Attitude Control of Underactuated Small Satellites

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Dedicated

To the memory of my father,

And to my mother,
Abstract

Actuator failures onboard satellites have caused severe and even disastrous consequences on several space missions. In this thesis, the problem of the attitude control of a LEO satellite, subject to actuator failures, is addressed. The three axis stabilisation of a satellite with the two remaining control torques on the principal axes, is a challenging problem since the control system is nonholonomic. Such a system has been proven by Brockett to be non-stabilisable using smooth (continuous and time invariant) control laws.

Different non-smooth stabilizing control laws for the underactuated attitude control of a satellite are investigated here using pairs of thrusters, and also using reaction wheels. Using two pairs of thrusters, known singular or time varying approaches are applied with a systematic study of the effects of the torque saturation, PWM, singularity avoidance, noise, external disturbances, sampling and angular velocity tracking that intervene in a realistic case. Using two reaction wheels, a novel control strategy based on a singular nonlinear control approach, is mathematically proven and demonstrated by simulation. The 3-axis stability is proven using Rodriguez parameters and then using quaternions. The study of the symmetrical satellite case using thrusters, and the investigation of the effect of a non-zero total momentum using wheels, are done separately.

Practical difficulties of the underactuated attitude control of small satellites using two pairs of on/off thrusters are pointed out. Conversely, using two reaction wheels, the possibility of decisive 3-axis manoeuvres is demonstrated (under realistic assumptions). Indeed, using two wheels, the 3-axis stabilisation is achieved with acceptable torque levels and very satisfactory performance. The activation of the non-smooth controller must be done under small momentum conditions. A complete control strategy, (in case of a high initial bias) including a detumbling phase with magnetorquing, and avoiding the non-smooth controller to start from a singularity, is presented. Following the encouraging results from the SSTL’s UoSAT-12 simulator, (accounting for noises and external disturbance torques) in-orbit testing of an underactuated control strategy using two wheels has been successfully achieved on UoSAT-12 (by restricting the attitude to sun tracking due to power consumption problems on UoSAT-12). Another in orbit experiment on UK-DMC, for nadir pointing, has been even more successful. Practical results therefore confirm the possibility of using only two control torques for the 3-axis stabilisation of a satellite. One of many interesting consequences of
these results is that a fully redundant 3-axis control can be practically envisaged using a 3-wheel configuration.

Key words: Actuator failure, non-smooth, underactuated, singular, time varying, reaction wheels, thrusters, detumbling, UoSAT-12.
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LIST OF SYMBOLS

A: Attitude transformation matrix
X, Y, Z: Principal satellite’s body axes: roll, pitch and yaw axes
φ, θ, ψ: Roll, pitch and yaw angles respectively
C, S: Cosine and sine functions respectively
q: Attitude quaternion vector with respect to the orbital frame
q₁, q₂, q₃, q₄: Attitude quaternion components in the orbital frame
qₑ: Orbit-referenced attitude quaternion error vector
qₑ₁, qₑ₂, qₑ₃, qₑ₄: Attitude quaternion error components in the orbital frame
qₑ: Attitude quaternion command vector
qₑ₁, qₑ₂, qₑ₃, qₑ₄: Attitude quaternion command components
w, z: Attitude components expressed in the Tsiontras-Longusky parameters
w₁, w₂: Real and imaginary parts of the attitude parameter w
p: Attitude vector expressed in Rodriguez parameters
p₁, p₂, p₃: Attitude components expressed in Rodriguez parameters
ω: Inertially referenced body angular velocity vector
ω₁, ω₂, ω₃: Inertially referenced body angular velocity components
n: Mean orbital angular rate
e: Orbit eccentricity
I: Satellite’s Moment of inertia matrix (including wheels)
I₁, I₂, I₃: Moment of inertia components in X, Y and Z axes respectively
Iₛ: Inertia matrix of the satellite (without wheels)
Iᵦᵢ: Diagonal moment of inertia tensor of the iᵗʰ wheel
Iᵦᵢ: Scalar amount of the moment of inertia of the iᵗʰ wheel
ε, a₁, a₂: Asymmetry parameters of the satellite
Ω: Skew symmetric angular velocity based matrix in the kinematic equation
H: Total momentum vector of the satellite in the inertial frame
H₁, H₂, H₃: Satellite’s Total momentum components in the inertial frame
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>Total momentum vector of the satellite in the body frame</td>
</tr>
<tr>
<td>$L_1, L_2, L_3$</td>
<td>Total momentum components in the body frame</td>
</tr>
<tr>
<td>h</td>
<td>Relative angular momentum of the reaction wheels</td>
</tr>
<tr>
<td>$h_1, h_2, h_3$</td>
<td>Reaction wheels angular momentum components</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular velocity vector resulting from a constant bias momentum. (introduced for the modelling of a bias momentum satellite)</td>
</tr>
<tr>
<td>$\omega_1, \omega_2, \omega_3$</td>
<td>Components of the angular velocity $\omega$ (due to a constant bias momentum).</td>
</tr>
<tr>
<td>$N_m$</td>
<td>Magnetic torque vector</td>
</tr>
<tr>
<td>M</td>
<td>Magnetic control dipole moment of magnetorquers</td>
</tr>
<tr>
<td>$M_1, M_2, M_3$</td>
<td>Magnetic control dipole moment components</td>
</tr>
<tr>
<td>$M_{\text{max}}$</td>
<td>The saturation value of the magnetic moment</td>
</tr>
<tr>
<td>B</td>
<td>Magnetic field vector in the satellite body coordinates</td>
</tr>
<tr>
<td>$B_1, B_2, B_3$</td>
<td>Magnetic field vector components in the body frame</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Magnetic field vector in the local orbital frame</td>
</tr>
<tr>
<td>e</td>
<td>Error vector to be minimised by magnetorquing</td>
</tr>
<tr>
<td>N</td>
<td>Control torque vector</td>
</tr>
<tr>
<td>$N_1, N_2, N_3$</td>
<td>Control torque components on the X, Y and Z body axes</td>
</tr>
<tr>
<td>$N_d$</td>
<td>External disturbance torque vector</td>
</tr>
<tr>
<td>$N_{\text{id}}, N_{\text{zd}}, N_{\text{zd}}$</td>
<td>Disturbance torques on the X, Y and Z axes respectively</td>
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<tr>
<td>$\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3$</td>
<td>Wheel speed commands on the X, Y and Z axes respectively</td>
</tr>
<tr>
<td>$N_{\text{wh}}$</td>
<td>Torque applied by the satellite on the wheels</td>
</tr>
<tr>
<td>$u_1, u_2$</td>
<td>Redefined control inputs of the underactuated satellite</td>
</tr>
<tr>
<td>$x$</td>
<td>Full state vector of the satellite</td>
</tr>
<tr>
<td>$x_k$</td>
<td>Discrete full state vector from the EKF at $t = t_k$</td>
</tr>
<tr>
<td>$\hat{x}_k$</td>
<td>Discrete estimate of the full state vector from the EKF</td>
</tr>
<tr>
<td>$\partial x_k$</td>
<td>Estimation error of the modelled or real attitude (perturbation)</td>
</tr>
<tr>
<td>$f, h$</td>
<td>Nonlinear transformation and observation functions respectively</td>
</tr>
<tr>
<td>F</td>
<td>Jacobian matrix for the linearization of the function $f$</td>
</tr>
</tbody>
</table>
Transition matrix of the Extended Kalman Filter

Observation matrix of the EKF

Process noise vector at time $t_k$

Measurement noise vector at time $t_k$

Mathematical expectation function

Estimation error covariance vector at time $t_k$

Process noise covariance at time $t_k$

Measurement noise covariance vector at time $t_k$

Azimuth and elevation Sun sensor readings

Desired angular velocity commands on the X and Y axes

Mathematical saturation function

sliding surface considered for sliding modes control

Lyapunov function used in stability analysis

Function of $p_1, p_2$, defined to simplify stability analysis.

Function of $q_1, q_2$, defined to simplify stability analysis.

Homogenous norm function of the Morin controller

Nonlinear functions of the 2 wheels time-varying controller (CTVC)

Two functions defined to simplify the stability analysis using the “2 wheels” controller based on quaternions

Unit vector of the $i^{th}$ wheel rotation axis

Cross product of two vectors

Neperian logarithm function

$3 \times 3$ identity matrix

Constant control parameters

Constant parameters of the spin-axis stabilising controller with two torques

Parameters of the Tsiotras nonlinear singular controller

Control parameter of the PD law in the Tsiotras controller

Constant parameter of the saturation function

Constant Parameters of the Kim nonlinear singular controller
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\mu_1, \mu_2, \mu_3$</td>
<td>Constant Parameters of the Kim angular velocity trajectory controller</td>
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<tr>
<td>$\eta_1, \eta_2, \eta_3$</td>
<td>Constant parameters of the Morin time varying controller</td>
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<tr>
<td>$\eta_3, \eta_4$</td>
<td>Constant parameters of the Moring angular velocity trajectory controller</td>
</tr>
<tr>
<td>$k_{TV}, \beta, \omega$</td>
<td>Constant parameters of the two wheels time-varying controller (CTVC)</td>
</tr>
<tr>
<td>$k, g, a$</td>
<td>Constant Parameters of the 2 wheels nonlinear singular controller (NLCS)</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Smoothing parameter to improve the performance of the NLSC controller</td>
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<tr>
<td>$\lambda_1, \lambda_2, \lambda_3, \lambda_4$</td>
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</tr>
<tr>
<td>$k_{p1}, k_{d1}$</td>
<td>Proportional and derivative gains of the PD controller (X-wheel)</td>
</tr>
<tr>
<td>$k_{p2}, k_{d2}$</td>
<td>Proportional and derivative gains of the PD controller (Y-wheel)</td>
</tr>
<tr>
<td>$k_{D1}, k_{D2}, k_{D3}, k_{p3}$</td>
<td>Parameters of the magnetorquer cross product control law</td>
</tr>
<tr>
<td>Acronym</td>
<td>Definition</td>
</tr>
<tr>
<td>---------</td>
<td>------------</td>
</tr>
<tr>
<td>ADCS</td>
<td>Attitude Determination and Control System</td>
</tr>
<tr>
<td>VSC</td>
<td>Variable Structure Control</td>
</tr>
<tr>
<td>PD</td>
<td>Proportional Derivative</td>
</tr>
<tr>
<td>PWM</td>
<td>Pulse Width Modulation</td>
</tr>
<tr>
<td>CMG</td>
<td>Control Moment Gyroscope</td>
</tr>
<tr>
<td>DCM</td>
<td>Direct Cosine Matrix</td>
</tr>
<tr>
<td>GPS</td>
<td>Global Positioning System</td>
</tr>
<tr>
<td>UAC</td>
<td>Underactuated Attitude Controller</td>
</tr>
<tr>
<td>CTVC</td>
<td>Continuous Time Varying Controller</td>
</tr>
<tr>
<td>NLSC</td>
<td>Nonlinear singular Controller</td>
</tr>
<tr>
<td>IGRF</td>
<td>International Geomagnetic Reference Field</td>
</tr>
<tr>
<td>UoSAT</td>
<td>University of Surrey Satellite</td>
</tr>
<tr>
<td>FUSE</td>
<td>Far Ultraviolet Spectroscopic Explorer</td>
</tr>
<tr>
<td>DLR</td>
<td>German Space Agency</td>
</tr>
<tr>
<td>SSC</td>
<td>Surrey Space Centre</td>
</tr>
<tr>
<td>SSTL</td>
<td>Surrey Satellite Technology Limited</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration in USA</td>
</tr>
<tr>
<td>LEO</td>
<td>Low Earth Orbiting</td>
</tr>
<tr>
<td>MTQ</td>
<td>Magnetorquer</td>
</tr>
<tr>
<td>LVLH</td>
<td>Local Vertical Local Horizontal</td>
</tr>
<tr>
<td>OBC</td>
<td>Onboard Computer</td>
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Chapter 1

1 Introduction

In this thesis, the problem of the attitude control will be addressed in the case of underactuated satellites, or in other words, satellites undergoing actuator failures. Prior to the presentation of the detailed control strategies, the problem formulation of the underactuated spacecraft control, and the motivating reasons to develop such new control software are explained, in the context of small satellites.

1.1 Problem formulation

The Attitude Determination and Control System (ADCS) module is one of the major subsystems for any type of space mission of a satellite. The ADCS module task is to control the orientation of the spacecraft to the desired attitude using the available onboard hardware. In practice, this means pointing the antennae, solar panels, payloads and tracking either the Sun or the Earth for various purposes. The ADCS hardware mainly consists of a combination of different sensors used for the attitude determination, and actuators used for the passive or active attitude control.

The recent advances in satellites attitude control systems have succeeded in improving space missions capabilities from several aspects such as precision pointing, optimal slew manoeuvres (see references [Chen 2000], [Schaub 1996], [Dalsmo 1997], [Vadali 1984]), formation flying (see [Vadali 1999] and [Palmer 2002]), robust control, (see references [Lam 1996], [Lintereur 1997], [Boskovic 1999]) for both rigid and flexible spacecrafts (see reference [Dodds 1986] for the case of flexible spacecrafts), etc ...

A large variety of control systems techniques have been envisaged, depending on the assumed type of actuators, to improve the control performances, such as nonlinear, adaptive, optimal, robust and the so-called “intelligent” control techniques generally based on neural networks or fuzzy logic. Most of these techniques are relevant to 3-axis control, which is a condition for many spacecraft applications requiring fine pointing. Recent advances in all fields of control theory have taken the conventional control performances to the highest limits of precision, rapidity and robustness.
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However, most of these results assume that the spacecraft is actively controlled with a number of actuators at least equal to the number of the degrees of freedom of the system.

For conventional attitude control systems, it is generally assumed that all three axes of the spacecraft are actively controlled. The assumption is only true in absence of any satellite actuator failures. However, the risk of the failure of one actuator (or even more than one) during the space mission is not negligible. Having redundant actuators is an expensive alternative that does not solve the problem completely since these may also fail (one actuator failure sometimes increases the probability of more failures in practice). The mass of the satellite inevitably increases by adding redundancies, which is not an advantage for satellites constellations.

It is not only optimistic but also risky to assume that a spacecraft will not experience any kind of actuator failures in the pre-launch design of the ADCS software. This assumption appears to be more and more invalid.

There are more and more examples of recent expensive space missions undergoing severe if not disastrous consequences due to actuator failures.

Consequences of actuators failures during recent space missions:

One way to show the evidence of the severe consequences of actuator failures is from the available examples of recent space missions undergoing one or more actuator failures:

**The FUSE (Far Ultraviolet Spectroscopic Explorer), funded by NASA, was launched on June 24, 1999. From launch, the LEO satellite had lost two of four reaction wheels by Dec 10th 2001, and it became impossible to point telescopes and perform science operations in a safe manner. Science operations on this satellite have consequently been suspended (see [Ake 2002] and [Robertson 2003]).**

**The SSTL mini-satellite UoSAT-12 lost one of its three onboard reaction wheels. Since the wheel failure on UoSAT-12, the 3-axis control performance using a combined linear control with two wheels and magnetorquing is very limited with very low control authority on the underactuated Z-axis. Later after that event, UoSAT-12 also suffered from power supply due to battery charging problems, which are made significantly worse by the use of magnetorquing. The failure is mentioned in [Chen 2000].**
The **BIRD** micro-satellite (built by the German space Centre DLR) lost one wheel after 14 months. The effect of a second wheel failure would require a new control code to be uploaded (see [Sat-index]).

**Radarsat1** (Built by the Canadian space agency), also suffered a pitch wheel failure (then used as a momentum wheel providing gyroscopic stiffness), followed by a redundant wheel failure, causing the imaging operations to be temporarily suspended.

Actuators failures are also happening at the larger scale of large satellites and even space stations: **GOES-9, GPS BII-07, Echostar V, Mir, Galaxy IV, Hubble** ... (See [Sat-index 2004] and [Robertson 2003])

For instance, the **Hubble** Space Telescope is equipped with six gyroscopes, three operational and three spares. They were manufactured by Honeywell Technology Solutions. Over the past 13 years of operation, there have been nine gyroscope (CMGs) failures (or, statistically, one every one and a half years). On one occasion (year 1999), the spacecraft has been left with only two active CMGs after the failures of 3 redundant ones and an original one. The failure left Hubble in safe mode for several weeks until the completion of an expensive servicing mission to fix the problem.

**GOES-9** (NASA weather spacecraft manufactured by Space Systems/Loral) suffered severe irreversible consequences of actuator failures. A momentum wheel failed on July 1998 (the wheel became dangerously hot). The redundant wheel was used in an attempt to continue providing Earth imagery, but similar problems happened to it. Due to these events, Goes-9 is no longer operational and has been replaced by Goes-10.

In reference [Robertson 2003], it was statistically demonstrated that the largest number of GNC (Guidance, Navigation and control) anomalies during the satellite’s design lifetime are due to wheels failures. CMG failures were found to be more likely to occur after the design lifetime. Thrusters failures are rarer but even more severe.

The close-failures of thrusters (no firing) also happened as in the case of Echostar V (which also lost a wheel). There are generally more possible redundancies in this case than using reaction wheels. The open-failure of thrusters (still firing) is generally dramatic and the only way to recover from it is by suspending operations and switching to a safe restrained tumbling mode.
In the following, the actuator failures assumed are either from thrusters in close-failure or from reaction wheels. Small low Earth orbiting satellites will be considered for the study of the underactuated spacecrafts.

**Necessity for underactuated control programs in the ADCS module**

Actuator failure may then have severe and even catastrophic consequences on the spacecraft mission. Therefore, in the design of a space mission, and more precisely in the design of the ADCS module, different attitude control scenarios should be planned, depending on the actuators likely to fail. The UASAT microsatellite developed at university of Arizona for instance allows for actuator failures in the design (see reference [Lewicki 1997]). Steering control laws are proposed (not 3-axis stabilizing control laws). Steering control laws (generally based on some kind of open loop control strategy) are used to slew the satellite from an initial orientation about the unactuated axis to a desired orientation. However, the attitude on the actuated axes diverges slowly after the desired orientation about the unactuated axis is attained. Therefore the desired orientation is not maintained for long enough using steering techniques. In this thesis, only stabilising (not simply steering) control strategies are considered.

The challenging problem of controlling (and stabilising) the attitude of a spacecraft subject to actuator failures has been dealt with in the recent literature. Preliminary encouraging results and control laws have been proposed. Various control laws (generally in ideal cases) have been proposed for different purposes: angular velocity control, detumbling maneuver, attitude control of a symmetry axis, and three-axis attitude control.

**Brief insight into non-holonomic systems**

The control system of an underactuated satellite is known to be nonlinear, but we need a more precise classification for that system. The design of a control strategy generally depends on the class of the nonlinear system. The underactuated satellite has a particularity that will be discussed in more detail later in the thesis, which is the presence of a so-called non-holonomic constraint in the satellite’s model (constraint on the kinematic model).
Before investigating the control design problem for non-holonomic systems, a clear understanding of the meaning of "non-holonomic system" or "non-holonomic constraint" is needed.

To make this concept clear, we can simply start by considering the following dynamical system:

\[
\begin{align*}
\dot{x}_1 &= ux_2 \\
\dot{x}_2 &= -ux_1
\end{align*}
\]  

(1.1)

where \((x_1, x_2)\) is the state vector and \(u\) is the input.

We notice that system (1.1) contains the following constraint on the velocities:

\[x_1\dot{x}_1 + x_2\dot{x}_2 = 0\]  

(1.2)

This constraint is a so-called holonomic constraint, since it can be integrated to obtain:

\[\frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 = \text{constant}\]  

(1.3)

It is now clear that the controllability of this system is severely limited, it only being possible to control the state to any point on the circle defined by equation (1.3) via manipulation of \(u\), the radius depending on the initial state.

The situation differs when considering the dynamical system:

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1u_2 - x_2u_1
\end{align*}
\]  

(1.4)

where \((x_1, x_2, x_3)\) is the state vector and \((u_1, u_2)\) is the input. The system (1.4) also contains the following constraint on the state derivatives:

\[x_1\dot{x}_2 - x_2\dot{x}_1 - \dot{x}_3 = 0\]  

(1.5)

However, contrary to the case of the constraint (1.2), the constraint (1.5) can not be integrated to have a relation between the state variables without their derivatives. The equation (1.5) is called a non-holonomic constraint, which also means that all the three
variables are necessary for modelling this system, and that the constraint is inherently part of the dynamics.

The possibility of the asymptotic stabilisation for this class of control system has been extensively studied in the recent literature since the fundamental asymptotic stability condition proven by Brockett in [Brockett 1983]. Definitions of the meanings of asymptotic, global and local stability are given in appendix D.

**Brockett's necessary condition for the stability of non-holonomic systems:**

It has been proven in [Brockett 1983] that non-holonomic systems cannot be stabilized by smooth control laws. The Brockett's necessary condition for the stability of non-holonomic systems is that the stabilizing control law must be non-smooth.

Examples of well known non-holonomic mechanical systems are: the two wheeled mobile robot (unicycle type), underactuated robotic manipulators, the acrobot, the cart pole system, flexible link robots, underactuated ship, underactuated underwater vehicles, space robots, multi-body or flexible spacecrafts, and more importantly underactuated rigid spacecraft.

The nature of the problem to be addressed may be clearly illustrated by means of the aforementioned two-wheeled mobile robot. In that case, there are two control variables, i.e., the wheel motor drive inputs, but three degrees of freedom to be controlled, the two translational coordinates and the yaw rotation angle. It would be impossible to design a conventional control law to respond independently and simultaneously to three reference inputs corresponding to these three degrees of freedom. The problem is then to design a non-conventional control algorithm that can accomplish this automatically.

An underactuated attitude control system is a nonlinear system, for which the linearisation is not stabilisable. It is also a so-called non-holonomic system (*system for which the constraint is not integrable*). According to the Brockett's necessary condition, only non-smooth control laws can potentially stabilize the full attitude of of an underactuated satellite.
The existing literature and research in the control of non-holonomic systems can therefore be seen as a possible source of inspiration in the design of attitude controllers for the particular case of underactuated spacecrafts.

General investigations into particular classes of non-holonomic systems have also been made by several authors. In references [Murray & Walsh 1992], [Khennouf 1995], the stabilization using time varying control for the so-called chained form systems (a class of non-holonomic systems) has been presented. The same problem has been solved using nonlinear singular control techniques in [Astolfi 1996], [Astolfi 1998]. One early important investigation into the promising nonlinear singular control strategy had in fact been made by Fliess in [Fliess 1991]. This paper inspired a fresh start in considering nonlinear singular control strategies for the control of non-holonomic mechanical systems.

Some particularly interesting investigations (possible to reformulate for spacecrafts) in non-holonomic systems control have been made in robotics. A survey of the main techniques to control non-holonomic robots such as underactuated robots, acrobots, cart pole systems and mobile robots, has been given in reference [Spong 1998]. In references [Hespanha 1999], [Sampei 1999] nonlinear singular control has been used for the motion control of underactuated mobile robots. In reference [Rosas 2000], an interesting investigation involving non-smooth control has also been made for a class of underactuated robots. In reference [Toussaint 2001], non-smooth $H_{\infty}$ control has been used for motion planning of non-holonomic robots. In reference [Tanner 2002], nonlinear singular backstepping was used to solve the same control problem.

The trajectory control of underactuated surface vessels has also been addressed in references such as [Pettersen 1996], [Mazenc 2002], and [McClamroch 2003]. Nonsmooth techniques have also been used for the optimal reorientation of a multi-body spacecraft through joint motion in [Cerven & Coverstone 2001]. Interestingly, McClamroch, Coverstone and Mazenc reformulated the techniques used in robotics and surface vessels to deal with underactuated spacecrafts. The main authors of the known control strategies in the case of underactuated satellites will be cited later in the chapter.

All the control laws proposed in the literature of non-holonomic systems control are either nonlinear singular and time invariant (nonlinear singular) or continuous and time varying
(time varying). Since the underactuated satellite is a particular example of a non-holonomic system, the control laws in this case are also necessarily non-smooth (either nonlinear singular or time varying). The 3-axis stabilisation of an underactuated satellite using only two control torques is consequently one of the challenging research problems in the ADCS area.

The spacecraft attitude problem to be addressed

In this thesis, we restrict our attention, without a real loss of generality, to the small low Earth orbiting satellites.

Reaction wheels and thrusters failures can happen on both small satellites and large platforms.

CMG’s are now studied at the Surrey Space Centre as possible actuators for small satellites (see [Lappas 2002]). The results drawn from the study of the two reaction wheels control should be extensible to the CMG’s in similar satellites if the use of this new small satellite technology is confirmed. Indeed, both CMG’s and wheels are momentum exchange devices based on the same directing principles. However, CMG’s are still generally part of the ADCS hardware for large spacecrafts or space stations, and are therefore out of the direct focus of the thesis. Using geometric control, local 3-axis attitude stabilisation (for a region of initial orientations) using only two CMGs has been proven in [Kwon 1998], but the approach needs to be reformulated for global stability.

The study of the satellite’s attitude in the case of thrusters close-failures, and in the case of a reaction wheel failure will both be dealt with in detail.

There is clearly need for investigation into a more general ADCS design that specifically accounts for actuator failures but the first question to answer is whether or not the 3-axis attitude control of a satellite is possible using only two active control torques.

1.2 Conventional ADCS of small low earth orbiting satellites

At the University of Surrey, and more precisely at the Surrey Satellite Technology Ltd, there is considerable experience in manufacturing small satellites ranging from the pico-satellites, nano-satellites, to the micro-satellites and mini-satellites.
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The great potential of these low cost small satellites, which has for long been realised at the Surrey Space Centre, has been understood by the large space industry. There is a growing interest for the design and future launch of small satellites with innovative designs for all sorts of missions, satellites in constellations, and formation flying satellites.

The concepts underlying the ADCS hardware of small satellites differ little from the more expensive large spacecrafts, except in the use of smaller amounts of torques, from the micro gas jets, or reaction wheels, which are built with a specific small satellite technology. Another difference is that magnetorquing can only be efficient on “small” low Earth orbiting satellites.

Most small satellites, and particularly the Surrey Satellite Technology Ltd (SSTL) satellites are nearly symmetrical about one axis (nearly axisymmetric). SSTL has accumulated experience after having successfully manufactured the series of the UoSAT micro-satellites (first satellites of SSTL).

Traditional UoSAT ADCS was initially established by M.S.Hodgart to maintain the attitude control, partly by exploiting the effect of the gravity gradient from the deployed boom and also by convenient implementing of magnetorquer control. SSTL satellites have proven relatively good attitude control accuracy for gravity gradient controlled satellites (up to 1° roll, 0.5° pitch and 3° yaw for 3-axis control). However, several recent space missions require 3-axis control with even higher accuracy (especially on the yaw axis), and the gravity gradient controlled satellites have shown limits to what can be achieved.

UoSAT-12, which is the first Earth pointing mini-satellite manufactured by SSTL, has been designed for high performance 3-axis attitude determination and control, using a 3-axis reaction wheel configuration (with no need for boom deployment). UoSAT-12 was launched into a 650km nearly circular orbit 64.6deg inclination, on April 1999.
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A model based on UoSAT-12 will be considered for most of the study of the attitude control of underactuated small satellites using two reaction wheels. The reason, as indicated before, is that UoSAT-12 has lost a reaction wheel along the Z-axis, making good performance 3-axis control by conventional controllers no longer possible. A nominally working UoSAT-12 is also a good candidate for in-orbit experimental study of underactuated satellites. The main specifications of UoSAT-12 are given in table 1.1:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Nearly Circular orbit</td>
<td>Altitude = 650 Km</td>
</tr>
<tr>
<td></td>
<td>Inclination = 64.6°</td>
</tr>
<tr>
<td></td>
<td>Eccentricity = 0.0026</td>
</tr>
<tr>
<td>Orbit period</td>
<td>6000 seconds</td>
</tr>
<tr>
<td>Mass</td>
<td>320 Kg</td>
</tr>
<tr>
<td>Reaction wheels</td>
<td>Maximum torque = 0.02 Nm</td>
</tr>
<tr>
<td></td>
<td>Maximum momentum = 4 Nms</td>
</tr>
<tr>
<td>Magnetorquers</td>
<td>Maximum magnetic moment = 36 A.m²</td>
</tr>
<tr>
<td>Thrusters</td>
<td>Maximum torque = 0.035 N.m</td>
</tr>
<tr>
<td></td>
<td>Minimum firing time 0.05 sec</td>
</tr>
<tr>
<td>Moment of inertia</td>
<td>[40.45 0 0]</td>
</tr>
<tr>
<td></td>
<td>[0 42.09 0]</td>
</tr>
<tr>
<td></td>
<td>[0 0 40.36]</td>
</tr>
<tr>
<td>Sampling time period</td>
<td>10 seconds</td>
</tr>
<tr>
<td>Satellite's attitude</td>
<td>Typically Nadir pointing or Sun tracking</td>
</tr>
</tbody>
</table>

Table 1.1: Main Specifications of SSTL's mini-satellite UoSAT-12

Two views of the geometrical configuration of the mini-satellite UoSAT-12 are shown in Figure (1.1).
1.2.1 ADCS Conventions

Coordinate systems

Representations of the attitude can only be defined in convenient coordinate systems. Three coordinate systems are considered in the thesis: body coordinates, inertial coordinates, and orbital coordinates (also known as LVLH, or local vertical local horizontal).

The body Z-axis is defined along the symmetry axis (to be towards Nadir facet), the body X-axis points towards the harness side of the satellite, and the Y-axis is chosen to form a right-handed orthogonal reference system. The inertial reference frame is Earth centred.

The inertial Z-axis points towards the Earth's celestial pole. The Y-axis points towards the orbit anti-normal (not exactly inertial if we take the slow precession of orbital plane into account). The inertial X-axis is chosen to complete the orthogonal set.

The orbital Z-axis is defined in the Nadir direction. The orbital Y-axis points towards the orbit anti-normal. The orbital X-axis is chosen to form the complete orthogonal set. The
orbital coordinates therefore rotate once per orbit. During nadir pointing, the objective is to keep the attitude in the orbital LVLH frame equal to zero.

**Attitude representation**

The attitude or orientation of a spacecraft can be completely defined (in a well-chosen coordinates frame) using a sequence of three Euler angles called: roll, pitch and yaw.

These angles are obtained from a series of three right-handed positive rotations from a reference frame to the body frame.

Using the Euler angles representation, care must be taken to clearly define the rotational sequence considered in three-dimensional space. There are 12 possible rotational sequences of Euler angles. These angles are obtained from an ordered series of right hand positive rotations from a referenced $X_0Y_0Z_0$ to a $X Y Z$ set of the satellite's body axes.

In this thesis, we adopt a 1-2-3 Euler rotational sequence, with a first rotation $\phi$ (roll) about the initial $X_0$ axis, followed by a rotation $\theta$ (pitch) about the $Y'$ axis formed after the first rotation, and finally a rotation $\psi$ (yaw) about the $Z$-axis. The rotational sequence is represented in figure (1.2).

![Figure 1.2: Representation of a 1-2-3 Euler angles rotational sequence](image)

For Nadir pointing satellites such as UoSAT-12, the direction cosine matrix (DCM) transforms any vector from the LVLH frame to the body frame. In this case, the roll angle is the positive rotation about the LVLH $X$-axis, the pitch angle is the positive rotation about the LVLH $Y$-axis, and the yaw angle is the positive rotation about the LVLH $Z$-axis. The figure (1.3) describes the Euler angles of an Earth pointing satellite:
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The attitude matrix $A$, which transforms the vector from the referenced $X_0Y_0Z_0$ coordinates to the final spacecraft's body coordinates $X Y Z$, after a 1-2-3 rotation, is then given by:

$$
A = \begin{bmatrix}
C\psi C\theta & C\psi S\theta S\phi + S\psi C\phi & -C\psi S\theta C\phi + S\psi S\phi \\
-S\psi C\theta & -S\psi S\theta C\phi + C\psi S\phi & S\psi S\theta C\phi + C\psi S\phi \\
S\theta & -C\theta S\phi & C\phi C\theta
\end{bmatrix}
$$

(1.6)

where $Cx = \cos(x)$, $Sx = \sin(x)$, $x = \phi, \theta, \psi$.

Using the assumed 1-2-3 rotation sequence, the kinematic equations of motion using Euler angles, are given by:

$$
\dot{\phi} = (\omega_1 \cos\psi - \omega_2 \sin\psi) \sec\theta \\
\dot{\theta} = \omega_2 \cos\psi + \omega_1 \sin\psi \\
\dot{\psi} = \omega_3 - (\omega_1 \cos\psi - \omega_2 \sin\psi) \tan\theta
$$

(1.7)

Euler angles are a good analysis tool, often specified at the inputs and outputs of the system, and even a good design tool for small rotations. However, attitude quaternions, and possibly other special parameterisations of the attitude, are generally preferred for both control design and numerical computation.
1.2.2 ADCS hardware:

In this thesis, the model of a standard typical micro-satellite, with moments of inertia around 1.5 Kg.m\(^2\) (moments of inertia in the same range as for the satellite Sunsat) will first be considered for the study of a thrusters based control system. Using pairs of thrusters, an asymmetric satellite model will be adopted in some simulations, and an axisymmetric satellite model in others.

For most of the reaction wheels system study, the UoSAT-12 model, (with moments of inertia around 40 kg.m\(^2\)) will be considered because of the possibility of in-orbit implementation.

UoSAT-12 was chosen as a first platform for in-orbit experimental tests of underactuated attitude control. Sun-tracking for UoSAT-12 with two wheels will indeed be demonstrated in orbit. In orbit tests for nadir pointing on UoSAT-12 have been impossible due to battery charge related power supply problems. The control authority should be even better for nadir pointing experiments, but the availability of such a microsatellite for on-orbit tests became possible only recently with the launch of UK-DMC on September 2003 (for which the boom is now deployed). The attitude control of a satellite with a boom is a slightly more difficult problem because of an inertia gap, which means a big difference in the wheels torques demand leading to an increased possibility of torque saturation if the controllers are not modified. However, the required modifications have been made, and a program for underactuated attitude control has been uploaded on UK-DMC, followed by successful in-orbit demonstration of nadir pointing on this satellite.

In this section, we describe the structure of the typical ADCS hardware, in the case of small satellites. The particular structure in the case of UoSAT-12 will also be presented.

Attitude sensors

The precise control of the orientation of a satellite is only possible when the attitude is accurately determined. Attitude sensors provide measurements, which can then be used to estimate the spacecraft's full attitude and angular rates.

