SIAM J. MATH. ANAL.
Vol. 49, No. 5, pp. 3854–3892

A RIGOROUS TREATMENT OF MOIST CONVECTION IN A SINGLE COLUMN*

BIN CHENG†, JINGRUI CHENG‡, MICHAEL CULLEN§, JOHN NORBURY¶, AND MATTHEW TURNER†

Abstract. We study a single-column model of moist convection in the atmosphere. We state the conditions for it to represent a stable steady state, then evolve the column by subjecting it to an upward displacement which can release instability, leading to a time-dependent sequence of stable steady states. We propose a definition of measure-valued solutions to describe the time dependence and prove its existence.

Key words. rearrangement, measure-valued solutions, Lagrangian equations

AMS subject classifications. 35B45, 35D30, 35Q86

DOI. 10.1137/16M1092647

1. Introduction.

This paper studies a simple mathematical model of moist convection in the atmosphere set out in Bokhove et al. [2]. Moist convection is responsible for much of the severe weather in the extratropics and is the main driver of tropical circulation, which is a fundamental part of the climate system. While convective storms have a very complicated structure, in which the physics of water in various phases is critical, the essential process can be captured by a one-dimensional model which only takes into account the saturation of air parcels with the associated release of latent heat. Such a model is used routinely by weather forecasters in interpreting the likely weather that will result from a given vertical profile of temperature and moisture; see [6, Chapter 4]. It also forms a key component of many theoretical studies of moist convection in the atmosphere; see, for instance, Holt [4], Lock and Norbury [5], and Shutts [7].

The model expresses conservation of heat and moisture, together with the change of phase of moisture from vapor to and from liquid and the associated release or absorption of latent heat. This takes place at a moisture concentration which depends on temperature and pressure, introducing a strong nonlinearity into the problem. Moist convection results from an instability of the vertical profile, which can be triggered by the upward bulk motion of the vertical profile generated by extratropical weather systems. In our model we represent the effect of this by making the saturation moisture content a monotonically decreasing function of time. This allows the model to be solved in a fixed vertical domain, which simplifies the presentation.

*Received by the editors September 6, 2016; accepted for publication (in revised form) July 12, 2017; published electronically October 3, 2017.
http://www.siam.org/journals/sima/49-5/M109264.html
Funding: This work was supported by the EPSRC Living with Environmental Change - Maths Foresees Network and the NERC Probability, Uncertainty and Risk in the Environment (PURE) Network. The work of the second author was supported in part by the National Science Foundation under grant DMS-1401490.
†Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK (B.Cheng@surrey.ac.uk, M.Turner@surrey.ac.uk).
‡Department of Mathematics, University of Wisconsin, Madison, WI 53706 (jrcheng@math.wisc.edu).
§Met Office, Exeter Devon EX1 3PB, UK (mike.cullen@metoffice.gov.uk).
¶Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK (John.Norbury@lincoln.ox.ac.uk).

The conservation properties are expressed in Lagrangian form, so that a discrete version of the problem can be solved by rearranging fluid parcels as in Bokhove et al. [2], Holt [4], and Lock and Norbury [5]. These conservation properties have been shown to be quite accurate even in more complicated models, e.g., by Shutts and Gray [8]. The rearrangement procedure is designed to reflect the underlying physics of the problem.

The first attempt to rigorously study this model was made by Dorian Goldman in his master's thesis [3], where he considered a particular choice of moisture content and initial data and proved the existence of weak solutions in Lagrangian variables. However, there seems be certain gaps in the proofs, and the solution was not completely characterized. In addition, his proof does not generalize to broader choice of moisture content and initial data, which can be physically interesting.

The aim of this paper is to show that the discrete problem converges to a limit solution as the number of parcels is increased and to interpret the resulting solution as a weak Lagrangian solution of the governing equations. We take a probabilistic approach in this paper, which is completely different from [3] and allows us to deal with a more general choice of moisture content function and initial data, which is physically meaningful.

The plan of the paper is as follows. In section 2 we present the problem to be solved and write it as a set of Lagrangian evolution equations. We note that we can only expect a probabilistic solution for general choices of initial data. In section 3, we describe the procedure to construct approximate (discrete) solutions given some deterministic discrete initial data, and we show that they satisfy the physical constraints. In section 4, we establish necessary estimates about these discrete solutions. In section 5, we come up with the notion of measure-valued solutions and show that this coincides with a natural definition of the solution when the initial data and evolution are deterministic. In section 6, we take the limit of the discrete solutions as the time/space step size tends to zero and obtain the existence of measure-valued solutions.

2. Definition of the problem. The problem to be studied (see Bokhove et al. [2]) is

$$D_t(\theta + q) = 0 \text{ in } (z,t) \in [0,1] \times (0,T), \tag{1}$$

$$D_t\theta = \begin{cases} 0 & \text{if } q < Q^{sat}(\theta,z,t), \\ [D_t(Q^{sat}(\theta,z,t))]^- & \text{if } q = Q^{sat}(\theta,z,t) \end{cases} \quad \text{in } (z,t) \in [0,1] \times (0,T), \tag{2}$$

$$\frac{\partial u}{\partial z} = 0 \text{ in } (z,t) \in [0,1] \times (0,T), \tag{3}$$

$$q(z,t) \le Q^{sat}(\theta(z,t),z,t). \tag{4}$$

As usual, we denote $D_t = \partial_t + u \cdot \partial_z$. Equation (3) should be interpreted as the divergence-free condition with respect to the space variable z; namely, its flow is measure preserving. The unknown functions are the potential temperature $\theta(z,t)$ and the moisture content $q(z,t)$. Equation (4) expresses the physical constraint that the moisture content is limited by the known saturation value Q^{sat}, which is time dependent. The interesting case, which we study, is where Q^{sat} is monotonically decreasing in time. However, this is not needed in the subsequent argument. In the above, $Q^{sat} : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function in its variables, and the following strict monotonicity conditions hold:

$$\partial_\theta Q^{sat} > 0, \ \partial_z Q^{sat} < 0 \text{ for any } (\theta,z,t) \in \mathbb{R}^3. \tag{5}$$

Physical solutions to (1)–(4) should also satisfy the following constraint:

(6) $$z \longmapsto \theta(z,t) \text{ is monotone increasing in } z \text{ for any } t \in (0,T).$$

The reason for imposing such a constraint is that physical solutions should minimize the energy functional

$$E(\bar{\theta}) = -\int_{[0,1]} z\bar{\theta}(z)dz,$$

where $\bar{\theta}(z)$ is a bounded Borel function on $[0,1]$, among all the possible rearrangements of the particles. It is easy to see that a function $\theta(z)$ achieves the minimum of E among all the functions $\bar{\theta}$ with the same distribution as θ if and only if $\theta(z)$ is monotone increasing. Rearranging the parcels is a measure-preserving map which does not change the distribution.

It is not hard to see that, in general, the solution does not have good regularity. Indeed, if everything is smooth, then let $F : [0,T) \times [0,1] \to [0,1]$ be the flow map. From (3), we get for each fixed $t \in [0,T)$ that $F_t(\cdot)$ preserves $\mathcal{L}^1_{[0,1]}$. If F is continuous, then we can obtain that $F(z) = z$ or $F(z) = 1 - z$. But $F_0 = \text{id}$, hence by continuity in t, we would be getting $F(t,z) = z$ for all t. This is not compatible with (4) and (6), except in trivial cases. Hence F cannot be continuous. Therefore, the velocity u is defined only as a measure. It is not clear how to define weak solutions to (1)–(4) in a standard way since the set $\{q = Q^{sat}(\theta, z, t)\}$ is only a general Borel set and may not have nice regularity.

We next define the solution in Lagrangian variables. Letting $F_t(z)$ be the flow map, we then get a reformulation of (1)–(4):

(7)
$$\partial_t(\hat{\theta} + \hat{q}) = 0 \text{ in } (z,t) \in [0,1] \times (0,T),$$
(8)
$$\partial_t\hat{\theta} = \begin{cases} 0 & \text{if } \hat{q} < Q^{sat}(\hat{\theta}, F_t(z), t), \\ [\partial_t(Q^{sat}(\hat{\theta}, F_t(z), t))]^- & \text{if } \hat{q} = Q^{sat}(\hat{\theta}, F_t(z), t) \end{cases} \quad \text{in } (z,t) \in [0,1] \times (0,T),$$
(9)
$$F_t \# \mathcal{L}^1_{[0,1]} = \mathcal{L}^1_{[0,1]} \quad \text{for any } t \in [0,T),$$
(10)
$$\hat{q}(z,t) \le Q^{sat}(\hat{\theta}(z,t), F_t(z), t).$$

In the above, $\hat{q}$ and $\hat{\theta}$ denote the corresponding variables in Lagrangian coordinates, namely, $\hat{q}(t,z) = q(t, F_t(z))$ and $\hat{\theta}(t,z) = \theta(t, F_t(z))$. Here we remark that the equations can be interpreted in a natural way. Indeed, (7) means that $\hat{\theta} + \hat{q}$ is conserved along flow lines. As for (8), notice that the right-hand side is nonnegative, and hence $\partial_t\hat{\theta}$ will be a nonnegative measure. If we can show $t \longmapsto F_t(z)$ has bounded variation, then $\partial_t(Q^{sat}(\hat{\theta}, F_t(z), t))$ is a well-defined finite signed measure, and its negative part can be defined. Therefore, (8) can be naturally defined as an equality of measures.

It will be convenient to consider the function $\Theta(w,z,t)$ as the solution θ to the equation

(11) $$\theta + Q^{sat}(\theta, z, t) = w.$$

This function is well defined thanks to the assumed strict monotonicity of Q^{sat} about θ. Also we know that Θ is smooth and satisfies the strict monotonicity

$$\partial_w \Theta > 0, \ \partial_z \Theta > 0. \tag{12}$$

This is clear from (5). First we make a simple observation whose proof is elementary.

LEMMA 2.1. *Define $\theta^M(t,z) = \theta(t,z) + q(t,z)$. Then $q(t,z) \le Q^{sat}(\theta(t,z), z, t)$ is equivalent to $\theta(t,z) \ge \Theta(\theta^M(t,z), z, t)$. Equivalence holds true also if we replace the above by a strict inequality.*

We assume the initial data satisfies the physical constraint. Namely, we are given $\theta_0(z), q_0(z) \in L^\infty([0,1])$, such that $z \longmapsto \theta_0(z)$ is monotone increasing, and $q_0(z) \le Q^{sat}(\theta_0(z), z, 0)$ for a.e. z. Inspired by the previous discussions, we propose the following definition of weak Lagrangian solutions.

DEFINITION 2.2. *Let $\theta_t(z)$, $q_t(z) \in L^\infty([0,T)\times[0,1]) \cap C([0,T), L^1([0,1]))$, and let $F, F^* : [0,T)\times[0,1] \to [0,1]$ be Borel measure-preserving maps such that $F_t(\cdot), F_t^*(\cdot) \in C([0,T); L^1([0,1]))$ and $F_\cdot(z) \in L^\infty([0,1]; BV([0,T)))$. Let $\theta_0(z), q_0(z)$ be as in the previous paragraph. Denote $\hat\theta_t(z) = \theta_t(F_t(z))$ and $\hat q_t(z) = q_t(F_t(z))$. Then we say (q_t, θ_t, F_t) is a weak Lagrangian solution to initial data θ_0, q_0 if the following hold:*

(i) *$\theta_t \to \theta_0$, $q_t \to q_0$ in $L^1([0,1])$, $F_t \to \mathrm{id}$ in $L^1([0,1])$ as $t \to 0$.*

(ii) *$z \longmapsto \theta_t(z)$ is monotone increasing for each $t \in [0,T)$.*

(iii) *For any $t \in [0,T)$, $F_t \circ F_t^*(z) = z$, $F_t^* \circ F_t(z) = z$ for $\mathcal{L}^1$-a.e. $z \in [0,1]$.*

(iv) *For $\mathcal{L}^1$-a.e. $z \in [0,1]$, $\hat\theta_t(z) + \hat q_t(z) = \theta_0(z) + q_0(z)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$.*

(v) *For $\mathcal{L}^1$-a.e. $z \in [0,1]$, $t \longmapsto \hat\theta_t(z) \le \hat\theta_{t'}(z)$ for $\mathcal{L}^2$-a.e.-(t,t') with $t < t'$.*

(vi) *$\partial_t \hat\theta(\cdot, z) = [\partial_t(Q^{sat}(\hat\theta(\cdot,z), F_\cdot(z), \cdot))]^- \lfloor E_z$, where $E_z = \{t \in (0,T) : \hat q_t^*(z) = Q^{sat}(\hat\theta_t^*(z), F_t(z), t)\}$, and $\hat q_t^*(z)$, $\hat\theta_t^*(z)$ is the monotone, left continuous version of $\hat q_t(z)$, $\hat\theta_t(z)$, chosen according to Remark 2.4.*

Remark 2.3. Let $f : (0,T) \to \mathbf{R}$ be Borel measurable, such that $f = \tilde f$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, with $\tilde f \in BV(0,T)$. Then $\partial_t f$ is a finite signed measure, defined by

$$\int_0^T f(t)\partial_t \zeta(t) dt = -\int_0^T \zeta(t)(\partial_t f)(dt) \quad \forall \zeta \in C_c^1((0,T)).$$

If we choose $\tilde f$ such that it is left continuous, then

$$\partial_t f([a,b)) = \tilde f(b) - \tilde f(a) \quad \text{ for any } [a,b) \subset (0,T).$$

Remark 2.4. Let $f : (0,T) \to \mathbb{R}$ be Borel measurable such that for $\mathcal{L}^2$-a.e. $(t,t') \in (0,T)^2$ with $t < t'$, we have $f(t) \le f(t')$. Then there exists a unique function $f^* : (0,T) \to \mathbb{R}$, monotone increasing, continuous from the left, such that $f = f^*$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$.

It turns out that the above definition of weak Lagrangian solutions is still too strong, and one cannot expect the existence of solutions in the sense defined above except for some special choice of the function Q^{sat} and the initial data.

One difficulty with the system (1)–(4) is that we do not have much regularity in space. The only regularity in space comes from the monotonicity of θ and, in general, no regularity in space for q, as well as the flow maps F_t, F_t^*. This means we lack the necessary compactness to get a function $q_t(z)$, or a measure-preserving flow map F, F^* in the limit.

The evolution of θ and q is highly unstable under small perturbations of the initial data, which can be seen from the construction of the discrete problem. This suggests the use of a probabilistic description of the solution. Under this description, heuristically for each time t, we have a certain probability distribution for $\{\theta, q(z)\}_{z\in[0,1)}$, and we make a random choice of θ, which is a monotone increasing function on $[0, 1)$, also make a random choice for $q(z)$ for each z, according to this probability distribution, and then evolve. This determines the probability distribution for $\{\theta, q(z)\}_{z\in[0,1)}$ at later times. In this spirit, we need to prescribe some probability distribution as initial data.

On the other hand we need the correct equation to be satisfied (point (vi) of Definition 2.2); this suggests considering some "path spaces" which describe all the possible paths of some parcel. Inspired by the probabilistic approach of the transport equation, we wish to obtain the solution as a measure in some path space, and the correct probability distribution is obtained by projecting to each t.

We will make the above heuristic discussions rigorous in section 5.

3. Solution of the discrete problem. In this section, we construct discrete solutions following the method of Bokhove et al. [2] and derive some estimates concerning them.

The discrete procedure is designed to reflect the underlying physics of the problem, as expressed in [6, Chapter 4]. It is based on a representation of the fluid as discrete parcels, so that $\hat\theta$ and $\hat q$ are piecewise constant. The initial values satisfy the physical constraints (10) and (6) at $t = 0$. We define Q^{sat} to be a monotonically decreasing function of time and discretize the time variation. Thus after some time interval the constraint (10) will be violated.

The Lagrangian form of (7)–(10) is solved by representing the flow map F_t as a rearrangement of the fluid parcels. The evolution of $\hat\theta$ and $\hat q$ on each parcel is computed using (7) and (8). If (10) is violated for any parcel, then (8) is used to update $\hat\theta$ and set q equal to Q^{sat}. The update to $\hat\theta$ may result in the constraint (6) being violated, in which case the parcels have to be rearranged to restore the constraint. $\hat\theta^M = \hat\theta + \hat q$ is conserved for each parcel under the rearrangement as required by (7), and $\hat\theta \geq \Theta(\hat\theta^M, z, t)$ at the final positions because of Lemma 2.1.

As found by Bokhove et al. [2], determining this rearrangement is nontrivial because of the dependence of Q^{sat} on θ and z, and because (10) may be violated on several parcels simultaneously. We call these "wet" parcels. In this case there may be many ways to satisfy the constraints. The physics of the problem requires that the final position of the wet parcel with the largest $\hat\theta^M$ be determined first. This is done by moving it upward, thus increasing $\hat\theta$ until it encounters a larger value of $\hat\theta$ at some $z = z_t$. We refer to this as the parcel "beating" all other parcels with $z < z_t$. All overtaken parcels have to move down to compensate for the upward displacement. Extreme care is required in showing that this procedure has a well-defined limit as the time step tends to zero.

We now define this procedure precisely. Denote $z_i = \frac{i}{n}$, $J_i = [\frac{i-1}{n}, \frac{i}{n})$ for any integer $1 \leq i \leq n$. Let $\{\theta_j^n\}_{j=1}^n$ and $\{q_j^n\}_{j=1}^n$ be given, such that $\theta_j^n \leq \theta_{j+1}^n$, and $q_j^n \leq Q^{sat}(\theta_j^n, z_j, 0)$ for any $1 \leq j \leq n$. This means that discrete versions of (6) and (10) are satisfied. It follows from Lemma 2.1 that $\theta_j^n \geq \Theta(\theta_j^n + q_j^n, z_j, 0)$. Let $\delta t = \frac{1}{Cn}$ for some large constant $C > 0$ to be determined later. This will be chosen so that it depends only on the function Q^{sat}, T, and the initial data. Define the wet set at time step 0 to be $W_n = \{1 \leq j \leq n : \theta_j^n < \Theta(\theta_j^n + q_j^n, z_j, \delta t)\}$. We will also denote

$\theta_j^{M,n} = \theta_j^n + q_j^n$. First we decide which parcels move to z_n. Define

$$W_n' = \{j_0 \in W_n : \text{for any } j > j_0 \text{ with } j \notin W_n,\ \theta_j^n < \Theta(\theta_{j_0}^{M,n}, z_j, \delta t), \text{ and if } j \in W_n,\ \theta_{j_0}^{M,n} > \theta_j^{M,n}\}.$$

Here we make the convention that $n \in W_n'$ if and only if $n \in W_n$. The set W_n' is exactly the set of parcels which are "wet" and can beat all other parcels above up to n. We will sometimes call them "eligible." First assume $W_n' \neq \emptyset$. Let $j_0 \in W_n'$ be the parcel with the largest $\theta_j^{M,n}$ among W_n' (if there are more than one such parcels, simply choose j_0 to be largest possible) and define the first rearrangement:

$$\sigma_n(k) = \begin{cases} k & \text{if } 1 \leq k < j_0, \\ n & \text{if } k = j_0, \\ k-1 & \text{if } j_0 < k \leq n. \end{cases}$$

To explain this in English, a parcel can jump to z_n only if it is wet and has the largest $\theta^{M,n}$ among all "eligible" parcels.

We also update θ_j^n after the first rearrangement in the following way:

$$\theta_j^{n,n-1} = \begin{cases} \theta^n_{\sigma_n^{-1}(j)} & \text{if } j \neq n, \\ \Theta(\theta_{j_0}^{M,n}, z_j, \delta t) & \text{if } j = n. \end{cases}$$

That is, we update the θ of parcels which jumped according to their final position, and leave the θ of other parcels unchanged. In Lagrangian coordinates, define $\hat{\theta}_j^{n,n-1} = \theta^{n,n-1}_{\sigma_n(j)}$, $\hat{q}_j^{n,n-1} = \theta_j^n + q_j^n - \hat{\theta}_j^{n,n-1}$. This is consistent with (7). Define the new wet set

(13)
$$W_{n-1} = \{1 \leq j \leq n : \theta_j^{n,n-1} < \Theta(\theta^{M,n}_{\sigma_n^{-1}(j)}, z_j, \delta t)\},$$
$$W_{n-1}' = \{j_0 \in W_{n-1} : \text{for any } j_0 < j \leq n-1 \text{ with } j \notin W_{n-1},\ \theta_j^{n,n-1} < \Theta(\theta^{M,n}_{\sigma_n^{-1}(j_0)}, z_j, \delta t). \text{ or if } j \in W_{n-1},\ \theta^{M,n}_{\sigma_n^{-1}(j_0)} > \theta^{M,n}_{\sigma_n^{-1}(j)}\}.$$

Notice that $W_{n-1} \subset \{1, 2, \ldots, n-1\}$.

If $W_n' = \emptyset$, then simply take $\sigma_n = \text{id}$ and take $\theta_j^{n,n-1} = \theta_j^n$, $q_j^{n,n-1} = q_j^n$. Then we have $W_{n-1} = W_n$.

Next we repeat the above procedure to $\{\theta_j^{n,n-1}\}_{j=1}^{n-1}$, $\{q_j^{n,n-1}\}_{j=1}^{n-1}$, and the new wet set defined by (13). Let σ_{n-1} be the resulting rearrangement of the first $n-1$ parcels. Let $\sigma_{n-1}(n) = n$, so that it becomes the rearrangement for n parcels.

