Dynamical friction on star clusters: implications for the Galactic Centre

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Abstract

I present a new semi-analytic dynamical friction model built upon Chandrasekhar’s formalism (Petts et al. 2015, 2016), and its first scientific application regarding the origin of the young stellar populations in the Galactic Centre (Petts and Gualandris 2017). The model is accurate for spherical potentials of varying inner slope, $\gamma = [0, 2]$, due to a few key novelties. Firstly, I use physically motivated, radially varying maximum and minimum impact parameters, that describe the range over which interactions are important. Secondly, I use the self-consistent velocity distribution as derived from the distribution function of the galactic potential, including the effect of stars moving faster than satellite. Finally, I reproduce the core-stalling effect seen in simulations of cored galaxies with a “tidal-stalling” prescription, which describes when the satellite disrupts the galaxy and forms a steady-state. I implemented dynamical friction analytically in the direct summation $N$-body code, NBODY6, excellently reproducing the orbital decay of clusters as compared with full $N$-body models. Since only cluster stars need be modelled in an $N$-body fashion, my method allows for simulation possibilities that were previously prohibited (e.g. Contenta et al. 2017; Inoue 2017; Cole et al. 2017).

Using this new method, I explore the scenario in which the young stellar populations in the central parsec of the Milky Way were formed by infalling star clusters. I find that clusters massive enough to reach the central parsec within the lifetime of these populations form very massive stars via collisions. Using up to date – yet conservative – mass loss recipes, I find that these very massive stars lose most of their mass via strong stellar winds, forming large stellar mass black holes incapable of bringing stars to the central parsec. A star cluster infalling in the Galactic Centre within the last 15 Myr would leave an observable population of massive stars from $\sim 1 - 10$ pc, contradicting observations. Thus, I rule out the star cluster inspiral scenario, favouring in-situ formation and/or binary disruption for the origin of the young stars.
Dedication

This thesis is dedicated to all those who have supported me along the way, from the frustration of hitting seemingly dead ends, through to the euphoria of discovery. Namely, my darling Marija and my supportive family: Colin, Marja and Hanna.
Declaration

“I confirm that the submitted work is my own work and that I have clearly identified and fully acknowledged all material that is entitled to be attributed to others (whether published or unpublished) using the referencing system set out in the programme handbook. I agree that the University may submit my work to means of checking this, such as the plagiarism detection service Turnitin UK. I confirm that I understand that assessed work that has been shown to have been plagiarised will be penalised.”

The majority of the work in this thesis has appeared in the following publications: Petts et al. (2015), Petts et al. (2016) and Petts and Gualandris (2017).

1This dedication is not in my own words but was downloaded from https://www.surrey.ac.uk/learningandteaching/regulations/Contents/2013-14_section_e-regulations_for_academic_integrity_final.pdf
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Chapter 1

Introduction

1.1 The Galactic Centre - A Unique Laboratory

Figure 1.1: Composite colour infrared image of the Galactic Centre. The image combines observations by Hubble Space Telescope’s Near Infrared Camera and Multi-Object Spectrometer (NICMOS) with colour Spitzer Space Telescope images taken by its Infrared Astronomy Camera (IRAC). The image is available in the public domain and credit goes to NASA and STScI.

The centre of our Galaxy is one of the most studied yet enigmatic regions of the Universe, with many open questions about its structure, formation and evolution. Its proximity to us, $8.27 \pm 0.09 \pm 0.1$ kpc (Chatzopoulos et al., 2015), makes the Galactic Centre a unique astronomical laboratory, as the only galactic nucleus where we may resolve the motions of individual stars in the vicinity of a supermassive black hole (SMBH). Fig. 1.1 shows a combined Hubble/Spitzer infrared image of the Galactic Centre region, which shows rich morphology in both its stellar and gaseous components. At the very centre lives a nuclear star cluster (NSC) comprised of both old and young stellar populations, as well as regions of molecular and ionized gas. The NSC harbours a powerful radio source deep within its core, SgrA*, produced by synchrotron emission from relativistic electrons spiralling around magnetic field lines in the presence of a SMBH (see Chapter 1.1.1). Fig. 1.2 shows a schematic view of the central region surrounding SgrA*.
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large gas clouds of mass $\sim 5 \times 10^5 M_\odot$, M-0.02-0.07 and M-0.13-0.08, are projected $\sim 7$ and $\sim 13$ pc from SgrA*, respectively. Overlapping with SgrA* are two regions of ionized gas, SgrA East and SgrA West (Novak et al. 2000; Zhao et al. 2009). SgrA East is a shell structure elongated along the Galactic Plane, and is generally thought to be a supernova remnant (Novak et al. 2000). SgrA West (also known as the “Mini-Spiral”) is a spiral shaped feature with three main arms, the Northern Arm, Eastern Arm and Western Arc (Ekers et al. 1983; Lo and Claussen 1983; Scoville et al. 2003). The Mini-Spiral is thought to be comprised of streams of gas that are spiralling towards SgrA* (Lo and Claussen 1983; Zhao et al. 2009). A circumnuclear ring (CNR) of molecular gas with inner radius $\sim 1.5$ pc surrounds SgrA*. The outer radius is less defined. Molecular gas has been observed out to $\sim 7$ pc, but recent studies suggest the CNR itself only extends to $3-4$ pc (Wright et al. 2001). The CNR does not appear to be currently star-forming, as it would need to be an order of magnitude more dense in order to overcome the tidal forces of SgrA* and fragment into stars (Becklin et al. 1982).

Rather surprisingly, in the immediate vicinity of SgrA* there exists two young populations which are discussed in detail in section §1.1.2 and are one of the main focii of this thesis. Two young dense star clusters reside near the NSC, Arches (Nagata et al. 1995; Cotera et al. 1996; Figer et al. 1999) and Quintuplet (Okuda et al. 1990; Nagata et al. 1990; Glass et al. 1990; Figer et al. 1999) projected only $\sim 30$ pc from the NSC (see Fig. 1.1). The luminous Wolf-Rayet and O stars present within these clusters, such as the Quintuplet’s famous Pistol Star (Glass et al. 1999; Figer et al. 1995), are unequivocal evidence of ongoing star formation in this crowded region, despite the extreme tidal forces induced by the presence of the NSC. The Galactic Centre has been the target of many observational and theoretical studies, and I refer the reader to (Mapelli and Gualandris 2016) for a more comprehensive review.

1.1.1 The supermassive black hole at the centre of our Galaxy

The first strong evidence for a compact dark mass in the centre of the Milky Way arose from radial velocity measurements of stars projected within 1 pc of SgrA*, obtained by their near-infrared spectra (McGinn et al. 1989; Sellgren et al. 1990; Haller et al. 1996). These works measured $\sim 3 \times 10^6 M_\odot$ within 0.1 pc, with a corresponding average density of $\sim 3 \times 10^9 M_\odot$ pc$^{-3}$, consistent with a compact cluster of stellar remnants (see Maoz 1998). However, measurements of the proper motions of these stars (Genzel et al. 1997; Eckart and Genzel 1997; Ghez et al. 1998) tightened the constraints, measuring $\sim 2.6 \times 10^6 M_\odot$ confined within $\sim 0.01$ pc, implying a density $\geq 10^{12} M_\odot$ pc$^{-3}$. A cluster of dark remnants was ruled out in this case, and such a density was indicative of the presence of a SMBH.

More recently, mass measurements of SgrA* were calculated by tracing the orbit of the bright young star S0-2 (or simply, S2), which has an orbital period of just 15.9 yrs (Schödel et al. 2002; Ghez et al. 2003, 2008; Gillessen et al. 2009a). The most recent SMBH mass derived from the orbits of 28 S-stars (see Chapter §1.1.2) is $4.3 \pm 0.2|_{\text{stat}} \pm 0.3|_{\text{sys}} \times 10^6 M_\odot$ (Gillessen et al. 2009a).
Figure 1.2: Schematic diagram of the Galactic Centre region, showing rough sizes and shapes of the features in the central few parsecs. This figure is inspired by fig. 1 from Novak et al. (2000) and fig. 12 from Mapelli and Gualandris (2016).
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1.1.2 The young stellar populations within the central parsec

The central parsec of the NSC in the Galactic Centre hosts two enigmatic populations of hot young stars. Firstly, there exists a thin disk which extends from $0.04 \lesssim r \lesssim 1 \, \text{pc}$ (Eckart et al., 1999), which consists of almost two dozen He-1 emission-line stars (Krabbe et al., 1995; Paumard et al., 2001) and a population of many other OB stars, with 90% being projected within $0.5 \, \text{pc}$ of SgrA* (Feldmeier-Krause et al., 2015). Krabbe et al. (1995) estimate the He-1 line emission stars to be only $\sim 3 - 7 \, \text{Myr}$ old. The disk exhibits a peculiarly top-heavy mass function, with a slope of $\alpha \sim 1.7$ (Lu et al., 2013), significantly shallower than the typical slope of $\alpha \sim 2.3$ (Salpeter, 1955). The disk rotates clockwise with respect to the sky and is known in the literature – and hereafter – as the “Clockwise Disk”.

A fainter disk has also been reported to be rotating anti-clockwise on the sky (Paumard et al., 2006), however recent proper motion measurements of the stars in this region only find the presence of the Clockwise Disk to be statistically significant (Lu et al., 2009; Yelda et al., 2014).

Interior to the Clockwise Disk exists a tight, spatially isotropic distribution of B-stars around SgrA*, with a super-thermal eccentricity distribution (Gillessen et al., 2009b; Mapelli and Gualandris, 2016). These “S-stars” have semi-major axes less than $0.04 \, \text{pc}$, with S0-102 having the shortest period of just $11.5 \pm 0.3 \, \text{yrs}$, with a pericentre approach of only $\sim 260 \, \text{AU}$ (Meyer et al., 2012). The brightest S-star, S2, is a main sequence B0-B2.5V star with an age $< 15 \, \text{Myr}$ (Martins et al., 2008). The other S-stars have spectra consistent with main sequence stars (Eisenhauer et al., 2005), and observational limits require them to be less than $20 \, \text{Myr}$ old in order to be visible, implying that it is likely that at least all the S-stars originate from the same star formation event. These age constraints mean that the S-stars could either have formed at the same time as the stars in the Clockwise Disk, or from an earlier – yet still very recent – star formation event $\sim 8 - 12 \, \text{Myr}$ prior to the formation of the disk.

The fact that stars are forming today so close to a SMBH is very puzzling and is a huge challenge for current star formation theories. The tremendous tidal forces present in the vicinity of a SMBH make it difficult for a giant molecular cloud (GMC) to remain bound long enough for gas to cool and fragment (Phinney, 1989; Morris, 1993; Genzel et al., 2003; Levin and Beloborodov, 2003). For example, even at a distance $\gtrsim 1.5 \, \text{pc}$, the CNR would need to be an order of magnitude denser in order to overcome the strong tidal forces and fragment into stars (Becklin et al., 1982).

1.1.2.a In-situ formation

One model suggests that a GMC on a near radial orbit which is tidally disrupted could spiral towards the Galactic Centre, forming a small gaseous disk. When the cloud hits the black hole, gas with opposite angular momentum to the orbit collides downstream, leading to a redistribution of angular momentum and loss of kinetic energy. This would result in a compact sub-parsec disk, dense enough to become Jeans unstable and fragment into stars (Bonnell and Rice, 2008; Hobbs and Nayakshin, 2009; Alig et al., 2011; Mapelli et al., 2012; Alig et al., 2013).

Two large gas clouds of mass $\sim 5 \times 10^5 \, \text{M}_\odot$, M-0.02-0.07 and M-0.13-0.08, are seen projected at $\sim 7$ and $\sim 13 \, \text{pc}$ respectively from the Galactic Centre (Solomon et al., 1972; Mapelli et al., 2012) show that an infalling cloud requires a mass of $\sim 10^5 \, \text{M}_\odot$ in order to
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reproduce the observations, and that the top heavy mass function of the Clockwise Disk can be reproduced so long as the gas temperature is $> 100$ K, both constraints of which are consistent with the the Galactic Centre clouds. A caveat of the in-situ formation model is that it requires near radial orbits with sub-parsec impact parameters incident upon SgrA*, perhaps requiring finely tuned cloud-cloud collisions in order to dissipate most of their angular momentum (Wardle and Yusef-Zadeh 2008; Hobbs and Nayakshin 2009).

The rotation axis of the Clockwise Disk has shown a strong transition from the inner to outer edge (Lu et al. 2009; Bartko et al. 2009), suggesting that the disk is either strongly warped, or is comprised of a series of stellar streams with significant variation in their orbital planes (Bartko et al. 2009). As an infalling star cluster (see section 1.1.2.B) would likely form a disk with a constant rotation axis (Perets and Gualandris 2010), in-situ formation of the Clockwise Disk has been favoured. However, this conclusion has recently been debated by (Yelda et al., 2014), whom showed that the Clockwise Disk is confirmed to high significance only in the central $\sim 0.13 \text{ pc}$. This gives the possibility that the outer radius may be as small as $0.13 \text{ pc}$, in which case it is not significantly warped.

In-situ formation alone cannot explain the presence of the S-stars, as they orbit so close that they must have formed further out and migrated in later. Chen and Amaro-Seoane (2014) suggest that secular evolution of the disk may be able to produce a population of S-stars from the disk within a few Myr. The authors show that stars from the clockwise disk can be brought very close to SgrA* via global Kozai-Lidov (KL) like resonances (Kozai 1962; Lidov 1962), if the Clockwise Disk was originally more massive and extended down to $\sim 10^{-6} \text{ pc}$ (the lowest general relativistically stable circular orbit around SgrA*). In addition, the authors also showed that O/WR stars would be tidally disrupted within the region of the observed S-star cluster due to their large stellar radii, whereas B-stars could survive, in agreement with observations.

Recently, Šubr and Haas (2016) showed that a clockwise disk with 100% primordial binarity can produce $\sim 20$ S-stars in $\leq 4 \text{ Myr}$, without requiring the disk to extend further down than $0.04 \text{ pc}$. Three confirmed O/WR eclipsing binaries are currently observed within the Clockwise Disk (Ott et al., 1999; Martins et al., 2006; Pfuhl et al., 2014). Gautam et al. (2017) predict that the present day binary fraction must be $> 32\%$ at 90% confidence. Šubr and Haas (2016) showed that KL oscillations could efficiently drag Clockwise Disk binaries close to SgrA*, producing an S-star and a hyper velocity star via the Hills mechanism (Hills, 1991). However, this mechanism produces an S-star population with orbits oriented in the same plane as the disk. In order randomise the inclination of the S-star orbits within their lifetime, the model requires the existence of $\sim 500 \text{ M}_\odot$ of dark remnants confined within the S-star cluster.

Another possibility is that the S-stars are a distinct population to the Clockwise Disk, originating from binaries formed outside the central parsec, which are scattered onto low angular momentum orbits by some mechanism and are disrupted by the Hills mechanism. However the captured stars typically have eccentricities higher than 0.97 (Miller et al., 2005; Bromley et al., 2006), and would again require a cusp of dark remnants to thermalize them (Perets et al., 2009).
1.1. THE GALACTIC CENTRE - A UNIQUE LABORATORY

1.1.2.b Star Cluster Inspiral with the aid of an Intermediate Mass Black Hole

There appears to be very few He I line emission stars outside the central parsec, other than inside/around the Arches and Quintuplet clusters at $\sim 30$ pc. This led Gerhard (2001) to postulate that the high density of the region could induce efficient dynamical friction on star clusters forming a few pc from SgrA* (see Chandrasekhar, 1943, and section §1.3.2), where GMCs may be able to more easily cool and fragment. Through this mechanism the star cluster’s orbit may decay to the centre of the NSC within the lifetime of the He-1 population, and is an alternative to the difficulties posed by in-situ formation.

Kim and Morris (2003) performed collisionless simulations of star clusters inspiraling in the Galactic Centre with the tree-code GADGET2 [Springel et al., 2001]. The authors showed that in order for a star cluster to transport stars to the Galactic Centre via dynamical friction from a distance $\geq 10$ pc, clusters either need to be very massive ($\sim 10^6 M_\odot$) or very dense (with a central density of $\sim 10^8 M_\odot pc^{-3}$). This experiment only considered the mass loss due to the shrinking tidal limits as the cluster falls toward SgrA*, as the internal dynamics of the clusters were not resolved. Rapid relaxation due to the high densities of these clusters would also cause the clusters to expand and become Roche overfilling (see Chapter §1.3). In Kim et al. (2004) the authors performed additional simulations in which the cluster contained an intermediate mass black hole (IMBH). The authors found that this lowered the core density required to transport stars to the central parsec, however this was only the case if the IMBH contained $\geq 10\%$ of the mass of the cluster, much more than was expected from the runaway collisions of massive stars (Portegies Zwart and McMillan, 2002).

Fujii et al. (2009) (hereafter F09) revisited this problem using the tree-direct hybrid code, BRIDGE (Fujii et al., 2007), allowing the internal dynamics of the star cluster to be resolved with direct summation, and the background potential to be integrated with a tree-code. The small tidal limits imposed by the background potential meant the clusters had core densities greater than $10^7 M_\odot pc^{-3}$, leading to runaway collisions on a mass segregation timescale (Portegies Zwart and McMillan, 2002; Portegies Zwart et al., 2004). During collisions, the age of this very massive star (VMS) was rejuvenated using extrapolated mass transfer formalisms from Meurs and van den Heuvel (1989) and was collapsed to an IMBH at the end of its main sequence lifetime – extrapolated from the results of Belkus et al. (2007). The authors found that by allowing the formation of a $3 - 16 \times 10^3 M_\odot$ IMBH (see also Fujii et al., 2010), some stars could be carried very close to SgrA* via a 1:1 mean resonance with the infalling IMBH, even after the star cluster had completely dissolved. The orbits of these “Trojan stars” were randomised by 3-body interactions with the SMBH and IMBH, constructing a spatially isotropic “S-star cluster”. F09’s simulation “LD64k” transported 23 stars to the central 0.1 pc, however, the resolution of the simulation was $\sim 0.2$ pc, set by the force softening of SgrA*. The spatial and thermal distributions of the “S-stars” would likely have been significantly different in the presence of an un-softened point mass potential. The simulation also brought 354 stars within 0.5 pc of SgrA*, 16 being more massive than 20 M$_\odot$, analogous to clockwise disk stars. The IMBH formed in LD64k is more massive than the observational upper limit of $\sim 10^4 M_\odot$, derived from VLBA measurements of SgrA* (Reid and Brunthaler, 2004). However, Fujii et al. (2010) state that an IMBH of 1500 M$_\odot$ is sufficient for the randomisation of orbits (see also Merritt et al., 2009).

Despite the successes of the F09 model, IMBH formation in young dense star clusters...
may be prohibited. VMSs of the order $10^3 \, M_\odot$ are expected to have luminosities $\geq 10^7 L_\odot$ \cite{Kudritzki2002, NadyozhinRazinkova2005, Belkus2007}, driving strong stellar winds. F09 assumed the mass loss rate of stars more massive than $300 \, M_\odot$ to be linear with mass, however, recent work on VMS winds show steeper relations for stars that approach the Eddington limit \cite{Kudritzki2002, Vink2006, Vink2011}. Additionally F09’s models also neglected the effect of the evolving chemical composition on the luminosity of the VMS \cite{NadyozhinRazinkova2005}. I show in Chapter §4 that effect is very important, as the strong stellar wind acts as an energy source for the cluster, causing it to rapidly expand over its tidal radius.

The Salpeter initial mass function (IMF) \cite{Salpeter1955} from $1-100 \, M_\odot$ used in the F09 simulations – although employed due to numerical constraints – meant there were initially ten times more massive stars than expected from a Kroupa IMF \cite{Kroupa2001}. This is important for two reasons. Firstly, massive stars have higher cross sections, and reach the core faster due to dynamical friction, both effects of which cause the collision rate to be overestimated. Secondly, the star cluster initially contains $10\times$ as many stars to deposit in the Central Parsec. Perhaps the LD64k model would only have produced $\sim 2$ S-stars – instead of 23 – had they been modelled with a Kroupa IMF. In addition to this, the evolution of stars less massive than $300 \, M_\odot$ is ignored. However, massive O-stars may lose a large fraction of their mass before they collide with the VMS, further inhibiting its growth. Finally, F09 do not deeply discuss the observational imprint the dissolved star cluster would leave in the Galactic Centre, which I explore in Chapter §4.

No conclusive evidence for the existence of IMBHs in star clusters has yet been found \cite{Lutzgendorf2013, Lutzgendorf2016} for a comprehensive review on IMBHs in globular clusters). Sufficiently high mass loss may cause VMSs to end their lives as stellar mass black holes or pair-instability supernovae at low metallicity \cite{HegerWoosley2002}. Pair-instability supernovae candidates have recently been found at metallicities as high as $\sim 0.1 \, Z_\odot$ \cite{Gal-Yam2009, Cooke2012}, with expected progenitors of several hundred solar masses \cite{Chen2015a}.

The most massive star observed, R136a1, is a $265^{+80}_{-35} \, M_\odot$ star in the 30 Doradus region of the Large Magellanic Cloud (LMC) \cite{Crowther2010, Crowther2016}, with metallicity $Z = 0.43 \, Z_\odot$. \cite{Crowther2010} suggest that it could be a very rare main sequence star, with a zero age main sequence mass of $320^{+100}_{-40} \, M_\odot$. However, it could be the collision product of a few massive stars. R136a1 has a large inferred mass loss rate of $(5.1^{+0.9}_{-0.8}) \times 10^{-5} \, M_\odot \, yr^{-1}$, $\sim 0.1$ dex larger than the theoretical predictions of \cite{Vink2001}. \cite{Belkus2007} predict that the evolution of all stars more massive than $300 \, M_\odot$ is dominated by stellar winds, with similar lifetimes of $\sim 2 - 3 \, Myr$. As such, it is not surprising that R136a1 is the most massive star currently observed, as more massive VMSs should be rare and short lived.

Whilst it may be unlikely for an IMBH to form at solar metallicity, a VMS could potentially transport stars to SgrA* within its lifetime. In Chapter §4 I explore the feasibility of the cluster inspiral model whilst modelling the evolution of the star cluster and VMS in realistic manner, and identify any observational signature such a scenario would leave in the sky.
1.2. THE GRAVITATIONAL $N$-BODY PROBLEM

1.2 The gravitational $N$-body problem

Originating from Newton’s *Principia* ([Newton](#)), the gravitational $N$-body problem is a long-standing fundamental problem in astrophysics and cosmology, yet the nature of the problem is simple enough to express in a single sentence: for given initial data $x_i(0)$, $\dot{x}_i(0)$, $i = 1, ..., n$ (with $x_i(0) \neq x_j(0)$ for mutually distinct $i$ and $j$), find the solution of the second order system:

$$\ddot{x}_i = \sum_{j \neq i}^n \frac{m_j (x_i - x_j)}{|x_i - x_j|^3}, \quad i = 1, ..., n,$$

where $m_1, m_2, ..., m_N$ are constants representing the masses of $N$ point masses, and $x_1, x_2, ..., x_N$ are 3-dimensional vector functions of time describing the positions of the point-masses. A complete solution to the $N = 2$ problem was solved by Johann Bernoulli in 1710 (see [Diacu](#) [1996] and references therein for a more complete overview), yet the $N \geq 3$ case baffled the scientific community for more than 150 years after Bernoulli’s $N = 2$ solution.

Interest in an $N \geq 3$ solution grew in the late 1800’s when *Acta Mathematica*, volume 7 1885/86 announced a prize in honor of Sweden and Norway’s King Oscar II’s 60th birthday. The competition posed 4 unsolved problems, of which finding a power-series solution to the $N \geq 3$ gravitational $N$-body problem was the first. There were 12 submissions to the competition, of which 5 focused on the $N$-body problem. However, none of these entries obtained a power-series solution as requested by the competition. Given this fact, the jury decided upon awarding the prize to Henri Poincaré, for his valuable contribution to the understanding of dynamical systems ([Poincaré](#) 1892). Poincaré’s contribution offered the first example of chaotic behaviour in a deterministic system.

It was already established that an $N$-body system had ten integrals of motion, functions that remain constant over time: three for the centre of mass, three for the linear momentum, three for the angular momentum, and one for the total energy of the system. These integrals allow one to constrain the problem to $6N - 10$ variables. ($6N$ because of three position and velocity components per particle). Jacobi showed that through symmetries and a method known as his *reduction of nodes*, the dimensionality of the problem could be reduced to $6N - 12$ ([Jacobi](#) 1843). This demonstrates why the two-body problem is analytically solvable, as $6 \times 2 - 12 = 0$, or in words: the orbit is completely described by the integrals of motion.

Poincaré’s prize winning paper proved that there are no integrals with respect to time, other than the 10 known global integrals of the system. Such that for even just $N = 3$, the system appears to be an unsolvable six-dimensional first order problem. However, it is a common misconception that Poincaré’s proof means that there is no analytic solution to the $N \geq 3$ problem, just that it cannot be solved through the use of integrals. [Sundman](#) (1913) solved the $N = 3$ problem in the form of a series solution in powers of $t^{1/3}$, so long as the total angular momentum of the system isn’t zero. By employing an analytic trick Sundman called *regularisation*, he showed that the series was convergent for all $t$, even if elastic collisions took place. Sundman’s solution proved impossible to generalise for $N > 3$. However, the series solution of the $N$-body problem was eventually solved in 1991. [Wang](#) (1991) derived a convergent series solution of the $N$-body problem for all $N$ and also the $N = 3$ case with zero angular momentum. Unfortunately, however, the series
solution is completely unusable in practice. Wang (1991)'s series solution converges extraordinarily slowly, requiring millions of terms in order to accurately determine the evolution of the particle properties over non-negligible amounts of time. (For a more complete history of attempts to solve the $N$-body problem see Diacu [1996].

1.2.1 The need for a numerical approach

Figure 1.3: Holmberg (1941)'s experimental setup. The left panel shows tidal deformations and clockwise rotations, due to a collision with an impact parameter equal to the diameters of the nebulae. The right panel shows the same setup for anti-clockwise rotations, where the spiral arms point in the direction opposite to the rotation. This figure was taken from Holmberg (1941): "On the clustering tendencies among the nebulae: II. A study of encounters between laboratory models of stellar systems by a new integration procedure", published in ApJ, vol. 94 p. 385. (© AAS. Reproduced with permission.)

By numerically integrating Eq. (1.2.1) for all bodies with small enough time steps, one can approximate the evolution of a system of $N$-particles. The first numerical simulation was performed by Holmberg (1941), prior to the invention of the transistor in 1947. Holmberg found a strategy to calculate forces between particles by using light as an analog of gravity, as both the gravitational force and intensity of light fall off as $1/r^2$. By representing point masses as light bulbs (where the candle power is analogous to the mass), he could estimate the gravitational force at any point with the combination of a photocell and a galvanometer. In the experimental setup the galvanometer gave a reading that was directly proportional to the acceleration of a particle at the location of the photocell. The scientific context of the experiment was to see if the loss of orbital energy due to tidal disturbances between two nebulae could be enough to effect the chance of capture. To
1.2. THE GRAVITATIONAL $N$-BODY PROBLEM

test this Holmberg used two clusters of 37 light bulbs arranged in a concentric circles, with each cluster representing a nebula. The experimental setup and the results of the integration are shown in Fig. 1.3 (© AAS. Reproduced with permission). He estimated the acceleration of each particle with the use of the photocell and galvanometer, then integrated the positions and velocities with respect to time in discreet steps (by physically moving the light bulbs). This preliminary $N = 74$ experiment worked remarkably well, with tidal features becoming prominent when the two nebulae interacted.

Holmberg’s experiment took weeks to setup and perform, and it wasn’t until the rise of modern computing that $N$-body simulations started to become common place. Aarseth (1963) pioneered the computational gravitational $N$-body problem by performing simulations of $25 \leq N \leq 100$ particles on a digital computer for the first time. Sverre Aarseth has continued to work on state-of-the-art direct-summation $N$-body simulation software, with the most recent GPU parallel version of his code, NBODY6 (Nitadori and Aarseth 2012), still being considered the “defacto standard” for integration via direct summation, with simulations of $\sim 10^5$ stars computed on a single GPU becoming commonplace in astrophysical research.

In the next section, §1.3 I briefly review the effects of relaxation and dynamical friction, before discussing modern computational $N$-body techniques in section §1.4.
1.3 Collisional Dynamics

1.3.1 Relaxation

The most important timescale for a stellar system is its relaxation time, \( t_{\text{rel}} \), which is defined as the time for the velocity of a typical star in the system to change by order unity due to the cumulative effect of many two-body encounters. In other words, \( t_{\text{rel}} \) is the time over which the system loses memory of its initial state.