The most commonly used attitude sensors for micro-satellites are:
- \textit{Flux gate magnetometers}

Magnetometers are low-cost sensors used to measure the strength and direction of the geomagnetic field vector in the satellite’s body coordinates. This type of sensor can only be effective at low Earth orbit, when used for attitude determination.

UoSAT-12 (as most recent small satellites) uses 3-axis magnetometers measurements to evaluate the magnetic torque vector. The estimation of the full attitude and angular rates is achieved after comparing the measurements to the well-known IGRF (International Geomagnetic Reference Field) model, and by applying an extended Kalman filter. In practice, a 3-axis magnetometer does not provide full 3-axis attitude determination by itself, even with knowledge of the IGRF. That is the reason why magnetometers are generally used in combination with other sensors (Sun sensors ...)

- \textit{Sun sensors}

Sun sensors are the most widely used sensor type due to the simplicity of design. Another advantage of sun sensors is that attitude determination models are considerably simplified as a consequence of the validity of the point approximation.

UoSAT-12 uses four 2-axis (azimuth and elevation Sun sensors to measure the Sun vector angle to a high accuracy.

- \textit{Horizon sensors}

For low Earth orbiting satellites, the Earth is the second brightest celestial object (covering up to 40% of the sky). Horizon sensors are the principal means for directly determining the orientation of a spacecraft with respect to the Earth.

UoSAT-12 uses a two-axis infrared horizon sensor to measure small roll and pitch angles during nominal Nadir pointing or for small pitch and roll off pointing. Horizon sensors are not available on every small satellite (none are available on UK-DMC for instance).
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- Star sensor

Star sensors are the most accurate sensors for attitude determination. The relationship between the star sensor measurements and the catalog star positions can be used to determine the attitude very precisely.

UoSAT-12 uses a dual set of opposite looking star sensors for that purpose. The attitude of UoSAT-12 can be determined to a precision of 0.02 degrees using star sensors.

- GPS receiver:

UoSAT-12 carries a newly developed Space GPS receiver, that is designed to provide an experimental testbed for orbit and attitude determination. It has 4 antennas to accurately calculate the spacecraft attitude by applying interferometry techniques to the phase difference measurements. Attitude determination from GPS has been achieved with a precision between 1° and 2° at SSTL. Further work at the university of Surrey is being undertaken to mitigate the sources of error in GPS attitude determination, due to multipath and other error sources (See [Hodgart&Puriviapung 1999]).

<table>
<thead>
<tr>
<th>Sensors</th>
<th>Magnetometer</th>
<th>Sun sensors</th>
<th>Horizon sensors</th>
<th>Star sensors</th>
<th>Rate gyro</th>
<th>GPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturer</td>
<td>SSTL</td>
<td>SSTL(2) Ultra(1)</td>
<td>Servo-MiDES</td>
<td>SSTL</td>
<td>BEI</td>
<td>SSTL</td>
</tr>
<tr>
<td>Quantity</td>
<td>3 units</td>
<td>4×2axis</td>
<td>1×2axis</td>
<td>2 units</td>
<td>1 unit</td>
<td>1 unit</td>
</tr>
<tr>
<td>Range</td>
<td>± 60 µTesla</td>
<td>±50°</td>
<td>±5.5°</td>
<td>14.4°</td>
<td>± 5°/sec</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Accuracy</td>
<td>30 nTesla</td>
<td>0.2°</td>
<td>0.06°</td>
<td>0.02°</td>
<td>0.02°</td>
<td>1°</td>
</tr>
<tr>
<td>Power</td>
<td>&lt;0.8 W</td>
<td>&lt;0.1 W</td>
<td>2.8 W</td>
<td>4 W</td>
<td>1.4 W</td>
<td>5-7 W</td>
</tr>
</tbody>
</table>

Table 1.2: Attitude determination sensors on SSTL's mini-satellite UoSAT-12
Attitude actuators

Attitude actuators are necessary to provide the required torques for the control of the attitude. The actuator types differ in their accuracy, range, energy consumption, and torque capability. Every type is particularly convenient for specific on-orbit operations or specific phases of the space mission.

Magnetorquers:

Magnetorquers are generally used for momentum management on low Earth orbiting satellites. Dipole moments generated by the magnetorquer interact with the Earth’s magnetic field to generate small torques on the satellite. Since the magnetic torque is always perpendicular to the local magnetic field vector, it is not possible to generate any required torque instantaneously as demanded by a full 3-axis control system.

A combination of magnetorquing and passive control from the gravity gradient torque due to a deployed boom constitutes a viable minimal control system (see [Hodgart1989]).

The sole use of magnetorquers for the attitude control without a deployed boom (or a damaged boom as in the case of the Cerise satellite) is a current research problem, similar in some aspects to the research in the attitude control of underactuated spacecrafts (see [Guelman2002]).

UoSAT-12 uses 3-axis magnetorquing for the purposes of detumbling manoeuvres, possibly disturbance compensation and momentum dumping.

Reaction/momentum wheels:

Momentum and reaction wheels are devices designed for the storage or exchange of angular momentum, used on the spacecraft to stabilise against the known disturbance torques, to provide variable momentum to allow operation at 1 revolution per orbit when Nadir pointing is required, or to transfer the required momentum to the satellite’s body for slewing manoeuvres.

A momentum wheel nominally operates at a nonzero momentum bias. It provides a variable momentum storage capability about it’s rotation axis. The reaction wheel is only different in that it nominally operates at a zero momentum.
UoSAT-12 uses 3 momentum/reaction wheels in a 3-axis configuration to enable full control of the attitude or angular momentum. For conventional ADCS, one wheel provides for the control of one axis. However, one property to remember is that the angular velocities of the satellite on two axes have an impact on the attitude rates of all three axes. The consequence is that there is a way to control attitude on all three axes with only two wheels under particular momenta conditions that will be discussed later.

UoSAT-12 uses a 3-axis reaction wheel configuration for:

- Full 3-axis pointing and slow slew manoeuvres during imaging. One other function is - Ground target tracking.

- Nadir pointing, Sun tracking, or inertial pointing of the payloads by using the gyroscopic stiffness property of wheels operating as momentum wheels (at nonzero momentum bias).

- Cancellation of disturbance torques caused by the gas jet based propulsion system during orbit control.

- Fast spin up or spin down of the satellite’s body.

Reaction wheels operating at nominally zero momentum can also be used to absorb cyclic torques and momenta from the body during slew manoeuvres. However, the secular disturbance torques, which are about the same capacity, can cause the saturation of the momentum storage capacity. Therefore, provision is made for periodic momentum dumping through external torques produced by gas jets or more often by low cost magnetorquing.

**Thrusters:**

Thrusters, also known as gas jets, produce a force by expelling propellant mass. A control torque (that can be used for attitude control) can be obtained from a pair of thrusters firing in opposite directions.

UoSAT-12 uses a three-axis cold-gas thrusters system, providing relatively large torques for fast attitude control, or even for orbit control.
Possible attitude control applications of thrusters are:

- Agile attitude control.
- Large angle manoeuvres.
- Momentum dumping of the reaction/momentum wheels.
- Spin rate and nutation control.

The main advantages of thrusters over other actuators are the high torque capability and high control authority for conventional attitude control.

However, the obvious disadvantage is the consumption of the precious propellant also needed for all sorts of orbit control manoeuvres, or for satellites encounters.

Another disadvantage is that only On/Off thrusters can be available on small satellites, with the additional requirement of a PWM (Pulse Width Modulation). This often causes chattering problems, which will be more severe on the attitude stabilisation algorithms with two control torques.

Variable thrusters, providing a continuous thrust, are in theory much more convenient for attitude control (especially for underactuated spacecrafts). However, they appear to be much more difficult and expensive to manufacture in practice, and subject to leakage problems because of the often regular opening of the valves to produce small control torques, causing dirt particles to stick in the valve openings, and never close completely (see [Sidi 1997]).
Chapter 1. Introduction

<table>
<thead>
<tr>
<th>Actuators</th>
<th>Magnetorquers</th>
<th>Reaction/Momentum wheels</th>
<th>Cold-gas thrusters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturer</td>
<td>SSTL</td>
<td>SSTL (2), Ithaco (1)</td>
<td>SSTL &amp; Polyflex</td>
</tr>
<tr>
<td>Quantity</td>
<td>12</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>Operation range</td>
<td>±15 Am²</td>
<td>±4 Nms, ±5000 rpm</td>
<td>± 0.02 N.m</td>
</tr>
<tr>
<td>Power</td>
<td>20 W maximum</td>
<td>2.8-14.6 W</td>
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</tr>
<tr>
<td>Operation</td>
<td>PWM</td>
<td>Speed controlled</td>
<td>PWM</td>
</tr>
<tr>
<td>Accuracy</td>
<td>20 msec min pulse</td>
<td>± 1 rpm</td>
<td>10 msec min pulse</td>
</tr>
</tbody>
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Table 1.3: Attitude actuators of SSTL’s mini-satellite UoSAT-12

Disturbance torques:

The main disturbance torques acting on a spacecraft are due to: Gravity gradient, aerodynamic drag, solar pressure, and Earth magnetic field.

The gravity gradient torque is a torque experienced by low Earth orbiting satellites, and caused by the asymmetry of the mass distribution. This asymmetry creates a slight difference in the gravity forces acting on the body, and the result is a torque around the centre of mass.

As explained in references [El-Bordany 2001] and [Chen 2000], UoSAT-12 presents a nearly 3-axis symmetrical configuration, and the inherent gravity gradient can be neglected. The gravity gradient torque is indeed significant for either asymmetric or satellites with only one symmetry axis (as in case of a boom deployment).

Generally, the solar disturbance, and magnetic disturbances can also be neglected. In reference [El-Bordani 2001], it has been shown that the maximum solar disturbance torque on UoSAT-12 (when the Sun vector is perpendicular to the front side of the solar
panels for maximum power delivery) is $4.25 \times 10^{-7}$ Nm on the Y-axis and $10^{-9}$ Nm on the Z-axis.

At low Earth orbit, the aerodynamic disturbance torque is also present. This torque is caused by the atmospheric drag acting on the satellite. We can very reasonably assume, following references [El-Bordany 2001] and [Chen 2000], that the main disturbance torque acting on UoSAT-12 is the aerodynamic disturbance torque, which is still small on average (not severely affecting the spacecraft), with a mean value at perigee of $1.9 \times 10^{-7}$ Nm, but with an absolute maximum at perigee of $1.6 \times 10^{-6}$Nm, which is one order of magnitude higher than the maximum solar disturbance torque (reference [Chen 2000]). A maximum disturbance torque of that order will be assumed here, when evaluating the effect of external disturbance torques on UoSAT-12.

**1.2.3 ADCS software**

The ADCS hardware is used to achieve the functions initially defined in the ADCS code stored in the OBC (UoSAT-12 uses a 186 OBC as the ADCS processor).

All ADCS functions are defined in the ADCS processor (with the possibility of uploading new simulator-proven functions from the ground for further applications).

The ADCS processor includes all attitude determination procedures (IGRF model, Kalman filtering, sensor models), orbital data related to the attitude (orbit propagator), all attitude control functions (attitude parameterisations, detumbling, momentum dumping, spin stabilisation, 3-axis stabilisation, tracking (for Sun and Earth and effect of the moon), using different actuators, actuator models...).

Therefore, the ADCS processor can be seen as the “brain” of the ADCS module, which coordinates the use of the sensor measurements in order to provide the required actuator commands.

The ADCS processor of UoSAT-12, as most other small satellites, did not include any specific code to deal with the case of actuator failures. The need for new underactuated control code for UoSAT-12 has been understood only since the failure of the Z-axis reaction wheel. This failure onboard UoSAT-12 is clearly not an isolated case (several similar examples were presented in section 1.1).
1.3 Research background

Research has been progressing over the recent years to prove that two control torques can in fact be sufficient for the three-axis stabilisation of a spacecraft, but only using non-smooth feedbacks. The reason why non-smooth feedbacks are used is that underactuated rigid spacecrafts behave as nonholonomic systems (as explained earlier).

Although the problem of the attitude control of a spacecraft using two control torques has been investigated in several papers, there are still open problems and important issues to investigate.

In most papers, authors have investigated the case when the two control torques are provided by two pairs of thrusters. In this case, it will be seen that most authors have focused on the kinematic control of the spacecraft's attitude using angular velocity commands. Often, the proposed angular velocity commands cannot in practice be generated by admissible control torques. In all the proposed papers, the thrusters are supposed to generate torque continuously with the assumption that actuators are variable thrusters. There have been no investigations into on/off thrusters (used for nearly all small satellites missions) in this case. Other new issues, which will be developed later in this thesis concern the presence of a small angular momentum and external disturbance torques. Such issues have never been investigated on either axis-symmetrical or asymmetric satellites.

For two reaction wheels, a few papers have addressed problems such as yaw manoeuvres, bias momentum stabilisation, but the problem of the three axis stabilisation in this case has only been studied in one paper ([Yamada 1998]). The three-axis stability using two reaction wheels in the presence of a momentum bias has never been studied and is still a problem to solve.

Generally, the underactuated attitude controller can only be started assuming some small initial momentum on start-up. If the satellite is initially in a fast tumbling state, then a preceding controller consisting of a detumbling manoeuvre must also be assumed to achieve the small momenta conditions required to start the controller with two torques only (low cost magnetorquing is convenient for detumbling). Conversely, if there were no initial momentum, a preceding controller is needed to initiate some movement if the
satellite's initial attitude causes controller singularity (if nonlinear singular control is used, not needed for time varying control).

**Literature review, progress in the research and remaining problems:**

To clearly describe the progress in the attitude control of underactuated satellites, we give an ordered review of the main results of the research in underactuated spacecraft control.

The important paper of Crouch [Crouch 1984], was the first attempt to investigate the dynamic and kinematic equations of a rigid body in the case of one, two or three independent control torques. In the case of momentum exchange devices (reaction wheels), he established that, in the non-restricted case (when no restriction is considered or imposed, more precisely for an arbitrary bias momentum), the stabilization of the rigid body is impossible (without prior detumbling) even for a small time. In the same paper [Crouch 1984], but in the case of gas jet actuators (thrusters), it has been established that in the general non-restricted sense, the stabilization is possible for a small time.

However, using restrictions that can be imposed after a preceding control phase, results were more encouraging in ref [Krishnan 1992] for both thrusters (zero angular velocity along unactuated axis) and reaction wheels (zero total angular momentum satellite restriction). Krishnan seems to be the first author who understood the possibility of attitude stabilisation by two wheels for a zero total momentum satellite.

Following these fundamental stability investigations, control laws for the detumbling of a satellite on all three axes, with two control torques (pairs of thrusters) have been proposed by different authors based on different advanced control theories. The issue of angular velocity control with two control torques (from thrusters) has been studied in [Coverstone 1996], [Reyhanoglu 1996], [Aeyels 1998], [Mazenc 2000] and [Astolfi 2002]. Angular velocity control laws with two control torques have even been proven robust to general model errors in references [Morin 1996], [Astolfi 1997] and [Mazenc 2000]. Strategies to restrain the tumbling of underactuated spacecrafts with disturbance due to open failure have also been proposed in references [Zhang 1999a] and [Zhang 1999b].

Using a new parameterisation technique in reference [Tsiotras 1994], Tsiotras has proposed an approach to solve the spin stabilisation problem with two pairs of thrusters for axis-symmetrical spacecrafts. The full details of the novel parameterisation of attitude
kinematics have been published in [Tsiotras 1995]. Using his novel parameterisation, Tsiotras has largely contributed to a better understanding of the attitude control of axis-symmetrical spacecrafts with two pairs of thrusters (assuming however that $\omega_3(0) = 0$), by developing reduced effort control laws in [Tsiotras 1996], optimal and time optimal control laws in [Tsiotras 1997] and [Tsiotras 1999] and control laws with bounded inputs in [Tsiotras 2000b]. A survey of these techniques (developed by Tsiotras) as well as the first steps in an attempt to generalise the results of Mazenc to the reorientation of rigid bodies has also been given by [Fauske 2003]. The optimal control problem has also been addressed in the same case of asymmetric spacecrafts in reference [Ruggs 1998] using quaternion modelling.

Recently, the 3-axis attitude stabilisation of asymmetric satellites using only two pairs of thrusters has been dealt with in [Walsh 1995], [Coron 1996], [Morin 1997], using the Rodriguez attitude parameterisation. The same 3-axis stabilisation problem has been solved using the Tsiotras parameterisation in [Tsiotras 2000a] and the quaternion modeling in [Godhavn 1996], [Kim 2000], and [Behal 2002] (Behal only gave a theoretical stability proof based on a backstepping technique without any kind of simulation demonstration). The authors proposed stabilising control laws with the actuated angular velocities as virtual inputs, for axis symmetrical satellites with $\omega_3(0) = 0$. The authors also proposed Stabilising control laws for asymmetric satellites when $\omega_3(0) \neq 0$. However, all the proposed control laws were designed at the kinematic level only, without any proof that the required control torque to generate the angular velocities trajectories was admissible. Another problem was the assumption of continuous and variable thrust (not applicable to small satellites).

In the papers [Tsiotras 1999], [Tsiotras 2000a] and [Tsiotras 2000b], the spacecraft has been considered as a cascade system, (interconnection of two interdependent control systems). Using this approach, the control torque has to make the angular velocities along the actuated axes follow a prescribed path. When this desired path is followed, proof is given that the satellite is three axis-stabilized. However, the problem with that kind of approach, which will be demonstrated in this thesis, is the high control torque expenditure and the need for torque saturation, as well as the restrictive assumption that the thrusters can deliver variable amounts of control torque.
In the case of momentum exchange devices, S.Kim and Y.Kim have proposed (see ref [Kim 2001]) a control law for the spin axis stabilisation of a spacecraft (stabilisation around a revolute motion) using two reaction wheels. The approach considered in that paper [Kim 2001] was based on a nonlinear Lyapunov stability design, and formulated using the Tsiotras attitude parameterisation.

The use of two reaction wheels for the three axis stabilization has only been proven, for a zero total angular momentum satellite in [Yamada 1998]. However, the system went through undesired oscillations, and the performance of that controller will be demonstrated very limited for the example of UoSAT-12. Another inconvenience using that technique is that it only guaranteed local stability for regions of initial conditions, not global stability.

In fact, there is an analogy between the restriction of zero total angular momentum $H$ in the case of reaction wheels and the restriction $\omega_z(0) = 0$ in the case of thrusters. There is a similarity between the resulting kinematic models in both cases, although the laws relating control torques to redefined control inputs are totally different.

The momentum bias satellite case, or even the case of a satellite with a small total angular momentum is still a remaining problem. Stability in a strict sense is in fact impossible (because a small momentum is still a non zero momentum), but the practical possibility of bringing the system to a neighbourhood of the desired reference requires a more thorough investigation. One trial for the 3-axis stabilisation of a bias momentum satellite using two wheels has been made in ref [Terui 2000], but only roll and pitch control have been proven (no three axis stabilisation).

The problem of reorienting the satellite about the unactuated axis (yaw manoeuvre without Z wheel) of an underactuated satellite with two wheels and magnetorquing (combined control) has also been adressed in references [Ake 2002], [Tillier 2000] and [Pal 2000]. Successful in-orbit results of the combined control for the Fuse mission have been given in [Alee 2002] although control accuracy was deteriorated.

Steering control laws (not 3-axis stabilising control laws) have also been considered in references [Walsh 1993] based on open loop planning, and based on the so-called periodic forcing in [Leonard 1996]. The meaning of "steering" is that the attitude is
slewed from a prescribed initial orientation to a prescribed final orientation, but without stability or in other words without maintaining the attitude at a neighbourhood of the desired attitude. Using that technique the satellite can be reoriented about the unactuated axis but the orientations about the other axes start to slowly diverge after reaching the desired attitude (The slow divergence is inherent to the steering strategy, even in the free disturbance case). In this thesis, it has been decided that steering was not the best control strategy to investigate because the divergence will be even worse in the presence of disturbance torques and because of the intent to maintain a stable orientation.

Despite the research giving now a very good understanding of all sorts of control objectives such as detumbling, steering, spin axis stabilisation, reorientation about the symmetry axis in the case of one actuator failure (either thruster or reaction wheel), the full 3-axis stabilisation still needs to be investigated from several aspects. There are several significant remaining problems.

Using thrusters, an insight to the practical aspect (in the presence of a torque saturation, PWM, sampling) is still needed. Generally, the redefined inputs needed to stabilize the attitude by underactuated controllers lead to very large control torques that cannot be provided in practice. Then again, the effect of sampling has never been considered in the proofs for the different control laws. Moreover, the very practical requirement of ON/OFF thrusters and the need for PWM has never been discussed in the underactuated case.

Research is also required in order to understand clearly what does really happen in the case of symmetrical satellites (in the realistic case when \( \omega_3(0) \neq 0 \)).

The possibility of meeting the restrictions considered (of zero initial angular velocity about the unactuated axis or zero total momentum) also needs to be studied for either thrusters or reaction wheels. In fact, the controllers proposed in the recent literature to detumble spacecrafts using only two control torques in [Coverstone 1996] and other references can be used during a first detumbling phase. Using those controllers in a first phase, we can approach the restriction \( \omega_3(0) = 0 \) in the case of thrusters or \( H(0) = \mathbf{0}_{3\times3} \) in the case of reaction wheels, before applying the attitude stabilising controller in a second phase. The detumbling can also be achieved via low cost magnetorquing.
Chapter 1. Introduction

The effect of external disturbance torques (bias) on the satellite (especially on the underactuated axis) is also a practical problem that must be studied.

In the case of reaction wheels, a clear investigation of the stabilisation for nonzero small total angular momentum satellites is needed. It will be demonstrated that a small angular momentum would only cause small residual oscillations about the desired attitude, by implementing any 3-axis stabilising control law initially designed for a zero total angular momentum.

To conclude, very encouraging results have been presented in the literature of spacecrafts underactuated attitude control. However, many issues have still to be investigated before we can really account for the actuator failures in the design of an ADCS module for a space mission.

1.4 Outline of the thesis

In this thesis, we first start in chapter 2 by the presentation of the main features and the structure of the underactuated satellite control system. The general dynamic model of an underactuated satellite will be analysed using either two pairs of thrusters (case of a thruster failure) or two reaction wheels (case of a wheel failure). The kinematic model will then be expressed, first using Euler angles, then using the well-known quaternion modelling, and then using even more subtle parameterisations of the attitude (particularly convenient for the study and control design of an underactuated satellite). The controllability conditions of the underactuated satellite will be discussed in both cases of thrusters and reaction wheels failures. Controllability by smooth feedbacks is impossible in the case of nonholonomic systems, and non-smooth feedbacks will be considered as the only viable technique.

The structure of the control system strategy, in two phases, of the underactuated satellite will be illustrated in detail via block diagrams.

In chapter 3, the state of the art of the different attitude control strategies, proposed in the literature, for underactuated satellites is presented. It will be shown that the spin-axis stabilisation is achievable using smooth PD feedbacks, using two pairs of thrusters, then using two reaction wheels as well. The non-smooth control strategies will be presented as
a solution to 3-axis stabilisation, in both cases of reaction wheels and thrusters. The different approaches will be based on either singular or time varying feedbacks using the different parameterisations of the attitude discussed earlier. The cases of axis-symmetrical or asymmetric satellites are both investigated in detail. The restrictions that the underactuated system needs to meet to make the system stabilisable will be given.

In chapter 4, the contributions of the thesis to the current knowledge in underactuated attitude control, are given in two distinct cases: using pairs of thrusters, and then using two reaction wheels.

Using pairs of thrusters, the control strategies presented in chapter 3 are applied to the complete (kinematic + dynamic model) first by assuming continuous thrust (as always assumed in the known literature of underactuated spacecrafts). Practical problems of torque availability, and the torque saturation effect, using thrusters will be highlighted. Another investigation, never done before, into the kind of stability that can be obtained for symmetrical satellites spinning about the unactuated axis is achieved. The practical effect of an external disturbance torque on the Z-axis is also shown for the same micro-satellite model. Finally, other practical issues such as the effect the required PWM using ON/OFF thrusters, are discussed. These results will demonstrate that the attitude control of small satellites using two pairs of on/off thrusters is not practical nor achievable using all known underactuated control strategy based on thrusters (assuming a realistic amount of torque availability).

Using reaction wheels, a novel nonlinear singular control strategy is compared to the only known 3-axis stabilising control law from the literature, first assuming a zero total momentum satellite. The nonlinear singular control (never proposed before to solve the reaction wheel underactuated control problem) proves very efficient with a high control authority and relatively low control demand. The case of a small nonzero momentum (possible result of a detumbling manoeuvre), is then investigated.

It appears that the attitude of the system can be maintained at a neighbourhood of the desired reference, with residual constant amplitude oscillations (where the oscillations amplitude directly depends of the initial momentum when the nonlinear singular controller is started).
The effects of the sampling on both controllers are shown. The proposed nonlinear singular control strategy works efficiently with 10 seconds sampling, while the known time varying controller only achieves stability for sampling times up to 3 seconds. The environment of the underactuated satellite also includes external disturbance torques, which the effect is studied.

In chapter 5, a complete underactuated attitude control strategy, in two phases, is presented. The first phase is proposed to meet the required restrictions that the underactuated control system demands, before the already proposed control strategy with two control torques can be started. Using pairs of thrusters, it is shown how the detumbling manoeuvre can be effectively achieved with two control torques.

In the case of a reaction wheel failure, another detumbling technique based on the use of low-cost magnetorquing (together with wheels to avoid a singular state) is proposed as a first phase.

The simulations of the complete control in two phases are then illustrated under different start-up conditions (under free noise, and free disturbance assumptions).

In chapter 6, the results of the UoSAT-12 simulator, for the underactuated control strategy using two reaction wheels are presented under the most realistic conditions (including the effect of noise, estimation errors, modelled disturbance torques...). The simulations are carried out for both cases of nadir pointing and Sun tracking.

The simulator results are obtained for a control strategy divided in two phases:

- A first phase with 2 wheels (used as PD controller) and magnetorquing (used as a cross product law), is considered for the detumbling of the total momentum, and at the same time make sure that the initial attitude is non-singular before the second phase.

- The second phase is set to start after less than one orbit, when the momentum is small enough (attitude is far from the zero roll and pitch singularity). During this phase, the novel “two wheels” nonlinear singular control strategy is applied to the attitude control of the underactuated system.
Chapter 1. Introduction

The UoSAT-12 simulator results of the “two wheels” control strategy in two phases, can be seen as a strong simulation proof of the possibility of a 3-axis control with two wheels starting from nonzero momentum conditions.

The encouraging results from the UoSAT-12 simulator, and the novelty of the “two wheels” control strategy after a proposed detumbling phase, have been a good reason to upload a code of “underactuated UoSAT-12 attitude control” to the OBC of UoSAT-12.

Despite the critical low energy status of UoSAT-12, on-orbit experiments of the two wheels underactuated controller have been achieved, but only for Sun tracking, due to severe battery discharge problems (nadir pointing experiment was unsafe).

Among the on-orbit results, the underactuated control strategy will first prove more power-budget effective than the combination of wheels with magnetorquing in a linear control law. One particular on-orbit test (particular choice of control parameters) will clearly demonstrate 3-axis stabilisation, especially on the unactuated yaw axis. Roll and pitch angles will be successfully controlled to the required periodic functions for Sun tracking, with more or less accuracy. Unfortunately, practical testing on UoSAT-12 has been impossible since July 2003, making it temporarily impossible to take the experimental part even further to improve the performance, control authority and possibly minimise control demand even more.

On the other hand, it will be shown using SSTL’s UoSAT-12 simulator results that nadir pointing testing (which has never been possible on a sick UoSAT-12) would be achieved rapidly and with much greater precision, and with less torque expenditure than Sun tracking experiments. The two wheels controller would be very appropriate for constant reference angles (which is the case with nadir pointing).

One first experiment has been recently possible on UK-DMC, (equipped with a pitch and yaw wheel only) and nadir pointing (and of course 3-axis stability) has been successfully demonstrated during that experiment. It has indeed proven better results that the case of Sun-tracking, as expected from the simulator.

The 3-axis attitude control of underactuated satellites principles will therefore be proven in this thesis for the case of reaction wheels, and 3-axis stability properties shown in this case, even on orbit. The nonlinear singular underactuated control strategy is potentially
very efficient, probably the most efficient way to achieve manoeuvres on the unactuated axis, by maintaining stability on the actively controlled axes.

SSTL’s Simulator results will suggest that the precision and rapidity of the control about the unactuated axis (after a slewing manoeuvre) are even much better than what could be achieved by magnetorquing on the unactuated axis.

Further testing (preferably on UK-DMC or possibly on other recently launched small satellites,), is still needed to demonstrate the maximum efficiency and control authority that might be expected for underactuated satellites, especially in the case of Earth pointing and imaging experiments.

The last in-orbit test on UK-DMC, which will be presented in the thesis, has demonstrated a very good control authority during most of each orbit, except during particular periods of time. A slight deterioration of attitude control performance occurs periodically when the external disturbance torque is more significant, presumably due to a higher atmospheric drag (at particular regions of the orbit). Uploading new code to deal with the periodic “small attitude jump” phenomenon, repeating itself at a period of exactly 1 orbit, due to a higher external disturbance torque should solve this problem. This might be achieved via magnetorquing during particular periods, and should improve the attitude control performance.

In practice, there is a trade-off between rapidity and precision to make in the empirical choice of the control parameters. All indications are that the precision should be convenient for different applications such as imaging as long as non-diagonal elements in the inertia matrix can be ignored.
Chapter 2

2 The underactuated satellite control system

In the case of loss of one actuator, we assume without any loss of generality that the unactuated axis is the Z (yaw) body axis. When only two of the three body axes of the satellite are actively controlled, the situation can be pictured as in figure (2.1):

![Space representation of a satellite actively controlled on two body axes](image)

**Figure 2.1:** Space representation of a satellite actively controlled on two body axes

During the last years, one of the active research challenging problems in ADCS is the design of new nonlinear controllers to stabilize this underactuated system.

Before one can address the stabilisation problem, an insight must first be given into the dynamic equation of motion, which describes the satellite's rotational motion.

2.1 Dynamic equation of motion

2.1.1 Case of two pairs of thrusters

The rotational motion of a rigid spacecraft under the influence of three body fixed torques on each principal axis, is described by Euler's equation of motion of the form:

\[
\begin{align*}
I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3 + N_1 \\
I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_3 \omega_1 + N_2 \\
I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2 + N_3
\end{align*}
\] (2.1)
Chapter 2. The underactuated satellite control system

In the case of one actuator failure, by assuming that the failure happens for instance on the Z-axis, the control torque component on the unactuated Z axis will disappear ($N_3=0$).

We have to consider a separate study for the case of axis-symmetrical satellites (including fully 3-axis symmetrical satellites) and for the case of asymmetric satellites:

2.1.1.1 Asymmetric satellite case

In the case of one actuator failure, the torque component on the unactuated axis disappears from equation (2.1). In this case, we can assume without any loss of generality that the unactuated axis is the Z (yaw) axis and the Euler's rotational equation is:

$$
\begin{align*}
I_x \dot{\omega}_1 &= (I_z - I_y) \omega_2 \omega_3 + N_1 \\
I_y \dot{\omega}_2 &= (I_z - I_x) \omega_3 \omega_1 + N_2 \\
I_z \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2 
\end{align*}
$$

From the above system of equations (2.2), we observe that we do not lose entirely the control of $\omega_3$ despite the missing control torque component on the Z-axis. This is because we still have an interconnection term (product) $\omega_1 \omega_2$ in the third equation of (2.2), which is controlled using the remaining actuators.

Therefore, the rotations (angular velocities) on the X and Y axes in this case will involve an angular acceleration on the Z axis. That is the reason why the complete attitude can at least be affected using only two actuators for asymmetric satellites. (the symmetrical case exception will be explained in the following).

2.1.1.2 Axis symmetrical satellite case:

For an axis-symmetrical satellite, in the general case $\omega_3(0) \neq 0$ non-zero initial velocity on Z axis), we face a particular control difficulty, especially when the unactuated Z-axis is the symmetry axis of the satellite. If we have no control on the symmetry axis, we will obtain a constant angular velocity along that axis. In this case, we also have no means to have any impact on that angular velocity simply because $I_1 = I_2$.

In this case, the Euler's dynamic equation is:
Chapter 2. The underactuated satellite control system

\[ I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3 + N_1 \]
\[ I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_3 \omega_1 + N_2 \]
\[ \dot{\omega}_3 = 0 \]  

(2.3)

We clearly notice from equation (2.3) that we have absolutely no impact on the yaw angular rate using the two actuator control inputs \( N_1, N_2 \). It has also been shown that a necessary condition for the controllability of the system (in terms of full attitude) is that \( \omega_3(0) = 0 \), and as \( \omega_3 \) is a constant according to the third equation in (2.3), this condition can be written as: \( \omega_3 = 0 \) for all times. Without assuming this condition, the best one could expect in this case with only two control torques is a partial attitude stabilization, but it will be shown in this report that we can still approach the concept of three-axis stabilization in this case.

The dynamic equation of the axis-symmetrical satellite controlled with two thrusters, when we consider the assumption \( \omega_3(0) = 0 \iff \omega_3 = 0 \) for all times, reduces to:

\[ I_1 \dot{\omega}_1 = N_1 \]
\[ I_2 \dot{\omega}_2 = N_2 \]
\[ \omega_3 = 0 \]  

(2.4)

Remark: If a failure occurs on one axis, which is not a symmetry axis then the control problem becomes very similar to that of controlling an asymmetric satellite. For instance, if the Y wheel fails, then \( N_2 = 0 \) and \( N_2 \neq 0 \), and \( \omega_1, \omega_3 \) can then be regarded as the virtual inputs because the coupling term \( \omega_1 \omega_3 \) can be used to control \( \omega_2 \) in this case. The required control torques \( N_1, N_2 \) are then used to generate the desired \( \omega_1, \omega_3 \) trajectories.

2.1.2 Case of two reaction wheels:

This case differs from the control using two thrusters in the fact that we can consider that the restriction to meet here is \( \mathbf{H} = 0 \). In this case, the new variable that intervenes in the dynamic model is the angular momentum of the wheels.

The stabilization has only been proven in the literature for a zero total angular momentum satellite. One attempt has been made in the case of a bias momentum satellite (see ref [Terui 2000]) but the stabilization has not been obtained on the three axes.
Chapter 2. The underactuated satellite control system

In this report, we will study the case of a small nonzero total angular momentum. This case has got a significantly practical meaning since the total momentum can be made small but nonzero in a previous control phase.