In general, let σ_k be the rearrangement when we decide which parcel moves to z_k with $\sigma_k(l) = l$ for $l > k$. Denote $\beta_k = \sigma_{k+1} \circ \cdots \circ \sigma_n$, with $\beta_n = \text{id}$. Let $\{\theta_j^{n,k}\}_{j=1}^n$, $\{q_j^{n,k}\}_{j=1}^n$ be the updated θ and q after σ_{k+1}. We also denote $\theta_j^n = \theta_j^{n,n}$ and $q_j^n = q_j^{n,n}$. The wet set at this stage is given by

$$W_k = \{1 \leq j \leq n : \theta_j^{n,k} < \Theta(\theta^{M,n}_{\beta_k^{-1}(j)}, z_j, \delta t)\},$$
$$W_k' = \{j_0 \in W_k : \text{ for any } j_0 < j \leq k \text{ with } j \notin W_k,\ \theta_j^{n,k} < \Theta(\theta^{M,n}_{\beta_k^{-1}(j_0)}, z_j, \delta t), \text{ or if } j \in W_k,\ \theta^{M,n}_{\beta_k^{-1}(j_0)} > \theta^{M,n}_{\beta_k^{-1}(j)}\}. \tag{14}$$

As before, we make the convention that if $j_0 \geq k$, then $j_0 \in W_k'$ if and only if $j_0 \in W_k$. The sets W_k, W_k' determine the evolution when we decide which parcel moves to z_{k-1}.

The following inductive formula holds when $W'_{k+1} \neq \emptyset$. Let $j_* \in W'_{k+1}$ be such that $\theta^{M,n}_{\beta^{-1}_{k+1}(j_*)} \geq \theta^{M,n}_{\beta^{-1}_{k+1}(j)}$ for any $j \in W'_{k+1}$. Then we move this parcel to z_{k+1}, namely,

$$\sigma_{k+1}(j) = \begin{cases} j & \text{if } j < j_* \text{ or } j > k+1, \\ k+1 & \text{if } j = j_*, \\ j-1 & \text{if } j_* < j \leq k+1. \end{cases} \tag{15}$$

We update θ^n_j accordingly:

$$\theta^{n,k}_j = \begin{cases} \theta^{n,k+1}_{\sigma^{-1}_{k+1}(j)} & \text{if } j \neq k+1, \\ \Theta(\theta^{M,n}_{\beta^{-1}_k(k+1)}, z_{k+1}, \delta t) & \text{if } j = k+1. \end{cases} \tag{16}$$

If $W'_{k+1} = \emptyset$, then simply put $\sigma_{k+1} = \mathrm{id}$ and $\theta^{n,k}_j = \theta^{n,k+1}_j$.

Let $\hat{\theta}^{n,k}_j = \theta^{n,k}_{\beta_k(j)}$. Define $q^{n,k}_j = q^{n,k+1}_{\sigma^{-1}_{k+1}(j)} + \theta^{n,k+1}_{\sigma^{-1}_{k+1}(j)} - \theta^{n,k}_j$. $\hat{q}^{n,k}_j = q^{n,k}_{\beta_k(j)}$. We observe some useful properties of the above rearrangement algorithm.

LEMMA 3.1. *For each index $j \in \{1, 2, \ldots, n\}$, one of the following must hold:*

(i) *There exists a unique $k_1 \in \{1, 2, \ldots, n\}$, such that $\beta_{k_1-1}(j) = \sigma_{k_1}(\beta_{k_1}(j)) = k_1 > \beta_{k_1}(j)$. In addition, for any $k \geq k_1$, $\beta_k(j) \leq \beta_{k+1}(j)$, and for any $k \leq k_1 - 1$, $\beta_k(j) = k_1$.*

(ii) *$\beta_k(j) \leq \beta_{k+1}(j)$ for any $0 \leq k \leq n-1$.*

Moreover, if for some k_2, $\beta_{k_2}(j) \leq k_2$ and $\notin W_{k_2}$, then the second alternative must hold. On the other hand, if for some j_1, j_2, and some k_3 it holds that $\beta_{k_3}(j_1) < \beta_{k_3}(j_2)$, but $\beta_{k_3-1}(j_1) > \beta_{k_3-1}(j_2)$, then the first alternative above holds for j_1 with $k_1 = k_3$.

This lemma says that for any given parcel, either it experiences no lifts at all among the σ_k's or there is a unique σ_k which lifts this parcel and it stays there in the latter rearrangements of the same time step. If a parcel becomes dry in a certain time step, then it will stay dry in the latter arrangement. The only way the order of two parcels can change is for the lower parcel to experience a jump.

Proof. Fix an index j. Suppose there exists some $0 \leq k_1 \leq n-1$ for which $\beta_{k_1-1}(j) > \beta_{k_1}(j)$. From the definition of σ_k given in (15), we see that if $\sigma_k(j) > j$ for some j, k, it must hold that $\sigma_k(j) = k$. Hence from $\beta_{k_1-1}(j) = \sigma_{k_1}(\beta_{k_1}(j)) > \beta_{k_1}(j)$, we see that $\sigma_{k_1}(\beta_{k_1}(j)) = k_1$. If, for some $k \geq k_1$, $\beta_k(j) > \beta_{k+1}(j)$, then the same argument shows that $\beta_k(j) = \sigma_{k+1}(\beta_{k+1}(j)) = k+1$. Now for any $k' \leq k$, we have $\sigma_{k'}(\beta_k(j)) = \beta_k(j)$, a contradiction. This proves, for $k \geq k_1$, that $\beta_k(j) \leq \beta_{k+1}(j)$. The case for $k \leq k_1 - 1$ follows directly from the definition of σ_k.

To see the "moreover" part of the lemma, observe that for any $k \geq k_2$ we must have $\beta_k(j) \leq \beta_{k+1}(j)$. If not, by the argument given in the first part, we can then conclude that $\beta_{k'}(j) = k+1$ for any $k' \leq k$. In particular, this means $\beta_{k_2}(j) = k+1 > k_2$, a contradiction. Since $\beta_{k_2}(j) \notin W_{k_2}$, from (15), we see $\beta_{k_2-1}(j) = \sigma_{k_2}(\beta_{k_2}(j)) \leq \beta_{k_2}(j)$. This means that

$$\theta^{n,k_2-1}_{\beta_{k_2-1}(j)} = \theta^{n,k_2}_{\beta_{k_2}(j)} \geq \Theta(\theta^{M,n}_j, z_{\beta_{k_2}(j)}, \delta t) \geq \Theta(\theta^{M,n}_j, z_{\beta_{k_2-1}(j)}, \delta t).$$

It follows that $\beta_{k_2-1}(j) \notin W_{k_2-1}$. Hence one concludes $\beta_{k_2-2}(j) = \sigma_{k_2-1}(\beta_{k_2-1}(j)) \leq \beta_{k_2-1}(j)$. The same argument as above applies and shows that $\beta_{k_2-2}(j) \notin W_{k_2-2}$. One can apply the same argument and show that $\beta_k(j)$ is monotone decreasing in

k. The "on the other hand" part follows directly from the definition of σ_k given in (15). ∎

We want to show that the above define algorithm preserves a discrete version of the physical constraint.

Lemma 3.2.
(i) $\theta_j^{n,k} \leq \theta_{j+1}^{n,k}$ *for any* $1 \leq j \leq n-1$.
(ii) $\hat{q}_j^{n,k} + \hat{\theta}_j^{n,k} = q_j^n + \theta_j^n$.
(iii) $\hat{\theta}_j^{n,k} \geq \hat{\theta}_j^{n,k+1}$.
(iv) $q_j^{n,k} \leq Q^{sat}(\theta_j^{n,k}, z_j, 0)$ *for* $1 \leq j \leq k$, *and* $q_j^{n,k} \leq Q^{sat}(\theta_n^{n,k}, z_j, \delta t)$ *for* $k+1 \leq j \leq n$.

Proof. First we prove point (ii). From our definition, we know that $q_j^{n,k} + \theta_j^{n,k} = q^{n,k+1}_{\sigma_{k+1}^{-1}(j)} + \theta^{n,k+1}_{\sigma_{k+1}^{-1}(j)}$. From this it immediately follows that $\hat{\theta}_j^{n,k} + \hat{q}_j^{n,k} = \hat{\theta}_j^{n,k+1} + \hat{q}_j^{n,k+1}$.

We prove the other three statements by induction on n. First observe that statements (i)–(iv) are true for $k = n$. (Point (iii) is empty when $k = n$.) Now, assuming these are true for $k+1$ and above with $1 \leq k+1 \leq n$, we wish to prove these for k.

Now we prove point (iii) for k, assuming $W'_{k+1} \neq \emptyset$. One can see point (iii) is equivalent to $\theta^{n,k}_{\sigma_{k+1}(j)} \geq \theta_j^{n,k+1}$ by our definition of $\hat{\theta}_j^{n,k}$. If $\sigma_{k+1}(j) \neq k+1$, then one has $\theta^{n,k}_{\sigma_{k+1}(j)} = \theta_j^{n,k+1}$ from (16). Now if $\sigma_{k+1}(j_*) = k+1$, then

$$\theta_{k+1}^{n,k} = \Theta(\theta^{M,n}_{\beta_{k+1}^{-1}(j_*)}, z_{k+1}, \delta t) \geq \theta_{k+1}^{n,k+1} \geq \theta_{j_*}^{n,k+1}.$$

The first inequality above used the fact that $j_* \in W'_{k+1}$, and hence it must "beat" the parcel originally at z_{k+1}. The second inequality used the induction hypothesis that point (i) holds with $k+1$. If $W'_{k+1} = \emptyset$, then we simply have $\sigma_{k+1} = \mathrm{id}$, and $\theta_j^{n,k} = \theta_j^{n,k+1}$, so there is nothing to prove.

Then we prove point (i). We only consider the case when $W'_{k+1} \neq \emptyset$; otherwise nothing is changed by σ_{k+1} and the proof is trivial. Let $j_* \leq k+1$ be such that $\sigma_{k+1}(j_*) = k+1$.

The only nontrivial cases to check is when $j = k$ and $j = k+1$; the rest of the cases will follow from (15), (16) and the induction hypothesis that (i) holds for $k+1$. So it boils down to proving

$$\theta_{k+1}^{n,k+1} \leq \Theta(\theta^{M,n}_{\beta_{k+1}^{-1}(j_*)}, z_{k+1}, \delta t) \leq \theta_{k+2}^{n,k+1}. \tag{17}$$

The first part of the inequality follows from $j_* \in W'_{k+1}$; that is, it needs to "beat" the parcel originally at z_{k+1} in order to rise to z_{k+1}. To be precise, suppose that $k+1 \notin W_{k+1}$; then we know from $j_* \in W'_{k+1}$ and the formula for W'_k in (14) that $\theta_{k+1}^{k+1} < \Theta(\theta^{M,n}_{\beta_{k+1}^{-1}(j_*)}, z_{k+1}, \delta t)$. This is exactly what we want. Now if $k+1 \in W_{k+1}$, then again from $j_* \in W_{k+1}$ one concludes that $\theta^{M,n}_{\beta_{k+1}^{-1}(j_*)} \geq \theta^{M,n}_{\beta_{k+1}^{-1}(k+1)}$. Hence

$$\Theta(\theta^{M,n}_{\beta_{k+1}^{-1}(j_*)}, z_{k+1}, \delta t) \geq \Theta(\theta^{M,n}_{\beta_{k+1}^{-1}(k+1)}, z_{k+1}, \delta t) \geq \theta_{k+1}^{n,k+1}.$$

The first inequality used the monotonicity of Θ with respect to θ^M, and the second inequality used that $k+1 \in W_{k+1}$. This proves the first part of (17).

The reason why the second part of the inequality holds is that if it were not true, then j_* would have risen to z_{k+2} instead of z_{k+1} in the rearrangement σ_{k+2}. To make this precise, let $\sigma_{k+2}(j_{**}) = j_*$. Then we must have $j_{**} \geq j_*$. If not, we will have $\sigma_{k+2}(j_{**}) = k+2 = j_*$, which is not possible. Since $j_* \in W_{k+1}$, we have

$$\theta^{n,k+2}_{j_{**}} = \theta^{n,k+1}_{j_*} < \Theta(\theta^{M,n}_{\beta^{-1}_{k+1}(j_*)}, z_{j_*}, \delta t) \leq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_{**})}, z_{j_{**}}, \delta t).$$

This implies $j_{**} \in W_{k+2}$. Now we claim that for any j with $j_{**} < j \leq k+1$ and $j \notin W_{k+2}$, then $\theta^{n,k+2}_j < \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_{**})}, z_j, \delta t)$. Indeed, since $j \notin W_{k+2}$, we have $\sigma_{k+2}(j) \leq j$, $\sigma_{k+2}(j) \notin W_{k+1}$, and $\theta^{n,k+2}_j = \theta^{n,k+1}_{\sigma_{k+2}(j)}$. If the claim is not true, then

$$\Theta(\theta^{M,n}_{\beta^{-1}_{k+1}(j_*)}, z_{\sigma_{k+2}(j)}, \delta t) \leq \Theta(\theta^{M,n}_{\beta^{-1}_{k+1}(j_*)}, z_j, \delta t) \leq \theta^{n,k+2}_j = \theta^{n,k+1}_{\sigma_{k+2}(j)}.$$

Notice that $\sigma_{k+2}(j) \leq k+1$, and also $\sigma_{k+2}(j) > \sigma_{k+2}(j_{**}) = j_*$; this contradicts $j_* \in W'_{k+1}$.

Let j_1 be the maximal j such that $j \geq j_{**}$, $j \in W_{k+2}$, and $\theta^{M,n}_{\beta^{-1}_{k+2}(j_{**})} \leq \theta^{M,n}_{\beta^{-1}_{k+2}(j)}$. From the induction hypothesis with $k+2$ and point (iv), we know $j_1 \leq k+2$. Consider the following two cases.

Case 1: $j_1 \notin W'_{k+2}$. First observe that for any $j > j_1$ and $j \in W_{k+2}$ we must have $\theta^{M,n}_{\beta^{-1}_{k+2}(j)} < \theta^{M,n}_{\beta^{-1}_{k+2}(j_1)}$. Otherwise it will contradict the maximality of j_1. Also for any j with $j_1 < j \leq k+1$ and $j \notin W_{k+2}$, we conclude from the claim that $\theta^{n,k+2}_j < \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_{**})}, z_j, \delta t) \leq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_1)}, z_j, \delta t)$. The only remaining possibility is that $k+2 \notin W_{k+2}$ and $\theta^{n,k+2}_{k+2} \geq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_1)}, z_{k+2}, \delta t)$. That is, $k+2$ is a dry parcel and cannot be beaten by j_1. Hence

$$\theta^{n,k+1}_{k+2} \geq \theta^{n,k+2}_{k+2} \geq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_1)}, z_{k+2}, \delta t) \geq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_{**})}, z_{k+1}, \delta t).$$

This is what we want.

Case 2: $j_1 \in W'_{k+2}$. Let $j_2 \leq k+2$ be such that $\sigma_{k+2}(j_2) = k+2$. From the definition of the procedure, we have $\theta^{M,n}_{\beta^{-1}_{k+2}(j_2)} \geq \theta^{M,n}_{\beta^{-1}_{k+2}(j_1)}$, since the parcel that actually jumps up should have the largest θ^M among all eligible parcels. From the inductive formula (16), we see

$$\theta^{n,k+1}_{k+2} = \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_2)}, z_{k+2}, \delta t) \geq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_1)}, z_{k+2}, \delta t) \geq \Theta(\theta^{M,n}_{\beta^{-1}_{k+2}(j_{**})}, z_{k+1}, \delta t).$$

So far we finished the proof of point (i).

It only remains to show point (iv). This is equivalent to showing $\theta^{n,k}_j \geq \Theta(\theta^{n,k}_j + q^{n,k}_j, z_j, 0)$ for $1 \leq j \leq k$, and $\theta^{n,k}_j \geq \Theta(\theta^{n,k}_j + q^{n,k}_j, z_j, \delta t)$ for $k+1 \leq j \leq n$. To see the first part, we know for $1 \leq j \leq k$ that $\theta^{n,k}_j = \theta^{n,k+1}_{\sigma^{-1}_{k+1}(j)}$, and we also know from point (ii) already proved that $\theta^{n,k}_j + q^{n,k}_j = \theta^{n,k+1}_{\sigma^{-1}_{k+1}(j)} + q^{n,k}_{\sigma^{-1}_{k+1}(j)}$. Also $\sigma^{-1}_{k+1}(j) \leq k+1$, since the rearrangement σ_{k+1} never moves down a parcel by 2. Applying the induction hypothesis that (iv) holds for $k+1$, we obtain

$$\theta^{n,k}_j = \theta^{n,k+1}_{\sigma^{-1}_{k+1}(j)} \geq \Theta(\theta^{n,k+1}_{\sigma^{-1}_{k+1}(j)} + q^{n,k+1}_{\sigma^{-1}_{k+1}(j)}, z_{\sigma^{-1}_{k+1}(j)}, 0) \geq \Theta(\theta^{n,k}_j + q^{n,k}_j, z_j, 0).$$

To see the second part, consider first when $W'_{k+1} \neq \emptyset$, and if $j = k+1$, then we know from (16) that

$$\theta^{n,k}_{k+1} = \Theta(\theta^{M,n}_{\beta^{-1}_k(k+1)}, z_{k+1}, \delta t) = \Theta(q^{n,k}_{k+1} + \theta^{n,k}_{k+1}, z_{k+1}, \delta t).$$

In the second equality above, we used point (ii), already proved, and also the definition of $\theta_j^{M,n}$ given in the beginning of this section. If instead $W'_{k+1} = \emptyset$, then we know in particular $k+1 \notin W_{k+1}$, and hence

$$\theta_{k+1}^{n,k} = \theta_{k+1}^{n,k+1} \geq \Theta(\theta^{M,n}_{\beta_k^{-1}(k+1)}, z_{k+1}, \delta t) \geq \Theta(\theta_{k+1}^{n,k} + q_{k+1}^{n,k}, z_{k+1}, \delta t).$$

If $k+2 \leq j \leq n$ (note $\sigma_{k+1}(j) = j$), we use the induction hypothesis and (15) to conclude that

$$\theta_j^{n,k} = \theta_j^{n,k+1} \geq \Theta(\theta_j^{n,k+1} + q_j^{n,k+1}, z_j, \delta t) = \Theta(\theta_j^{n,k} + q_j^{n,k}, z_j, \delta t).$$

This finishes the proof. ∎

Denote $\theta_j^n(\delta t) = \theta_j^{n,0}$, $q_j^n(\delta t) = q_j^{n,0}$. Then we have $q_j^n(\delta t) \leq Q^{sat}(\theta_j^n(\delta t), z_j, \delta t)$ for any $1 \leq j \leq n$, and $j \longmapsto \theta_j^n(\delta t)$ is monotone increasing by Lemma 3.2. Define the flow map at the first time step $\tilde{F}_{\delta t}^n : [0,1) \to [0,1)$ such that it shifts J_i to $J_{\beta_0(i)}$ by translation, that is, $\tilde{F}_{\delta t}^n(z) = z - z_i + z_{\beta_0(i)}$ for $z \in J_i$. Then $\tilde{F}_{\delta t}^n \# \mathcal{L}^1_{[0,1]} = \mathcal{L}^1_{[0,1]}$. Apply the previous procedure to $\{\theta_j^n(\delta t)\}_{j=1}^n$, $\{q_j^n(\delta t)\}_{j=1}^n$, but with Q^{sat} evaluated at δt to get $\{\theta_j^n(2\delta t)\}_{j=1}^n$, $\{q_j^n(2\delta t)\}_{j=1}^n$, and the corresponding flow map $\tilde{F}_{2\delta t}^n : [0,1) \to [0,1)$. Repeating the procedure, we get a sequence of solutions at discrete times $\{\theta_j^n(k\delta t)\}_{j=1}^n$, $q_j^n(k\delta t)\}_{j=1}^n$, and a sequence of flow maps $\tilde{F}_{k\delta t}^n$ connecting $k\delta t$ and $(k+1)\delta t$. Denote $\theta_j^{M,n}(k\delta t) = \theta_j^n(k\delta t) + q_j^n(k\delta t)$. Here k is an integer with $0 \leq k \leq \frac{T}{\delta t} + 1$. Define $F_{k\delta t} = \tilde{F}_{k\delta t} \circ \cdots \circ \tilde{F}_{\delta t}$. We will also denote $\alpha_{k\delta t}$, $\tilde{\alpha}_{k\delta t}$ to be the corresponding rearrangement map on the discrete indices $\{1, 2, \ldots, n\}$. Denote $\theta^n(t,z) = \theta_j^n(k\delta t)$, $q^n(t,z) = q_j^n(k\delta t)$ if $z \in J_j$ and $k\delta t \leq t < (k+1)\delta t$. Also $F^n(t,z) = F_{k\delta t}$ if $k\delta t \leq t < (k+1)\delta t$. Define $\theta_0^n(z) = \theta_j^n$, $q_0^n(z) = q_j^n$, and $\theta_0^{M,n}(z) = \theta_0^n(z) + q_0^n(z)$ for $z \in J_j$. We deduce an immediate corollary of Lemma 3.2.

COROLLARY 3.3.
(i) *$z \longmapsto \theta^n(t,z)$ is monotone increasing for any $t \in [0,T)$.*
(ii) *Denote $\hat{\theta}^n(t,z) = \theta^n(t, F^n(t,z))$, $\hat{q}^n(t,z) = q^n(t, F^n(t,z))$. Then we have $\hat{\theta}^n(t,z) + \hat{q}^n(t,z) = \theta_0^n(z) + q_0^n(z)$.*
(iii) *$t \longmapsto \hat{\theta}^n(t,z)$ is monotone increasing for any $z \in [0,1)$.*
(iv) *$q^n(t,z) \leq Q^{sat}(\theta^n(t,z), z, k\delta t)$, where k is the integer such that $k\delta t \leq t < (k+1)\delta t$.*

Now we can make a more precise description of the motion of a single particle.