Consider a star of velocity, \( v \), which has an encounter with a stationary star of equal mass, with an impact parameter, \( b \), as demonstrated in Fig. 1.4. If we assume that the perpendicular change in velocity, \( \delta v \), due to the interaction is small (i.e. \( \delta v/v \ll 1 \), and the point of closest approach is \( \sim b \)), then from Fig. 1.4 it is evident that:

\[
F_\perp = \frac{GM^2}{r^2} \cos \theta = \frac{GM^2}{x^2 + b^2} \cos \theta. \tag{1.3.1a}
\]

As \( \cos \theta = b/r = b/(b^2 + x^2)^{1/2} \), and \( x(t) = vt \):

\[
F_\perp(t) = \frac{GM^2b}{(x^2 + b^2)^{3/2}} = \frac{GM^2}{b^2} \left[ 1 + \left( \frac{vt}{b} \right)^2 \right]^{-3/2}. \tag{1.3.1b}
\]

By Newton’s second law, \( F = m \ddot{v} \):

\[
\delta v = \frac{1}{m} \int_{-\infty}^{\infty} F_\perp(t) \, dt, \tag{1.3.2a}
\]

and it follows that:

\[
\delta v = \frac{Gm}{b^2} \int_{-\infty}^{\infty} \frac{dt}{[1 + (vt/b)^2]^{3/2}}. \tag{1.3.2b}
\]

Substituting for \( s = vt/b \), and performing a standard integral over all space:

\[
\delta v = \frac{Gm}{bv} \int_{-\infty}^{\infty} \frac{ds}{[1 + s^2]^{3/2}} = \frac{2Gm}{bv}. \tag{1.3.2c}
\]
1.3. COLLISIONAL DYNAMICS

Next, if the surface density of stars in the system is of the order of $N/\pi R^2$, where $N$ is the number of stars and $R$ is the radius of the galaxy, then by crossing the galaxy once the subject star has:

$$\delta n = \frac{N}{\pi R^2} 2\pi b \, d\beta = \frac{2N}{R^2} b \, d\beta$$

encounters with impact parameters between $b$ and $b + d\beta$. If the system comprised of a smooth mass distribution (i.e. $N \to \infty$), then the sum of all the perturbations would be zero, as the planes of the velocity perturbations are randomly oriented. As we are considering systems where $N$ is finite and the integral over all space isn‘t guaranteed to be zero, we consider the mean-square velocity change, which is not zero:

$$\sum \delta v^2 \delta n = \left( \frac{2Gm}{b \nu} \right)^2 \frac{2N}{R^2} b \, d\beta$$

Integrating over all space gives logarithmic divergence if we integrate from zero. Instead we integrate from a minimum impact parameter, $b_{\text{min}}$, where our straight line approximation breaks down, which is approximately the impact parameter for a 90 degree deflection. We take the upper cutoff, $b_{\text{max}}$, to be $\sim R$, the order of the size of the system. Integrating gives:

$$\Delta v^2 = \int_{b_{\text{min}}}^{b_{\text{max}}} \sum \delta v^2 \delta n = 8N \left( \frac{Gm}{R \nu} \right)^2 \log(\Lambda)$$

where:

$$\log(\Lambda) = \log \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right)$$

is known as the Coulomb logarithm. If the typical speed of a field star is roughly that of a particle on a circular orbit at the edge of the system:

$$v^2 \approx \frac{GNm}{R},$$

then substituting this into Eq. 1.3.5 gives the relation:

$$\frac{\Delta v^2}{v^2} \approx \frac{8 \log(\Lambda)}{N},$$

and as $v^2$ changes by $\sim \Delta v^2$ every time it crosses the system, the number of crossings needed for $v$ to change by the order of itself is:

$$n_{\text{rel}} = \frac{N}{8 \log(\Lambda)}.$$

As the time for each crossing, $t_{\text{cross}} \sim R/v$, the time that this takes is:

$$t_{\text{rel}} \sim \frac{N}{8 \log(N)} t_{\text{cross}} \sim \frac{N}{8 \log(N)} \frac{R}{v},$$

where $t_{\text{rel}}$ is the relaxation time and $\Lambda \sim N$ as (using Eq. 1.3.7):
1.3. COLLISIONAL DYNAMICS

\[ \Lambda = \frac{b_{\text{max}}}{b_{\text{min}}} \sim \frac{R}{Gm/v^2} \sim N. \]  (1.3.11)

From this we can see that star clusters – \( N \sim 10^{4-6} \) – with \( t_{\text{cross}} \sim 1 \) Myr, can have relaxation times significantly shorter than their age, making them collisional systems in which relaxation strongly influences their structure. In contrast spiral galaxies may contain \( \sim 10^{11} \) stars (and countless particles of dark matter), thus their relaxation time can be much longer than the age of the universe, making them collisionless systems in which relaxation effects are negligible. A more rigorous derivation can yield the relaxation time in physical quantities (see Binney and Tremaine, 2008):

\[ t_{\text{rel}} = 0.34 \frac{\sigma^3}{G^2 m \rho \log(\Lambda)}. \]  (1.3.12)

However, the relaxation time can vary by many orders of magnitude in different regions of a stellar system. It is often useful to characterise a system by a single relaxation time. For this reason one defines the “half-mass relaxation time”, \( t_{\text{rh}} \), as the relaxation time at the half-mass radius of a stellar system (see Spitzer, 1987; Binney and Tremaine, 2008):

\[ t_{\text{rh}} = 0.17 \frac{N}{\log(\lambda N)} \sqrt{\frac{r^3_{\text{h}}}{GM}}. \]  (1.3.13)

where \( \lambda = 0.11 \) (Giersz and Heggie, 1994).

1.3.1.a The Gravitational Catastrophe, Core Collapse and Physical Collisions

Gravitationally bound systems have the interesting property of negative heat capacity. This can be demonstrated easily if one considers an isolated binary system. If one injects energy into the binary the separation between the stars increases, and the system becomes dynamically colder. If one removes energy from the binary instead, the binary hardens and becomes dynamically hotter.

In star clusters, hot stars in the core transfer kinetic energy to cold stars in the outskirts. The hot star thus falls to a lower energy orbit within the hot core – perhaps surprisingly – causing its velocity to increase. Due to the increased temperature of the core, this process repeats in a runaway fashion leading to “the gravitational catastrophe”, a sharp increase in the core density, also known as core collapse. The diffusion of energy throughout the cluster causes the core to contract, and by conservation of energy, the outer regions of the cluster expand. However, when an energy source within the core becomes available, the gravitational catastrophe is halted. This usually occurs when the density becomes high enough to dynamically form a binary system, as these can then be used as an energy source. A likely result of a 3-body encounter between a single star and a binary is for the binary to harden, which liberates energy which gets transferred to the other stars in the system. As such, post core-collapse the core radius “bounces”, and begins to expand again.

If the core density becomes high enough at core collapse that the mean separation of stars in the core is of the order of the radii of the most massive stars, physical collisions between stars can become important. A collision involving two massive stars forms a star with a larger radius than the primary, and thus a larger cross section for subsequent
1.3. COLLISIONAL DYNAMICS

collisions. This leads to a “runaway collision” (Portegies Zwart and McMillan, 2002; Portegies Zwart et al., 2004), forming a very massive star. However, these massive stars have powerful line-driven winds, and mass loss from such stars can be very significant, reducing the mass of this star (Kudritzki, 2002; Belkus et al., 2007; Vink et al., 2011). Physical collisions and mass loss due to the stellar wind both act as an energy source and cause the cluster to expand, halting core collapse. This scenario is very important when considering the cluster inspiral model as the origin of the young stellar populations in the Galactic Centre, as I will show in Chapter 4.

1.3.2 Dynamical friction

During relaxation the orbit of a particle is deflected by two-body encounters, however there is no preferred direction for the velocity change from any encounter, i.e. when summed over many encounters, the change in velocity of the satellite in the plane perpendicular to its direction of motion is approximately zero. However, if one particle is significantly more massive than the majority of the particles (e.g. an O-star amongst a distribution of low mass stars), Chandrasekhar (1943) showed that the net force along the direction of motion always results in a deceleration of the satellite, a dynamical friction. This means that massive objects tend to lose energy to their host environments, and sink to the centre of the system. This is a very interesting and important astrophysical phenomenon as the dynamical friction timescale for a single massive object can be many orders of magnitude shorter than the relaxation time of the system. This has far reaching implications, as dynamical friction is likely responsible for galactic mergers (Gan et al., 2010; Peirani et al., 2010), the accretion of satellites and globular clusters on the Milky Way (Gan et al., 2010; Arca-Sedda and Capuzzo-Dolcetta, 2014), mass segregation within star clusters and even (in part) the coalescence of supermassive black holes (Begelman et al., 1980).

I briefly outline Chandrasekhar’s original derivation here, but direct the reader to Chandrasekhar (1943) for the complete derivation and the accompanying book Chandrasekhar (1942) for the fine details. It can be shown that when a field star of mass, $m_*$, and velocity, $v_*$, flies by a satellite of mass, $M_s$, and velocity, $v_s$, the relative velocity, $V = v_s - v_*$, is deflected by an angle $\pi - 2\psi$, where (Chandrasekhar, 1942, eq. 2.301):

$$\cos \psi = \frac{1}{\sqrt{1 + b^2/b_{\text{min}}^2}}, \tag{1.3.14}$$

$b$ is the impact parameter, and $b_{\text{min}}$ is the impact parameter for a 90 degree deflection:

$$b_{\text{min}} = \frac{G (M_s + m_*)}{V^2}. \tag{1.3.15}$$

The resulting change in velocity of the satellite parallel to its direction of motion is given by (Chandrasekhar, 1942, eq. 5.721):

$$\Delta v_s = -\frac{2m_*}{m_* + M_s} [(v_s - v_\star \cos \theta) \cos \psi + v_\star \sin \theta \cos \Theta \sin \psi] \cos \psi, \tag{1.3.16}$$

where $\theta$ is the angle between vectors $v_\star$ and $v_s$, and $\Theta$ is the angle between the orbital plane of the satellite and the plane containing vectors $v_\star$ and $v_s$.
1.3. COLLISIONAL DYNAMICS

The satellite experiences a total change of velocity per time interval, $dt$, due to a sea of background particles:

$$\frac{dv_s}{dt} = \int_0^{\infty} dv_s \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{b_{\text{max}}} db \int_0^{2\pi} d\Theta \frac{2 \pi N(v_s, \theta, \phi) V(v_s) b \Delta v_{\parallel}(v_s, \theta, \Theta, b)}{2\pi} .$$

(1.3.17)

where $\phi$ is the azimuthal angle of $v_s$ when the Z-axis is aligned along $v_s$, and $b_{\text{max}}$ is the maximum impact parameter, a cutoff chosen such that the integral does not diverge. The maximum impact parameter, $b_{\text{max}}$, is usually taken to be of the order of the size of the galaxy, however, I will go deeper into this subject in Chapter §2.

Substituting Eq. (1.3.14) into Eq. (1.3.16) and integrating Eq. (1.3.17) over the inclination of the fundamental plane to the orbital plane, $\Theta$, one obtains:

$$\frac{dv_s}{dt} = -4\pi \frac{m_s}{m_s + M_s} \int_0^{\infty} dv_s \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{b_{\text{max}}} db$$

$$\times \left[ N(v_s, \theta, \phi) V(v_s) \left[ v_s - v_s \cos \theta \right] \frac{1}{\sqrt{1 + \frac{b_{\text{max}}^2 V^4}{G^2 (M_s + m_s)^2}}} \right] .$$

(1.3.18)

Next integrating over $b$, one obtains:

$$\frac{dv_s}{dt} = -2\pi G^2 (m_s + M_s) m_s \int_0^{\infty} dv_s \int_0^{\pi} d\theta \int_0^{2\pi} d\phi$$

$$\times \left[ N(v_s, \theta, \phi) \frac{1}{V^3} [v_s - v_s \cos \theta] \log \left( 1 + \frac{b_{\text{max}}^2 V^4}{G^2 (M_s + m_s)^2} \right) \right] .$$

(1.3.19)

Now, assuming the velocity distribution of the galaxy is spherical around the vicinity of the satellite, we can rewrite:

$$N(v_s, \theta, \phi) = N(v_s) \frac{1}{4\pi} \sin \theta,$$

(1.3.20)

$$N(v_s) = 4\pi Nf(v_s)v_s^2,$$

(1.3.21)

where $N(v_s)$ is the number density of velocity species $v_s$ in velocity space at the location of the satellite, $N$ is the stellar number density and $f(v_s)$ is the velocity distribution function (Chandrasekhar 1942, eq. 2.336). Substituting Eq. (1.3.20) into Eq. (1.3.19) and performing the – now simple – integration over $\phi$:

$$\frac{dv_s}{dt} = -\pi G^2 (m_s + M_s) m_s \int_0^{\infty} dv_s \int_0^{\pi} d\theta$$

$$\times \left[ N(v_s) \frac{\sin \theta}{V^3} [v_s - v_s \cos \theta] \log \left( 1 + \frac{b_{\text{max}}^2 V^4}{G^2 (M_s + m_s)^2} \right) \right] .$$

(1.3.22)
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In order to integrate over \( \theta \) it is useful to instead substitute for an integration over the relative velocity, \( V \). Consider:

\[
V = v_\ast - v_s, \tag{1.3.23a}
\]

\[
V^2 = (v_\ast - v_s)^2; \tag{1.3.23b}
\]

\[
V^2 = v_\ast^2 + v_s^2 - 2(v_\ast \cdot v_s), \tag{1.3.23c}
\]

\[
v_\ast^2 + v_s^2 - 2v_\ast v_s \cos \theta. \tag{1.3.23d}
\]

Now differentiating Eq. 1.3.23d gives:

\[
V \frac{dv_\ast}{dt} = v_\ast v_s \sin \theta, \tag{1.3.24a}
\]

and Eq. 1.3.23d can also be rearranged to give:

\[
v_s - v_\ast \cos \theta = \frac{1}{2v_\ast} \left(V^2 + v_\ast^2 - v_s^2\right). \tag{1.3.24b}
\]

Substituting Eqs. 1.3.24a and 1.3.24b into Eq. 1.3.22 gives:

\[
\frac{dv_\ast}{dt} = -\frac{\pi}{2} G^2 (m_\ast + M_s) m_\ast \int_0^\infty \frac{1}{v_\ast} N(v_\ast) J(v_\ast) dv_\ast.
\]

\[
J(v_\ast) = \int_{|v_\ast - v_s|}^{v_\ast + v_s} \left(1 + \frac{v_s^2}{V^2} \right) \log \left(1 + \frac{b_{\max}^2 V^4}{G^2 (M_s + m_\ast)^2} \right) dV,
\]

where \( J(v_\ast) \) is a factor defining the strength of interactions from velocity species, \( v_\ast \), across all possible relative interaction velocities and impact parameters. This is not the end point of the original Chandrasekhar (1943) derivation, however I will show in Chapter §3 that Chandrasekhar’s formula in the form of Eq. 1.3.25 has very useful practical applications when modelling the physics behind the rapid orbital decay of satellites in cored galaxies, as well as shedding light on the “core stalling” phenomena seen in simulations of satellites inspiraling in such cored potentials (Read et al., 2006; Goerdt et al., 2006, 2010; Inoue, 2009, 2011; Cole et al., 2012).

For now, we continue on with the derivation by finding approximate solutions for the remaining integrals. Performing an integration by parts for \( J(v_\ast) \):

\[
J(v_\ast) = \left[ \left(V - \frac{v_\ast^2 - v_s^2}{V} \right) \log \left(1 + \Lambda(V)^2\right) \right]_{v_\ast - v_s}^{v_\ast + v_s} - 4 \int_{|v_\ast - v_s|}^{v_\ast + v_s} \left(1 - \frac{v_s^2 - v_\ast^2}{V^2} \right) \frac{\Lambda(V)^2}{1 + \Lambda(V)^2} dV, \tag{1.3.26}
\]

where:

\[
\Lambda(V) = \frac{b_{\max} V^2}{G (M_s + m_\ast)}; \tag{1.3.27}
\]

defines the relative strength of close and distant encounters. If one considers that typically \( \Lambda \gg 1 \), we can drop the latter term of the remaining integral in \( J(v_\ast) \) and easily evaluate the remaining integral:
1.3. COLLISIONAL DYNAMICS

\[ J(v_s) = \left[ \left( V - \frac{v_s^2 - v^*^2}{V} \right) \log \left( 1 + \Lambda(V) \right)^2 - 4 \left( V + \frac{v_s^2 - v^*^2}{V} \right) \right]^{(v_s + v_s)}_{|v_s - v_s|}. \]  

(1.3.28)

By assuming either \( v_s < v_s \) or \( v_s > v_s \), and again considering that \( \Lambda \gg 1 \) (see Chandrasekhar, 1943):

\[ J(v_s) = \begin{cases} 
8v_s \log \left( \frac{b_{\text{max}}(v_s^2 - v^*^2)}{G(M_s + m_s)} \right) & \text{if } v_s < v_s, \\
4v_s \log \left( 4 \frac{b_{\text{max}}^2 v_s^4}{G(M_s + m_s)} \right) - 8v_s & \text{if } v_s = v_s, \\
8v_s \log \left( \frac{v_s + v_s}{v_s - v_s} \right) - 16v_s & \text{if } v_s > v_s.
\end{cases} \]  

(1.3.29)

Here it becomes obvious that the dominate contribution to \( J(v_s) \) comes from stars moving slower than the satellite. To good approximation, one usually ignores the sub-dominant term coming from stars moving faster than the satellite. However, I note that in Eq. 1.3.25, \( J(v_s) \) is multiplied by \( N(v_s) \), the available number of stars of velocity species \( v_s \).

As such, if one constructs a system whereby very few slow moving stars are available, the friction can be dominated by stars moving faster than the satellite [Antonini and Merritt, 2012; Petts et al., 2016]. I will explore such a configuration which arises naturally in dwarf galaxies containing a large dark matter core in Chapter 3.

Next, assuming the slow moving stars are the dominant contribution to the friction, recalling Eq. 1.3.21, and assuming \( M_s \gg m_s \):

\[ \frac{dv_s}{dt} = -4\pi G^2 \rho M_s \log (\Lambda) \frac{1}{v_s^2} \int_0^{v_s} 4\pi f(v_s) v_s^2 dv_s, \]  

(1.3.30)

where:

\[ \rho = N m_s; \]  

(1.3.31)

is the mass density of stars in the galaxy, and:

\[ \log (\Lambda) = \log \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) = \log \left( \frac{b_{\text{max}}}{GM_s/V_{\text{rel}}^2} \right), \]  

(1.3.32)

Is the Coulomb logarithm, where \( V_{\text{rel}} \) is the average relative interaction velocity of the stars with the satellite.

By assuming the velocity distribution function is Maxwellian with width, \( \sigma \) (taken to be the velocity dispersion of the galaxy), one obtains the Chandrasekhar formula in the form most often encountered in the literature:

\[ \frac{dv_s}{dt} = -4\pi G^2 \rho M_s \log (\Lambda) f(v_s < v_s) \frac{1}{v_s^2}, \]  

(1.3.33)

\[ f(v_s < v_s) = \text{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2), \]  

(1.3.34)

where \( f(v_s < v_s) \) is the fraction of stars moving slower than the satellite.
1.4 $N$-body simulation in the modern era

1.4.1 Star Clusters

Starting with his initial $N$-body experiments in 1963 (Aarseth [1963]), Sverre Aarseth has pioneered the development of direct summation $N$-body codes, with the latest version of his simulation code, NBODY6, being the “defacto standard” in the field of star cluster simulation. Although, I note that the first published computational simulations were performed by von Hoerner (1960), with up to $N = 16$ particles. Recent developments into parallel processing on GPUs (Nitadori and Aarseth [2012]) has made simulating $\sim 10^5$ stars over cosmological timescales possible on a desktop computer with one dedicated GPU for computation. Whilst parallelisation across multiple nodes of a super computer (with NBODY6++ Wang et al., 2015) has recently enabled simulations of globular clusters containing $10^6$ stars to be computed in less than a year (Wang et al., 2016).

Whilst the computational capability of hardware continues to rise with time, so has the efficiency of the numerical schemes implemented in $N$-body integrators. NBODY6 uses a fourth-order Hermite scheme which calculates the acceleration and its time derivative explicitly, and constructs a third-order interpolation polynomial using two points in time. Each particle has individual time-steps, so that particles moving fast through the core can be resolved with fine temporal resolution, whilst distant particles at the edge of the system can be computed with coarser time stepping (Aarseth, 1985). Additionally, NBODY6 utilises an Ahmad-Cohen neighbour scheme (Ahmad and Cohen, 1973), in which the force on a particle is split into two components: a slowly varying “regular” force from distant particles, and a rapidly fluctuating “irregular” force from the nearby particles. The “regular” forces are calculated on the GPU, whilst the “irregular” forces are calculated more frequently, on the CPU. Finally, two-body encounters are computed with the KS-regularisation formalism (Kustaanheimo and Stiefel, 1965) – in which a co-ordinate transformation treats the two body Kepler problem as a four dimensional harmonic oscillator – whilst multiple encounters are treated by the chain regularisation formalism of Mikkola and Aarseth (1990) (see also Mikkola and Aarseth, 1993).

1.4.2 Galaxies

On larger scales the calculation of forces by direct summation is prohibited. A single force calculation for all particles in the system using Eq. 1.2.1 takes $O(N^2)$ operations. Thus, it it would require $> 10^{11}$ stars, plus countless dark matter particles, to model the Milky Way. Such simulations are an impossibility for any direct summation $N$-body code available at present or conceivable within the foreseeable future.

One can get around this issue to some extent by modelling large $N$-systems with low-$N$ analogs. With this method, each particle does not represent a single point mass, but a large population of stars and/or dark matter particles. To avoid artificial scattering, the point mass potential of each particle is usually replaced by that of a Plummer sphere (Plummer, 1911) of scale length, $\epsilon$, known as the “softening length”, such that Eq. 1.2.1 becomes:
1.4. \textit{N}-BODY SIMULATION IN THE MODERN ERA

\[ \ddot{x}_i = \sum_{j \neq i}^{n} \frac{m_j (x_i - x_j)}{(|x_i - x_j|^2 + \epsilon^2)^{3/2}}, i = 1, \ldots, n. \quad (1.4.1) \]

By “softening” the force in this way, collisions with impact parameters \( b \lesssim \epsilon \) are suppressed, mimicking the interaction of two extended mass distributions. The dynamics on scales \( \lesssim \epsilon \) are unresolved by such simulations, including the tidal distortions of the Plummer spheres themselves during interactions.

Integration by direct summation coupled with softening can allow one to study the dynamics of galaxies (see e.g. Arca-Sedda and Capuzzo-Dolcetta, 2014; Bortolas et al., 2016). However, unrealistic relaxation is still a problem. Contrary to widespread belief, gravitational softening does not significantly reduce the effects of relaxation (Hernquist and Barnes, 1990; Dehnen, 2001) (see section 1.3). This can be demonstrated by considering the works of Chandrasekhar (1942) and Spitzer and Hart (1971). These authors show that relaxation is driven by both close and distant encounters, with each octave in distance contributing equally in a homogeneous system. Although softening reduces the contributions from encounters at scales \( \sim \epsilon \), most of the contributions to relaxation come from interactions on larger scales, and thus cannot be significantly reduced by softening techniques (see also the experiments of Hernquist and Barnes, 1990, for numerical verification).

Therefore, in addition to softening of the gravitational potential, one must also employ a method of approximating gravity in such a way that the computational effort required scales slower than \( O(N^2) \). There are many methods to compute the gravitational evolution of a system in an approximate manner, such as a Monte-Carlo statistical approach (Hénon, 1971) (e.g. Giersz, 1998, 2001, 2006; Giersz et al., 2008), particle-mesh/grid codes (Fellhauer et al., 2000; Teyssier, 2002, e.g.), tree-codes (Barnes and Hut, 1986, 1989) (e.g. Springel et al., 2001) and hybrid methodologies (e.g. Fujii et al., 2007; Hypki and Giersz, 2013). For a recent comprehensive review on both collisional and collisionless \textit{N}-body simulation I direct the reader to Dehnen and Read (2011).

In this thesis I primarily use the direct summation code NBODY6(df) (Nitadori and Aarseth, 2012; Petts et al., 2015) as well as the tree-code, GADGET2 (Springel et al., 2001). So I focus here on the methodology of a tree-code. The tree algorithm is a gridless gravitational algorithm that groups particles in cells in order to calculate an approximation of the potential (Barnes and Hut, 1986, 1989). Particles are first organised in a hierarchy of cells in a tree-like structure, where each cell (starting with the whole simulation space) is subdivided into 8 cells, until each deepest level contains only one particle. The acceleration on a particle is calculated by walking down the tree (starting from the largest volume) until size of the subdivision in question satisfies the specified criterion:

\[ \frac{s}{d} \leq \theta, \quad (1.4.2) \]

where \( s \) is the length of one side of the cell, \( d \) is the distance between the particle and the cell and \( \theta \) is called the opening angle and specifies the accuracy of the integration. Nearby particles are computed in direct summation, yet distant cells satisfying Eq. 1.4.2 will be computed as one particle-cell interaction. If \( \theta = 0 \) then the force from every particle is calculated in direct summation. Approximating the force this way reduces the
computational time to $O(N \log N)$, with newer, advanced tree-codes scaling as $O(N)$ \cite{Dehnen2001, Dehnen2014}.

1.5 Thesis Overview

In this research I revisit the star cluster inspiral model to see if it can still reproduce Clockwise Disk and S-stars, whilst including all known physical effects: dynamical friction, stellar evolution, regularised binary evolution, physical collisions, and state-of-the-art evolution of any dynamically formed VMSs.

In Chapter 2 \cite{Petts2015}, I develop a semi-analytic model for dynamical friction that can accurately reproduce the inspiral of a live $N$-body star cluster that can evolve due to internal $N$-body evolution and tidal stripping. This removes the need to model the galaxy in an $N$-body fashion, as the decay of the orbit can be calculated analytically.

In Chapter 3 I generalise this dynamical friction model so that one can accurately model the rapid inspiral and stalling of a satellite orbit in cored dwarf galaxies analytically, for the first time. Although this development wasn’t explicitly required in the context of the Galactic Centre problem, the Petts et al. (2016) model is very useful and has already been utilised in scientific works \cite{Contenta2017, Inoue2017, Cole2017}.

In Chapter 4 I use my modified version of Sverre Aarseth’s NBODY6 \cite{Aarseth1999, Nitadori2012}, NBODY6df, to explore the star cluster inspiral model for the origin of the young stars within the central parsec of the Milky Way. Thanks to my analytic dynamical friction model, I was able to explore the parameter space of this model extensively. I find that any cluster dense enough to reach the central parsec forms a very massive star via physical collisions in its core, which ends its life as a $20 - 400 M_\odot$ black hole due to strong stellar winds. The resulting black hole is not massive enough to bring a population of massive stars to the Galactic Centre, contradicting observations. Additionally, the presence of a star cluster inhabiting the central $\sim 10$ pc in the last 15 Myr would leave an observational signature which is not seen. I thus conclude that the star cluster inspiral scenario is highly unlikely.

In Chapter 5 I discuss my conclusions and further research that could be performed using the new methods outlined in this work.
Chapter 2

Dynamical Friction on Star Clusters

This chapter is based on (Petts et al., 2015), published in Monthly Notices of the Royal Astronomy Society, Volume 454, Issue 4.

2.1 Introduction

In a seminal work, Chandrasekhar (1943) showed that massive objects moving through an infinite homogeneous stellar medium will experience a drag force that he called ‘dynamical friction’ (see Binney and Tremaine, 2008, and §2.2). This frictional force is likely responsible for galactic mergers (Gan et al., 2010; Peirani et al., 2010), the accretion of satellites and globular clusters onto the Galaxy (Gan et al., 2010; Arca-Sedda and Capuzzo-Dolcetta, 2014), and even (in part) the coalescence of supermassive black holes (Begelman et al., 1980).

Numerical simulations using both direct summation codes and collisionless codes have shown that Chandrasekhar’s formula (Eq. 2.2.1) works remarkably well, despite the fact that it likely misses important physics like resonant interactions between the infalling body and the background (Tremaine and Weinberg, 1984; Inoue, 2009; Weinberg, 1986). Nonetheless, there are some situations in which it has been reported to perform poorly. Most notably, in the case of a constant density background (Read et al., 2006; Inoue, 2009; Goerdt et al., 2010), or when the mass of the infalling body approaches the enclosed background mass (Gualandris and Merritt, 2008). The former is perhaps surprising given that the original derivation assumes a homogeneous sea of background stars. But it has been thought that this failure could owe to the extreme resonance of such harmonic potentials (Read et al., 2006).