- Equations of motion (Dynamic model):

The equation of the satellite's attitude if we have no external torque is:

\[ \dot{L} + \omega L = 0 \]  \hspace{1cm} (2.5)

And the equation of the total angular momentum is:

\[ L = I \omega + h \]  \hspace{1cm} (2.6)

Where, the vector \( h \) is given by:

\[ h = \sum_{i=1}^{2} I_w \dot{\alpha}_i z_i \]  \hspace{1cm} (2.7)

By replacing \( h \) from the equation (2.7) into the equation (2.6), and by considering another satellite's inertia momentum without wheels, we have:

\[ L = A_b^1 H = I \omega + \sum_{i=1}^{2} I_w \dot{\alpha}_i z_i \]  \hspace{1cm} (2.8)

with:

- \( A_b^1 \): Attitude matrix (or direction cosine matrix) from the inertial to the body frame.
- \( I \): Inertia matrix of the of the satellite (including wheels) about it’s center of mass.
- \( I_s \): Inertia matrix of the of the satellite(without wheels) about it’s center of mass.
- \( \omega \): Vector of the inertial referenced body angular rates.
- \( \dot{\alpha}_i \): Speed command of the \( i^{th} \) wheel.
- \( z_i \): Unit vector along the rotational axis of the \( i^{th} \) wheel.
- \( h = [h_1, h_2, h_3]^T \): Relative angular momentum generated by the wheels in the body frame.
- \( H \): Total angular momentum in the inertial frame.
- \( L \): Total angular momentum in the body frame.
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We can note that in practice, the difference between $I_s$ and $I$ is often considered negligible. Another practical fact is that the matrix $I_{wi}$ of the $i^{th}$ wheel can be replaced by $I_{wi} \cdot I_{3 \times 3}$ where $I_{3 \times 3}$ is the identity matrix and $I_{wi}$ is a scalar such that $h_i = I_{wi} \alpha_i$.

By substituting $L$ by its expression (2.8), into the equation (2.5), we obtain:

$$I\dot{\omega} = -h - \dot{\omega} \times (I\omega + h)$$ \hspace{1cm} (2.9)

And, we can rewrite this equation for each component:

$\begin{align*}
I_1 \dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + N_1 - \omega_2 h_3 + \omega_3 h_2 \\
I_2 \dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + N_2 - \omega_3 h_1 + \omega_1 h_3 \\
I_3 \dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + N_3 - \omega_1 h_2 + \omega_2 h_1
\end{align*}$ \hspace{1cm} (2.10)

Where:

$I_1, I_2, I_3$: The principal moments of inertia of the body.

$\omega_1, \omega_2, \omega_3$: The inertially referenced body angular velocities.

$h_1, h_2, h_3$: The relative angular momentum of the reaction wheels.

$N_1, N_2, N_3$: The input control torques provided by the reaction wheels.

$N_1, N_2, N_3$ are not to be confused with the external disturbance torques (as we are considering a free disturbance torque case at this stage).

The vectors $N = [N_1, N_2, N_3]^T$, $h = [h_1, h_2, h_3]^T$ are in fact simply related. Indeed, from the derivative of the angular momentum vector, we have:

$\dot{h} = N_{\text{wheel}}$ represents the torque applied to the wheels by the spacecraft's body, and by the Newton's third law of motion, $-h$ represents the control torque applied to the spacecraft body by the wheels, given by:

$$N_i = -h_i = -I_{wi}\alpha_i, \hspace{0.5cm} i = 1, 3$$ \hspace{1cm} (2.11)

If the third wheel fails, then $N_3 = 0$. We can also assume that $h_3 = 0$, which is a realistic assumption under the conditions detailed in the next page (as a remark). Therefore, if the $Z$-wheel fails, (2.10) simply reduces to:
\[ I_i \dot{\omega}_i = (I_{2i} - I_{3i}) \omega_2 \omega_3 + N_i + \omega_i h_2 \]
\[ I_j \dot{\omega}_j = (I_{3j} - I_{2j}) \omega_3 \omega_1 + N_2 - \omega_j h_1 \]
\[ I_k \dot{\omega}_k = (I_{1k} - I_{2k}) \omega_1 \omega_2 - \omega_k h_2 + \omega_k h_1 \]

**Remark**

The usual assumption \( h_j = 0 \) after the Z-wheel failure, (also assumed in different references such as [Yamada 1998]) must not necessarily be considered as an immediate condition at the time of the failure. The Z-wheel momentum \( h_3 \) will in theory remain constant if no external torque is applied, or it might also decrease, but then the residual Z-wheel angular momentum will have been transferred to the spacecraft body (assuming that the total momentum is conserved). In both cases, de-spin by other actuators such as magnetorquers is required. It is only after de-spin about the Z-axis or any other axis where the momentum has been transferred, that the control strategy (based on \( h_j = 0 \) assumption) using only two reaction wheels can be implemented. The de-spin is generally the first safety measure taken after detection of a reaction wheel failure (as in the case of UoSAT-12, FUSE, ...) and that is why the assumption \( h_j = 0 \) is realistic.

**2.2 Kinematic model using Euler angles**

Using the Euler angles convention, we assume a 1-2-3 rotational sequence (Roll, Pitch, Yaw) to describe the orientation of the satellite. In this case, kinematic equation of the satellite is:

\[ \dot{\phi} = (\omega_1 \cos \psi - \omega_2 \sin \psi) \sec \theta \]
\[ \dot{\theta} = \omega_2 \cos \psi + \omega_3 \sin \psi \]
\[ \dot{\psi} = \omega_3 - (\omega_1 \cos \psi - \omega_2 \sin \psi) \tan \theta \]

where \( \phi, \theta, \psi \) denote respectively the roll, pitch and yaw angles, and \( \omega_1, \omega_2, \omega_3 \) are the components of the angular velocity vector \( \omega \) along the principal axis in the body referenced frame.

The equations of the system (2.13) are nonlinear and nonholonomic. The control of such a system using simple linear controllers or smooth controllers (continuous and time invariant) has been proven in [Brockett 1983] to be impossible. The control design would
be very complicated using the equations (2.13) (although done in [Krishnan 1992]) but the physical meaning of Euler angles can be convenient for study purposes. That is the reason why we choose in practice other parameterizations of the system, which will still be nonholonomic but more appropriate for applying control techniques.

2.3 The cascade system (Dynamic model + Kinematic model)

2.3.1 Case of thrusters:

For the stabilization of the complete system, we have to consider a separate study for the symmetrical and asymmetric cases:

2.3.1.1 Asymmetric case:

We rewrite the dynamic + kinematic models in this case

\[ \begin{align*}
\dot{\omega}_1 &= u_1 \\
\dot{\omega}_2 &= u_2 \\
\dot{\omega}_3 &= \epsilon \omega_1 \omega_2 \\
\dot{\phi} &= (\omega_1 \cos \psi - \omega_2 \sin \psi) \sec \theta \\
\dot{\theta} &= \omega_2 \cos \psi + \omega_1 \sin \psi \\
\dot{\psi} &= \omega_3 - (\omega_2 \cos \psi - \omega_1 \sin \psi) \tan \theta
\end{align*} \] (2.14)

Where we used the following redefined control inputs to simplify the model:

\[ \begin{align*}
u_1 &= a_1 \omega_2 \omega_3 + \frac{N_1}{I_1} \\
u_2 &= a_2 \omega_2 \omega_1 + \frac{N_2}{I_2}
\end{align*} \] (2.15)

\[ \begin{align*}
a_1 &= \frac{I_2 - I_3}{I_1} , \quad a_2 &= \frac{I_3 - I_1}{I_2} , \quad \epsilon = \frac{I_1 - I_2}{I_3}
\end{align*} \]

In the asymmetric case, stability is not conditioned by any restriction or constraint on the initial angular velocity. New parameterisations of the attitude are needed to simplify the kinematic part of the model. Non-smooth control laws are required to stabilize this nonholonomic cascade system. However, to enhance control performances, it is still appropriate to consider a detumbling maneuver (using the two available control torques) to make \( \omega_3(0) \) as small as possible before applying any attitude stabilising controller.
2.3.1.2 Axis symmetrical case:

In this case, we first deal with the controllable case (restriction): \( \omega_3(0) = 0 \)

The system's equation will considerably simplify as follows:

\[
\begin{align*}
\dot{\omega}_1 &= u_1 \\
\dot{\omega}_2 &= u_2 \\
\dot{\omega}_3 &= 0 \\
\phi &= (\omega_1 \cos \psi - \omega_2 \sin \psi) \sec \theta \\
\dot{\theta} &= \omega_2 \cos \psi + \omega_1 \sin \psi \\
\psi &= (\omega_1 \cos \psi - \omega_2 \sin \psi) \tan \theta
\end{align*}
\]  

(2.16)

where:

\[
\begin{align*}
u_1 &= \frac{N_1}{I_1} + a_1 \omega_1 \omega_3, \quad u_2 = \frac{N_2}{I_2} + a_2 \omega_1 \omega_3 \quad (a_1=a_2=0 \text{ if the satellite is fully symmetrical})
\end{align*}
\]

The system is still nonholonomic. So, we still have to look for non-smooth control laws. However, the control laws for this system, in the assumed case \( \omega_3(0) = 0 \), are relatively easier to design than in the asymmetric case. The design of control laws for that case has even been dealt with using the Euler angles formulation in reference [Krishnan 1992]. It is however preferable to design control laws using appropriate parameterisations of the attitude, as already done by Tsiotras in reference [Tsiotras 1994].

In both cases of symmetrical or asymmetric satellites, we can assert that the complete attitude control system of the satellite may be represented as a cascade interconnection of two important subsystems: the dynamic model and the kinematic model.

The outputs of the dynamic model are also inputs for the kinematic block. That is what we call a cascade system in the control terminology.

![Figure 2.2: Cascade structure of the complete system (dynamic model+Kinematic model)](image-url)
We can notice from equations (2.6) and (2.8), and also from the last block diagram, that the variables $\omega_1, \omega_2$ can be seen as virtual attitude control inputs.

The spacecraft is therefore a cascade system, such that the control torques generate the desired angular velocities, which are designed to have full attitude + angular velocity stabilisation.

**Axis symmetrical satellite with: $\omega_3(0) \neq 0$:**

If the failure happens on the Z-axis an axis-symmetrical satellite when the yaw angular velocity is different from zero $\omega_3(0) \neq 0$, it is obvious that the detumbling cannot be ensured by the two control torques $N_1, N_2$ as they have no impact on the third axis (unactuated axis).

We then have to look for other actuators (possibly magnetorquers...) to ensure the detumbling of this angular velocity component by bringing it as close as possible to the desired attitude (typically zero).

We cannot strictly stabilize the system to the zero attitude in this case, but we can bring it average zero with a constant oscillation. More generally, we can also bring the system to the neighborhood of any non-zero reference, with a constant oscillation. The magnitude of this constant oscillation will depend on the effectiveness of the detumbling maneuver.

**2.3.1.3 The two required phases for underactuated control**

In practice, the restriction “$\omega_3(0) = 0$” must be approached as much as possible in the case of pairs of thrusters, to improve the control performance using the stabilising non-smooth controller with two torques.

In other words, a detumbling phase is necessary to bring the system within the required restrictions before the stabilising controllers with two torques can be implemented.

As already mentioned before, the detumbling manoeuvre can be achieved using two pairs of thrusters if the satellite is asymmetric (by exploiting the coupling on the Z-axis). However, for axis symmetrical satellites, another source of torque would be required for
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the detumbling (because thrusters have no impact on the Z-axis in this case), namely low cost magnetorquers.

![Diagram showing the two-phases of the underactuated attitude controller (Detumbling+Stabilisation)](image)

**Figure 2.3:** The two-phases of the underactuated attitude controller (Detumbling+Stabilisation)

2.3.2 Case of reaction wheels

Using reaction wheels, there is no distinction in the control design between symmetrical and asymmetric satellites. It can be shown that the restriction \( H(0) = 0_{3x1} \) (where \( H \) is the total momentum in the inertial frame), is equivalent to the restriction \( \omega_3(0) = 0 \) using pairs of thrusters (for an axis-symmetrical satellite).

By assuming a diagonal moment of inertia matrix \( I \), the components of the total angular momentum vector are given by:

\[
L_1 = h_1 + I_1 \omega_1 \\
L_2 = h_2 + I_2 \omega_2 \\
L_3 = h_3 + I_3 \omega_3 = I_3 \omega_3 \quad (Z \text{ wheel failed } \Rightarrow h_3 = 0)
\]

Where \( L = A.H = [L_1, L_2, L_3]^T \), where \( A \) is the direction cosine matrix, \( H \) is a constant vector (total momentum in inertial space), and \( L \) is a variable vector (total momentum in body frame) but with a constant norm.

In a disturbance free case, \( H(0) = 0_{3x1} \) will imply \( H(t) = L(t) = 0_{3x1} \) at all times.

The condition \( H = 0_{3x1} \) for the underactuated satellite, can therefore be written as:
\[ h_1 = -I_1 \omega_1 \]
\[ h_2 = -I_2 \omega_2 \]
\[ \omega_3 = h_3 = 0 \]  

(2.18)

then, by replacing \( \omega_1, \omega_2, \omega_3 \) from equation (2.18) into the dynamic equation (2.12), we have:

\[ I_1 \dot{\omega}_1 = N_1 \]
\[ I_2 \dot{\omega}_2 = N_2 \]
\[ \omega_3 = 0 \]  

(2.19)

The above equation, consequence of the zero total momentum assumption (without any symmetry assumptions) applied to a satellite with two active wheels, is similar to equation (2.4) using two pairs of thrusters for axis-symmetrical satellites (where the restriction \( \omega_3(0) = 0 \), using two pairs of thrusters has been assumed). The kinematic model using Euler angles for a zero total momentum is also similar to the case of thrusters with \( \omega_3(0) = 0 \), but only for axis-symmetrical satellites.

Therefore, techniques based on non-smooth control design might be proposed for both nonholonomic systems (for axis-symmetrical satellites) under the two specified restrictions. However, this similarity is only real for perfectly axis-symmetrical satellites, and there is also an additional difficulty using reaction wheels because the control strategy is to be employed in practice with angular velocity inputs (not control torques), which must be chosen to deliver the required satellite angular velocities to track (under the zero total momentum assumption). Retrieving the control commands from a virtual control input also differs completely from the case of thrusters to the case of reaction wheels.

Using reaction wheels, the control laws will not differ from symmetrical to asymmetric satellites, which is not the case using thrusters.

2.4 Alternative parameterisations of the attitude kinematics

Some particular problems in attitude control require alternatives to the Euler angles parameterization of the attitude (based on three successive rotations). The most common alternative is the use of the Euler symmetric parameters, better known as quaternion
parameters. These parameters have the advantage of the absence of any singularities, but the control design for an underactuated satellite using quaternions is generally complicated, and that is the reason why, in the control design, other parameterizations should be considered. Once the control design (and proof of stability) has been achieved using any convenient parameterisation, it is easy to rewrite the control laws into quaternions to avoid possible singularities.

The new parameterization proposed by Panagiotis Tsiotras and James M. Longusky [Tsiotras 1995] (based on two perpendicular rotations) for instance shows to be more convenient for the spacecraft underactuated control problem using thrusters (especially for the spin axis stabilization problem).

Another possible representation for the attitude, which is particularly convenient in the case of the two reaction wheels control, is the Euler-Rodriguez parameterization (based on one rotation) simply referred to as Rodriguez parameters in most references.

In fact, Euler angles provide a complete and well-known framework for the dynamics of the rotational motion. However, for the kinematics, other alternatives could be more appropriate due to the fact that the attitude matrix (also known as the rotation matrix or direction cosine matrix), which determines the relative orientation between two reference frames can be parameterized in more than one way. The best choice of the parameterization generally depends on the specific control problem.

2.4.1 Euler symmetric parameters (quaternion modeling)

Euler's theorem states that any finite rotation of a rigid body can be expressed as a rotation of angle $\phi$ about a fixed axis $e$ (generally known as Euler axis). In other words, the transformation matrix $A$ can be obtained by rotating by the angle $\phi$ about the fixed axis $e$. The Euler symmetric parameters $q_1, q_2, q_3, q_4$ in terms of angle $\phi$ and rotation axis $e$ are given by:
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\[ q_1 = e_x \sin \left( \frac{\phi}{2} \right) \]
\[ q_2 = e_y \sin \left( \frac{\phi}{2} \right) \]
\[ q_3 = e_z \sin \left( \frac{\phi}{2} \right) \]
\[ q_4 = \cos \left( \frac{\phi}{2} \right) \]  

(2.20)

where:

\[ \mathbf{q} = [q_1, q_2, q_3, q_4] \]: Components of the attitude quaternion vector with respect to the chosen reference frame (orbital frame in case of Nadir pointing).

\[ \mathbf{e} = [e_x, e_y, e_z] \]: Components of the unit Euler axis vector with respect the chosen reference frame (orbital frame in case of Nadir pointing).

The quaternion can in fact be seen as a complex vector with and a scalar real part and a vector imaginary part.

The quaternion components are not independent, but satisfy the following "normalising" constraint:

\[ q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \]  

(2.21)

The attitude matrix in terms of quaternion parameters is given by:

\[ \mathbf{A} = \begin{bmatrix}
q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\
2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\
2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2
\end{bmatrix} \]  

(2.22)

The expression of the attitude matrix is very convenient to transform from any attitude parameterisation to another one. For instance, roll, pitch and yaw angles for a 1-2-3 rotational sequence can easily be obtained, by identifying the terms of the third line of the matrix \( \mathbf{A} \).
2.4.2 The \((w_1, w_2, z)\) parameterisation: [Tsiotras 1995]

The new kinematic formulation for describing the rotational motion of a rigid body proposed by Tsiotras and Longusky provides a three dimensional parameterisation of the group using two perpendicular rotations (the main idea could be seen as a compromise between the three rotations using Euler angles and the single rotation considered with Euler-Rodriguez parameters.

There is in fact an analogy between this formulation and the more familiar Euler angle formulation that would be obtained from a 3-1-3 rotational sequence, although the last two rotations are expressed using a single complex variable using the new parameterisation.

Two of the three parameters used to describe the motion (namely \(w_1,w_2\)) can be combined into one single complex variable \(w\), thus reducing the number of differential equations used to describe the kinematics from three real to two complex equations (convenient for the design of control laws for underactuated satellites, especially for single axis or spin stabilisation).

This complex coordinate designates one of the two rotations; it describes in fact the location of the third axis (the yaw axis in the case of a satellite) in the inertial frame.

The real parameter \(z\) stands for the previous rotation about this axis. This parameter gives, along with the complex rotation, a complete description of the attitude and forms a new coordinates set on \(SO(3)\) (special orthogonal group of 3×3 rotations) so that we can properly consider an acceptable parameterization.

The rotation matrix from a reference frame \((\hat{i}_1, \hat{i}_2, \hat{i}_3)\) (inertial frame for the sake of simplicity) and the body frame \((\hat{b}_1, \hat{b}_2, \hat{b}_3)\), is decomposable using two rotations as as follows:

\[
A = A_2(w)A_1(z) \tag{2.23}
\]

For these two rotations we assume that \(A_1, A_2 \in SO(3)\) so that these are valid rotation matrices, or in other words matrices which are orthogonal and have determinant 1.
In this representation, $A_1$ multiplies $A_2$ on the right, so it represents an initial rotation from the inertial frame. The consequence of this property is that the variable $z$, which represents the initial rotation about the Z-body axis, will not appear on the right hand side of the kinematic equations (as it is also the case with any first rotation of an Euler angles rotational sequence).

We adopt a reference frame of unit vectors $(\hat{i}_1, \hat{i}_2, \hat{i}_3)$ before the first rotation. Now, assuming that the new reference frame resulting from the rotation $A_1$ is $(\hat{i}'_1, \hat{i}'_2, \hat{i}'_3)$, we have:

$$\begin{bmatrix} \hat{i}'_1 \\ \hat{i}'_2 \\ \hat{i}'_3 \end{bmatrix} = A_1(z) \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix}$$

(2.24)

And by assuming that $z$ is the positive rotation about the Z axis, we can write:

$$A_1(z) = \begin{bmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2.25)

Now that the first rotation $A_1$ gives the initial rotation about one of the body axes (here taken to be the Z axis), we consider the second rotation providing the orientation of this axis (more precisely its unit vector) in $\mathbb{R}^3$. Let us to this end consider the two reference frames associated with the unit vectors $(\hat{\hat{i}}_1, \hat{\hat{i}}_2, \hat{\hat{i}}_3)$ (formed after the first rotation) and $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ (final result of the second “complex” rotation), then we have:

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = A_2(w) \begin{bmatrix} \hat{\hat{i}}'_1 \\ \hat{\hat{i}}'_2 \\ \hat{\hat{i}}'_3 \end{bmatrix}$$

(2.26)

The rotation sequence is represented on figure (2.4). Here, we aim to characterize the rotation of $\hat{\hat{b}}_2$ with respect to $\hat{\hat{i}}'_2$ (in terms of magnitude and axis of rotation). To this end, we let the location of $\hat{\hat{i}}'_2$ in the $\hat{b}$ reference frame be described by the direction cosines $(a, b, c)$, thus we have:
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\[ \hat{\mathbf{b}}_3 = a\hat{\mathbf{b}}_1 + b\hat{\mathbf{b}}_2 + c\hat{\mathbf{b}}_3 \]  
\[ (2.27) \]

and it can be shown that, we have in the inverse relation is:

\[ \hat{\mathbf{b}}_3 = -a\hat{\mathbf{b}}_1 + b\hat{\mathbf{b}}_2 + c\hat{\mathbf{b}}_3 \]  
\[ (2.28) \]

Further calculations give (using Euler’s formula):

\[ A_2(w) = A_2(a, b, c) = \begin{bmatrix} c + \frac{b^2}{1 + c} & -\frac{ab}{1 + c} & a \\ -\frac{ab}{1 + c} & c + \frac{a^2}{1 + c} & b \\ -a & -b & c \end{bmatrix} \]  
\[ (2.29) \]

Tsiotras and Longusky have then introduced the so-called stereographic projection, which includes the necessary information about the location of the \( \hat{\mathbf{i}}_3 \) axis in the \( \hat{\mathbf{b}} \) frame, or equivalently the position of \( \hat{\mathbf{b}}_3 \) axis in the \( \hat{\mathbf{i}}' \) frame. As \( a, b, c \) are the coordinates of a unit vector, they must be included in the set \( S^2 = \{ (x_1, x_2, x_3) \in IR^3 : x_1^2 + x_2^2 + x_3^2 = 1 \} \).

We define as in [Tsiotras 1995], the stereographic projection \( \sigma \), which is a projection from the unit sphere (except the singular vector \( (0,0,-1) \)) to the set of complex numbers \( C \). For \( (a, b, c) \in S^2 \), we define the stereographic projection \( \sigma \) as:

\[ \sigma^S = \{(0,0,-1)\} \rightarrow C \]
\[ (a, b, c) \rightarrow w = \sigma(a, b, c) \]
\[ w = w_1 + iw_2 = \sigma(a, b, c) = \frac{b - ia}{1 + c} \]  
\[ (2.30) \]

Some further calculations (using the inverse projection) give:

\[ A_2(w) = \frac{1}{1 + w_1^2 + w_2^2} \begin{bmatrix} 1 + w_1^2 - w_2^2 & 2w_1w_2 & -2w_2 \\ 2w_1w_2 & 1 - w_1^2 + w_2^2 & 2w_1 \\ 2w_2 & -2w_1 & 1 - w_1^2 - w_2^2 \end{bmatrix} \]  
\[ (2.31) \]

Using the complex formulation, we can write the previous equation more compactly as follows:
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\[
A_z(w) = \frac{1}{1+|w|^2} \begin{bmatrix}
1 + \text{Re}(w^2) & \text{Im}(w^2) & -2\text{Im}(w) \\
\text{Im}(w^2) & 1 - \text{Re}(w^2) & 2\text{Re}(w) \\
2\text{Im}(w) & -2\text{Re}(w) & 1 - |w|^2
\end{bmatrix}
\] (2.32)

The attitude matrix (or direction cosine matrix) obtained using this parameterization is given by:

\[
A(w,z) = \frac{1}{1+ w_1^2 + w_2^2} \begin{bmatrix}
(1 + w_1^2 - w_2^2)Cz - 2w_1w_2Sz & (1 + w_1^2 - w_2^2)Sz + 2w_1w_2Cz & -2w_2 \\
2w_1w_2Cz - (1 - w_1^2 + w_2^2)Sz & 2w_1w_2Sz + (1 - w_1^2 + w_2^2)Cz & 2w_1 \\
2w_2Cz + 2w_1Sz & 2w_2Sz - 2w_1Cz & 1 - w_1^2 - w_2^2
\end{bmatrix}
\] (2.33)

Where \(Cz\) denotes \(\cos z\) and \(Sz\) denotes \(\sin z\).

Using this expression of the direction cosine matrix, it becomes possible to find the relation between the Euler angles, or in other words between the roll, pitch and yaw rotations of a satellite and the \((w,z)\) parameterization.

In fact, we can identify \((a,b,c)\) with the third column of the Eulerian angles (1-2-3) rotation matrix \(A(\phi, \theta, \psi)\) which is given by:

\[
A(\phi, \theta, \psi) = \begin{bmatrix}
C\psi C\theta & C\psi S\theta S\phi + S\psi C\phi & -C\psi S\theta C\phi + S\psi S\phi \\
S\psi C\theta & -S\psi S\theta C\phi + C\psi C\phi & S\psi S\theta C\phi + C\psi S\phi \\
S\theta & -C\theta S\phi & C\phi C\theta
\end{bmatrix}
\] (2.34)

The differential equations of the attitude are given simply by differentiating the direction cosine matrix as follows:

\[
\dot{A}(w,z) = -\omega^* A(w,z)
\] (2.35)

Where \(\omega^*\) is the skew symmetric cross product matrix:

\[
\omega^* = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
\] (2.36)

By expanding the differential equation (2.35) (see [Tsiotras 1995]), we have:
Chapter 2. The underactuated satellite control system

\[
\dot{\omega} = -i \omega_z \omega + \frac{\omega}{2} + \frac{\overline{\omega}}{2} \omega^2 \\
\dot{z} = \omega_z + \frac{i}{2} (\overline{\omega} w - \omega \overline{w})
\]  

(2.37)

And in terms of real and imaginary parts of our complex variables, we finally have the attitude kinematic equations:

\[
\begin{align*}
\dot{w}_1 &= \omega_3 w_2 + \omega_1 w_1 w_2 + \frac{\omega_1}{2} (1 + w_1^2 - w_2^2) \\
\dot{w}_2 &= -\omega_3 w_1 + \omega_3 w_1 w_2 + \frac{\omega_2}{2} (1 + w_2^2 - w_1^2) \\
\dot{z} &= \omega_3 - \omega_1 w_2 + \omega_2 w_1
\end{align*}
\]

(2.38)

One very interesting property of the Tsiotras-Longusky parameterisation is that the variable \(z\), representing the attitude of the satellite about the unactuated axis, does not appear on the right hand side of the kinematic equations (2.37) and (2.38). This is a consequence of the fact that the first rotation in a sequence of rotations can be ignored (as we can notice even in equation (1.7)). This property contributes to the simplicity of the control design and stability analysis using the Tsiotras-Longusky parameterisation.

![Figure 2.4: Attitude representation of the Tsiotras-Longusky parameterization](image.png)
2.4.3 Rodriguez parameters:

In this representation, the attitude is expressed by the difference between the orientation of the body fixed coordinates and the inertial coordinates (see [Hughes 1986], [Yamada 1998]). In fact, this approach can be seen as an improvement of the well-known quaternion parameterisation with a reduced number of 3 parameters instead of 4 quaternions (we consider in fact a parameter \( p_1 = \frac{q_1}{q_4} \)).

The Rodriguez parameters will prove convenient for control design problems, but the main disadvantage is the presence of singularities, which is not the case with the quaternions.

Assuming that such a rotation can be represented by an angle \( \phi \) around a unit vector \( \mathbf{a} \), we define the Rodriguez parameters for the spacecraft attitude expression as:

\[
\mathbf{p} = [p_1, p_2, p_3]^T = \mathbf{a} \tan \frac{\phi}{2}
\]

(2.39)

The differential equation using the Rodriguez parameterisation is:

\[
\dot{\mathbf{p}} = \frac{1}{2} \left( \mathbf{1}_{3 \times 3} + \mathbf{p}^\times + \mathbf{p} \mathbf{p}^T \right) \omega
\]

(2.40)

where \( \mathbf{1}_{3 \times 3} \) is the 3×3 identity matrix and the cross product matrix \( \mathbf{p}^\times \) is defined similarly to the cross product matrix defined in (2.36) by replacing the \( \omega \) vector components by the \( \mathbf{p} \) vector components.

The above equation stands for the attitude differential equation in Rodriguez parameters.

The equation can be written on each axis as:

\[
\begin{align*}
\dot{p}_1 &= \frac{1}{2} \left( \omega_1 - (p_3 - p_1 p_2) \omega_2 + (p_2 + p_1 p_3) \omega_3 + p_1^2 \omega_1 \right) \\
\dot{p}_2 &= \frac{1}{2} \left( \omega_2 + (p_3 + p_1 p_2) \omega_1 - (p_1 - p_3 p_2) \omega_3 + p_2^2 \omega_2 \right) \\
\dot{p}_3 &= \frac{1}{2} \left( \omega_3 - (p_2 - p_1 p_3) \omega_1 + (p_1 + p_2 p_3) \omega_2 + p_3^2 \omega_3 \right)
\end{align*}
\]

(2.41)
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We are still able to describe the attitude in terms of roll, pitch and yaw angular rotations. The direction cosine matrix is given by (can be deduced from the rotation matrix with quaternions):

\[
A(p_1, p_2, p_3) = \frac{1}{1 + p_1^2 + p_2^2 + p_3^2} \begin{bmatrix}
1 + p_1^2 - p_2^2 - p_3^2 & 2(p_1p_2 + p_3) & 2(p_1p_3 - p_2) \\
2(p_1p_2 - p_3) & 1 - p_1^2 + p_2^2 - p_3^2 & 2(p_2p_3 + p_1) \\
2(p_1p_3 + p_2) & 2(p_2p_3 - p_1) & 1 - p_1^2 - p_2^2 + p_3^2
\end{bmatrix}
\]

(2.44)

The elements of the third column of the direction cosine matrix provide the attitude of the satellite in terms of yaw, pitch and roll angles as it has been previously seen for the \((w, \zeta)\) parameterisation.

\[2.5 \text{ Conclusions}\]

In this chapter, the dynamic and kinematic models of an underactuated satellite have been investigated in detail using two pairs of thrusters then using two reaction wheels.

Three parameterisations of the attitude of a satellite have been presented as alternatives to the Euler angles parameterization for the kinematics of a satellite.

The first one, based on the well-known quaternion modelling has the main advantage of having no singularities. The second, based on a series of two rotations (initial rotation of angle \(z\), and a second rotation characterized using a complex variable \(w\)) is very convenient for control design of underactuated satellites (used in many papers for the control of a satellite using two pairs thrusters).

The third formulation known as Rodriguez parameterization, based on a unique rotation of a unit vector using three real variables \((p_1, p_2, p_3)\), has already been used in [Yamada 1998] and [Terui 2000] for the attitude control of a satellite using two reaction wheels.

Using all different parameterisations, the attitude of the satellite in terms of roll, pitch and yaw angles can simply be retrieved from the rotation matrix of the rigid body (direction cosine matrix of the satellite). In the next section, we show how a convenient choice of attitude parameterisation can considerably simplify both control design and stability analysis.
Chapter 3

3 State of the art of the underactuated control strategies:

3.1 Case of two pairs of thrusters:

3.1.3 Spin-axis stabilization

The 3-axis attitude of an underactuated rigid spacecraft can not be stabilized by a time-invariant smooth state feedback because of the nonholonomic constraint in the kinematic model of underactuated satellites. However, the partial stabilization to an equilibrium manifold (around a revolute motion), or in other words the spin-axis stabilization, remains possible using smooth control laws. This property was first shown by Byrnes and Isidori in reference [Byrnes 1991] where the attitude was stabilized to a so-called circular attractor about the origin (revolute motion). Before dealing with the 3-axis stabilization problem, it is convenient to start with the more simple problem of spin-axis stabilization with two control torques.

The smooth spin-stabilising controller presented in this section is designed using Tsiotras-Longusky $(w,z)$ parameterisation, which has already been presented in chapter 2 as a new parameterisation of the attitude based on the composition of two rotations. The complex variable $w = w_1 + iw_2$ describes the location of the Z-axis, and the real variable $z$ describes the previous rotation about that axis.

The stabilization to an equilibrium manifold with constant spin can be achieved with very simple smooth and linear control laws when using the $(w,z)$ parameterization. To simplify the stability analysis, an axisymmetric satellite model is assumed as in reference [Tsiotras 1994].

The objective of the spin-axis stabilization is to design control laws that achieve $w_1 = w_2 = \omega_1 = \omega_2 = 0$ ($\omega_3 = \omega_0 = \text{constant, where } \omega_0 = \omega_2(t=0)$). The equations of an underactuated axisymmetric satellite $(I_1 = I_2)$ using the $(w,z)$ parameterisation are:
\[
\begin{align*}
\dot{\omega}_1 &= \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_4}{2} (1 + w_1^2 - w_2^2) \\
\dot{\omega}_2 &= -\omega_3 w_1 + \omega_1 w_1 w_2 + \frac{\omega_4}{2} (1 + w_2^2 - w_1^2) \\
\dot{\omega}_1 &= u_1 \\
\dot{\omega}_2 &= u_2
\end{align*}
\]

(3.1)

Where we can recall that \( u_1 \) and \( u_2 \) are simply related to the control torques by the equation (2.15) of chapter 2, \( \omega_1 \), \( \omega_2 \), \( \omega_3 \) stand for the angular velocities of the satellite (with \( \omega_3 = \omega_0 = \) constant for an axis symmetrical satellite), \( w_1 \) and \( w_2 \) are two attitude components defining the orientation of the unactuated axis (orientations with respect to the 2 actuated axes). The third component of the attitude, representing the rotation about the unactuated axis is denoted by the variable \( z \) (not yet considered at this stage of spin stabilisation).

The angular velocity \( \omega_3 \) is a constant for an axisymmetric spacecraft. The \((w,z)\) parameterization is particularly convenient for the control design of that specific problem.