LEMMA 3.4. *Suppose that $n\delta t < \frac{\inf \partial_z \Theta}{\sup |\partial_t \Theta|}$, where both* sup *and* inf *are taken on the set $\{(w,z,t) : |w| \leq \max_j |\theta_j^{M,n}|, z \in [0,1], t \in [0,T]\}$. Then one of the following must hold:*
(i) *There exists a unique $k_1 \in \{1, 2, \ldots, n\}$, such that $\beta_{k_1-1}(j) = \sigma_{k_1}(\beta_{k_1}(j)) = k_1 > \beta_{k_1}(j)$. In addition, for any $k \geq k_1$, $\beta_k(j) = \beta_{k+1}(j)$, and any $k \leq k_1 - 1$, $\beta_k(j) = k_1$.*
(ii) *$\beta_k(j) \leq \beta_{k+1}(j)$ for any $0 \leq k \leq n-1$.*

This lemma says that if $n\delta t$ is small enough, then for any parcel experiencing a jump, the rearrangements before and after the jump will fix this parcel. In particular, if a parcel gets pushed down ($\beta_{k-1}(j) < \beta_k(j)$ for some k), then we must be in the second alternative, and by Lemma 3.1 it cannot overtake any other parcel.

Proof. The only difference between this lemma and Lemma 3.1 is that in the first alternative we can now conclude $\beta_k(j) = \beta_{k+1}(j)$ for any $k \geq k_1$. Suppose we are in the first alternative of Lemma 3.1, and $\beta_k(j) < \beta_{k+1}(j)$ for some $k \geq k_1$, we will show that $\beta_k(j) \notin W_k$. Clearly we have $\beta_{k+1}(j) \leq k+1$, since σ_{k+1} fixes all index strictly bigger than $k+1$. From Lemma 3.2, we know $\theta^{n,k+1}_{\beta_{k+1}(j)} \geq \Theta(\theta^{M,n}_j, z_{\beta_{k+1}(j)}, 0)$. Since σ_{k+1} moves $\beta_{k+1}(j)$ down, it does not change the value of θ, and hence $\theta^{n,k+1}_{\beta_{k+1}(j)} = \theta^{n,k}_{\beta_k(j)}$, and $\beta_{k+1}(j) \geq \beta_k(j) + 1$. It follows that

$$\begin{aligned}\theta^{n,k}_{\beta_k(j)} &= \theta^{n,k+1}_{\beta_{k+1}(j)} \geq \Theta(\theta^{M,n}_j, z_{\beta_{k+1}(j)}, 0)\\ &\geq \Theta(\theta^{M,n}_j, z_{\beta_k(j)}, \delta t) + (\inf \partial_z \Theta) n^{-1} - \sup |\partial_t \Theta| \delta t > \Theta(\theta^{M,n}_j, z_{\beta_k(j)}, \delta t).\end{aligned}$$

The last step used the smallness of $n\delta t$. Hence $\beta_k(j) \notin W_k$. It follows that $\beta_{k-1}(j) = \sigma_k(\beta_k(j)) \leq \beta_k(j)$. Repeating the argument shown in the proof of the "moreover" part of Lemma 3.1, we see that $\beta_k(j)$ will keep decreasing starting from k, and then no jump up is possible. ∎

To conclude this section, we make a simple observation which will be useful in the next section.

Lemma 3.5. *Suppose for some pair of indices $j_1, j_2 \in \{1, 2, \ldots, n\}$ and for some k we have $\alpha_{k\delta t}(j_1) < \alpha_{k\delta t}(j_2)$ and $\alpha_{(k+1)\delta t}(j_1) > \alpha_{(k+1)\delta t}(j_2)$; then $\theta^{M,n}_{j_1} > \theta^{M,n}_{j_2}$. In particular, $\alpha_{l\delta t}(j_1) > \alpha_{l\delta t}(j_2)$ for all $l > k$.*

In plain English, this lemma says that if the index j_1 is initially below j_2, then a necessary condition for j_1 to overtake j_2 is to have strictly larger θ^M. This means, in particular, that j_2 cannot overtake j_1 again since θ^M is invariant along parcels.

Proof. Here we write $\tilde{\alpha}_{(k+1)\delta t} = \sigma_1 \circ \cdots \circ \sigma_n$ and $\beta_k = \sigma_{k+1} \circ \cdots \circ \sigma_n$. Let m_0 be the maximal integer m for which $\beta_{m-1}(\alpha_{k\delta t}(j_1)) > \beta_{m-1}(\alpha_{k\delta t}(j_2))$. Then $\beta_{m_0}(\alpha_{k\delta t}(j_1)) < \beta_{m_0}(\alpha_{k\delta t}(j_2))$ and $\beta_{m_0}(\alpha_{k\delta t}(j_1)) \in W'_m$. If $\beta_{m_0}(\alpha_{k\delta t}(j_2)) \in W_m$, then we immediately have $\theta^{M,n}_{j_1} > \theta^{M,n}_{j_2}$. Otherwise,

$$\Theta(\theta^{M,n}_{j_1}, z_{\beta_{m_0}(\alpha_{k\delta t}(j_2))}, (k+1)\delta t) > \theta^{n,m_0}_{\beta_{m_0}(\alpha_{k\delta t}(j_2))}(k\delta t) \geq \Theta(\theta^{M,n}_{j_2}, z_{\beta_{m_0}(\alpha_{k\delta t}(j_2))}, (k+1)\delta t).$$

The first inequality used the definition of W'_m, while the second used the definition of W_m. ∎

4. Estimates on the discrete solution. Next we do some estimates on the discrete solutions. Denote $M' = ||\theta^n_0||_{L^\infty(0,1)} + ||q^n_0||_{L^\infty(0,1)}$. In the following, we say a constant is universal if it depends only on M', T, and Q^{sat}. We will derive estimates for the discrete solutions in this section. They are collected in the following theorem.

Theorem 4.1.

(i) $||\theta^n||_{L^\infty((0,1)\times(0,T))} + ||q^n||_{L^\infty((0,1)\times(0,T))} \leq C_1$ *for some universal constant* C_1.

(ii) *There exists a universal constant $C_2 > 0$, such that for any $\varepsilon > \frac{C_2}{n}$ and any $t \in [0,T)$, if $\hat{\theta}^n(t,z) > \Theta(\theta^{M,n}(t,z), F^n_t(z), t) + \varepsilon$, then we have $\hat{\theta}_{t'}(z) = \hat{\theta}_t(z)$, and $F_{t'}(z) \leq F_t(z)$ holds for any $t' - t < \frac{\varepsilon}{C_2}$.*

(iii) *For any $z \in [0,1]$, $TV_{t\in[0,T)}(F^n_t(z)) \leq C_3$.*

If $n\delta t < \frac{1}{C'_4}$ for some universal constant $C'_4 > 0$, then the following hold:

(iv) *For any $\varepsilon > 0$ and any $[t-\varepsilon, t+\varepsilon] \subset [0,T)$, we have $|\hat{\theta}^n_{t+\varepsilon}(z) - \hat{\theta}^n_{t-\varepsilon}(z) - (\Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon))^+| \leq 2C_4(\varepsilon + \delta t)$.*

(v) *For any $s < t \in [0,T)$, we have $||\theta^n(t,\cdot) - \theta^n(s,\cdot)||_{L^1([0,1])} = ||\hat{\theta}^n(t,\cdot) - \hat{\theta}^n(s,\cdot)||_{L^1([0,1])} \leq C_5\sqrt{t-s+\delta t}$.*

(vi) *For any* $s < t \in [0, T)$*, we have* $||F^n(t,\cdot) - F^n(s,\cdot)||_{L^1([0,1])} \le C_6\sqrt{t-s+\delta t}$ *for some universal constant* C*.*

Throughout this section, we make the following conventions: when we write expressions like $\sup|\partial_t\Theta|$ and so on, they are assumed to be taken over the set $\{(w,z,t)\in\mathbb{R}^3 : |w|\le M',\, z\in[0,1],\, t\in[0,T]\}$ unless otherwise stated.

We start with point (i) of the above theorem.

LEMMA 4.2. *There exists a universal constant* $C_1 > 0$*, such that*

$$||\theta^n||_{L^\infty((0,1)\times(0,T))} + ||q^n||_{L^\infty((0,1)\times(0,T))} \le C_1.$$

Proof. We need to go back to the construction of the discrete solution. First we know from Corollary 3.3(ii) that for any $z\in[0,1]$

$$\theta^n(t,z)+q^n(t,z) = \hat\theta^n(t,(F^n_t)^{-1}(z))+\hat q^n(t,(F^n_t)^{-1}(z)) = \theta^n_0((F^n_t)^{-1}(z))+q^n_0((F^n_t)^{-1}(z)).$$

Therefore $||\theta^n+q^n||_{L^\infty}\le M'$.

On the other hand, from the construction of $\theta^n_l(k\delta t)$, we know that either $\theta^n_l(k\delta t) = \theta^n_j((k-1)\delta t)$ for some $1\le j\le n$, or $\theta^n_l(k\delta t) = \Theta(\theta^{M,n}_j, z_l, k\delta t)$. In the former case, we have $|\theta^n_l(k\delta t)| \le \max_j|\theta^n_j((k-1)\delta t)$. In the latter case, we have $|\theta^n_l(k\delta t)|\le\sup|\Theta|$. Here sup is taken over the set $\{(w,z,t) : |w|\le M',\, z\in[0,1],\, t\in[0,T]\}$. But Θ is determined via (11) in terms of Q^{sat}, and hence $\sup|\Theta|$ satisfies a universal bound. In any case, we have $\max_j|\theta^n_j(k\delta t)|\le\max(\max_j|\theta^n_j((k-1)\delta t)|,\sup|\Theta|)$. It then follows easily by induction that $||\theta^n||_{L^\infty}\le\max(M',\sup|\Theta|)$. The bound for q then follows automatically. ∎

Next we prove point (ii). Roughly speaking, point (ii) says that if a parcel is "strictly" dry, then it will remain dry and go down for a while; the length of time this state lasts depends in a universal way on how dry this parcel is.

LEMMA 4.3. *There is a universal constant* $\tilde C_2 > 0$*, such that for any* $\varepsilon > 0$*, if for some integer* k, j *it holds that* $\hat\theta^n_j(k\delta t) > \Theta(\theta^{M,n}_j, z_{\alpha_{k\delta t}(j)}, k\delta t)+\varepsilon$*, then we have* $\hat\theta^n_j(l\delta t) = \hat\theta^n_j(k\delta t)$*, and* $\alpha_{l\delta t}(j)\le\alpha_{k\delta t}(j)$ *for any* l *with* $0\le(l-k)\delta t\le\frac{\varepsilon}{\tilde C_2}$*.*

Proof. Actually, we will see that one can take $\tilde C_2 = 2\sup|\partial_t\Theta|$. We prove this by induction on l. First observe that the statement is trivial if $l = k$. Assume this is true for some l with $(l+1-k)\delta t\le\frac{\varepsilon}{2\sup|\partial_t\Theta|}$. We need to show this is true also for $l+1$. Using the induction hypothesis, we can calculate

$$\begin{aligned}\hat\theta^n_j(l\delta t) = \hat\theta^n_j(k\delta t) &> \Theta(\theta^{M,n}_j, z_{\alpha_{k\delta t}(j)}, k\delta t)+\varepsilon\\ &\ge \Theta(\theta^{M,n}_j, z_{\alpha_{l\delta t}(j)}, (l+1)\delta t)+\varepsilon-\sup|\partial_t\Theta|(l+1-k)\delta t\\ &\ge \Theta(\theta^{M,n}_j, z_{\alpha_{l\delta t}(j)}, (l+1)\delta t).\end{aligned}$$

The first equality is the induction hypothesis. In the second inequality, we used the induction hypothesis that $\alpha_{l\delta t}(j)\le\alpha_{k\delta t}(j)$.

The above calculation shows that at time $l\delta t$ the parcel $\alpha_{l\delta t}(j)$ is still "dry" by taking one more time step forward. Hence we know $\alpha_{(l+1)\delta t} = \tilde\alpha_{(l+1)\delta t}(\alpha_{l\delta t}(j))\le\alpha_{l\delta t}(j)$, and $\hat\theta^n_j((l+1)\delta t) = \hat\theta^n_j(l\delta t)$ from the procedure. ∎

Now we can deduce point (ii) as a corollary of the previous lemma.

COROLLARY 4.4. *Let* $\varepsilon > 0$*,* $n\delta t\le 1$*. Then there exists a universal constant* $C_2 > 0$*, such that if* $\varepsilon > \frac{C_2}{n}$ *and* $\hat\theta^n(t,z) > \Theta(\theta^{M,n}(z), F^n_t(z), t)+\varepsilon$ *for some* $t\in[0,T)$*, we have* $\hat\theta^n_{t'}(z) = \hat\theta^n_t(z)$*, and* $F^n_{t'}(z)\le F^n_t(z)$ *holds for any* $t'-t < \frac{\varepsilon}{C_2}$*.*

Proof. First we can find integers k, j, such that $k\delta t \le t < (k+1)\delta t$ and $z \in J_j$. It then follows from the definition of $\hat\theta^n$ that

$$\begin{aligned}\hat\theta^n(t,z) = \hat\theta^n_j(k\delta t) &> \Theta(\theta^{M,n}(t,z), F^n_t(z), t) + \varepsilon \\ &\ge \Theta(\theta^{M,n}_j, z_{\alpha_{k\delta t}(j)}, k\delta t) + \varepsilon - \sup|\partial_z\Theta| n^{-1} - \sup|\partial_t\Theta|\delta t.\end{aligned}$$

Since $\delta t \le n^{-1}$, we will have $\varepsilon - \sup|\partial_z\Theta|n^{-1} - \sup|\partial_t\Theta|\delta t > \frac{\varepsilon}{2}$ if $\varepsilon > \frac{C_2'}{n}$ for some universal constant n. With such a choice of ε, we then have

$$\hat\theta^n_j(k\delta t) > \Theta(\theta^{M,n}_j, z_{\alpha_{k\delta t}(j)}, k\delta t) + \frac{\varepsilon}{2}.$$

Now we can conclude from Lemma 4.3 that $\hat\theta^n_j(l\delta t) = \hat\theta^n_j(k\delta t)$, and $\alpha_{l\delta t}(j) \le \alpha_{k\delta t}(j)$ for any integer l with $0 \le (l-k)\delta t \le \frac{\varepsilon}{C_2''}$ for some universal constant C_2''. This means precisely that $\hat\theta^n(t', z) = \hat\theta^n(t,z)$ and $F^n_{t'}(z) \le F^n_t(z)$ for any $t' - t < \frac{\varepsilon}{C_2''}$. ∎

Next we wish to prove point (iii). For this, we need to establish a lemma which gives control over the total variation of $t \longmapsto F^n_t(z)$ in terms of the absolute bound of θ.

LEMMA 4.5. *There exists a universal constant $C_3' > 0$, such that for any indices $j \in \{1, 2, \dots, n\}$,*

$$\sum_{0\le k\le \frac{T}{\delta t}} \frac{1}{n}(\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+ \le C_3' ||\hat\theta^n||_{L^\infty((0,T)\times(0,1))},$$
$$\sum_{0\le k\le \frac{T}{\delta t}} \frac{1}{n}|\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)| \le 2C_3' ||\hat\theta^n||_{L^\infty((0,T)\times(0,1))} + 2.$$

Proof. First we observe that the second estimate follows from the first. Indeed, we just need to notice

$$\begin{aligned}&\sum_{0\le k\le \frac{T}{\delta t}} \frac{1}{n}|\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)| \\ &= \sum_{0\le k\le \frac{T}{\delta t}} \frac{2}{n}(\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+ - \sum_{0\le k\le \frac{T}{\delta t}} \frac{1}{n}(\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)) \\ &\le 2C_3' ||\hat\theta^n||_{L^\infty((0,T)\times(0,1))} + 2.\end{aligned}$$

Now we only need to focus on the first estimate. Fix some k such that $\alpha_{(k+1)\delta t}(j) > \alpha_{k\delta t}(j)$. Then we know $\alpha_{(k+1)\delta t}(j) = \tilde\alpha_{(k+1)\delta t}(\alpha_{k\delta t}(j)) > \alpha_{k\delta t}(j)$. This means the parcel $\alpha_{k\delta t}(j)$ is "wet" at $k\delta t$, or $\hat\theta^n_j(k\delta t) < \Theta(\theta^{M,n}_j, z_{\alpha_{k\delta t}(j)}, (k+1)\delta t)$. After the time step, we know $\hat\theta^n_j((k+1)\delta t) = \Theta(\theta^{M,n}_j, z_{\alpha_{(k+1)\delta t}(j)}, (k+1)\delta t)$. Therefore

$$\begin{aligned}&\hat\theta^n_j((k+1)\delta t) - \hat\theta^n_j(k\delta t) \ge \Theta(\theta^{M,n}_j, z_{\alpha_{(k+1)\delta t}(j)}, (k+1)\delta t) - \Theta(\theta^{M,n}_j, z_{\alpha_{k\delta t}(j)}, (k+1)\delta t) \\ &\quad \ge (\inf \partial_z\Theta)(z_{\alpha_{(k+1)\delta t}(j)} - z_{\alpha_{k\delta t}(j)})^+ = \frac{1}{n}(\inf\partial_z\Theta)(\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+.\end{aligned} \tag{18}$$

Now we sum (18) over k, and the first estimate follows. ∎

Then we can deduce point (iii) as a corollary.

LEMMA 4.6. *For any* $z \in [0,1)$

$$PV_{t\in[0,T)}(F_t^n(z)) \le C_3' \|\hat{\theta}^n\|_{L^\infty((0,T)\times(0,1))},$$
$$TV_{t\in[0,T)}(F_t^n(z)) \le 2C_3' \|\hat{\theta}^n\|_{L^\infty((0,T)\times(0,1))} + 2.$$

Here PV *denotes the positive variation, and* TV *denotes the total variation.* C_3' *is the same constant as in the previous lemma.*

Proof. Find indices $j \in \{1,2,\dots,n\}$ such that $z \in J_j$. Then

$$PV_{t\in[0,T)}(F_t^n(z)) = \sum_{0\le k\le \frac{T}{\delta t}} (z_{\alpha_{(k+1)\delta t}(j)} - z_{\alpha_{k\delta t}(j)})^+ = \sum_{0\le k\le \frac{T}{\delta t}} \frac{1}{n}(\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j))^+,$$
$$TV_{t\in[0,T)}(F_t^n(z)) = \sum_{0\le k\le \frac{T}{\delta t}} |z_{\alpha_{(k+1)\delta t}(j)} - z_{\alpha_{k\delta t}(j)}| = \sum_{0\le k\le \frac{T}{\delta t}} \frac{1}{n}|\alpha_{(k+1)\delta t}(j) - \alpha_{k\delta t}(j)|. \qquad \square$$

It only remains to prove points (iv) and (v). For this we need the following key lemma, which concludes that any given parcel can only be overtaken at a finite rate.

LEMMA 4.7. *Fix* $j_0 \in \{1,2,\dots,n\}$. *Let* $0 \le k < l \le \frac{T}{\delta t}$. *Define the set*

$$J_{k,l} = \{j \in \{1,2,\dots,n\} : \alpha_{k\delta t}(j) < \alpha_{k\delta t}(j_0),\ \alpha_{l\delta t}(j) > \alpha_{l\delta t}(j_0)\}.$$

Then there exists a universal constant $C_4' > 0$*, such that if* $n\delta t < \frac{1}{C_4'}$*, we have*

$$\#J \le 2(l-k).$$

Proof. We will prove this statement with the choice of constant $C_4' = \frac{\sup|\partial_t\Theta|}{\inf \partial_z \Theta}$. Here sup, inf are taken over the set $\{(w,z,t) : |w| \le M, z \in [0,1], t \in [0,T]\}$. With this choice, Lemma 3.4 applies. For $k \le s \le l-1$, we may define

$$A_s = \{j \in \{1,2,\dots,n\} : \alpha_{s'\delta t}(j) < \alpha_{s'\delta t}(j_0) \text{ for any } s' \text{ with } k \le s' \le s, \text{ and } \alpha_{(s+1)\delta t}(j) > \alpha_{(s+1)\delta t}(j_0)\}.$$

Then we have $J_{k,l} = \cup_{s=k}^{l-1} A_s$. That $J_{k,l} \subset \cup_{s=k}^{l-1} A_s$ is clear, and the reverse inclusion follows from Lemma 3.5. Therefore it suffices to show $\#A_s \le 2$ for each s, when $n\delta t$ is small. Here we use the notation of section 2 and write $\tilde{\alpha}_{(s+1)\delta t} = \sigma_1 \circ \cdots \circ \sigma_n$. Here σ_k is the rearrangement map of the indices when we decide which parcel moves to z_k. Denote $\beta_k = \sigma_{k+1} \circ \cdots \circ \sigma_n$. Without loss of generality, we may assume that between time steps $s\delta t$ and $(s+1)\delta t$, the second alternative of Lemma 3.4 holds for $\alpha_{s\delta t}(j_0)$. Indeed, if the first alternative holds, we will have $A_s = \emptyset$, and there is nothing to prove in this case.