While dynamical friction can be accurately modelled using N-body simulations, it can often be prohibitively expensive. To achieve accurate results, the background ”stars” (I shall call them ”stars”, though these could be any distribution of gravitating masses, e.g. dark matter particles) must be substantially less massive than the infalling body. Otherwise, the body will simply be stochastically buffeted around, experiencing little friction (Baumgardt et al., 2006). If we wish to self consistently model the internal dynamics of a globular cluster, for example, falling onto the Galaxy the numerical requirements rapidly become extreme. For example a globular cluster of $10^5 M_\odot$ (O(10^5) particles), would need $10^{7−12}$ background particles to accurately model the inspiral in a massive host. If a lower mass resolution is used for the background, and gravitational softening is employed
to keep the relaxation time the same, the dynamical friction force is under-predicted (see §2.5.3). For this reason, semi-analytic models of dynamical friction have become invaluable (Hashimoto et al., 2003; Just and Peñarrubia, 2005; Just et al., 2011; Arca-Sedda et al., 2015). These significantly speed up the simulations since only the internal dynamics of the satellite needs to be integrated self-consistently, and the effects of a particular background distribution are modelled in an analytic way. Such models have been well-tested for point mass satellites in steep power-law density backgrounds, giving a good match to full N-body simulations (Just et al., 2011). However, in shallow or constant density backgrounds, dynamical friction stalls (Read et al., 2006) – an effect that has so far not been captured by semi-analytic models. This has led some authors to move away from semi-analytic modelling, towards particle-mesh codes calibrated by direct N-body and tree-codes (Spinnato et al., 2003), mixed collisional/collisionless methodologies (Fujii et al., 2006, 2007, 2009), and accurate advanced tree-codes (Dehnen, 2014). While these are exciting new approaches, they remain computationally expensive. This begs the question of whether the semi-analytic models cannot be improved. Such improvements would not only open up new classes of astrophysically interesting problems, but also shed more light on the interesting physics of dynamical friction and core stalling.

In Petts et al. (2015) I presented a new semi-analytical dynamical friction model accurate for extended objects, based on Chandrasekhar’s formalism (Chandrasekhar, 1943). The motivation of this work was to create an accurate, fast and reliable analytic prescription of the drag force induced on a satellite by its host galaxy, and to implement it in the direct-summation N-body code, NBODY6 (Aarseth, 1999; Nitadori and Aarseth, 2012). This was the first step required to model young star clusters in our Galactic Center in a new and novel way (see Chapter §4).

The key novelty of the model is that I present a new physically motivated, and radially varying minimum impact parameter ($b_{\text{min}}$), which when combined with the maximum impact parameter ($b_{\text{max}}$) from Just and Peñarrubia (2005), gives my semi-analytic model a remarkable match to a large range of full N-body simulations, both cuspy and shallow, without any need for fine tuning of the model parameters. In particular, I was able to reproduce the dramatic core-stalling phenomenon that occurs as a satellite approaches the center of its host galaxy. I showed that for Dehnen models (Dehnen, 1993) of asymptotic inner slope $\gamma = 0.2$, the core stalling occurs in the limit that $b_{\text{min}}$ approaches $b_{\text{max}}$. However, in an erratum I showed that this does not hold for galaxies with a large, constant density core (see section §2.6.2). In Petts et al. (2016) I generalised the model to correctly capture the infall and stalling of satellites in large cores, showing that the physics is the same as for a cusped galaxy (see Chapter §3).

The second novelty is that the mass term of the dynamical friction formula and the minimum impact parameter are functions of the live cluster properties, given by the N-body dynamics. For extended objects, the mass term is well represented by the mass enclosed within the Roche volume, including potential escapers (Fukushige and Heggie, 2000; Baumgardt, 2001). The minimum impact parameter is defined to be the maximum of the live half-mass radius and the impact parameter for a 90 degree deflection by a point mass. When the drag force is only applied to stars within the Roche volume, the semi-analytic models show excellent agreement with full N-body simulations. The model being free of any fitting parameters makes it useful to study a variety of astrophysical systems, including young star clusters orbiting near the Galactic Centre; globular clusters
2.2. Theory

Chandrasekhar’s dynamical friction formula for a satellite of mass $M_S$ is often written as (Binney and Tremaine, 2008):

$$\frac{dv_S}{dt} = -4\pi G^2 M_S \rho \log(\Lambda) f(v_*) \frac{v_S}{v_S^3},$$ \hspace{1cm} (2.2.1)

where $\log(\Lambda)$ is the Coulomb logarithm equal to $\log\left(b_{\text{max}}/b_{\text{min}}\right)$, $v_S$ is the satellite velocity, $v_S = |v_S|$ and $\rho$ is the local background density. If we assume a Maxwellian distribution of velocities:

$$f(v_*) = \text{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2),$$ \hspace{1cm} (2.2.2)

where $X = v_S/\sqrt{2}\sigma$.

Eq. (2.2.1) is derived under the assumption of an infinite homogeneous background. Despite this assumption, the Coulomb logarithm, $\log(\Lambda)$, takes into account the finite size of a real system through the maximum and minimum impact parameters. Typically $b_{\text{max}}$ is of the order of the size of the host system, and $b_{\text{min}}$ is defined as the impact parameter for a 90 degree deflection.

From a theoretical standpoint, it is surprising that Chandrasekhar’s formula is so successful at reproducing the effects of dynamical friction. In real systems, dynamical friction almost certainly results from discrete resonances with background stars (Tremaine and Weinberg, 1984; Weinberg, 1986). Chandrasekhar’s formula does not model these resonances, however it likely works because in most systems when an infalling body migrates from one radius to the next it experiences a whole new set of resonances. Tremaine and Weinberg (1984) show that if the resonances are assumed to form a continuum, Chandrasekhar’s approximation is reproduced. In this way, the infalling body behaves similarly to a massive body moving through an infinite homogeneous medium that encounters each background star only once. If we construct a special system, however, where by moving
2.2.2. from one radius to the next we do not encounter new resonances then Chandrasekhar’s formula has been known to fail. An example of this is given by the harmonic core (where the background density is constant: $\rho(r) \sim \rho_0$), which exhibits a short burst of what initially appears to be “super-Chandrasekhar” dynamical friction, followed by rapid stalling of the cluster orbit (Read et al. [2006]; Inoue [2009]; Goerdt et al. [2010]). I show that infall in a cored potential is in fact not “super-Chandrasekhar” in chapter §3.

2.2.1 Maximum impact parameter

Although a constant $b_{\text{max}}$ has been traditionally used to estimate inspiral timescales, Hashimoto et al. [2003] computed semi-analytic approximations of $N$-body models and found that a spatially dependent $b_{\text{max}}$ better reproduces simulation results. The physical motivation for this comes from the local approximation under which Eq. 2.2.1 is derived, which assumes the density distribution is constant up to $b_{\text{max}}$. Therefore $b_{\text{max}}$ should be a local property of the galaxy’s geometry. The authors took $b_{\text{max}}$ to be the distance from the Galactic Centre, $R_g$, which seems like a reasonable order of magnitude estimate, as the density of particles with impact parameters larger than $R_g$ is low compared to the local density distribution around the subject cluster. However for sufficiently cuspy profiles ($\gamma > 1$, where $\gamma$ is the asymptotic slope of the distribution) this approach will typically over estimate the dynamical friction effect near the centre of the background distribution. The slope of the density distribution is difficult to account for, however Just et al. [2011] show that:

$$b_{\text{max}} = \frac{\rho'(R_g)}{|\frac{d\rho}{dr}|_{R_g}},$$

is a better maximum cutoff to compensate for a cuspy density profile (i.e. the local density over the local density gradient). This impact parameter gives a length scale for which the density is approximately constant, giving a more accurate representation of the local approximation (see also Just and Pena-Rubia [2005]). This makes intuitive sense if one considers the two extreme cases of density distribution. If $\rho$ is a constant over all space, $b_{\text{max}} \to \infty$, and the force logarithmically diverges, as in Chandrasekhar’s original derivation. On the other hand if the distribution is infinitely cuspy (i.e. $\rho(r) \sim r^{-\infty}$), $b_{\text{max}} \to 0$, and the satellite effectively orbits a point mass in a Keplerian orbit with no decay.

From Eq. 2.2.3 Just et al. [2011] approximate the potential as a power-law, so that $b_{\text{max}} \sim R_g/\gamma$, however I keep the full expression so that the denominator can vary locally. In this chapter I assume that the background density distribution is a Dehnen model [Dehnen 1993], however Eq. 2.2.3 is valid for any distribution. For a Dehnen model the density and its derivative are both analytic, and Eq. 2.2.3 can be expressed as:

$$b_{\text{max}} = \frac{R_g(R_g + a)}{a\gamma + 4R_g},$$

which indeed reduces to $b_{\text{max}} = R_g/\gamma$ for $R_g < a$. Note that an attractive feature of Eq. 2.2.4 is that it is well-behaved in the limit $\gamma \to 0$, tending to a constant $b_{\text{max}} = a/4$. This is not the case if we use instead $b_{\text{max}} = R_g/\gamma$ for which $b_{\text{max}} \to \infty$. This led Just et al. [2011] to adopt $b_{\text{max}} = R_g$ for $\gamma < 1$. It would first seem, then, that my Eq. 2.2.4 is superior in this regard. However, this finite $\gamma \to 0$ limit is peculiar to the
assumed background Dehnen profile. It is straightforward to show that split power law profiles that transition from the inner to the outer slope more steeply than the Dehnen profile will produce divergent $b_{\text{max}}$ in the limit $\gamma \to 0$, if we assume the ansatz $b_{\text{max}} = \rho(R_g) / \Delta \rho(R_g)$. For this reason, I take $b_{\text{max}}$ to be the maximum of $R_g$ and Eq. 2.2.4.

### 2.2.2 Minimum Impact Parameter

The minimum impact parameter (i.e. the impact parameter corresponding to a 90 degree deflection) of extended objects is roughly of the order of the half mass radius of the object [Binney and Tremaine 2008]. Hashimoto et al. (2003) found, for infalling satellites of a Plummer density profile, that $b_{\text{min}}$ is well approximated by $1.4 \epsilon_s$, where $\epsilon_s$ is the Plummer scale radius, $a$. In terms of the half-mass radius this corresponds to $b_{\text{min}} = (1.4/1.3) r_{\text{hm}}$.

It should be noted that even though Hashimoto et al. (2003)'s $b_{\text{min}}$ was fit for a Plummer sphere, it is a reasonable approximation for other stellar distributions. Just and Peñarrubia (2005) show in their discussion about $b_{\text{min}}$ that the minimum impact parameter for Plummer, King and singular isothermal sphere models are very similar in terms of the half mass radius. Similarly to Just and Peñarrubia (2005), I take $b_{\text{min}}$ of extended objects to be $r_{\text{hm}}$ instead of Hashimoto et al. (2003)'s $\sim 1.07 r_{\text{hm}}$, to keep my formalism physically motivated rather than calibrated by $N$-body models.

At any epoch, I take $b_{\text{min}}$ to be the maximum of $r_{\text{hm}}$ and the minimum impact parameter of a point mass, which is typically taken to be $GM/v_{\text{typ}}^2$ (where $M$ and $v_{\text{typ}}$ are the bound mass and “typical” velocity for an encounter, respectively) [Binney and Tremaine 2008]. In general, $v_{\text{typ}}$ (and thus $b_{\text{min}}$) are poorly constrained. Just and Peñarrubia (2005) take $v_{\text{typ}}^2 = 2 \sigma^2 + v_S^2$. If one considers that $b$ will be minimised for the highest velocity encounter, this seems a reasonable choice at first glance. However, with this formulation $v_{\text{typ}}$ is the maximum relative velocity of any encounter (i.e. a head on collision), not the maximum velocity of a typical encounter that contributes to dynamical friction.

By recalling that two assumptions of Chandrasekhar’s formula are isotropy, and that only stars orbiting slower than the satellite contribute to the frictional force (but see chapter §3), it is easy to show that $v_{\text{typ}} = v_S$. Consider that a single background star with velocity $v_s$ can minimally and maximally interact with the satellite at velocity:

$$V_{\text{min}} = v_s - v_s \text{ and } V_{\text{max}} = v_s + v_s. \quad (2.2.5a)$$

In an isotropic distribution the mean interaction velocity of species $v_s$ with the satellite is:

$$\bar{V} = \frac{1}{2}(V_{\text{max}} + V_{\text{min}}). \quad (2.2.5b)$$

It follows that if $v_s \leq v_S$, $\bar{V} = v_S$. Integrating over all velocities slower than the satellite:

$$v_{\text{typ}} = v_s \int_0^{v_S} 4\pi f(v_s, r)v_s^2 dv_s, \quad (2.2.5c)$$

where $f(v_s, r)$ is the probability density of a star having velocity $v_s$ at $r$. As we are only considering the PDF of the slow stars, the PDF is defined as $\int_0^{v_S} 4\pi f(v_s)v_s^2 dv_s = 1$. Thus, from Eq. 2.2.5c $v_{\text{typ}} = v_S$. 

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This calculation only holds for isotropic backgrounds, for stars moving slower than the infalling satellite and when one ignores the velocity dependence in $\log(\Lambda)$. If one wants to include the effects of the stars moving faster than the satellite, $v_{\text{typ}} \neq v_S$, and Eq. 2.2.1 is inadequate. In this case one must integrate a more general form of Chandrasekhar’s formula. I discuss this in chapter §3.

2.2.3 The Coulomb logarithm and core stalling

My prescriptions for $b_{\text{max}}$ and $b_{\text{min}}$ give us the following functional form for the Coulomb logarithm:

$$\log(\Lambda) = \log \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) = \log \left( \frac{\min(\rho(R_g)/\Delta\rho(R_g), R_g)}{\max (r_{\text{hm}}, GM_S/v_S^2)} \right). \quad (2.2.6)$$

Eq. 2.2.6 shows that my prescription for $\log \Lambda$ is a function of the radial distance to the centre of the background potential, the slope of the background distribution and the half mass radius of the cluster. If during inspiral $b_{\text{min}} \geq b_{\text{max}}$, the dynamical friction term is set to zero. This ansatz is reasonable since this means there are no particles available to scatter off the satellite in a way that would reduce its orbital energy (see §3.3.1 for a more elegant solution). I now show that this ansatz is equivalent to the well-known result that friction ceases if the satellite mass approaches the enclosed mass of the background ($\text{Binney and Tremaine, 2008}$), as:

$$v_{\text{typ}}^2 \sim \frac{GM_g(R_g)}{R_g^2} \sim \frac{GM_g(R_g)}{b_{\text{max}}}, \quad (2.2.7a)$$

$$b_{\text{min}} \sim \frac{GM}{v_{\text{typ}}^2} \sim \frac{M}{M_g(R_g)} b_{\text{max}}, \quad (2.2.7b)$$

$$b_{\text{min}} \sim \frac{M}{b_{\text{max}}} \sim \frac{M_g(R_g)}{M_g(R_g)}, \quad (2.2.7c)$$

where $M_g(R_g)$ is the galaxy mass enclosed at $R_g$. Stalling occurs at this scale because perturbations from individual stars dominate over the mean field effects, making dynamical friction less efficient ($\text{Gualandris and Merritt, 2008}$, $\text{Inoue, 2011}$, $\text{Petts et al., 2016}$).

For the case of a Dehnen model an approximate analytic Eq. for the stalling radius can be derived. Equating the argument of the Coulomb logarithm to unity and assuming a circular orbit:

$$\frac{b_{\text{max}}}{b_{\text{min}}} = \frac{R_g(R_g + a)/4R_g}{GM_S/v_S^2}, \quad (2.2.8a)$$

$$v_S^2 = \frac{GM_g(R_g)}{R_g}, \quad (2.2.8b)$$

$$b_{\text{max}} = \frac{R_g(R_g + a)/4R_g}{M_gR_g/M_g(R_g)^2}, \quad (2.2.8c)$$

$$\frac{b_{\text{max}}}{b_{\text{min}}} = \left( \frac{R_g + a}{4R_g} \right) \left( \frac{M_g(R_g)}{M_g} \right) = 1. \quad (2.2.8d)$$
Recalling the formula for $M_g(R_g)$ (Dehnen, 1993):

$$M_g(R_g) = M_g \left( \frac{R_g}{R_g + a} \right)^{3-\gamma},$$  \hspace{1cm} (2.2.8e)

and inserting this into Eq. 2.2.8d and rearranging:

$$\frac{M_S}{M_g} = \left( \frac{R_g}{R_g + a} \right)^{3-\gamma} \left( \frac{R_g + a}{4R_g} \right),$$  \hspace{1cm} (2.2.8f)

$$\frac{M_S}{M_g} = \frac{r^{3-\gamma}}{(r + a)^{2-\gamma}(a\gamma + 4r)}.$$  \hspace{1cm} (2.2.8g)

If we take the limit of $r < a$:

$$\frac{M_S}{M_g} = \frac{r^{3-\gamma}}{a^{2-\gamma} + a\gamma}.$$  \hspace{1cm} (2.2.8h)

Therefore:

$$R_S = \left( \frac{M_S}{M_G} (a^{2-\gamma} + a\gamma) \right)^{\frac{1}{1-\gamma}},$$  \hspace{1cm} (2.2.8i)

where $R_S$ is the stalling radius of the satellite. Note that this is the theoretical stalling radius for a point particle. If the particle loses mass, $M_S/M_G$ will shrink and the cluster can potentially reach further in, but of course the timescale for inspiral will be longer. I will show in section §2.5.3.b that this shrinking log($\Lambda$) captures the core stalling phenomenon in galaxies with an asymptotic inner slope $\gamma = 0.2$. For a profile with an intrinsically flat core, stalling occurs even farther out. The stalling phenomenon in large cores is examined in §2.6.2 and resolved in Chapter §3, where I derive a more accurate and generalised stalling radius.

### 2.2.4 Velocity Dispersion

The fraction of background stars moving slower than the satellite (Eq. 2.2.2) is obtained from the underlying density distribution. Given a particular analytic density distribution, the velocity dispersion as a function of $R_g$ can be derived from the Jeans equation. For a Dehnen model $\sigma(R_g)$ is analytic if $4\gamma$ is an integer, and in the current implementation of NBODY6df a selection of analytic results have been implemented for various values of gamma. Once derived this allows for a quick analytical calculation in the code (see Appendix §A.1 for the full derivation). To use a non-integer value of $4\gamma$ one would need to implement a numerical solver of the Jeans equation in the code. However, for the sake of speed, I suggest instead to look for a degenerate model by modifying the scale radius, $a$, and mass, $M_G$, of the Dehnen model so that an integer $4\gamma$ may be used. I chose to use Dehnen models in the initial implementation due to their versatility for modelling spherical systems. If one would like to implement a different density distribution, I urge authors to take great care with the definition of $b_{max} = \rho(r)/\Delta\rho(r)$, however this impact parameter is calculable for any density distribution with a continuous derivative (see also Just and Peñarrubia (2005); Just et al. (2011)). I also note that in Chapter §3 I derive a more accurate model, which is also available in the current version of NBODY6df.
2.3 Implementation

In NBODY6df (see Chapter 2), Eq. 2.2.1 is applied as an external analytical acceleration. Dynamical friction is applied on the regular integration step, and added to the $N$-body forces which are computed in parallel on the GPU.

Fujii et al. (2006) fit semi-analytical models to $N$-body simulations of dwarf galaxies experiencing dynamical friction in larger parent galaxies. The authors show that simulations undergo enhanced dynamical friction as compared with the standard Chandrasekhar formula due to two effects. The first is direct gravitational interactions with escaped particles. This effect is included naturally in NBODY6df by integrating tidally stripped material self consistently. The second is the enhanced gravitational wake induced by energetically unbound particles that remain within the tidal radius of the system for an extended period before escaping. In an attempt to replicate this effect, the mass term, $M_S$, in Eq. 2.2.1 is taken to be the total mass of the particles contained within the tidal radius of the satellite. The tidal radius of a satellite on a circular orbit is given by (King, 1962; Binney and Tremaine, 2008):

$$r_t^3 = \frac{GM_s}{\Omega^2 - \frac{d^2\Phi}{dr^2}},$$

(2.3.1)

where $\frac{d^2\Phi}{dr^2}$ is the second derivative of the host galaxy’s potential at the satellite’s position, and $\Omega$ is the rotational velocity of the satellite. For a circular orbit, by definition:

$$\Omega^2 = \frac{GM_{\text{enc}}}{r^3}.$$  

(2.3.2)

Although this definition is for a circular orbit, for eccentric orbits I take $r_t$ to be the tidal radius of a circular orbit at apocentre, to make sure I include the drag on all nearby stars at pericentre. I show that this approximation for $M_S$ and cluster membership reproduces full $N$-body results excellently in section 2.5, successfully reproducing the findings of Fujii et al. (2006).

For the remainder of the this thesis I will refer to particles inside the tidal radius as “bound” for brevity, even though this includes potential escapers. In NBODY6df, particles experience no dynamical friction whilst unbound, but feel dynamical friction once again if they re-enter the Roche volume. NBODY6 includes the regularisation of binaries and close encounters (Mikkola and Aarseth, 1998) in which the system is replaced by a centre of mass (CoM) particle during the regular step. The regularised system is considered under the dynamical friction regime if its CoM particle is bound. The dynamical friction force is then applied directly to the CoM particle, and the differential force on each particle is handled by the KS regularisation scheme.

Collecting all of the scalar terms in Eq. 2.2.1 allows it to be rewritten as:

$$a_{df} = -C_{\text{fric}}\left(\frac{v_S}{v_S^3}\right),$$

(2.3.3)

where:

$$C_{\text{fric}} = 4\pi M_{\text{Roche}} \log(\Lambda) \rho(R_g) \left[ \text{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2) \right].$$

(2.3.4)

The cluster velocity, $v_S$, is taken to be the average velocity of the particles in the cluster core with respect to the galactocentric rest frame. Eq. 2.3.4 is calculated every...
2.4 Simulations

2.4.1 Initial Conditions

2.4.1.a Cluster

The clusters modelled in this chapter are initially Plummer models of mass $M_S = 10^5 M_\odot$ and half mass radius $r_{hm} = 0.1$ pc, similar to the clusters modelled in Kim and Morris (2003) and Fujii et al. (2009). The mass of a cluster particle is $m_S = M_S/N_S$, where $N_S$ is the number of cluster particles.

2.4.1.b Background

I adopt single component Dehnen models (Dehnen, 1993), representing the central region of the Galaxy. I use a slope $\gamma = 1.5$, scale radius $a = 8.625$ pc and mass $M_g = 5.9 \times 10^7 M_\odot$ to represent the density distribution in the central few tens of parsecs in the Milky Way. This closely represents the observed broken power-law profile obtained by Genzel et al. (2003) for the central 10 parsecs of the Galaxy. For runs with different $\gamma$ I use the same parameters as stated above. It should be noted that such a set of units was was originally used in Petts et al. (2015) with science goals in mind (see [3]), but as I performed simulations with gravity only, the significance of these scales is arbitrary in the chapter, as Newtonian Dynamics are scale-independent. The density as a function of radius for the Dehnen model is:

$$\rho(r) = \frac{(3 - \gamma)M_g}{4\pi} \frac{b}{r^{\gamma}(r + b)^{4-\gamma}}.$$  \hspace{1cm} (2.4.1)

It should be noted that any time-independent analytical spherical background potential can be included in the code, such as the addition of a central SMBH, and a dark halo component (although for some models the density and velocity dispersion functions may need to be calculated numerically). I have adopted a simple model here to ease comparison of the code with full $N$-body models with low-$N$. Choosing a single spherical component for the test simulations also gives more applicability to larger scale simulations. Models

1github.com/JamesAPetts/NBODY6df/blob/master/NBODY6df.pdf
### 2.4. SIMULATIONS

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Table 2.1: Initial conditions of simulations. Column 1 lists the names of the simulations, where the prefixes: df, nb and gt indicate the code used, NBODY6df, NBODY6 and GADGET, respectively. This is also stated in column 2. Columns 3 to 6 display the particle numbers and masses for both the cluster and the background, subscripts $c$ and $bg$ respectively. Column 7 lists the initial distance of the cluster from the Galactic Centre, all runs start at apocentre. Column 8 states the asymptotic slope used in the background Dehnen model. Column 9 shows the initial velocity of the cluster in units of the circular velocity, $v_c$. Column 10 displays which chapter each group of simulations first appears in.
2.5. RESULTS

where the scale radius and mass of the host are larger (i.e. globular clusters in a dwarf spheroidal) will behave similarly, as integration in NBODY6 is performed in scale independent Henon units internally, \( G = 1 = M = R_v = -4E \) (where \( M \) is the total mass, \( R_v \) is the virial radius and \( E \) is the total energy) (Nitadori and Aarseth 2012). A more realistic treatment of clusters near the Galactic Centre is presented in chapter §4.

2.4.2 Models

I compare results of NBODY6df with results from fully self-consistent NBODY6 and GADGET runs, where the background distribution is granular. NBODY6 is a direct-summation collisional code without softening and as such I use equal particle masses for the cluster and background to reduce spurious scattering effects. When using the tree-code GADGET however, I use a smaller mass for the cluster particles, as gravitational softening can be employed to reduce low-\( N \) relaxation effects. I use softening parameters of \( 0.025 \) pc for the cluster particles and \( 0.1 \) pc for the bulge particles in all GADGET simulations.

The maximum initial cluster distance I test is \( 10 \) pc (greater than the scale radius of the Dehnen model). I thus truncate the galaxy potential in the GADGET and NBODY6 runs at \( 50 \) pc (6 scale lengths) in order to reduce the number of particles that need to be computed. The truncation is at a large enough distance that friction induced by more distance particles is negligible. The models are summarised in Table 2.1.

2.5 Results

2.5.1 Comparison with NBODY6

2.5.1.a Orbit Comparison

Simulations df1k and nb1k are compared in Fig. 2.1 which shows the radial position of the cluster with respect to the Galactic Centre as a function of time. Fig. 2.2 shows the bound mass of the clusters in the different simulations. The agreement between the two models is excellent. After \( \sim 2 \) Myrs nb1k experiences stochastic changes in its orbit due to the low-\( N \) background, this is because the low-\( N \) cluster has nearly dissolved by this time. Prior to this epoch, when the clusters are not close to dissolution, the radial distance travelled by the cluster in the two codes differs by less than 2%.

The stochastic changes in the cluster orbit of simulation nb1k come from \( N \)-body sampling of the distribution function. This introduces chaotic effects on both large and small scales, compared to the equivalent analytic distribution (Arca-Sedda and Capuzzo-Dolcetta 2014). At small scales the granularity of the background induces stochastic changes in the orbit if \( N \) of the background is low (in nb1k each background particle represents the mass of an ensemble of stars). On large scales the system may deviate from spherical symmetry, inducing moderate eccentricity and precession. These effects accumulate over time and cause the eccentricity in nb1k once the cluster has almost dissolved.

Fig. 2.3 shows snapshots of the simulations at different times. Only after \( \sim 3 \) Myrs are the models distinguishable, and it can be seen that the structure and distribution of the tidally stripped material is well reproduced in NBODY6df.
Figure 2.1: Distance of the cluster with respect to the Galactic Centre as a function of time for simulations df1k (blue line) and nb1k (red dashed line). The clusters are comprised of 1k particles, at an initial distance of 10 pc, on an initially circular orbit in a $\gamma = 1.5$ Dehnen galaxy.
Figure 2.2: Mass enclosed within the tidal radius as a function of time for simulations df1k (blue line) and nb1k (red dashed line). The clusters are comprised of 1k particles, at an initial distance of 10 pc, on an initially circular orbit in a $\gamma = 1.5$ Dehnen galaxy.
Figure 2.3: Snapshots of the models df1k (blue) and nb1k (red) at $T = 0, 1, 2, 3$ Myr. The clusters are comprised of 1k particles, at an initial distance of 10 pc, on an initially circular orbit in a $\gamma = 1.5$ Dehnen galaxy. For nb1k only particles originating from the cluster are plotted.
Figure 2.4: Distance of the cluster with respect to the Galactic Centre as a function of time for df1ke (blue line) and nb1ke (red dashed line). The cluster models are comprised of 1k particles, at an initial distance of 10 pc, with an initial velocity of $(1 - 0.3)v_c$ in a $\gamma = 1.5$ Dehnen galaxy (where $v_c$ is the circular velocity at 10 pc).
Figure 2.5: Distance of the cluster with respect to the Galactic Centre as a function of time for df1ke0.75 (blue line) and nb1ke0.75 (red dashed line). The cluster models are comprised of 1k particles, at an initial distance of 10 pc, with an initial velocity of \((1 - 0.75)v_c\) in a \(\gamma = 1.5\) Dehnen galaxy (where \(v_c\) is the circular velocity at 10 pc).
2.5. RESULTS

Figure 2.6: Distance of the cluster with respect to the Galactic Centre as a function of time for df1kg175 (blue line) and nb1kg175 (red dashed line). The cluster models are comprised of 1k particles, at an initial distance 10 pc, on an initially circular orbit in a $\gamma = 1.75$ Dehnen galaxy.

