta) \text{Im.} (Q_{11} Q_{21}^*) + 46 \left \{ 9\cos^2(\theta) - 4 \right \} \text{Im.} (Q_{20} Q_{11}^*) \right \} \\
+ 6 \left \{ 3\cos^2(\theta) - 2 \right \} \text{Im.} (Q_{00} Q_{11}^*) \} - 6 \left \{ 3\cos^2(\theta) - 2 \right \} \text{Im.} (Q_{21} Q_{22}^*) . \tag{C.20} \]

Since we know that the \( Q_{2m} \)'s are related, as shown by Eqn.(C.4), it is therefore possible to further simplify Eqn.(C.20) by cancelling certain combinations. Thus, the expression between "curly" brackets in the first line of Eqn.(C.20) becomes \( \text{Im.}(BA^*) = - \text{Im.}(AB^*) \):

\[ 346 \left \{ \cos^2(\theta) \text{Im.} \left \{ Q_{20} \left [ (1/2) \tan(\theta) \left ( Q_{22}^* - (43/2) Q_{20}^* \right ) \right ] \right \} - \sin(2\theta) \text{Im.} (Q_{20} Q_{22}^*) \right \} \]

\[ = \left \{ (946)/4 \right \} \sin(2\theta) \text{Im.} (Q_{20} Q_{22}^*) . \tag{C.21} \]

Now let us consider the second line of Eqn.(C.20). If we replace the amplitude \( Q_{21}^* \) that appears in the last term of this line by its equivalent from Eqn.(C.4), then we get
\[ \left( \sin(2\theta) - \tan(\theta)\cos(2\theta) \right) \left\{ 343 \ \text{Im.} (Q_{00}Q_{22}^*) - \left(\frac{9}{4} \right) 2 \ \text{Im.} (Q_{00}Q_{20}^*) \right\} = \]
\[ 343 \\tan(\theta) \left\{ \text{Im.} \left[ Q_{00} \left( Q_{22}^* - \left(\frac{13}{2} \right) Q_{20}^* \right) \right] \right\} = 643 \ \text{Im.} (Q_{00}Q_{21}^*) \quad \text{(C.22)} \]

Next we consider the third line of Eqn.(C.20). By extracting the amplitude \( Q_{21}^* \) from Eqn.(C.4) and then substituting it into the first term of this line, we obtain for the first term of this line

\[ 18 \sin(2\theta) \ \text{Im.} (Q_{11}Q_{21}^*) = 18 \sin^2(\theta) \left\{ \text{Im.} (Q_{11}Q_{22}^*) - \left(\frac{13}{2} \right) \text{Im.} (Q_{11}Q_{20}^*) \right\}, \]

now the third line of Eqn.(C.20) takes on this form

\[ 6 \ \text{Im.} (Q_{11}Q_{22}^*) + 946 \left[ \sin^2(\theta) + \cos^2(\theta) \right] \ \text{Im.} (Q_{20}Q_{11}^*) - 446 \ \text{Im.} (Q_{20}Q_{11}^*) \]
\[ = 6 \ \text{Im.} (Q_{11}Q_{22}^*) + 546 \ \text{Im.} (Q_{20}Q_{11}^*) \quad \text{(C.23)} \]

The last line of Eqn.(C.20) transforms into the following expression when we substitute \( Q_{22}^* \) by its equivalent from Eqn.(C.4),

\[-6 \left\{ 3 \cos^2(\theta) - 2 \right\} \ \text{Im.} (Q_{21}Q_{22}^*) = -\left(\frac{13}{2} \right) \left\{ 3 \cos^2(\theta) - 2 \right\} \ \text{Im.} (Q_{21}Q_{20}^*) \quad \text{(C.24)} \]

Replacing the expressions of each line in Eqn.(C.20) by their respective simplified versions, Eqns.(C.21-4), one obtains
\[ 3 I^y_{0} K^x_{y} =\]
\[ -243 \text{ Im}(Q_{00}^* Q_{11}^*) + 643 \text{ Im}(Q_{00}^* Q_{21}^*) + 6 \text{ Im}(Q_{11}^* Q_{22}^*) + 546 \text{ Im}(Q_{20}^* Q_{11}^*) + \]
\[ \left\{ -\frac{(946)}{4} \sin(2\theta) \text{ Im}(Q_{20}^* Q_{22}^*) \right\} + \left( \frac{43}{2} \right) \left[ 18 \cos^2(\theta) - 12 \right] \text{ Im}(Q_{20}^* Q_{21}^*) \].