If $\#A_s \le 2$ fails, then let $k_1 > k_2 > k_3$ be the 3 biggest integers for which there exists some index j such that $\alpha_{s\delta t}(j) < \alpha_{s\delta t}(j_0)$ and $\beta_{k-1}(\alpha_{s\delta t}(j)) > \beta_{k-1}(\alpha_{s\delta t}(j_0))$. Let j_1, j_2, j_3 be the indexes corresponding to k_1, k_2, k_3. Namely $\alpha_{s\delta t}(j_i) < \alpha_{s\delta t}(j_0)$, but $\beta_{k_i-1}(\alpha_{s\delta t}(j_i)) > \beta_{k_i-1}(\alpha_{s\delta t}(j_0))$, $i = 1,2,3$. It is clear that $\beta_{k_i}(\alpha_{s\delta t}(j_i)) < \beta_{k_i}(\alpha_{s\delta t}(\alpha_{s\delta t}(j_0)))$, $1 \le i \le 3$. It is also clear that such index j_i must be unique since each σ_k lifts at most one index. From Lemma 3.4, we can see it must hold that $\alpha_{s\delta t}(j_3) < \alpha_{s\delta t}(j_2) < \alpha_{s\delta t}(j_1) < \alpha_{s\delta t}(j_0)$. The last inequality is clear, since $\alpha_{s\delta t}(j_0)$ will not jump. If, say, $\alpha_{s\delta t}(j_2) > \alpha_{s\delta t}(j_1)$, then $\alpha_{s\delta t}(j_1)$ overtakes $\alpha_{s\delta t}(j_2)$

since $\alpha_{s\delta t}(j_1)$ jumps first (under σ_{k_1}) and ends up above $\alpha_{s\delta t}(j_0)$. Thus $\alpha_{s\delta t}(j_2)$ will get pushed down by σ_{k_1}. But then, according to Lemma 3.4, it cannot jump later on and cannot overtake any other parcel.

The plan is to show $\theta^{M,n}_{j_3} \leq \theta^{M,n}_{j_2} \leq \theta^{M,n}_{j_1}$, and also show that this implies $\beta_{k_3}(\alpha_{s\delta t}(j_3)) \notin W'_{k_3}$. This will be a contradiction since we will have that $\beta_{k_3}(\alpha_{s\delta t}(j_3))$ cannot jump under σ_{k_3}.

First we observe that for any k with $k_2 < k \leq k_1$, and any index j with $\beta_{k_1}(\alpha_{s\delta t}(j_1)) < \beta_{k_1}(\alpha_{s\delta t}(j)) \leq k_1$, it holds that $\beta_{k-1}(\alpha_{s\delta t}(j)) = \beta_{k_1}(\alpha_{s\delta t}(j)) - 1$, while for index j with $\beta_{k_1}(\alpha_{s\delta t}(j)) < \beta_{k_1}(\alpha_{s\delta t}(j_1))$, it must hold that $\beta_{k-1}(\alpha_{s\delta t}(j)) = \beta_{k_1}(\alpha_{s\delta t}(j))$. Observe that this is clear with $k = k_1$. For $k < k_1$ and for j satisfying $\beta_{k_1}(\alpha_{s\delta t}(j_1)) < \beta_{k_1}(\alpha_{s\delta t}(j)) \leq k_1$, they cannot jump under σ_k since they are already overtaken by $\alpha_{s\delta t}(j_1)$. For j satisfying $\beta_{k_1}(\alpha_{s\delta t}(j)) < \beta_{k_1}(\alpha_{s\delta t}(j_1))$, they cannot jump up, because once they jump up under σ_k, they will jump to z_k, hence overtaking $\alpha_{s\delta t}(j_0)$ and contradicting our choice of k_2. They also cannot be pushed down, since if this happens, some parcel below needs to jump, again contradicting the choice of k_2.

Now we wish to prove $\theta^{M,n}_{j_2} \leq \theta^{M,n}_{j_1}$. If not, we will show below that $\beta_{k_1}(\alpha_{s\delta t}(j_2)) \in W'_{k_1}$. This will give us a contradiction since the parcel $\beta_{k_1}(\alpha_{s\delta t}(j_1))$ does not have the largest θ^M among the parcels in the set W'_{k_1}, and hence cannot jump up under σ_{k_1}. First it is clear from the "moreover" part of Lemma 3.1 that $\beta_{k_1}(\alpha_{s\delta t}(j_2)) \in W_{k_1}$. For any index j with $\beta_{k_1}(\alpha_{s\delta t}(j_2)) < \beta_{k_1}(\alpha_{s\delta t}(j)) < \beta_{k_1}(\alpha_{s\delta t}(j_1))$, and any k with $k_2 < k \leq k_1$, we conclude from the last paragraph that $\sigma_k = \mathrm{id}$, and $\theta^{n,k-1}_{\beta_{k-1}(\alpha_{s\delta t}(j))} = \theta^{n,k_1}_{\beta_{k_1}(\alpha_{s\delta t}(j))}$. Since $\beta_{k_2}(\alpha_{s\delta t}(j_2)) \in W'_{k_2}$, and no changes happen for these j's under σ_k with $k_2 < k \leq k_1$, we see $\beta_{k_1}(\alpha_{s\delta t}(j_2))$ will beat them under rearrangement σ_{k_1}. Since we assumed $\theta^{M,n}_{j_2} > \theta^{M,n}_{j_1}$, we know $\alpha_{s\delta t}(j_2)$ beats $\alpha_{s\delta t}(j_1)$ as well. Now consider index j satisfying $\beta_{k_1}(\alpha_{s\delta t}(j_1)) < \beta_{k_1}(\alpha_{s\delta t}(j)) \leq k_1$; if $\beta_{k_1}(\alpha_{s\delta t}(j)) \in W_{k_1}$, it can be beaten by $\beta_{k_1}(\alpha_{s\delta t}(j_1))$, which means $\theta^{M,n}_{j} < \theta^{M,n}_{j_1}$. Since $\theta^{M,n}_{j_2} > \theta^{M,n}_{j_1}$, it can also be beaten by $\alpha_{s\delta t}(j_2)$. If $\beta_{k_1}(\alpha_{s\delta t}(j)) \notin W_{k_1}$, we know from Lemma 3.1 that $\beta_{k_2}(\alpha_{s\delta t}(j)) \notin W_{k_2}$. Hence from $\beta_{k_2}(\alpha_{s\delta t}(j_2)) \in W'_{k_2}$, we see that $\theta^{n,k_2}_{\beta_{k_2}(\alpha_{s\delta t}(j))} < \Theta(\theta^{M,n}_{j_2}, z_{\beta_{k_2}(\alpha_{s\delta t}(j))}, (s+1)\delta t)$. Since they are "dry" parcels, we know their θ does not change, namely, $\theta^{n,k_2}_{\beta_{k_2}(\alpha_{s\delta t}(j))} = \theta^{n,k_1}_{\beta_{k_1}(\alpha_{s\delta t}(j))}$; we also know from the observation made in the previous paragraph with $k = k_2 + 1$ that $\beta_{k_2}(\alpha_{s\delta t}(j)) = \beta_{k_1}(\alpha_{s\delta t}(j)) - 1$. Hence they will be beaten by $\alpha_{s\delta t}(j_2)$ in the rearrangement σ_{k_1}. This shows that $\beta_{k_2}(\alpha_{s\delta t}(j_2)) \in W'_{k_1}$.

By the same argument as above, one can also conclude $\theta^{M,n}_{j_3} \leq \theta^{M,n}_{j_2}$; following the same logic for $\theta^{M,n}_{j_3} > \theta^{M,n}_{j_2}$, we will then conclude $\beta_{k_2}(\alpha_{s\delta t}(j_3)) \in W'_{k_2}$, and thus $\alpha_{s\delta t}(j_2)$ will not jump under σ_{k_2}. So we have shown $\theta^{M,n}_{j_3} \leq \theta^{M,n}_{j_2} \leq \theta^{M,n}_{j_1}$.

Next we show $\beta_{k_3}(\alpha_{s\delta t}(j_3)) \notin W'_{k_3}$. Let j_4 be the index such that $\beta_{k_1}(\alpha_{s\delta t}(j_4)) = \beta_{k_1}(\alpha_{s\delta t}(j_1)) + 1$. Since it is overtaken by j_1 under σ_{k_1} it will remain "dry" for all later rearrangements, that is, $\beta_m(\alpha_{s\delta t}(j_4)) \notin W_m$ for any $1 \leq m < k_1$. If $\beta_{k_3}(\alpha_{s\delta t}(j_3)) \in W'_{k_3}$, in particular, one should have $\Theta(\theta^{M,n}_{j_3}, z_{\beta_{k_3}(\alpha_{s\delta t}(j_4))}, (s+1)\delta t) > \theta^{n,k_3}_{\beta_{k_3}(\alpha_{s\delta t}(j_4))} = \theta^{n,k_1}_{\beta_{k_1}(\alpha_{s\delta t}(j_4))}$. On the other hand, using Corollary 3.3(i),(iv), we get

$$\theta^{n,k_1}_{\beta_{k_1}(\alpha_{s\delta t}(j_4))} \geq \theta^{n,k_1}_{\beta_{k_1}(\alpha_{s\delta t}(j_1))} \geq \Theta(\theta^{M,n}_{j_1}, z_{\beta_{k_1}(\alpha_{s\delta t}(j_1))}, s\delta t). \tag{19}$$

Observe that $\beta_{k_3}(\alpha_{s\delta t}(j_4)) \leq \beta_{k_1}(\alpha_{s\delta t}(j_4)) - 2 = \beta_{k_1}(\alpha_{s\delta t}(j_1)) - 1$, since there are at least 2 parcels (j_1 and j_2) overtaking j_4. Therefore

(20)
$$\theta^{n,k_1}_{\beta_{k_1}(\alpha_{s\delta t}(j_4))} < \Theta(\theta^{M,n}_{j_3}, z_{\beta_{k_3}(\alpha_{s\delta t}(j_4))}, (s+1)\delta t) \leq \Theta(\theta^{M,n}_{j_3}, z_{\beta_{k_1}(\alpha_{s\delta t}(j_1))-1}, (s+1)\delta t)$$
$$\leq \Theta(\theta^{M,n}_{j_1}, z_{\beta_{k_1}(\alpha_{s\delta t}(j_1))}, s\delta t) - (\inf \partial_z \Theta) n^{-1} + (\sup |\partial_t \Theta|)\delta t.$$

In the second inequality above, we used the above observation, and in the third inequality we used that $\theta^{M,n}_{j_3} \leq \theta^{M,n}_{j_2} \leq \theta^{M,n}_{j_1}$. Now combining (19) and (20) gives a contradiction. ∎

From the above discrete estimate, we can get the lemma of "finite speed of penetration."

Lemma 4.8. *Let $z_{i_0} \in [0,1)$. Choose n, δt such that $n\delta t = \frac{1}{2C_4'}$, where C_4' is the constant given by previous lemma. Let $t_0 \in [0,T)$, $\varepsilon > 0$ and define the set*

$$J = \{z \in [0,1) : F_{t_0}(z) < F_{t_0}(z_{i_0}), F_{t_0+\varepsilon}(z) > F_{t_0+\varepsilon}(z_{i_0})\}.$$

Then $\mathcal{L}^1(J) \leq C(\varepsilon + \delta t)$ for some universal constant C.

Proof. Choose $j_0 \in \{1, 2, \ldots, n\}$, such that $z_{i_0} \in J_{j_0}$. Choose integers k, l such that $k\delta t \leq t_0 < (k+1)\delta t$, $l\delta t \leq t_0 + \varepsilon < (l+1)\delta t$. Then we know that $J = \cup_{j \in J_{k,l}} J_j$. Here $J_{k,l}$ is defined as in the previous lemma. Hence

$$\mathcal{L}^1(J) = n^{-1} \# J_{k,l} \leq \frac{2(l-k)}{n} = \frac{2(l-k)\delta t}{n\delta t} \leq 4C_4'(\varepsilon + \delta t).$$

As an application of this lemma, we can prove point (iv) of Theorem 4.1.

Proposition 4.9. *Let n, δt be chosen as in Lemma 4.8. There exists a universal constant $C > 0$, such that for any $\varepsilon > 0$, and $t \in [0,T)$, with $[t-\varepsilon, t+\varepsilon] \subset [0,T)$, we have*

(21)
$$|\hat{\theta}^n_{t+\varepsilon}(z) - \hat{\theta}^n_{t-\varepsilon}(z) - \big(\Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon)\big)^+| \leq C(\varepsilon + \delta t).$$

Proof. Let κ be the quantity in the absolute value above. First we show $\kappa \leq C(\varepsilon + \delta t)$. Without loss of generality, we can then assume $\kappa > 0$. Then we know that $\hat{\theta}^n_{t+\varepsilon}(z) - \hat{\theta}^n_{t-\varepsilon}(z) \geq \kappa$. It follows from Corollary 4.4 that

$$\hat{\theta}^n_{t-\varepsilon}(z) \leq \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon) + C\varepsilon. \tag{22}$$

On the other hand,

$$\begin{aligned} \hat{\theta}^n_{t+\varepsilon}(z) - \hat{\theta}^n_{t-\varepsilon}(z) &\geq \Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon) + \kappa \\ &\geq \Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \hat{\theta}^n_{t-\varepsilon}(z) - \sup |\partial_t \Theta| \delta t + \kappa. \end{aligned} \tag{23}$$

In the first inequality, we only used the definition of κ, and in the second inequality, we used Corollary 3.3(iv). If we let $t' \in [t-\varepsilon, t+\varepsilon]$ be such that $F^n_{t'}(z) = \max_{\tau \in [t-\varepsilon, t+\varepsilon]} F^n_\tau(z)$, we have $\hat{\theta}^n_{t+\varepsilon}(z) \leq \Theta(\theta^{M,n}(z), F^n_{t'}(z), t+\varepsilon) + \sup |\partial_t \Theta| \varepsilon$. Hence it follows from (23) that

(24)
$$\Theta(\theta^{M,n}(z), F^n_{t'}(z), t+\varepsilon) + \sup |\partial_t \Theta| \varepsilon \geq \Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \sup |\partial_t \Theta| \delta t + \kappa.$$

Noticing $\delta t \leq \varepsilon$, we obtain

$$F^n_{t'}(z) - F^n_{t+\varepsilon}(z) \geq \frac{1}{\inf \partial_z \Theta}(\kappa - 2\sup|\partial_t \Theta|\varepsilon).$$

Now consider the set $E' = \{z' \in [0,1] : F^n_{t'}(z') < F^n_{t'}(z), F^n_{t+\varepsilon}(z') > F^n_{t+\varepsilon}(z)\}$. Then we know $\mathcal{L}^1(E') \geq \frac{1}{\inf \partial_z \Theta}(\kappa - 2\sup|\partial_t\Theta|\varepsilon)$. But it follows from Lemma 4.8 that $\mathcal{L}^1(E') \leq C(\varepsilon + \delta t)$. Hence $\kappa \leq C'(\varepsilon + \delta t)$ for some universal constant C'.

Next we derive a lower bound. We consider two cases. First, if $\hat{\theta}^n_{t+\varepsilon}(z) > \hat{\theta}^n_{t-\varepsilon}(z)$, then, as observed in (22), we know that

$$\hat{\theta}^n_{t+\varepsilon}(z) - \hat{\theta}^n_{t-\varepsilon}(z) \geq \Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon) - C\varepsilon. \tag{25}$$

If $\Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon) \geq 0$, then we can immediately conclude $\kappa \geq -C\varepsilon$. If it is negative, then we can calculate

$$\begin{aligned} &\Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) - \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon) \\ &\geq -\sup|\partial_z\Theta|(F^n_{t+\varepsilon}(z) - F^n_{t-\varepsilon}(z))^- - \sup|\partial_t\Theta|2\varepsilon. \end{aligned} \tag{26}$$

Define $\tilde{E} = \{z' \in [0,1) : F^n_{t-\varepsilon}(z') < F^n_{t-\varepsilon}(z), F^n_{t+\varepsilon}(z') > F^n_{t+\varepsilon}(z)\}$. Then Lemma 4.8 shows $\mathcal{L}^1(\tilde{E}) \leq C(\varepsilon + \delta t)$. But this means $F^n_{t+\varepsilon}(z) - F^n_{t-\varepsilon}(z) \geq -C(\varepsilon + \delta t)$. Thus from (25), (26) we know $\hat{\theta}^n_{t+\varepsilon}(z) - \hat{\theta}^n_{t-\varepsilon}(z) \geq -C(\varepsilon + \delta t)$. The conclusion follows as well.

If $\hat{\theta}^n_{t+\varepsilon}(z) = \hat{\theta}^n_{t-\varepsilon}(z)$, this means no jumps happen during $[t-\varepsilon, t+\varepsilon]$. Therefore $F^n_{t+\varepsilon}(z) \leq F^n_{t-\varepsilon}(z)$. In this case

$$\Theta(\theta^{M,n}(z), F^n_{t+\varepsilon}(z), t+\varepsilon) \leq \Theta(\theta^{M,n}(z), F^n_{t-\varepsilon}(z), t-\varepsilon) + C\varepsilon.$$

Therefore, we also have the quantity $\geq -C\varepsilon$ as well. ∎

As a second application of Lemma 4.8, we finally prove point (v) of the theorem. First we derive an obvious corollary of the above lemma.

Fix some $t_0 \in [0,T)$, take $\varepsilon > \delta t$, $\kappa > 0$, and define

$$\begin{aligned} J_1 &= \Big\{z \in [0,1) : \sup_{t\in[t_0,t_0+\varepsilon]} F^n_t(z) - F^n_{t_0}(z) \geq \kappa\Big\}, \\ J_2 &= \Big\{z \in [0,1) : F^n_{t_0+\varepsilon}(z) - F^n_{t_0}(z) \geq \frac{\kappa}{2}\Big\}. \end{aligned}$$

First we observe the following lemma.

LEMMA 4.10. *Let n, δt be chosen as in Lemma 4.8. Then there exists a universal constant $C > 0$, such that if $\varepsilon + \delta t \leq \frac{\kappa}{C}$, then $J_1 \subset J_2$.*

Proof. Suppose there exists $z_0 \in J_1 - J_2$. Let $t' \in [t_0, t_0+\varepsilon]$ be such that $F^n_{t'}(z_0) = \max_{t\in[t_0,t_0+\varepsilon]} F^n_t(z_0)$. Then we know $F^n_{t'}(z_0) - F^n_{t_0}(z) \geq \kappa$. It follows that $F^n_{t'}(z_0) - F^n_{t_0+\varepsilon}(z_0) \geq \frac{\kappa}{2}$. Consider the set

$$\hat{J}_{z_0,\varepsilon,\kappa} = \{z \in [0,1] : F^n_{t'}(z) < F^n_{t'}(z_0); F^n_{t_0+\varepsilon}(z) > F^n_{t_0+\varepsilon}(z_0)\}.$$

Then we know that $\mathcal{L}^1(\hat{J}_{z_0,\varepsilon,\kappa}) \geq \frac{\kappa}{2}$. On the other hand, it follows from Lemma 4.8 that $\mathcal{L}^1(\hat{J}_{z_0,\varepsilon,\kappa}) \leq C(\varepsilon + \delta t)$ with a universal constant C. Hence we have that $\frac{\kappa}{2} \leq C(\varepsilon + \delta t)$. ∎

The next lemma estimates $\mathcal{L}^1(J_2)$.

Lemma 4.11. *Let ε, n, δt be chosen as in the previous lemma. Let $\kappa \geq C\varepsilon$, where C is the constant given by the previous lemma. Then for some universal constant C' we have $\mathcal{L}^1(J_2) \leq \frac{C'\varepsilon}{\kappa}$.*

Proof. Let $\delta = \mathcal{L}^1(J_2)$. Since F^n is measure preserving, we know

$$\int_{[0,1)} F^n_{t_0+\varepsilon}(z) - F^n_{t_0}(z)dz = 0.$$

On the other hand, from the definition of J_2,

$$\int_{J_2} F^n_{t_0+\varepsilon}(z) - F^n_{t_0}(z)dz \geq \frac{\delta\kappa}{2}.$$

Therefore, there exists $z_1 \in [0,1]$, such that $F^n_{t_0+\varepsilon}(z_1) - F^n_{t_0}(z_1) \leq -\frac{\delta\kappa}{2}$. Now consider

$$\hat{J}_{z_1,\varepsilon} = \{z \in [0,1] : F^n_{t_0}(z) < F^n_{t_0}(z_1), F^n_{t_0+\varepsilon}(z) > F^n_{t_0+\varepsilon}(z_1)\}.$$

Then we have $\mathcal{L}^1(\hat{J}_{z_1,\varepsilon}) \geq \frac{\delta\kappa}{2}$. But by Lemma 4.8 we know $\mathcal{L}^1(\hat{J}_{z_1,\varepsilon}) \leq C(\varepsilon + \delta t) \leq 3C\varepsilon$. Hence $\frac{\delta\kappa}{2} \leq 3C\varepsilon$. This completes the proof. ∎

With the above preparation, we can obtain the following continuity estimate.

Lemma 4.12. *Let n, δt be chosen as in Lemma 4.8. Let $\varepsilon > \frac{\delta t}{2}$. Then there exists a universal constant C such that if $\varepsilon < \frac{1}{C}$, we have*

$$||\hat{\theta}^n_{t_0+\varepsilon} - \hat{\theta}^n_{t_0}||_{L^1(0,1)} = ||\theta^n_{t_0+\varepsilon} - \theta^n_{t_0}||_{L^1(0,1)} \leq C'\sqrt{\varepsilon}$$

for some universal constant C'.