Figs. 2.4 and 2.5 show simulations which have the same initial conditions as df1k and nb1k, but with initial cluster velocities of $(1 - x)v_c$ with $x = 0.3$ and 0.75, respectively – where $v_c$ is the circular velocity (I note that these are erroneously labelled in Petts et al. (2015), where we mistakenly state these runs as having an eccentricity of 0.3 and 0.75). The agreement is excellent for both eccentricities. For $x = 0.75$ the simulations diverge near the end as the clusters have lost the majority of their mass. The agreement is so good because of my prescription for $b_{\text{min}}$, which is dependent on both position and velocity. At pericentre the cluster moves fastest, giving a smaller $b_{\text{min}}$ and a stronger dynamical friction force, at apocentre the opposite is true, decreasing the force. Meanwhile $b_{\text{max}}$ varies across the length of the orbit due to its radial dependence. The result is an accurate calculation of the force along the entire orbit. The excellent agreement of both these models shows that the semi-analytic dynamical friction model in NBODY6df can accurately reproduce the force experienced on an live $N$-body cluster for a range of eccentricities.

Figs. 2.6 and 2.7 compare models which have the same initial conditions as df1k and nb1k, but with asymptotic slopes of $\gamma = 1.75$ and $\gamma = 1$ respectively. Both show good agreement. The NBODY6 runs gain some eccentricity from the granularity of the low-$N$ background distributions. This common problem with the low $N$-models is addressed in sections 2.5.1.b and 2.5.3.
Figure 2.7: Distance of the cluster with respect to the Galactic Centre as a function of time for df1kg1 (blue line) and nb1kg1 (red dashed line). The cluster models are comprised of 1k particles, at an initial distance 10 \text{ pc}, on an initially circular orbit in a $\gamma = 1$ Dehnen galaxy.
2.5. RESULTS

2.5.1. Angular Momentum Comparison

During inspiral the dynamical friction force is coupled with the relaxation of the cluster, meaning $b_{\text{min}}(R_g, v_S) = b_{\text{min}}(R_g, v_S, t)$ and $M_{\text{cl}} = M_{\text{cl}}(t)$. Therefore different realisations of low-$N$ simulations can significantly deviate from each other by using a different random seed, as the mass loss from dynamical ejections is very much a stochastic process for low-$N$ simulations, where the relaxation time is short.

Attempting to isolate each effect can give some indication of how accurate an approximation NBODY6df is. In the limit of negligible dynamical friction NBODY6df is identical to NBODY6, and the relaxation timescales will be similar.

I ran a series of short simulations to try to isolate the dynamical friction effect from the relaxation process as much as possible. In Fig. 2.8 I plot the total angular momentum of the bound material perpendicular to the orbital plane (i.e. $L_z$) as a function of time for half an orbit, for different initial cluster orbits (see table 2.1 for initial conditions). Fig. 2.9 shows the same for more eccentric orbits. Over a time of only half an orbit the clusters lose no more than 10 per cent of their mass, and as such the orbital evolution is only weakly dependant on relaxation/tidal effects. In Fig. 2.9 the full $N$-body models lose a bit more mass than NBODY6df at pericentre. After further investigation this appeared to be due to stars near the tidal radius of the cluster being stripped more aggressively due to two body scattering with the low-$N$ background, rather than deviation from Chandrasekhar’s formula.

I tested a specific case ($a=5.0, v_0=, v_c(1 - 0.75)v_c$) in models where the cluster is comprised of 2k, 4k and 8k particles, whilst keeping the total mass constant. I did not redo the entire grid of initial radii and eccentricities as the full $N$-body models are very numerically expensive. The models were run until the cluster in the NBODY6df simulation had lost 10% of its mass. Fig. 2.10 shows these sets of models, where very good agreement is found. The higher $N$ models retain their mass for longer due to slower two-body relaxation. In all the NBODY6 simulations, the high ratio of cluster particle mass to background particle mass causes a few of the less tightly bound cluster particles to be stripped very early on in the simulation. Therefore the angular momentum of the bound particles is systematically lower after $\sim 0.2$ Myr in all cases. This is not significant, as these stripped particles contribute very little to the total mass of the cluster, and so the gradient of the $L_z(t)$ curves still show very good agreement.

The agreement is excellent over the range of orbits tested, which validates that NBODY6df can reproduce the expected angular momentum loss at different radii for both circular and eccentric orbits. The orbital evolution over many orbits can be considered accurate because the dynamical friction coefficient only depends on the instantaneous cluster properties, i.e. the prior history of the cluster is irrelevant. At any epoch, $t_1$, $M_S = M_S(t_1)$ and $R_g = R_g(t_1)$. If I assume that my limited number of models in Fig. 2.8 indicate that the dynamical friction coefficient is initially correct compared to full $N$-body models at any $M_{S0}, R_{g0}, e_0$, then an entire inspiral can be thought of as traversing a grid of these models, and as such the dynamical friction coefficient, when decoupled from relaxation, can be considered correct. The models in Fig. 2.8 have an asymptotic slope of $\gamma = 1.5$, but similar agreement was found for a few test cases with $\gamma = 1$ and $\gamma = 1.75$. 

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Figure 2.8: Z-component of the angular momentum (perpendicular to the orbital plane) as a function of time for half a cluster orbit at different initial distances: 10, 5 and 2.5 pc, with initial velocities of either $v_c$ or $(1 - 0.3)v_c$ (where $v_c$ is the circular velocity). In all models the galaxy is a $\gamma = 1.5$ Dehnen model.
Figure 2.9: Z-component of the angular momentum (perpendicular to the orbital plane) as a function of time for half a cluster orbit at different initial distances: 10, 5 and 2.5 pc, with initial velocities of $(1 - 0.75)v_c$ (where $v_c$ is the circular velocity). In all models the galaxy is a $\gamma = 1.5$ Dehnen model.
2.5. RESULTS

Figure 2.10: Z-component of angular momentum (perpendicular to the orbital plane) as a function of time until cluster loses 10% of its mass. The clusters are initially at 5 pc, with initial velocities of either $v_c$ or $(1 - 0.75)v_c$ (where $v_c$ is the circular velocity). From top to bottom the cluster consists of 2k, 4k and 8k particles, and have equal total mass. In all models the galaxy is a $\gamma = 1.5$ Dehnen model.
2.5. RESULTS

2.5.2 N-dependence study with NBODY6df

I ran a series of simulations to see if the expected $N$-dependence of cluster relaxation, and its effects on inspiral, are well reproduced by NBODY6df. Simulations df1k-df64k have the same initial cluster mass split evenly amongst their cluster particles, and otherwise have the same initial conditions. Low-$N$ systems should lose their mass faster than an equivalent realistic cluster due to shorter relaxation timescales. This behaviour is illustrated in Fig. 2.11, which shows the bound mass as a function of time for simulations with different $N$.

With 32k particles the cluster initially has a relaxation timescale of $\sim 2$Myr, and the mass loss is mostly dominated by the tides during inspiral (Fig. 2.11). In simulations df32k and df64k, most of the mass is lost when the cluster reaches the centre of the potential, where the remaining mass is deposited in a disk around the galactic center.

If the cluster was modelled in a realistic fashion with a mean mass of $0.58 \, M_\odot$ (Kroupa, 2001) the relaxation timescale would be longer than the inspiral time ($T_{\text{relax}} \sim 9$Myr), and the mass loss would be dominated by the shrinking tidal radius. The low-$N$ models show accelerated mass loss due to increased dynamical ejections as expected.

Fig. 2.12 shows how this mass loss drastically alters the evolution of the orbit. If the cluster loses significant mass, its inspiral will stall due to a continually decreasing dynamical friction coefficient.

2.5.3 Comparison with GADGET

2.5.3.a Cuspy Models

The rapid relaxation of low-$N$ models means that even if the dynamical friction force exerted on the cluster is correct at any epoch (as shown in section 2.5.1.b), different cluster realisations will diverge in agreement due to the stochasticity of the relaxation process. As such, one would ideally like to perform NBODY6 runs with higher particle number to reduce this effect, but the computational cost is too high at the time of writing. Simulation nb1k took 7 days to run on 4 GeForce GTX 780 GPUs and 16 CPU cores. Increasing the particle number by a factor of 10 would take over 2 years to compute. As an alternative I used the softened tree-code GADGET to simulate a larger particle number. I would like to stress that in agreement with Kim and Morris (2003), I cannot accurately describe the internal dynamics of these clusters with GADGET. However tidal stripping still occurs in a natural way, and I can compare the bulk properties (i.e. mass and position) in order to test the validity of my dynamical friction perturbation.

Softened simulations are expected to exhibit weaker dynamical friction than collisional simulations. Whilst softening helps with numerical stability and computational speed, it suppresses close interactions that contribute to the friction. It is true that the GADGET simulations have a greater mass resolution for the background than nb1k, yet the eccentricity of the orbit grows faster due to numerical inaccuracies in the integration. GADGET’s integrator is accurate to 2nd order, and so accumulates the errors discussed in Section 2.5.1.a faster than NBODY6’s integrator, which is accurate to 4th order. The tree’s force calculation is also not as accurate as direct summation, with smoothing effects causing the calculated potential to deviate from spherical symmetry. This deviation is evident in Fig. 2.13, which shows the inspiral of cluster models at 10, 5 and 2.5 parsecs...
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Figure 2.11: Mass enclosed in the Roche volume as a function of time for simulations df1k-df64k, which have the same total mass and initial orbit, but contain a different number of particles.
Figure 2.12: Distance of the cluster with respect to the Galactic Centre as a function of time for simulations df1k-df32k, which have the same total mass and initial orbit, but contain a different number of particles.
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Figure 2.13: Distance of the cluster with respect to the Galactic Centre as a function of time for NBODY6df (blue line) and GADGET (green dashed line) simulations. The cluster models are comprised of 10k particles, on initially circular orbits in a $\gamma = 1.5$ Dehnen galaxy.

in NBODY6df and GADGET. The correspondence is much better between simulations dfa2.5 and gta2.5 because the number density of the background at 2.5 pc is approximately 30 times greater than at 10 pc, and appears more spherical when calculated with the tree. With the limitations of these simulations taken into account, the agreement between the NBODY6df and GADGET is rather satisfying.

Fig. 2.14 shows simulations with the same initial apocentres as dfa10 and gta10 in Fig. 2.13, but with an initial velocity of $(1 - 0.3)v_c$, where $v_c$ is the initial velocity. The same general trend is seen as compared with the circular case, in which GADGET inspirals slower, but the first few orbits give very good agreement.

2.5.3.b Shallow Models

In this section I look at how well the Petts et al. (2015) model (with radially varying impact parameters and the assumption of a Maxwellian velocity distribution) can reproduce the infall and stalling phases in shallow cusps, and where it fails in galaxies with a large core. Fig. 2.15 shows the comparison of simulation dfg0.5 and the self-consistent GADGET simulation, gtg0.5, in which a cluster orbits in a shallow cusp with asymptotic inner slope, $\gamma = 0.5$. I choose to only compare GADGET simulations for the shallow models...
Figure 2.14: Distance of the cluster with respect to the Galactic Centre as a function of time for dfa10e (blue line) and gta10e (green dashed line). The cluster models are comprised of 10k particles, at an initial distance of $10 \text{ pc}$, with an initial velocity of $(1 - 0.3)v_c$ in a $\gamma = 1.5$ Dehnen galaxy (where $v_c$ is the circular velocity at $10 \text{ pc}$).
Figure 2.15: Distance of the cluster with respect to the Galactic Centre as a function of time for dfg0.5 (blue line) and gtg0.5 (green line). The cluster models are comprised of 10k particles, at an initial distance of 5 pc, on initially circular orbits in a $\gamma = 0.5$ Dehnen galaxy.
2.5. RESULTS

Figure 2.16: Distance of the cluster with respect to the Galactic Centre as a function of time for dfg0.0 (blue line) and gtg0.0 (green line). The cluster models are comprised of 10k particles, at an initial distance of 5 pc, on initially circular orbits in a $\gamma = 0$ Dehnen galaxy.

because I cannot compare the full inspiral in NBODY6 due to the fast relaxation effects at low $N$, as seen in previous sections. Studying the full inspiral for the shallow profiles is important as they show interesting deviations from the standard Chandrasekhar’s formula. Even in this shallow cusp the Petts et al. (2015) model shows excellent agreement with the $N$-body simulation.

In Fig. 2.16 I compare the simulations dfg0.0 and gtg0.0, in which the asymptotic inner slope, $\gamma = 0$. The Petts et al. (2015) model cannot fully reproduce the inspiral of the cluster during was has previously been described as the “super-Chandrasekhar” phase that occurs near the centre of cored galaxies (Read et al., 2006; Goerdt et al., 2010). I explain why the Petts et al. (2015) under-predicts the force in Chapter 3. Interestingly the stalling radius is very well reproduced, this owes to the core in this model slowly asymptotes to $\gamma = 0$, and the stalling radius is still of the order $M_S \sim M_G(R_G)$ (Goerdt et al., 2010).

The physical interpretation of the stalling that occurs when $b_{\text{min}}$ approaches $b_{\text{max}}$ is that there are no background particles available to scatter in such a way as to decrease the satellites velocity, suppressing any further inspiral. Read et al. (2006) showed an analytic solution for this behaviour in an idealised harmonic potential, in which any background star in the core effectively moves on epicycles around the satellite and the galactic centre.
This static solution approximates what happens when a satellite falls into the galactic centre and tidally disrupts the cusp (Goerdt et al., 2010). This core formation occurs at a larger $M_S/M_G(R_G)$ for shallower cusps, as the material enclosed within the satellite’s orbit is less tightly bound. Fig. 2.17 shows how the Petts et al. (2015) model captures this core formation from steep cusps down to the $\gamma = 0$ Dehnen model. Note that if one instead defines the minimum impact parameter by $v_{\text{typ}}^2 = 2\sigma^2 + v_s^2$, as in Just et al. (2011), the stalling effect is not correctly captured in the $\gamma = 0$ model.

2.6 Discussion

In this section I consider the inspiral of a point mass object, and as such take my dynamical friction formalism and implement it as an external force in a 2nd-order integrator, which integrates the motion of a point particle in a Dehnen potential.
2.6. DISCUSSION

Simulation Code $R_a$ $v_0$ $v_c$ $\gamma$

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Code</th>
<th>(pc)</th>
<th>$v_c$</th>
<th>$\gamma$</th>
</tr>
</thead>
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<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
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<td>GADGET</td>
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<td>0.7</td>
<td>1.5</td>
</tr>
<tr>
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<td>GADGET</td>
<td>5.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>gtpt1.0e</td>
<td>GADGET</td>
<td>5.0</td>
<td>0.7</td>
<td>1.0</td>
</tr>
<tr>
<td>sapt1.5</td>
<td>Semi-Analytic Integrator</td>
<td>5.0</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td>sapt1.5e</td>
<td>Semi-Analytic Integrator</td>
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<td>Semi-Analytic Integrator</td>
<td>5.0</td>
<td>0.7</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 2.2: Simulations in which the cluster is modelled by a point mass particle. Simulations are performed in a self consistent way in GADGET, the mass resolution of the GADGET simulations is $30 M_\odot$. The properties of the cluster and background are the same as in section 2.4.

2.6.1 Comparison with Arca-Sedda and Capuzzo-Dolcetta (2014)

Arca-Sedda and Capuzzo-Dolcetta (2014) (hereafter AC14) studied dynamical friction in cuspy galaxies and presented a new treatment for massive objects near the centre of their host systems. The authors derive a semi-analytical formula for the inspiral time of massive point particles orbiting Dehnen models, calibrated by N-body models in the GPU-parallel direct N-body code HiGPUs (Capuzzo-Dolcetta et al., 2013). In their semi-analytic fitting process, they use an exponential interpolation between Chandrasekhar’s formula with a constant $b_{\text{max}}$ and varying $b_{\text{min}}$, and their detailed evaluation of the frictional force near the centre of host systems. The authors do not fix $b_{\text{min}}$ and instead let it be a fitting parameter, along with $r_{\text{cr}}$, which they define as the critical radius at which they switch to the new regime.

In my model both $b_{\text{max}}$ and $b_{\text{min}}$ vary along the orbit as a function of the local background and satellite properties. Arca-Sedda and Capuzzo-Dolcetta (2014) state that the local approximation overestimates the effects of dynamical friction in the innermost cuspy regions of galaxies, however in my approach the maximum impact parameter tends to zero at small radii, reducing the range at which the local approximation acts over. For cuspy distributions this scale length can be smaller than the distance to the galactic center. By setting my $b_{\text{max}}$ to Eq. 2.2.3 I ensure that the local approximation is valid, but likely slightly underestimate the frictional force at the very centre of the systems (see also Just and Peñarrubia, 2005).

Arca-Sedda and Capuzzo-Dolcetta (2014) model satellites as Plummer-softened point particles. Simulations with a satellite consisting of a cluster of particles take longer to reach the centre of their host, due to mass loss and expansion due to relaxation. For this reason I cannot directly test NBODY6df against their timescale formula. Instead I use my semi-analytical integrator. I performed GADGET simulations with the same initial conditions to test the comparison. The list of simulations is presented in Table 2.2.

I compared the inspiral time of these simulations with results from AC14 and found significant discrepancy with their dynamical friction timescale, which was calibrated mostly on radial orbits (eq. 20 in AC14). However a good agreement is found with
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Figure 2.18: Distance of the cluster with respect to the Galactic Centre as a function of time for simulations gtpt1.5, gtpt1.5e, sap1.5 and sap1.5e. The clusters are modelled as analytic plummer spheres, at an initial distance of 5 pc, with initial velocities of either $v_c$ (gtpt1.5 and sap1.5) or $(1 - 0.3)v_c$ (gtpt1.5e and sap1.5e), in a $\gamma = 0.5$ Dehnen galaxy (where $v_c$ is the circular velocity).

an improved formula calibrated on a wider range of models, given in Arca-Sedda et al. (2015).

Both my semi-analytic approach and GADGET simulations show good agreement with the revised timescale formula, (see Fig. 2.19 and 2.18 for the $\gamma = 1.0$ and $\gamma = 1.5$ cases, respectively). The radial trajectory of the inspiral in GADGET is very well reproduce by my semi-analytic formula and I can validate my approach for the inner cuspy regions. In the $\gamma = 1.0$ models, the semi-analytic approach diverges slightly from the GADGET simulation, as the live background distribution is slightly shallowed by the inspiraling body (Goerdt et al. 2010), however the match is still reasonably good, with the inspiral time being well captured. A mechanism for feeding the energy lost by the satellite back into the analytic background distribution would be able to correct for this effect for massive satellites in weaker cusps.

It should be noted that gtpt1.5e agrees much better with my semi-analytic model than gta10e does with NBODY6df. This is because the effect of close encounters (i.e. $b \sim b_{\text{min}}$) is completely resolved in the point mass case. Whereas with a cluster comprised of particles, the effect of close encounters with background stars at the edge of the cluster...
Figure 2.19: Distance of the cluster with respect to the Galactic Centre as a function of time for simulations gtpt1.0, gtpt1.0e, sapt1.0 and sapt1.0e. The clusters are modelled as analytic plummer spheres, at an initial distance of 5 pc, with initial velocities of either $v_c$ (gtpt1.0 and sapt1.0) or $(1 - 0.3)v_c$ (gtpt1.0e and sapt1.0e), in a $\gamma = 0.5$ Dehnen galaxy (where $v_c$ is the circular velocity).
2.6. DISCUSSION

Figure 2.20: Density profiles of Hénon’s Isochrone (solid red line), and Dehnen’s model with $\gamma = 0$ (dashed green line); $r$ is normalised in units of $b$, and $\rho$ in units of and $\rho_0$, the central density.

is underestimated. The NBODY6 simulations treat these encounters accurately, and thus excellent agreement is found.

Chandrasekhar’s local approximation is inaccurate near the centre of cuspy host systems, and AC14’s approach is more representative of the true force in the very central region of the background distribution. I recommend the use of AC14’s numerically calculated Coulomb logarithm when a very accurate representation of inspiral is required in the very inner region of cuspy profiles. However, my varying Coulomb logarithm can reasonably approximate the force experienced throughout the cuspy region, only slightly underestimating the inspiral time, without the need for two free parameters.

2.6.2 Failure of the model in large cores

The Petts et al. (2015) model seems to be able to capture the onset of core formation in both strong cusps and cusps as shallow as the $\gamma = 0$ Dehnen model. But what if a galaxy has a large pre-existing core, as is expected for dwarf galaxies due to baryonic feedback within a $\Lambda$CDM (Lambda cold dark matter) cosmology (Navarro et al., 1996; Read and Gilmore, 2005; Pontzen and Governato, 2012, Governato et al., 2012, Pontzen and Governato, 2014, Di Cintio et al., 2014, Read et al., 2016a,b), or as is expected in self-interacting dark matter models (Rocha et al., 2013, Elbert et al., 2015)?
Figure 2.21: Inspiral of a satellite on an initially circular orbit in a cored galaxy modelled by Hénon’s Isochrone model. The dashed green line is the result from the N-body model, and the magenta line is computed using the semi-analytic model from Petts et al. (2015).
In an erratum of Petts et al. (2015), I showed that the Petts et al. (2015) dynamical friction model fails in galaxies with a constant density core. I presented an additional simulation of a satellite inspiraling in Hénon’s Isochrone model (Henon, 1959; Hénon, 1960), which has a large constant density core. The density profile of Hénon’s Isochrone model is given by:

\[ \rho(r) = M_g \left[ \frac{3(b + a)a^2 - r^2(b + 3a)}{4\pi(b + a)^3a^3} \right], \quad (2.6.1) \]

where \( M_g \) is the total galaxy mass, \( b \) is the scale radius and \( a = \sqrt{b^2 + r^2} \). Fig. 2.20 shows that the density profile of the Hénon model is substantially flatter than the \( \gamma = 0 \) Dehnen model. The Hénon model was composed of \( 2^{24} \) equal mass particles, sampled with isotropic velocities. The units were \( G = M_g = R_V = 1 \), where \( G \) is the gravitational constant and \( R_V \) is the virial radius of the galaxy, defined as \( R_V = -GM_g^2/2U \), where \( U \) is the total potential energy of the system. The satellite had a mass of \( 1.6 \times 10^{-4} \) and was initially placed on a circular orbit at the scale radius of the galaxy (\( \sim 0.237R_V \)). Fig. 2.21 shows the results of the N-body simulation and the semi-analytic model from Petts et al. (2015). Fig. 2.21 shows that the Petts et al. (2015) model does predict stalling, but at a smaller radius than the N-body model shows.

In section 2.2 I discussed how Chandrasekhar’s dynamical friction formula has proven to fail in galaxies with a large core (i.e. when the density is approximately constant). The reason for this failure has previously been thought to be due to the “super-resonance” of harmonic potentials (for the link between Chandrasekhar’s formula and the resonant nature of dynamical friction see also Tremaine and Weinberg, 1984; Weinberg, 1986), which cause satellites to fall to the centre of its host galaxy rapidly, followed by an abrupt stalling near density region (Read et al., 2006; Inoue, 2009; Goerdt et al., 2010). In Chapter 3 I generalise the Petts et al. (2015) model and show that the interpretation of resonance is valid for the stalling phase, but that the “super-Chandrasekhar” inspiral does not owe to resonance, and can be reproduced by more accurate treatment of Chandrasekhar’s formula (Petts et al., 2016).

2.7 Conclusions

In this chapter I present a modification to Sverre Aarseth’s GPU-enabled direct summation code NBODY6, dubbed NBODY6df, to include the effects of dynamical friction on the inspiral of a star cluster in a smooth background particle distribution. In this approach, the dynamical friction force on each cluster particle is computed analytically and added to the \( N \)-body forces exerted by the other cluster particles. In this way, only the cluster needs to be modelled in a \( N \)-body fashion, while the effect of the background stars is included in an approximated but reliable way. This significantly reduces computational time with respect to a full \( N \)-body modelling of the cluster and the background system.

It should be emphasised that the dynamical friction treatment I have implemented in NBODY6df is physically motivated rather than calibrated on \( N \)-body simulations, and thus has predictive power, owing to the physically motivated maximum and minimum impact parameters. The predictive power of NBODY6df allows for quick modelling of a large parameter space of initial conditions without prior calibration. The mass term in
Chandrasekhar’s formula for extended objects is found to be well represented by the mass enclosed within the tidal radius, rather than the just the energetically bound stars. This is due to potential escapers enhancing the gravitational wake whilst they remain inside the cluster. NBODY6df can be used to simulate young cluster inspiral in the Galactic Centre, or the inspiral of globular clusters in dwarf galaxies. It should be noted that dynamical friction in a disk or other highly non-spherical systems cannot yet be reliably modelled with NBODY6df. This is due to the maximum impact parameter being smaller perpendicular to the disk than parallel to it. Accurately modelling inspiral in a disk would likely require an angular dependence in the summation of possible impact parameters, and is beyond the scope of this work. The first published scientific application of NBODY6df (Petts and Gualandris, 2017) is discussed in Chapter §4.

For a direct summation code, computational time scales with $N^2$ for an integration of one $N$-body time unit. A full $N$-body simulation of a $10^5 M_\odot$ cluster with mean mass of $0.58 M_\odot$ and a Kroupa mass function would require $\sim 9 \times 10^7$ background particles for a 10:1 ratio of $M_{bg} : M_c$. On the other hand, a simulation with NBODY6df would only require the $1.73 \times 10^5$ cluster particles to be modelled as $N$-body particles, reducing the computational time by several orders of magnitude.

While the current implementation adopts a Dehnen model for the background system, any static model can be implemented in order to follow the evolution of star clusters in which dynamical friction of the orbit is important. NBODY6df is publicly available on GitHub².

The Petts et al. (2015) model has also been shown to fail in large cores. This issue is resolved in §3 (Petts et al., 2016), where I generalise the model to work accurately in galaxies with a large constant density core. With the updated Petts et al. (2016) model, NBODY6df can also be used to study dynamical friction in constant density cores.

²github.com/JamesAPetts/NBODY6df
Chapter 3

A semi-analytic dynamical friction model for cored galaxies

This chapter is based on (Petts et al., 2016), published in Monthly Notices of the Royal Astronomy Society, Volume 463, Issue 1.

3.1 Introduction

In Chapter §2 (Petts et al., 2015) I introduced a semi-analytic model that reproduces the inspiral and correct stalling radii of satellites orbiting a Dehnen background (Dehnen, 1993), including the case where the asymptotic logarithmic slope approached zero. However, the model fared less well for large constant density cores. It also failed to reproduce the rapid “super-Chandrasekhar” phase reported in Read et al. (2006) (hereafter R06).

In this chapter I present a generalisation of the Petts et al. (2015) model, that reproduces the fast inspiral and stalling experienced by satellites orbiting galaxies with a large constant density core. I show that the fast inspiral phase does not owe to resonance. Rather, it owes to the background velocity distribution function for the constant density core being dissimilar from the usually-assumed Maxwellian distribution. By including the correct background velocity distribution function and the semi-analytic model from Petts et al. (2015), I am able to correctly reproduce the infall rate in both cored and cusped potentials. However, in the case of large cores, my model is no longer able to correctly capture core-stalling. I show that this stalling owes to the tidal radius of the satellite approaching the size of the core. By switching off dynamical friction when \( r_t(r) = r \) (where \( r_t \) is the tidal radius at the satellite’s position) I arrive at a model which reproduces the \( N \)-body results remarkably well. Since the tidal radius can be very large for constant density background distributions, my model recovers the result that stalling can occur for \( M_s/M_{\text{enc}} \ll 1 \), where \( M_s \) and \( M_{\text{enc}} \) are the mass of the satellite and the enclosed galaxy mass, respectively. Finally, I include the contribution to dynamical friction that comes from stars moving faster than the satellite. This next-to-leading order effect becomes the dominant driver of inspiral near the core region, prior to stalling.

The idea that core stalling is tidally induced was already explored in Goerdt et al. (2010) for massive satellites infalling within relatively cuspy background distributions. Here, I show that this same idea can be generalised to large constant density cores in which the tidal radius of the satellite approaches the size of the cored region. I also address the...
“Super-Chandrasekhar” friction phase observed in R06 previously thought to be “super-resonance” of the harmonic core. The idea was that as the angular frequency in a perfectly flat core is the same for every star, some global resonant effects may drive the friction in a way that cannot be correctly described by considering only two-body interactions (R06). As real systems are never truly harmonic – especially if one considers the back reaction of the satellite – I argue here that the friction cannot owe to “super-resonance” (i.e. a proposed efficient resonant interaction that dominates the friction, see R06). Instead, I show that this phase of rapid infall is due to previously invalid assumptions about the velocity distribution in the core, and can be explained entirely through local friction via two-body interactions.

The chapter is organised as follows. In section 3.2 I describe the galaxy models used in this study. In section 3.3 I explain the theory and necessary improvements to my model. In section 3.4 I describe the simulations used to test my model. In section 3.5 I compare the results of my new model to N–body results. In section 3.6 I discuss the stalling mechanism and the potentially related problem of “dynamical buoyancy” reported recently in Cole et al. (2012). Finally, in 3.7 I present my conclusions.