In reference [Tsotras 1994], it was demonstrated that the spin-axis stabilization is guaranteed using a simple smooth PD control law of the form:

\[
\begin{align*}
u_1 &= -k_1 \omega_1 - k_2 w_1 \\
u_2 &= -k_1 \omega_2 - k_2 w_2
\end{align*}
\]

(3.2)

The variables \( u_1 \), \( u_2 \) are the redefined control inputs related to the control torques by equation (2.15) of chapter 2.

The control law (3.2) with \( k_1 > 0 \) and \( k_2 > 0 \), asymptotically and globally stabilizes the system (3.1) to the equilibrium manifold \( w_1 = w_2 = \omega_1 = \omega_2 = 0 \). A simple proof of that proposition can be made using the Lyapunov criterion. As in [Tsotras 1994], we can consider the following Lyapunov function:

\[
V = \frac{1}{2} (\omega_1^2 + \omega_2^2) + k_2 \ln(1 + w_1^2 + w_2^2)
\]

(3.3)

where \( k_2 > 0 \) is the constant positive control parameter used in equation (3.2).
Chapter 3. State of the art of the underactuated control strategies

By differentiating the Lyapunov function along the trajectories of (3.1) and (3.2), we have:

$$\dot{V} = \omega_1 u_1 + \omega_2 u_2 + k_z (\omega_3 w_2 + \omega_4 w_1)$$  \hspace{1cm} (3.4)

Fortunately, we can note that, $a_{3a}$ does not appear in the derivative of the Lyapunov function, thanks to the use of the $(w,z)$ parameterization. By substituting the expressions of $u_1$ and $u_2$ from equation (3.2) into equation (3.4), we have:

$$\dot{V} = -k_1 (a_1^2 + a_2^2) \leq 0$$  \hspace{1cm} (3.5)

where $k_1 > 0$ is a constant positive control parameter.

The derivative of the Lyapunov function from equation (3.5) is negative (semi-definite) and the system is stable. However, the more demanding global asymptotic stability is only guaranteed for a negative definite Lyapunov function. A further subtle analysis of the case $\dot{V} = 0$ by Tsiontros using LaSalle's theorem proves that $V$ is radially unbounded, which is the only further necessary condition (in addition to (3.5) of course) of global asymptotic stability.

The problem of the spin-axis stabilization of an underactuated satellite using only two control torques (from thrusters) can therefore be seen as a solved problem. There is, in fact, no need to derive any unconventional control laws to solve the problem. Using the $(w,z)$ parameterization, stability is guaranteed with reasonable performances (see reference [Tsiontros1994]) using the standard PD controller (which represents a smooth control law) of equation (3.2). Using reaction wheels, a similar Lyapunov-based approach also solves the spin stabilization problem (see reference [Kim2002]).

A similar smooth but time varying control technique has also been proposed by Tsiontros in [Tsiontros 2000a] to solve the so-called partial stabilization problem, where the state variable $(w_1, w_2, \omega_1, \omega_2, \omega_3)$ was brought to (0,0,0,0,0), without controlling the $z$ to a desired value though. That kind of control can be useful in situations when a particular instrument must point in some direction but that the fixed orientation about the axis of the instrument does not matter. However, for many applications such as imaging of particular areas, the orientations must not only be fixed but also precisely determined with respect to each axis.
Different problems of partial stabilization of underactuated spacecrafts, including spin stabilization, can therefore be considered solved. The main remaining challenging problem to be explored in different aspects (using both wheels and thrusters) is the 3-axis stabilization problem with only two control torques. That is the reason why the focus of the thesis is mainly on the 3-axis stabilization as a primary objective.

### 3.1.1 Tsiotras nonlinear attitude controller

The problem of the 3-axis attitude stabilization has also been addressed by Tsiotras in reference [Tsiotras 2000a], where the analysis included both axisymmetric and asymmetric spacecrafts.

The kinematic model using the \((w,z)\) formulation of the attitude is:

\[
\begin{align*}
\dot{w}_1 &= \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2} \left( 1 + w_1^2 - w_2^2 \right) \\
\dot{w}_2 &= -\omega_3 w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2} \left( 1 + w_2^2 - w_1^2 \right) \\
\dot{z} &= \omega_3 - \omega_1 w_2 + \omega_2 w_1
\end{align*}
\]

(3.6)

We notice that the angular velocities of the satellite \(\omega_1, \omega_2\) can be seen as virtual control inputs to the attitude equations. Indeed, from the first two equations of (2.14), the control inputs \(u_1, u_2\) directly affect the angular velocity components \(\omega_1, \omega_2\), which can then be used to control the third component \(\omega_3\) as well as the complete attitude. This is done by using the coupling term \(\omega_1 \omega_2\), which appears on the third equation of the dynamic model (equation (2.2)). The control inputs \(u_1, u_2\) must then be designed to ensure that \(\omega_1, \omega_2\) track the desired angular velocity components \(\omega_{1d}, \omega_{2d}\), designed to achieve the full state stabilization.

A possible control law for angular velocity tracking is:

\[
\begin{align*}
u_i &= -\gamma (\omega_i - \omega_{id}) + \dot{\omega}_{id} \\
u_2 &= -\gamma (\omega_2 - \omega_{2d}) + \dot{\omega}_{2d}
\end{align*}
\]

(3.7)

The previous control law will force \(\omega_i\) to track \(\omega_{id}\) exponentially (for \(i=1,2\)), with rate of convergence \(\gamma\). The larger we will set the value of \(\gamma\), the smaller will be the tracking error.
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\(\omega_t - \omega_{td}\), and the higher will be the required torque demand. In the case of a torque saturation, the situation could however differ.

- We first consider the general asymmetric case \((s \neq 0)\):

In this case the restriction \(\omega_t(0) = 0\) is not needed because the asymmetry of the system is such that the two controlled rotations on the roll and pitch axes will induce an acceleration on the yaw axis, as we can observe in equation: \(\dot{\omega}_3 = \epsilon \omega_1 \omega_2\).

From the third equation in (3.6), we notice that if we bring \((w_1, w_2) \to (0, 0)\), and \(z \to 0\) as \(t \to \infty\), this implies since the control laws for \(\omega_1\) and \(\omega_2\) are bounded, that \(\omega_3 \to 0\).

The control laws proposed in [Tsiotras 2000a] is:

\[
\begin{align*}
\omega_{1d} &= -k_w w_1 + k_z \frac{z - k_u \omega_2}{w_1^2 + w_2^2} w_2 \\
\omega_{2d} &= -k_w w_2 - k_z \frac{z - k_u \omega_3}{w_1^2 + w_2^2} w_1
\end{align*}
\]

The control laws in equation (3.8) could become singular and lead to large control torques, unless the control parameters are chosen to ensure the convergence of the numerator before the convergence of the denominator.

The linear parts in these feedback laws \(-k_w w_1, -k_w w_2\) are used to stabilize the attitude on the X and Y axes, whereas the non-smooth nonlinear term containing the interconnections contributes to the stability on the unactuated axis. A proof of stability using an equation similar to (3.8) but assuming the more simple case of \(\omega_t(0) = 0\) (and that the satellite is axisymmetric) is given in appendix A. For the formal proof using the control law (3.8), please refer to [Tsiotras 2000a].

We have to consider a saturated version of this control law (as in [Tsiotras 2000a]) in order to avoid singularities (which may happen if \(w_1 \to 0, w_2 \to 0\)) and to ensure that the control inputs \(\omega_1, \omega_2\) are indeed bounded. The saturated control law is given by:
\[
\begin{align*}
\omega_d &= -k_w \frac{w_1}{\sqrt{1 + w_1^2 + w_2^2}} + k_z \text{ sat} \left( \frac{z - k_w \omega_1}{\sqrt{w_1^2 + w_2^2}}, a \right) \frac{w_2}{\sqrt{w_1^2 + w_2^2 + 1}} \\
\omega_{zd} &= -k_w \frac{w_2}{\sqrt{1 + w_1^2 + w_2^2}} - k_z \text{ sat} \left( \frac{z - k_w \omega_1}{\sqrt{w_1^2 + w_2^2}}, a \right) \frac{w_1}{\sqrt{w_1^2 + w_2^2 + 1}}
\end{align*}
\]  
(3.9)

with: 
\[
\text{sat}(x, a) = \begin{cases} 
x & \text{if } |x| \leq a \\ 
a & \text{if } |x| > a
\end{cases}
\]

The control law (3.9) has been proven by extensive numerical simulation, according to reference [Tsiotras 2000a]. Stability was therefore conjectured but not proven mathematically.

In fact, as for the symmetrical case, we have considered here that \( \omega_i = \omega_d \), for \( i = 1, 2 \), and this condition is verified using the control law given in equation (3.7).

If the actual angular velocities follow these trajectories (3.9), it is conjectured by Tsiotras that we can always find appropriate controller parameters \( k_w, k_z \) and \( k_\omega \) that bring the attitude from any arbitrary attitude to zero roll, pitch and yaw angles and also bring \( \omega_b \) to zero.

We then consider an angular velocity trajectory controller such that the control torque makes the actual angular velocity follow the desired trajectory. To this purpose, Tsiotras suggests a PD type controller as in equation (3.7).

And following the definitions of \( u_1, u_2 \) in (2.14), (2.15), the corresponding control torques are given by:

\[
\begin{align*}
u_1 &= a_1 \omega_2 \omega_3 + \frac{N_1}{I_1} \Rightarrow N_1 = I_1 (u_1 - a_1 \omega_2 \omega_3) \\
u_2 &= a_2 \omega_2 \omega_3 + \frac{N_2}{I_2} \Rightarrow N_2 = I_2 (u_2 - a_2 \omega_2 \omega_3)
\end{align*}
\]  
(3.10)

It will be demonstrated that the control torques must be saturated in practice.
3.1.2 General Block diagram of the 3-axis underactuated control system:

The attitude control technique presented in the last subsection is based on a cascade control system configuration. The first stage of the cascade control is the computation of desired angular velocity trajectories. The second stage is the determination of the required torques to generate angular velocities along the desired path. This cascade structure can be seen as a typical and general attitude control scheme for underactuated satellites.

It might be convenient to represent the typical block diagram valid for any attitude control technique with two control torques from pairs of thrusters. A general block diagram for a satellite’s 3-axis attitude control strategy with two control torques (from thrusters) can be represented as in figure (3.1). The block diagram represents a very general case, where the parameterisation, stabilising non-smooth control law, and angular velocity trajectory controller, can vary depending on the particular control strategy considered to solve the 3-axis control problem (The Tsiotras controller could be a particular case).

The purpose of the underactuated attitude controller is to drive the attitude to a desired reference, having three-axis stability. The desired reference to follow is typically zero attitude, but the nonzero reference tracking will also be proven in this thesis.

Using two pairs of thrusters, the basic principle is to bring the angular velocity along the actively controlled axes (X and Y) to a desired path. This desired path is designed according to nonlinear non-smooth control laws, such that the complete attitude is stabilized when we have a good tracking of that path.

The angular velocity trajectory controller, generally based on proportional (P) or proportional derivative (PD) control laws, is then used to make the actual angular velocities follow the desired path. This controller generates the redefined inputs $u_1$ and $u_2$ (used instead of the control torques to make the control design easier). The corresponding control torques are then computed via a simple change of variable.

When the convenient control torque is applied, the complete attitude is stabilized. All control laws (using thrusters) presented in this thesis are applied to the same block diagram of the complete system, which is given below:
3.1.3 The Kim nonlinear Controller: [Kim 2000]

The mathematical model of an underactuated satellite, controlled with two pairs of thrusters, can also be expressed using quaternion modelling for the formulation of the attitude kinematics ([Kim 2000]). In this case, the dynamic model is still given by equation (2.2). We have to recall here that the quaternion defines the spacecraft's attitude as an Euler-axis rotation from a reference frame that can be inertial or non-inertial. The quaternion has already been defined in chapter 2 as a complex vector with a scalar real part and a vector imaginary part.

By expressing the attitude of the satellite in the orbital frame, the kinematic equations for the quaternion are well known to be:

$$\dot{q} = \frac{1}{2} \Omega q$$  \hspace{1cm} (3.11)
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\[
\Omega = \begin{bmatrix}
0 & \omega_{30} & -\omega_{20} & \omega_{10} \\
-\omega_{30} & 0 & \omega_{10} & \omega_{20} \\
\omega_{20} & -\omega_{10} & 0 & \omega_{30} \\
-\omega_{10} & -\omega_{20} & -\omega_{30} & 0
\end{bmatrix}
\]  
(3.12)

where \( \omega_{10} \) denote the components of the body angular rate vector referenced to the inertial coordinates frame if \( \mathbf{q} \) is expressed in the inertial frame.

In fact, in the non-inertial reference case, we have:

\[
\begin{bmatrix}
\omega_{10} \\
\omega_{20} \\
\omega_{30}
\end{bmatrix} = \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} - A \begin{bmatrix}
0 \\
n(1 + 2e \cos(n(t))) \\
0
\end{bmatrix}
\]  
(3.13)

where \( n \) represents the mean orbital angular rate of the satellite, \( e \) represents the eccentricity of the orbit, and the matrix \( A \) transforms any vector from the inertial frame to the orbital frame.

In the following, we will assume an inertial reference frame for the sake of simplicity. We consider that the orbital and inertial referenced body angular velocities are equal in the control design \( (\omega_{10} \approx \omega_i) \) as the difference is in fact small during the implementation of underactuated control (ref [Kim 2000]).

**Stabilizing control law to the origin:**

In ref [Kim 2000], the author proposed following time invariant singular control laws for a spacecraft controlled with two pairs of thrusters:

\[
\omega_{1d} = -g_1q_1 + g_2 \frac{q_2q_3}{q_1^2 + q_2^2}
\]

\[
\omega_{2d} = -g_1q_2 - g_2 \frac{q_1q_3}{q_1^2 + q_2^2}
\]  
(3.14)

where \( g_1, g_2 \) are constant positive controller parameters.

This control law is defined on the manifold:
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\[ \Omega = \{ (q_1, q_2, q_3, q_4) \in \mathbb{R}^4 / q_1^2 + q_2^2 \neq 0 \} \] (3.15)

The global stability on the unactuated axis (and by conjecture on the other axes) occurs when the stabilizing controller parameters satisfy the condition:

\[ g_2 > 2g_1 > 0 \] (3.16)

The control law will in fact move the attitude (quaternions) from any initial condition \( q(0) \in \Omega \), to the final state: \((q_1, q_2, q_3, q_4) = (0,0,0,1)\).

In the control law (3.14), the state feedbacks \(-g_1 q_3, -g_1 q_4\) are used to stabilize the x and y axes attitude. The nonlinear singular interconnections terms are used to ensure the stabilization of the remaining unactuated axis.

For the stabilization of the complete system, the following VSC control law has been proposed in reference [Kim 2000], based on physical intuition and extensive numerical simulations, not on a formal stability proof of the complete system. This controller brings the angular velocity to the desired angular velocity commands. The equations of the angular velocity trajectory controller are:

\[ \begin{align*}
    u_1 &= -\mu_1 (\omega_1 - \omega_d) - \mu_3 \varepsilon \omega_2 s \\
    u_2 &= -(\mu_2 - \mu_1) (\omega_2 - \omega_{2d}) - \mu_3 \varepsilon \omega_3 s
\end{align*} \] (3.17)

With the sliding surface:

\[ s = \varepsilon \omega_1 \omega_2 + \mu_2 \omega_3 \] (3.18)

where \( s \) is the asymmetry parameter of the satellite.

The constant controller parameters must be chosen such that \( \mu_2 > \mu_1 > 0, \mu_3 > 0 \), and the redefined control inputs \( u_1, u_2 \) are simply related to the control torques by equations (2.15) of chapter 2.

There is one remarkable difference between the Kim approach and the Tsiotras approach. For the former, the control of the angular velocity \( \omega_3 \) is directly controlled with the inputs \( u_1 \) and \( u_2 \). In the latter case (Tsiotras case), \( \omega_3 \) is controlled by the feedback non-smooth angular velocity commands equations.
Reference following control law: [Kim 2000]

The control laws given by equations (3.14) and (3.17) can easily be generalized to a reference following control using quaternion modelling. Indeed, it will be shown that the differential equation of the quaternion error is the same as that of the quaternion.

The purpose of the reference following controller is to bring the attitude to the desired reference or commanded attitude vector:

\[ \mathbf{q}_e = (q_{1e}, q_{2e}, q_{3e}) \]  

(3.19)

To this end, we first define the error quaternion vector:

\[
\begin{bmatrix}
q_{1e} \\
q_{2e} \\
q_{3e} \\
q_{4e}
\end{bmatrix} =
\begin{bmatrix}
q_{1e} & q_{2e} & -q_{2e} & -q_{1e} \\
-q_{3e} & q_{4e} & q_{1e} & -q_{2e} \\
q_{2e} & -q_{1e} & q_{4e} & -q_{3e} \\
q_{3e} & q_{1e} & q_{2e} & q_{4e}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
\]  

(3.20)

In this case we use the previous control law to bring the vector part of the error quaternion to zero.

The angular velocity tracking control law is the same as (3.17):

\[
u_1 = -\mu_1 (\omega_1 - \omega_{1d}) - \mu_3 \omega_3 \omega_2 s
\]
\[
u_2 = - (\mu_3 - \mu_1) (\omega_2 - \omega_{2d}) - \mu_5 \omega_1 \omega_3 s
\]

(3.21)

and the new desired angular velocity command is given by:

\[ s = \varepsilon \omega_1 \omega_2 + \mu_2 \omega_3 \]  

(3.22)

\[
\omega_{1d} = -g_1 q_{1e} + g_2 \frac{q_{2e} q_{3e}}{q_{1e}^2 + q_{2e}^2}
\]
\[
\omega_{2d} = -g_2 q_{2e} - g_2 \frac{q_{1e} q_{3e}}{q_{1e}^2 + q_{2e}^2}
\]

(3.23)

And the kinematic equation using the quaternion error is:

\[ \dot{\mathbf{q}}_e = \frac{1}{2} \Omega \mathbf{q}_e \]  

(3.24)
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\[
\Omega = \begin{bmatrix}
0 & \omega_3 & -\omega_2 & \omega_1 \\
-\omega_3 & 0 & \omega_1 & \omega_2 \\
\omega_2 & -\omega_1 & 0 & \omega_3 \\
-\omega_1 & -\omega_2 & -\omega_3 & 0 \\
\end{bmatrix}
\]

3.1.4 Morin & Samson time varying controller [Morin 1997]

Rodriguez parameters are used here for the control. In this approach, we consider a rotation matrix \( A \) of angle \( \phi \in [-\pi, \pi[ \) about an axis which is defined by the unit vector \( a \), and the attitude vector is then given by:

\[
p = [p_1, p_2, p_3]^T = \tan(\phi/2) a
\]

(3.25)

The kinematic equation of a rigid body (satellite) using Rodriguez parameters is:

\[
\dot{p} = \frac{1}{2}(q_{33} + p^* + pp^T)\omega
\]

(3.26)

where \( p^* \) is an cross product matrix given by:

\[
p^* = \begin{bmatrix}
0 & -p_3 & p_2 \\
p_3 & 0 & -p_1 \\
-p_2 & p_1 & 0 \\
\end{bmatrix}
\]

(3.27)

The equation (3.26) can also be written for each component as:

\[
\begin{align*}
\dot{p}_1 &= \frac{1}{2}(\omega_1 - (p_3 - p_1 p_2)\omega_2 + (p_2 + p_1 p_3)\omega_3 + p_1^2 \omega_1) \\
\dot{p}_2 &= \frac{1}{2}(\omega_2 + (p_3 + p_1 p_2)\omega_1 - (p_1 - p_2 p_3)\omega_3 + p_2^2 \omega_2) \\
\dot{p}_3 &= \frac{1}{2}(\omega_3 - (p_2 - p_1 p_3)\omega_1 + (p_1 + p_2 p_3)\omega_2 + p_3^2 \omega_3)
\end{align*}
\]

(3.28)

and the dynamic equations have been written in [Morin1997] as:

\[
\begin{align*}
\dot{\omega}_1 &= a_1 \omega_2 \omega_3 + u'_1 \\
\dot{\omega}_2 &= a_2 \omega_2 \omega_3 + u'_2 \\
\dot{\omega}_3 &= \varepsilon \omega_1 \omega_2
\end{align*}
\]

(3.29)
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with: \( u'_1 = \frac{N_1}{I_1}, \quad u'_2 = \frac{N_2}{I_2} \) and \( a_1, a_2, c \) are asymmetry parameters defined after equation (2.15).

The objective carried out by Morin and Samson was to find a continuous time varying control law that exponentially stabilizes the system.

The control laws proposed by Morin and Samson in [Morin 1997] are:

\[
\begin{align*}
\omega_{1d}(p, \omega_3, t) &= -\eta_1 p_1 - \rho(p, \omega_3)\sin(t/\beta_1) \\
\omega_{2d}(p, \omega_3, t) &= -\eta_2 p_2 + \frac{1}{\rho(p, \omega_3)}(p_3 + \omega_3)\sin(t/\beta_1)
\end{align*}
\] (3.30)

where \( \eta_1, \eta_2 \) are positive constants, \( \rho \) is a homogenous norm of class \( C^4 \) on \( R^4 \)-\{0\}, associated with the dilation:

\( \delta_j^i(p, \omega_3) = (\lambda p_1, \lambda p_2, \lambda^2 p_3, \lambda^3 \omega_3, t) \) (see appendix D for some useful definitions about homogeneity, dilation parameter ...).

The homogenous norm \( \rho'_4(p, \omega_3) \) (of order 4) associated to the dilation \( \delta_j^i \) is:

\[ \rho'_4(p, \omega_3) = (p_1^4 + p_2^4 + p_3^4 + \omega_3^3)^{1/4} \] (3.31)

The stability proof provided by Morin and Samson was based on homogeneity theory. For a better understanding of the notions of dilation parameter, homogenous norm and homogenous functions properties, see appendix D (or reference [Morin 1997] for even more details).

We also have to consider a continuous control law to have \( \omega_i = \omega_{id} \), for \( i = 1, 2 \). In other words, the control torques are delivered to force the virtual control \( \omega_i \) to track the desired angular velocity trajectory.

This objective can be met with the following control torques:

\[
\begin{align*}
N_1(p, \omega, t) &= -\eta_1 I_1(\omega_1 - \omega_{1d}(p, \omega_3, t)) \\
N_2(p, \omega, t) &= -\eta_2 I_2(\omega_2 - \omega_{2d}(p, \omega_3, t))
\end{align*}
\] (3.32)

and we notice that this is simply the form of a proportional controller.
And it has been proven by Morin and Samson that: For any system equations, we can find positive parameters \( \eta_1, \eta_2, \eta_3, \eta_4 \) such that the feedback control laws (3.30) and (3.32) locally asymptotically and exponentially stabilize the origin of the system.

For this controller, we have the feedback terms \(-\eta_1 p_1, \eta_2 p_2\) to stabilize the attitude on the X and Y axes and the time varying terms use the interconnections to bring the Z axis attitude to zero.

### 3.2 Case of two reaction wheels

#### 3.2.1 Spin-axis stabilisation [Kim 2001]

Although the objective of the thesis is the 3-axis stabilisation, it is convenient to start with the more simple problem of spin-axis stabilisation with two reaction wheels, as already done in the case of thrusters.

The spin stabilisation using a smooth PD feedback has been shown using two pairs of thrusters. One first question is whether or not the same is possible using two reaction wheels. Intuitively, we can suggest that a similar approach to the case of thrusters must be applicable, but the difference is that the coupling between satellite's angular velocity and wheels speeds must be compensated in the expression of the controller.

A simple smooth control law for the spin-axis stabilisation with only two wheels has been proposed by S.Kim and Y.Kim in [Kim 2001] using the \((w,z)\) parameterisation.

The equations of the satellite on the actuated axes are:

\[
\begin{align*}
\dot{\omega}_1 &= \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_4}{2} \left(1 + w_1^2 - w_2^2 \right) \\
\dot{\omega}_2 &= -\omega_3 w_1 + \omega_2 w_1 w_2 + \frac{\omega_4}{2} \left(1 + w_2^2 - w_1^2 \right) \\
I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3 + \omega_3 h_2 + N_1 \\
I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_2 \omega_3 - \omega_3 h_1 + N_2 \\
I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_2 \omega_2 - \omega_1 h_2 + \omega_2 h_1
\end{align*}
\]
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We observe that if a control law brings the system to the state $(w_1, w_2, \omega_1, \omega_2) = (0,0,0,0)$ then the angular velocity $\omega_3$ will stop varying and be constant. In this case, the system will be stabilised about a revolute motion.

The control law proposed by Kim is:

\[ N_1 = -I_z \omega_2 \omega_3 - \omega_3 h_2 + k_1 w_1 + k_2 \omega_1 \]
\[ N_2 = I_1 \omega_1 \omega_3 + \omega_2 h_1 + k_1 w_2 + k_2 \omega_2 \] (3.34)

The control law (3.34) is based on a PD approach as in the case of thrusters, with an additional term for the compensation of the coupling between wheels momenta and satellites angular velocities (the compensation of the other term was already present in the case of thrusters because of equation (2.15) in chapter 2).

By considering exactly the same Lyapunov function as in (3.3), and by applying Lasalle’s principle (trajectories are bounded by considering constant total angular momentum), stability was proven in reference [Kim 2001]. In the same reference, it was shown by simulation that the control torque and wheels speeds were oscillating periodically to conserve the total angular momentum of the satellite.

The problem of spin-axis stabilisation is therefore fairly simple and a solved problem by considering the $(v,z)$ parameterisation. The problem of 3-axis stabilisation remains the most challenging because smooth feedbacks are no longer of any use in this case.

### 3.2.1 The zero total angular momentum satellite

The 3-axis stabilisation is not possible in a general non-restricted sense. However, under the restriction of a zero total momentum satellite, the now known possibility of 3-axis stabilisation will be demonstrated. In practice, a more flexible condition of small not necessarily zero total angular momentum at start-up will be sufficient.

The total momentum can be made small initially (if it is not already small) if we “disturb” the system by low cost magnetorquing in a first phase. When the “only two wheels” controller is activated, the total momentum will remain constant because the wheels torques (becoming sole control torques) are internal torques. Of course, this is true by
assuming a free environmental disturbance case for simplicity, but environmental disturbances are easily handled as explained later in this thesis).

We now express the zero total angular momentum condition: $\mathbf{H} = 0_{3 \times 1}$

The dynamic equations simply reduce to:

$$\dot{\mathbf{H}} = \mathbf{L} = \dot{\mathbf{H}} = \dot{\mathbf{L}} = \mathbf{0}$$  \hspace{1cm} (3.35)

In Appendix A, it is shown that the attitude kinematic equation using Rodriguez parameters (2.43) in the case ($\mathbf{H} = \mathbf{0}$) can be simplified to:

$$\dot{p}_3 = -p_2 \dot{p}_1 + p_1 \dot{p}_2$$ \hspace{1cm} (3.36)

It is then appropriate to introduce the new input variables $u_1 = \dot{p}_1$ and $u_2 = \dot{p}_2$ in order to obtain a reduced kinematic model as follows:

$$\dot{p}_1 = u_1$$
$$\dot{p}_2 = u_2$$
$$\dot{p}_3 = -p_2 u_1 + p_1 u_2$$ \hspace{1cm} (3.37)

Therefore, the attitude control problem simply turns to the problem of controlling the so-called Brockett integrator in equation (3.37).

It has also been shown in appendix A that the redefined control inputs are functions of the attitude and the wheels speeds commands.

The control of the attitude kinematics (or the control of the Brockett integrator) has involved some research in the recent literature (see ref [Fliess1991]). It has been shown (ref [Brockett 1983]) that this system cannot be stabilized using smooth continuous control laws.

As for the case of thrusters, we distinguish two control strategies proposed in the literature: nonlinear singular control laws (NLS) and continuous time varying control laws (CTV).
3.2.2 (Yamada, Yoshikawa, Yamaguchi) time varying control law

In ref [Yamada 1998], the following time varying control law has been proposed:

\[
\begin{align*}
    u_1 &= -k_T v p_1 + f_1(p_3) \cos \omega t \\
    u_2 &= -k_T v p_2 + f_2(p_3) \sin \omega t
\end{align*}
\]  \tag{3.38}

Where \( k > 0 \) is a control gain and \( f_1, f_2 \) are functions of \( p_3 \), which satisfy:
\( f_1(0) = 0, f_2(0) = 0 \).

The functions proposed in order to ensure exponential convergence of \( p_3 \) are:

\[
\begin{align*}
    f_1 &= \beta \text{sign}(p_3) \sqrt{|p_3|} \\
    f_2 &= -\beta \sqrt{|p_3|}
\end{align*}
\]  \tag{3.39}

where \( \beta \) is a positive constant control parameter.

This control law is the only 3-axis stabilising controller with two reaction wheels known to the author. In the following, it is convenient to give a stability proof in order to show that this technique can be seen (after a change of variable) as a feedback linearising strategy. Trigonometric time varying functions make it easy to transform a system into a more simple and linear equivalent model.

**Stability of the control law [Yamada 1998]**

We introduce the following new variable:

\[
q = \text{sign}(p_3) \sqrt{|p_3|}
\]  \tag{3.40}

Then, \( f_1 \) and \( f_2 \) in equation (3.39) are given by:

\[
\begin{align*}
    f_1 &= \beta q \\
    f_2 &= -\beta \text{sign}(q) q
\end{align*}
\]  \tag{3.41}

and the following equation holds when \( q \neq 0 \):

\[
\dot{p}_3 = 2q \dot{q}
\]  \tag{3.42}
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By substituting the last relations into equation (3.37) and by eliminating $p_3$, we obtain the following equation for $q \neq 0$:

\[
\begin{align*}
\dot{p}_1 &= -k_{TV} p_1 + \beta q \cos \omega t \\
\dot{p}_2 &= -k_{TV} p_2 - \beta \text{sign}(q) q \sin \omega t \\
\dot{q} &= -\frac{1}{2} \beta \text{sign}(q) p_1 \sin \omega t - \frac{1}{2} \beta p_2 \cos \omega t
\end{align*}
\]  
(3.43)

Following [Yamada 1998], let's introduce the following transformation:

\[
\begin{bmatrix}
    p_1 \\
    p_2
\end{bmatrix} =
\begin{bmatrix}
    \cos(\omega t) & \sin(\omega t) \\
    -\text{sign}(q) \sin(\omega t) & \text{sign}(q) \cos(\omega t)
\end{bmatrix}
\begin{bmatrix}
    r_1 \\
    r_2
\end{bmatrix}
\]  
(3.44)

By substituting the above equation (3.44) into equation (3.43), we have following linear differential equation:

\[
\begin{bmatrix}
    \dot{r}_1 \\
    \dot{r}_2 \\
    \dot{q}
\end{bmatrix} =
\begin{bmatrix}
    -k_{TV} & -\omega & \beta \\
    \omega & -k_{TV} & 0 \\
    0 & \frac{\beta}{2} & 0
\end{bmatrix}
\]  
(3.45)

The last transformed system turns out to be linear. Linear stability theory can therefore be applied in this case. If the eigenvalues are all real numbers, the system is always stable (because proof can be made that they would be negative) and the same conclusion can be made for Rodriguez parameters.

In the case when the system has got two complex eigenvalues and a real one, a more complex analysis done in reference [Yamada 1998] shows that the stability is subject to the following initial condition:

\[
\sqrt{|p_{30}|} > \frac{\beta (\omega p_{10} + k_{p_{20}})}{\omega^2 + k_{TV}^2}
\]  
(3.46)

And the stability using this time varying control depends not only on the controller parameters but also on the initial conditions.
Therefore, the stabilization that can be guaranteed in this case is local, not global. In this case, an interesting question is to determine if this condition is generally met at the initial time when the system switches from a fully actuated satellite (three actuators) to the two actuators control after the failure.

In this thesis, one objective is to propose a globally stabilising underactuated controller that can achieve slew manoeuvres from any initial attitude to any desired attitude.

3.3 Conclusions

In this section, non-smooth control strategies have been presented for the 3-axes attitude control of underactuated spacecrafts with two control torques (the easier problem of spin-axis stabilisation was solved with smooth feedbacks). In the case of thrusters, three different 3-axes stabilising approaches have been addressed, based on either singular or time varying feedbacks, and also based on three different parameterisations of the attitude. Using two reaction wheels, the satellite is only strictly controllable in the zero total momentum case. The only known 3-axis stabilising control law using two reaction wheels, based on a time varying approach, has also been presented. The development of a globally 3-axis stabilising control law using two reaction wheels will be addressed later in the thesis.

In both cases of thrusters and reaction wheels, the control strategies need to be analysed in detail under realistic assumptions (of fuel consumption, sampling, ...) . Practical problems such as the presence of a momentum bias, external disturbance torques, and others issues also need to be addressed.
Chapter 4

4 Practical analysis and contributions to underactuated attitude control

In this chapter, a first contribution to the current state of knowledge in underactuated attitude control is given in both cases of thrusters and reaction wheels. Before presenting the results, it is convenient to clearly describe what is meant by ‘contribution’ in this section. Apart from that, it is worth mentioning that the simulation study in this section is entirely original (simulations of cases and conditions never tested or simulated before with known or novel control laws of underactuated satellites).

Using pairs of thrusters, the contribution is not by the design of a totally novel control law (although practical changes such as formulating a saturated version of the Kim controller etc might be used). Three typical and elegant control laws for 3-axis stabilisation from the recent literature are investigated with an unprecedented complete and practical analysis. The undertaken analysis provides answers to rather crucial practical questions, which had not been addressed before in the case of underactuated satellites with thrusters.

Tsiotras had only tested and simulated his controller at the kinematic level (assuming that the generation of desired angular velocities trajectories was achievable with a PD control law). It will be demonstrated by analysis of the complete cascade control system (kinematic +dynamic model) that the angular velocity tracking causes a high control torque demand, which the profile will be analysed (tracking possible in theory but not practical). Morin and Kim had proven stability in theory by simulation using redefined control input trajectories. The redefined input trajectories will also be demonstrated to be equivalent to high control torques in the case of small satellites.

The case of axis-symmetric or fully symmetrical satellites had never been properly understood in the case of a non-zero angular velocity along the unactuated axis ($\omega_3 \neq 0$). Although the purely mathematical conditions of asymptotic stability are not satisfied in this critical case, it will be demonstrated using a standard 3-axis thruster underactuated
controller (Kim controller) that the practical impact of any residual initial angular velocity is a constant amplitude residual oscillation, performed around the desired attitude. The attitude can therefore be slewed to any desired orientation in this case with residual oscillations.

The case of small satellites with on/off thrusters had never been specifically addressed in any reference. It will be demonstrated that the fuel expenditure in this case is too high, due to the fact that angular velocity tracking requires permanent firing.