Proof. From the discrete procedure, we know that if $\hat{\theta}^n_{t_0+\varepsilon}(z) > \hat{\theta}^n_{t_0}(z)$,

$$\hat{\theta}^n_{t_0+\varepsilon}(z) \leq \max_{t'\in[t_0,t_0+\varepsilon]} \Theta(\theta_0^{M,n}(z), F^n_{t'}(z), t_0+\varepsilon) + \sup|\partial_t\Theta|\varepsilon.$$

Since $\hat{\theta}^n_{t_0}(z) \geq \Theta(\theta^{M,n}(z), F^n_{t_0}(z), t_0) - \sup|\partial_t\Theta|\delta t$, by Corollary 3.3(iv) we know

$$\hat{\theta}^n_{t_0+\varepsilon}(z) - \hat{\theta}^n_{t_0}(z) \leq \sup\partial_z\Theta \cdot \max_{t\in[t_0,t_0+\varepsilon]}(F^n_t(z) - F^n_{t_0}(z)) + \sup|\partial_t\Theta|\cdot(\varepsilon+\delta t).$$

Let $\kappa \geq C\varepsilon$, where C is the universal constant given by Lemma 4.10. Combining Lemmas 4.10 and 4.11, we conclude that $\mathcal{L}^1(J_1) \leq \mathcal{L}^1(J_2) \leq \frac{C'\varepsilon}{\kappa}$. Hence

$$\begin{aligned}\int_{[0,1]} \hat{\theta}_{t_0+\varepsilon}(z) - \hat{\theta}_{t_0}(z)dz &= \int_{J_1} \hat{\theta}_{t_0+\varepsilon}(z) - \hat{\theta}_{t_0}(z)dz + \int_{J_1^c} \hat{\theta}_{t_0+\varepsilon}(z) - \hat{\theta}_{t_0}(z)dz \\ &\leq ||\hat{\theta}||_{L^\infty}\cdot\frac{C'\varepsilon}{\kappa} + \sup|\partial_z\Theta|\kappa + 3\sup|\partial_t\Theta|\cdot\varepsilon.\end{aligned}$$

Now we take ε small enough such that $\sqrt{\varepsilon} \leq \frac{1}{C}$, where C is given by Lemma 4.10, and, letting $\kappa = \sqrt{\varepsilon}$, we obtain the continuity estimates

$$\int_{[0,1]} |\theta_{t_0+\varepsilon}(z) - \theta_{t_0}(z)|dz = \int_{[0,1]} \hat{\theta}_{t_0+\varepsilon}(z) - \hat{\theta}_{t_0}(z)dz \leq C'\sqrt{\varepsilon}.$$ ∎

Next we derive the continuity estimate for the flow maps, which follows from the continuity estimate for θ.

LEMMA 4.13. *Let s, $t \in [0, T)$. Then for some universal constant C*

$$(F^n(t,z) - F^n(s,z))^+ \leq C(\hat{\theta}^n(t,z) - \hat{\theta}^n(s,z)). \tag{27}$$

Proof. As before, this follows from the discrete estimate. For any m, l integers with $0 \leq m \leq l \leq \frac{T}{\delta t}$, $(F^n_{l\delta t}(z) - F^n_{k\delta t}(z))^+ \leq C(\hat{\theta}^n_{l\delta t}(z) - \hat{\theta}^n_{k\delta t}(z))$. This follows from summing up both sides of (18) for k ranging from m to $l-1$. ∎

COROLLARY 4.14. *Let s, $t \in [0, T)$. Then for the same constant as in the previous lemma, we have*

$$\int_{[0,1]} |F^n(t,z) - F^n(s,z)|dz \leq 2C \int_{[0,1]} (\hat{\theta}^n(t,z) - \hat{\theta}^n(s,z))dz.$$

Proof. Since F^n is measure preserving, we know $\int_{[0,1]}(F^n(t,z) - F^n(s,z))^+dz = \int_{[0,1]}(F^n(t,z) - F^n(s,z))^-dz$. Then the result follows by integrating (27) into z. ∎

Point (vi) follows from the previous corollary and the continuity estimate for θ.

5. Definition of measure-valued solution. In this section, we wish to define the measure-valued solutions. As suggested in the discussion in the first section, we need to consider "path spaces" which represents all the possible trajectories of an arbitrary parcel. Thanks to point (iii) of Theorem 4.1, such paths take value in $[0, 1]$ and should have bounded variation, with uniform bound on BV.

Let $B_1 > 0$, and let X_{B_1} be the set of functions $f : (0, T) \to [0, 1]$ which is left continuous and has total variation no bigger than B_1; that is to say, for any partition of the interval $[0, T]$, denoted $0 < t_0 < t_1 < t_2 < \cdots < t_m < T$, we have

$$\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})| \leq B_1.$$

Let d be the L^2-distance for functions in X_{B_1}, that is,

$$d(f,g) = \left(\int_0^T |f(t) - g(t)|^2 dt \right)^{\frac{1}{2}}.$$

It is not hard to see that d is indeed a distance, since we required continuity from the left. Also one can check whether (X_{B_1}, d) is a complete separable metric space by Helly's selection principle. The physical meaning of such a space X_{B_1} is the space of all possible paths of the parcels. The reason such paths have bounded variation is due to point (iii) of Theorem 4.1.

Let $B_2 > 0$, and let Y_{B_2} be the space of monotone increasing functions on $[0, 1]$, right continuous on $[0, 1)$, and with absolute bound $\leq B_2$, equipped with the L^2-distance. That is, given $h, k \in Y_{B_2}$, define their distance to be $d'(h, k) = ||h - k||_{L^2(0,1)}$. The physical meaning of this space is all the possible profiles of potential temperature θ. We have incorporated the physical constraint that they must be monotone increasing.

Let $B_3 > 0$, and put $Y = C([0, T); Y_{B_3})$; that is, Y is the space of continuous maps from $[0, T)$ to Y_{B_3}. B_3 will be determined later on. We can define a metric on the space Y: given $h, k : [0,T) \to Y_{B_3}$, define $d(h,k) = \max_{t\in[0,T)} d'(h(t), g(t))$. The physical meaning of the space is the possible evolutions of the potential temperature profile.

To avoid confusion, we will denote a generic element from the space Y_B to be θ, while a generic element from the space Y will be denoted by $\tilde{\theta}$. It is easy to check that both Y_B and Y are complete separable metric spaces for any fixed $B > 0$.

First we specify the class of initial data we will be considering. Since we will be considering solutions in some "probabilistic" sense, our initial data will also be "probabilistic," namely, some probability distributions. Let $B_2 > 0$ be an arbitrary positive constant.

Definition 5.1. *Let* $\zeta_0 \in \mathcal{P}(Y_{B_2} \times \mathbb{R} \times [0,1])$. *We say* ζ_0 *is an admissible data if the following hold:*

(i) $\pi_{13}\#\zeta_0 = \mu_0 \times \mathcal{L}^1_{[0,1]}$ *for some* $\mu_0 \in \mathcal{P}(Y_{B_2})$, *and* $\pi_2\#\zeta_0$ *has compact support.*
(ii) $Q^{sat}(\theta(z), z, 0) \geq s - \theta(z)$ *for* ζ_0*-a.e.* (θ, s, z).

Remark 5.2. Heuristically, ζ_0 can be thought of as prescribing the probability distribution of $\{\theta, \theta^M(z)\}_{z\in[0,1]}$, where $\theta^M = \theta + q$. Indeed, using that $\pi_{13}\#\zeta_0 = \mu_0 \times \mathcal{L}^1_{[0,1]}$ and also the disintegration theorem, we can write $\zeta_0 = \int_{Y_{B_2}\times[0,1]} d(\mu_0 \times \mathcal{L}^1_{[0,1]})(\theta, z) \int_{\mathbb{R}} d\zeta_{\theta,z}(s)$. Here $(\theta, z) \longmapsto d\zeta_{\theta,z}(s)$ is a Borel family of probability measures on $\mathbb{R}$. This describes the probability distribution of θ^M, hence q, given θ and z. The second point simply says the physical constraint is satisfied with probability 1.

We can define the following evaluation maps for the spaces X_{B_1} and Y. Fix any $t \in (0,T)$, and define $e_t : X \to [0,1]$ by $\gamma \longmapsto \gamma(t)$. Similarly, define $e'_t : Y \to Y_{B_3}$ by $\tilde{\theta} \longmapsto \tilde{\theta}(t)$. We will frequently write $e_t(\gamma) = \gamma_t$ and $e'_t(\tilde{\theta}) = \tilde{\theta}_t$ to simplify the notation. We see from the definition of the space Y that e'_t is a continuous map. Also we can observe e_t is Borel, even if it is not continuous in general. Here we observe the following.

Lemma 5.3. *The set* $\{(\theta, s, z) \in Y_{B_2} \times \mathbb{R} \times [0,1] : Q^{sat}(\theta(z), z, 0) \geq s - \theta(z)\}$ *is a Borel subset of* $Y_{B_2} \times \mathbb{R} \times [0,1]$. *Also the map* e_t *defined in the previous paragraph is a Borel map.*

Proof. We just need to show that the evaluation map $A : (\theta, z) \longmapsto \theta(z)$ is Borel. Fix $\varepsilon > 0$, and let $A_\varepsilon(\theta, z) = \varepsilon^{-1} \int_z^{z+\varepsilon} \theta(w)dw$. Here we extended $\theta(w) \equiv C$ for $w > 1$. Then for each fixed $\varepsilon > 0$, $A_\varepsilon(\theta, z)$ is continuous, and for any fixed (θ, z), $A_\varepsilon(\theta, z) \to A(\theta, z)$, since θ is right continuous after extension. This proves A is Borel measurable.

That e_t is Borel is proved in a similar way. First we can define $f(t) = f(0)$ for $t \leq 0$. With this extension, $f(t)$ is defined on $(-\infty, T)$ and is left continuous. Hence the map $f(t) \longmapsto \varepsilon^{-1} \int_{t-\varepsilon}^t f(s)ds$ will converge to $f(t)$ as ε tends to 0. ∎

A "deterministic" initial data takes the form $\zeta_0 = \delta_{\theta^0} \times ((\theta^0 + q^0) \times id)\#\mathcal{L}^1_{[0,1]}$. In this case, there is only one possible choice, θ^0, and for each fixed z, θ^M takes a deterministic value $\theta^0(z) + q^0(z)$.

Here we can make a definite choice of the constants B_1 and B_3 which are involved in defining the spaces X_{B_1} and Y. By point (i) of Definition 5.1, we can assume $\pi_2\#\zeta_0 \subset [-M+1, M-1]$ for some $M > 0$. We will determine B_1 and B_3 such that they depend only on B_2, M, T and also the function Q^{sat}. Now take B_1 to be the constant C_3 given by point (iii) of Theorem 4.1 if we have the bound $||\theta_0^n||_{L^\infty} + ||q_0^n||_{L^\infty} \leq M + 2B_2$. Let B_3 be the constant C_1 given by point (i) of Theorem 4.1 if we have the bound $||\theta_0^n||_{L^\infty} + ||q_0^n||_{L^\infty} \leq M + 2B_2$. Without loss of generality we can assume $B_3 > B_2$ so that $Y_{B_2} \subset Y_{B_3}$.

With such a choice of constants in place, we propose the following definition of measure-valued solutions.

DEFINITION 5.4. *Let* $\lambda \in \mathcal{P}(Y \times \mathbb{R} \times [0,1] \times X_{B_1})$, *and denote* $\eta_t = (e'_t \times \mathrm{id} \times \mathrm{id} \times e_t)\#\lambda$, $\zeta_t = \pi_{124}\#\eta_t \in \mathcal{P}(Y_{B_3} \times \mathbb{R} \times [0,1])$. *Then we say* λ *is a measure-valued solution to admissible initial data* ζ_0 *if the following are satisfied:*

(i) $\zeta_t \to \zeta_0$, $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}^1_{[0,1]}, \mathcal{L}^1_{[0,1]}) \to (\mathrm{id} \times \mathrm{id})\#\mathcal{L}^1_{[0,1]}$ *narrowly as* $t \to 0$, *and* $t \longmapsto \eta_t$ *narrowly continuous.*

(ii) *For any* $t \in (0,T)$, $\pi_{13}\#\zeta_t = \mu_t \times \mathcal{L}^1_{[0,1]} \in \mathcal{P}(Y_{B_3} \times [0,1])$, $\pi_2\#\zeta_t$ *has compact support. In addition,* $Q^{sat}(\theta(z), z, t) \geq s - \theta(z)$ *for* ζ_t*-a.e.* (θ, s, z).

(iii) *For* λ*-a.e.* $(\tilde{\theta}, s, z, \gamma)$, *we have* $\tilde{\theta}_t(\gamma_t) \leq \tilde{\theta}_{t'}(\gamma_{t'})$ *for* $\mathcal{L}^2$*-a.e.* $(t, t') \in (0,T)^2$ *and* $t < t'$.

(iv) *For* λ*-a.e.* $(\tilde{\theta}, s, z, \gamma)$, *we have the equality of measures*

$$\partial_t(\tilde{\theta}_t(\gamma_t)) = \left[\partial_t(Q^{sat}(\tilde{\theta}_t(\gamma_t), \gamma_t, t))\right]^- \lfloor E_{\tilde{\theta}, s, \gamma},$$

where $E_{\tilde{\theta},s,\gamma}$ *is the wet set given by* $E_{\tilde{\theta},s,\gamma} = \{t \in (0,T) : (\tilde{\theta}(\gamma))^*(t) = s - Q^{sat}((\tilde{\theta}(\gamma))^*(t), \gamma_t, t)\}$.

In the above, $(\tilde{\theta}(\gamma))^*$ is the monotone increasing, left continuous version of $\tilde{\theta}_t(\gamma_t)$ chosen according to Remark 2.4. This is possible due to point (iii) in the above definition. The notation $\Gamma(\mathcal{L}^1_{[0,1]}, \mathcal{L}^1_{[0,1]})$ in point (i) denotes the set of Radon probability measures on $[0,1]^2$ whose projections on both components are equal to $\mathcal{L}^1_{[0,1]}$.

Point (i) simply specifies in what sense the initial data is satisfied. ζ_t gives the probability distribution of $\{\theta, \theta^M(z)\}_{z\in[0,1]}$ at time t and narrowly converges to ζ_0 as $t \to 0$. The second convergence simply means the "flow map" converges to identity as $t \to 0$. That $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}^1_{[0,1]}, \mathcal{L}^1_{[0,1]})$ is a reformulation of the measure-preserving property of the flow map, namely, the incompressibility.

Point (ii) shows that the ζ_t obtained satisfies the same conditions as required by the "admissibility" of the data. Therefore, one can take any ζ_t as initial data and evolves the solution forward.

Point (iii) shows that for all the possible choices of evolution of $\tilde{\theta}$ and the fluid path γ, $t \longmapsto \tilde{\theta}_t(\gamma_t)$ is always monotone increasing in t (up to some set of Lebesgue measures 0).

Point (iv) shows that for possible choices of evolution of $\tilde{\theta}$ and the fluid path γ, the correct equation is satisfied.

Next we show that if the random evolution of the solution happens to be deterministic, then Definitions 2.2 and 5.4 are consistent.

LEMMA 5.5. *Let* λ *be a measure-valued solution to admissible initial data* ζ_0. *Assume* $\zeta_0 = \delta_{\theta_0} \times \big((\theta_0 + q_0) \times \mathrm{id}\big)\#\mathcal{L}^1_{[0,1]}$. *Assume also that* $\lambda = \delta_{\theta(t)} \times \big((\theta_0 + q_0) \times \mathrm{id} \times \Phi_F\big)\#\mathcal{L}^1_{[0,1]}$ *for some Borel map* $F : [0,T) \times [0,1] \to [0,1]$. *Let* $\Phi_F(z)$ *associate each* z *to its path* $t \longmapsto F_t(z)$. *We also assume that there exists an inverse map* $F^* : [0,T) \times [0,1] \to [0,1]$ *with* $F^*_t \circ F_t = \mathrm{id}$ *and* $F_t \circ F^*_t = \mathrm{id}$ *for* $\mathcal{L}^1_{[0,1]}$*-a.e.* z *and any* $t \in (0,T)$. *Then* $(\theta(t,z), \theta_0(F^*_t(z)) + q_0(F^*_t(z)) - \theta(t,z), F)$ *is a weak Lagrangian solution in the sense of Definition* 2.2.

Proof. From the definition of admissible data, we know that for $\mathcal{L}^1$-a.e. $z \in [0,1]$, it holds that $Q^{sat}(\theta_0(z), z, 0) \geq q_0(z)$ for $\mathcal{L}^1$-a.e. $z \in [0,1]$. Since $\operatorname{supp} \pi_2\#\zeta_0 = (\theta_0 + q_0)\#\mathcal{L}^1_{[0,1]} \subset [-M, M]$, this means $||\theta_0 + q_0||_{L^\infty} \leq M$.

That $\theta_t \in L^\infty([0,T)\times[0,1])\cap C([0,T),L^1([0,1]))$ follows from $\theta_t \in C([0,T];Y_{B_2})$. Let $q_t(z) = \theta_0(F_t^*(z)) + q_0(F_t^*(z)) - \theta(t,z)$. From the boundedness of $\theta_0 + q_0$, we immediately get $q_t \in L^\infty([0,T)\times[0,1])$.

That $F_t(\cdot) \in C([0,T);L^1([0,1]))$ follows from $\pi_{34}\#\eta_t = (\mathrm{id}\times F_t)\#\mathcal{L}^1_{[0,1]}$ is narrowly continuous. From $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}^1_{[0,1]},\mathcal{L}^1_{[0,1]})$, we see $F_t\#\mathcal{L}^1_{[0,1]} = \mathcal{L}^1_{[0,1]}$, namely, F_t is measure preserving. Combined with the assumption, $F_t^*\circ F_t = \mathrm{id}$ shows that $(\mathrm{id}\times F_t)\#\mathcal{L}^1_{[0,1]} = (F_t^*\times \mathrm{id})\#\mathcal{L}^1_{[0,1]}$, and hence F_t^* is also measure preserving. As before, the narrow continuity of $t \longmapsto \pi_{34}\#\eta_t$ implies $F_t^*(\cdot) \in C([0,T);L^1([0,1]))$. This in turn implies $q_t \in C([0,T);L^1([0,1]))$.

From the definition of the space X_{B_1}, we see $F_\cdot(z) \in L^\infty([0,1];BV(0,T))$, with total variation $\le C$. Next we will check through points (i)–(vi) in Definition 2.2.

To see point (i), we observe that the measure $\pi_1\#\zeta_t \to \pi_1\#\zeta_0$ narrowly. In other words, we have $\delta_{\theta_t} \to \delta_{\theta_0}$ narrowly in $\mathcal{P}(X_0)$. This implies $\theta_t \to \theta_0$ in $L^2([0,1])$. That $F_t \to \mathrm{id}$ follows from $\pi_{34}\#\eta_t = (\mathrm{id}\times F_t)\#\mathcal{L}^1_{[0,1]} \to (\mathrm{id}\times\mathrm{id})\#\mathcal{L}^1_{[0,1]}$.

Point (ii) immediately follows from the definition of spaces Y and Y_{B_2}.

Point (iii) follows from the assumption of F made in this lemma.

To see point (iv), recall $q_t(z) = \theta_0(F_t^*(z)) + q_0(F_t^*(z)) - \theta(t,z)$. Hence for any $t \in (0,T)$, $\hat{q}_t(z) = q_t(F_t(z)) = \theta_0(F_t^*\circ F_t(z)) + q_0(F_t^*\circ F_t(z)) - \theta(t,F_t(z)) = \theta_0(z) + q_0(z) - \hat{\theta}_t(z)$ for $\mathcal{L}^1$-a.e. z.

Point (v) of Definition 2.2 follows from point (iii) of Definition 5.4. Indeed, from point (iii), we know that for $\pi_{14}\#\lambda$-a.e. $(\tilde{\theta},\gamma)$, it holds that $\tilde{\theta}_t(\gamma_t) \le \tilde{\theta}_{t'}(\gamma_{t'})$ for $\mathcal{L}^2$-a.e. $(t,t') \in (0,T)^2$ with $t < t'$. But $\pi_{14}\#\lambda = \delta_{\theta(t)} \times \Phi_F\#\mathcal{L}^1_{[0,1]}$. Hence for $\mathcal{L}^1$-a.e. z, it holds that $\theta_t(F_t(z)) \le \theta_{t'}(F_{t'}(z))$ for $\mathcal{L}^2$-a.e. $(t,t') \in (0,T)^2$ with $t < t'$.

To see the last point, we know from our assumption on λ that for λ-a.e. $(\tilde{\theta},s,z,\gamma)$, it holds that $s = \theta_0(z) + q_0(z)$ and $\gamma_t = F_t(z)$. For $(\tilde{\theta},s,z,\gamma)$ such that this holds, we know

$$
\begin{aligned}
E_{\theta,s,\gamma} = E_{\theta,\theta_0(z)+q_0(z),F_\cdot(z)} &= \{t \in (0,T) : (\theta(F))_t^*(z) \\
&= \theta_0(z) + q_0(z) - Q^{sat}((\theta(F))_t^*(z), F_t(z), t)\}.
\end{aligned}
$$

We have seen in the above proof that for $\mathcal{L}^1$-a.e. z, we have $\hat{q}_t(z) = \theta_0(z) + q_0(z) - \theta_t(F_t(z))$. Hence if we choose the monotone and left continuous representative, we have $\hat{q}_t^*(z) = \theta_0(z)+q_0(z)-(\theta(F))_t^*(z)$. Hence for λ-a.e. $(\tilde{\theta},s,z,\gamma)$, $E_{\theta,s,\gamma} = E_z$, where E_z is given in point (vi) of Definition 2.2. Finally the measure theoretic equation holds λ-a.e. Since $\pi_2\#\lambda = \mathcal{L}^1_{[0,1]}$, we see that point (vi) of Definition 2.2 holds. ∎

The main existence theorem we will prove will be the following.

THEOREM 5.6. *Let ζ_0 be an admissible initial data. Then there exists a measure valued solution to* (1)–(4) *with initial data ζ_0.*

6. Existence of measure-valued solutions.

In this section, we will show the existence of measure-valued solutions to any admissible data defined in the previous section.