### 3.2 Models

In this paper, I primarily consider the inspiral of a massive body moving in an isotropic distribution of stars described by Hénon’s Isochrone model (Henon, 1959; Henon, 1960) (see Eq. 2.6.1). This galaxy model has a particularly large constant density core that leads to dynamical friction stalling much further out than predicted in the Petts et al. (2015) semi-analytic model.

In addition, in order to understand the “super-Chandrasekhar” phase that precedes stalling in large cores like the Hénon model, above, I consider also an isotropic Dehnen model background (Dehnen, 1993; Saha, 1993; Tremaine et al., 1994) (see Eq. 2.4.1). I consider a model with $\gamma = 0$, in which satellites in Petts et al. (2015) exhibited the “super Chandrasekhar” phase but did not exhibit stalling at $M_s \gg M_{enc}$; and a cuspy model with $\gamma = 1.0$ which shows neither a “super Chandrasekhar” phase nor unexpected stalling. The Dehnen model, for both $\gamma = 0$ and especially $\gamma = 1.0$, is well fit by the semi-analytic dynamical friction model from Petts et al. (2015). The Hénon Isochrone and Dehnen model background density distributions are shown in Fig. 2.20.

### 3.3 Theory

#### 3.3.1 A more accurate formula when $b_{\text{min}} \sim b_{\text{max}}$

When studying the entire inspiral of satellites using Chandrasekhar’s formalism, $b_{\text{min}}$ can approach and exceed $b_{\text{max}}$. It is therefore necessary to relax the assumption that $b_{\text{max}} \gg b_{\text{min}}$. Neglecting the velocity dependence of eq. 27 from Chandrasekhar (1943), Eq. 2.2.1 originates from assuming $\Lambda \gg 1$:

$$\log(\Lambda^2 + 1) \simeq \log(\Lambda^2) = 2 \log(\Lambda). \quad (3.3.1)$$
3.3. THEORY

The factor of 2 is included in the coefficient of Eq. 2.2.1, such that if the approximation is not made then Eq. 2.2.1 becomes:

$$\frac{dv_S}{dt} = -2\pi G^2 M_S \rho \log(\Lambda^2 + 1) f(v_* < v_s) \frac{v_S}{v_S^3},$$  

(3.3.2)

Therefore when $\Lambda < 1$, $\Lambda^2 \ll 1$ and the logarithm sharply tends to 0. In Petts et al. (2015) I used Eq. 2.2.1 and simply set $\log(\Lambda)$ to 0 if $\Lambda \leq 1$. The quantitative difference of the two approaches during inspiral is negligible, but Eq. 3.3.2 is more elegant, with no arbitrary cutoff. The reader will notice that by relaxing the assumption that $b_{\text{max}} \gg b_{\text{min}}$, one loses the core-stalling mechanism of the Petts et al. (2015) model. However, I replace this stalling formalism with a superior one that reproduces the stalling in both cusps and large cores in section 3.3.3.

3.3.2 Reproducing the inspiral: the importance of velocity structure

To first order, only stars moving slower than the satellite contribute to the friction ( Chandrasekhar 1943) (but see Antonini and Merritt 2012 for an example where the next to leading order term is important), and $f(v_* < v_s)$ is taken to be the fraction of stars moving slower than the satellite. Usually a Maxwellian distribution of velocities is assumed, which leads to the simple expression:

$$f(v_* < v_s) = \text{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2),$$  

(3.3.3)

where $X = v_S/\sqrt{2}\sigma$ and $\sigma$ is the velocity dispersion.

This Maxwellian assumption can fail for two reasons. Firstly, it is typically assumed that the velocities of background stars extend to infinity, whereas in realistic backgrounds they will be truncated at the local escape velocity, $v_{\text{esc}}$. Secondly, the shape of the local velocity distribution function can deviate significantly from the Maxwellian form.

Fig. 3.1 shows the fraction of stars moving slower than the circular velocity as computed by the Maxwellian approximation and by the distribution function (i.e. the true fraction) for Hénon’s Isochrone and Dehnen’s model with $\gamma = 0$ and $\gamma = 1$. As can be seen, the assumption of a locally Maxwellian velocity distribution works reasonably well (at the $\sim 10\%$ level) for both Dehnen models, apart from in the very centre. However, for Hénon’s Isochrone model, it gives a very poor match. In particular, the Maxwellian assumption severely under-predicts the fraction of slow moving stars. This, then, is a promising first place to look for understanding why the semi-analytic model in Petts et al. (2015), that assumes a Maxwellian velocity distribution function, fails for a Hénon Isochrone background.

3.3.3 Correctly capturing the stalling effect in large cores

In Petts et al. (2015), I argued that core stalling occurs when $b_{\text{min}} \geq b_{\text{max}}$ and/or the fraction of slow moving stars at the satellite’s position approaches zero as it inspirals. Indeed, this gave an excellent match to core-stalling in a Dehnen background, even for $\gamma = 0$. However, for large and particularly flat cores, the Petts et al. (2015) model fails...
Figure 3.1: The fraction of stars moving slower than a satellite on a circular orbit, \( f(v_* < v_{s,c}) \), as calculated from the distribution function (solid), and by assuming a Maxwellian distribution of velocities (dashed); as a function of the scale radius for Hénon’s Isochrone Model (red) and Dehnen’s Model with \( \gamma = 0, 1.0 \) (blue and green respectively), where \( v_{s,c} \) is the circular velocity. As \( \sigma \) is larger than the circular velocity inside the entire scale radius of Hénon’s Isochrone model, the Maxwellian velocity distribution function severely under-predicts the number of slow moving stars. The radius, \( r \) is normalised in units of the scale radius, \( b \).
3.3. THEORY

(see Chapter §2.6.2). This owes in part to the poor approximation of a Maxwellian velocity distribution function for the Hénon Isochrone background. However, as we shall show in section §3.5, this is only part of the story.

To obtain the correct stalling radii for large cores, I extend an idea originally presented in Goerdt et al. (2010) that core-stalling occurs when the infalling satellite tidally disrupts the cusp and forms its own small core. The authors claim that the stalling occurs after core creation due to the mechanism described in R06, whereby the stars move in epicycles in the combined potential of the galaxy and satellite without a net change in energy when averaged over the orbit. They state that this transformation must occur at approximately the tidal radius, where the satellite itself tidally strips the cusp of the galaxy. The authors found the empirical relation:

$$r_s \sim (2 - \gamma) r_t,$$

(3.3.4)

where \(r_s\) is the stalling radius of satellite, and \(r_t\) is the tidal radius of the satellite. The coefficient was derived empirically for inner slopes of \(\gamma = 0.5, 1.75\). However, I show that although this coefficient gave an excellent fit, it is an artifact arising from an inaccurate definition of the tidal radius. The formal definition of the tidal radius for a point mass on a circular orbit is (King, 1962; Binney and Tremaine, 2008):

$$r_t^3 = \frac{GM_s}{\Omega^2 - \frac{d^2\Phi}{dr^2}},$$

(3.3.5)

where \(d^2\Phi/dr^2\) is the second derivative of the host galaxy’s potential at the satellite’s position, and \(\Omega\) is the rotational velocity of the satellite. For a circular orbit, by definition:

$$\Omega^2 = \frac{GM_{enc}}{r^3}.$$

(3.3.6)

Eq. 3.3.5 highlights why the tidal radius becomes very large near the centre of galaxies with a large core, as both terms in the denominator tend to zero as \(r \rightarrow 0\). Conversely, for very cuspy distributions the mass is very centrally concentrated and thus the denominator greatly increases towards the centre of the system.

Inside a homogeneous spherical galaxy \(\Omega^2 = d^2\Phi/dr^2 = (4/3)\pi G \rho_0\) and \(r_t = \infty\) everywhere, independent of the satellite orbit. With an infinite tidal radius, any star in the galaxy is formally bound to the satellite and transfers no net energy when time averaged over its orbit, thus the satellite experiences no friction (consistent with the similar analytic argument of R06). However, such a configuration would be unstable due to the influence of the satellite on the background. If the assumptions of homogeneity are relaxed and one considers a realistic and finite density profile with a large core, such as Hénon’s Isochrone model, the tidal radius is now only infinite at the very centre of the system and finite everywhere else. However, the presence of the large core can cause the tidal radius to grow very large as the cluster migrates towards the centre and \(\gamma \rightarrow 0\), causing the cusp/core transformation described in Goerdt et al. (2010) to occur when \(M_s \gg M_{enc}\).

The dynamics of stars within the tidal radius are dominated by the satellite as opposed to the background and the phase-space distribution of the background will be drastically disrupted from its original state when \(r = r_t\). At this scale the galactic centre is tidally disrupted by the satellite, reshaping the velocity distribution of the core and stalling the orbit. The probability of stars scattering off of the satellite at a specific angle is no longer
uniform and $v_{\text{typ}} \neq v_s$, as some relative interaction velocities become more probable than others. Thus, the assumption of an isotropic pristine core is broken and Eq. [3.3.2] fails. Calculating $v_{\text{typ}}$ of the resulting distribution is far from trivial, especially because the combined potential is not spherical and is only stationary in the co-rotating frame of the satellite (and only for a circular orbit).

R06 explain the stalling behaviour by showing that for a harmonic core there exist states where no net energy is transferred to the satellite. Inoue (2011) showed that in this regime a few particles have orbits that feed energy to the satellite over a few satellite orbits. These so called “horn” particles have orbits that stay close to the tidal radius of the satellite for extended periods (namely between the L2 and L3 equipotential surfaces, see fig. A1 of Inoue, 2011). These horn particles counter-act most of the dynamical friction due to a complex 3-body interaction with the satellite and the galaxy. The horn particles evidently play a vital role in keeping the satellite buoyant. However, particles occupying this region of phase space do eventually move away from the satellite and other particles enter the horn trajectory (see fig. 10 and table 1 of Inoue, 2011). I consider that the analytic estimate of R06 may be degenerate with the presence of horn particles if most stars that come close to the tidal radius of the satellite in this new distribution will be horn trajectory for at least part of their orbits.

As an ansatz, I put a constraint on the frictional force such that:

$$
\dot{v}_{df} = \begin{cases} 
\frac{d v_S}{dt}, & \text{if } r_t (r_a) < r_a \\
0, & \text{if } r_t (r_a) \geq r_a,
\end{cases}
$$

(3.3.7)

where $\frac{d v_S}{dt}$ is the frictional model employed (either Eq. [3.3.2] or [3.3.8]). Hereafter, I call this mechanism “tidal stalling”. In section §3.5, I show that with this additional constraint core stalling can be captured remarkably well in both large cores and cuspy galaxies. This suggests that the stalling in large cores occurs via the same “cusp disruption” mechanism that occurs in cuspy profiles, in agreement with Goerdt et al. (2010). The only “unique” aspect of a large pre-existing core is that the extended tidal influence of the satellite in the shallow region means that the satellite can disrupt the galactic centre at $M_{\text{enc}} \gg M_S$.

3.3.4 The effect of fast moving stars

When deriving Eq. [2.2.1] Chandrasekhar (1943) assumed that only stars moving slower than the satellite contribute to the frictional force. In most distributions this is a good approximation as there is an abundance of slow moving stars that all contribute to, and dominate, the friction. The effect of interactions with faster moving stars is fundamentally different, which I demonstrate by considering the general Chandrasekhar formula (eqs. 25 and 26 in Chandrasekhar (1943)):

$$
\frac{d v_S}{dt} = -4 \pi G^2 M_s \rho(r) \frac{v_s}{v_{\ast}^3} \int_0^{\sqrt{-2\phi(r)}} \frac{J(v_s)}{8 v_s} 4 \pi f(v_s) v_s^2 d v_s, \\
J(v_s) = \int_{|v_s - v_*|}^{v_s + v_*} \left(1 + \frac{v_s^2 - v_*^2}{V^2}\right) \log \left(1 + \frac{b_{\text{max}}^2 V^4}{G^2 M_s^2} \right) d V,
$$

(3.3.8) (3.3.9)
where \( V \) is the relative velocity of the encounter, and \( J(v_s) \) is a function describing the interaction strength of a single velocity species integrated over all possible relative velocities and impact parameters. \( J(v_s) \) is positive for all \( V \) if \( v_s < v_s \), therefore all slow moving stars in the system remove energy from the satellite.

Intriguingly, Eq. 3.3.9 predicts “dynamical buoyancy” (Cole et al., 2012) from a portion of the stars moving faster than the satellite. If \( v_s > v_s \) then \( J(v_s) \) is negative if:

\[
\frac{v_s^2 - v_s^2}{V^2} < -1.
\] (3.3.10)

Such interactions feed energy into the satellite producing a buoyancy effect opposing the frictional force of other fast moving stars. However, as the minimum impact parameter is smaller for interactions with a higher relative velocity, when summed over all impact parameters there is a net residual frictional force from the fast moving stars. This residual force is usually small compared to the friction coming from the slow moving stars, and is ignored in deriving Eq. 2.2.1. However, [Antonini and Merritt (2012)] showed that in situations where the density of fast moving stars is much greater than that of the slow moving stars this residual force can become non-negligible or even dominate in extreme conditions (such as deep in the potential well a of shallow stellar cusp around a super massive black hole). Subsequently, [Arca-Sedda and Capuzzo-Dolcetta (2014)] modelled the dynamical friction on satellites in galaxies of various inner log-slope, \( \gamma \), taking into account the non-locality of the friction in cusps as well as the contribution of the fast stars, showing improved agreement in galactic centres compared with using Eq. 2.2.1.

Similarly, inside a large core there are very few stars moving slower than the circular velocity, and the residual friction from the fast stars could be important in this case.

In section §3.5 I solve Eq. 3.3.8 in addition to Eq. 3.3.2 to quantify the effect of these fast moving stars. The possible role of the fast moving stars in the stalling phase is discussed in section 3.6.2.

### 3.3.5 Summary of the updated model

In Table 3.1 I briefly summarise the differences in my updated models as compared to the model presented in Chapter 2 (Petts et al., 2015, hereafter P15). The new models use \( b_{\text{max}} \) and \( b_{\text{min}} \) as described in P15, with the exception of the P16f model which includes the relative velocity dependence of interactions in \( \log(\Lambda) \). In general, the P16f model will give the most physically accurate results, however it requires a little more computation than the P16 model. Although I stated in [Petts et al., 2016] that the P16f model required a double integral, [Inoue (2017)] derived an analytical solution to Eq. 3.3.9 which I reproduce in appendix A.2. For most scenarios the P16 model is adequate, however the fast moving stars can make up a significant portion of the friction in regions where there are few slow moving stars available, as I show in section 3.5. In this chapter I sometimes turn off tidal stalling to demonstrate the model, however, for practical application is should always be employed. I notate the inclusion of tidal stalling in the models with the addition of “+TS” to the name (i.e. with tidal stalling switched on P16 and P16f become P16+TS and P16f+TS, respectively).
3.4 Simulations

In order to test my analytic predictions, I use the tree-code GADGET2 (Springel 2005) to simulate the inspiral of satellites in Hénon’s Isochrone model and Dehnen’s model. I use units of $G = M_g = b = 1$, where $G$ is the gravitational constant, $M_g$ is the total galaxy mass and $b$ is the scale radius of the given galaxy. I use point mass satellites with masses of multiples of $1.595 \times 10^{-4} M_g$, which corresponds to $2 \times 10^5 M_\odot$ in the Hénon model when normalised to the same central density as the simulations in R06. If the stalling mechanism is independent of the velocity structure I should obtain similar ratios of $M_s/M_{\text{enc}}$ at the stalling radius to R06. The initial conditions of the simulations are displayed in Table 3.2.

I compare the GADGET2 simulations to my semi-analytical model where I use a static analytic model for the background galaxy and perturb the orbit with different friction models as described in Table 3.1. I use a leap-frog integrator with variable time-step to integrate the perturbed orbit, which conserves energy to a relative error of $< 10^{-7}$ in the absence of dynamical friction over the timescales considered. With dynamical friction switched on, if I sum up the removed orbital energy and add it to the final energy of the satellite I obtain the same relative error.

I use the following naming convention for simulations: for $N$-body models computed with GADGET2 I name the simulation $gt_{\langle IC\rangle}$, where $\langle IC\rangle$ are the initial conditions described in Table 3.2. For semi-analytic models I name the simulation $df_{\langle X\rangle_{\langle IC\rangle}}$, where $\langle X\rangle$ is the dynamical friction model used, as described in Table 3.1.
3.4. SIMULATIONS

Figure 3.2: Evolution of the satellite orbit with time for simulations $gt_H$ (green line) and $df\langle X \rangle_H$. The $df\langle X \rangle_H$ simulation is shown with the standard Maxwellian approximation (P15; dot-dashed red line), with the true $f(v_s < v_s)$ (P16; dashed red line), with the true $f(v_s < v_s)$ including the effects of the fast moving stars (P16f; dotted red line), and the same model with tidal stalling (see section §3.3.3) turned on (P16f+TS; solid red line)
Figure 3.3: Evolution of the satellite orbit with time for simulations computed with GADGET2 ($g_{t_H}$, $g_{t_H2}$, $g_{t_H4}$; dashed lines) and my model considering only the slow moving stars (P16+TS; dotted lines) and considering all the stars (P16f+TS; solid lines). The satellite masses are $1.595 \times 10^{-4}$, $3.19 \times 10^{-4}$ and $6.38 \times 10^{-4}$ for simulations $g_{t_H}$, $g_{t_H2}$ and $g_{t_H4}$, respectively. The simulation initial conditions are described in table 3.2.
3.5. RESULTS

Table 3.2: Initial conditions of the simulations. Column 1 lists the name of the initial conditions. Column 2 states the mass of the satellite in units of $1.595 \times 10^{-4} M_g$. Column 3 shows the initial position of the satellite, where $r_0$ is in units of the scale length, $b$, in Eqs. 2.6.1 and 2.4.1. Column 4 shows the initial satellite velocity, $v_0$, in units of $v_c$, the circular velocity at $r_0$. Column 5 shows the inner asymptotic slope of the Dehnen models and column 6 shows the number of particles used to simulate the halo in the GADGET2 simulations. (*H2multi includes two satellites, initially orbiting in the x-y and x-z planes.)

<table>
<thead>
<tr>
<th>IC Name</th>
<th>$M_s$</th>
<th>$r_0$</th>
<th>$v_0$</th>
<th>$\gamma$</th>
<th>$N_{bg}$</th>
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<td>-2</td>
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</tr>
<tr>
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<td>-</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>H2e4</td>
<td>2</td>
<td>0.4</td>
<td>-</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>H2multi*</td>
<td>2</td>
<td>0.5</td>
<td>1.0</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>H4</td>
<td>4</td>
<td>1.0</td>
<td>-</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>H1.5b</td>
<td>1</td>
<td>1.5</td>
<td>-</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>H0.3b</td>
<td>1</td>
<td>0.295</td>
<td>1.0</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>H0.17b</td>
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<td>1.0</td>
<td>-2</td>
<td>24</td>
</tr>
<tr>
<td>D0</td>
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<td>0.5</td>
<td>1.0</td>
<td>0</td>
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<tr>
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<td>1.0</td>
<td>1</td>
<td>22</td>
</tr>
</tbody>
</table>

3.5 Results

3.5.1 Circular inspiral

Fig. 3.2 shows N-body simulation $gt_H$ and 4 realisations of the semi-analytic model, $df_\langle X \rangle_H$, with different force models. The dot-dashed red line shows the result obtained by the standard Maxwellian approximation (P15), and gives an extremely poor fit to the N-body data. When the correct $f(v_*)$ for Hénon’s isochrone is used (P16; dashed red line) then the inspiral is reproduced excellently for the majority of the orbital evolution, as the velocity distribution has the correct shape and the fraction of slow moving stars is no longer under-predicted. Deep in the core, prior to stalling, Eq. 3.3.2 slightly underestimates the friction experienced by the satellite in the N-body model. Solving Eq. 3.3.8 (P16f; dotted red line) shows that the discrepancy originates from ignoring the residual friction from the fast moving stars, which becomes significant in this region. Including the fast moving stars gives an excellent fit right down to the stalling radius.

In the semi-analytic models P16+TS and P16f+TS, dynamical friction stops when $r_t(r_a) = r_a$, when the satellite can tidally disrupt the core. At this point the satellite stalls, with inspiral being much slower than one would estimate if the core is assumed to resemble its initial conditions (marked with a red dashed line and a red dotted line for Eqs. 3.3.2 and 3.3.8 respectively). This simple model for the tidal stalling gives a very good fit to the N-body data, which stalls at $M_s/M_{enc} = 0.03$. This is only a factor of 2 smaller than in R06, which is expected as Hénon’s Isochrone has a shallower core than the model in R06. The semi-analytic model stalls at $M_s/M_{enc} = 0.04$, slightly farther in. It is not surprising that I underestimate the tidal radius with Eq. 3.3.5 as it is derived under the assumption that $r_t \ll r$, which allows one to linearise the forces. Nevertheless, the approximate tidal radius gives a satisfying fit.
3.5. RESULTS

Fig. 3.3 shows how the $N$-body simulations and semi-analytic models vary as a function of the satellite mass. The semi-analytic models reproduce the inspiral excellently. The stalling radii scale in the same way as the $N$-body models, with larger masses stalling further out. Although the tidal radii scale as $M_S^{1/3}$, the stalling radii have a sub $M_S^{1/3}$ scaling, as the other terms in Eq. 3.3.5 are also functions of $r$.

One could fit a free parameter that scales with the mass to better reproduce the stalling radii, however, any such free parameter would be dependant on the galaxy model. I choose to keep the model free of any free parameters to ensure its predictive power in general spherical potentials.

3.5.1.a Effect of initial distance

Simulations $gt_{\text{H}1.5b}$, $gt_{\text{H}0.3b}$ and $gt_{\text{H}0.17b}$ have the same initial conditions at $gt_{\text{H}}$, except the satellites are initially on circular orbits at $1.5\,\text{b}$, $0.295\,\text{b}$ and $0.168\,\text{b}$, respectively.

In $gt_{\text{H}1.5b}$ the satellite is initially far outside the core, where the local density slope, $\gamma = -2.4$. Fig. 3.4 shows that in models using the self-consistent $f(v_*)$, the inspiral is very well reproduced throughout the satellite’s migration to the cored region. As well as using the correct distribution function, the success of the model owes also to my radially varying $b_{\text{max}}$, which is smaller than $r$ in the cuspy outer regions (see Just et al., 2011; Petts et al., 2015). The satellite stalls at the same radius as in $gt_{\text{H}}$, verifying that the stalling radius is independent of the initial distance if the satellite originates from outside the core region, in agreement with Goerdt et al. (2010).

Simulation $gt_{\text{H}0.3b}$ starts just outside of where the satellite stalls in simulations $gt_{\text{H}}$ and $gt_{\text{H}1.5b}$, but stalls slightly further in. This is because the distribution function is in its pristine state and the satellite feels friction until it has enough time for the background and satellite to settle into the R06 state with no net momentum exchange. The same is true for $gt_{\text{H}0.17b}$, where the satellite is initially below the stalling radius of the satellites in $gt_{\text{H}}$ and $gt_{\text{H}1.5b}$. In both $gt_{\text{H}0.3b}$ and $gt_{\text{H}0.17b}$ the semi-analytic model including all stars is initially in great agreement with the $N$-body results, as the distribution function assumed in the model is initially correct. This strengthens the idea that a shift in the distribution function is why Eqs. 3.3.2 and 3.3.8 fail without including the tidal stalling prescription. Interestingly, I do not see the “dynamical buoyancy” effect discovered in Cole et al. (2012) in simulation $gt_{\text{H}0.17b}$, this is discussed in section 3.6.3.

One should note that although the semi-analytic formula will be poor at reproducing $N$-body results if $r_i \sim r_1(r_1)$, where $r_1$ is the initial apocentre, this is a purely numerical effect. In the real universe, initial conditions such as $gt_{\text{H}0.3b}$ and $gt_{\text{H}0.17b}$ are impossible as the galaxy potential will be self-consistent with the presence of the satellite upon its formation, meaning that if the satellite is born deep in the cored region it will initially be in a stalled state. As such, the semi-analytic model for $gt_{\text{H}0.17b}$ in which the satellite simply has no orbital evolution, is justified.

I would also like to note that in theory one may have no need to employ Eq. 3.3.7 if one could model the self consistent velocity distribution that includes the effect of the satellite on the distribution function of the background. However such a model is far from trivial to calculate since the satellite is off centre from the background distribution, thus the distribution function is highly aspherical around the satellite. Such a model is
Figure 3.4: Evolution of the satellite orbit with time for simulations \( g\Gamma \text{H1.5b} \) (green line) and \( \text{df} \langle x \rangle \text{H1.5b} \). The \( \text{df} \langle x \rangle \text{H1.5b} \) simulation is shown with the standard Maxwellian approximation (P15; dot-dashed red line), with the true \( f(v_s < v_a) \) (P16+TS; dashed red line), and with the true \( f(v_s < v_a) \) including the effects of all stars (P16f+TS; solid red line). The later two models include the effects of tidal stalling.
beyond the scope of this work. I use my tidal stalling prescription as a simple physically motivated model to capture the stalling radius at which the distribution function should be heavily perturbed by the satellite.

### 3.5.2 Elliptical inspiral

I re-ran simulations \( g_{t,H2} \) and \( d_{f,H2} \) with an initial tangential velocity of \( 0.4 v_c \) (simulations \( g_{t,H2e4} \) and \( d_{f,H2e4} \)). Taking the stalling radius to be the same as the radius at which a circular orbit at apocentre would stall gives good agreement to the \( N \)-body model. This makes intuitive sense if one considers that stalling is a result of tidal disruption of the core. Prior to stalling, the satellite experiences friction when passing through pericentre, as the satellite moves quickly in and out of the core. However, once the entire orbit is inside the core the satellite can tidally disrupt the core over the course of a few orbits. For a spherical potential the apocentre can easily be calculated from any point of the orbit by solving the equation for the turning points of the orbit (Binney and Tremaine, 2008):

\[
\frac{r^{-2} + \frac{2.0 [\Phi(r) - E]}{L^2}}{r} = 0,
\]

where \( \Phi \), \( E \) and \( L \) are the potential, specific orbital energy and specific angular momentum, respectively. The largest and smallest solutions are the apocentre and pericentre of the orbit, respectively.

### 3.5.3 Updated friction model in weak and strong cusps

In Petts et al. (2015) my dynamical friction model used the Maxwellian approximation and satellites stalled when \( b_{\text{min}} \leq b_{\text{max}} \). As I have improved my model in (Petts et al., 2016), I ran two simulations in which the satellite orbits a Dehnen model with \( \gamma = 0 \) and \( \gamma = 1 \). Fig. 3.7 shows that using the self-consistent distribution function greatly improves the accuracy of the model for the \( \gamma = 0 \) case, and including the fast stars improves it further. There is still some discrepancy in the infall and this is most likely because the \( \gamma = 0 \) Dehnen model has a local log-slope of the density which varies rapidly over its scale radius. In such a distribution the frictional force will always be slightly underestimated, as the locality of the density distribution assumed in Eqs. 3.3.2 and 3.3.8 is the limiting assumption. If one wanted to more accurately reproduce the infall one would need to use a friction model that does not suffer from this assumption, such as the approach employed in Arca-Sedda and Capuzzo-Dolcetta (2014).

Fig. 3.8 shows the cuspy case \( \gamma = 1 \), which was already described well by the Maxwellian model in Petts et al. (2015), as \( f(v_s) \) more closely resembles Maxwellian form in cuspy models. Interestingly, the model only considering the slow stars slightly over-predicts the force, and the model with all the stars reproduces the inspiral excellently. This shows that in the cuspy case the importance of the velocity dependence of the Coulomb logarithm is small, but non-negligible. However, I would like to stress that the Maxwellian approximation, although justified in this case, performs as well as the full model by coincidence. Fig. 3.1 shows that it over-predicts the number of slow moving stars down to \( \sim 0.2 b \). In general the Maxwellian approximation will not perform as well as using the self-consistent distribution function for general cuspy distributions.
3.5. RESULTS

Figure 3.5: Evolution of the satellite orbit with time for simulations $gt_{H0.3b}$ (green dashed line), $df_{\langle X\rangle H0.3b}$ (red lines), $gt_{H0.17b}$ (magenta dashed line) and $df_{\langle X\rangle H0.17b}$ (blue lines). Simulations $df_{\langle X\rangle H0.3b}$ and $df_{\langle X\rangle H0.17b}$ are realised with different dynamical friction models, as specified in the legend.
Figure 3.6: Evolution of the satellite orbit with time for simulations $gt_{H2e4}$ (green line), $df_{P16} + TS_{H2e4}$ (red dotted line) and $df_{P16f} + TS_{H2e4}$ (solid red line).
3.5. RESULTS

Figure 3.7: Evolution of the satellite orbit with time for simulations $\text{gt}_D0$ (green line) and $\text{df}_\langle \langle X \rangle \rangle_D0$. The $\text{df}_\langle \langle X \rangle \rangle_D0$ simulation is shown with the standard Maxwellian approximation (P15; dot-dashed red line), with the true $f(v_s < v_b)$ (P16+TS; dashed red line), and with the true $f(v_s < v_b)$ including the effects of all stars (P16f+TS; solid red line). The later two models include the effects of tidal stalling.
Figure 3.8: Evolution of the satellite orbit with time for simulations $gt_{D1}$ (green line) and $df_{\langle X \rangle D1}$. The $df_{\langle X \rangle D1}$ simulation is shown with the standard Maxwellian approximation (P15; dot-dashed red line), with the true $f(v_s < v_*)$ (P16+TS; dashed red line), and with the true $f(v_s < v_*)$ including the effects of all stars (P16f+TS; solid red line). The later two models include the effects of tidal stalling.
3.6. DISCUSSION

In both the $\gamma = 0$ and $\gamma = 1$ case the stalling is very well captured by my tidal stalling mechanism. The P15 model reproduces the stalling in $\gamma = D1$ identically, but slightly under-predicts the stalling radius in $\gamma = D0$. This leads to the conclusion that there is only one type of stalling, the tidal stalling first described in Goerdt et al. (2010). It just so happens that for distributions without a large core $b_{\text{max}} \sim b_{\text{min}}$ when $r_t(r) \sim r$, which explains the success of the P15 model in these galaxies.