(C.25)

Eqn.(C.25) can be further simplified. The first term of the second line (i.e. expression between curly brackets) of this equation can be transformed by replacing \( Q_{22}^* \) by its equivalent from Eqn.(C.4), thus

\[-\frac{(946)}{4} \sin(2\theta) \text{ Im}(Q_{20}^* Q_{22}^*) = -\frac{(946)}{2} \sin(2\theta) \cot(\theta) \text{ Im}(Q_{20}^* Q_{21}^*) \],

therefore, the second line of Eqn.(C.25) becomes

\[ \left\{ -\frac{(946)}{2} \sin(2\theta) \cot(\theta) + \left( \frac{43}{2} \right) \left[ 18 \cos^2(\theta) - 12 \right] \right\} \text{ Im}(Q_{20}^* Q_{21}^*) \]
\[ = -646 \text{ Im}(Q_{20}^* Q_{21}^*) \].

(C.26)

Replacing the second line of Eqn.(C.25) by its simplified version, namely, Eqn.(C.26) one obtains the final result of the expansion of \( K^x_{y} \) in terms of the \( Q^* \)'s as given by Eqns.(3.19-21) of section (3.2). Therefore we have

\[ 3 I^y_{0} K^x_{y} = 643 \text{ Im}(Q_{00}^* Q_{21}^*) - 243 \text{ Im}(Q_{00}^* Q_{11}^*) + 6 \text{ Im}(Q_{11}^* Q_{22}^*) + \]
\[ 546 \text{ Im}(Q_{20}^* Q_{11}^*) - 646 \text{ Im}(Q_{20}^* Q_{21}^*) \]
APPENDIX-D

INTEGRALS WITH WEINBERG STATES

A ) - NORMALIZATION INTEGRALS

In Section (7.2) of Chapter VII, we mentioned that the orthonormalization integrals of Eqn.(6.10b) can be determined analytically when using the expression (7.16a) of the Weinberg states. The normalization integral we would like to evaluate is of the following form ($V_{np}^H$ is the Hulthén potential of Eqn.(7.2))

\[ I_i = \int dr \phi_i^*(r) V_{np}^H(r) \phi_i(r), \quad (D.1) \]

where (with $\rho_p = k + p\beta$ and $\gamma^{00} = (4\pi)^{-1/2}$)

\[ \phi_i(r) = \gamma^{00} \sum_{p=0}^{\infty} a_p^i e^{i \alpha_p^i \rho_p^r} / r, \quad (D.2) \]

are the S-wave solutions of ($\tau_{np} = -\hbar^2 \nabla^2 / 2\mu_{np}$)

\[ \left[ \tau_{np} + \alpha_i^i V_{np}^H(r) \right] \phi_i(r) = -\epsilon_i \phi_i(r), \quad i=1,2,\ldots \quad (D.3) \]

To guarantee the uniqueness of the power series, the summation of the $a_p^i$ should be equal to zero, thus

\[ \sum_{p=0}^{\infty} a_p^i = 0, \quad (D.4) \]

We use Eqn.(D.3) in (D.1) to eliminate $V_{np}^H$ ($\epsilon_d = \hbar^2 \rho_d^2 / 2\mu_{np}$)

\[ I_i = \frac{\gamma^{00}}{\alpha_i^i} \sum_{p=0}^{\infty} a_p^i \int dr \phi_i^*(r) \left[ -\frac{\hbar^2 \rho_{p0}^2}{2\mu_{np}} + \frac{\hbar^2 \nabla_{p}^2}{2\mu_{np}} \right] \frac{e^{i \alpha_p^i \rho_p^r}}{r}. \quad (D.5) \]
We make use of the following general formula (c is a constant)

\[
\left[ \nabla^2_r - c^2 \right] \frac{e^{-cr}}{r} = -4\pi \delta(r),
\]