The plan is as follows: Let ζ_0 be an admissible initial data. Then we approximate ζ_0 by convex combinations of discrete and "deterministic" initial data. For each measure appearing in the convex combination, we can run the discrete procedure described in section 3. The question is to show how one can take the limit of this approximation and how the properties given in Definition 5.4 hold in the limit.

6.1. Discretizing the initial data.

We assume $\pi_2\#\zeta_0 \subset [-M+1, M-1]$ for some $M > 0$. Write $K = M-1$. As a preliminary step, we note the following.

LEMMA 6.1. *Let ζ_0 be an admissible data. Then there exists a Borel family of probability measures $\{\alpha_\theta\}_{\theta\in Y_{B_2}} \subset \mathcal{P}(\mathbb{R}\times[0,1])$, such that for μ_0-a.e. θ, supp $\pi_1\#\alpha_\theta \subset [-K,K]$, $\pi_2\#\alpha_\theta = \mathcal{L}^1_{[0,1]}$, $\theta(z) \geq \Theta(s,z,0)$ for α_θ-a.e. (s,z) and for any bounded Borel function $f(\theta,s,z) : Y_{B_2}\times\mathbb{R}\times[0,1] \to \mathbb{R}$, it holds that*

$$\int_{Y_{B_2}\times\mathbb{R}\times[0,1]} f(\theta,s,z)d\zeta_0(\theta,s,z) = \int_{Y_{B_2}} d\mu_0(\theta)\left[\int_{\mathbb{R}\times[0,1]} f(\theta,s,z)d\alpha_\theta(s,z)\right]. \tag{28}$$

Proof. The existence of this family of probability measures satisfying the integral identity (28) is the standard disintegration theorem; see Theorem 5.3.1 of [1]. We just need to check that supp $\pi_1\#\alpha_\theta \subset [-K,K]$, $\pi_2\#\alpha_\theta = \mathcal{L}^1_{[0,1]}$, $\theta(z) \geq \Theta(s,z,0)$ for α_θ-a.e. (s,z).

To see that supp $\pi_1\#\alpha_\theta \subset [-K,K]$, take $f(\theta,s,z) = f_1(\theta)\chi_{[-K,K]^c}(s)$, and $f_1 : Y_{B_2} \to \mathbb{R}$ bounded and Borel measurable. Then we obtain

$$\left|\int f_1(\theta)\chi_{[-K,K]^c}(s)d\zeta_0(\theta,s,z)\right| \leq \sup|f|\int \chi_{[-K,K]^c}(s)d\zeta_0(\theta,s,z) = 0.$$

From (28), we know that for any choice of bounded Borel function f_1 one has

$$\int f_1(\theta)d\mu_0(\theta)\left[\int \chi_{[-K,K]^c}d\alpha_\theta(s,z)\right] = 0.$$

This implies $\int \chi_{[-K,K]^c}(s)d\alpha_\theta(s,z) = 0$ for μ_0-a.e. θ. For such θ, we have supp $\pi_1\#\alpha_\theta \subset [-K,K]$.

To see $\pi_2\#\alpha_\theta = \mathcal{L}^1_{[0,1]}$, one can similarly take $f(\theta,s,z) = f_1(\theta)f_2(z)$. Using that $\pi_{13}\#\zeta_0 = \mu_0\times\mathcal{L}^1_{[0,1]}$, one concludes for any choice of $f_2(z)$ bounded and Borel on $[0,1]$ that $\int_{\mathbb{R}\times[0,1]} f_2(z)d\alpha_\theta(s,z) = \int f_2(z)dz$ for μ_0-a.e. θ holds. One just needs to choose a countable dense subset $\{f_2^n\}_{n\geq1}$ of $C([0,1])$, apply this argument with $f_2 = f_2^n$, and conclude for μ_0-a.e. θ, $\pi_2\#\alpha_\theta = \mathcal{L}^1_{[0,1]}$.

To see that $\theta(z) \geq \Theta(s,z,0)$ for α_θ-a.e. (s,z), we integrate $\chi_{\{(\theta,s,z):\theta(z)\geq\Theta(s,z,0)\}}$ and use (28). By Lemma 5.3, such a function is bounded and Borel, and hence its integral is well defined. ∎

Due to Lemma 6.1, we can write $\zeta_0 = \int_{Y_{B_2}} \delta_\theta\times\alpha_\theta(s,z)d\mu_0(\theta)$, and for μ_0-a.e. θ, $\alpha_\theta(s,z)$ satisfy the "correct" condition mentioned in Lemma 6.1. Next we will construct the discretization of α_θ for each fixed such θ.

Recall that we have shown supp $\alpha_\theta \subset [-K,K]\times[0,1]$. Define $K_j = [-K + \frac{(j-1)2K}{n}, -K + \frac{j\cdot 2K}{n}]$ for $1 \leq j \leq n$ and call $w_j = -K + \frac{j\cdot2K}{n}$. Suppose $\alpha_\theta(K_j\times J_i) > 0$. Since we assumed $\theta(z) \geq \Theta(s,z,0)$ for α_θ-a.e. (s,z), we have, for some $(\tilde{\alpha}_{ij}, z_{ij}) \in K_j\times J_i$, $\theta(z_{ij}) \geq \Theta(\tilde{\alpha}_{ij}, z_{ij}, 0)$. Therefore, for some universal constant $C > 2K + \frac{\sup|\partial_z\Theta|}{\inf\partial_w\Theta}$, if we define $\alpha_{ij} = w_j - \frac{C}{n}$, we have

$$\theta(z_i) \geq \theta(z_{ij}) \geq \Theta(\tilde{\alpha}_{ij}, z_{ij}, 0) \geq \Theta(\tilde{\alpha}_{ij}, z_i, 0) - \frac{\sup|\partial_z\Theta|}{n} \geq \Theta\left(w_j - \frac{C}{n}, z_i, 0\right).$$

There is no loss of generality to assume n is chosen sufficiently large so that $\frac{C}{n} < 1$. Now we can define a measure α_θ^n which is an approximation to α_θ by putting

$$\alpha_\theta^n = \sum_{i=1}^n \sum_{j\in H_{\theta,i}} \chi_{J_i}(z)dz\cdot n\alpha_\theta(K_j\times J_i)\delta_{\alpha_{ij}}(s).$$

Here we denote

$$H_{\theta,i} = \{j : 1 \le j \le n,\, \alpha_\theta(K_j \times J_i) > 0\}. \tag{29}$$

LEMMA 6.2. *$\alpha_\theta^n \to \alpha_\theta$ narrowly in $\mathcal{P}(\mathbb{R}\times[0,1])$.*

Proof. Let $f \in C_b(\mathbb{R}\times[0,1])$ with Lipschitz constant 1, and denote $H_i = H_{\theta,i}$ for simplicity.

$$\begin{aligned}
\int f(s,z)d\alpha_\theta(s,z) &= \sum_{i=1}^n \sum_{J\in H_i} \int_{K_j\times J_i} f(z,s)d\alpha_\theta(s,z)\\
&= \sum_{i=1}^n \sum_{j\in H_i} \int_{K_j\times J_i} \left(f(z,s) - \left(z_i, w_j - \frac{C}{n}\right)\right) d\alpha_\theta(s,z)\\
&\quad + \sum_{i=1}^n \sum_{j\in H_i} f\left(z_i, w_j - \frac{C}{n}\right) \alpha_\theta(K_j\times J_i)\\
&= \sum_{i=1}^n \sum_{j\in H_i} \int_{J_i} f\left(z, w_j - \frac{C}{n}\right) dz \cdot n\alpha_\theta(J_i\times K_j)\\
&\quad + \sum_{i=1}^n \sum_{j\in H_i} \int_{J_i\times K_j} \left(f(z,s) - f\left(z_i, w_j - \frac{C}{n}\right)\right) d\alpha_\theta(z,s)\\
&\quad + \sum_{i=1}^n \sum_{j\in H_i} \int_{J_i} f\left(z_i, w_j - \frac{C}{n}\right) - f\left(z, w_j - \frac{C}{n}\right) dz \cdot n\alpha_\theta(J_i\times K_j).
\end{aligned}$$

The first term above is exactly the integral of f with respect to α_0^n. The last two terms will go to zero, because in each term of the sum, the integrand is controlled by $\frac{C}{n}$. ∎

For any $B > 0$ and each $\theta \in Y_B$, we can define $D^n : Y_B \to Y_B$ by putting $D^n(\theta) = \sum_{i=1}^n \theta(z_i)\chi_{J_i}$. Then for each fixed θ, it holds that $D^n(\theta) \to \theta$ in Y_B. Now we choose $B = B_2$. By putting $\zeta_0^n = \int_{Y_{B_2}} \delta_{D^n(\theta)} \times \alpha_\theta^n(s,z)d\mu_0(\theta)$, we then have $\zeta_0^n \to \zeta_0$ narrowly in $\mathcal{P}(Y_{B_2}\times\mathbb{R}\times[0,1])$ as $n\to\infty$. In addition, ζ_0^n satisfies the following properties:

(i) $\pi_{13}\#\zeta_0^n = (D^n\#\mu_0)\times\mathcal{L}^1_{[0,1]}$, supp $\pi_2\#\zeta_0^n \subset [-K-1, K+1] = [-M, M]$.

(ii) $\theta(z_i) \ge \Theta(w_j - \frac{C}{n}, z_i, 0)$, whenever $j \in H_{\theta,i}$, or $\theta(z) \ge \Theta(s,z,0)$ for ζ_0^n-a.e. (θ, s, z).

For simplicity of notation, we will write

$$\alpha_\theta(s,z) = \sum_{i=1}^n \sum_{j\in H_{\theta,i}} n\chi_{J_i}(z)\cdot \mu_{ij}^\theta \delta_{\alpha_{ij}}(s), \text{ with } \mu_{ij}^\theta = \alpha_\theta(K_j\times J_i), \text{ and } \alpha_{ij} = w_j - \frac{C}{n}. \tag{30}$$

6.2. Construction of approximate solutions and passage to limit. The measures ζ_0^n determine a sequence of discrete probability distributions. For each choice of θ, α_θ prescribes the probability distribution of $\theta^M(z)$ for each fixed z. More precisely, when $z \in J_i$, the possible values of θ^M are given by α_{ij}, with probability μ_{ij}.

In order to apply the discrete procedure, we will make a random choice of α_{ij} on each J_i, and this gives us a "deterministic" and discrete initial θ^M. Then we run the discrete procedure, with this θ^M as initial data, and it gives us an evolution, with probability determined by the choice of α_{ij}.

First we make a random choice of the α_{ij}, allowed by the physical constraint. Denote S_θ to be the set of functions $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, such that for each i, $\sigma(i) \in H_{\theta,i}$. Determine a discrete initial $\theta_0^{M,n}$ from σ by prescribing

$$\theta_{0,\sigma}^{M,n}(z) = \sum_{i=1}^{n} \alpha_{i\sigma(i)} \chi_{J_i}(z), \quad \text{or} \quad \theta_{0,\sigma,i}^{M,n} = \alpha_{i\sigma(i)}.$$

The probability of such a choice is given by $n^n \mu_{1\sigma(1)} \mu_{2\sigma(2)} \cdots \mu_{n\sigma(n)}$. Notice here that $\sum_j \mu_{ij} = \frac{1}{n}$, and hence $\sum_\sigma \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta = n^{-n}$. It is straightforward to check that α_θ given by (30) is equal to

$$\alpha_\theta = \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta (\theta_{0,\sigma}^{M,n} \times id) \# \mathcal{L}_{[0,1]}^1. \tag{31}$$

After we make such a choice, we apply the discrete procedure described in section 3 to $\{\theta(z_i)\}_{i=1}^n$, and $\{\theta_{\sigma,i}^{M,n} - \theta(z_i)\}_{i=1}^n$, with time step size $\delta t = \frac{1}{nC_4'}$; here C_4' is given by Theorem 4.1 in order for all the estimates in that theorem to hold. Notice that $\theta(z_i) \le \theta(z_{i+1})$, and $\theta(z_i) \ge \Theta(\theta_{\sigma,i}^{M,n}, z_i, 0)$, hence satisfying the assumptions made at the beginning of section 2. We denote $\{\theta_{\sigma,j}^n(k\delta t)\}_{1 \le j \le n, 0 \le k \le \frac{T}{\delta t}}$, $\{\theta_{\sigma,j}^{M,n}(k\delta t)\}_{1 \le j \le n, 0 \le k \le \frac{T}{\delta t}}$ to be the discrete solutions constructed according to section 2 and adopt similar notation, but here with dependence on σ. Define

$$\theta_0^n(z) = \sum_i \theta(z_i) \chi_{J_i}, \tag{32}$$

$$\theta_\sigma^n(t, z) = \theta_{\sigma,j}^n(k\delta t), \tag{33}$$

$$\theta_\sigma^{M,n}(t, z) = \theta_{\sigma,j}^{M,n}(k\delta t), \tag{34}$$

$$\bar{\theta}_\sigma^n(t, z) = \frac{(k+1)\delta t - t}{\delta t} \theta_{\sigma,j}^n(k\delta t) + \frac{t - k\delta t}{\delta t} \theta_{\sigma,j}^n((k+1)\delta t) \tag{35}$$

for $k\delta t \le t < (k+1)\delta t$ and $z \in J_j$.

Denote $F_\sigma^n(t, z)$ to be the discrete flow map constructed in section 3. Since we know $\theta \in Y_{B_2}$, we see $||\theta_0^n||_{L^\infty} \le B_2$, and $||\theta_{0,\sigma}^{M,n}||_{L^\infty} \le M$. Hence $||\theta_0^n||_{L^\infty} + ||q_0^n||_{L^\infty} \le M + 2B_2$. We know from point (iii) of Theorem 4.1, as well as from the choice of the constant B_1 made in the paragraph before Definition 5.4, that $F_\sigma(\cdot, z) \in X_{B_1}$, namely, $TV_{t \in (0,T)}(F_\sigma^n(t, z)) \le B_1$. Similarly, we have $||\theta_\sigma^n||_{L^\infty} \le B_3$, according to point (i) of Theorem 4.1, as well as from the choice of B_3 made before Definition 5.4. In particular, for each fixed $t \in [0, T)$, it holds that $\bar{\theta}_\sigma^n(t, \cdot) \in Y_{B_3}$. Hence $\bar{\theta}_\sigma^n \in C([0, T); Y_{B_3}) = Y$.

Let $\Phi_\sigma^n : [0, 1] \to X_{B_1}$ be defined by $\Phi_\sigma^n(z) = F_\sigma^n(\cdot, z)$. From the discussion in the previous paragraph, we know Φ_σ^n indeed maps $[0, 1]$ into the space X_{B_1} and is easily seen to be a Borel map.

Now we form the probability measure:

$$\lambda^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)}^\theta \cdots \mu_{n\sigma(n)}^\theta \big(\delta_{\bar{\theta}_\sigma^n} \times (\theta_{0,\sigma}^{M,n} \times \mathrm{id} \times \Phi_\sigma^n) \# \mathcal{L}_{[0,1]}^1\big). \tag{36}$$

Then we see $\lambda^n \in \mathcal{P}(Y \times \mathbb{R} \times [0, 1] \times X_{B_1})$, by our previous construction, and this will be our approximate solution. Following the notation in Definition 5.4, defining

$\eta_t^n = (e_t \times \mathrm{id} \times \mathrm{id} \times e_t')\#\lambda^n$, we have

$$\eta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \big(\delta_{\bar\theta^n_\sigma(t)} \times (\theta^{M,n}_{0,\sigma} \times \mathrm{id} \times F^n_{\sigma,t})\#\mathcal{L}^1_{[0,1]}\big). \tag{37}$$

In order to take the limit, we need to show that λ^n is tight. For this, we just need to show that $\pi_k\#\lambda^n$ is tight for $k = 1, 2, 3, 4$, where π_k is the projection map onto the kth component. Indeed, we have

$$\pi_1\#\lambda^n = \int_{Y_C} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \delta_{\theta^n_\sigma}.$$

Because of point (v) in Theorem 4.1, we see that for any $\sigma \in S_\theta$ and $s, t \in [0, T)$, with $t - s > \frac{\delta t}{2}$ we have

$$||\theta^n_\sigma(s, \cdot) - \theta^n_\sigma(t, \cdot)||_{L^1} \leq C_5 \sqrt{3(t - s)}.$$

The constant C_5 is given by point (v) of Theorem 4.1 when we have the bound $||\theta^n_0||_{L^\infty} + ||q^n_0||_{L^\infty} \leq M + 2B_2$. Recalling (35), it follows that $\bar\theta^n_\sigma$ satisfies the same estimate, but without the restriction $t - s > \frac{\delta t}{2}$. Therefore, $\pi_1\#\lambda^n$ is concentrated on the compact set:

$$F = \{f \in Y : ||f(t, \cdot) - f(s, \cdot)||_{L^2} \leq C_5 \sqrt{3(t - s)} \text{ for any } s < t\}.$$

That this is a compact set follows from the Arzelà–Ascoli theorem and the compactness of the space Y_{B_3}. Note that the functions in Y has bound in L^∞, so convergence in L^p are equivalent for any $p < \infty$.

As for $\pi_2\#\lambda^n$, since for all $\sigma \in S_\theta$, we know from the construction of discrete solutions that $||\theta^{M,n}_{0,\sigma}||_{L^\infty} \leq M$, hence $\operatorname{supp} \pi_2\#\lambda^n \subset [-M, M]$ and is therefore tight.

There is nothing to prove for $\pi_3\#\lambda^n$ and $\pi_4\#\lambda^n$, as $[0, 1]$ and X_{B_1} are compact spaces.

So up to extracting a subsequence, we can pass to the limit for the measure defined in (36), and we denote a limit measure to be $\lambda \in \mathcal{P}(Y \times \mathbb{R} \times [0, 1] \times X_{B_1})$. It only remains to show λ is a solution. Finally we record a lemma which will be useful in the next subsection.

Lemma 6.3. *Define $\eta^n_t = (e'_t \times \mathrm{id} \times \mathrm{id} \times e_t)\#\lambda^n$. Then $\eta^n_t \to \eta_t$ for any $t \in (0, T)$ as $n \to \infty$.*

Proof. First we remind the reader that this is not completely obvious as the evaluation map e_t in the last component is not continuous. Instead, we will use the continuity estimates in point (vi) of Theorem 4.1.

For $\gamma \in X_{B_1}$, extend $\gamma(t) = \gamma(0+)$ for $t \leq 0$, and define the operator: $A'_\varepsilon : X_{B_1} \to X_{B_1}$, given by $\gamma \longmapsto \varepsilon^{-1} \int_{t-\varepsilon}^t \gamma(s)ds$. Then it is straightforward to check that A'_ε is a continuous map for each fixed ε, and $A_\varepsilon(\gamma) \to \gamma$ for each fixed γ as $\varepsilon \to 0$.

Let $f \in C_b(Y_{B_3} \times \mathbb{R} \times [0, 1] \times [0, 1])$ and be 1-Lipschitz. Then we can compute

$$\begin{aligned}
&\int f(\theta, s, z, z') d\eta^n_t(\theta, s, z, z') = \int f(\tilde\theta_t, s, z, \gamma_t) d\lambda^n(\tilde\theta, s, z, \gamma) \\
&= \int (f(\tilde\theta_t, s, z, \gamma_t) - f(\tilde\theta_t, s, z, (A'_\varepsilon(\gamma))_t)) d\lambda^n(\tilde\theta, s, z, \gamma) \\
&\quad + \int f(\tilde\theta_t, s, z, (A_\varepsilon(\gamma))_t) d\lambda^n(\tilde\theta, s, z, \gamma).
\end{aligned} \tag{38}$$

Now observe that for each fixed $\varepsilon > 0$, the map $(\tilde{\theta}, s, z, \gamma) \longmapsto (\tilde{\theta}_t, s, z, (A'_\varepsilon(\gamma))_t)$ is continuous. Hence for each fixed $\varepsilon > 0$, the following convergence holds as $n \to \infty$:

$$\int f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t) d\lambda^n(\tilde{\theta}, s, z, \gamma) \to \int f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t) d\lambda(\tilde{\theta}, s, z, \gamma). \tag{39}$$

To estimate the first term of (38), observe

(40)
$$\begin{aligned}
&\left| \int (f(\tilde{\theta}_t, s, z, \gamma_t) - f(\tilde{\theta}_t, s, z, (A_\varepsilon(\gamma))_t)) d\lambda^n(\tilde{\theta}, s, z, \gamma) \right| \leq \int |\gamma_t - (A_\varepsilon(\gamma))_t| d\lambda^n(\tilde{\theta}, s, z, \gamma) \\
&\leq \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{[0,1]} |F^n_{\sigma,t}(z) - (A_\varepsilon F^n_{\sigma,\cdot}(z))(t)| dz \\
&\leq \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \cdot \varepsilon^{-1} \int_{t-\varepsilon}^t \int_{[0,1]} |F^n_{\sigma,t}(z) - F^n_{\sigma,s}(z)| dz ds \leq C_6 \sqrt{\varepsilon + \delta t}.
\end{aligned}$$

In the last inequality, we used point (vi) of Theorem 4.1.

Combining (38)–(40), it follows that

$$\limsup_{n \to \infty} \left| \int f(\theta, s, z, z') d(\eta^n_t - \eta_t) \right| \leq C_6 \sqrt{\varepsilon} + \int |\gamma_t - (A_\varepsilon(\gamma))_t| d\lambda(\tilde{\theta}, s, z, \gamma).$$

Now using the bounded convergence theorem, we can conclude that the integral on the right-hand side tends to zero as $\varepsilon \to 0$ since $(A_\varepsilon(\gamma))_t \to \gamma_t$ as $\varepsilon \to 0$ and γ is left continuous. ∎

6.3. The limit is a solution. In this section, we will show that the limit λ obtained in the previous subsection is a measure-valued solution. Some preparations are needed before we proceed.