The effects of external disturbance torques (which cannot be compensated on the unactuated axis), had also never been investigated before in the case of unactuated satellites. This effect is also studied. The divergence under the effect of a worst-case disturbance torque is shown, and low cost magnetorquing proposed for the disturbance compensation.

Using reaction wheels, the performance of the only 3-axis stabilising controller from the literature, proposed in [Yamada 1998] and based on a time varying approach is assessed and compared to a singular approach. The model of UoSAT-12 is adopted for both controllers, which both designs are based on Rodriguez parameterisation. Singular control had never been envisaged to solve the “two wheels” control problem). It will also be explained that singular control can be seen as a way of applying feedback linearisation.

An original 3-axis stability proof is provided for a singular control law, designed for a zero total momentum satellite using the Rodriguez parameterisation (very convenient to simplify control design with two wheels). The singular control law has a familiar form compared to another singular controller such as the Tsiotras or Kim controller, but the difficulty is in finding the appropriate formulation or parameterisation of the design problem and to have a formal proof of the validity of singular control strategy using two wheels.

In other words, the difficulty is in transforming the complicated system, into a more simple system, stabilizable using singular control with redefined control inputs. Rodriguez parameters had been used in [Yamada 1998], but the control law designed using the same parameterisation in this thesis is clearly more effective than the Yamada time varying control law. The control design is very significantly simplified using Rodriguez
parameters, but the law relating redefined control inputs (virtual controls) to the control torques is proven to be much better by transforming the Rodriguez formulation of attitude kinematics (which is in fact a quaternion normalisation) into the more familiar quaternion formulation.

The performance using the singular control strategy for reaction wheels will be demonstrated to be impressive in both rapidity and precision in comparison to the known time varying approach. Using this approach, fast and decisive manoeuvres are possible on all three axes using two reaction wheels.

The performance of singular control with two wheels is near enough to perfection (from an engineering point of view) in the case of a zero total momentum satellite. Following the very good performance of the nonsmooth singular approach, it does not appear necessary to develop any more control laws for two wheels based on other theories, because it is unlikely that a significantly better performance is achievable with other techniques. It has then been decided that the best direction is to adopt the singular approach as the candidate for further implementation and adapt it to the most realistic assumptions (controller singularities, sampling, bias momentum, disturbance torques, etc...). Another important issue is to determine the cases when a previous phase (detumbling, momentum dumping or singularity avoidance) might be necessary.

The sampling, which had never been considered in the study of underactuated spacecrafts is also investigated, and the stabilisation with the requirement of a 10 seconds sampling is only achieved using singular control.

The case of a small but non-zero total angular momentum satellite (small momentum bias), which had never been investigated using two wheels is analysed with a formulation of the kinematics in Rodriguez parameters. An encouraging result is that the desired attitude is still obtained with small residual constant amplitude oscillations about it, for a small momentum bias (that can be guaranteed after detumbling if the previous phase was bias momentum control).

The effect of an external disturbance torque on the 'two wheels' control system (zero total momentum for simplicity) will also be analysed. The attitude divergence in this case is
very slow (less than a degree per orbit) and low cost magnetorquing might only be necessary for long time stabilisation applications.

The practical achievement of attitude control in two phases for bias momentum satellites (Detumbling + stabilisation), is another issue that had never been addressed before. The influence of sensor noises has also never been considered. However, these two problems are left to the last two chapters of the thesis, which are more practically oriented towards in-orbit applications.

4.1 Case of Two Pairs of Thrusters

4.1.1 Simulations of the Controllers Presented in the Asymmetric Case:

In this sub-section, three 3-axis stabilising control laws (based on angular velocity virtual inputs) proposed by Tsiotras, Morin and Kim are analysed in a complete cascade satellite model (kinematic + dynamic model). The objective is to assess the required thrusters control torques, study the possibility of on/off thrusters, and other practical issues on a standard small micro-satellite model.

Unless specified elsewhere, the micro-satellite parameters used for the study and the simulations using two pairs of thrusters are:

\[ I_1 = 1.5 \text{kg.m}^2, I_2 = 1.3 \text{kg.m}^2, I_3 = 1 \text{kg.m}^2 \] : Let’s call this small satellite used for the study **microsat1**.

\[ w_1(0) = -1, w_2(0) = -0.2, \omega_3(0) = 10^{-4} \text{rad.s}^{-1}, z(0) = 0.7 \text{rad} \]

\[ \omega_1(0) = 0.01 \text{rad.s}^{-1}, \omega_2(0) = 0.01 \text{rad.s}^{-1} \]

The initial attitude converted into Euler angles will be: \( \phi_0 = 10^\circ, \theta_0 = -23^\circ, \psi_0 = 41^\circ \)

The Topsat model \( (I_1 = 10 \text{kg.m}^2, I_2 = 9.5 \text{kg.m}^2, I_3 = 9.1 \text{kg.m}^2) \) will be used in only one simulation with the same parameters to show the results on a larger satellite.

We will use the same initial condition for all the controllers (for comparison purposes).

The simulations in chapters 4 and 5 will be generated using Matlab 6.1.
Chapter 4. Practical analysis and contributions to underactuated attitude control

Full attitude controllers simulations:

Simulation results show the stabilisation for a torque saturation at 5 mN.m (thrusters torque capability of Sunsat which had almost similar moments of inertia to those considered here). For the controller parameters (determined empirically) see table (4.1).

- Tsiotras controller:

The simulations are first carried out without any torque saturation (see Figure (4.1)). In this case the stabilisation is rapidly achieved with transient oscillations, (about 20° for the roll, pitch angles and 40° for the yaw) so we have a good tracking far from the singularities. However the control torque reaches 1 N.m, which cannot be supported by the available thrusters on the microsatellite considered. The control torque was high because of the extensive torque demand for the tracking of the desired angular velocity trajectories.

Using a torque saturation of 5 milli N.m, the transient oscillations are higher (we reach about 80° for the roll and pitch angles), and we are closer to the singularities (not too far from the 90°singularity on the pitch axis). However, we still have a precise tracking rapidly (after 500 seconds). The torque is first oscillating (between the saturation levels) but starts fading towards zero after more than 1200 seconds. (see Figure 4.2)

- The (Sungpil Kim- Youdan Kim) controller:

Control rapidity in this case is not as good as using the Tsiotras controller . It takes less than 1000 seconds to reach a good tracking precision.

The torque also oscillates between the saturation levels, but starts fading to zero after less than 1000 seconds (faster than using the Tsiotras controller). For the controller parameters see table 1. A torque saturation at 5 $10^{-3}$ N.m has been considered as well. (see Figure 4.3).

Reference following control:

The reference following control is also achieved (see figure 4.4) for:

initial condition: $(q_{10}, q_{20}, q_{30}, q_{40}) = (-0.1, 0, 0, 0.99)$

Commanded (desired) attitude quaternion: $(q_{1c}, q_{2c}, q_{3c}, q_{4c}) = (0.3, 0.3, 0.3, 0.8544)$
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- The Morin & Samson controller:

In figure (4.5), figure (4.6), we have an exponential stabilization of the attitude to zero for the case study microsat1 (torque saturation at 0.005 N.m), as well as for Topsat ($I_1 = 10\text{kg.m}^2, I_2 = 9.5\text{kg.m}^2, I_3 = 9.1\text{kg.m}^2$, and torque saturation at 0.03 N.m). The stabilisation takes longer for Topsat, which has got higher inertia parameters than microsat1. The torque starts fading towards zero later as well in the case of Topsat. The stabilisation for Topsat has been achieved using the same controller parameters but the saturation level is higher (30 mNm for Topsat data from SSTL’s Topsat datasheet, 5mNm for microsat1). (Controller parameters in table 4.1.)

The precision of the different controllers:

The root mean square error (RMS), is generally computed when the steady state is reached. We calculate this criterion during the second orbit for the variable $x$ (that can be either roll, pitch or yaw angle):

$$\text{RMS}(x) = \sqrt{\text{mean}(x)^2 + \text{std}(x)^2}$$  \hspace{1cm} (4.1)

The RMS criterion is used to evaluate the precision of the controller used. We notice that the CTVC (Morin& Samson [Morin 1997]) control law is the most precise in the steady state. The comparison between the NLSC controllers shows that the Kim controller is more precise than the Tsiotras controller in the steady state (See table 4.2). Of course, the precision in practice will never be in this range (effects of noises, disturbance torques, ...) but all controllers are compared under the same assumption of a perfect disturbance free case.

The RMS criterion is convenient for the comparison at steady state, but in the transient state, we can see that the Kim controller is the best (no high frequency oscillations, and not as close to the singularity as the Tsiotras controller). 

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Figure 4.1: Tsiotras attitude controller without torque saturation. Good tracking after only 500 sec but the control torque produced is too high.

Figure 4.2: Tsiotras attitude controller for micorosat1, torque saturation at 5 milli Nm

Figure 4.3: Kim singular controller for micorosat1, torque saturation at 5 milli Nm.
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Figure 4.4: Reference following control using the Kim singular controller, Torque saturation at 5 milli Nm, \( (q_{1c}, q_{2c}, q_{3c}, q_{4c}) = (0.3, 0.3, 0.3, 0.8544) \)

Figure 4.5: Morin and Samson time varying controller for microsat1, torque saturation at 5 milli Nm

Figure 4.6: Morin and Samson time varying controller for Topsat, torque saturation at 30 milli Nm
Chapter 4. Practical analysis and contributions to underactuated attitude control

<table>
<thead>
<tr>
<th>STABILIZING CONTROLLER</th>
<th>CONTROLLER PARAMETERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tsiotras controller</td>
<td>$k_x = \frac{3}{8} s, k_z = 1 s^{-1}, k_w = 0.1 \text{rad.s}^{-1}, \gamma = 10 \text{s}^{-1}$, controller saturation at $\pm 0.2$ (avoid singularity)</td>
</tr>
<tr>
<td>Kim controller</td>
<td>$(g_1, g_2) = (0.04, 0.4) \text{rd.s}^{-1}, (\mu_1, \mu_2, \mu_3) = (2, 20, 20) \text{s}^{-1}$, controller saturation at $\pm 0.2$ (avoid singularity)</td>
</tr>
<tr>
<td>Morin &amp; Samson controller</td>
<td>$(\eta_1, \eta_2) = (1, 1) \text{rad.s}^{-1}, \beta = 2, (\eta_3, \eta_4) = (2, 2) \text{s}^{-1}$</td>
</tr>
</tbody>
</table>

Table 4.1: Controller parameters used for the simulations (determined empirically)

<table>
<thead>
<tr>
<th>CONTROLLER EULER ANGLES</th>
<th>TSIOTRAS</th>
<th>KIM</th>
<th>MORIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS(ROLL)</td>
<td>0.0041</td>
<td>7.6857 $10^{-7}$</td>
<td>3.2521 $10^{-10}$</td>
</tr>
<tr>
<td>RMS(PITCH)</td>
<td>0.0042</td>
<td>7.4228 $10^{-7}$</td>
<td>2.9794 $10^{-10}$</td>
</tr>
<tr>
<td>RMS(YAW)</td>
<td>1.777 $10^{-6}$</td>
<td>7.2358 $10^{-14}$</td>
<td>3.0402 $10^{-20}$</td>
</tr>
</tbody>
</table>

Table 4.2: RMS criterion for the different underactuated controllers using two pairs of thrusters

4.1.2 Problem of the torque availability and the torque saturation effect

The control torque directly computed using the non-smooth control laws proposed by Tsiotras, Kim and Morin has been proven by their authors to ensure the stabilisation to the origin, assuming that angular velocity trajectories were generated (without any study of the torque demand of such angular velocity tracking).

However, the torque that can be produced by thrusters is limited. If the torque is not saturated, the controller then demands very high control torque values that cannot be
provided by the available thrusters (see simulation in Figure (4.1)). A controller allowing for a torque saturation is then needed.

Constrained by a torque saturation, the stabilisation of the attitude is slower. When the value of the saturation level decreases (in absolute value), the convergence to zero is slower. This fact is illustrated in Figure (4.7) where the effect of the saturation level on the yaw angle using the Tsiostras controller is shown.

We cannot decrease the torque saturation levels arbitrarily. If the saturation levels considered are too small, we then lose the characteristics of the controllers and undesired oscillations will occur. In the example below, if we saturate the torque under 1mN\text{m} for instance the yaw angle will diverge and the system will not stabilise.

Using a torque saturation at 5 mN\text{m} (black line on figure 4.7) we still have the stabilization after approximately 500 seconds. If a bigger saturation level of 8 mN\text{m} (red line on figure 4.7) is considered, the stabilisation is obtained even faster (400 seconds). The fastest convergence is with 20 mN\text{m} saturation (blue line on figure 4.7). The stabilisation is faster if the actuator provides more torque.

![Figure 4.7: Effect of the torque saturation (on the yaw angle) using the Tsiotras controller](image-url)
4.1.3 Investigation of the symmetrical satellite case

If \( \omega_3(0) = 0 \), the stabilization of the symmetrical satellite is carried out using the same techniques used for the asymmetric satellites. The control design in this case is simple using the Tsiotras controller (we simply choose \( k_\omega = 0 \) in equation (3.9)).

However, this restriction is never strictly met in practice before applying the stabilizing controller. Using a detumbling maneuver, we can only bring the angular velocity close to zero but the initial small nonzero value of the angular velocity has an impact on the stabilization that must be studied. We then have to consider the general case when the initial angular velocity along the unactuated axis is not zero.

- if \( \omega_3(0) \neq 0 \):

In this case, the dynamic equations of the satellite are:

\[
\begin{align*}
\dot{\omega}_1 &= u_1 \\
\dot{\omega}_2 &= u_2 \\
\dot{\omega}_3 &= 0 \Rightarrow \omega_3 = \omega_{30} = \text{constant} \neq 0
\end{align*}
\]  

and the kinematic attitude equations are:

\[
\begin{align*}
\dot{w}_1 &= \omega_{30} w_2 + \omega_2 w_1 w_2 + \frac{\omega_3}{2} \left( 1 + w_1^2 - w_2^2 \right) \\
\dot{w}_2 &= -\omega_{30} w_1 + \omega_1 w_1 w_2 + \frac{\omega_3}{2} \left( 1 + w_2^2 - w_1^2 \right) \\
\dot{\gamma} &= \omega_{30} - \omega_1 w_2 + \omega_2 w_1
\end{align*}
\]  

As we are in the case of an axis symmetrical satellite, it becomes impossible to bring the system to the equilibrium manifold:

\[
D = \{ (\omega_1, \omega_2, \omega_3, w_1, w_2, z) \in IR^6 / \omega_1 = \omega_2 = \omega_3 = w_1 = w_2 = z = 0 \}
\]  

The stabilisation to the manifold \( D \) is impossible only when we have a perfect symmetry, which is never the case in practice since there is always at least a small asymmetry operator \( \varepsilon \).
We can however (even for a perfectly symmetrical satellite) have a partial stabilization of the system to the submanifold (as seen in section 3.1.3):

\[ D' = \{(\omega_1, \omega_2, \omega_3, w_1, w_2, z) \in IR^6 / \omega_1 = \omega_2 = w_1 = w_2 = 0\} \]

In other words, it is possible to control the orientation of the Z axis of the satellite (yaw axis which is unactuated). So we can in this case ensure a reorientation of the symmetry axis of the satellite to a desired position (Z axis pointing).

We can even do better in this case by controlling five states. The five states will not be defined as usual with respect to the fixed inertial frame but with respect to the rotating frame with a constant velocity \( \omega_{20} \) in the Z axis.

In reference [Walsh 1995], it was proven that stability of 5 out of the 6 degrees of freedom of the system, (2 angular velocities and full attitude defined in a new frame) can still be controlled to zero in a new reference frame, rotating at the constant angular velocity \( \omega_{20} \). The notion of stability can therefore be seen as a relative notion since it depends on the adopted reference frame.

In this section, we provide a proof, based on numerical simulations, that the effect of a small initial angular velocity about the Z-axis of an axisymmetric will cause constant oscillations about the desired reference. In other words, using any underactuated controller designed for the case of a zero angular velocity about the Z-axis, the desired attitude will still be obtained, without any significant loss in rapidity, but with a small residual oscillation of constant amplitude about the desired reference. Therefore, after detumbling about the unactuated axis, it becomes possible to bring the attitude on that axis to any desired orientation with small residual oscillations about it.

**Case of a small non zero initial angular velocity on the Z axis**

In fact, the small nonzero value of \( |\omega_{30}| \) will involve a constant oscillation about the equilibrium. The stabilization is achieved about this constantly and slowly rotating frame. As a consequence, we can have the stabilization of the complete attitude into a small neighbourhood around zero (or about the reference angle if a reference following controller is used).
Numerical simulation for a symmetrical satellite:

Using the Kim controller with different initial conditions for $|\omega_{30}|$, simulations are given in figure (4.8), figure (4.9) and figure (4.10) for $I_1 = I_2 = I_3 = 1.5 \text{ kg.m}^2$. We notice constant oscillations at the steady state. The amplitude of the oscillations about the reference to follow (zero in this example), becomes smaller when $|\omega_{30}|$ decreases. The amplitude of the oscillations in the torque also decreases in this case.

One surprising result is that the transient oscillations are of small magnitude compared to the asymmetric case.

The value of $\omega_{30}$ depends on the effectiveness of the controller used for the detumbling maneuver. In practice, it is then possible to bring the symmetrical satellite into a desired reference with only a small remaining constant oscillation, even when $\omega_{30} \neq 0$. We then have a stabilisation into a small neighbourhood around the desired reference (The stabilisation into one equilibrium point is impossible in this case).

\[ \text{Figure 4.8: Simulations (Attitude, Torque), } \omega_{30}(0) = 0.01, I_1 = 1.5, I_2 = 1.5, I_3 = 1.5 \]
4.1.4 Effect of the external disturbance torques:

In the realistic conditions, the satellite is subject to both modelled and unknown unmodelled disturbance torques. When three control torques are used, the known modelled disturbance torques are simply compensated in the expression of the control torques. The unmodelled disturbances also affect the stabilization but very slightly because of the tiny value of their magnitude, and they are consequently neglected.
One particular problem occurs when one actuator fails. In this case, the modelled disturbance on the unactuated Z axis cannot be compensated. The modelled disturbance torques are in general due to the aerodynamic drag, solar radiation and gravity gradient torque. In particular, the gravity gradient torque is only significant in the case of asymmetric satellites.

The modelled disturbance torque is generally time varying (depends on the position in the orbit). However, for the sake of simplicity in this study of the robustness, we only consider constant disturbances (biases) on each axis of the satellite.

As far as the disturbance on the Z axis can be modelled (known), another source of torque can be used to compensate the known disturbance torque on the unactuated Z axis. Numerical simulations have been carried out to evaluate the effect of a non-compensated external disturbance torque on the Z unactuated axis. The value of the disturbance torques on the unactuated axis, used for the simulations is the constant bias: $10^6$ N.m.

Using two pairs of thrusters, the undesired effect of a disturbance torque will be shown to be more important in the case of a symmetrical satellite. In the asymmetric satellite case, the attitude is still included in a small neighbourhood of the origin (desired reference) after one orbit (Figure 4.11), whereas for the symmetrical satellite, we face a relatively fast divergence (Figure 4.12).

![Figure 4.11: Simulations of the attitude and corresponding torque if $T_d = 10^6$ N.m, Asymmetric satellite: $I_1 = 1.5, I_2 = 1.3, I_3 = 1$ (Kim controller)](image)
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Figure 4.12: Simulations of the attitude and corresponding torque if $T_d=10^{-6}$ N.m, symmetrical satellite: $I_1 = I_2 = I_3 = 1.5 \text{kg.m}^2$ (Kim controller)

Necessity of a third control torque for the disturbance compensation:

From the above robustness study, there is a need for an additional source of control torque to compensate the modelled (known) disturbance torque on the unactuated axis, especially for a symmetrical satellite (fast divergence without compensation).

For the asymmetric satellite, this third torque would improve the results, but is not strictly required for realistic disturbances except for very long term applications (slow divergence without compensation).

4.1.5 Effect of the sampling and the PWM (On/Off thrusters)

When On/Off thrusters are used, we must generally consider a constant sampling time. Let's try to investigate what happens to the fuel consumption and the attitude stabilisation in this case.

A numerical simulation of the LEO satellite (microsat1) using the Tsiotras controller with a torque saturation at 5 milliNm and a sampling time of 1 sec is given below in figure (4.13). We notice that the control torque does not fade, but keeps oscillating at a high frequency between the maximum and minimum allowed values for the torque. So, if on/off thrusters are used this way for a long time, the fuel consumption will be too high.
However, the attitude still converges to a small neighbourhood around zero. Therefore, using the PWM, the torque expenditure and the fuel consumption are increased but we still have the stabilization into a small neighbourhood around zero. It appears in figure (4.13) that using the Tsiotras controller, a major problem is presented by the fuel expenditure. Similar observations can be made using the other controllers using thrusters. This observation means that using thrusters with controllers published in the literature to date for underactuated control, the fuel consumption would be unacceptable.

![Figure 4.13: Simulation of the effect of the PWM on the underactuated control system (T_c=1sec)](4-17)

### 4.1.6 Conclusions

The three-axis stabilization (with tracking of a desired reference) using two pairs of thrusters has been demonstrated for an asymmetric microsatellite first. The stabilization into a single equilibrium in this case is then possible using realistic torque saturation levels.

For a symmetrical satellite, we still have the stabilization into a small neighbourhood of the desired reference. The remaining problem is still with the torque that must be generated for the stabilization. The torque still ensures the stabilization into a small neighbourhood of the desired reference (zero in most simulations) using the PWM.

The computed torque is limited but the fuel expenditure is too high (at a high frequency). Indeed, we notice in Figure (4.13) that torque is no longer fading but keeps oscillating. As a consequence, the underactuated attitude control using thrusters seems not to be
practically feasible in the presence of a PWM and assuming a realistic torque limitation. The underactuated control system using thrusters is however convenient for a theoretical investigation of the underactuated satellite motion. Therefore, it is appropriate to have a look into other actuators for the stabilisation with only two control torques, namely momentum exchange devices and particularly reaction wheels.

4.2 Case of two reaction wheels

The case $H = 0_{3x1}$ (zero total angular momentum) has been investigated in chapter 2 and a time varying controller has been proposed to solve this problem using the Rodriguez parameterisation. In this section, a novel control strategy based on the same attitude parameterisation, will be presented for the case of a two wheels controller. The novel approach, proposed to achieve 3-axis control with two wheels, is based on nonlinear singular control theory. The basic principle is to use a mathematically singular control law, to ensure convergence towards the singular region. Another remaining problem is to ensure a similar stabilisation when $H \neq 0_{3x1}$ (with $|H|$ small) using the proposed control laws. A study of the effect of a non-zero total momentum will also be investigated in this chapter.

4.2.1. Novel Nonlinear singular controller with two wheels

The controller is designed for a zero total momentum satellite, unactuated on the Z-axis. We need to control the system having derived the appropriate kinematics from (3.37). The nonlinear singular controller presented here is based on advanced nonlinear control theory.

Singular control has already been proposed in order to stabilize nonlinear and nonholonomic systems such as underactuated satellites using pairs of thrusters [Tsiotras 2000a]. Our parameterisation of the problem here in the two reaction wheels case is however different to simplify control design and stability analysis.

For convenience, we recall the kinematic equation of the zero total momentum satellite (see appendix B for proof), expressed in Rodriguez parameters:
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\[ \begin{align*}
\dot{p}_1 &= u_1 \\
\dot{p}_2 &= u_2 \\
\dot{p}_3 &= -p_2 u_1 + p_1 u_2
\end{align*} \]  \hfill (4.7)

For our two reactions wheels control system, we want to design a novel controller that first stabilizes the unactuated Z-axis.

\[ \begin{align*}
\dot{p}_3 &= -p_2 u_1 + p_1 u_2 \\
&= -g p_3
\end{align*} \]  \hfill (4.8)

where \( g \) is a positive constant of the controller.

The relation (4.8) is ensured using the control law:

\[ \begin{align*}
u_1 &= -k p_1 + g \frac{p_2 p_3}{p_1^2 + p_2^2} \\
u_2 &= -k p_2 - g \frac{p_1 p_3}{p_1^2 + p_2^2}
\end{align*} \]  \hfill (4.9)

where \( k \) is another positive constant of the controller.

Using physical intuition, we can justify this form of the control law by the superposition of a continuous term in the feedbacks \((-k p_1, -k p_2)\) used to stabilize the attitude on the actuated axes (X and Y), and interconnection singular terms used to complete the stabilization of the unactuated axis (Z-axis).

This control law is singular when \( p_1 = p_2 = 0 \).

We have now to show that the above control law, not only stabilizes the unactuated Z-axis, but also brings the attitude along the X and Y axes to zero.

By substituting the control law from equation (4.9) into equation (4.7), the resulting closed loop control system is:

\[ \begin{align*}
\dot{p}_1 &= -k p_1 + g \frac{p_2}{p_1^2 + p_2^2} p_3 \\
\dot{p}_2 &= -k p_2 - g \frac{p_1}{p_1^2 + p_2^2} p_3 \\
\dot{p}_3 &= -g p_3
\end{align*} \]  \hfill (4.10)
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The asymptotic stability of the variable $p_3$ is trivial. The convergence of $p_3$ to zero is also trivial.

Before we can check the stability for the two remaining variables, we notice that any trajectory starting in $\Omega = \{(p_1, p_2, p_3) \in \mathbb{R}^3 / p_1^2 + p_2^2 \neq 0\}$ will remain defined in $\Omega$.

Indeed if we consider a variable: $V_p = p_1^2 + p_2^2$, simple calculations yield:

$$\dot{V}_p = -k V_p \quad (4.11)$$

Therefore, the manifold defined by: $D = \{(p_1, p_2, p_3) \in \mathbb{R}^3 / p_1^2 + p_2^2 = 0\}$, is also exponentially attractive and stability is consequently ensured for the complete attitude to the zero equilibrium point.

**Remark**

From equations (4.11) and (4.8), the singular control approach is can be seen as a feedback linearization approach. The singular feedbacks have been chosen to have the linear exponentially and asymptotically stable system formed by equations (4.8) and (4.11). In fact, by solving equations (4.8) and (4.11) for $u_1, u_2$ with a model as in (4.7), the unique solution is equation (4.9).

It is also worth mentioning that if the failure happens on the X axis for instance, then $u_1$ and $u_2$ are replaced by $u_3$ and $u_3$ respectively and $p_1, p_2, p_3$ are replaced by $p_2, p_3, p_1$ respectively in equation (4.9). The case of a failure on the Y axis is also dealt with by replacing $u_1$ by $u_3$ and $u_2$ by $u_1$ and by the permutation of $p_1, p_2, p_3$ into $p_3, p_1, p_2$ respectively in equation (4.9). It is therefore easy by simple permutation to rewrite the control laws for a failure on any axis other than the Z axis. Hence, a fully redundant configuration using any two of three orthogonal reaction wheels is possible.

**Reference following control**

We can bring the system to any other desired attitude just by replacing the attitude by the attitude error in the expression of the controller.
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Singularity problem

However, singularities may happen in practice when \( p_1, p_2 \) are simultaneously too small, we then have to consider a saturated version of the controller as follows:

\[
\begin{align*}
\dot{u}_1 &= -kp_1 + g \text{sat}\left( \frac{p_2}{p_1^2 + p_2^2} \right) \\
\dot{u}_2 &= -kp_2 + g \text{sat}\left( \frac{p_1}{p_1^2 + p_2^2} \right)
\end{align*}
\]

(4.12)

where the saturation function is given by:

\[
\text{sat}(x) = \begin{cases} 
  x & -a < x < a \\
  a & x \geq a \\
  -a & x \leq -a
\end{cases}
\]

(4.13)

Non-zero momentum case

The previous theory applies exactly to the case of zero momentum. A more practical and significant case to study is the one of a small nonzero momentum. The effect of a small non-zero total angular momentum needs to be studied by simulations. The required modification in this case to the equation (4.7) is:

\[
\begin{align*}
\dot{p}_1 - h'_1 &= u_1 \\
\dot{p}_2 - h'_2 &= u_2 \\
\dot{p}_3 - h'_3 &= -p_2 u_1 + p_1 u_2
\end{align*}
\]

(4.14)

where it can be shown that:

\[
h' = [h'_1, h'_2, h'_3]^T
\]

\[
= \frac{1}{2}(I_{3x3} + p^x + pp^T)(I_{3x3} + I_{w1} + I_{w2})^{-1} A_w^1 H
\]

(4.15)

where the total angular momentum \( H \) is a constant vector (in the disturbance free case), \( h' \) is just a function of the total momentum, which components intervene in the kinematic model (not to be confused with satellite’s momentum or wheel’s momentum in any reference frame ...)
For the computation of the speed commands when \( u_1, u_2 \) are given by a control law, refer to appendix B (equations B.3, B.4 and B.14). We can show via simulations that the controller designed for a zero total angular momentum satellite still brings the system to a neighbourhood of the zero reference to follow, provided that the momentum of the satellite can be made small enough initially.

**Reformulation of the control laws using quaternion modelling**

The Rodriguez parameters are very convenient for the design and stability analysis of underactuated attitude controllers using two wheels. However, the transition to quaternions is required in practice because the estimated attitude from the Kalman filter is in quaternion parameters. The Rodriguez parameters are given by the normalisation of the quaternions: \( p_i = q_i / q_4 \). Problems happen in practice when \( q_4 = 0 \). That is why the control laws using the Rodriguez parameterization can cause singularity problems in practice.

The transition between the redefined control inputs \( u_1 \) and \( u_2 \) and the wheels speeds commands is computationally heavy, and also subject to possible singularities due to the Rodriguez parameterization. Therefore, the control laws need to be rewritten using quaternion modelling for practical implementation.

The model based on Rodriguez parameters (based on original proof) is used for all the simulations of the “two wheels” control of chapter 4, but a reformulation into quaternion modelling is adopted in chapters 5 and 6 (where attitude determination is taken into account). Therefore, we want to generalise the original proof made using Rodriguez parameters into the case of quaternions. This is also part of the novel contribution made in the thesis.

For a zero total momentum satellite, we have \( \omega = 0 \) (for a diagonal inertia matrix as in the case of UoSAT-12), because of the wheel failure along that axis. On the X and Y axes, we simply have: \( h_1 = -I_1 \omega_1 \), \( h_2 = -I_2 \omega_2 \), which means that the momentum of the satellite without wheels is equal to the momentum of the wheels.

For a zero total momentum, \( \omega = 0 \) and the kinematic equation in quaternion parameters is given by:
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\[
\begin{align*}
\dot{q}_1 &= \frac{1}{2} q_4 \omega_1 - \frac{1}{2} q_3 \omega_2 \\
\dot{q}_2 &= \frac{1}{2} q_3 \omega_1 + \frac{1}{2} q_4 \omega_2 \\
\dot{q}_3 &= -\frac{1}{2} q_2 \omega_1 + \frac{1}{2} q_1 \omega_2 \\
\dot{q}_4 &= -\frac{1}{2} q_1 \omega_1 - \frac{1}{2} q_2 \omega_2 
\end{align*}
\] (4.16)

Strictly speaking, the angular velocities in equation (4.16) should be expressed in the local orbital frame, which is given by:

\[
\omega_o = \omega - A n
\]

Where \(A\) is the transformation matrix from inertial to orbital frame, and \(n\) represents the mean orbital angular rate of the satellite.

In fact, the elements of the matrix \(A\) are all smaller than 1 at any time. Therefore, we also have: \(|\omega_t - \omega_{o_t}| \leq n (\approx 0.001 \text{ rad/sec}) \) \(t=1,2,3\). The difference between the local orbital and inertially referenced angular velocity is smaller than \(n\) on each axis at all times. This difference is also small in comparison to the angular velocity on each axis during the underactuated control manoeuvre. In other words, we can replace the local orbital angular velocity by the inertially referenced angular velocity in the control design (To simplify the stability analysis as done in several references dealing with the problem). The control design in this case is significantly simplified.

It has been checked by simulations that the effect of the orbital rate on the control laws has no significant impact during the implementation of the underactuated control. To control the attitude in the orbital frame, (which is the required in practice for nadir pointing) we simply need to use the orbit-referenced quaternions in the feedbacks (instead of the inertially-referenced quaternions adopted for the stability proof) without explicitly using \(n\).

Therefore, the control law has been designed without the effect of \(n\) and has been successfully applied to the true satellite’s model, which includes the effect of \(n\) in the attitude propagation. In fact, the control system appears to be robust to this intentional
model error assumed in the control design and stability analysis.

It can be shown that a quaternion control law similar to the form of equation (4.12), will stabilise the system (4.16) and bring it to the equilibrium.

Using physical intuition (from the results derived using Rodriguez parameters, which are just a normalisation of the quaternions), we can try to have a formal proof of the following expression for the singular controller in quaternion formulation:

\[
\begin{align*}
\dot{q}_1 &= -kq_1 + g \frac{q_2 q_3}{q_1^2 + q_2^2} \\
\dot{q}_2 &= -kq_2 - g \frac{q_1 q_3}{q_1^2 + q_2^2}
\end{align*}
\]  

(4.17)

where \( k, g \) are positive constant control parameters (to be determined empirically for best performance).

By replacing \( \dot{q}_1 \) and \( \dot{q}_2 \) from equation (4.17), into the kinematic model of the underactuated zero total momentum satellite in quaternion (equation (4.16)), after straightforward calculations, we obtain:

\[ \dot{q}_3 = -\frac{g}{2} q_3 \]  

(4.18)

Therefore, the stability along the unactuated Z axis is guaranteed (result known by Kim in the case of thrusters).

The convergence of \( q_1^2 + q_2^2 \) to zero (which implies the convergence of \( q_1 \) and \( q_2 \) to zero) has been conjectured (by Kim in the case of thrusters, which differs from the case of wheels in some respects) but never formally proven using quaternion modelling. The original proof of singular control has been made earlier in this thesis but Rodriguez parameters were adopted. Developing a formal mathematical proof of the convergence on the actuated axes is more complex using quaternion modelling than using the Rodriguez parameterisation. However, the following (original) analysis will demonstrate this stability property even using quaternion modelling.
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The same singularity issues, as in the case of the Tsiotras 3-axis underactuated controller of chapter 3, are still present if \( q_1 \) and \( q_2 \) converge to zero before \( q_3 \). In this case the control torques would be too high. The only way of singularity avoidance is by ensuring that the convergence of \( q_3 \) to zero precedes the convergence of \( q_1 \) and \( q_2 \) to zero.