(D.6)
to transform Eqn.(D.5) into

\[
I_i = \frac{Y^{00}}{\alpha_i} \sum_{p=0}^{i} a_p^* \int dr \phi_i^*(r) \frac{\hbar^2}{2\mu_{np}} \left\{ \rho_p^2 - \rho_0^2 \right\} \frac{e^{-p/r}}{r} - 4\pi \delta(r).
\]

(D.7)

Since the summation of all the coefficients \( a_p^* \) should be equal to zero (see Eqn.(D.4)), then the term with \( \delta(r) \) in Eqn.(D.7) yields a zero contribution to the integral. Therefore Eqn.(D.7) becomes

\[
I_i = \frac{Y^{00}}{\alpha_i} \sum_{p=0}^{i} a_p^* \int dr \phi_i^*(r) \frac{e^{-p/r}}{r} ,
\]

(D.8)

where

\[
\int dr \phi_i^*(r) \frac{e^{-p/r}}{r} = Y^{00} \sum_{q=0}^{i} a_q^* \int dr \frac{e^{-(\rho_q + \rho_p)r}}{r^2} ,
\]

(D.9)

= \frac{4\pi Y^{00}}{\rho_q + \rho_p} \sum_{q=0}^{i} a_q^* .

Replacing Eqn.(D.9) into (D.8) yields the final expression of \( I_i \):

\[
I_i = \frac{\hbar^2}{2\mu_{np}} \sum_{p=0}^{i} \sum_{q=0}^{i} \frac{a_p^* a_q^* (\rho_p^2 - \rho_0^2)}{\rho_q + \rho_p} .
\]

(D.10)
B ) - THE $\beta_{ij}$ INTEGRALS

The analytical result of the $\beta_{ij}$'s, as given by Eqn. (7.23) in section (7.3), was obtained using the technique adopted above. The $\beta_{ij}$ element, as defined by Eqn. (6.14d), has the form

\[
\beta_{ij} = \int dr \, \phi_i^*(r) \left( V_{np}^H(r) \right)^2 \phi_j(r),
\]

which is equivalent to

\[
\beta_{ij} = \int dr \left\{ V_{np}(r) \phi_i(r) \right\}^* \left\{ V_{np}(r) \phi_j(r) \right\}.
\]

Similarly, we use Eqn. (D.3) to remove $V_{np}$ from Eqn. (D.12)

\[
\beta_{ij} = \left[ \frac{Y^{00}}{2 \mu_{np}^2} \right] \frac{1}{\alpha_i \alpha_j} \int dr \left\{ \sum_m a_m^i \left( -\rho_0^2 + \nabla^2 \right) \frac{e^{P_m r}}{r} \right\} \left\{ \sum_n a_n^j \left( -\rho_0^2 + \nabla^2 \right) \frac{e^{P_n r}}{r} \right\}.
\]

The result of Eqn. (D.6) is used twice in Eqn. (D.13) to produce (terms with $\delta(r)$ give zero contribution to the integral)

\[
\beta_{ij} = \left[ \frac{Y^{00}}{2 \mu_{np}^2} \right] \frac{1}{\alpha_i \alpha_j} \sum_{m=0}^{i} \sum_{n=0}^{j} a_m^i a_n^j (\rho_m^2 - \rho_n^2)(\rho_m^2 - \rho_n^2) \int dr \, e^{-(P_m + P_n) r},
\]

and finally, we obtain the analytical expression of $\beta_{ij}$:

\[
\beta_{ij} = \left[ \frac{\hbar^2}{2 \mu_{np}} \right] \frac{1}{\alpha_i \alpha_j} \sum_{m=0}^{i} \sum_{n=0}^{j} a_m^i a_n^j \frac{(\rho_m^2 - \rho_n^2)(\rho_n^2 - \rho_m^2)}{(\rho_m + \rho_n)}. \]

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where for a fixed $N$

$$i, j = 1, 2, \ldots, N.$$
APPENDIX-E

NUMERICAL SOLUTIONS OF THE TWO COUPLED CHANNELS EQUATIONS

In this appendix, we explain in detail the numerical procedure we used within our computer code TWOCEQ to solve the two-coupled channel Schrödinger equations (7.34) of section (7.4). The example of the main text (i.e. \( N = 15 \) and \( E_d = 88.2 \text{ MeV} \)) involves two open channels whose energies are given by \( E_i = E_d - \lambda_i \ (i=1,2) \).