For any $B > 0$, we may define the "averaging" operator: $A_\varepsilon : Y_B \to Y_B$, given by $A_\varepsilon(\theta)(z) = \varepsilon^{-1} \int_z^{z+\varepsilon} \theta(w) dw$. Here we extended the definition of θ so that $\theta(z) = B$ for $z \geq 1$. It is clear that for any θ_1, $\theta_2 \in Y_B$, one has $||A_\varepsilon(\theta_1) - A_\varepsilon(\theta_2)||_{L^2} \leq ||\theta_1 - \theta_2||_{L^2}$. Hence A_ε is a continuous map for each fixed $\varepsilon > 0$. Also it is clear that for any $\theta \in Y_B$, we have $A_\varepsilon \theta \to \theta$ in Y_B as $\varepsilon \to 0$. We can define a map $A_\varepsilon : Y \to Y$ by the same formula, namely, $A_\varepsilon(\tilde{\theta})(t, z) = \varepsilon^{-1} \int_z^{z+\varepsilon} \tilde{\theta}(w, t) dw$. Also one can check that $A_\varepsilon : Y \to Y$ is a continuous map for each $\varepsilon \to 0$, and $A_\varepsilon(\tilde{\theta}) \to \tilde{\theta}$ in Y as $\varepsilon \to 0$.

Now, choosing $B = B_3$, we prove the following estimate about A_ε.

Lemma 6.4. *Let $\varepsilon > 0$. For any $n \geq 1$ and any $t \in (0, T)$, the following holds:*

$$\int_{Y \times X_{B_1}} |\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t)| d\pi_{14} \# \lambda^n(\theta, \gamma) \leq B_3 \varepsilon.$$

Proof. From the definition of the measure λ^n, we can calculate

$$\begin{aligned}
&\int_{Y \times X_{B_1}} |\tilde{\theta}_t(\gamma_t) - (A_\varepsilon \tilde{\theta})_t(\gamma_t)| d\pi_{14} \# \lambda^n(\tilde{\theta}, \gamma) dt \\
&= \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{[0,1]} |\bar{\theta}^n_{\sigma,t}(F^n_{\sigma,t}(z)) - (A_\varepsilon \bar{\theta}^n_\sigma)_t(F^n_{\sigma,t}(z))| dz \\
&= \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{[0,1]} |\bar{\theta}^n_{\sigma,t}(z) - (A_\varepsilon \bar{\theta}^n_\sigma)_t(z)| dz.
\end{aligned}$$

In the last equality, we used the measure-preserving property of $F^n_{\sigma,t}$. For each fixed θ, σ, we can estimate

$$\int_{[0,1]} |\bar\theta^n_{\sigma,t}(z) - (A_\varepsilon \bar\theta^n_\sigma)_t(z)| dz = \varepsilon^{-1} \int_0^\varepsilon ds \int_{[0,1]} \bar\theta^n_{\sigma,t}(z+s) - \bar\theta^n_{\sigma,t}(z) dz$$
$$= \varepsilon^{-1} \int_0^\varepsilon ds \left[\int_1^{1+s} \bar\theta^n_{\sigma,t}(z) dz - \int_0^s \bar\theta^n_{\sigma,t}(z) dz \right] ds \le \varepsilon^{-1} \int_0^\varepsilon 2B_3 s ds \le B_3 \varepsilon.$$

In the first equality above, we used the monotonicity of $\bar\theta^n_\sigma$ in z. ∎

Next we will check the properties listed in Definition 5.4 one by one.

LEMMA 6.5. *Following the notation of Definition* 5.4, *we have that* $t \longmapsto \eta_t \in \mathcal{P}(Y_{B_3} \times \mathbb{R} \times [0,1] \times [0,1])$ *is narrowly continuous, and* $\zeta_t \to \zeta_0$, $\pi_{34}\#\eta_t \in \Gamma(\mathcal{L}^1_{[0,1]}, \mathcal{L}^1_{[0,1]}) \to (\mathrm{id} \times \mathrm{id})\#\mathcal{L}^1_{[0,1]}$ *as* $t \to 0$.

Proof. First we check the continuity. Let $f \in C_b(Y_{B_3} \times \mathbb{R} \times [0,1] \times [0,1])$ be 1-Lipschitz. We compute

$$\int f(\theta, s, z, z') d\eta^n_t(\theta, z, s, z')$$
$$= \int_{Y_{B_2}} d\mu_0(\theta) \Bigg[\sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int f(\bar\theta^n_\sigma(t), z, \theta^{M,n}_{0,\sigma}(z), F^n_{\sigma,t}(z)) dz \Bigg].$$

Now, choosing t, t', with $t' > t$, one has for each fixed $\theta \in Y_{B_2}$ and each $\sigma \in S_\theta$

$$\int |f(\bar\theta^n_\sigma(t), \theta^{M,n}_{0,\sigma}(z), z, F^n_{\sigma,t}(z)) - f(\bar\theta^n_\sigma(t'), \theta^{M,n}_{0,\sigma}(z), z, F^n_{\sigma,t'}(z))| dz$$
$$\le ||\bar\theta^n_\sigma(t) - \bar\theta^n_\sigma(t')||_{L^2([0,1])} + \int |F^n_{\sigma,t}(z) - F^n_{\sigma,t'}(z)| dz \le (C_5 B_3 + C_6)\sqrt{t' - t + \delta t}.$$

Here the constants C_5 and C_6 are given by the points (v) and (vi) of Theorem 4.1. Now since $\sum_{\sigma \in S} n^n \mu_{1\sigma(1)} \cdots \mu_{n\sigma(n)} = 1$, we obtain the following for any $t < t'$:

$$\left| \int f(\theta, s, z, z') d(\eta^n_t - \eta^n_{t'})(\theta, s, z, z') \right| \le (C_5 B_3 + C_6)\sqrt{t' - t + \delta t}. \tag{41}$$

The continuity now follows by sending $n \to \infty$ in (41) and using Lemma 6.3. To show that $\zeta_t \to \zeta_0$, we show that for each $f \in C_b(Y_{B_3} \times \mathbb{R} \times [0,1])$, and 1-Lipschitz, one has $\sup_n |\int f d\zeta^n_t - \int f \zeta^n_0| \le C\sqrt{t}$, with C universal. Indeed we calculate

$$\int f(\theta, s, z') d\zeta^n_t(\theta, s, z') = \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int f(\bar\theta^n_\sigma(t), \theta^{M,n}_{0,\sigma}(z), F^n_{\sigma,t}(z)) dz.$$

On the other hand, from (31), we know that

$$\int f(\theta, s, z') d\zeta^n_0(\theta, s, z') = \int d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int f(\theta^n_0, \theta^{M,n}_{0,\sigma}(z), z) dz.$$

Using points (v) and (vi) in Theorem 4.1 once more, we see that for any choice of $\theta \in Y_{B_2}$, $s \in \mathbb{R}$, and $\sigma \in S_\theta$ we have

$$\int_{[0,1]} |f(\theta^n_0, s, z) - f(\bar\theta^n_\sigma(t), s, F^n_{\sigma,t}(z))| dz \le ||\theta^n_0 - \bar\theta^n_{\sigma,t}||_{L^2} + \int_{[0,1]} |z - F^n_{\sigma,t}(z) dz|$$
$$\le (C_5 B_3 + C_6)\sqrt{t + \delta t}.$$

Then we can proceed in a similar way as before. That $\pi_{34}\#\eta_t^n \in \Gamma(\mathcal{L}^1_{[0,1]}, \mathcal{L}^1_{[0,1]})$ follows readily from (37). The convergence for $\pi_{34}\#\eta_t^n$ is similar. ∎

Next we check point (ii) of Definition 5.4.

LEMMA 6.6. *For any $t \in (0,T)$, $\pi_{13}\#\zeta_t = \mu_t \times \mathcal{L}^1_{[0,1]}$, $\pi_2\#\zeta_t$ has compact support. In addition, $\theta(z) \geq \Theta(s,z,t)$ for ζ_t-a.e. (θ,s,z).*

Proof. From (37) we conclude

$$\zeta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \delta_{\bar{\theta}^n_\sigma(t)} \times (\theta^{M,n}_{0,\sigma} \times F^n_{\sigma,t})\#\mathcal{L}^1_{[0,1]}. \tag{42}$$

Hence we conclude

$$\pi_{13}\#\zeta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \delta_{\bar{\theta}^n_\sigma(t)} \times \left(F^n_{\sigma,t}\#\mathcal{L}^1_{[0,1]}\right) = \mu_t^n \times \mathcal{L}^1_{[0,1]}. \tag{43}$$

In the above, we used the measure-preserving property of the map $F^n_{\sigma,t}$, and here $\mu_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \delta_{\bar{\theta}^n_\sigma(t)} \in \mathcal{P}(Y_{C_1})$. Lemma 6.3 implies $\zeta_t^n \to \zeta_t$. On the other hand, we may assume $\mu_t^n \to \mu_t$. Passing to the limit in (43), we see $\pi_{13}\#\zeta_t = \mu_t \times \mathcal{L}^1_{[0,1]}$.

Now from (42), one also calculates

$$\pi_2\#\zeta_t^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \theta^{M,n}_{0,\sigma}\#\mathcal{L}^1_{[0,1]}.$$

From the construction given in the last subsection, we have $||\theta^{M,n}_{0,\sigma}||_{L^\infty} \leq M$, and hence supp $\pi_2\#\zeta_t^n \subset [-M,M]$. Passing to the limit, the same will hold for $\pi_2\#\zeta_t$.

It remains to check that $\theta(z) \geq \Theta(s,z,t)$ for ζ_t-a.e. (θ,s,z). It suffices to show $\int(\theta(z)-\Theta(s,z,t))^- d\zeta_t(\theta,s,z) = 0$. Now we calculate

$$\begin{aligned}
&\int_{Y_{B_3}\times\mathbb{R}\times[0,1]} (\theta(z)-\Theta(s,z,t)) d\zeta_t^n(\theta,s,z)\\
&= \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{[0,1]} (\bar{\theta}^n_\sigma(t, F^n_{\sigma,t}(z)) - \Theta(\theta^{M,n}_\sigma(z), F^n_{\sigma,t}(z), t))^- dz\\
&\leq \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{[0,1]} |\bar{\theta}^n_\sigma(t, F^n_{\sigma,t}(z)) - \theta^n_\sigma(t, F^n_{\sigma,t}(z))| dz + \sup|\partial_t\Theta|\delta t\\
&\leq (C_5 + \sup|\partial_t\Theta|)\sqrt{\delta t}.
\end{aligned}$$

In the first inequality above, we used that $\theta^n_\sigma(t, F^n_{\sigma,t}(z)) \geq \Theta(\theta^{M,n}_\sigma(z), F^n_{\sigma,t}(z), k\delta t)$, where $t \leq k\delta t < (k+1)\delta t$. This follows from point (iv) of Corollary 3.3. In the last inequality, we used point (v) of Theorem 4.1 and the measure-preserving property of the map $F^n_{\sigma,t}$. Passing to the limit as $n\to\infty$, the conclusion follows. ∎

Now we check point (iii) of Definition 5.4.

LEMMA 6.7. *Fixing $\varepsilon > 0$, define $I_\varepsilon : Y \times X_{B_1} \to \mathbb{R}$, given by*

$$(\tilde{\theta}, \gamma) \longmapsto \int_{(0,T)^2} ((A_\varepsilon\tilde{\theta})_{t_1}(\gamma_{t_1}) - (A_\varepsilon\tilde{\theta})_{t_2}(\gamma_{t_2}))^+ \chi_{t_1<t_2} dt_1 dt_2.$$

Then this function is continuous.

Proof. Letting $(\tilde{\theta}^k, \gamma^k) \to (\tilde{\theta}, \gamma)$ in $Y \times X_{B_1}$, we need to show that $I(\tilde{\theta}^k, \gamma^k) \to I(\tilde{\theta}, \gamma)$. We prove this by showing that for $\mathcal{L}^1$-a.e. $t \in (0,T)$ we have pointwise convergence: $(A_\varepsilon \tilde{\theta}^k)_t(\gamma^k) \to (A_\varepsilon \tilde{\theta})_t(\gamma_t)$. Then the desired convergence follows from the dominated convergence theorem.

By Helly's selection principle, we know that $\gamma_t^k \to \gamma_t$, except for a countable set of t. Since $|A_\varepsilon(\tilde{\theta}^k)_t(z) - A_\varepsilon(\tilde{\theta})_t(z)| \le \varepsilon^{-1/2}||\tilde{\theta}_t^k - \tilde{\theta}_t||_{L^2}$, we know that $A_\varepsilon(\tilde{\theta}^k)_t \to A_\varepsilon(\tilde{\theta})_t$ uniformly for each $t \in (0,T)$. Hence for any t with $\gamma^k(t) \to \gamma(t)$, it holds that $(A_\varepsilon \tilde{\theta}^k)_t(\gamma_t^k) \to (A_\varepsilon \tilde{\theta})_t(\gamma_t)$. It follows that $((A_\varepsilon(\tilde{\theta}^k))_{t_1}(\gamma_{t_1}^k) - (A_\varepsilon \tilde{\theta}^k)_{t_2}(\gamma_{t_2}^k))^+ \to ((A_\varepsilon(\tilde{\theta}))_{t_1}(\gamma_{t_1}) - (A_\varepsilon \tilde{\theta})_{t_2}(\gamma_{t_2}))^+$ for $\mathcal{L}^2$-a.e. (t_1, t_2). Then the result follows from the bounded convergence theorem. ∎

LEMMA 6.8. *For any n, the following estimate holds:*

$$\int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{14}\#\lambda^n(\theta, \gamma) \le 2C_5 T^2 \sqrt{\delta t}.$$

Here C_5 is the universal constant given in point (v) *of Theorem* 4.1.

Proof. According to the definition of λ^n, we find

$$\begin{aligned}
&\int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{14}\#\lambda^n(\theta, \gamma) \\
&= \int_{Y_{B_2}} \mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{(0,T)^2} \int_{[0,1]} (\bar{\theta}^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z)) - \bar{\theta}^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)))^+ \\
&\quad \chi_{t_1 < t_2} dt_1 dt_2 dz.
\end{aligned}$$

For each fixed σ and θ, it holds that

$$\begin{aligned}
&\int_{(0,T)^2} \int_{[0,1]} (\bar{\theta}^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z)) - \bar{\theta}^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)))^+ \chi_{t_1 < t_2} dt_1 dt_2 dz \\
&\le 2T \int_0^T \int_{[0,1]} |\bar{\theta}^n_{\sigma,t}(F^n_{\sigma,t}(z)) - \theta^n_{\sigma,t}(F^n_{\sigma,t}(z))| dt dz \\
&+ \int_{(0,T)^2} \int_{[0,1]} (\theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z)) - \theta^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)))^+ \chi_{t_1 < t_2} dt_1 dt_2 dz \\
&= 2T \int_0^T \int_{[0,1]} |\bar{\theta}^n_{\sigma,t}(z) - \theta^n_{\sigma,t}(z)| dt dz \le 2T^2 C_5 \sqrt{\delta t}.
\end{aligned}$$

In the above calculation, we used point (iii) of Corollary 3.3; hence for any $\sigma \in S_\theta$, $\theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}) \le \theta^n_{\sigma,t_2}(F^n_{\sigma,t_2})$ for any $t_1 < t_2$. When estimating $\bar{\theta}^n_\sigma - \theta^n_\sigma$, we used (35) and point (v) of Theorem 4.1. ∎

LEMMA 6.9. *For λ-a.e. $(\tilde{\theta}, s, z, \gamma) \in Y \times \mathbb{R} \times [0,1] \times X_{B_1}$, we have*

$$\tilde{\theta}_{t_1}(\gamma_{t_1}) \le \tilde{\theta}_{t_2}(\gamma_{t_2}) \text{ for } \mathcal{L}^2\text{-a.e. } (t_1, t_2) \in (0,T)^2 \text{ with } t_1 < t_2.$$

Proof. It suffices to show that

$$\int_{(0,T)^2} \int_{Y \times X_{B_1}} (\tilde{\theta}_{t_1}(\gamma_{t_1}) - \tilde{\theta}_{t_2}(\gamma_{t_2}))^+ \chi_{t_1 < t_2} d\mathcal{L}^2(t_1, t_2) d\pi_{1,4}\#\lambda(\tilde{\theta}, \gamma) = 0. \tag{44}$$

For any $\varepsilon > 0$, we can write

$$\begin{aligned}
&\int_{(0,T)^2}\int_{Y\times X_{B_1}} (\tilde\theta_{t_1}(\gamma_{t_1}) - \tilde\theta_{t_2}(\gamma_{t_2}))^+ \chi_{t_1<t_2} d\mathcal{L}^2(t_1,t_2) d\pi_{1,4}\#\lambda(\tilde\theta,\gamma)\\
&\le 2T\int_0^T \int_{Y\times X_{B_1}} (\tilde\theta_t(\gamma_t) - (A_\varepsilon\tilde\theta)_t(\gamma_t))^+ dt d\pi_{14}\#\lambda(\tilde\theta,\gamma)\\
&+\int_{(0,T)^2}\int_{Y\times X_{B_1}} ((A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) - (A_\varepsilon\tilde\theta)_{t_2}(\gamma_{t_2}))^+ \chi_{t_1<t_2} d\mathcal{L}^2(t_1,t_2) d\pi_{14}\#\lambda(\tilde\theta,\gamma).
\end{aligned}$$

For the first term, it goes to zero as $\varepsilon \to 0$, since the integrand tends to 0 for each fixed $(t, \tilde\theta, \gamma)$ and is clearly bounded. For the second term, we estimate

$$\begin{aligned}
&\int_{(0,T)^2}\int_{Y\times X_{B_1}} ((A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) - (A_\varepsilon\tilde\theta)_{t_2}(\gamma_{t_2}))^+ \chi_{t_1<t_2} d\mathcal{L}^2(t_1,t_2) d\pi_{14}\#\lambda^n(\tilde\theta,\gamma)\\
&\le 2T\int_0^T \int_{Y\times X_{B_1}} |\tilde\theta_t(\gamma_t) - (A_\varepsilon\tilde\theta)_t(\gamma_t)| dt d\pi_{14}\#\lambda^n(\tilde\theta,\gamma)\\
&\quad + 2C_5T^2\sqrt{\delta t} \le 2B_3T^2\varepsilon + 2C_5T^2\sqrt{\delta t}.
\end{aligned}$$

The last inequality follows from Lemma 6.4, while the first inequality used Lemma 6.8. Now, sending $n \to \infty$, Lemma 6.7 allows us to conclude the following:

$$\int_{(0,T)^2}\int_{Y\times X_{B_1}} ((A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) - (A_\varepsilon\tilde\theta)_{t_2}(\gamma_{t_2}))^+ \chi_{t_1<t_2} d\mathcal{L}^2(t_1,t_2) d\pi_{14}\#\lambda(\tilde\theta,\gamma) \le 2B_3T^2\varepsilon.$$

The proof is completed by sending $\varepsilon \to 0$. ∎

Up to now, we have checked points (i)–(iii) in Definition 5.4. It only remains to check point (iv). Our first goal will be to show that the measure $\partial_t(\tilde\theta_t(\gamma_t))$ is concentrated on the "wet" set. As preparation, we prove the following.

Lemma 6.10. *For any ε, ε_1, $\varepsilon_2 > 0$, define the function $K_i : Y \times \mathbb{R} \times X_{B_1} \to \mathbb{R}$, $i = 1, 2$, given by*

$$\begin{aligned}
&K_1(\tilde\theta, s, \gamma)\\
&= \int_{(0,T)^2} \chi_{\{0<t_2-t_1<\varepsilon_1\}} |(A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) - (A_\varepsilon\tilde\theta)_{t_2}(\gamma_{t_2})| \chi_{\{(A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+\varepsilon_2\}} dt_1 dt_2,\\
&K_2(\tilde\theta, s, \gamma) = \int_{(0,T)^2} \chi_{\{0<t_2-t_1<\varepsilon_1\}} (\gamma_{t_2} - \gamma_{t_1})^+ \chi_{\{(A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+\varepsilon_2\}}.
\end{aligned}$$

Then K_i is lower semicontinuous for $i = 1, 2$.

Proof. Letting $(\tilde\theta^k, s^k, \gamma^k) \to (\tilde\theta, s, \gamma)$, we need to show that $K_i(\tilde\theta, s, \gamma) \le \liminf_{k\to\infty} K_i(\tilde\theta^k, s^k, \gamma^k)$.