We provide a mathematical proof that singularity problems can in fact be avoided if a convenient choice of the control parameters is made. In other words, there is a mathematical condition on the control parameters, which makes the underactuated control possible without a large torque consumption. This is due to the fact that the numerator of the nonlinear term of the control law (4.17) converges to zero before the denominator.

To make our proof, we will exploit the fact that \( q_4(t) \) is a monotonous increasing function. Indeed, by replacing the angular velocity components \( \omega_1, \omega_2 \) by their expression from the control law equation (4.17), we have:

\[
\begin{align*}
\dot{q}_4 &= \frac{1}{2} q_1 \omega_1 - \frac{1}{2} q_2 \omega_2 \\
\Rightarrow \dot{q}_4 &= \frac{1}{2} q_1 \left( -k q_1 + g \frac{q_2 q_3}{q_1^2 + q_2^2} \right) - \frac{1}{2} q_2 \left( -k q_2 - g \frac{q_1 q_3}{q_1^2 + q_2^2} \right) \\
\Rightarrow \dot{q}_4 &= \frac{1}{2} k (q_1^2 + q_2^2) > 0
\end{align*}
\] (4.19)

The case \( q_1 = q_2 = 0 \) is singular. The fact that we replace \( \omega_1, \omega_2 \) by the expressions in equation (4.17) means that we assume a non-singular state in the proof. It will appear that a non-singular initial state involves a non-singular state at all times if the control parameters are conveniently chosen.

We first assume that \( q_4(0) \geq 0 \) to make the proof. It will then be explained that the case \( q_4(0) < 0 \) does not differ much.

Let's define the function \( V_q \) as follows:

\[
V_q = q_1^2 + q_2^2
\] (4.20)

By differentiating \( q_1^2 + q_2^2 \) with respect to time, we have:
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\[
\dot{V}_q(t) = \frac{d}{dt} \left( q_1^2 + q_2^2 \right) = 2q_1 \dot{q}_1 + 2q_2 \dot{q}_2 = 0
\]

\[
\dot{V}_q(t) = -kq_4 V_q(t) + gq_3^2(0)e^{-\sigma}
\]

and since the function \( q_4(t) \) is monotonous and increasing \( (q_4(t) > q_4(0)) \), we also have:

\[
\dot{V}_q(t) < -kq_4(0)V_q(t) + gq_3^2(0)e^{-\sigma}
\]

Then, we can introduce a function \( U(t) \), with \( U(0) = q_1^2(0) + q_2^2(0) = V_q(0) \), which satisfies:

\[
\dot{U}(t) = -kq_4(0)U(t) + gq_3^2(0)e^{-\sigma}
\]

The solution of the above equation (4.20) is of the form:

\[
U(t) = c_1 e^{-kq_4(0)t} + c_2 e^{-\sigma t}
\]

where \( c_1 \) and \( c_2 \) are constants (functions of the control parameters and initial conditions).

From equation (4.23), we have:

\[
\lim_{t \to \infty} U(t) = 0, \quad \lim_{t \to \infty} \dot{U}(t) = 0
\]

From equations (4.22) and (4.23), we have:

\[
\dot{V}_q(t) + kq_4(0)V_q(t) < \dot{U}(t) + kq_4(0)U(t) \quad \forall t \geq 0
\]

and by writing the limits at infinity, we have:

\[
\lim_{t \to \infty} \left( \dot{V}_q(t) + kq_4(0)V_q \right) \leq 0 \quad \forall t \geq 0
\]

and since \( k > 0, q_4(0) > 0 \) and \( V_q > 0 \), we also have:

\[
\lim_{t \to \infty} \dot{V}_q \leq 0
\]

and \( V_q = q_1^2 + q_2^2 \) is bounded, therefore \( V_q \) is also convergent and we have:

\[
\lim_{t \to \infty} \dot{V}_q(t) = 0
\]
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Hence, from equation (4.29), the function \( V_q(t) \) is convergent, but not necessarily to zero. The convergence of \( V_q(t) \) to zero is demonstrated in the following analysis.

One property that will be used to prove that \( V_q(t) \) converges to zero is that, for any two functions of time \( x(t) \) and \( y(t) \), we have:

\[
\forall x(t), y(t): x(t) < y(t), \forall t \geq 0 \Rightarrow \lim_{t \to \infty} x(t) \leq \lim_{t \to \infty} y(t) \tag{4.30}
\]

The consequence of equation (4.30) for the function \( V_q(t) \) is:

\[
V_q(t) = q_1^2 + q_2^2 > 0, \forall t \geq 0 \Rightarrow \lim_{t \to \infty} V_q(t) \geq 0 \tag{4.31}
\]

On the other hand, from equation (4.22), we have:

\[
\dot{V}_q(t) + kq_4(0)V_q(t) < gq_3^2(0)e^{-\alpha t} \Rightarrow V_q(t) < \tau(t) \tag{4.32}
\]

where the new function \( \tau(t) \) is defined as:

\[
\tau(t) = \frac{gq_3^2(0)e^{-\alpha t} - \dot{V}_q(t)}{kq_4(0)}
\]

From equation (4.28), we have:

\[
\lim_{t \to \infty} \tau(t) = 0 \tag{4.33}
\]

Therefore, from the general property (4.30), equations (4.32) and (4.33) imply that:

\[
\lim_{t \to \infty} \dot{V}_q(t) \leq 0 \tag{4.34}
\]

and from the two equations (4.31) and (4.34), we simply have:

\[
\lim_{t \to \infty} V_q(t) = 0 \tag{4.35}
\]

Finally, we can conclude that \( V_q = q_1^2 + q_2^2 \) converges to zero. Another important conclusion from equation (4.35) and equation (4.18) is that the control law (4.17) ensures the 3-axis stability of the satellite. However, this conclusion about stability is only valid if \( q_4(0) > 0 \) (which is the case for an Euler rotation \( \phi \in [-\pi, \pi] \)). If \( q_4(0) \leq 0 \) (which corresponds to an equivalent of the case \( \phi \in [\pi, 3\pi] \)), then we can reach similar
conclusions by replacing $q_4(0)$ by $|q_4(0)|$ in equations (4.22), (4.23). Another assumption was that the singular state ($q_1 = q_2 = 0$) is never reached if it does not occur initially. We prove this property in the following, provided that $g > 2k$.

Since $U(t) = c_1 e^{-k_1(t)} + c_2 e^{-k_2} > c_1 e^{-k_1} + c_2 e^{-k_2}$ and from equation (4.18), on the singular part of the controller, we have:

$$\frac{q_3}{q_1^2 + q_2^2} < \frac{q_3(0)e^{-k_1}}{c_1 e^{-k_1} + c_2 e^{-k_2}}$$

(4.36)

It is obvious from equation (4.36) that the numerator converges to zero before the denominator with $g > 2k$, and we can conclude that the mathematical singularity never occurs in theory if it does not happen initially. The only condition to avoid the mathematical singularity, is to chose $g > 2k$. In this case, the control torques are not too large and are bounded.

To avoid singularities in practice (which are likely to happen under the effect of noise and other phenomena during the transient stage), we introduce a saturated version of the controller similar to the one in equation (4.12). The saturated control law in quaternion parameters is given by:

$$\omega_1 = -kq_1 + g \text{sat}\left(\frac{q_2}{q_1^2 + q_2^2}, a\right)$$

$$\omega_2 = -kq_2 - g \text{sat}\left(\frac{q_1}{q_1^2 + q_2^2}, a\right)$$

(4.37)

For a zero total momentum satellite ($H = 0_{3 \times 1}$), we have by definition:

$$h_1 = -I_1 \omega_1$$

$$h_2 = -I_2 \omega_2$$

(4.38)

Therefore, for the stabilisation of the complete system, we set:

$$h_1 = -I_1 \omega_{id}$$

$$h_2 = -I_2 \omega_{2id}$$

(4.39)

From equation (4.28), by applying the commands as in (4.29), we have:
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\[ \begin{align*}
\dot{\omega}_1 &= \omega_{1d} \\
\dot{\omega}_2 &= \omega_{2d}
\end{align*} \]  
\tag{4.40}

Under a zero total momentum condition, the required wheels speeds commands are simply obtained from the equation:

\[ \begin{align*}
\dot{\alpha}_1 &= -\frac{I_1}{I_{W1}} \omega_{1d} \\
\dot{\alpha}_2 &= -\frac{I_2}{I_{W2}} \omega_{2d}
\end{align*} \]  
\tag{4.41}

Therefore, although the design of control laws has been made easier using Rodriguez parameters, the computation of the equivalent wheels speeds commands is much easier using the quaternion modelling. Singularities do not happen using quaternions modelling.

However, formal stability proofs are significantly easier to develop using the Rodriguez parameterisation.

The simulations of the control response using the quaternion formulation are given in chapter 5. The quaternion modelling has also been chosen for the simulator and in-orbit applications of underactuated control, presented in chapter 6. However, Rodriguez parameters are used for the simulations of chapter 4 (without the Detumbling phase and without Kalman filtering or sensor noise).

**Remark**

The control laws (4.27) and (4.12) are singular at (0,0) (a straight line in 3 dimensional space). If we consider a smoothed control law as follows:

\[ \begin{align*}
\omega_{1d} &= -kq_1 + g \operatorname{sat}\left( \frac{q_2}{q_1^2 + q_2^2 + v} \right) \\
\omega_{2d} &= -kq_2 - g \operatorname{sat}\left( \frac{q_1}{q_1^2 + q_2^2 + v} \right)
\end{align*} \]  
\tag{4.42}
Where the parameter $\nu$ is a positive constant, then the attitude will converge and the command will be smoothed, but the convergence will not be to the desired reference and a small static error will occur. The function $s$ has been defined earlier for equation (3.9).

4.2.2 Comparative Simulation Study of the 'Two Wheels' Controllers (Zero Total Momentum Case):

Using the two reaction wheels control system, the LEO satellite considered (using two wheels) is UoSAT-12.

Since the time varying approach presented in chapter 3 was designed using Rodriguez parameters, the comparative study is made with the nonlinear singular control strategy in Rodriguez parameters as well.

The inertia parameters of UoSAT-12 are adopted:

$$I = \text{Diag}(40.36, 42.09, 40.45) \text{kg.m}^2$$

$$I_{w1} = I_{w2} = 0.008 \text{kg.m}^2$$

The initial conditions considered were:

$$p_1(0) = -1, p_2(0) = -1, p_3(0) = -1$$

In figure(4.14), the attitude stabilisation of the zero momentum satellite using the NLSC controller from an almost upside down initial configuration (almost worst case) to zero is very fast and goes through few oscillations. A yaw control precision about the unactuated $Z$ axis of $0.5^\circ$ is obtained after 300 seconds (5 minutes).

The stabilisation is also achieved using the CTVC controller but the same precision is obtained after about 1000 seconds and the system goes through high frequency oscillations.


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<table>
<thead>
<tr>
<th>CONTROLLER</th>
<th>CONTROLLER PARAMETERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>CTV C controller</td>
<td>$\omega = 0.1, k_{TV} = 0.1 \text{rad.s}^{-1}, \beta = 0.05 \text{rad.s}^{-1}$</td>
</tr>
<tr>
<td>NLSC controller</td>
<td>$k = 0.01 \text{rad.s}^{-1}, g = 0.025 \text{rad.s}^{-1}, a = 0.1$</td>
</tr>
</tbody>
</table>

Table 4.3: Controller parameters using the two reaction wheels controllers

![Attitude and control torque](image)

Figure 4.14: Attitude and control torque using two reaction wheels – NLSC controller
(control law in equation (4.12))

![Attitude and control torque](image)

Figure 4.15: Attitude and control torque using two reaction wheels – CTV C controller
(control law in equation (3.38))
4.2.3 **Investigation of the non zero total angular momentum case**

We recall here the angular momentum equation (\( \dot{\omega} \) is the rotation speed of the wheels):

\[
L = I \omega + h
\]

\[
L = A_B^T H = I_S \omega + \sum_{i=1}^{2} J_i \omega + \sum_{i=1}^{2} I_i \omega_i z_i
\]

where \( A_B^T \) is the transformation matrix from inertial to body coordinates, and where \( L, H, \omega, h, I, I_S, \omega_i \), and \( z_i \) have already been defined in chapter 2 (just after equation (2.8)).

We assume in the following that the matrix \((I + I_{w_1} + I_{w_2})\) is invertible (it is a realistic assumption as the matrix is generally nearly diagonal):

\[
\omega = -(I_S + I_{w_1} + I_{w_2})^{-1} \left( (I_{w_1} \dot{\omega}_1) z_1 + (I_{w_2} \dot{\omega}_2) z_2 + L \right)
\]

and equation (4.43) can also be written as:

\[
\omega = -(I_S + I_{w_1} + I_{w_2})^{-1} I_{w_1} \dot{\omega}_1 z_1 - (I_S + I_{w_1} + I_{w_2})^{-1} I_{w_2} \dot{\omega}_2 z_2 + H_0
\]

where \( H_0 = (I_S + I_{w_1} + I_{w_2})^{-1} L \)

And we have the following equations for the rotational motion of each wheel:

\[
\begin{align*}
N_1 &= \dot{h}_1 = I_{w_1} \dot{\omega}_1 \\
N_2 &= \dot{h}_2 = I_{w_2} \dot{\omega}_2
\end{align*}
\]

The accelerations values provide the corresponding control torques from the equation (4.45).

**Attitude expression:**

We still use the Rodriguez parameters in order to describe the attitude kinematics. In this case, the total angular momentum has to be taken into account (as in [Terui 2000]).

\[
\dot{p} = -\frac{1}{2} (p_1 \omega_1 + p_2 \omega_2) p - \frac{1}{2} (z_1 \omega_1 + z_2 \omega_2) - \frac{1}{2} p \times (z_1 \omega_1 + z_2 \omega_2) + \omega
\]
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Where:

\[ \omega = \frac{1}{2} (1_{3 \times 3} + p^* + pp^T)(I + I_{w_1} + I_{w_2})^{\dagger} A_h^i H \]  \hspace{1cm} (4.47)

where \(1_{3 \times 3}\) is the 3 dimensional identity matrix, \(A_h^i\) is the transformation matrix from inertial to body coordinates, \(z_i\) for \(i = 1, 2, 3\) is a unit vector along the \(i^{th}\) body axis (which is also the wheel's rotation axis).

And with \(\omega = \omega_1 z_1 + \omega_2 z_2\), and \(\omega_3 = 0\), the equations for each component become:

\[ \dot{\omega}_1 - \omega_1 = (p - \omega) \cdot z_1 = \frac{1}{2} (1 + p_1^2) \omega_1 - \frac{1}{2} (p_3 - p_1 p_2) \omega_2 \]
\[ \dot{\omega}_2 - \omega_2 = (p - \omega) \cdot z_2 = \frac{1}{2} (p_3 + p_1 p_2) \omega_1 + \frac{1}{2} (1 + p_2^2) \omega_2 \]
\[ \dot{\omega}_3 = (p - \omega) \cdot z_3 = \frac{1}{2} (p_3 - p_1 p_2) \omega_1 + \frac{1}{2} (p_1 + p_2 p_3) \omega_2 \]  \hspace{1cm} (4.48)

From the first and second equations of (4.48), we can simply express \(\omega_1, \omega_2\) by \(\dot{\omega}_1, \dot{\omega}_2\) as follows:

\[ \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \frac{2}{1 + p_1^2 + p_2^2 + p_3^2} \begin{bmatrix} 1 + p_2^2 & p_3 - p_1 p_2 \\ -(p_3 + p_1 p_2) & 1 + p_3^2 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 - \omega_1 \\ \dot{\omega}_2 - \omega_2 \end{bmatrix} \]  \hspace{1cm} (4.49)

Substituting this result (4.40) into the third equation of (4.39), and simply by defining the new variables: \(u_1 = \dot{\omega}_1 - \omega_1, u_2 = \dot{\omega}_2 - \omega_2\).

The simplified form of the control system is:

\[ \begin{align*}
\dot{\omega}_1 - \omega_1 &= u_1 \\
\dot{\omega}_2 - \omega_2 &= u_2 \\
\dot{\omega}_3 &= -p_3 u_1 + p_1 u_2
\end{align*} \]  \hspace{1cm} (4.50)

If no disturbance torque is applied on the system, then we have a constant total angular momentum \(H = H(0)\). Instead of the ideal case \(H = 0_{3 \times 1}\), we rather consider the case \(0 < \|H\| = \|H(0)\| << 1\) corresponding to a very slowly spinning satellite.
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The results are shown and for the case study \(0 \leq \|\mathbf{H}\| = \|\mathbf{H}(0)\| < 1\) using the controller (4.12) in figure (4.16), figure (4.17) and using the controller (3.32) in figure (4.18). The results in this case show that we can bring the system to the desired reference with a constant amplitude rotation about it. The amplitude of the rotation is small using the NLSC with small initial conditions (a small initial momentum means small residual oscillations). The control torque (not shown) also oscillates but the amplitude can be made very small as well.

The rapidity and precision of the CTVC have been considerably deteriorated in case of the presence of an angular momentum as small as 0.0001 rad/sec (as we can see on figure 4.18). Hence, the NLSC is much more convenient than the CTVC in the presence of a small bias momentum (see figures (4.17) and (4.18)).

![Figure 4.16: simulations of the NLSC controller, \(H(0) = [0.1,0.1,0.1]^T\)](image-url)
Chapter 4. Practical analysis and contributions to underactuated attitude control

Figure 4.17: simulations of the NLSC controller, $H(0) = [0.01, 0.01, 0.01]^T$

Figure 4.18: simulations of the CTVC controller, $H(0) = 0.0001.1_{3x1}$
4.2.4 Effect of the sampling:

The feedback control laws are applied in practice with a sampling time since the measured or estimated data are only available at those sampling times. The sampling time period can have an influence on the results and the controller allowing the highest sampling period will be preferred. The sampling period must be small enough to maintain stability.

However, the sampling time in practice cannot be too small, as this would involve data handling problems. Using the CTVC controller, the attitude is not controlled and diverges with a sampling time $T_s = 10$ sec (in Figure (4.19), we simulated the Rodriguez parameters to show the divergence without being confused by any singularities). Using the same controller, the sampling time must be reduced to $T_s = 3$ sec (which is not even possible on UoSAT-12 for instance) in order to achieve the stabilisation (see Figure (4.20)). Using the NLSC controller, the stabilization is still guaranteed without any problem for the same sampling time of $T_s = 10$ sec (see Figure (4.21)).

![Figure 4.19: Attitude in Rodriguez parameters using the CTVC, $T_s$=10 sec](image-url)
Chapter 4. Practical analysis and contributions to underactuated attitude control

Figure 4.20: Attitude in Euler angles using the CTVC, $T_s=3$ sec

Figure 4.21: Attitude using the NLSC, $T_s=10$ sec
4.2.5 Effect of the external disturbance torque:

If any external disturbance torque is applied, then the total angular momentum of the satellite is no longer conserved. The variations of the total angular momentum are expressed by the following differential equation:

\[ \dot{L} + \omega \times L = T_d \]  

(4.51)

If we add a non-compensated disturbance torque on the Z axis, divergence may follow. A slow divergence (if small disturbance torques are applied) is admissible in practice. However, the figure (4.22) shows the attitude of the satellite when a constant disturbance torque of \(10^{-5}\) N.m (which corresponds to a worst case) is applied on all three axes of the satellite. The attitude divergence is so slow that it is hard to observe, but the slow torque divergence means that the momentum needs to be dumped periodically if we intend to leave the controller for a long time (see figure (4.22)).

Contrary to the case of the two pairs of thrusters, we do not need a third control torque on the unactuated Z axis for the disturbance compensation, although it might be needed for momentum dumping if the control law is used for a long time.

Figure 4.22: Simulations (attitude, torque) using the NLSC controller, \(T_d=10^{-5}\) N.m
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The improvement when replacing thrusters by reactions wheels:

The advantage using reaction wheels to provide two control torques instead of thrusters is that the restriction assumed for reaction wheels $\mathbf{H}=0_{3\times1}$ (valid for all times if there is no disturbance torque) considerably simplifies the control system design (especially using Rodriguez parameters).

Conversely, using thrusters, the system remains a cascade interconnection between the dynamic and kinematic models, and the complete system is still complicated whatever restriction is used. The required control torque to achieve the tracking of the desired angular velocities using thrusters is very high. The torque saturation is needed using gas jet actuators, which is not the case using reaction wheels. The control torque demand using reaction wheels is very reasonable because the wheels torques are directly controlling the satellite’s attitude (not controlling angular velocities as virtual inputs).

Furthermore, the X and Y axes angular velocities cannot be subject to restrictions for all times in the case of thrusters since we use $\omega_1, \omega_2$ as virtual inputs. Therefore, using reaction wheels, we can consider restrictions that further reduce the system and cannot be considered using thrusters. The reduced system to control is quite simple and that is why the results and the stabilization are achieved with an impressive efficiency.

4.2.6 Conclusions

Using two reaction wheels, the 3-axis stabilisation has been first demonstrated for a zero total angular momentum satellite. Two 3-axis controllers based on non-smooth control laws have been used. The attitude control of a satellite using two reaction wheels is practically possible (with a sampling of 3 sec using the CTVC and 10 sec using the NLSC).

Using the singular control approach (NLSC), the control performance is by far better than the control authority using the time varying approach (CTVC), which was the only known 3-axis stabilising technique using two wheels in the available literature. Using this technique for a satellite controlled with only two reaction wheels, large angle decisive manoeuvres can be achieved on all three axes with very a good rapidity and precision and a very reasonable amount of control torque expenditure.
In the case of a small non-zero angular momentum at start up, the convergence is achieved towards a small neighbourhood of the desired reference. In other words, the desired attitude is still rapidly reached, but residual oscillations about the desired orientation persist.

The disturbance torque on the Z-axis, mainly due to aerodynamic disturbance torque in the case of a symmetrical satellite like UoSAT-12, has no important impact on the response of the underactuated spacecraft to the “two wheels” controller. One consequence is that the presence of a third control torque is no longer strictly required during three axis manoeuvres with two reaction wheels (no need for magnetorquer based compensation of disturbance torques as suggested in the case of thrusters). A third torque might only be required for very long term applications.
5. The complete control strategy in two phases

5.1 Case of a thruster failure

Before stabilisation, reorientation or slew maneuver of a satellite, control algorithms are also needed to detumble the satellite in the case of thrusters. In the case of reaction wheels, we have to control the total momentum near zero. In this chapter, we deal with the detumbling in the case of thrusters in order to show that the remaining two control torques can be used for this purpose.

For an asymmetric satellite, the two remaining external control torques provided by thrusters on the actuated axes can be used for the detumbling.

![Diagram](image)

*Figure 5.1: The two-phases underactuated attitude controller (Detumbling + Stabilisation)*

Many control algorithms have been proposed in the literature to detumble spacecrafts using only two control torques ([Tsiotras 2000a], [Morin 1996], [Astolfi 2002]). For easy implementation reasons, we give a particular attention to the control algorithm that has been proposed by Victoria Coverstone Caroll in a recent publication [Coverstone 1995].
5.1.1 Detumbling Control laws

We recall that for an underactuated asymmetric satellite, which is unactuated on the Z axis, the Euler’s dynamic equation is:

\[ \dot{\omega}_1 = a_1 \omega_1 \omega_3 + \frac{N_1}{I_1} \]
\[ \dot{\omega}_2 = a_2 \omega_1 \omega_3 + \frac{N_2}{I_2} \]
\[ \dot{\omega}_3 = \varepsilon \omega_1 \omega_2 \]  \hspace{1cm} (5.1)

Where: \( a_1 = \frac{I_2 - I_3}{I_1} \), \( a_2 = \frac{I_1 - I_2}{I_3} \), \( \varepsilon = \frac{I_1 - I_2}{I_3} \).

The control algorithm used to detumble the satellite is a sliding mode controller, also known as variable structure controller (VSC).

In this approach, we have to construct a sliding surface that will be reached by the control algorithm. In ref [Coverstone 1995], V.C Coverstone has designed this surface so that the angular velocity about the passive unactuated axis exponentially decays.

This sliding surface \( s \), is represented as follows:

\[ s = \dot{s} + \lambda s = \dot{s} + \lambda \omega_1 \omega_2 + \lambda_3 \omega_3 \]  \hspace{1cm} (5.2)

where \( \lambda \) is a positive constant.

The control torques \( N_1, N_2 \) are designed to drive the system to the surface \( (s=0) \) and to keep it there. The stabilizing controller is designed using the Lyapunov function:

\[ V(s) = \frac{s^2}{2} \]  \hspace{1cm} (5.3)

which is positive definite except for \( s = 0 \).

The control torques \( N_1, N_2 \) are designed to force the derivative of the Lyapunov function to be negative along the system trajectories.

One appropriate choice of the control torques is:
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\[ N_1 = -I_1 \dot{\alpha}_1 \omega_2 \omega_3 - \lambda_1 I_1 \dot{\omega}_1 + I_1 \dot{\lambda}_2 \text{sat}(s \omega_2, c) \]
\[ N_2 = I_2 (\lambda_1 - \lambda_3) \omega_2 - I_2 \dot{\alpha}_2 \omega_1 \omega_3 + I_2 \dot{\lambda}_3 \text{sat}(s \omega_1, c) \]

(5.4)

where:

\[ \text{sat}(x, c) = \begin{cases} 
\text{sign}(x) & \text{for } |x| > c \\
\frac{x}{c} & \text{for } |x| < c 
\end{cases} \]

(5.5)

with: \( \text{sign}(x) = \begin{cases} 
1 & x \geq 0 \\
-1 & x \leq 0 
\end{cases} \)

Using the control torques in (5.4), the derivative of the Lyapunov function is:

\[ \dot{V}(s) = s \ddot{s} = -(s \omega_1) \lambda_2 \text{sat}(s \omega_1, c) -(s \omega_2) \lambda_3 \text{sat}(s \omega_2, c) \]

(5.6)

which is a negative definite function, as the product of the function and it's saturation is positive.

Therefore, the sliding surface \( (s = 0) \) will be reached by applying the control torques in (5.4) because the condition of attractivity \( s \dot{s} < 0 \) is achieved. The sliding surface has been designed to ensure the exponential decay of the passive axis angular velocity.

We can show that the active axes angular velocities will also exponentially decay:

Once the sliding manifold \( (s = 0) \) is reached by applying the control torques in (5.4), equations (5.4) simplify to the following continuous feedback control laws:

\[ N_{1eq} = -I_1 \dot{\alpha}_1 \omega_2 \omega_3 - \lambda_1 I_1 \dot{\omega}_1 \]
\[ N_{2eq} = I_2 (\lambda_1 - \lambda_3) \omega_2 - I_2 \dot{\alpha}_2 \omega_1 \omega_3 \]

(5.7)

as the right hand term in each equation of (5.4) is zero when \( s = 0 \). \( N_{1eq} \) and \( N_{2eq} \) in equation (5.7) are known as equivalent control torques. \( N_{i eq}, i = 1,2 \), represents the continuous function, which is equivalent to a switching on/off stabilising torque (see [Coverstone 1995]).

We can simply substitute the control torques (when \( s = 0 \)) into the dynamic equation (5.1) to show the linear angular velocity equation, which is:
Chapter 5. The complete control strategy in two phases

\[
\begin{align*}
\dot{\omega}_1 &= -\lambda_1 \omega_1 \\
\dot{\omega}_2 &= (\lambda_1 - \lambda_2) \omega_2 \tag{5.8}
\end{align*}
\]

From equation (5.8), we clearly have exponential convergence of the angular velocity along the X and Y axes when \( \lambda_1 > 0, \lambda_2 > 0 \) and \( \lambda_1 > \lambda_2 \). As a consequence, once the sliding manifold is reached, all the angular velocity components will exponentially decay to zero.

### 5.1.2 Detumbling maneuver simulation and conclusions:

The angular velocities converge to zero very soon. The convergence takes less than 500 seconds. We then can ensure fast detumbling maneuvers before the stabilisation.

![Figure 5.2: Angular velocities and control torques of the underactuated detumbling controller with two pairs of thrusters](image)

Using two pairs of thrusters, it has been shown how the two control torques from thrusters can be used to detumble the satellite, as required before starting the 3-axis attitude controller.

It is shown on figure 5.2, that the angular velocities are driven to zero with a very good rapidity (within 3 minutes). This is however only possible for an asymmetric satellite. If the satellite is axis symmetrical, there is no remedy other than to use low cost magnetorquing to achieve the detumbling manoeuvre (when detumbling is needed).
Chapter 5. The complete control strategy in two phases

It is well-known that the thrusters are rarely regarded as an actuator for small satellites attitude control because fuel loss is generally tolerated only for orbit control applications. However, the thrusters based control system is convenient for the study of the underactuated satellite system.

In the following, a similar detumbling phase, also needed in the case of reaction wheels failures, will be presented. The detumbling in this case is achievable using low cost magnetorquing (eventually in combination with the active wheels).

5.2 Case of a reaction wheel failure

During the first phase of the underactuated control with two wheels, we must ensure that the total momentum of the satellite is made as small as possible before the activation of the non-smooth underactuated ‘two wheels’ controller (in practice, we consider in this thesis that the norm of the total momentum must be smaller than 0.02 rad/sec). During the control of the total momentum, it is also preferred to ensure that the attitude is initially non-singular (not simultaneously zero on roll and pitch axes) when the ‘two wheels’ controller is activated.

If the total momentum is reasonably small (as in the case of a zero momentum control mode) and if the initial attitude is non singular (non zero simultaneously in roll and pitch), then the two wheels controller can be activated immediately without any intermediate detumbling phase.

However, the total momentum in the general case can be non-zero, especially for a bias momentum mode. In this case, there is no remedy other than to adopt a detumbling phase to ensure the small momenta conditions and start the controller with only two control torques. A typical detumbling manoeuvre strategy (not necessarily the only way) that can be considered to accommodate high initial momenta is described in this section.

5.2.1 The detumbling phase

One way to achieve a successful first phase (detumbling + singularity avoidance) in the case of a wheel failure is to use a combined controller (magnetorquing + both wheels initially).
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During the detumbling phase, the two wheels are used as a PD control law to steer the satellite to a non-singular attitude before the activation of the underactuated nonlinear controller. The singular state to avoid is when both roll and pitch angles are zero.

The equation of the linear PD controller with two wheels is:

\[ N_{w1} = k_{p1} \cdot q_{1e} + k_{d1} \cdot \omega_1 + h_2 \omega_3 \]
\[ N_{w2} = k_{p2} \cdot q_{2e} + k_{d2} \cdot \omega_2 - h_2 \omega_3 \]

where \( q_{1e} = q_1 - q_{1e}, q_{2e} = q_2 - q_{2e} \) and \( (q_{1e}, q_{2e}) \neq (0,0) \)

One consequence of using this control law is that the satellite’s angular velocity on the X and Y axes will be very small. The attitude will also be given a non-singular value before the activation of the underactuated controller.

At low Earth orbit, the most practical way to ensure a detumbling manoeuvre and to dump the momentum of the satellite is by low cost magnetorquing. A magnetic control torque results from the interaction of the magnetorquer dipole moment with the Earth’s geomagnetic field.

The magnetic field vector entirely depends on the orbital location. There are in the literature two ways of modelling the geomagnetic field vector. A moment known as the dipole moment is often considered for its simplicity (dependence on only two orbit parameters: orbit’s inclination, right ascension from the ascending node). However, this model is only convenient for preliminary studies of magnetorquing. The model used to generate the magnetic field in this section is a spherical harmonic model, called the IGRF (International Geomagnetic Reference Field), which is more rigorous and suitable for evaluation or implementation of magnetorquing controllers. The coefficients of the IGRF model are updated every 4 years and supplied with secular variation times due to the secular drift and magnitude decrease of the geomagnetic field. A standard 10th order IGRF model is used for the simulations.
Chapter 5. The complete control strategy in two phases

Figure 5.3: Typical Earth’s geomagnetic field in local orbital frame for UoSAT-12

The magnetorquing control is based on the well-known cross product control law, proven to generate the most favourable magnetorquer torque. This control law is used to minimise an error vector. The error vector is chosen to have a small total momentum. Magnetorquing minimises the momenta of the wheels, and the momentum of the satellite on the unactuated Z-axis. The attitude control from magnetorquing on the Z-axis is very limited (due to the limited performance of magnetorquing for attitude control or attitude manoeuvres), but still convenient to avoid a totally uncontrolled initial attitude. The error vector to be minimized by magnetorquing is:

\[
e = \begin{bmatrix}
  k_{d1} h_1 \\
  k_{d2} h_2 \\
  k_{d3} (q_3 - q_3^d) + k_{d3} (\omega_3) \omega_b
\end{bmatrix}
\] (5.10)

When this error vector is made very small, the momentum due to the wheels is obviously small. The momentum due to the satellite, on the Z-axis is also small in this case. The angular momentum of the satellite on the X and Y axes are also small because of the
attitude control on those axes. In other words, the total momentum is small when this error vector is controlled to zero.

The most favourable cross product magnetorquer command is given by (see [Steyn 1995] for instance):

$$M = \frac{B \times e}{||B||}$$  \hspace{1cm} (5.11)

The generated magnetorquer control torque is:

$$N_m = M \times B$$  \hspace{1cm} (5.12)

When wheels and magnetorquing are used simultaneously as suggested in this section, 3-axis control is achievable, but with limited control authority along the Z axis. This combined control can therefore be seen as a way of achieving 3-axis control. However, the performance of manoeuvres on the unactuated axis using combined linear control provides very limited performances (in terms of precision and rapidity). The reason is that on/off magnetorquing control on the Z-axis is not precise. In other words, the combination between two wheels PD linear controller and the magnetorquing linear cross product law is not suitable for precise 3-axis control, but only useful for detumbling and momentum dumping purposes (and to avoid eventual singularity of the nonlinear controller).

Alternatively, it will be shown that, under small momentum conditions, the non-smooth nonlinear attitude control with only two wheels (no magnetorquing) can achieve a high precision and fast 3-axis control. In particular, manoeuvres about the unactuated axis can not be achieved with comparable efficiency using any other controller.

5.2.2 Simulations of the control strategy in two-phases (noise free, disturbance free case)

We now assume a complete control strategy to achieve attitude control with two reaction wheels from any initial condition. The working assumptions are a free noise and free disturbance environment. The attitude control is achieved in two phases.