3.5.4 Comparison with Goerdt et al. (2010) and Petts et al. (2015)

Fig. 3.9 shows the stalling radii predicted by Goerdt et al. (2010), p15 and Eq. 3.3.7 for Dehnen models as a function of $\gamma$. Also displayed are the $N$-body results from simulations $\gamma = D0$, $\gamma = D1$ and $\gamma = H2$. Although all predictions agree in the cuspy regime, it is clear that Eq. 3.3.7 best reproduces the $\gamma = 0$ case. Eq. 3.3.7 also well reproduces the stalling radius in the Hénon profile (green circle on Fig. 3.9), whereas both the Goerdt et al. (2010) relation and P15 model fail. A caveat is that the Goerdt prediction was fit only down to $\gamma = 0.5$, and I extrapolate their fit down to $\gamma = 0$. However for galaxies with large cores, only Eq. 3.3.7 is good at approximating the stalling radius.

3.6 Discussion

Despite the successes of the analytic R06 model, it cannot be the full story. Firstly, the galactic potential is never truly harmonic in a realistic system, therefore there should be stars that do not orbit on epicycles that can contribute some frictional force. Secondly, Cole et al. (2012) report an extreme example where a satellite initially inside the core actually moves outwards – a process that authors call “dynamical buoyancy”. Such buoyancy is not captured by my semi-analytic model, nor by the R06 stalling state.

In this section, I discuss the nature of the stalling phase. In section 3.6.1 I generalise the analytical R06 stalling state and show it is consistent with and the numerical work of Inoue (2009) (hereafter I09). In section 3.6.2 I discuss the role of the fast moving stars in the stalling phase and the related work of Inoue (2011) (hereafter I11). Finally, in section 3.6.3 I discuss the fast stars in the context of “dynamical buoyancy”.

3.6.1 Generalisation of the R06 model to systems with multiple satellites

I09 performed simulations of cored dwarf galaxies containing multiple point mass globular clusters inspiraling simultaneously. The clusters perturb each others’ orbits significantly throughout I09’s simulations, yet the clusters still appear to stall at $M_{\text{enc}} > M_s$. I09 stated that if the co-rotating model of R06 were correct, then perturbations from other globular clusters should break the anisotropic velocity distribution found in R06, and one would expect the clusters to reach the galactic centre. In this section, I show that perturbations from other satellites are sub-dominant by generalising the R06 analysis to include multiple satellites. By starting from eq. 10 in R06 and including a perturbation from $N$ other satellites, one arrives at the following equation of motion:
Figure 3.9: Stalling radii predicted by Goerdt et al. (2010) (dashed red line), Petts et al. (2015) (dashed blue line) and Eq. 3.3.7 (solid blue line) for Dehnen models as a function of \( \gamma \). N-body results from \( \text{gt}_D0 \) and \( \text{gt}_D1 \) are marked by blue vertical and diagonal crosses respectively. The stalling radius of \( \text{gt}_H2 \) is marked by a green cross, and the estimate by Eq. 3.3.7 with a green circle. The Goerdt et al. (2010) fit is extrapolated below \( \gamma = 0.5 \). The mass of the particle in all calculations and simulations is \( 3.19 \times 10^{-4} M_G \).
3.6. DISCUSSION

\[ \ddot{r}_p + \Omega^2 r_p = \frac{G M_1 (r_1 - r_p)}{|r_1 - r_p|^3} + \sum_{i=2}^{N} \frac{G M_i (r_i - r_p)}{|r_i - r_p|^3}, \]  

(3.6.1)

where \( r_p \) is the vector position of a star orbiting the combined potential of the harmonic core and system of satellites, and \( M_1 \) and \( r_1 \) are the mass and vector position of the \( i^{th} \) satellite. If \( M_g(r_1) \gg \sum_{i=2}^{N} M_i \) (i.e. the gravitational potential that \( M_1 \) experiences is dominated by the galaxy), then:

\[ \ddot{r}_1 + \Omega^2 r_1 \approx 0. \]  

(3.6.2)

Combining Eqs. 3.6.1 and 3.6.2 and substituting \( r_d = r_p - r_1 \):

\[ \ddot{r}_d + \Omega^2 r_d = \frac{G M_s r_d}{|r_d|^3} - \sum_{i=2}^{N} \frac{G M_i (r_p - r_i)}{|r_p - r_i|^3}, \]  

(3.6.3)

\[ \ddot{r}_d + \left( \Omega^2 + \frac{GM_1}{|r_1|^3} \right) r_d = - \sum_{i=2}^{N} \frac{G M_i (r_p - r_i)}{|r_p - r_i|^3}. \]  

(3.6.4)

Hence when \( |r_p - r_i| \gg r_d \) the \( i^{th} \) satellite is sub-dominant and solutions exist where the satellite moves in approximate epicycles around \( M_1 \). For any close encounter of a star with \( M_1 \) this is satisfied for all \( N - 1 \) perturbations. It follows that only distant particles may contribute to the friction of satellite 1. However, by being distant from satellite 1, these particles are likely dominated by the potential of a different satellite, and will not interact with satellite 1 in the straight line as assumed by Chandrasekhar’s formula. Therefore, the energy transfer between the distant star and satellite 1 will likely be small, if not negligible. If a satellite \( M_i \) comes close to \( M_1 \), our assumptions are broken until the scattering event is complete, but after scattering \( |r_p - r_i| \gg r_d \) is satisfied again and \( M_s \) experiences no friction from local stars once again. This extension of the analytic R06 model is in qualitative agreement with the simulations of I09.

In simulation \( H_2 \text{mult} \) I test the prediction of the improved analytic R06 model by setting up a fiducial case where two satellites are initially on circular orbits in the same halo. I set one satellite at 0.5bx with its velocity vector in the positive y-direction, and the other to be at -0.5bx with its velocity vector in the z-direction. This setup ensures the satellites are maximally distant from each other during inspiral so that \( |r_p - r_i| \gg r_d \). From Eq. 3.6.4 I predict that the stalling of each satellite should be similar to the single satellite case, as the satellites should not strongly interact. Fig. 3.10 shows that this is indeed the case, verifying the validity of the multi-satellite R06 formula in reproducing the results of I09.

I would like to note that for real satellites, during close encounters tidal distortions would become dominant, leading to significant distortions of the subject bodies. Satellite 1 will change shape, size and mass, but after the encounter the satellite will again find itself in a steady state with the background, as the right hand side of Eq. 3.6.4 will againc
Figure 3.10: Evolution of the satellite orbits with time for simulations $gt_{H2multi}$ (blue and magenta dashed lines), the late evolution of $gt_{H2}$ (green dashed line) and $df_{P16f+TS_{H2multi}}$.
3.6. DISCUSSION

Figure 3.11: Cumulative energy transfer from the fast moving stars to the satellite prior to stalling in simulations $df_{P16f,H}$ and $gt_{H}$. For $df_{P16f,H}$ the contributions from stars that remove energy from (red dashed line) or feed energy to (red dot-dashed line) the satellite, and the net effect (solid red line) are displayed. Analogously for $gt_{H}$, the cumulative energy transfer to the satellite from the P-horn (dashed blue line), H-horn (dot-dashed blue line) and P+H horn particles (solid blue line) are shown. A dotted black line marks the x-axis for reference. Note the resemblance to fig. 6 in Inoue (2011).

be negligible. Although $M_1$ will have changed, solutions with negligible net changes of energy would still exist. The model will of course fail when the satellite is of comparable size/mass of the core, such that the assumption of a point mass satellite is invalid. In this case the object will experience negligible friction regardless, as $b_{\text{min}}$ will approach $b_{\text{max}}$.

3.6.2 Fast stars as the origin of stalling

I11 showed that strongly interacting “horn particles” both decelerate (P-horn) and accelerate (N-horn) the satellite. In the stalling phase, Inoue (2011) finds that the net effect of the horn particles is a transfer of orbital energy to the satellite, opposing the frictional force from other stars. During the infall phase, however, the net effect of the P and N-horn particles is a drag on the satellite, similar to the interaction of the fast moving stars predicted by Eq. 3.3.8.

Fig. 3.11 shows the cumulative energy transfer between fast stars and the satellite during the infall phase of simulation $df_{P16f,H}$, as predicted by Eq. 3.3.8. Also shown
is the exchange of energy between the P and N-horn particles extracted from simulation \( g_t \cdot H \) in the same fashion as in I11. The absolute energy transferred from each horn is larger than expected from the fast stars, however this is not surprising. The cut-off energy to define the horns is somewhat arbitrary, and one could tweak the cut-off to more closely resemble the model. However, particles artificially classified as horn particles this way (that didn’t strongly interact with the satellite) should be equally numerous in each horn if their change in energy is instead due to two-body relaxation. Indeed, the net effect of the P and N-horn particles in \( g_t \cdot H \) and all the fast stars in \( df \cdot P_{16} f \cdot H \) is remarkably similar, which is strong evidence that in the inspiral phase the horn particles are the fast moving stars.

It is logical that the P and N-horn stars are synonymous with the fast frictional and buoyant stars in the stalling phase also. In isotropic distributions there is residual drag as interactions with low \( V_{\text{rel}} \) have a larger \( b_{\text{min}} \), so less of these interactions can occur. However, fig. A1 of I11 shows that during the stalling phase, the potential that stars orbit is far from spherical. It is intuitive that configurations exist where the buoyant stars can outweigh the fast frictional stars when horn-like orbits exist (those shown in fig. 10 of I11). In the stalling state, horn particles stay very close to the cluster for an extended period, allowing each interaction to occur numerous times. However, the N-horn particles transfer more energy than the P-horn particles in this state, as the strength of each individual interaction is stronger due to the \( 1/V^2 \) dependence in Eq. 3.3.9. I note that if all stars interact on a horn-like orbit at some point over many orbital times, this mechanism is analogous to the R06 model, whereby the time averaged contribution is zero. However, considering the effects of horn particles/fast moving stars emphasises how the stalling phenomenon can occur even if the potential is never truly harmonic.

### 3.6.3 A remark on the origin of dynamical buoyancy

Cole et al. (2012) explored different mass distributions of the Fornax dwarf spheroidal and modelled the orbital evolution of its globular clusters in a suite of 2800 \( N \)-body simulations. They discovered a curious effect whereby a globular cluster originating inside a constant density core is pushed outwards before stalling similarly to clusters originating further out, the authors described this as “dynamical buoyancy”. Convergence tests ruled it out as numerical error.

The origin of this effect may owe to an increased phase space density of allowed horn orbits when the satellite is placed deep inside the pristine core. When migrating to the core from the outside, the region in which these orbits can exist expands until the satellite stalls. If the satellite originates from inside the core, it is possible that the phase-space density of horn orbits could be large enough that the buoyancy provided by N-horn stars outweighs all of the friction. In fact, since it has been verified in I11 and in section \([3.6.2]\) that the stalling owes to orbits of individual stars which transfer energy into the satellite, one must be able to construct a fiducial system whereby the N-horn particles dominate over the frictional particles. This setup will be unstable and the net effect would be “dynamical buoyancy” as discussed in Cole et al. (2012). The satellite orbit would stop increasing as the phase-space density of these orbits tends towards zero. A limitation of the Petts et al. (2016) model is that it cannot capture this effect since I do not explicitly model the effect of N-horn versus P-horn interactions. As discussed in section \([3.5.1.a]\)
3.7 Conclusions

In this chapter I have shown that Chandrasekhar’s dynamical friction model considering only two-body encounters is sufficient to explain the inspiral of satellites in constant density profiles, so long as one takes care to use the self-consistent velocity distribution function instead of the usually assumed Maxwellian distribution. In particular, I show that we can reproduce the “super-Chandrasekhar” phase, suggesting that it does not owe to resonance. The Chandrasekhar formula probably works so well because the potential is never truly harmonic in any physically reasonable distribution. The agreement is improved further by including the usually neglected contribution of the fast moving stars, which contributes a non-negligible portion of the drag inside the core.

However, even after including the correct background distribution function and the effect of fast moving stars, one finds that the model cannot reproduce the stalling observed in large constant density cores such as in the Hénon Isochrone Model. Following Goerdt et al. (2010), I show that such large-core stalling occurs in the same manner as it does for cusps. The infalling satellite tidally disrupts the core when \( r_t(r_a) = r_a \). For cusped background models this occurs at \( M_s \sim M_{\text{enc}} \). However, for cored galaxies, the satellite tidal radius can become very large indeed. This leads to stalling at many times the radius at which the mass in background stars approaches the satellite mass. In my model, \( f(v_s) \) is derived from the distribution function of the background density distribution and my model has no free or tuned parameters. As such, it should be general for any model with a cored or cusped centre.

Finally, I suggest that the dynamical friction core-stalling can be understood in two different ways. For a perfectly harmonic background with a single point mass satellite, R06 demonstrated that there exist stable solutions with no net momentum exchange between the satellite and the background. I generalised this model in section 3.6, showing that the same should be true when multiple satellites are present. While the satellite and the background likely reach an approximation to this state, the correspondence cannot be perfect. Secondly, the “dynamical buoyancy” effect discussed in Cole et al. (2012) is not captured by my semi-analytic model, nor by the R06 stalling state. Instead, the answer may lie in the interactions with stars moving faster than the satellite. Inoue (2011) showed that strongly interacting “Horn” stars can both decelerate (P-horn) and accelerate (N-horn) the satellite depending on their relative orbital phase. For a large-cored background, the cumulative effects of P and N-horn stars approximately cancel the friction experienced in the core region, leading to core-stalling as in the R06 model. However, configurations can exist where it is possible for N-horn stars to dominate over P-horn ones, if a satellite begins its life deep inside a constant density core. This is a possible explanation for Cole et al. (2012)'s dynamical buoyancy; however, a full proof would require further investigation beyond the scope of this work.
Chapter 4

Infalling Young Clusters in the Galactic Centre: implications for IMBHs and young stellar populations

This chapter is based on (Petts and Gualandris, 2017), published in Monthly Notices of the Royal Astronomy Society, Volume 467, Issue 4.

4.1 Introduction

The central parsec of the Milky Way hosts two puzzlingly young stellar populations, a tight isotropic distribution of B stars around SgrA* (the S-stars) and a disk of OB stars extending to $\sim 0.5\,\text{pc}$. Using NBODY6df (my modified version of Sverre Aarseth’s direct summation code NBODY6), I explore the scenario in which a young star cluster migrates to the Galactic Centre within the lifetime of the OB disk population due to dynamical friction. Star clusters massive and dense enough to theoretically reach the central parsec form a very massive star (VMS) via physical collisions in the core on a mass segregation timescale. I follow the evolution of the merger product by evolving the chemical composition of the VMS, coupled with the most up to date, yet conservative, mass loss recipes for VMSs. Over a large range of initial conditions, I find that the very massive star expels most of its mass via a strong stellar wind, eventually collapsing to form a black hole of mass $\sim 20 - 400\,M_\odot$, incapable of bringing massive stars to the Galactic Centre. No massive intermediate mass black hole can form in this scenario, owing to the strong line-driven winds at high metalicity. The presence of a star cluster in the central $\sim 10\,\text{pc}$ within the last $15\,\text{Myr}$ would also leave a $\sim 2\,\text{pc}$ ring of massive stars, contradicting observations. Thus, I conclude that the star cluster migration model is highly unlikely to be the origin of either young population, and in-situ formation models or binary disruptions are favoured.

The central parsec of the Milky Way hosts almost two dozen He-1 emission-line stars (Krabbe et al., 1995; Paumard et al., 2001) and a population of many other OB stars in a thin clockwise disk extending from $\sim 0.04 - 1.0\,\text{pc}$ (Eckart et al., 1999). Feldmeier-Krause et al. (2015) recently extended the range of observations up to $\sim 4\,\text{pc}^2$ centred on SgrA*; the radio source associated with the supermassive black hole (SMBH) at the centre of the Milky Way. The authors show that the OB population is very centrally
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concentrated, with 90% projected within the central 0.5 pc. The clockwise disk exhibits a top heavy mass function \((\alpha \sim 1.7, \text{Lu et al. (2013)})\). Krabbe et al. (1995) estimate the He-1 stars to be only \(\sim 3 - 7\) Myr old, which is puzzling as the tremendous tidal forces in this region make it difficult for a giant molecular cloud (GMC) to remain bound long enough for gas to cool and fragment (Phinney 1989; Morris 1993; Genzel et al. 2003; Levin and Beloborodov 2003).

There appears to be very few He-1 stars farther than the central parsec of the Galactic Centre, other than inside/near the young Arches (Nagata et al. 1995; Cotera et al. 1996; Figer et al. 1999) and Quintuplet (Okuda et al. 1990; Nagata et al. 1990; Glass et al. 1990; Figer et al. 1999) clusters at \(\gtrsim 30\) pc. This led Gerhard (2001) to postulate that efficient dynamical friction on star clusters forming a few parsecs from SgrA*, where GMCs could more easily cool and fragment, could bring a dense core of massive stars to the central parsec within the age of the He-1 population.

Another model suggests that in-situ formation of the clockwise disk is possible if a tidally disrupted GMC spirals in to form a small gaseous disk, which can be dense enough to become Jeans unstable and fragment into stars (Bonnell and Rice 2008; Alig et al. 2011; Mapelli et al. 2012; Alig et al. 2013). The infalling cloud needs to be \(\sim 10^5 M_\odot\) in order to reproduce observations (Mapelli et al. 2012). Two large gas clouds of mass \(\sim 5 \times 10^5 M_\odot\), M-0.02-0.07 and M-0.13-0.08, are seen projected at \(\sim 7\) and \(\sim 13\) pc from the Galactic Centre, respectively (Solomon et al. 1972). The top heavy mass function can be reproduced by the in-situ model so long as the gas has a temperature greater than 100 K, consistent with observations of the Galactic clouds. The rotation axis of the clockwise disk shows a strong transition from the inner to outer edge (Lu et al. 2009; Bartko et al. 2009; Lu et al. 2013), suggesting that the disk is either strongly warped, or is comprised of a series of stellar streamers with significant variation in their orbital planes (Bartko et al. 2009). In-situ formation is currently favoured for the clockwise disk, as an infalling cluster would likely form a disk with a constant rotation axis (Perets and Gualandris 2010). A caveat of in-situ formation is that it requires near radial orbits incident upon SgrA*, perhaps requiring cloud-cloud collisions (Wardle and Yusef-Zadeh 2008; Hobbs and Nayakshin 2009; Alig et al. 2011).

Interior to the disk lies a more enigmatic population of B-stars in a spatially isotropic distribution around SgrA*, with a distribution of eccentricities skewed slightly higher than a thermal distribution (Gillessen et al. 2009b; Mapelli and Gualandris 2016). These “S-stars” have semi-major axes less than 0.04 pc, with S0-102 having the shortest period of just 11.5 \(\pm\) 0.3 yrs, and a pericentre approach of just \(\sim 260\) AU (Meyer et al. 2012). The S-star population could potentially be older than the disk population, as the brightest star in this population, S0-2, is a main sequence B0-B2.5 V star with an age less than 15 Myr (Martins et al. 2008). The other stars in this population have spectra consistent with main sequence stars (Eisenhauer et al. 2005), and observational limits require them to be less than 20 Myr old in order to be visible.

The tidal forces in this region prohibit standard star formation, so the S-stars must have formed farther out and migrated inwards. A possible formation mechanism of the S-stars is from the tidal disruption of binaries scattered to low angular momentum orbits, producing an S-star and a hyper-velocity star via the Hills mechanism (Hills 1991). The captured stars would have initial eccentricities greater than 0.97 (Miller et al. 2005; Bromley et al. 2006), but the presence of a cusp of stellar mass black holes around SgrA*...
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could efficiently reduce the eccentricities of these orbits via resonant relaxation within the lifetime of the stars (Perets et al., 2009). Additionally, Antonini et al. (2010) show that if a binary is not tidally disrupted at first pericentre passage, the Kozai-Lidov (KL) resonance (Kozai, 1962; Lidov, 1962) can cause the binary to coalesce after a few orbital periods, producing an S-star and no hyper velocity star.

Alternatively, Chen and Amaro-Seoane (2014) show that stars from the clockwise disk can be brought very close to SgrA* via global KL like resonances, if the clockwise disk of gas originally extended down to $\sim 10^{-6}$ pc (the lowest stable circular orbit around SgrA*). The authors also show that O/WR stars would be tidally disrupted within the region of the observed S-star cluster due to their large stellar radii, whereas B-stars could survive, in agreement with observations. Recently, Subr and Haas (2016) showed that a clockwise disk with 100% primordial binarity can produce $\sim 20$ S-stars in less than 4 Myr. KL oscillations can efficiently drag binaries close to SgrA*, producing an S-star and a hyper-velocity star. This mechanism produces S-stars with eccentricities lower than from the disruption of binaries originating from outside the disk. However, in order to thermalize the S-stars, $\sim 500 M_\odot$ in dark remnants are still required around SgrA* in order to match observations, consistent with Fokker-Planck models (Hopman and Alexander, 2006). Three confirmed eclipsing binaries are observed within the clockwise disk, all being very massive O/WR binaries (Ott et al., 1999; Martins et al., 2006; Pfuhl et al., 2014). Pfuhl et al. (2014) estimate the present day binary fraction of the disk to be $0.3^{+0.24}_{-0.21}$ at 95% confidence, with a fraction greater than 0.85 ruled out at 99% confidence. More recently, Gautam et al. (2017) predict that the binary fraction must be greater than 32% at 90% confidence.

An additional popular scenario is the transport of stars from young dense star clusters that migrate to the Galactic Centre via dynamical friction, with the aid of an intermediate mass black hole (IMBH). Kim and Morris (2003) showed that to survive to the central parsec from a distance $\geq 10$ pc, clusters either need to be very massive ($\sim 10^6 M_\odot$) or very dense (central density, $\rho_c \sim 10^8 M_\odot pc^{-3}$). Kim et al. (2004) showed that including an IMBH in the cluster means the the core density can be lowered, but only if the IMBH contains $\sim 10\%$ of the mass of the entire cluster, far greater than is expected from runaway collisions (Portegies Zwart and McMillan, 2002).

Fujii et al. (2009) (hereafter F09) revisited this problem using the tree-direct hybrid code, BRIDGE (Fujii et al., 2007), allowing the internal dynamics of the star clusters to be resolved. The small tidal limits imposed by SgrA* meant the clusters had core densities greater than $10^7 M_\odot pc^{-3}$, leading to runaway collisions on a mass segregation timescale (Portegies Zwart and McMillan, 2002; Portegies Zwart et al., 2004). During collisions, the resulting very massive star (VMS) was rejuvenated using the formalism of Meurs and van den Heuvel (1989), and collapsed to an IMBH at the end of its main sequence lifetime, extrapolated from the results of Belkus et al. (2007). The authors found that by allowing the formation of a $3 - 16 \times 10^4 M_\odot$ IMBH (see also Fujii et al., 2010), some stars could be carried very close to SgrA* via a 1:1 mean resonance with the infalling IMBH. The orbits of these “Trojan stars” were randomised by 3-body interactions with the SMBH and IMBH, constructing a spatially isotropic S-star cluster. F09’s simulation “LD64k” transported 23 stars to the central 0.1 pc, however, the resolution of the simulation is $\sim 0.2$ pc, set by the force softening of SgrA*. The simulation also brought 354 stars within 0.5 pc of SgrA*, 16 being more massive than 20 $M_\odot$, analogous to clockwise
disk stars. The IMBH formed in LD64k is more massive than the observational upper limit of \( \sim 10^4 \, M_\odot \), derived from VLBA measurements of SgrA* (Reid and Brunthaler, 2004). However, Fujii et al (2010) state that an IMBH of 1500 \( M_\odot \) is sufficient for the randomisation of stars (see also Merritt et al. 2009).

Despite the successes of the F09 model, IMBH formation in young dense star clusters may be prohibited. VMSs of the order \( 10^3 \, M_\odot \) are expected to have luminosities greater than \( 10^7 \, L_\odot \) (Kudritzki, 2002; Nadyozhin and Razinkova, 2005; Belkus et al., 2007), driving strong stellar winds. F09 assumed the mass loss rate of stars more massive than \( 300 \, M_\odot \) to be linear with mass, however, recent work on VMS winds show steeper relations for stars that approach the Eddington limit (Kudritzki, 2002; Vink, 2006; Vink et al., 2011). F09’s model also neglected the effect of the evolving chemical composition on the luminosity, and hence the mass loss, of the VMS (Nadyozhin and Razinkova, 2005). I note that the initial mass function (IMF) used in F09, although employed due to numerical constraints, meant there were ten times more massive stars than expected from a full Kroupa IMF, leading to an increased collision rate and buildup of the VMS mass.

No conclusive evidence for the existence of IMBHs in star clusters has yet been found (See Lützgendorf et al., 2013, 2016 for a comprehensive review on IMBHs in globular clusters). Sufficiently high mass loss could cause VMSs to end their lives as stellar mass black holes or pair-instability supernovae at low metallicity (Heger and Woosley, 2002). Pair-instability supernovae candidates have recently been found at metallicities as high as \( \sim 0.1 \, Z_\odot \) (Gal-Yam et al., 2009; Cooke et al., 2012), with expected progenitors of several hundred solar masses (Chen et al., 2015a).

The most massive star observed, R136a1, is a 265\( ^{+80}_{-35} \, M_\odot \) star in the 30 Doradus region of the Large Magellanic Cloud (LMC) (Crowther et al., 2010, 2016), with metallicity \( Z = 0.43 \, Z_\odot \). Crowther et al. (2010) suggest that it could be a very rare main sequence star, with a zero age main sequence mass of 320\( ^{+100}_{-40} \, M_\odot \). However, it could be the collision product of a few massive stars. R136a1 has a large inferred mass loss rate of \( (5.1^{+0.9}_{-0.8}) \times 10^{-5} \, M_\odot \, yr^{-1} \), \( \sim 0.1 \) dex larger than the theoretical predictions of Vink et al. (2001). Belkus et al. (2007) predict that the evolution of all stars more massive than \( 300 \, M_\odot \) is dominated by stellar winds, with similar lifetimes of \( \sim 2 − 3 \, Myr \). As such, it is not surprising that R136a1 is the most massive star currently observed, as more massive VMSs should be rare and short lived.

Whilst it may be unlikely for an IMBH to form at solar metallicity, a VMS could transport stars to SgrA* within its lifetime. In this chapter I test the feasibility of the star cluster migration scenario as the origin of either young population in the Galactic Centre.

I evolve direct N-body models of star clusters in the Galactic Centre, using the GPU-accelerated code NBODY6df, a modified version of Sverre Aarseth’s NBODY6 (Aarseth 1999; Nitadori and Aarseth, 2012) which includes the effects of dynamical friction semi-analytically (Petts et al., 2015, 2016). In section \( \S 4.2 \) I describe the theory behind my dynamical friction and stellar evolution models. In section \( \S 4.3 \) I describe the numerical implementation. Section \( \S 4.4 \) discusses prior constraints on the initial conditions and describes the parameters of the simulations performed. In sections \( \S 4.5 \) and \( \S 4.6 \) I present my results and discuss their implications for the origin of the young populations. Finally, I present my conclusions in section \( \S 4.7 \).
4.2 Theory

4.2.1 Dynamical friction

In this chapter I use the Petts et al. (2015) dynamical friction model, with the assumption of a Maxwellian velocity distribution. The distribution function for a Dehnen model with a black hole is only analytic in the approximation that the black hole dominates the dynamics, which is only valid at $\lesssim 2\,\text{pc}$ for my model (see section §4.4). As the model I use is very cuspy, including the black hole’s influence on the velocity dispersion of the Maxwellian approximation (see A.1) gives a very good representation of the dynamical friction force. A comparison to GADGET-2 $N$-body simulations is given in Appendix §A.3. I do not use the Petts et al. (2016) model in this chapter, as the Petts et al. (2015) model is sufficient and avoids the need for a double integration to obtain $f(v^*)$.