If we define the radial solutions of Eqns.(7.34) as

\[
U_i^{(e)}(R) = R F_i^{(e)}(R),
\]

then Eqns.(7.34) become for \( U_1^{(e)} \)

\[
\begin{pmatrix}
U_1^{(e)}(R) \\
U_2^{(e)}(R)
\end{pmatrix}
'' = X(R) U_1^{(e)}(R) + Z(R) U_2^{(e)}(R),
\]

where \( M = m_a + m_p \)

\[
X(R) = \frac{2M}{\hbar^2} \left[ W_{11}(R) - E_1 + \frac{\hbar^2}{2M} \frac{L(L+1)}{R^2} \right],
\]

\[
X(R) = \frac{2M}{\hbar^2} \left[ W_{22}(R) - E_2 + \frac{\hbar^2}{2M} \frac{L(L+1)}{R^2} \right],
\]

\[
Z(R) = \frac{2M}{\hbar^2} W_{21}(R),
\]

\[
Z(R) = \frac{2M}{\hbar^2} W_{12}(R).
\]

In order to solve the above pairwise coupled second order differential equations, we used the modified Numerov or modified Fox-Goodwin algorithm

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[Me66,Ra71,Ro67]. In this method, one requires the values of $U_1^{(s)}$ and $U_2^{(s)}$ at two starting points, say $R=h$ and $R=2h$ ($h$ is a constant step size), in order to be able to generate the values of both solutions at $R=3h$. This scheme can be repeated up to a certain $R_{\text{max}} (= nh)$ beyond which the effects of the coupling potentials are negligible. The algorithm of this method takes the following matrix form

$$
\begin{pmatrix}
A(R+h) & B(R+h) \\
B(R+h) & \tilde{A}(R+h)
\end{pmatrix}
\begin{pmatrix}
U_1^{(+)}(R+h) \\
U_2^{(+)}(R+h)
\end{pmatrix}
= 
\begin{pmatrix}
\mathcal{A}(R) & \tilde{B}(R) \\
\mathcal{B}(R) & \mathcal{A}(R)
\end{pmatrix}
\begin{pmatrix}
U_1^{(+)}(R) \\
U_2^{(+)}(R)
\end{pmatrix},
$$

(E.4)

where

$$
A(R \pm h) = 1 - h^2 X(R \pm h) / 12 ,
$$

(E.5a)

$$
B(R \pm h) = - h^2 Z(R \pm h) / 12 ,
$$

(E.5b)

$$
\mathcal{A}(R) = 2 + 5 h^2 X(R) / 6 ,
$$

(E.5c)

$$
\mathcal{B}(R) = 5 h^2 Z(R) / 6 ,
$$

(E.5d)

and similarly for $\tilde{A}$, $B$, $\tilde{B}$ and $\mathcal{B}$ in terms of $X$ and $Z$.

In order to extract the physical solutions which have the appropriate asymptotic forms, we require two linearly independent solutions of Eqns.(E.2). A linear combination of the latter is in turn used to produce the desired physical solutions. The mathematical vector solutions denoted here by (omitting the $^{(+)}$ symbol for clarity)

$$
U^{(1)} = \begin{pmatrix} U_1^{(1)} \\ U_2^{(1)} \end{pmatrix} \quad \text{and} \quad U^{(2)} = \begin{pmatrix} U_1^{(2)} \\ U_2^{(2)} \end{pmatrix},
$$

(E.6)
are obtained from the following linearly independent starting conditions

\[
\begin{align*}
\begin{pmatrix}
U_1^{(1)}(h) \\
U_2^{(1)}(h)
\end{pmatrix}
&= 
\begin{pmatrix}
 j(hk_1) \\
 3j(hk_2)
\end{pmatrix}, \\
\begin{pmatrix}
U_1^{(1)}(2h) \\
U_2^{(1)}(2h)
\end{pmatrix}
&= 
\begin{pmatrix}
 j(2hk_1) \\
 3j(2hk_2)
\end{pmatrix}, \\
\begin{pmatrix}
U_1^{(2)}(h) \\
U_2^{(2)}(h)
\end{pmatrix}
&= 
\begin{pmatrix}
 3j(hk_1) \\
 j(hk_2)
\end{pmatrix}, \\
\begin{pmatrix}
U_1^{(2)}(2h) \\
U_2^{(2)}(2h)
\end{pmatrix}
&= 
\begin{pmatrix}
 3j(2hk_1) \\
 j(2hk_2)
\end{pmatrix},
\end{align*}
\]