We consider the mathematical model of UoSAT-12 for the simulations.
Chapter 5. The complete control strategy in two phases

We assume the following control scenario (noise free, disturbance free case):

- The first phase starts from nonzero total momentum. During this phase, the magnetorquing based control laws given by equations (5.10), (5.11) and (5.12) are applied to detumble the satellite. The two wheels during this phase are used as a PD controller (equation (5.9)) to avoid singularities before starting the second phase and to have an off pointing of 20 degrees on each axis before the slew manoeuvre to Nadir pointing. The desired attitudes at the end of that phase are \( q_1 = 0.1, q_2 = 0.1, q_3 = -0.15 \), and more importantly the satellite’s total momentum must be near zero.

- The second phase starts after the end of the detumbling phase (after 4000 seconds in the simulations). The controller used after the detumbling phase is the quaternion based nonlinear singular controller from equations (4.23) and (4.25). A quaternion formulation can be used to avoid any singularities related to the attitude parameterization.

We rewrite the quaternion version of the underactuated control laws here to avoid any possible confusion:

\[
\begin{align*}
\dot{w}_1 &= -kq_1 + g \text{sat} \left( \frac{q_2}{q_1^2 + q_2^2} q_3 \right) \\
\dot{w}_2 &= -kq_2 - g \text{sat} \left( \frac{q_1}{q_1^2 + q_2^2} q_3 \right) \\
\dot{h}_1 &= -I_1 \omega_{h_1} \\
\dot{h}_2 &= -I_2 \omega_{h_2}
\end{align*}
\]

(5.14)  (5.15)

The control parameters used are: \((k_{d1}, k_{d2}) = (1.4, 2)\) N.m.s/rad, \((k_{p1}, k_{p2}) = (0.02, 0.05)\) N.m, \(k_{D1} = k_{D2} = 40\) s\(^{-1}\), \(k_{D3} = 40\) N.m.s/rad, \(k_{p3} = 1\) N.m/rad.

The simulations of the attitude of UoSAT-12 during the two phases are shown in figure (5.4), for a high momentum initial condition of \( H(0) = [0.1, 0.05, 0.1]^T \), and in figure (5.5) for a low momentum initial condition of \( H(0) = [0.01, 0.005, 0.01]^T \). In figures (5.6) and (5.7), the control torques for the respective cases \( H(0) = [0.1, 0.05, 0.1]^T \) and \( H(0) = [0.01, 0.005, 0.01]^T \) are shown. The satellite’s momentum is simulated in both cases of initial momentum conditions in figures (5.8) and (5.9). The magnetic torque is also simulated in both cases of high then small initial momentum, in figures (5.10) and (5.11).
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By comparing figures (5.4) and (5.5) representing the attitude for two different initial momenta conditions, we notice, not surprisingly, that a smaller initial total momentum condition implies a better attitude control precision, better rapidity, and better detumbling manoeuvre.

Both the detumbling maneuver (first phase) based on magnetorquing, and the two wheels nonlinear singular controller (second phase), are achieved with better efficiency for a smaller momentum before the activation of the complete underactuated control strategy.

The initial total momentum can be high and unknown, and that is the reason for considering a first phase in the control for the detumbling.

During the first phase, the wheels were used successfully (as a PD controller), to bring the roll and pitch angles to a nonsingular nonzero reference before applying the underactuated two wheels nonlinear singular controller. The second phase of underactuated control is switched on after 4000 seconds.

For an initial condition \( \mathbf{H}(0) = [0.01, 0.005, 0.01]^T \), the yaw angle is controlled to a precision of 0.5° within 5 minutes (300 seconds), but the roll and pitch angles reach a 2° precision after 2000 seconds. For that initial condition the detumbling manoeuvre was in fact not really needed, but it has been simulated for comparison purposes.

The results are clearly worse when the initial condition is set to the bigger values (\( \mathbf{H}(0) = [0.1, 0.05, 0.1]^T \)). In this case, the yaw control to an accuracy of 0.5° takes about (1000 seconds). A roll and pitch control precision of 2° is reached after almost one orbit (100 minutes).

Therefore, in the presence of a non-small bias momentum, the effectiveness of the detumbling maneuver is a major requirement to achieve the two wheels attitude control efficiently with reasonable rapidity and precision requirements. The effectiveness of the detumbling depends on the amount of the initial state to detumble, (generally resulting from the previous control mode), the torque capability of the magnetorquers, as well as the good choice of the control parameters for the cross product magnetorquing controller.
Clearly, the 3-axis stability is guaranteed after the underactuated control strategy in two phases (whatever the initial operating mode an initial momentum can be), and the nonlinear singular approach is very efficient for small momenta conditions.

Figure 5.4: Attitude quaternion of UoSAT-12 during the 2 phases of the control
\( \mathbf{H}(0) = [0.1, 0.05, 0.1] \)

Figure 5.5: Attitude of UoSAT12 during the 2 phases of the control
\( \mathbf{H}(0) = [0.01, 0.005, 0.01] \)
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Figure 5.6: Wheels torques of UoSAT12 during the 2 phases of the control 
\( H(0) = [0.1, 0.05, 0.1]^T \)

Figure 5.7: Wheels torques of UoSAT12 during the 2 phases of the control 
\( H(0) = [0.01, 0.005, 0.01]^T \)
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Figure 5.8: Satellite's momentum of UoSAT12 during the 2 phases of the control 
\( \mathbf{H}(0) = [0.1, 0.05, 0.1]^T \)

Figure 5.9: Satellite's momentum of UoSAT12 during the 2 phases of the control 
\( \mathbf{H}(0) = [0.01, 0.005, 0.01]^T \)
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Figure 5.10: Required magnetorquer torque for UoSAT12 during the 2 phases of the control (initial momentum $H(0) = [0.1, 0.05, 0.1]^T$)

Figure 5.11: Required magnetorquer torque for UoSAT12 during the 2 phases of the control (initial momentum $H(0) = [0.01, 0.005, 0.01]^T$)
5.3 Conclusions

In this section, the case of a satellite working at a high bias momentum mode has been addressed. Given a high total angular momentum, the activation of the underactuated controller with two control torques is not possible directly. A first phase is required to detumble the spacecraft’s angular velocities and to dump the wheels momenta. The result of both actions is the reduction of the total angular momentum by using an external control torque because the total momentum can decrease under the effect of external torques.

Using pairs of thrusters, it has been shown that two control torques are sufficient to detumble the satellite as required before the activation of the underactuated control. However, thrusters can not be seen as a regular attitude control actuator because the precious fuel is generally saved for orbit manoeuvres. Using two reaction wheels, the use of low cost magnetorquing has been presented as the most practical and efficient way to achieve the required detumbling manoeuvre.

By assuming different scenarios in two phases (the detumbling phase + underactuated control phase), we have demonstrated that the attitude can be controlled to the desired reference, with a satisfactory precision on all 3 axes, even in the case of a high momentum prior to the detumbling phase.
Chapter 6

6. Results from UoSAT-12 simulator and in-orbit experiments

The UoSAT-12 simulator of SSTL, written in C, is a software allowing very realistic simulations by taking into account the sensor noises, sensor calibration, sensor models, attitude estimation models, orbit propagation models, actuator dynamics, external disturbances, control modes and so fourth.

Most functions and programs of the UoSAT-12 simulator are already present onboard the satellite’s ADCS processor. The difference between the ADCS processor programs and the simulator’s programs is that the simulator also includes a realistic model of the satellite’s kinematics and dynamics.

In practice, any newly developed function must first pass the test results on that simulator (containing the modelled satellite’s kinematics and dynamics in presence of noise and disturbance), before the decision to upload the code to the onboard ADCS processor can be taken.

In-orbit experiments can therefore only be planned after successful simulator results. Once a function is uploaded, we can easily activate it using telecommands at a favourable satellite’s pass. A control scenario can also easily be decided by telecommand.

The transition from simulator results to the in-orbit implementation is not always straightforward. The mathematical model is sometimes subject to non-negligible model errors (presence of unaccounted non-diagonal elements in the inertia matrix in some cases, unexpected power loss in other cases, ...). For these reasons, the control parameters often (if not always) need to be conveniently modified to have the expected response. To have the expected in-orbit results for a given function, a certain number of in-orbit tests might then be needed to determine the best control parameters (or estimation parameters depending on the application).
6.1 Extended Kalman filter:

In the previous chapters, we have assumed perfect measurements of the full attitude for the feedbacks. Obviously, measurements are subject to noise, and the attitude in the feedback is in fact an estimate of the real attitude.

In the simulator, a model of the real attitude is considered, so that we can evaluate the attitude estimation errors. Therefore, the control strategy needs to be tested together with the attitude estimation procedure. On some occasions, controllers can “upset” the attitude estimator, and the result would be ADCS performance deterioration.

The attitude estimator is an extended Kalman filter used for the estimation of the full satellite state vector: \( x = [q_1, q_2, q_3, \omega_1, \omega_2, \omega_3]^T \). The extended Kalman filter (see [Kalman 1960]) is a well-known recursive estimation algorithm applied to the nonlinear satellite’s dynamic and kinematic model (in the presence of noise).

The equation of the system (dynamic + kinematic model) is nonlinear, and by assuming the presence of process noise, we can write:

\[
\begin{align*}
\dot{x}(t) &= g_1(x, t) + g_2(x, t) u(x, t) + w(t) \\
\ddot{x} &= f(x, t) + w(t) 
\end{align*}
\]  

(6.1)

where:

\( w(t) \): is the process white noise at time \( t \).
\( f \): is the process nonlinear function of the full state vector (including the control input).
\( x(7 \times 1) \): is the full state vector of the satellite.

By defining \( \hat{x} \) as the estimated state vector, and a state perturbation \( \delta x \) as the estimation error, we have (see [Steyn 1995]):

\[
\delta x(t) = x(t) - \hat{x}(t) 
\]  

(6.2)

In this case, a simple first order Taylor expansion of the function \( f \) can be given:

\[
f(x, t) = f(\hat{x}, t) + \left. \frac{\partial f}{\partial x} \right|_{x=\hat{x}} \delta x(t) 
\]  

(6.3)
We can then define the Jacobian matrix as:

\[
F(\mathbf{x}, \mathbf{0}) = \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{0}}
\] (6.4)

More precisely, \( F \) is the following 7x7 matrix:

\[
F = \begin{bmatrix}
\frac{\partial \mathbf{q}}{\partial \mathbf{q}} & \frac{\partial \mathbf{q}}{\partial \mathbf{0}} \\
\frac{\partial \mathbf{q}}{\partial \mathbf{0}} & \frac{\partial \mathbf{0}}{\partial \mathbf{0}} \\
\frac{\partial \mathbf{q}}{\partial \mathbf{0}} & \frac{\partial \mathbf{0}}{\partial \mathbf{0}} \\
\end{bmatrix}
\] (6.5)

The Jacobian matrix expression can then be used to have the following linearized perturbation state model:

\[
\partial \mathbf{x} = F(\mathbf{x}, t) \partial \mathbf{x}(t) + \mathbf{w}(t)
\] (6.6)

The discrete version of the EKF is used onboard the satellite, and the discrete perturbation model equivalent to equation (6.6) is:

\[
\partial \mathbf{x}_{k+1} = e^{F(\mathbf{x}, \mathbf{t}) \Delta t} \partial \mathbf{x}_k + \mathbf{w}_k
\] (6.7)

A first order Taylor expansion of the exponential matrix gives:

\[
\partial \mathbf{x}_{k+1} = \Phi_k \partial \mathbf{x}_k + \mathbf{w}_k
\] (6.8)

Where the matrix \( \Phi_k \) is:

\[
\Phi_k = I_{7x7} + F \Delta t
\] (6.9)

where \( \Delta t \) is the sampling time period.

The matrix \( \Phi_k \) is the transformation matrix of the extended Kalman filter.

The observation equation is:

\[
\mathbf{z}_k = h_k(\mathbf{x}_k, t_k) + \mathbf{v}_k
\] (6.10)

A simple linearization technique, similar to the one used to extract the transformation matrix can then be used to rewrite the system as:
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

\[ z_k = T_k x_k + v_k \]  \hspace{1cm} (6.11)

where:

- \( z_k \): is the measurement vector at time \( t_k \).
- \( h_k , T_k \): are the nonlinear observation function, and observation matrix of the full state vector.
- \( v_k \): is the measurement white noise at time \( t_k \).

The white noise covariance matrices are defined as:

\[ E(w_i w_j^T) = \begin{cases} Q_i & i = k \\ 0 & i \neq k \end{cases} \quad E(v_i v_j^T) = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases} \quad E(w_i v_j^T) = 0 : \text{independance } \forall i, k \]

- \( Q_k \): Covariance vector of the process noise.
- \( R_k \): Covariance vector of the measurement noise.

The observation matrix \( T_k \) is computed depending on the sensor model. We present here the case of magnetometers and Sun sensors, which are used on most UoSAT satellites.

**Observation in the case of magnetometers**: [Hashida 1997]

If the attitude sensor used is a magnetometer, then the observation model is:

\[ B_B = A_B^B \hat{B}_O + v \]

In this case, the observation nonlinear function \( h_k \) is transformed to the following Jacobian matrix:

\[ T_{\text{magnetometer}} = \begin{bmatrix} \frac{\partial B_B}{\partial q} & \frac{\partial B_B}{\partial \omega} \end{bmatrix} = \begin{bmatrix} \frac{\partial A_B^B}{\partial q} & \hat{B}_O \end{bmatrix} \begin{matrix} 0_{3 \times 3} \end{matrix} \]  \hspace{1cm} (6.12)

where:

- \( B_B \): magnetic field measured vector at time \( t_k \) in the body fixed frame.
- \( \hat{B}_O \): magnetic field vector at time \( t_k \) in the local orbital frame from the IGRF model.
- \( A_B^B \): Transformation matrix from the local orbital frame to the body frame.
- \( v \): is a measurement white noise.
**Observation in the case of Sun sensors:** [Hashida 1997]

If the attitude sensor used is an azimuth-elevation Sun sensor, then we have:

\[ S_0 = A^S_t (r_{\text{sun}} - r_{\text{sat}}) \]  \hspace{1cm} (6.13)

A 'measured' unit solar vector can be computed from the azimuth and elevation Sun sensor readings as follows:

\[ S_s = \begin{bmatrix} 1 \\ \frac{\tan \lambda}{\sqrt{1 + \tan^2 \lambda + \tan^2 \delta}} \\ \frac{\tan \delta}{\sqrt{1 + \tan^2 \lambda + \tan^2 \delta}} \end{bmatrix} \]  \hspace{1cm} (6.14)

where \( \lambda, \mu \) denote the azimuth and elevation sun sensor readings.

The measured Sun sensor unit vector is simply related to the solar position in local orbit coordinates:

\[ S_g = A^S_B A^B_O S_0 + v_s \]  \hspace{1cm} (6.15)

In this case, the observation nonlinear function \( h_k \) is transformed to the following Jacobian matrix:

\[ T_{\text{sensor}} = \begin{bmatrix} \frac{dS_s}{dq} \\ \frac{dS_s}{d\omega} \end{bmatrix} = \begin{bmatrix} A^S_B \frac{dA^B_O}{dq} \hat{B}_O & 0_{3x1} \end{bmatrix} \]  \hspace{1cm} (6.16)

where:

- \( r_{\text{sun}}, r_{\text{sat}} \): Position vectors of the Sun, and the satellite respectively, in the inertial frame.
- \( B_B \): measured magnetic field vector at time \( t_k \) in the body fixed frame.
- \( \hat{B}_O \): magnetic field vector at time \( t_k \) in the local orbital frame from the IGRF model.
- \( A^O_t \) is the Transformation matrix from the inertial to the local orbital frame.
- \( A^B_O \): Transformation matrix from the local orbital to the body fixed frame.
- \( A^S_B \): Transformation matrix from the body frame to the Sun sensor coordinate frame.
- \( v_s \): is a measurement white noise from the Sun sensor.
**EKF equations**: ([Grewal 1993])

The equation in (6.8) is the process equation, and the one in (6.11) is known as observation equation.

The elements of $F$ (from equation (6.5)) can easily be computed from the known dynamic and kinematic models (already presented in previous chapters). The transformation matrix $\Phi_k$ of the extended Kalman filter is computed in equation (6.9).

Then the estimation procedure can easily be implemented now that the discrete state model has been established.

The estimation procedure for the full state EKF is as follows:

- We initialise $\hat{x}_0, P_0$: prior estimate and prior error covariance.
- We compute $k_0, \hat{x}_0, P_0, \hat{x}_1, P_1$.

Then for each time sample $t_k$:

- We inject a prior knowledge of $\hat{x}_k, P_k$ from the previous step.
- Update equations: We compute $k_k, \hat{x}_k, P_k$ as follows:

\[
\dot{x}_k = \hat{x}_k + k_k \left( z_k - T_k \hat{x}_k \right)
\]

\[
k_k = P_k T_k^T \left( T_k P_k T_k^T + R_k \right)^{-1}
\]

\[
P_k = \left( I - k_k T_k \right) P_k \left( I - k_k T_k \right)^T + k_k R_k k_k^T
\]

- Projecting ahead equations: $\hat{x}_{k+1}, P_{k+1}$

\[
\hat{x}_{k+1} = \dot{x}_k + \int_{t_k}^{t_{k+1}} f(\hat{x}_k, t_k) dt
\]

\[
P_{k+1} = \Phi_k P_k \Phi_k^T + Q_k
\]

So at every time $t_k$, after updating the state estimate vector, and calculating Kalman gain, we can project ahead equations to have a prior state estimate and a prior covariance matrix at the beginning of the next time sample.

It is essential to demonstrate that the underactuated control strategy can still work in the presence of estimation errors, process noise and measurement noise. In fact, we need to empirically address the robustness of the control strategy with respect to estimation errors.
6.2 UoSAT-12 Simulator results

In this part, we no longer assume perfect measurements in the implementation of the underactuated control strategy. We use SSTL’s UoSAT-12 simulator (programmed in C) to demonstrate the 3-axis control capability with only two wheels. Simulator results will be presented in 6 different cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>Control mode</th>
<th>Control Parameters</th>
<th>Control law (2wheels)</th>
<th>Process noise and Measurement noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Nadir pointing</td>
<td>((k,g) = (0.01, 0.05))</td>
<td>Nonlinear</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(\text{rad.s}^{-1}), (a = 0.025)</td>
<td>equation(4.23)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Nadir pointing</td>
<td>((k,g) = (0.015, 0.03))</td>
<td>Nonlinear</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(\text{rad.s}^{-1}), (a = 0.2)</td>
<td>equation(4.23)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Nadir pointing</td>
<td>((k,g) = (0.012, 0.038))</td>
<td>Nonlinear</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(\text{rad.s}^{-1}), (a = 0.035)</td>
<td>equation(4.23)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Nadir pointing</td>
<td>((k,g) = (0.01, 0.05))</td>
<td>Smoothed</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(a = 0.025), (v = 0.5)</td>
<td>equation(4.28)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Nadir pointing</td>
<td>((k,g) = (0.04, 0.01))</td>
<td>Smoothed</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(a = 0.01), (v = 0.05)</td>
<td>equation(4.28)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Sun tracking</td>
<td>((k,g) = (0.01, 0.05))</td>
<td>Nonlinear</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(\text{rad.s}^{-1}), (a = 0.025)</td>
<td>equation(4.23)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Sun tracking</td>
<td>((k,g) = (0.04, 0.01))</td>
<td>Nonlinear</td>
<td>Proc.noise: (10^{-4}[1,1,1]) (\text{meas.noise: }2.10^{-3}[1,1,1])</td>
</tr>
<tr>
<td></td>
<td>(\text{rad.s}^{-1}), (a = 0.01)</td>
<td>equation(4.23)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Choices of control parameters and control modes for UoSAT-12 simulator
6.2.1 Nadir Pointing Case

The underactuated controller is based on a non-smooth nonlinear control approach with two reaction wheels. The C function of the “two wheels” control strategy is incorporated in the UoSAT-12 simulator. A detumbling control phase is first assumed using a magnetorquing cross product control law.

The primary objective is to demonstrate the possibility of a manoeuvre towards Nadir pointing with only two reaction wheels using SSTL’s simulator (despite the effect of noise and estimation errors assumed in the UoSAT-12 simulator). The in-orbit demonstration of Nadir pointing is unfortunately unsafe due to UoSAT-12 power consumption problems.

The control study scenario is as follows:

- Initially the spacecraft operates in free torque mode and non-zero total momentum (up to 4000 seconds).

- At \( t = 4000 \) seconds, we activate the cross product controller given by equation (5.4), and we also activate the wheels as a PD controller to bring the satellite to an initial angle of approximately 20 degrees on each axis. We leave this controller for 10000 seconds, slightly less than two orbits (we could leave it for a shorter time) to have a small total momentum.

- At \( t = 14000 \) seconds (slightly more than two orbits), we activate the underactuated controller with only two wheels (based on a small total momentum assumption). Magnetorquing is stopped.

Nadir pointing UoSAT-12 simulator results, based on the above scenario, are shown on figures (6.1) to (6.4). The control law considered is a quaternion version of the nonlinear singular controller, given in equations (5.14) and (5.15). The control parameters for the figures (6.1) to (6.4) are \( k = 0.01 \text{ rad.s}^{-1}, \ g = 0.05 \text{ rad.s}^{-1}, \ a = 0.025 \).

On figure (6.1), we notice when the underactuated controller is activated, (at \( t = 14000 \)) that the estimated attitude is very rapidly slewed from about 20° on each axis to the required zero pointing mode. The manoeuvre takes about 200 seconds to slew the satellite on all axes to average zero oscillations. However, undesired bounded oscillations appear on all three axes almost as soon as soon as the desired orientation is obtained. These
oscillations are due to the combined effect of the residual angular momentum and the effect of noise, and the fact that the control law is non-smooth. In figure (6.2), we notice that the real (modelled) attitude’s profile from the simulator differs very slightly from the estimated attitude.

On figure (6.3), the wheels speeds oscillate at approximately 3 minutes pseudo-period, but the amplitude is very reasonable (less than 200 rpm). The speed maximum capability of both wheels is 5000 rpm, so the amplitude of 200 rpm is within admissible range. In figure (6.4), the magnetorquer torque is clearly zero when the underactuated controller is activated as required.

One surprising result with that first choice of control parameters is that the best attitude control authority is achieved on the unactuated Z axis. The yaw oscillations on that axis are within a precision of 1.5°, which is much better than the 9° amplitude of oscillations of the roll and pitch angles.

Another choice of the control parameters can help to decrease the amplitude of the roll and pitch oscillations. However, there is a rapidity-precision trade-off in this case. Indeed, we can see on figure (6.5) that, by choosing the gains \( k= 0.015 \text{ rad.s}^{-1}, g =0.03 \text{ rad.s}^{-1}, a =0.02 \), the roll and pitch control precision is significantly improved to 1.5°, but the control rapidity of the slew manoeuvre on the unactuated axis is significantly slower (stability takes several orbits). In figures (6.6) and (6.7), the wheel speed demand is admissible, and the magnetorquer torque is zero during the underactuated control with the same control parameters.

A trade-off to have the best compromise in terms of rapidity and precision must therefore be considered in the choice of the best control parameters. In figures (6.8) to (6.11), the control parameters considered were \((k, g) = (0.012, 0.038) \text{ rad.s}^{-1}, a = 0.035\). We clearly see in figure (6.8) that the attitude is controlled with a reasonable precision of less than 4° on roll and pitch axes, a precision of 0.8° on the yaw axis after 5 minutes (control rapidity is still reasonably good). This is the best case of UoSAT-12 simulator results so far.

The effect of a smoothing of the control laws can be seen on figures (6.11), (6.12) where the nonlinear singular controller is smoothed, with \( \nu = 0.5 \). On figure (6.11), the attitude is tracked with more precision and less chattering on the X and Y axes, but the yaw
control is slowed down and seems to follow a non zero reference (static error). The wheels speeds profiles are smooth on figure (6.12). On figures (6.13) and (6.14), with $\nu = 0.05$ the roll and pitch angles are controlled with an even higher precision, and with even less chattering on the wheels speeds, but the inconvenience is that the yaw control in this case is far too slow.

These stability results are also valid for other satellites. SSTL's recently launched and operational UK-DMC micro-satellite simulator (boom now deployed) have also proven similar 3-axis stability results. It can be observed on figures (6.15) and (6.16) (before boom deployment) and figures (6.17), (6.18) (after boom deployment), that the attitude is stabilised towards zero after 3-axis manoeuvres using the "two wheels controller" with reasonable wheels speeds bounds, after a bias momentum control mode. Control precision on the unactuated axis (X axis for UK-DMC) is $0.5^\circ$ after 3 minutes. This control performance on is even better than what was obtained from UoSAT-12 simulator.

To conclude, simulator results have demonstrated 3-axis control. A trade-off between control performance on the unactuated axis, and control performance on the actuated axes, is necessary. The best choice of the control parameters depends on the control objectives, which might prioritise yaw control over roll and pitch control and vice versa.

![Figure 6.1: UoSAT-12 simulator-Estimated Euler angles for case1](image)
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

Figure 6.2: UoSAT-12 Simulator-modelled Euler angles for case 1

Figure 6.3: UoSAT-12 simulator-wheels speeds commands for case 1
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

![Figure 6.4: UoSAT-12 simulator-magnetic dipole moment command](image)

![Figure 6.5: UoSAT-12 simulator-Euler angles for case 2](image)
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

Figure 6.6: UoSAT-12 simulator- Wheels speeds commands for case 2

Figure 6.7: UoSAT-12 simulator- magnetic moment command for case 2
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No control \( \rightarrow \) \( +2\text{wh}(PD) \rightarrow 2 \) wheels NLSC controller

Figure 6.8: UoSAT-12 simulator- Euler angles for case 3

Figure 6.9: UoSAT-12 simulator- Wheels speeds commands for case 3
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Figure 6.10: UoSAT-12 simulator- magnetic dipole moment command for case 3

Figure 6.11: UoSAT-12 simulator-Euler angles in case 4 with a smoothing variable $v=0.05$
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Figure 6.12: UoSAT-12 simulator-Wheels speeds in case 4 with a smoothing variable
\( v = 0.05 \)

Figure 6.13: UoSAT-12 simulator-Euler angles in case 5 with a smoothing variable \( v = 0.5 \)
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Figure 6.14: UoSAT-12 simulator-wheels speeds in case 5 with a smoothing variable \( v = 0.5 \)

Figure 6.15: UK-DMC simulator before boom deployment, X-axis unactuated- Euler angles with nonlinear singular controller
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

Figure 6.16: UK-DMC simulator before boom deployment, X-axis unactuated- wheels speeds with nonlinear singular controller.

Figure 6.17: UK-DMC simulator after boom deployment, X-axis unactuated- Euler angles with nonlinear singular controller.
6.2.2 Sun Tracking Case

A Sun tracking mode is required on all small satellite missions when the satellite is not in eclipse. During Sun tracking, the attitude is controlled to maximise the power provided by the solar panels to charge the batteries. In the case of UoSAT-12, Sun tracking has become the only safe control mode due to the excessive power losses related to battery charge problems.

The desired Sun tracking path can be described as periodic roll and pitch angles (with a period of one orbit), and a constant yaw reference (the value of that constant doesn’t matter much, it ranges from 0 to 360°, preferred angles in practice are 0 degree, 180°, or a small reference like 30°). During one orbit (period), the roll and pitch profiles look like trigonometric functions except during the eclipse time (during eclipse the desired roll and pitch angles are set to zero). The accurate mathematical models of the desired trajectories of Sun tracking are already implemented as a C function in the UoSAT-12 SSTL’s UoSAT-12 simulator.

Figure 6.18: UK-DMC simulator before boom deployment, X-axis unactuated- wheels speeds with nonlinear singular controller
Simulator tests of the underactuated controller in Sun tracking mode are achieved by assuming a similar scenario, (before the activation of the “two wheels” controller) to the case of Nadir pointing (in section 6.2.1).

The two wheels underactuated controller is activated at \( t = 14000 \) seconds, after a magnetorquer cross product based detumbling manoeuvre. The initial condition when the underactuated controller is activated is exactly the same as in the case of the Nadir pointing simulator tests.

On figure 6.19, we observe that the Sun tracking attitude control starts immediately on all 3 axes when the underactuated control is activated. The roll and pitch angles seem to be tracking the expected trigonometric reference with a period equal to 1 orbit (6000 seconds). There is still a bounded tracking error on roll and pitch desired angles, of a magnitude of \( 2^\circ \) (the desired attitude on roll and pitch axes are in fact very similar in form but smoother than roll and pitch profiles in figure 6.19).

From figure 6.19, we also observe that the yaw angle is controlled to the desired reference of \( 30^\circ \), but the yaw control is very slow (in comparison to the Nadir pointing case for instance), and the roll, pitch and yaw control precisions are all just under \( 5^\circ \). The reasons for this deterioration of the yaw control performance in the case of Sun tracking is that the roll and pitch control parameters must be increased to have trajectory control, and the consequence is a loss of control authority on the unactuated Z axis.

We can assert from figure (6.20) that the quaternion error is indeed controlled (when the underactuated controller is activated) to zero on the Z axis, and controlled average zero on the X and Y axes, with residual oscillations of magnitude 0.04 in quaternions (except instantaneous roll errors of transition to eclipse).

On figure (6.21), the wheels speeds start oscillating at higher frequency and magnitude when the underactuated is activated \( (t = 14000 \) seconds). However, the magnitude of less than 200 rpm is very reasonable (maximum admissible wheel speed is 5000 rpm), and the oscillations are of a 3 minutes pseudo period, which is also admissible (no torque saturation).

On figure (6.22), the control performance using a different choice of control parameters are shown. We notice that increasing the parameters \( k \), and decreasing \( g \) and \( a \), contribute
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

to the enhancement of roll and pitch control (the quaternion error is very close to zero, with a magnitude on the order of $2 \times 10^{-3}$, except during the transitions to eclipse). Unfortunately, the enhancement of roll and pitch control causes deterioration of the yaw control. In the case of figure (6.22), we completely lose control on the Z axis with an inappropriate choice of parameters.

![Diagram](image)

**Figure 6.19**: UoSAT-12 simulator- Euler angles during Sun tracking (case6)
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

Figure 6.20: UoSAT-12 simulator - quaternion error during Sun tracking (case 6)

Figure 6.21: UoSAT-12 simulator - wheels speeds during Sun tracking (case 6)
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6.3 In-orbit Results from UoSAT-12:

After the successful SSTL’s UoSAT-12 simulator results for the 3-axis attitude control using only two reaction wheels, the C function of the underactuated controller has been uploaded to the OBC. Unfortunately, only Sun tracking in-orbit experiments (which do not give the best performances), have been judged safe and possible due to the current power consumption problems of UoSAT-12. The risk was that excessive power loss could eventually cause the OBC to trip out and to be reset, and then all functions (not only ADCS would have to be uploaded again).

Nadir pointing experiments with two wheels would be achieved with significantly greater performance according to simulator results. However, that fact that UoSAT-12 is no longer safe for Nadir pointing means that the alternative is to have another satellite available for Nadir pointing experiments (UK-DMC after boom deployment).

The in-orbit implementation of the underactuated two wheels controller for Sun tracking has been achieved with the following parameter: $k = 0.01, g = 0.05, a = 0.05$. 

Figure 6.22: UoSAT-12 simulator - wheels speeds during Sun tracking (case 7)
6.3.1 Experiment 1

The result of a first in-orbit experiment is shown in figures (6.23), (6.24), and (6.25). The controller used from \( t=0 \) to \( t=t_0 \) was a combined linear controller, using two wheels in a PD controller, and magnetorquing in a cross product control law to detumble the spacecraft, dump the wheels momenta.

We notice on figure (6.23) that the roll and pitch angles were very well controlled to a Sun tracking trajectory using the combined “magnetorquing + 2wheels” linear controller. However, the yaw angle was oscillating and it was decided that an optimal orientation was \( \text{yaw} = 180^\circ \). At time \( t=t_0 \), under the required small momentum conditions, we switched to the underactuated 2 wheels controller, disabled the magnetorquing (as we can notice on figure (6.25)). We immediately obtained roll and pitch control, with a slightly higher error than using the linear 2-axis PD controller, but with even more precision than what was expected on the simulator. At the same time, the yaw angle seemed to be going with a good start towards the 180 degrees desired reference. However, an independent problem happened on UoSAT-12 the next day and no telemetry was available for the rest of the duration of underactuated control. The experiment shows though a good beginning with successful roll and pitch control and a good beginning in yaw control to 180°. The constraint in the case of Sun tracking is that the yaw control must unfortunately be slow to maintain roll and pitch profiles of Sun tracking.

On figure (6.24), we observe that the wheels speeds are within a reasonable range of magnitude and frequency, even during underactuated control (3 minutes period and less than 200rpm maximum wheel speed).
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

2 wheels only

2 wheels (PD) + MTQ Sun tracking

2 wheels only

Sun tracking

Figure 6.23: Euler angles during UoSAT-12 in-orbit experiment 1

Figure 6.24: Wheels speeds during UoSAT-12 in-orbit experiment 1
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6.3.2 Experiment 2

Following the encouraging results of the first in-orbit test, a second in-orbit experiment has been achieved to prove the control authority on the Z axis. A small but non-zero Y-wheel momentum bias of -80 rpm has been applied in combination with 3-axis magnetorquing up to \( t = t_0 \). The presence of a wheel momentum bias is known to be good for a better EKF estimation, and we obviously assumed a small momentum bias to maintain the small total angular momentum assumption.

On figure (6.26), we clearly observe that the yaw angle is controlled to zero using only the roll and pitch wheels after \( t = t_0 \). The yaw control is slow but this is expected for Sun tracking underactuated control application (not if we have the opportunity for Nadir pointing tests). The roll and pitch angles followed periodic periodical functions as required for purposes of Sun tracking. Neither the desired attitude nor the quaternion error was included in the log file, so it is difficult to give a very precise account of the roll and pitch control accuracies (although roll and pitch profiles look reasonable with moderate precision).
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

The objective of this in-orbit experiment was not the assessment of the control accuracy or rapidity of the underactuated control strategy for Sun tracking. It has already been explained that a precise roll and pitch trajectory control in the case of Sun tracking can only be achieved by deliberately slowing the yaw response. The experiment objective was to prove that the X and Y wheels can have a stabilising impact on the Z axis by bringing the attitude on that axis to zero in finite time. The 3-axis control is clearly proven in figure (6.26), and we clearly have a control authority on Z axis using only the X and Y wheels.

On figure (6.27), we notice that the wheels speeds have been reasonable (with a magnitude of less than 200 rpm and a 3 minutes pseudo period). We also see on figure (6.28) that the magnetorquing is aborted after the activation of the “two wheels” underactuated controller as required.