4.2.2 Evolution of very massive stars

Nadyozhin and Razinkova (2005) present similarity theory models of VMSs, for which the stellar properties can be calculated by solving a set of differential equations (Imshen- nik and Nadezhin 1968). VMSs are predicted to have large convective cores containing more than 85% of the mass, surrounded by a thin extensive radiative envelope. In such stars the opacity becomes larger than the electron scattering value, and can be considered to come from Thomson scattering alone. Utilising such approximations, the authors provide simple formulae to calculate the core mass and luminosity, as functions of stellar mass and chemical composition.

The luminosity of stars with $\mu^2 M \geq 100$ can be found by substituting eq. 36 of Nadyozhin and Razinkova (2005) into their eq. 34:

$$L \approx \frac{64826M \left(1 - 4.5/\sqrt{\mu^2M}\right)}{1 + X},$$

(4.2.1)

where $L$ is the luminosity, $M$ is the mass of the VMS, $X$ is the core hydrogen abundance, and $\mu$ is the mean atomic mass of the core. Assuming a fully ionised plasma, $\mu$ takes the form:

$$\mu = \frac{4}{6X + Y + 2},$$

(4.2.2)

where $Y$ is the core helium abundance. Eq. 4.2.1 shows that at very large masses $L \propto M$. However, unlike the F09 model, this formulation of the luminosity explicitly includes an $L \propto (1 + X)^{-1}$ dependence. As the mass loss rate depends on $L$, this leads to an increased mass loss rate in the late stages of main sequence evolution (see section §4.2.2.a).

Belkus et al. (2007) (hereafter B07) modelled the evolution of VMSs with zero age main sequence (ZAMS) masses of up to $1000\,M_\odot$, assumed to have formed via runaway collisions in a young dense star cluster. The authors numerically evolve the chemical composition of the star through the Core Hydrogen Burning (CHB) and Core Helium Burning (CHeB) phases via conservation of energy and mass loss from the stellar wind. In this section I briefly outline the model of Belkus et al. (2007) and describe how I include stellar collisions and their effect on VMS evolution.
As VMSs have large convective cores, one can reasonably approximate them as homogeneous (verified to be a good approximation down to $120 \, M_\odot$, B07). Applying conservation of energy, the hydrogen fraction in the core during CHB evolves as eq. 1 of B07:

$$M_{cc}(\mu, M) \frac{dX}{dt} = -\frac{L(\mu, M)}{\epsilon_H},$$

(4.2.3)

where $M_{cc}$ is the mass of the convective core and $\epsilon_H$ is the hydrogen burning efficiency (i.e. the energy released by fusing one mass unit of hydrogen to helium).

When the core is depleted of hydrogen, the VMS burns helium via eq. 4 of B07 (see also Langer (1989a)):

$$M_{cc}(\mu, M) \frac{dY}{dt} = -\frac{L(\mu, M)}{\epsilon_{\text{ratio}}},$$

(4.2.4a)

$$\epsilon_{\text{ratio}} = \left[ \left( \frac{B_Y}{A_Y} - \frac{B_O}{A_O} \right) + \left( \frac{B_C}{A_C} - \frac{B_O}{A_O} \right) C'(Y) \right],$$

(4.2.4b)

where $\epsilon_{\text{ratio}}$ accounts for the fact that C and O are produced in a non-constant ratio, affecting the energy production per unit mass of helium burnt. Here, $A$ and $B$ are the atomic weights and binding energy of nuclei; with subscripts $Y$, $C$ and $O$ representing helium, carbon and oxygen respectively. $C'(Y)$ is the derivative of the $C(Y)$ fit from Langer (1989b) with respect to $Y$ (see B07 for the derivation of eq. 4.2.4b). During CHeB, $\mu$ is defined as (Nadyozhin and Razinkova, 2005):

$$\mu = \frac{48}{36Y + 28C + 27O},$$

(4.2.5a)

which by assuming $Y + C + O = 1$ and using the fit to $C(Y)$ by Langer (1989b), can be rewritten solely as a function of $Y$ as:

$$\mu = \frac{48}{19Y + C(Y) + 27}.$$

(4.2.5b)

Subsequent stages of evolution are rapid and explosive. I assume that after core helium burning the remnant collapses to a black hole with no significant mass loss (i.e. the optimistic upper limit).

### 4.2.2.a Mass loss

The chemical evolution of the VMS is coupled to the mass evolution, as the luminosity of the star sets the wind strength. Vink et al. (2011) (hereafter V11) show that the wind strength is heavily dependent on the proximity to the Eddington limit, when gravity is completely counterbalanced by the radiative forces, i.e. $\frac{g_{\text{rad}}}{g_{\text{grav}}} = 1$, where $g_{\text{rad}}$ and $g_{\text{grav}}$ are the radiative and gravitational forces, respectively. For a fully ionised plasma, the Eddington parameter, $\Gamma_e$, is dominated by free electrons and is approximately constant throughout the star (V11):

$$\Gamma_e = \frac{g_{\text{rad}}}{g_{\text{grav}}} = 10^{-4.813} \left( 1 + X_s \right) \left( \frac{L}{L_\odot} \right) \left( \frac{M}{M_\odot} \right)^{-1},$$

(4.2.6)
where $X_s$ is the surface hydrogen abundance of the star. V11’s fig. 2 shows that the logarithmic difference between the empirical Vink et al. (2001) (hereafter V01) rates and the VMS rates follow a tight relation with $\Gamma_e$, almost independent of mass. The authors find that the mass loss rate is proportional to:

$$
\dot{M} \propto \begin{cases} 
\Gamma_e^{2.2}, & \text{if } 0.4 < \Gamma_e < 0.7 \\
\Gamma_e^{4.77}, & \text{if } 0.7 < \Gamma_e < 0.95.
\end{cases}
$$

(4.2.7)

During the CHB phase, I model the stellar wind of the VMS using the formulae from V01, whilst correcting for the proximity to the Eddington limit by fitting on the data from table 1 of V11. In this way I obtain a coefficient that allows us to convert the V01 rate to the $\Gamma_e$ enhanced rates of stars approaching the Eddington limit (similarly to Chen et al., 2015b). V11 modelled stars up to $300M_\odot$, however, as the logarithmic difference between the V11 and V01 rates shows little dependence on mass, I extrapolate this approach to higher masses. V11 state that their predicted wind velocities are a factor 2–4 less than derived empirically. The effect of rotation is also neglected. It should be noted that due to these two effects, and my extrapolation of the V11 models, I most likely underestimate the mass loss of VMSs. Therefore the masses of my VMSs and their resulting remnants should be taken as a conservative upper limit at solar metallicty.

During CHeB, VMSs are depleted of hydrogen and are expected to show Wolf-Rayet like features. I follow the approach of Belkus et al. (2007) and extrapolate the mass loss formula of Nugis and Lamers (2000):

$$
\log(\dot{M}) = -11 + 1.29 \log(L) + 1.7 \log(Y) + 0.5 \log(Z).
$$

(4.2.8)

B07 explored models with Wolf-Rayet like mass loss rates (arbitrarily) up to 4 times weaker, which only left a remnant twice as massive. The uncertainty arising from extrapolation of this formula should be of little significance to the transport of young stars to the central parsec, as post main sequence VMSs are not massive enough to experience substantial dynamical friction after the cluster is disrupted (B07). However, if sufficiently chemically rejuvenated, a CHB VMS may be capable of bringing stars to the central parsec before losing most of its mass. Thus the evolution during the CHB stage is of most interest.

I make sure that in both burning phases the predicted mass loss never exceeds the photon tiring limit, the maximum mass loss rate that can theoretically be achieved using 100% of the stars luminosity to drive the wind (Owocki et al., 2004):

$$
\dot{M}_{\text{tir}} = 0.032 \left( \frac{L}{10^6 L_\odot} \right) \left( \frac{R}{R_\odot} \right) \left( \frac{\dot{M}}{\dot{M}_\odot} \right)^{-1}
$$

(4.2.9)

Here, the radii, $R$, of stars are taken from the mass-radius relation of Yungelson et al. (2008), which is in excellent agreement with Nadyozhin and Razinkova (2005)’s similarity theory models of VMSs, but requires less computational resources to calculate. The OB disk population is less than 7 Myr old. Hence, I assume approximately solar abundances such that $X_0 = 0.7$, $Y_0 = 0.28$ and $Z_0 = 0.2$ (Pols et al., 1998). I note that although Eq. 4.2.9 is enforced, $\Gamma_e < 1$ in all models considered, and the photon tiring limit was never reached in any simulation.
4.3. NUMERICAL METHOD

4.2.2.b Rejuvenation following collisions

Nadyozhin and Razinkova (2005) show that VMSs have nearly all of their mass in their large convective cores. Repeated collisions can efficiently mix the core and the halo, keeping the star relatively homogeneous. Additionally, the stellar wind also ensures homogeneity, as the extended loosely bound envelope can be rapidly removed, leaving the surface with composition similar to the core.

I chemically rejuvenate a VMS following a collision with another star. I assume that stars colliding with the VMS efficiently mix with the convective core such that:

\[
X_{\text{new}} = \frac{X_{\text{star}} M_{\text{star}} + X_{\text{VMS}} M_{\text{VMS}}}{M_{\text{VMS+star}}}
\]  

(4.2.10)

Similarly for Y and Z. I approximate \(X_{\text{star}}(t)\) and \(Y_{\text{star}}(t)\) for main sequence stars by interpolating the detailed stellar models of Schaller et al. (1992), so that I do not overestimate the rejuvenation of the VMS if the infalling star is already evolved. If a CHeB VMS collides with a hydrogen rich main sequence star, I assume that CHB is reignited. If two VMSs collide their composition is also assumed to be well mixed.

4.3 Numerical Method

To model the effects of dynamical friction on self-consistent star cluster models I use the GPU-parallel direct N-body code NBODY6df (Petts et al., 2015), which is a modified version of Aarseth’s direct N-body code NBODY6 (Aarseth, 1999; Nitadori and Aarseth, 2012). In this chapter I model the background as an analytic stellar distribution with a central black hole (see section §4.4).

I introduced an additional modification to the code to model the evolution of a VMS, as described in section §4.2.2. When a physical collision creates a star greater than 100 M\(_\odot\) I flag it as a VMS and treat its evolution separately from the standard SSE package in NBODY6 (Hurley et al., 2000) via the method described in section §4.2.2. As the mass loss can be very large for VMSs, fine time resolution is needed to prevent overestimation of the mass loss. I introduce a new routine which integrates the mass and composition of the star between the N-body dynamical time steps using a time step of 0.1 years, sufficiently accurate to resolve the chemical and mass evolution. V11 predict terminal wind speeds of a few thousand \(\text{km s}^{-1}\) for VMSs, as such I assume that the stellar wind escapes the cluster and simply remove this mass from the VMS. An arbitrary number of VMSs can potentially form and evolve simultaneously, and coalesce in the simulation.

4.4 Initial Conditions

In NBODY6df the background potential is assumed static and analytic; an assumption valid over the short timescales considered here (less than 7 Myr). I adopt a Dehnen model (Dehnen, 1993), representing the central region of the Galaxy. I use a slope \(\gamma = 1.5\), scale radius \(a = 8.625\) pc and total mass \(M_g = 5.9 \times 10^7 M_\odot\), which closely reproduces the observed broken power-law profile obtained by Genzel et al. (2003) for the central region of the Galaxy, yet has simple analytic properties. I place a central fixed point mass of \(4.3 \times 10^6 M_\odot\) to represent SgrA* (Gillessen et al., 2009b).
4.4. INITIAL CONDITIONS

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<th>$W_0$</th>
<th>$N$</th>
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Table 4.1: Initial conditions of the isolated simulations. Column 1 lists the name of the simulation. The naming convention is described in section §4.4. Columns 2,3 show the mass and half mass radius of the cluster. Column 4 shows the dimensionless central potential of the King model. Column 5 shows the number of particles, and column 6 gives the lower mass limit of the IMF. The upper mass limit is $100 M_\odot$ for all models.

4.4.1 Physical and numerical constraints on the initial conditions

There are two constraints on the initial conditions of the clusters. Firstly, they must reach the Galactic Centre within the age of the young populations. I therefore wish to model clusters that can potentially reach the Galactic Centre in less than 7 Myr, so that I may test the migration model for both the clockwise disk and the S-stars. I first obtain tight constraints on the initial orbital parameters by integrating the orbits of point masses in the Galactic Centre potential including dynamical friction. Fig. 4.1 shows contours of equal inspiral time for different initial masses, apocentres and initial velocities. Initial conditions to the right of each line are such that the clusters can reach within 0.5 pc in less than 7 Myr. Arches like clusters (initial mass $4-6 \times 10^4 M_\odot$) could reach the Galactic centre in less than 7 Myr if they formed at $\sim 5$ pc, or from $7-10$ pc if large initial eccentricities were assumed. More massive clusters can easily migrate $\sim 10$ pc in 7 Myr. I note that these inspiral times are lower limits, as real clusters would lose mass from stellar winds and tides. I choose to model only those clusters for which a point mass object of the same mass can reach the Galactic Centre within $\sim 7$ Myr.

Secondly, the size of the clusters is limited by their small tidal limits when so close to SgrA*. Approximating the cluster as a point mass, the tidal radius is given by (Binney and Tremaine 2008):

$$r^3_t = \frac{GM_{cl}}{\Omega_p^2 + \left(\frac{d^2 \Phi}{dt^2}\right)_p}, \tag{4.4.1}$$

where $\Omega_p$ and $\left(\frac{d^2 \Phi}{dt^2}\right)_p$ are the angular velocity of a circular orbit and the second derivative of the potential at pericentre, respectively. The high mass requirement for fast inspiral, coupled with the small tidal limits, means that all models are inherently very dense and runaway mergers are expected. Although it is unknown whether such dense clusters are likely to form in the Galactic Centre, I explore these initial conditions in order to test the feasibility of the inspiral model.

4.4.2 Initial Mass function

I sample stars from a Kroupa initial mass function (IMF) with an upper limit of $100 M_\odot$ (Kroupa 2001). A lower mass limit of $0.08 M_\odot$ would yield the most physically realistic results, but at a computational cost unfeasible for a parameter study of such massive clusters at the current time (365k-730k particles for the most massive models explored).
4.4. INITIAL CONDITIONS

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<td>5</td>
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Table 4.2: Initial conditions of the simulations. Column 1 lists the name of the simulation. The naming convention is described in section §4.4. Columns 2 and 3 give the mass and half mass radius of the cluster. Column 4 gives the dimensionless central potential of the King model. Column 5 shows the average core density. Column 6 shows the number of particles, and column 7 the lower mass limit of the IMF. The upper mass limit is 100 $M_\odot$ for all models. Column 8 shows the initial binary fraction. Columns 9 and 10 show the initial position and velocity, in units of pc and the circular velocity, respectively. Models are set up to be Roche-filling at first pericentre passage; unless they are marked by an asterisk, in which case they are Roche-filling at their initial positions.
4.4. INITIAL CONDITIONS

Figure 4.1: Contours of $T_{df} = 7$ Myr as a function of cluster mass, initial distance, $R_{a,0}$, and initial velocity, $v_0$, given in units of the local circular velocity, $v_c$. Models to the right of each line approach within 0.5 pc of SgrA* in less than 7 Myr. The half mass radius of the cluster is assumed to be 0.1 pc.
Figure 4.2: Time evolution of the VMS mass formed in simulations with different lower mass cutoffs in the IMF. The solid blue, dashed red and dot-dashed green lines show the mass of the VMS in simulations $1lo$, $1hi$ and $1kr$, respectively. The diagonal crosses show the end of CHB, the vertical crosses show re-ignition of CHB, and the solid circles show where the remnants collapse to black holes.
However, truncating the low end of the IMF means that one samples too many massive stars as compared with a full Kroupa IMF. To quantify the difference this has on VMS formation, I ran three test simulations at different mass resolutions, in the absence of a tidal field. Simulations 1lo, 1hi and 1kr have lower cutoffs of 1.0, 0.16 and 0.08 M⊙, respectively. I model star clusters as King models with dimensionless central potential, $W_0 = 6$, and with no primordial mass segregation. The parameters of the isolated simulations are displayed in Table 4.1. Fig. 4.2 shows the VMS mass as a function of time for simulations 1lo, 1hi and 1kr, showing that better sampling of the low end of the IMF inhibits the growth of the VMS. This occurs because primarily high mass stars build up the VMS, due to their short dynamical friction timescales and large cross sections for collision. In simulation 1kr, although half the cluster mass is comprised of stars less massive than 0.58 M⊙, only 37 stars less massive than 0.58 M⊙ are consumed throughout the entire lifetime of the VMS. The VMS initially grows very rapidly. However, the late main sequence evolution is dominated by the strong stellar wind of the helium rich VMS. Throughout its lifetime, the VMS in simulation 1kr removes 2244 M⊙ of material from the cluster through its stellar wind, $\sim 2\%$ of the cluster mass. During CHeB, simulations 1lo and 1hi reignite CHB via collision with a massive main sequence star, resulting in a lower remnant mass at collapse. The late evolution is very stochastic, however this is not important for the migration of young stars to the Galactic Centre, as the VMS only provides gravitational binding energy comparable to normal cluster stars during its CHeB phase.

Fig. 4.2 shows that a lower limit of 0.16 M⊙ is sufficient to resolve the mass evolution of the VMS, and as I am only interested in the final distribution of OB stars, this IMF is sufficient for my simulations. I cannot evolve the most massive clusters at high mass resolution, as these models become too computationally expensive. As a compromise, I test a large range of initial conditions with a lower limit of 1 M⊙, and re-run a selection of initial conditions with a lower limit of 0.16 M⊙ to obtain more realistic results. I can simultaneously use the low resolution simulations to explore the possibility of an initially top heavy mass function for clusters forming close to SgrA*. A very top heavy function is observed for the clockwise disk (Lu et al., 2013), however, it is unknown whether a top heavy IMF is expected from the collapse of GMCs at $\sim 5 - 10$ pc from SgrA*.

4.4.3 Binary fraction

Some simulations include a population of primordial binaries. Binaries are initialised as follows. Firstly all stars more massive than 5 M⊙ are ordered by mass. The most massive star is then paired with the second most massive star, and so on. This choice is motivated by observational data showing that massive OB stars are more likely to form in binary systems with mass-ratios of order unity (Kobulnicky and Fryer [2007], Sana and Evans [2011]). Once all stars more massive than 5 M⊙ are in binaries, lower mass stars are paired at random until the specified binary fraction (the fraction of stars initially in a binary system) is reached (Kroupa [2008]). For stars more massive than 5 M⊙, the periods and eccentricities are drawn from the empirical distributions derived in Sana and Evans (2011), which show that short periods and low eccentricities are preferred in massive binaries. For lower mass stars the periods are drawn from the Kroupa (1995) period distribution and are assigned thermal eccentricities. The mass of a binary and its initial
4.5. RESULTS

position in the cluster are assumed to be independent.

4.4.4 Simulations

The initial conditions are described in Table 4.2 and are referred to by the following naming convention: \(< M > < M_f > < R_a >\), where \(< M >\) is the cluster mass in units of \(\sim 10^5 M_\odot\), \(< M_f >\) is the mass resolution of the simulation, and \(< R_a >\) is the initial galactocentric distance in pc. For most simulations I sample from a Kroupa IMF with an upper mass cut off, \(m_{up} = 100 M_\odot\). The \("lo"\) resolution models have a lower mass cut off, \(m_{low} = 1 M_\odot\) and mean mass, \(\langle m_\ast \rangle = 3.26 M_\odot\). The \("hi"\) resolution models have \(m_{low} = 0.16 M_\odot\) and \(\langle m_\ast \rangle = 0.81 M_\odot\). For simulations with \(< M_f > = lu\) I use an IMF identical to the mass function of the clockwise disk (Lu et al., 2013). The simulation name is followed by a suffix describing additional information about the simulation. The suffix \(W_4\) denotes that the dimensionless central potential, \(W_0\), is initially 4 instead of 6. The suffix \(vX\) indicates an eccentric orbit with initial velocity, \(v_c(\nu_c\text{v}_c)\) (where \(v_c\) is the circular velocity at the initial position). The Suffix \("ms"\) indicates that the cluster is primordially mass segregated. Finally, the suffix \("b"\) denotes the inclusion of primordial binaries (see section §4.4.3). Most models are Roche-filling at first pericentre passage, apart from runs marked with an asterisk, which are Roche-filling at their initial positions. The model with the suffix \("d"\) is extremely Roche under-filling at its initial position.

4.5 Results

In all models, the clusters are completely tidally disrupted in less than 7 Myr. Massive clusters migrate farther in than lower mass clusters on the same initial orbits, due to shorter dynamical friction timescales and less efficient tidal stripping. However any cluster that reaches \(\sim 3\) pc is rapidly dissolved by its shrinking tidal limit as it approaches SgrA*. Clusters on eccentric orbits inspiral faster, as they pass through denser regions of the cusp periodically. However, clusters on very eccentric orbits (e.g. \(21\alpha 10\nu 2^+\), \(e \sim 0.9\)) disrupt on the first few pericentre passages, depositing stars at large distances along the initial cluster trajectory.

Most simulations naturally form a VMS in less than 1 Myr due to their high initial densities. However, the initial rapid mass accretion soon loses to the increasing mass loss rate and relaxation of the cluster, causing the VMS to collapse to a black hole of \(\sim 20 – 250 M_\odot\) after \(2 – 5\) Myr \((300 – 400 M_\odot\) for models with a Lu et al. (2013) IMF), typically before their parent clusters completely disrupt. Table 4.3 shows the maximum mass, remnant mass and lifetime of the VMS formed in each simulation. The clusters completely unbind at \(\sim 2 – 3\) pc, and the IMBHs formed are not massive enough to experience significant dynamical friction and drag stars close to SgrA* (dynamical friction timescales for even a \(400 M_\odot\) IMBH are longer than 100 Myr). Conversely, the evolution of the VMS does not appear to significantly inhibit the inspiral of the cluster, as only \(\lesssim 2\%\) of the initial cluster mass is typically expelled by the VMS throughout its lifetime.

For each model, Table 4.4 shows the final distribution of stars after complete cluster dissolution and death of any VMSs. I show the distributions of semi-major axes for all stars and main sequence stars more massive than \(8 M_\odot\) at 7 Myr, as well as how many of these stars have final semi-major axes smaller than 1 pc. I use a \(8 M_\odot\) cut-off as these
4.5. RESULTS

<table>
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<th>Run Name</th>
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<th>$M_{\text{VMS, rem}}$</th>
<th>$t_{\text{VMS}}$</th>
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<td>-</td>
<td>-</td>
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Table 4.3: Properties of VMSs formed in the simulations. Column 1 lists the name of the simulation. Column 2 shows the maximum mass via stellar collisions, column 3 shows the resulting remnant mass after CHeB, and column 4 shows the epoch in the cluster evolution when the VMS collapses.
4.5. RESULTS

<table>
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<th>$\langle a \rangle_{\text{all}}$ (pc)</th>
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<th>$N(&lt; 1 \text{ pc})$</th>
<th>$N(&lt; 1 \text{ pc}, &gt;8 , M_\odot)$</th>
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<th>$\langle D_{2D} \rangle_{15 , \text{Myr}}$ (pc)</th>
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</table>

Table 4.4: Final distributions of stars originally from the cluster. Simulations are run until the cluster is completely unbound and any VMSs collapse, up to a maximum of 7 Myr. Column 1 shows the name of each simulation. Column 2 shows the mean semi-major axis of all stars remaining in the simulation and the standard deviation. Column 3 shows the same for stars more massive than $8 \, M_\odot$ that are still on the main sequence at 7 Myr. Column 4 shows the total number of stars with final semi-major axes less than 1 pc, and column 5 shows only those which are still on the main sequence and more massive than $8 \, M_\odot$ at 7 Myr. Columns 6 and 7 show the distributions of projected distances from SgrA* for main sequence stars visible at 7 and 15 Myr, respectively. The dissolved clusters are projected so that the resulting disk of stars rotates clockwise in the sky.
are the faintest main sequence stars spectroscopically observable in the Galactic Centre with current telescopes, $K \geq 15.5$ (Do et al. 2009, 2013; Lu et al. 2013; Feldmeier-Krause et al. 2015). Although photometric studies can see objects down to magnitudes of $K < 19 - 18 \sim 2 \, M_\odot$ main sequence stars, (Genzel et al. (2003)), it is impossible to determine whether these stars are young or old. I also show the projected distributions of visible main sequence stars at 7 Myr and 15 Myr (as the S-star population may be older than the disk population, see section §4.1).

4.5.1 Low Resolution Models

Fig. 4.3 shows the final distributions of the semi-major axes and projected positions of stars for a representative selection of the models with a lower mass cutoff of 1 $M_\odot$ ($< m_f > = 10$). In all models the final distributions are broad, with a standard deviation of $\sim 2$ pc. Other simulations show similar distributions, with less massive and less eccentric models dissolving farther out (see Table 4.4).

Models with very eccentric orbits (e.g. 21o10_v2*) can bring stars close to SgrA*, however, very few stars have final semi-major axes smaller than 1 pc. No stars more massive than 8 $M_\odot$ are scattered to semi-major axes smaller than 1 pc in either 11o10_v2* or 21o10_v2*. This is likely due to the preferential loss of low mass stars, whereas high mass stars remain inside the cluster for longer, and end up tracing the final cluster orbit.

Simulation 21o5 is the only non-radial model to bring stars to the central parsec, and the only model that brings a significant number of massive stars. However, one would expect to also see $\sim 3000$ massive stars in the range 1 – 10 pc, about 10 times more than reach the central parsec. The right side of Fig. 4.3 shows the distributions of projected distances of stars that are spectroscopically visible at 7 and 15 Myr. The amplitudes of the distributions are normalised to the expected number of stars had the simulation been run with a Kroupa IMF. The stars are projected to rotate clockwise in the sky. It can be seen that for all simulations, more than 1000 young stars are observed out to $\sim 10$ pc. Considering current observational limitations, if a cluster were present in the central $\sim 10$ pc within the last $\sim 15$ Myr, a large number of stars would be observable up to $\sim 10$ pc, suggesting it is unlikely that any clusters have inhabited this region in the last $\sim 15$ Myr.

Simulations 21o10 and 21o10_W4 have the same initial orbit and mass, yet 21o10_W4 is less concentrated. The lower central density and longer relaxation timescale cause 21o10_W4 to form a less massive VMS than 21o10. However, the VMS in 21o10_W4 lives longer as not all the most massive stars are consumed within $\sim 1$ Myr. The models end up with similar final distributions of the resulting disk (see Table 4.4). The same trend is seen for the less massive analogues, 11o10 and 11o10_W4. The two most massive simulations 41o15_W4 and 41o15_W4v75, are massive enough to reach the central parsec from 15 pc in $\sim 7$ Myr, but with central densities low enough to suppress the formation of VMSs. However, these simulations are more susceptible to tides, and are tidally disrupted at large radii (essentially reproducing the results of Kim and Morris, 2003).
Figure 4.3: Left: Final distributions of semi-major axes of stars in simulations 2lo10, 2lo10_v2* and 2lo5 at $T = 7$ Myr. The solid green histogram shows all the stars. The dashed blue histogram shows main sequence stars more massive than $8 \, M_\odot$ at $T = 7$ Myr. Right: Final distributions of the projected distances of stars from SgrA* in the same simulations. The dashed blue and dot-dashed red histograms show the distributions of main sequence stars more massive than $8 \, M_\odot$ at $T = 7$ Myr and $T = 15$ Myr, respectively, projected to rotate clockwise on the sky. The $y$-values of the projected distributions are re-normalised to the expected number of stars had the model been simulated with a full Kroupa IMF.
4.5. RESULTS

4.5.2 Higher resolution models

Fig. 4.4 shows a comparison between simulations 1lo10, v5 and 1hi10, v5, which have the same initial conditions, except 1hi10, v5 better samples the low mass end of the IMF. The panels on the left show the distributions of semi-major axes for all the stars and main sequence stars more massive than 8 M\(_\odot\) at 7 Myr. The distributions are very similar, however 1hi10, v5 has a smaller ratio of spectroscopically visible stars to all stars due to differences in the IMF. The panels on the right show the projected distributions of main sequence stars visible at 7 and 15 Myr. For the projected distributions, the number of stars is re-normalised to the expected number of stars had the simulation been run with a Kroupa IMF from 0.08 – 100 M\(_\odot\). Although massive stars are consumed to construct the VMS, this is a small fraction of the population. The distributions look very similar in shape and magnitude, indicating that models run with a lower limit of 1 M\(_\odot\) produce similar final distributions to simulations that better sample the IMF. This verifies the validity of the normalisation approach used on the projected visible distributions in Fig. 4.3.