(E.7a)

where \( j(Rk) \) is the regular spherical Bessel function. These starting conditions are obtained by neglecting, at small \( R \) values, the off-diagonal potentials and by considering the diagonal ones as complex constants. As a result of the above approximations, Eqns.(E.2) become, at small \( R \) values \( (R_s) \), identical to Bessel equations, i.e.

\[
\left( \begin{array}{c}
U_1^{(s)}(R_s) \\
U_2^{(s)}(R_s)
\end{array} \right)'' = \left[ -k_1^2 + \frac{L(L+1)}{h^2} \right] \left( \begin{array}{c}
U_1^{(s)}(R_s) \\
U_2^{(s)}(R_s)
\end{array} \right),
\]

(E.8a)

\[
\left( \begin{array}{c}
U_1^{(s)}(R_s) \\
U_2^{(s)}(R_s)
\end{array} \right)'' = \left[ -k_2^2 + \frac{L(L+1)}{h^2} \right] \left( \begin{array}{c}
U_1^{(s)}(R_s) \\
U_2^{(s)}(R_s)
\end{array} \right)
\]

(E.8b)

with

\[
k_1^2 = -\left[ \frac{2M}{h^2} \left( \text{Re} \ W_{11}(R_s) + V_c(R_s) - E_1 + i \text{Im} \ W_{11}(R_s) \right) \right],
\]

(E.9a)

\[
k_2^2 = -\left[ \frac{2M}{h^2} \left( \text{Re} \ W_{22}(R_s) + V_c(R_s) - E_2 + i \text{Im} \ W_{22}(R_s) \right) \right].
\]

(E.9b)

Since the physical vector solution \( U^{(s)} \) has the following asymptotic form

\[
U^{(s)}|_{R \to \infty} = \left( \begin{array}{c}
U_1^{(s)}(R) \\
U_2^{(s)}(R)
\end{array} \right) \equiv \left( \begin{array}{c}
H_L^{(-)}(L,R) - S_L H_L^{(*)}(L,R) \\
-(2/i) T_L H_L^{(*)}(L,R)
\end{array} \right),
\]

(E.10a)

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where we have defined the Hankel functions $H^{(\pm)}_L$ as

$$H^{(\pm)}_L(KR) = (i/2) e^{iQ_L} \left\{ G_L(KR) \pm i F_L(KR) \right\}, \quad (E.10b)$$

then the mathematical vector solutions (E.6) should contain proportions of these functions when $R$ is of sufficiently large $R$ values ($R \to \infty$). Thus

$$U^{(1)} = \begin{pmatrix} U_1^{(1)}(R) \\ U_2^{(1)}(R) \end{pmatrix}_{R \to \infty} = \begin{pmatrix} \alpha_1^1 H^{(3)}(K_1 R) + \beta_1^1 H^{(0)}(K_1 R) \\ \alpha_2^1 H^{(3)}(K_2 R) + \beta_2^1 H^{(0)}(K_2 R) \end{pmatrix}, \quad (E.11a)$$

$$U^{(2)} = \begin{pmatrix} U_1^{(2)}(R) \\ U_2^{(2)}(R) \end{pmatrix}_{R \to \infty} = \begin{pmatrix} \alpha_1^2 H^{(3)}(K_1 R) + \beta_1^2 H^{(0)}(K_1 R) \\ \alpha_2^2 H^{(3)}(K_2 R) + \beta_2^2 H^{(0)}(K_2 R) \end{pmatrix}, \quad (E.11b)$$