Figure 6.26: Euler angles during UoSAT-12 in-orbit experiment 2
Chapter 6. Results from UoSAT-12 simulator and in-orbit experiments

Figure 6.27: Wheels speeds during UoSAT-12 in-orbit experiment 2

Figure 6.28: Magnetic moment command during UoSAT-12 in-orbit experiment 2
6.3.3 Experiment 3

Following the successful 3-axis stability results on UoSAT-12, the underactuated control algorithm has also been uploaded to the ADCS processor of UK-DMC after slight modifications. In the case of UK-DMC, nadir pointing has been possible, unlike the case of UoSAT-12 where Sun tracking was the safe option.

UK-DMC is equipped with a yaw and pitch wheel, that is why the control law is very similar to the case of UoSAT-12, although the subscripts in the control law must be permuted. A smoothing parameter \( \nu \) has been used with parameters \( k_2, k_3, g_2, g_3 \) and a nonlinear singular quaternion control law of the form:

\[
\begin{align*}
\omega_{2d} &= -k_2 q_2 + g_2 \text{sat}\left( \frac{q_3}{q_2^2 + q_3^2 + \nu} q_1, a_2 \right) \\
\omega_{3d} &= -k_3 q_3 - g_3 \text{sat}\left( \frac{q_2}{q_2^2 + q_3^2 + \nu} q_1, a_3 \right)
\end{align*}
\]  

(6.19)

where the control parameters \( k_2, k_3, g_2, g_3, a_2, a_3, \nu \) are all positive , but unlike the mathematically proven control law, we have used \( k_2 \neq k_3, g_2 \neq g_3, a_2 \neq a_3 \). There is no theoretical proof of stability for the control law in equation (6.19) (which is the same as equation (4.42) only when \( k_2 = k_3, g_2 = g_3 \) and \( a_2 = a_3 \)). However, extensive simulator results, prior to the in-orbit experiment, have proven how a convenient choice of control parameters can ensure stability and good performance using the control law (6.19).

The reason for allowing for more control parameters during this experiment was due to the presence of a deployed gravity gradient boom on UKDMC. The effect of the boom is indeed a large amplification of the ratio between the desired angular velocities and the wheels speeds commands on the Y-axis, not on the Z-axis where the moment of inertia is small. Using high gains for the Y-wheel command, the high moment of inertia of that wheel could lead to very high wheels speeds, very possibly Y-wheel saturation (see equations (4.31) and replace subscript 1 by 2 and subscript 2 by 3). For the gains of the Z-wheel, the situation is different and the gains can be increased for better performance without leading to excessive torques on that axis. Therefore, the values of the control parameters for the Z-wheel can be significantly bigger than the ones used to control with the Y-wheel.
The inertia parameters of UK-DMC after the boom deployment are: \( I_1 = 103.2 \text{ kg.m}^2 \), 
\( I_2 = 102.8 \text{ kg.m}^2 \), \( I_3 = 5 \text{ kg.m}^2 \).

The control parameters used were: \( k_2 = 0.0035 \text{ rad.s}^{-1} \), \( g_2 = 0.0105 \text{ rad.s}^{-1} \), \( k_3 = 0.0105 \text{ rad.s}^{-1} \), \( g_3 = 0.021 \text{ rad.s}^{-1} \), \( a_2 = 0.04 \), \( a_3 = 0.12 \), \( k = 0.0012 \).

The in-orbit experimental results for a nadir experiment during 2 orbits are shown on figures (6.29), (6.30) and (6.31). The initial attitude was (-1.5° roll, -0.5° pitch and 10° yaw). On figure (6.30), we first observe that the satellite has been 3-axis stable during the 2 orbits. We also observe a very fast control initially, by bringing the roll angle (about the unactuated axis) to a precision of 0.75°, and the pitch and yaw angles to the desired 0 reference with a 2° precision, after only 2 minutes. The attitude remains in the (0.75° roll, 2° pitch and 2° yaw) precision during most of the 2 orbits, except during the periods of time marked by rectangles on figure (6.29).

During the periods of time marked by a rectangle on figure (6.29), the attitude control precision is temporarily deteriorated, and this phenomenon happens with a period of exactly one orbit. It therefore is an orbit related phenomenon, due to an external disturbance torque, which is particularly severe during particular parts of the orbit. This orbit-dependent disturbance torque, affecting the control system performance, (not dramatically though) is presumably due to atmospheric drag. Therefore, the only way to deal with it is by adding disturbance torque compensation (preferably via magnetorquing), only during the time of the limited attitude jump. This should be done by uploading dedicated new code to UK-DMC to fix the disturbance-related problem. The precision of (0.75° roll, 2°pitch, 2° yaw) should then be confirmed if not even improved. The “attitude jump” on the roll unactuated axis occurs with a delay after the simultaneous jump on the pitch and yaw axes. The fact that the attitude jump is accommodated means that the underactuated controller is robust to some extent to that disturbance, although not as robust as a controller with 3 torques.

The wheels speeds are also admissible as we can see on figure (6.30). The Y-wheel speed is high in comparison with the Z-wheel speed because of a high moment of inertia on the Y axis. However, the maximum wheel speed limit of 5000 RPM is far from being reached. The magnetic dipole moment is represented on figure (6.31), where we clearly
observe that magnetorquing stops firing during the implementation of the two wheels underactuated control.

![Graphs showing Euler angles and wheel speed commands](image)

**Figure 6.29:** Euler angles during UK-DMC in orbit experiment 3

**Figure 6.30:** Wheel speed commands during UK-DMC in orbit experiment 3
6.4 Conclusions

The underactuated non-smooth control strategy, with only two reaction wheels has been demonstrated from SSTL’s UoSAT-12 simulator results (including all UoSAT-12 ADCS processor functions + a complete satellite model accounting for process and measurement noises). Both Nadir pointing and Sun tracking are possible with a very good rapidity in the first case but not in the second one.

From the simulator results, we can also conclude that there is a trade-off between the precision (and rapidity) on the two actuated axes, and the precision (and rapidity) on the unactuated axis (Z axis).

A precision of 0.8° on the yaw angle has been obtained for a 5° precision on roll and pitch control, from UoSAT-12 simulator after a 20° to 25° manoeuvre to nadir pointing on all 3 axes, in no more than 5 minutes (see confirmation on figure (6.8)). Conversely, the control parameters can also be chosen to have a good roll, pitch pointing accuracy of 0.5° degrees, but the resulting yaw precision in this case is up to 5° and the yaw control is slow in this case (for confirmation see figure (6.13) or (6.11)).

The control parameters the best yaw control performance have been found not to be the best for roll and pitch pointing performances. The non-smooth “two wheels” controller is
the only way to have a precise yaw pointing (typically 0.8° yaw precision for UoSAT-12 simulator, and 0.5° yaw precision for UK-DMC simulator) after manoeuvres about the unactuated Z axis (or X axis in case of UK-DMC). Magnetorquers precision on the unactuated axis in practice is about 5° at best, and thrusters are not practical because of fuel consumption. The control precision along the unactuated axis under the same conditions on SSTL’s UK-DMC simulator (before boom deployment) is even better at 0.5° (see figure) and even rapidity is enhanced to 3 minutes instead of 5 minutes (UK-DMC only has pitch and yaw wheels). Before boom deployment, the problem of fairly high non-diagonal elements on UK-DMC makes the underactuated control code more appropriate and safer after boom deployment. The boom of UK-DMC has been deployed, and the new code after boom deployment has recently been uploaded.

Following the encouraging simulator results, the underactuated controller code has been uploaded to the UoSAT-12 ADCS processor (OBC) on May 2003. The in-orbit experiments on UoSAT-12 have only been possible for Sun tracking. The 3-axis stabilisation has been proven during in-orbit tests using only two control torques (from reaction wheels), with a clear control capability along the unactuated axis. The rapidity of the in-orbit Sun tracking control was not brilliant as expected from the Sun tracking simulator results.

All indications from the simulator study are that nadir pointing experiments should prove a significantly better control performance than Sun tracking on all 3 axes (control within 3 to 5 minutes minutes and yaw control precision of up to 0.8° (0.5° for UK-DMC simulator) depending on the desired roll and pitch precision). Nadir pointing experiments are not possible with UoSAT-12 due to power consumption problems, but could be possible on other operational SSTL’s small satellites such as UK-DMC (even with a deployed boom) with the only assumption of nearly diagonal inertia tensors.

Uploading new code is always a little risky. However, the code has been uploaded on UK-DMC. A successful in-orbit experiment on UK-DMC has recently been achieved, following the very promising results of SSTL’s UK-DMC simulator including most onboard functions (see figure 6.15 where the attitude is controlled with the ‘two wheels controller’ in presence of the deployed boom). The in-orbit experiment on UK-DMC has proven 3-axis control for nadir pointing during two orbits. This opportunity has never
been possible on UoSAT-12. The UK-DMC in-orbit experiment has even demonstrated a good control performance, (for both rapidity and precision) during most of the 2 orbits except the periodic durations when a more severe disturbance torques was acting on the satellite. There is now need to upload new code to compensate the orbit-dependant aerodynamic disturbance torque, which has presumably been causing the slight periodic deterioration problem.

To conclude, simulator results have proven 3-axis control without any ambiguity and with a very satisfactory control performance about the unactuated axis (not achievable by magnetorquing on that axis). In-orbit experiments on UoSAT-12 have proven good sun tracking on the actuated axes (on experiment 1) and slow but apparent yaw control about the unactuated Z axis (on experiment 2). The last in-orbit experiment, which has been achieved on UK-DMC has proven the potentially efficient 3-axis control for nadir pointing. However, future work should be undertaken to deal with the external disturbance torque problem.

Finally, the nonlinear singular control approach, with a few minor modifications, appears to be an elegant and efficient, if not the most efficient strategy to have 3-axis control after a wheel failure (or even if the satellite is built with only 2 wheels).
Chapter 7

7. Conclusions and future work

In this thesis, the 3-axis attitude stabilisation and control of underactuated small satellites has been investigated in detail, in both cases of a thruster failure and a reaction wheel failure. In both cases, the system appeared to be nonholonomic. This makes the control problem very challenging since it requires non-smooth and absolutely non-standard techniques. The attitude control problem has been analysed from a practical point of view by resolving problems that had never been addressed before in the case of underactuated satellites.

We have presented some of the latest theory dealing with the control problem using only two pairs of thrusters. In the control design, novel parameterisations of the attitude kinematics were employed to formulate the non-smooth time varying or singular control laws.

The basic principle using thrusters was the design of desired angular velocity trajectories along the actuated axes, which must be tracked to guarantee the full system stabilisation. It has been shown that the stabilisation of the complete system, (dynamic + kinematic cascade interconnection) which had never been simulated or presented in detail before, requires very high control torques in practice to track the desired path of the actively controlled angular velocities.

The degradation in the performances of three different non-smooth controllers with two pairs of thrusters, in presence of a realistic torque saturation level, has been illustrated. The 3-axis attitude control has been addressed in the cases of axis-symmetrical and asymmetric satellites. For symmetrical satellites initially spinning about the unactuated axis, we have demonstrated that the system can be slewed to a neighbourhood of the desired reference, with residual constant amplitude oscillations. The effect of an external disturbance torque on the unactuated axis has been analysed, and low cost magnetorquing on that axis has been proposed as a solution to the problem.
For variable thrusters, it has been shown that the control torque oscillates between maximum levels for some time before it starts fading towards zero. The case of two pairs of on/off thrusters with PWM (never addressed before) was even more alarming with a permanent firing between maximum levels to keep the attitude contained in a neighbourhood of the desired reference (even by assuming dead zones). It was concluded that the attitude control with two pairs of thrusters was not practically feasible with the present technologies of on/off thrusters used onboard small satellites.

Using momentum exchange devices, namely reaction wheels, the results have been much more encouraging. For a zero total momentum satellite, stability on all three axes, using two reaction wheels, has been demonstrated (with the Rodriguez parameterisation) using known and novel control strategies based on non-smooth feedbacks. The control authority has been considerably enhanced, in terms of precision, rapidity, and robustness, (to sampling, oscillations), by employing mathematically singular feedbacks. Using that approach, decisive control on all 3 axes within 3 minutes has been proven.

In the presence of a bias momentum, it has been demonstrated that the attitude can still be brought average zero (or to the desired reference), with constant amplitude residual oscillations. The effect of external disturbance torques has been demonstrated to be much smaller than using thrusters, and it was concluded that additional external disturbance torque monitoring was only necessary for very long time application of the underactuated attitude control strategy.

A detumbling manoeuvre has been judged necessary, prior to the activation of the underactuated control strategy, in the case of a high bias momentum mode. For a high bias momentum mode, a simple combined control technique, based on a magnetorquing cross product law and a PD two wheels controller, has been proposed as a way to ensure satisfactory conditions prior to the activation of the underactuated non-smooth controller with only two wheels. The 3-axis stabilisation, with two wheels only, has been demonstrated for such a general scenario (detumbling + stabilisation).

In the presence of process and measurement noises and estimation errors, it has also been demonstrated on SSTL’s UoSat-12 simulator (in C) that the attitude can be controlled to the desired reference on all three axes, with only two wheels, in very reasonable time and with reasonable torque expenditure. It appears that the attitude control on the unactuated...
axis is generally achieved with even better precision than the actively controlled axes (property of the non-smooth asymptotic stabilisation). We also observed that a trade-off between rapidity and precision is required. A compromise between precision on actuated axes, and precision on the unactuated axis is also needed to have the best control performance. The 3-axis attitude control has been demonstrated on the simulator for both nadir pointing and Sun tracking modes. Following the encouraging simulator results, the underactuated control strategy has been uploaded to UoSat-12, and in-orbit tests have been successfully carried out for Sun tracking (Sun tracking was the only safe mode for UoSat-12 due to power loss). The precision on the actuated axes has been shown to be a function of the initial momentum, and the attitude stabilisation on the unactuated axis has been proven in-orbit. The in-orbit control on the unactuated axis was slow, as expected for sun tracking via simulator results. The expected (via simulator) control performance for nadir pointing is considerably better (both in rapidity and precision).

The proposed nonlinear singular control strategy appears to be potentially the most efficient and practical way of controlling underactuated satellites. Nearly symmetrical micro-satellites or mini-satellites appear to be the best candidates for high performance attitude control with two wheels (especially for earth pointing or imaging applications). For instance, a precision of 5° on roll and pitch angles pointing, and 0.5° on yaw after a large manoeuvre on the yaw unactuated axis is feasible for typical Surrey SSTL’s micro-satellites or mini-satellites (assuming reasonably diagonal inertia tensor as for UoSat-12). There is even a possibility to improve roll and pitch angles pointing by smoothing the control laws, although this will lead to a small yaw static error.

The in-orbit experiment on UK-DMC has also successfully demonstrated nadir pointing (3-axis control). As expected from SSTL’s simulator study, the experiment has proven a better control performance than the sun tracking experiments, which were achieved on UoSAT-12. A rapid slew on all 3-axis has been observed during the in-orbit implementation of underactuated control with two wheels. The precision has also been very satisfactory (0.75° roll, 2° pitch and 2° yaw), except during a relatively small part of the orbit, where external disturbance torques have caused a slight performance deterioration.

For future applications, the potentially high performance of the underactuated control strategies should be confirmed even better. An improved code should be uploaded to
specifically compensate for the periodic disturbances during in-orbit underactuated control of micro-satellites (preferably with diagonal inertia tensor). The disturbance torque compensation would only be required during the particular durations when the disturbance torque is potentially high. On the other hand, other momentum exchange devices such as single gimbal CMGs can be regarded as possible actuators for the 3-axis attitude stabilisation with two control torques. The available control torque for the stabilisation is significantly higher in the case of CMGs, which should imply higher control performance. In fact, it is very likely that only two CMGs might be sufficient for agile 3-axis control. At present, Tubitak-Bilsat, which is equipped with 2 CMGs is a perfect example of a possible application platform. The only additional control difficulty using two CMGs would be the singularity avoidance (which is apparently more difficult with less actuators).

To summarize, the 3-axis attitude stabilisation and control of underactuated satellites with only two control torques has been successfully demonstrated using both thrusters and reaction wheels. Reaction wheels have been proven more practical to deal with the stabilisation of underactuated small satellites. Surprisingly high performance 3-axis attitude control has been obtained using two reaction wheels, using non-smooth strategies. The open problems of the presence of a bias momentum or external disturbances have been practically dealt with. Very promising and encouraging results have been obtained for zero or small momentum satellites. In practice, very satisfactory performances have been obtained on the UoSat-12 simulator (including effects of noise...). In-orbit tests have proven the stability on all three axes (including the unactuated axis) during sun-tracking mode. The 3-axis stability using only two control torques had never been proven in-orbit in any mission before, despite the existing theoretical investigations in the literature.

To conclude, one consequence of these results, if they are taken further, is that a fully redundant 3-axis control is possible using only 3 orthogonal reaction wheels, and that two wheels represent a sufficient non-redundant configuration for 3-axis control. This is clearly an advantage for small satellites constellations, where it is generally preferred to reduce the mass to the limit. The 3-axis control results can also be significantly enhanced using two single gimbal CMGs. Finally, underactuated control code, similar to the
techniques proposed in this research, should be included in the ADCS software of future space missions to avoid the often severe consequences of actuator failures.
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- A further journal paper (addressing underactuated control in two phases) is to be submitted to the icce transactions on aerospace and electronic systems.
Appendix A: Proofs of Stability

Appendix A

Proofs of stability for the controllers based on two pairs of thrusters

Tsiontras controller

For the sake of simplicity of the proof, we consider $\omega_z(0) = 0$ (the general case has been conjectured by P. Tsiontras based on extensive numerical simulations but not proven). The purpose here is to show how the stabilization works in that case. Concerning the more complex control law for asymmetric satellites from any initial condition, it was only conjectured by Tsiontras that the system would be stable, based on extensive numerical simulations.

A particular attention is given to the control of the orientation about the unactuated axis ($Z$ axis), which is stabilized using interconnection singular terms in the controller.

We restrict our attention to $w \neq 0$, the proposed control law is:

\[
\begin{align*}
\omega_1 &= -k_w w_1 + k_z \frac{z}{w_1^2 + w_2^2} w_2 \\
\omega_2 &= -k_w w_2 - k_z \frac{z}{w_1^2 + w_2^2} w_1
\end{align*}
\]  

(A.1)

with $k_w > 0, k_z > 0$.

We recall that the kinematic model using Tsiontras parameters is given by: [Tsiontras 2000a]
Appendix A: Proofs of Stability

\[
\begin{align*}
\dot{w}_1 &= \omega_2 w_1 w_2 + \frac{\omega_1}{2} (1 + w_1^2 - w_2^2) \\
\dot{w}_2 &= \omega_1 w_1 w_2 + \frac{\omega_2}{2} (1 + w_2^2 - w_1^2) \\
\dot{z} &= -\omega_1 w_2 + \omega_2 w_1
\end{align*}
\] (A.2)

In equation (A.1), the continuous parts of these feedback laws \(-k_w w_1, -k_w w_2\) are used to stabilize the attitude variables \(w_1, w_2\).

**Stability of z:**

The singular part in the feedbacks will have an impact on the \(z\) dynamic equation.

By replacing \(\omega_1, \omega_2\) into equation (A.2) by their expressions in (A.1), we obtain:

\[
\dot{z} = -k_z z
\] (A.3)

Integrating this equation, we simply have:

\[
z = z(0)e^{-k_z t}
\] (A.4)

Therefore, \(z\) is bounded and \(\lim_{t \to \infty} z(t) = 0\).

**Stability of \(w_1, w_2\):**

We need the complex formulation of the Tsiontas parameterization \((w = w_1 + iw_2)\).

In fact, it is convenient to prove that the complex \(|w|^2 = w_1^2 + w_2^2\) (with \(|w| \neq 0\)) is stabilized.

The complex form of the dynamic model is:

\[
\begin{align*}
\dot{w} &= \frac{\omega}{2} + \overline{\omega} w^2 \\
\dot{z} &= \frac{i}{2} (\overline{\omega} w - \omega \overline{w})
\end{align*}
\] (A.5)
where $\omega = \omega_1 + i\omega_2$ is a complex variable representing the angular velocity.

The controller equation in this complex form is:

$$\omega = -k_w w - i \frac{k_s}{w} \frac{\dot{w}}{w} \quad (A.6)$$

Using the norm variable $|w|$, we have:

$$\frac{d}{dt} |w|^2 = 2 \text{Re}(\dot{w}w) \quad (A.7)$$

By replacing the derivative of $w$ by its expression into (A.7), we have:

$$\frac{d}{dt} |w|^2 = -k_w |w|^2 (1 + |w|^2) \quad (A.8)$$

Equation (A.8) is a differential equation of the form $\dot{x} = -k_w x(x + 1)$ with $x = |w|^2$.

The solution of this equation (which can be obtained using Laplace transform techniques for instance), is:

$$|w(t)| = \left( \frac{1}{c_0 e^{k_w t} - 1} \right)^{1/2} \quad (A.9)$$

where $c_0 = \frac{1 + |w(0)|^2}{|w(0)|^2}$.

And we have:

$$\lim_{t \to \infty} w(t) = 0 \text{ with } w(t) \neq 0 \quad (A.10)$$

Therefore, from (A.9) and (A.4), the 3-axis stability is guaranteed.
Appendix A: Proofs of Stability

Kim controller:

The proof here is made for a symmetrical satellite. In this case \( \omega_3(0) = 0 \Rightarrow \omega_3(t) = 0 \ \forall t \), and the dynamic model using quaternion modeling is:

\[
\begin{align*}
\dot{q}_1 &= \frac{1}{2} q_4 \omega_1 - \frac{1}{2} q_3 \omega_2 \\
\dot{q}_2 &= \frac{1}{2} q_3 \omega_1 + \frac{1}{2} q_4 \omega_2 \\
\dot{q}_3 &= -\frac{1}{2} q_2 \omega_1 + \frac{1}{2} q_1 \omega_2 \\
\dot{q}_4 &= -\frac{1}{2} q_1 \omega_1 - \frac{1}{2} q_2 \omega_2 
\end{align*}
\]

(A.11)

The proposed angular velocity commands are:

\[
\begin{align*}
\omega_{1d} &= -g_1 q_1 + g_2 \frac{q_3 q_3}{q_1^2 + q_2^2} \\
\omega_{2d} &= -g_1 q_2 - g_2 \frac{q_1 q_3}{q_1^2 + q_2^2}
\end{align*}
\]

(A.12)

The interconnections terms in the control law will also contribute here to the stabilization of the third quaternion \( q_3 \).

The differential equation on the third axis is:

\[
\dot{q}_3 = -\frac{1}{2} q_2 \omega_1 + \frac{1}{2} q_1 \omega_2
\]

(A.13)

By replacing \( \omega_1, \omega_2 \) by the expressions of the desired angular velocities \( \omega_{1d}, \omega_{2d} \) (perfect angular velocity tracking) we obtain:

\[
\dot{q}_3 = -\frac{1}{2} q_2 \left( -g_1 q_1 + g_2 \frac{q_3 q_3}{q_1^2 + q_2^2} \right) + \frac{1}{2} \left( -g_1 q_2 - g_2 \frac{q_1 q_3}{q_1^2 + q_2^2} \right)
\]

(A.14)

By expanding (A.14), we simply have:
Appendix A: Proofs of Stability

\[ \dot{q}_3 = -\frac{\theta_2}{2} q_3 \quad (A.15) \]

The attitude on the unactuated axis is then simply given by:

\[ q_3(t) = q_3(0)e^{-\frac{\theta_2}{2} t} \quad (A.16) \]

Therefore \( q_3 \) exponentially converges to zero.

The proof of stability of the Kim controller on the X and Y axes in this case has been conjectured but not strictly proven (The X and Y axes stability will be strictly proven in the case of wheels in chapters 4 and 5). The control law proposed for the stabilization of the complete system (assuming a sliding surface) was also conjectured based on physical intuition and extensive numerical simulations.

- The proofs of stability using the Morin controller with pairs of thrusters are based on most complicated homogeneity and center manifold theory. They are available in reference [Morin 1997].

- The proofs of stability in the case of the reaction wheels control strategies on all 3 axes are given in the chapters 4 and 5.
Appendix B

The zero total angular momentum satellite

We make the proof that the kinematic equation of an underactuated satellite, which has lost a wheel on the Z-axis, using Rodriguez parameters (equation (2.42)), reduces to the Brockett integrator when $H=0$. (see ref [Yamada 1998]).

Zero total momentum condition

We now express the zero total angular momentum condition: $H=0$

$$H = L = 0_{3x1} \quad (B.1)$$

Where $H$ represents the total momentum in inertial frame, and $L$ is the total momentum in body frame.

Equation (B.1) can be written for each component in the simple form:

$$h_i = -I_i \omega_i$$

$$h_2 = -I_2 \omega_2$$

$$h_3 = \omega_3 = 0 \quad (B.2)$$

where $I_i$, $h_i$, and $\omega_i$ for $i=1, 2, 3$ are respectively the moments of inertia of the satellite (including wheels), the angular momentum of the $i^{th}$ wheel, and the $i^{th}$ angular velocity component of the satellite.

Let’s also recall that the angular momentum of the $i^{th}$ wheel is given by: $h_i = I_{wi} \dot{\omega}_i$, where $I_{wi}$ represents the moment of inertia of the wheel, and $\dot{\omega}_i$ is the wheel speed.

From the first and second equations of (B.2), the wheels speeds commands are given by:

$$\dot{\omega}_1 = \frac{-I_1}{I_{w1}} \omega_1$$

$$\dot{\omega}_2 = \frac{-I_2}{I_{w2}} \omega_2 \quad (B.3)$$
Rodriguez parameterisation

Using the Rodriguez parameterisation, the kinematic equation of a satellite is given by:

\[
\begin{align*}
\dot{\omega}_1 &= \frac{1}{2} \left( \omega_1 - (p_3 - p_1 p_2) \omega_2 + (p_2 + p_1 p_3) \omega_3 + p_1^2 \omega_1 \right) \\
\dot{\omega}_2 &= \frac{1}{2} \left( \omega_2 + (p_3 + p_1 p_2) \omega_1 - (p_1 - p_2 p_3) \omega_3 + p_3^2 \omega_2 \right) \\
\dot{\omega}_3 &= \frac{1}{2} \left( \omega_3 - (p_2 - p_1 p_3) \omega_1 + (p_1 + p_2 p_3) \omega_2 + p_2^2 \omega_3 \right)
\end{align*}
\]  

(B.4)

By assuming a zero total angular momentum, we have \( \omega = \omega_1 z_1 + \omega_2 z_2 \), where \( z_1 \) and \( z_2 \) are unit vectors along the X and Y body axes (also wheels axes) of the satellite, and we substitute \( \omega_3 \) by zero in equation (B.4). The kinematic model in this case is given by:

\[
\begin{align*}
\dot{\omega}_1 &= \frac{1}{2} \left( (1 + p_1^2) \omega_1 - (p_3 - p_1 p_2) \omega_2 \right) \\
\dot{\omega}_2 &= \frac{1}{2} \left( (1 + p_2^2) \omega_2 + (p_3 + p_1 p_2) \omega_1 \right) \\
\dot{\omega}_3 &= \frac{1}{2} \left( (p_1 p_3 - p_2) \omega_1 + (p_2 p_3 + p_3) \omega_2 \right)
\end{align*}
\]  

(B.5)

From the first an second equations of (B.5), we can simply express the equation relating \( \omega_1, \omega_2 \) to \( \dot{\omega}_1, \dot{\omega}_2 \) as follows:

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2
\end{bmatrix} = \frac{2}{1 + p_1^2 + p_2^2 + p_3^2} \begin{bmatrix}
1 + p_2^2 & - (p_3 + p_1 p_2) \\
-(p_3 + p_1 p_2) & 1 + p_1^2
\end{bmatrix} \begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2
\end{bmatrix}
\]  

(B.6)

By substituting the results of (B.6) into the third equation of (B.5), we obtain:

\[
p_3 = -p_2 \dot{\omega}_1 + p_1 \dot{\omega}_2
\]  

(B.7)

Brockett integrator

By defining the new input variables \( u_1 \) and \( u_2 \) as \( \dot{\omega}_1 = u_1 \) and \( \dot{\omega}_2 = u_2 \), we simply have:

\[
\begin{align*}
\dot{\omega}_1 &= u_1 \\
\dot{\omega}_2 &= u_2 \\
\dot{\omega}_3 &= -p_2 u_1 + p_1 u_2
\end{align*}
\]  

(B.8)
Which is the equation of the so-called Brockett integrator (nonholonomic integrator).

The Brockett integrator is not controllable using smooth feedbacks (result demonstrated in [Brockett 1983]. The control of the Brockett integrator is dealt with in chapter 4 using a singular nonlinear feedback approach.

**Computation of the wheels speeds commands**

When $u_1$ and $u_2$ are determined from a control law, $\omega_1$ and $\omega_2$ are also determined from equation (B.6) and by applying the first and second equations of (B.8). The equation (B.3) can then be used to determine the wheels speeds commands.
Appendix C

Sequence of orientations during a typical manoeuvre with two wheels

In the figures below, the blue colour corresponds to the X axis, the green colour to the Y axis and the red colour corresponds to the Z axis. The body axes are represented by the bold coloured lines, and the axes of the desired orientation (local orbital frame) are represented with the thin coloured lines.

The singular controller is used. The sequence starts from a configuration with nearly 90 degrees pointing error on each axis, and the order of the sequence is from left to right and from the top to the bottom. We observe on the last 7 figures that the orientation about the Z body axis is first controlled before the pointing of the Z body axis is obtained.
Appendix C: sequence of orientations during a typical manoeuvre with two wheels
Figure C.1: Sequence of transient orientations during a large angle manoeuvre to Nadir pointing with two reaction wheels (every 20 seconds)
Appendix D

Theory definitions

D.1 Stability definitions

For a general non-autonomous nonlinear system of the form:

\[ \dot{x} = f(x,t) \]  \hspace{1cm} (D.1)

We can define \( x \) as an equilibrium point if:

\[ f(x,0) = 0 \text{ } \forall t \]  \hspace{1cm} (D.2)

For the sake of simplicity of the definitions, we can generally assume the 0 as an equilibrium point (without any loss of generality because we can always return to 0 by change of variable).

- Local Stability of an equilibrium point:

An equilibrium point is stable if all initial conditions that start near the equilibrium point, stay near it. For a 0 equilibrium point, we have stability if and only if:

\[ \forall \epsilon > 0, \exists \delta(\epsilon) \text{ such that: } \|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \]  \hspace{1cm} (D.3)

where \( \| \| \) is a standard Euclidian vector norm.

An equilibrium point is unstable if some initial conditions cause divergence from the equilibrium point.
Appendix D. Theory definitions

- **Attractivity:**

The 0 equilibrium point is attractive if and only if:

\[ \exists \delta > 0 \text{ such that: } \|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} f(x,t) = 0 \]  \hspace{1cm} (D.4)

- **Local Asymptotic stability of an equilibrium point:**

An equilibrium point is asymptotically stable if all initial conditions converge to the equilibrium point.

0 is an asymptotically stable equilibrium point \( \iff \) The 0 equilibrium point is stable and attractive

By definition, asymptotic stability means that the process is a limiting one. So, the equilibrium point is reached at \( t = \text{infinity} \) (which is not defined). At no finite time will the state of the process actually be equal to the final point, although it can be said to be getting "closer" to the final point with time.

Exponential stability is a particular case of asymptotic stability when the state convergence of \( f(x,t) \) is exponential.

- **Global stability:**

In this thesis, non-smooth control laws are designed to ensure global asymptotic stability.

An equilibrium point (typically 0) is *globaly stable* if the condition (D.3) is true \( \forall \delta \).

The 0 equilibrium point is *globally asymptotically stable* if it is stable and attractive \( \forall \delta \).

**D.2 Homogeneity theory definitions**

Homogeneity theory has been extensively investigated by authors such as Tsiotras (reference [Tsiotras 2001]), Morin and Samsom (references [Morin 1997]) as a way to design continuous time varying stabilising feedbacks for underactuated satellites. The Morin and Samsom time varying control approach has been presented in chapter 3, and the control performance on a micro-satellite investigated in chapter 4.
Appendix D. Theory definitions

To have a better understanding of the control approach based on homogeneity theory, we present here some important and useful definitions.

- **Dilation parameter** \( \lambda > 0 \):

  For any \( \lambda > 0 \) and any set of real parameters \( r_i > 0 \) (\( i=1,\ldots,n \)), then the dilation operator \( \delta_\lambda \) is defined by:

  \[
  \delta_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n \\
  \delta_\lambda(x_1,x_2,\ldots,x_n) = (\lambda^{r_1}x_1, \lambda^{r_2}x_2,\ldots,\lambda^{r_n}x_n) \quad (D.5)
  \]

- **Homogenous norm**:

  We can then define what is called a homogenous norm associated with this dilation operator as:

  \[
  \rho_\lambda(x) = \left( \sum_{j=1}^{n} |x_j|^{\lambda r_j} \right)^{\frac{1}{\lambda}} \quad \text{with} \; \lambda > 0 \quad (D.6)
  \]

  And a continuous function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is homogenous of degree \( l \geq 0 \) with respect to the dilation \( \delta_\lambda \) if:

  \[
  \forall \lambda > 0, \quad f(\delta_\lambda(x)) = \lambda^l f(x) \quad (D.7)
  \]

- **Homogenous system**:

  A differential system \( \dot{x} = f(x) \), with \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is homogenous of degree \( l \geq 0 \) with respect to the dilation \( \delta_\lambda \) if:

  \[
  \forall \lambda > 0, \quad f_i(\delta_\lambda(x)) = \lambda^{rr_i} f_i(x) \quad i = 1,\ldots,n \quad (D.8)
  \]

  Using the properties of homogeneity, Morin and Samson used some results related to the exponential stabilization under homogeneity conditions.

  In fact, the first result used comes from the proposition of Pomet and Samson that establishes the existence of homogenous Lyapunov functions for time varying asymptotically stable systems, which are homogenous of degree zero with respect to some dilation.

D-3
A proposition has been proven by Morin and Samsom for a general non-homogenous system:

\[ \dot{x} = f_{\text{inh}}(x, t) \]  

(D.9)

It was proven that if we can write the system (D.9) as:

\[ \dot{x} = f(x, t) + g(x, t) \]  

(D.10)

where \( f \) is homogenous of degree zero with respect to a given dilation, and \( g \) is a continuous \( T \) periodic function and defines a sum of homogenous vector fields of degree strictly positive with respect to that dilation, then it becomes sufficient to show that the origin of the system:

\[ \dot{x} = f(x, t) \]  

(D.11)

is locally asymptotically stable to have the stability of the system (D.9). This property can be used when only a reduced part of the system is homogenous.