Fig. 4.5 demonstrates how simulations 1hi5, 1hi10, 1hi10, v5 and 1hi10, v2 evolve with time. The top two panels show the evolution of the Galactocentric distance of the cluster and the mass enclosed within the tidal radius. The bottom two panels show the evolution of the VMS mass and the half mass radius of the cluster. Simulations 1hi5, 1hi10 and 1hi10, v5 quickly form a VMS and expand due to rapid two body relaxation in the dense core. The expansion lowers the core density and thus the collision rate. The reduced collision rate allows the VMSs to rapidly burn their fuel and collapse without significant hydrogen rejuvenation. Simulation 1hi5 forms a more massive VMS than the other simulations as it is initially ∼10 times as dense, however the resulting increased luminosity decreases its lifetime. In simulation 1hi10, v2, the cluster disrupts before the massive stars can reach the centre of the cluster, however the initial density is high enough that a 235 M\(_\odot\) VMS forms by the first pericentre passage. The self-limiting nature of the VMS formation is discussed in section 4.6.

4.5.3 Models with extreme initial conditions

The young clockwise disk population exhibits a top heavy mass function, with power law index \(\alpha \sim 1.7\) (Lu et al., 2013). In the context of the cluster inspiral scenario this has been explained by mass segregation inside the cluster, with the most massive stars reaching the central parsec, and low mass stars being preferentially lost due to tides during inspiral. However, as I have shown in section 4.5.1 clusters lose massive stars as well as low mass stars throughout inspiral, via dynamical ejections and the shrinking tidal limits as the clusters approach SgrA*. In order to test the effect of mass segregation, I ran simulation 1hi5, ms, which I primordially mass segregated using the method described in Baumgardt et al. (2008) (see also Küpper et al., 2011). For simulations 1hi5 and 1hi5, ms, Fig. 4.6 shows the semi-major axes of all stars and main sequence stars more massive than 8 M\(_\odot\) at \(T = 7\) Myr, as well as the distributions of projected distances of spectroscopically visible stars at 7 and 15 Myr. Their distributions look similar, as simulation 1hi5 has an initial mass segregation timescale of \(t_{df} \sim 0.1\) Myr for the most massive stars, causing the cluster to rapidly mass segregate. As such, primordial mass segregation does not significantly enhance the transport of massive stars to the central parsec, as clusters of
Figure 4.4: Comparison between simulations 1lo10_v5* and 1hi10_v5*, which have almost identical initial conditions, the latter sampling better the low end of the IMF. The left panels show the distribution of semi-major axes of all stars (solid green line) and stars more massive than $8 M_\odot$ at $T = 7 \text{ Myr}$ (dashed blue line). The right panels show the distributions of projected distances of main sequence stars more massive than $8 M_\odot$ at $T = 7 \text{ Myr}$ (dashed blue) and $T = 15 \text{ Myr}$ (dot-dashed red). The stars are projected so that the disk rotates clockwise in the sky. The y-values of the projected distributions are re-normalised to the expected number of stars had the model been simulated with a full Kroupa IMF.
Figure 4.5: The evolution of galactocentric distance, cluster mass, VMS mass and cluster half mass radius as a function of time, for simulations 1hi5 (solid blue lines), 1hi10 (dashed red lines), 1hi10_v5* (dot-dashed green lines) and 1hi10_v2* (dotted magenta lines).
Figure 4.6: Comparison between simulations 1hi5 and 1hi5.ms, which have almost identical initial conditions, however the latter is primordially mass segregated. The left panels show the distribution of semi-major axes of all stars (solid green line) and stars more massive than $8\,M_\odot$ at $T = 7\,\text{Myr}$ (dashed blue line). The right panels show the distributions of projected distances of spectroscopically visible stars at $T = 7\,\text{Myr}$ (dashed blue) and $T = 15\,\text{Myr}$ (dot-dashed red). The stars are projected so that the disk rotates clockwise in the sky. The y-values of the projected distributions are re-normalised to the expected number of stars had the model been simulated with a full Kroupa IMF.
4.5. RESULTS

Figure 4.7: The evolution of galactocentric distance, cluster mass, VMS mass and cluster half mass radius as a function of time, for simulations $1\text{o5}$ (solid blue line), $1\text{lu5}$ (dashed cyan line), $2\text{o5}$ (solid red line) and $2\text{lu5}$ (dashed dark red line).

As star formation close to a SMBH is not well understood, I also test a model in which the cluster is born with the top heavy mass function derived in Lu et al. (2013). Fig. 4.7 shows the evolution of simulations $1\text{o5}$, $2\text{o5}$, $1\text{lu5}$ and $2\text{lu5}$, where the two latter clusters have stars sampled from the Lu et al. (2013) mass function. Models computed with the top-heavy mass function form VMSs of greater mass, as more massive stars are sampled, and their cross sections for collisions are larger. However, as the cluster mass is distributed amongst fewer stars, simulations $1\text{lu5}$ and $2\text{lu5}$ relax faster than $1\text{o5}$ and $2\text{o5}$ and dissolve more rapidly. A flatter mass function does not help bring stars closer to SgrA*, and leaves more visible stars spread across the central 10 pc.

As a final test of extreme initial conditions I re-run simulation $1\text{hi5}$ with an initial size and density corresponding to being Roche filling at 1 pc. In simulation $1\text{hi5}_W4d$, this cluster is placed initially on a circular orbit at 5 pc, so that it is initially very Roche underfilling. Fig. 4.8 shows the evolution of Galactocentric distance, cluster mass, VMS mass and half mass radius of simulations $1\text{hi5}$ and $1\text{hi5}_W4d$ as a function of time. Increasing the density shortens the relaxation time, and after $\sim 2\text{ Myr}$ the clusters in simulations
4.5. RESULTS

Figure 4.8: The evolution of galactocentric distance, cluster mass, VMS mass and cluster half mass radius as a function of time, for simulations 1hi5 (solid blue line) and 1hi5_W4d (dashed red line).
4.6. DISCUSSION

1hi\(15\) and 1hi\(15.4d\) are similar. The cluster is able to retain its mass for slightly longer in 1hi\(15.4d\), but after the cluster expands it ultimately gets disrupted by the tidal field in the same way as 1hi\(15\). As such, making clusters arbitrarily dense is self-defeating and does not help the cluster migration scenario.

4.5.4 Models with primordial binaries

I include a primordial binary population in three of my simulations 1hi\(10.1_b\), 1hi\(10.2b^*\), and 1hi\(15.b\). The inclusion of primordial binaries is interesting as the clockwise disk has three confirmed eclipsing binaries (Ott et al. 1999, Martins et al. 2006, Pfuhl et al. 2014), with a total binary population estimated to be greater than 30\% (Pfuhl et al. 2014, Gautam et al. 2017). Secondly, a popular formation scenario for the S-stars is from the tidal disruption of binaries by Sgr\(A^*\) via the Hills mechanism (Hills 1991, 1992), where one star is captured and the other is ejected as a hyper-velocity star.

Fig. 4.9 shows the final projected distances of binaries in simulations 1hi\(10.1_b\), 1hi\(10.2b^*\), and 1hi\(15.b\). In these models 5\% of the stars are initially in binary systems, the properties of which are described in section §4.4.3. A large number of binary systems survive, despite some being consumed during the formation of the VMS. The final distributions of binaries with main sequence primaries more massive than 8 M\(_\odot\) are very similar to the distribution of single stars more massive than 8 M\(_\odot\) in models with no primordial binaries.

In all three simulations, no binaries end up with semi-major axes less than 1 pc. In 1hi\(10.2b^*\) one massive binary of total mass 68.9 M\(_\odot\) came within 0.1 pc of Sgr\(A^*\). Fig. 4.10 shows the orbit of this star, which came 0.09 pc from Sgr\(A^*\) at its third pericentre passage. However, the binary remained bound, as its tidal disruption radius by Sgr\(A^*\) was equal to \(\sim 10 \text{ AU}\), \(\sim 2000\) times smaller than its distance. The binary coalesced at apocentre. Due to the scarcity of binaries that approach Sgr\(A^*\), and the fact that many binaries would be observed beyond the disk, I conclude that if the binary breakup scenario is the origin of the S-stars, it is unlikely that the progenitors originated from nearby star clusters.

4.6 Discussion

The formation and evolution of a VMS appears to have little effect on cluster inspiral as compared with the collisionless models of Kim and Morris (2003), yet the suppression of IMBH formation strongly inhibits the radial migration of a sub population of massive stars towards the central parsec (F09). However, even in the case of IMBH formation, one would still observe a broad distribution of massive stars out to \(\sim 10\) pc, making the scenario unlikely even if IMBH formation were efficient.

All simulations form a large disk of massive stars from \(\sim 1 - 10\) pc, contradicting observations. This implies that no cluster has been present in this region in the past \(\sim 15\) Myr, as a large population would still be visible with current telescopes. Two \(\sim 10^5 M\odot\) gas clouds, M-0.02-0.07 and M-0.13-0.08, are seen projected at \(\sim 7\) and \(\sim 13\) pc from Sgr\(A^*\) (Solomon et al. 1972), suggesting the presence of GMCs in this region is commonplace. However, the absence of young stars in this region suggests that perhaps GMC collapse is suppressed, and fragmentation into stars primarily occurs only when the
4.6. DISCUSSION

Figure 4.9: Final distributions of the projected distances of binaries from SgrA* in simulations 1hi10_b, 1hi10_v2b* and 1hi5_b. The green histograms show the distributions of projected distances of all binaries remaining at 7 Myr, and the dashed blue histograms show those which have a main sequence primary more massive than 8 M⊙ at T = 7 Myr. The stars are projected so that the disk rotates clockwise in the sky. The y-values are re-normalised to the expected number of stars had the model been simulated with a full Kroupa IMF.
Figure 4.10: The orbit of the only binary to reach $< 0.1\text{ pc}$ in simulation 1hi10,2b*. The top panel shows the orbit of the star in the x-y plane of the galactocentric rest frame. The x-y plane is defined such that the infalling cluster orbits clockwise, with SgrA* at the origin. The bottom panel shows the separation between SgrA* and the binary as a function of time. In both subplots the solid blue circle shows when the binary collides, forming a blue straggler star.
4.7. CONCLUSIONS

cloud forms a dense disk around SgrA* (Mapelli et al., 2012). Verifying this hypothesis would require further study of GMC collapse in the close vicinity of a massive black hole.

4.7 Conclusions

I ran $N$-body simulations of young dense star clusters that form at distances of $5 - 15$ pc from SgrA* and inspiral towards the Galactic Centre due to dynamical friction. Most models are dense enough that runaway collisions are inevitable, forming a very massive star in less than 1 Myr. However, careful treatment of the evolution of this very massive star shows that it is likely to lose most of its mass through its stellar wind and end its life as a $\sim 20 - 400 M_{\odot}$ black hole. As no significant intermediate mass black hole can form in this model, clusters dissolve a few pc from SgrA*, leaving a population of bright early type stars that would be observable for longer than the age of both the clockwise disk and S-star population, contradicting observations. It is therefore unlikely that a cluster has inhabited the central 10 pc in the last $\sim 15$ Myr, as such the S-stars are unlikely to have formed via disrupted binaries originating from star clusters. Instead, the clockwise disk likely formed in-situ, perhaps from a gas cloud on a radial orbit incident on SgrA* (Mapelli et al., 2012), and the S-star cluster is likely to be populated either by dynamical processes in the clockwise disk (Subr and Haas, 2016), or through the binary breakup of scattered binaries (Perets et al., 2009).
Chapter 5
Conclusions

5.1 The Young stars in the Galactic Centre

Fujii et al. (2009) (hereafter F09) demonstrated that the star cluster inspiral model could bring a sub-population of massive stars to the Galactic Centre within the age of the young populations situated within the central parsec, including ∼ 20 S-star analogues within 0.1 pc of SgrA*. However, the caveat is that F09’s model requires the formation of an intermediate mass black hole (IMBH) within the core of the cluster. F09’s favoured model (labelled “LD64k” in F09.) formed an IMBH of > 16000 M☉, which is greater than the observational upper limit of ∼ 10⁴ M☉ derived from VLBA measurements of SgrA* (Reid and Brunthaler, 2004), however (Fujii et al., 2010) state that an IMBH of ∼ 1500 M☉ is sufficient to randomise the S-star orbits (see also Merritt et al., 2009).

Despite the successes of the F09 model, the formation of an IMBH is speculative. The most successful model had a low mass loss rate for very massive stars (VMSs) as compared with other studies (Vink et al., 2001; Kudritzki, 2002; Belkus et al., 2007; Vink et al., 2011) and took the collision criteria to be twice the sum of the stellar radii. The most conservative models had a collision criteria of the sum of the stellar radii and a 5× higher mass loss rate. However, the authors did not take into account the effect of chemical evolution on the wind strength of the VMS. Additionally, the rejuvenation scheme used (Meurs and van den Heuvel, 1989) has not been tested for such high mass ratio collisions, and all IMBHs collapsed at similar times of ∼ 2.5 Myr.

Due to computational limitations at the time, the models in F09 had ten times too many massive stars, casting doubt on the small number statistics of the S-stars produced (∼ 20). The gravitational potential of SgrA* was also taken to be a Plummer sphere of softening length ε = 0.2 pc, whereas the S-star cluster exists at distances ≲ 0.04 pc from SgrA*, the dynamics of which are not well resolved in this region.

In Chapter §4 (Petts and Gualandris, 2017), I revisited the star cluster inspiral scenario using a newly developed method in which dynamical friction due the galactic potential is modelled analytically, and only the internal dynamics of the star cluster need be integrated in an N-body fashion (see Chapter §2, Petts et al., 2015). This development, along with recent GPU-parallelisation of NBODY6 (Nitadori and Aarseth, 2012), allowed me to extensively explore the parameter space of the star cluster inspiral scenario.

In addition to the N-body dynamics, I simultaneously modelled the formation and evolution of VMSs formed via runaway collisions in a state-of-the-art fashion (see §4.2.2).
5.1. THE YOUNG STARS IN THE GALACTIC CENTRE

I modelled the hydrogen and helium burning phases by considering conservation of energy as in (Belkus et al., 2007; Vanbeveren et al., 2009). I only collapsed VMSs once all of their fuel had been exhausted, requiring no ad-hoc life-time estimations/extrapolations. Stars collided when they came within the sum of their stellar radii, which could be modelled accurately with the aid of the (Kustaanheimo and Stiefel, 1965) and chain (Mikkola and Aarseth, 1990, 1998) regularisation schemes implemented in NBODY6, which allow accurate integration of binary systems and close encounters. I mixed the composition of the resulting collision product, which generally resulted in the rejuvenation of hydrogen in the VMS. The stellar wind prescription used is taken from (Vink et al., 2001, 2011), and – with the aid of similarity theory models of (Nadyozhin and Razinkova, 2005) – is dependent on the chemical composition of the VMS.

I found that across a large range of initial cluster masses, densities and initial mass functions; the VMS lives for $2.5 - 6 \text{ Myr}$ and ends its life as black hole of mass $20 - 400 \, M_\odot$, insufficient to bring a population of massive stars to the central parsec of the Milky Way. In all models, the clusters dissolve far from SgrA*, leaving a population of bright early type stars that would be observable for longer than maximum age of both the Clockwise Disk and S-star populations, making it very unlikely that a star cluster has inhabited the central $\sim 10 \, \text{pc}$ in the last $\sim 15 \, \text{Myr}$. If one allows for extreme initial conditions such as an initially top-heavy mass function, or makes the cluster initially Roche under-filling, one still cannot overcome these problems. This owes to the cluster’s relaxation time always being much shorter than its dynamical friction timescale in these cases. I thus rule out the star cluster inspiral scenario as the origin of the young populations with high confidence.

5.1.1 Future work on the origin of the young stellar populations in the Galactic Centre

By refuting the star cluster inspiral scenario, in-situ formation of the Clockwise Disk seems increasingly likely, despite the requirement of extremely radial orbits (Bonnell and Rice, 2008; Hobbs and Nayakshin, 2009; Alig et al., 2011; Mapelli et al., 2012; Alig et al., 2013), especially considering recent observational constraints becoming increasingly difficult to overcome by any migration scenario (Yelda et al., 2014; Feldmeier-Krause et al., 2015). Secular evolution processes in the disk can likely explain the formation of the S-star cluster, so long as the disk had an initial binary fraction close to unity, and a cusp of $\sim 500 \, M_\odot$ exists within the extent of the S-star cluster (Chen and Amaro-Seoane, 2014; Šubr and Haas, 2016). Very recently, Habibi et al. (2017) (accepted for publication in ApJ) have concluded that S2 has an age of just $6.6^{+5.4}_{-4.7} \, M_\odot$, strengthening the conclusion that the Clockwise Disk and the S-stars originate from the same star formation event.

Despite the successes of these models, however, no single model is currently able to reproduce the observed properties of all of the young stars. Recently (Yelda et al., 2014) have claimed that only $\sim 20\%$ of the stars in the central parsec belong to the Clockwise Disk, a factor of $\sim 2.5$ lower than previously estimated (Bartko et al., 2009). Orbits highly inclined with respect to the Clockwise Disk are found to be very difficult to populate through relaxation effects or secular processes in the disk (Šubr and Haas, 2016), indicating that star formation in the Galactic Centre may be significantly more complicated than previously thought. Indeed, a plethora of complex gaseous structures...
are observed, such as the molecular circumnuclear ring (Becklin et al., 1982) and the ionized gas reservoirs SgrA East (Novak et al., 2000) and SgrA West (also known as the Galactic Centre minispiral Ekers et al., 1983; Lo and Claussen, 1983; Scoville et al., 2003; Zhao et al., 2009). Future hydrodynamical simulations which are able to resolve the rich morphology of the Galactic Centre environment may shed light on the origin of the off-disk populations, as well as constrain the rarity of the radial orbits required for in-situ formation models.

5.2 The Petts et al. dynamical friction model.

I have shown that Chandrasekhar’s dynamical friction model considering only the effect of two-body encounters is sufficient to reproduce the infall of satellites in both cuspy and cored spherical potentials (Chapters §2 and §3 Petts et al., 2015, 2016). In particular, I have shown that one can reproduce the “super-Chandrasekhar” inspiral phase (previously reported by Read et al., 2006), by utilising a velocity distribution function that is self-consistent with the galactic potential. The agreement is improved further still by incorporating the non-negligible effect of the stars moving faster than the satellite. By considering the tidal radius at which the infalling satellite tidally disrupts the galactic centre – along with some analytical arguments – I can reproduce the stalling phenomenon in both large/shallow cores as well as cuspy potentials with an additional “tidal stalling” prescription. By coupling the (Petts et al., 2015, 2016) model to a live $N$-body model (utilising the live mass and half-mass radius of the satellite), one can accurately model the infall of star clusters in various potentials. An adaptation of the direct summation code NBODY6 that includes the dynamical friction model is freely available.\(^1\)

5.2.1 Future use of the model

The predictive power of the Petts et al. (2015, 2016) model comes from the fact it has been extensively tested to accurately reproduce the inspiral of satellites in galaxies of asymptotic inner slopes, $\gamma = 0.2$, yet it has no free parameters. As such, the model has already been used to constrain the gravitational potential of dwarf spheroidal galaxies and ultra-compact dwarfs (Inoue, 2017; Cole et al., 2017; Contenta et al., 2017).

In Contenta et al. (2017, submitted) my collaborators and I used NBODY6df to put constraints on the density profile of the recently discovered dwarf galaxy, Eridanus II (Bechtol et al., 2015; Koposov et al., 2015; Crnojević et al., 2016), by comparing simulations of Eridanus II’s lone star cluster evolving in a cusped or cored potential. Although this idea is not new (Goerdt et al., 2006; Cole et al., 2012; Amorisco, 2017), NBODY6df allowed us to use the internal properties of a star cluster to constrain the potential of a dwarf galaxy for the first time. We modelled star clusters infalling in a cusped and a cored potential, motivated by the results of hydro-dynamical simulations of Read et al. (2016a), whom showed that dark matter cores as large as the half mass radius of the stars can form in dwarf galaxies due to baryonic feedback from supernovae (see also Navarro et al., 1996; Read and Gilmore, 2005; Pontzen and Governato, 2012; Governato et al., 2012; Pontzen and Governato, 2014; Di Cintio et al., 2014; Read et al., 2016b).

\(^1\)github.com/JamesAPetts/NBODY6df
Observations of the star cluster show that it is projected $\sim 45 \text{ pc}$ from the centre of Eridanus II (which has a half light radius of $\sim 280 \text{ pc}$, Crnojević et al., 2016), meaning it is likely inside the core unless it is observed at a special epoch. By comparing the observed properties of the star cluster with the simulated properties – including projection effects – Contenta et al. (2017) showed that a cored potential is significantly more favourable than a cuspy potential at reproducing the observed properties of the Eridanus II star cluster.

The method presented by Contenta et al. (2017) has a lot of potential for massive dwarfs for which there are more globular clusters, e.g. the Fornax dwarf spheroidal with 5 (Buonanno et al., 1985; Demers et al., 1990; Dubath et al., 1992; Beauchamp et al., 1995; Smith et al., 1997; Jorgensen and Jimenez, 1997) and NGC 6822 with at least 7 old globulars (Hwang et al., 2011; Huxor et al., 2013; Hwang et al., 2014) as well as young star cluster candidates (Wilson, 1992; Chandar et al., 2000; Wyder et al., 2000). However, this would require a new method, as running $\sim 100$ realisations of each cluster for each galaxy model in NBODY6df (as in Contenta et al. (2017)) would be too computationally expensive, especially considering some of the Fornax globulars are significantly more massive than the one in Eridanus II. For this reason it may be useful to use an alternative fast star cluster evolution method, such as EMACSS (Evolve Me A Cluster of StarS) (Alexander and Gieles, 2012; Gieles et al., 2014; Alexander et al., 2014) to approximate the evolution of a single model in a fraction of a second on a single CPU core. With this approach one will be able to leave the mass, radius and asymptotic inner power-law slope of the galaxy model as free parameters to truly constrain the shape of the galactic potential. By including an analytic time evolving potential (see Read et al., 2016a), one may also see if baryonic feedback is enough to explain the present day globular cluster properties. If no cusped or baryonically formed core model can fit the data then perhaps the globular cluster systems of these dwarf galaxies may hold evidence for physics beyond the standard $\Lambda$CDM cosmology.
Bibliography


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Appendix A

Appendices

A.1 Velocity dispersion of Dehnen models with an optional black hole

Here we derive the velocity dispersion as a function of radius for Dehnen models. We also include the optional potential of a central black hole, as used in Chapter 4 [Petts and Gualandris, 2017]. The option for a black hole is implemented in the initial release of NBODY6df.

The velocity dispersion for an isotropic spherical system is [Binney and Tremaine, 2008]:

\[ v_r^2(r) = \frac{1}{\nu(r)} \int_r^\infty \nu(r') \frac{d\Phi}{dr'} \, dr' \]  

(A.1.1)

Where \( \nu(r) \) is the number density and \( \Phi \) is the potential. The number density of a Dehnen model is given by:

\[ \nu(r) = \frac{\rho(r)}{M_g} = \frac{(3 - \gamma)}{4\pi} \frac{a}{r^\gamma(r + a)^{4-\gamma}} \]  

(A.1.2)

And the potential by [Dehnen, 1993]:

\[ \Phi(r) = -\frac{GM_g}{a} \frac{1}{2-\gamma} \left[ 1 - \frac{r}{r + a} \right]^{2-\gamma} - \frac{G\mu M_g}{r} \]  

(A.1.3)

Where the second term is the additional potential due to an optional central black hole, and \( \mu \) is the mass ratio of the black hole and Dehnen model. It follows that:

\[ \frac{d\Phi}{dr} = \frac{GM_g}{a} \left( \frac{r}{r + a} \right)^{1-\gamma} + \frac{G\mu M_g}{r^2} \]  

(A.1.4)

By putting equations A.1.2 and A.1.4 into equation A.1.1 and making the substitution \( x = r/a \), equation A.1.1 becomes:

\[ v_r^2(x) = \frac{GM_g}{a} x^{\gamma} (x + 1)^{4-\gamma} \left[ f(x) + \mu h(x) \right] \]  

(A.1.5)
A.2. ANALYTIC SOLUTION OF EQUATION ??

Where:

\[ f(x) = \int_x^{\infty} \frac{(x+1)^{1-\gamma}}{x^\gamma(x+1)^{6-\gamma}} \, dx \quad (A.1.6) \]

\[ h(x) = \int_x^{\infty} x^{-\gamma-2}(x+1)^{\gamma-4} \, dx \quad (A.1.7) \]

Which must be evaluated for the desired value of \( \gamma \). \( f(x) \) and \( h(x) \) are analytic for integer values of \( 4\gamma \).

A.2 Analytic solution of equation 3.3.9

Shigeki Inoue kindly gave an analytic solution to equation 3.3.9 in the appendix of his recent paper, Inoue (2017). With this welcome addition, the full (Petts et al., 2016) model now only requires a single numerical integration over \( f(v_*) \). I reproduce his solution here for completeness of Chapter 3.

Let \( A = v_s^2 - v_*^2 \) and \( B = b_{\text{max}}/G^2 M_*^2 \), then:

\[ J(v_*) = \int_{|v_s-v_*|}^{v_s+v_*} \left( 1 + \frac{A}{V^2} \right) \log(1 + BV^4) dV \]

\[ = \left[ \left( V - \frac{A}{V} \right) \log(1 + BV^4) - 4V + \frac{I_1 + \text{Im}(I_2)}{\sqrt{2B^{1/4}}} \right]_{|v_s-v_*|}^{v_s+v_*}. \quad (A.2.2) \]

Where:

\[ I_1(V) = (A\sqrt{B} - 1) \log \left( \frac{\sqrt{BV^2 + 1 - \sqrt{2}B^{1/4}V}}{\sqrt{BV^2 + 1 + \sqrt{2}B^{1/4}V}} \right), \quad (A.2.3) \]

\[ I_2(V) = (A\sqrt{B} + 1) \log \left( \frac{\sqrt{BV^2 - 1 - \sqrt{2}iB^{1/4}V}}{\sqrt{BV^2 - 1 + \sqrt{2}iB^{1/4}V}} \right). \quad (A.2.4) \]

And finally, using the identity \( \text{Im}(\log(x + iy)) = \arctan(y/x) \) (Gradshteyn and Ryzhik, 2007):

\[ \text{Im}(I_2) = -2(A\sqrt{B} + 1) \arctan \left( \frac{\sqrt{2}B^{1/4}V}{\sqrt{BV^2 - 1}} \right) \quad (A.2.5) \]

A.3 Dynamical Friction Comparison with GADGET

In Petts et al. (2015), I only tested my dynamical friction formulation against N-body models of single component Dehnen profiles. In Chapter 4 (Petts and Gualandris, 2017) I include an additional central black hole. In the Maxwellian approximation, valid for cuspy distributions, this comes into Chandrasekhar’s formula via the black hole’s contribution to
Figure A1: Orbital evolution of a $10^5 M_\odot$ point mass cluster in the Galactic Centre potential described in section §4.4, on circular and eccentric orbits in GADGET and using the semi-analytic model from Petts et al. (2015).
A.3. DYNAMICAL FRICTION COMPARISON WITH GADGET

the velocity dispersion of the stars. The addition of a black hole to the model is described in the appendix of Petts et al. (2015).

Here I briefly show that the Petts et al. (2015) model is accurate in the vicinity of a black hole, by means of two $N$-body models of point mass clusters, with mass $10^5 M_\odot$, orbiting the potential described in section §4.4. The $N$-body models are computed using the mpi-parallel tree-code GADGET2 (Springel et al., 2001). The stellar background is comprised of $2^{24}$ particles of mass $3.5 M_\odot$, with a central black hole of $4.3 \times 10^6 M_\odot$. The softening of the cluster potential is $\epsilon = 0.0769$ pc, corresponding to $r_{hm} \sim 0.1$ pc. The same softening length is used for the background particles. The black hole is given a softening length, $\epsilon = 0.2$ pc to reduce numerical inaccuracies resulting from the large mass ratio of the black hole to background particles. In GADGET2 the force is exactly Newtonian at $2.8 \epsilon$, so the semi-analytic and $N$-body models should agree to $\sim 0.56$ pc. The cluster is initially 5 pc from SgrA*.

Fig. A1 shows the orbital evolution of the circular and eccentric cases computed semi-analytically and with GADGET2. The Petts et al. (2015) model agrees very well with the $N$-body models in both cases. In the eccentric case the circularisation appears to be slightly under-predicted. Dosopoulou and Antonini (2016) show that this is due to the assumed Maxwellian distribution, which always under-predicts the circularisation of the orbit in the vicinity of a black hole (see their fig. 3). Another contribution is that we neglect the drag force from stars moving faster than the cluster (see Petts et al. 2016).