By choosing two matching radii $R^m_1$ and $R^m_2$ such that Eqns.(E.10) and (E.11) are always satisfied, one can determine all the $\alpha, \beta$ constants using expressions of the following form (e.g. the case of $\alpha_1^1$ and $\beta_1^1$)

$$\begin{pmatrix} U_1^{(1)}(R^m_1) \\ U_2^{(1)}(R^m_2) \end{pmatrix} = \begin{pmatrix} H^{(3)}_L(K_1 R^m_1) & H^{(0)}_L(K_1 R^m_1) \\ H^{(3)}_L(K_2 R^m_2) & H^{(0)}_L(K_2 R^m_2) \end{pmatrix} \begin{pmatrix} \alpha_1^1 \\ \beta_1^1 \end{pmatrix}, \quad (E.12)$$

which can be repeated to obtain the rest of the $(\alpha, \beta)$ pairs.

Once all the $(\alpha, \beta)$ pairs are evaluated, we then consider a linear combination of the mathematical vector solutions as a solution of Eqns.(E.2). We therefore require that such a combination ought to be equal, when $R \to \infty$, to the RHS of Eqn.(E.10a). Thus

$$\omega \begin{pmatrix} U_1^{(1)} \\ U_2^{(1)} \end{pmatrix}_{R \to \infty} + \eta \begin{pmatrix} U_1^{(2)} \\ U_2^{(2)} \end{pmatrix}_{R \to \infty} = \begin{pmatrix} H^{(3)}_L(K_1 R) - S_L H^{(+)}_L(K_1 R) \\ - (2/i) T_L H^{(+)}_L(K_2 R) \end{pmatrix}, \quad (E.13)$$
The LHS of Eqn.(E.13) can be expressed, using Eqns.(E.11), in terms of the \((\alpha, \beta)\) pairs previously evaluated and the unknowns \(\omega\) and \(\eta\), viz

\[
\begin{pmatrix}
(\omega \alpha_1^1 + \eta \alpha_2^2) H^{(5)} + (\omega \beta_1^1 + \eta \beta_2^2) H^{(6)} \\
(\omega \alpha_2^1 + \eta \alpha_1^2) H^{(5)} + (\omega \beta_2^1 + \eta \beta_1^2) H^{(6)}
\end{pmatrix},
\]

(E.14)

by equating both sides of Eqn.(E.13), we obtain four equations for the unknowns \(\omega, \eta, S_L\) and \(T_L\), i.e.

\[
\begin{align*}
\omega \alpha_1^1 + \eta \alpha_2^2 &= +1, \\
\omega \alpha_1^1 + \eta \alpha_2^2 &= 0, \\
\omega \beta_1^1 + \eta \beta_1^2 &= -S, \\
\omega \beta_2^1 + \eta \beta_2^2 &= + (2/i) T_L.
\end{align*}
\]

(E.15a)  
(E.15b)  
(E.15c)  
(E.15d)

Combining Eqns.(E.15a) and (E.15b) we immediately obtain \(\omega\) and \(\eta\),

\[
\begin{align*}
\omega &= \frac{\alpha_2^2}{\Delta}, \\
\eta &= \frac{-\alpha_2^1}{\Delta}.
\end{align*}
\]

(E.16a)  
(E.16b)

and using the values of \(\omega\) and \(\eta\) in conjunction with the rest of Eqns.(E.15) we deduce the expressions of \(S_L\) and \(T_L\),

\[
\begin{align*}
S_L &= \left(\alpha_2^1 \beta_1^2 - \alpha_2^2 \beta_1^1\right) / \Delta, \\
T_L &= \left(i/2\right) \left(\alpha_2^2 \beta_2^1 - \alpha_2^1 \beta_2^2\right) / \Delta.
\end{align*}
\]

(E.16c)  
(E.16d)

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where $\Delta = \left( \alpha_1^2 \alpha_2^2 - \alpha_2^1 \alpha_1^2 \right)$. We finally obtain the desired physical solutions $U_1^{(+)}(R)$ and $U_2^{(+)}(R)$ for $R$ covering the interval $[h, R_{\text{max}} = nh]$

\[
U_1^{(+)}(R) = \omega U_1^{(1)}(R) + \eta U_1^{(1)}(R), \tag{E.17}
\]
\[
U_2^{(+)}(R) = \omega U_2^{(2)}(R) + \eta U_2^{(2)}(R). \tag{E.18}
\]
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