As explained in the proof of Lemma 6.7, for any t such that $\gamma^k(t) \to \gamma(t)$, we have $(A_\varepsilon\tilde\theta^k)_t(\gamma^k_t) \to (A_\varepsilon\tilde\theta)_t(\gamma_t)$. Hence for such t

$$\chi_{\{(A_\varepsilon\tilde\theta)_t(\gamma_t) > \Theta(s,\gamma_t,t)+\varepsilon_2\}} \le \liminf_{k\to\infty} \chi_{\{(A_\varepsilon\tilde\theta^k)_t(\gamma^k_t) > \Theta(s,\gamma^k_t,t)+\varepsilon_2\}}.$$

The integrand is lower semicontinuous with respect to $(\tilde\theta, s, \gamma)$ if we fix (t_1, t_2) such that $\gamma^k_{t_i} \to \gamma_{t_i}$, $i = 1, 2$. Then the lower semicontinuity of J_i follows by applying Fatou's lemma. ∎

LEMMA 6.11. *Let C_2 be the universal constant given by Theorem* 4.1(ii). *Then we have for any $\varepsilon > 0$*

$$\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+2\varepsilon\}} d\lambda(\tilde{\theta}, s, z, \gamma) = 0 \tag{45}$$

and

$$\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int (\gamma_{t_2} - \gamma_{t_1})^+ \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+2\varepsilon\}} d\lambda(\tilde{\theta}, s, z, \gamma) = 0. \tag{46}$$

Proof. We only prove (45). The proof of (46) follows along similar lines and is simpler. Fix $0 < \delta < \varepsilon$. Denote $\chi_{\delta,\varepsilon} = \chi_{\{(A_\delta\tilde{\theta})_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+\varepsilon\}}$. From the lower semicontinuity proved in the previous lemma, we conclude that

$$\begin{aligned} &\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |(A_\delta\tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\delta\tilde{\theta})_{t_1}(\gamma_{t_1})| \chi_{\delta,2\varepsilon} d\lambda(\theta, s, z, \gamma) \\ &\le \liminf_{n\to\infty} \int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |(A_\delta\tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\delta\tilde{\theta})_{t_1}(\gamma_{t_1})| \chi_{\delta,2\varepsilon} d\lambda^n(\tilde{\theta}, s, z, \gamma). \end{aligned} \tag{47}$$

We know the left-hand side of (47) will tend to the left-hand side of (45) as $\delta \to 0$. Next we estimate the right-hand side of (47):

$$\begin{aligned} &\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |(A_\delta\tilde{\theta})_{t_2}(\gamma_{t_2}) - (A_\delta\tilde{\theta})_{t_1}(\gamma_{t_1})| \chi_{\delta,2\varepsilon} d\lambda^n(\tilde{\theta}, s, z, \gamma) \\ &\le 2T \int_0^T \int_{Y\times X_{B_1}} |(A_\delta\tilde{\theta})_t(\gamma_t) - \tilde{\theta}_t(\gamma_t)| d\pi_{14}\#\lambda^n(\tilde{\theta}, \gamma) \\ &\quad + \int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \chi_{\delta,2\varepsilon} d\lambda^n(\tilde{\theta}, s, z, \gamma) \\ &\le 2B_3T^2\delta + \int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1})| \\ &\chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+1.5\varepsilon\}} d\lambda^n + B_3T \int_0^T dt_1 \int (\chi_{\delta,2\varepsilon} - \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1}) > \Theta(s,\gamma_{t_1},t_1)+1.5\varepsilon\}})^+ d\lambda^n. \end{aligned} \tag{48}$$

In the second inequality, we used Lemma 6.4. For the second term on the right-hand side above, we calculate the following:

$$\begin{aligned} &\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} dt_1 dt_2 \int |\tilde{\theta}_{t_2}(\gamma_{t_2}) - \tilde{\theta}_{t_1}(\gamma_{t_1}))| \chi_{\{\tilde{\theta}_{t_1}(\gamma_{t_1})) > \Theta(s,\gamma_{t_1},t_1)+1.5\varepsilon\}} d\lambda^n \\ &= \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma\in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{(0,T)^2} \int_0^1 \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}} |\bar{\theta}^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)) \\ &\quad - \bar{\theta}^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z))| \chi_{\{\bar{\theta}^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z)) > \Theta(\theta^{M,n}_{0,\sigma}(z), F^n_{\sigma,t_1}(z), t_1)+1.5\varepsilon\}} dt_1 dt_2 dz. \end{aligned} \tag{49}$$

We now define

$$\chi_1 = \chi_{\{\bar{\theta}^n_{\sigma,t_1}(F^n_{\sigma,t_1}) > \Theta(\theta^{M,n}_{0,\sigma}(z), F^n_{\sigma,t_1}(z), t_1)+1.5\varepsilon\}},$$

$$\chi_2 = \chi_{\{\theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}) > \Theta(\theta^{M,n}_{0,\sigma}(z), F^n_{\sigma,t_1}(z), t_1)+\varepsilon\}}.$$

We now estimate the right-hand side of (49). For any $\theta \in Y_{B_2}$ and $\sigma \in S_\theta$,

$$\begin{aligned}
&\int_{(0,T)^2}\int_0^1 \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}}|\bar\theta^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)) - \bar\theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z))|\chi_1 dt_1 dt_2 dz\\
&\le 2T\int_0^T\int_0^1 |\bar\theta^n_{\sigma,t}(F^n_{\sigma,t}(z)) - \theta^n_{\sigma,t}(F^n_{\sigma,t}(z))|dtdz\\
&\quad + \int_{(0,T)^2}\int_0^1 \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}}|\theta^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)) - \theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z))|\chi_1 dt_1 dt_2 dz\\
&\le 2T^2C_5\sqrt{\delta t} + \int_{(0,T)^2}\int_0^1 \chi_{\{0<t_2-t_1<\frac{\varepsilon}{C_2}\}}|\theta^n_{\sigma,t_2}(F^n_{\sigma,t_2}(z)) - \theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z))|\chi_2 dt_1 dt_2 dz\\
&\quad + 2B_1\int_{(0,T)^2}\int_0^1 (\chi_1-\chi_2)^+ dt_1 dt_2 dz.
\end{aligned}$$

In the last inequality, we used point (v) of Theorem 4.1. The second term on the right-hand side above is 0, due to point (iv) of Theorem 4.1. To estimate the last term, we notice that

$$\begin{aligned}
&\int_{(0,T)^2}\int_0^1 (\chi_1-\chi_2)^+ dt_1 dt_2 dz \le \int_{(0,T)^2}\int_0^1 \chi_{\{\bar\theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z)) - \theta^n_{\sigma,t_1}(F^n_{\sigma,t_1}(z))>0.5\varepsilon\}}\\
&\le \frac{2T}{\varepsilon}\int_0^T\int_0^1 |\bar\theta^n_{\sigma,t}(z) - \theta^n_{\sigma,t}(z)|dzdt \le \frac{2TC_5\sqrt{\delta t}}{\varepsilon}.
\end{aligned}$$

For the last term of (48), we have

$$\begin{aligned}
&\int_0^T dt_1\int (\chi_{\delta,2\varepsilon} - \chi_{\{\tilde\theta_{t_1}(\gamma_{t_1}))>\Theta(s,\gamma_{t_1},t_1)+\varepsilon\}})^+ d\lambda^n\\
&\le \int_0^T dt_1\int \chi_{\{|(A_\delta\tilde\theta)_{t_1}(\gamma_{t_1})-\tilde\theta_{t_1}(\gamma_{t_1})|\ge\varepsilon\}}d\lambda^n \le \varepsilon^{-1}\int_0^T dt_1\int |(A_\delta\tilde\theta)_{t_1}(\gamma_{t_1}) - \tilde\theta_{t_1}(\gamma_{t_1})|d\lambda^n\\
&\le \varepsilon^{-1}B_3T\delta.
\end{aligned}$$

In the last inequality, we use Lemma 6.4 again. Combining the calculations above, we obtain the left-hand side of (47) $\le \varepsilon^{-1}B_3T\delta + 2B_3T^2\delta$. The proof follows by sending $\delta \to 0$. ∎

By Remark 2.4 and Lemma 6.9, we know that for $\pi_{14}\#\lambda$-a.e. (θ,γ) one can determine a unique monotone increasing and left continuous function $(\theta(\gamma))^*$, which equals $\theta_t(\gamma_t)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$. We will simply denote this function by $\alpha(t)$ in the following lemma.

Lemma 6.12. *Let $\varepsilon > 0$. Then for λ-a.e. $(\tilde\theta, s, z, \gamma) \in Y \times X_{B_1}$, the following property holds:*

For any $t \in (0,T)$ such that $\alpha(t) > \Theta(s,\gamma_t,t)+\varepsilon$, it holds that $\alpha(t') = \alpha(t)$, and $\gamma_{t'} \le \gamma_t$ for any $0 < t'-t < \frac{\varepsilon}{2C_2}$. Here C_2 is the constant given by point (ii) *of Theorem* 4.1.

Proof. Let (θ,γ) be chosen so that the statement of Lemma 6.9 holds true, and so that we can define $\alpha(t)$. Let $(\tilde\theta, s, z, \gamma)$ also satisfy that

$$\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{2C}\}}|\tilde\theta_{t_2}(\gamma_{t_2}) - \tilde\theta_{t_1}(\gamma_{t_1})|\chi_{\{\tilde\theta_{t_1}(\gamma_{t_1})>\Theta(s,\gamma_{t_1},t_1)+\varepsilon\}}dt_1dt_2 = 0 \tag{50}$$

and

$$\int_{[0,T]^2} \chi_{\{0<t_2-t_1<\frac{\varepsilon}{2C}\}} dt_1 dt_2 \int (\gamma_{t_2}-\gamma_{t_1})^+ \chi_{\{\tilde\theta_{t_1}(\gamma_{t_1})>\Theta(s,\gamma_{t_1},t_1)+\varepsilon\}} = 0. \tag{51}$$

We know from Lemma 6.11 that (50) and (51) hold for λ-a.e. $(\tilde\theta, s, z, \gamma)$ and for λ-a.e. $(\tilde\theta, s, z, \gamma)$. Choosing some $(\tilde\theta, s, z, \gamma)$ so that (50) and (51) hold, then we have $\tilde\theta_{t_2}(\gamma_{t_2}) = \tilde\theta_{t_1}(\gamma_{t_1})$ and $\gamma_{t_2} \le \gamma_{t_1}$ for $\mathcal{L}^2$-a.e. (t_1, t_2) with $\tilde\theta_{t_1}(\gamma_{t_1}) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon$ and $0 < t_2 - t_1 < \frac{\varepsilon}{2C}$.

Fix any (t_1, t_2) with $0 < t_2 - t_1 < \frac{\varepsilon}{2C_2}$ and $\alpha(t_1) > \Theta(s, \gamma_{t_1}, t_1) + \varepsilon$. By left continuity, there exists $\delta > 0$, such that $\alpha(t_1') > \Theta(s, \gamma_{t_1'}, t_1') + \varepsilon$ for any $t_1' \in (t_1 - \delta, t_1)$. By Fubini's theorem, we know that for $\mathcal{L}^1$-a.e. $t_1' \in (t_1 - \delta, t_1)$ we have $\alpha(t_2') = \tilde\theta_{t_2'}(\gamma_{t_2'}) = \tilde\theta_{t_1'}(\gamma_{t_1'})$ and $\gamma_{t_2'} \le \gamma_{t_1'}$ for $\mathcal{L}^1$-a.e. $t_2' \in (t_1', t_1' + \frac{\varepsilon}{2C_2})$. By left continuity of $\alpha(t)$ and $\gamma(t)$, we conclude that $\alpha(t_2') = \tilde\theta_{t_1'}(\gamma_{t_1'})$ and $\gamma_{t_2'} \le \gamma_{t_1'}$ for *any* $t_2' \in (t_1', t_1' + \frac{\varepsilon}{2C_2})$. This is true for $\mathcal{L}^1$-a.e. $t_1' \in (t_1 - \delta, t_1)$. Hence we can find a sequence $\{t_1^n\}_{n=1}^\infty \subset (t_1 - \delta, t_1)$ such that this is true for t_1^n and $t_1^n \to t_1$ as $n \to \infty$. We can assume $\tilde\theta_{t_1^n}(\gamma_{t_1^n}) = \alpha(t_1^n)$ holds. Define $t_2^n = t_1^n + t_2 - t_1$; then $t_2^n \to t_2$, and we have $\alpha(t_2^n) = \alpha(t_1^n)$ and $\gamma_{t_2^n} \le \gamma_{t_1^n}$. Letting $n \to \infty$, and using left continuity of α and γ one more time, we can conclude that $\alpha(t_1) = \alpha(t_1)$ and $\gamma_{t_2} \le \gamma_{t_1}$. ∎

LEMMA 6.13. *For any $C > 0$, $\varepsilon > 0$, define the function $H_\varepsilon : Y \times \mathbb{R} \times X_{B_1} \to \mathbb{R}$, given by*

$$H_\varepsilon(\tilde\theta, s, \gamma) = \int_{(0,T)^2} \Big(|(A_\varepsilon\tilde\theta)_{t_2}(\gamma_{t_2}) - (A_\varepsilon\tilde\theta)_{t_1}(\gamma_{t_1}) - (\Theta(s,\gamma_{t_2},t_2) - \Theta(s,\gamma_{t_1},t_1))^+| - C(t_2-t_1) \Big)^+ \chi_{t_1<t_2} dt_1 dt_2.$$

Then H_ε is continuous.

Proof. The proof of this lemma is quite similar to that of Lemma 6.9. We already noted that $(\tilde\theta^k, \gamma^k) \to (\tilde\theta, \gamma)$ implies $(A_\varepsilon\tilde\theta^k)_t(\gamma_t^k) \to (A_\varepsilon\tilde\theta)_t(\gamma_t)$ for any t such that pointwise convergence of γ_t^k happens. The proof then follows from the dominated convergence since everything is bounded. ∎

LEMMA 6.14. *Let C_4 be the constant given in point* (iv) *of Theorem* 4.1. *Then we have*

$$\int_{(0,T)^2}\int \Big(|\tilde\theta_{t_2}(\gamma_{t_2}) - \tilde\theta_{t_1}(\gamma_{t_1})) - (\Theta(s,\gamma_{t_2},t_2) - \Theta(s,\gamma_{t_1},t_1))^+| - C_4(t_2-t_1) \Big)^+ \chi_{t_1<t_2} dt_1 dt_2 d\lambda = 0. \tag{52}$$

Proof. Write the left-hand side of (52) to be $\int H(\tilde\theta, s, \gamma) d\lambda$, with the definition of H similar to H_ε in the last lemma (without A_ε). Then we can estimate

$$\int_{Y\times\mathbb{R}\times[0,1]\times X_{B_1}} H(\tilde\theta, s, \gamma) d\lambda(\tilde\theta, s, z, \gamma) = \int (H - H_\varepsilon)(\tilde\theta, s, \gamma) d\lambda + \lim_{n\to\infty} \int H_\varepsilon(\tilde\theta, s, \gamma) d\lambda^n.$$

The first term will tend to zero as $\varepsilon \to 0$. For the second term, we have

$$\begin{aligned}\int H_\varepsilon(\tilde\theta, s, \gamma) d\lambda^n &\le \int |H_\varepsilon - H| d\lambda^n + \int H(\tilde\theta, s, \gamma) d\lambda^n \\ &\le 2T \int_0^T \int |\tilde\theta_t(\gamma_t) - (A_\varepsilon \tilde\theta)_t(\gamma_t)| d\lambda^n + \int H(\tilde\theta, s, \gamma) d\lambda^n \\ &\le 2T^2 B_3 \varepsilon + \int H(\tilde\theta, s, \gamma) d\lambda^n.\end{aligned}$$

In the last inequality above, we used Lemma 6.4. To deal with the remaining term, first we can write

$$\int H(\tilde\theta, s, \gamma) d\lambda^n = \int_{Y_{B_2}} d\mu_0(\theta) \sum_{\sigma \in S_\theta} n^n \mu^\theta_{1\sigma(1)} \cdots \mu^\theta_{n\sigma(n)} \int_{[0,1]} H(\bar\theta^n_\sigma, \theta^{M,n}_{0,\sigma}(z), F^n_\sigma(z)) dz.$$

Fixing some $\theta \in Y_{B_2}$ and $\sigma \in S_\theta$, we can calculate

$$\begin{aligned}\int H(\bar\theta^n_\sigma, \theta^{M,n}_\sigma(z), F^n_\sigma(z)) dz &\le 2T \int_0^T \int_{[0,1]} |\bar\theta^n_{\sigma,t}(F^n_{\sigma,t}(z)) - \theta^n_{\sigma,t}(F^n_{\sigma,t}(z))| dt dz \\ &+ \int H(\theta^n_\sigma, \theta^{M,n}_\sigma(z), F^n_\sigma(z)) dz \le 2T^2 C_5 \sqrt{\delta t} + T^2 C_4 \delta t.\end{aligned}$$

In the second inequality above, we used points (iv) and (v) of Theorem 4.1, and that $F^n_{\sigma,t}$ is measure preserving.

Now the proof is finished by first letting $n \to \infty$ and then letting $\varepsilon \to 0$. ∎

As a corollary, we deduce the following.

Corollary 6.15. *For λ-a.e. $(\tilde\theta, s, z, \gamma)$, it holds that*

$$|\alpha(t_2) - \alpha(t_1) - (\Theta(s, \gamma_{t_2}, t_2) - \Theta(s, \gamma_{t_1}, t_1))^+| \le C_4(t_2 - t_1)$$

for any $0 < t_1 < t_2 < T$. Here C_4 is the constant given in point (iv) *of Theorem* 4.1.

With above preparation, we can check point (iv) of Theorem 5.4.

Proposition 6.16. *For λ-a.e. $(\tilde\theta, s, z, \gamma)$, we have the equality of measures*

$$\partial_t(\theta_t(\gamma_t)) = \left[\partial_t(Q^{sat}(\theta_t(\gamma_t), \gamma_t, t))\right]^- \lfloor E_{\theta,s,\gamma}, \tag{53}$$

where $E_{\theta,s,\gamma}$ is the wet set given by

$$E_{\theta,s,\gamma} = \{t \in (0,T) : (\theta(\gamma))^*(t) = s - Q^{sat}((\theta(\gamma))^*(t), \gamma_t, t)\}.$$

Proof. We choose $(\tilde\theta, s, z, \gamma)$ such that the statements of Lemmas 6.9 and 6.12 and Corollary 6.15 hold. The plan is to apply Lemma 7.1 to the functions $f(t) = \alpha(t)$ and $g(t) = s - Q^{sat}(\alpha(t), \gamma(t), t)$. Here $\alpha(t)$ is the monotone increasing and left continuous version of $\theta_t(\gamma_t)$ chosen according to Remark 2.4. This is possible since Lemma 6.9 holds.

First we verify that for λ-a.e. $(\tilde\theta, s, z, \gamma)$, it holds that $\alpha(t) \ge s - Q^{sat}(\alpha(t), \gamma(t), t)$ for any $t \in (0,T)$. This is the same as $\alpha(t) \ge \Theta(s, \gamma(t), t)$. This follows from Lemma 6.6. Indeed, we have shown there that for any $t \in (0,T)$, $\theta(z') \ge \Theta(s, z', t)$ for ζ_t-a.e. (θ, s, z'). Recalling the definition of ζ_t, this is the same as saying that for any fixed $t \in (0,T)$, $\theta_t(\gamma_t) \ge \Theta(s, \gamma_t, t)$ for λ-a.e. $(\tilde\theta, s, z, \gamma)$. Now choose a countable dense subset D of the set $\{t \in (0,T) : \alpha(t) = \theta_t(\gamma_t)\}$. Then $D \subset (0,T)$ is also dense. Then for λ-a.e. $(\tilde{\theta}, s, z, \gamma)$, it holds that $\alpha(t_i) \geq \Theta(s, \gamma_{t_i}, t_i)$ for any $t_i \in D$. Since D is dense and both $\alpha(t)$, $\gamma(t)$ continuous from the left, we see it is true for all $t \in (0,T)$.

Next we verify points (i)–(iii) of Lemma 7.1.

Point (i) follows from Lemma 6.12. Indeed, if $\alpha(t) > s - Q^{sat}(\alpha(t), \gamma(t), t) + \varepsilon$, then we know $\alpha(t) > \Theta(s, \gamma(t), t) + \frac{\varepsilon}{C_7}$ for some universal constant C_7. Now Lemma 6.12 implies that $\alpha(t') = \alpha(t)$, $\gamma(t') \leq \gamma(t)$ for any $0 < t' - t < \frac{\varepsilon}{C_8}$. Hence $\alpha(t') = \alpha(t) > s - Q^{sat}(\alpha(t'), \gamma(t'), t') - \sup|\partial_t Q^{sat}|(t'-t) + \varepsilon$. If we still have $\varepsilon > 2\sup|\partial_t Q^{sat}|(t'-t)$, then we have $f(t') > g(t')$. Hence $f(t') = f(t)$ and $f(t') > g(t')$, as long as $0 < t' - t < \frac{\varepsilon}{C_8 + 2\sup|\partial_t Q^{sat}|}$. This verifies point (i).

Next we verify point (ii). This follows from Corollary 6.15. Indeed, from that corollary, we can deduce $\alpha(t^+) - \alpha(t) = (\Theta(s, \gamma(t^+), t) - \Theta(s, \gamma(t), t))^+$ by fixing any $t_1 = t$ and $t_2 \searrow t$. If $\alpha(t^+) = \alpha(t)$, we conclude that $\Theta(s, \gamma(t^+), t) \leq \Theta(s, \gamma(t), t)$. Therefore $\gamma(t^+) \leq \gamma(t)$ by strict monotonicity of Θ in the z variable. Then $Q^{sat}(\alpha(t^+), \gamma(t^+), t) \geq Q^{sat}(\alpha(t), \gamma(t), t)$. If $\alpha(t^+) > \alpha(t)$, we must have $\alpha(t) = \Theta(s, \gamma(t), t)$ or $\alpha(t) = s - Q^{sat}(\alpha(t), \gamma(t), t)$; otherwise it contradicts Lemma 6.12. But then $\alpha(t^+) = \Theta(s, \gamma(t^+), t)$. Hence $\alpha(t^+) = s - Q^{sat}(\alpha(t^+), \gamma(t^+), t)$. In any case, we have $f(t^+) - f(t) = (g(t^+) - g(t))^+$. This verifies point (ii).

Point (iii) again follows from Lemma 6.12. Indeed, from the monotonicity of Q^{sat}, we just need to show that for any $[a,b) \subset \{f > g\}$ it holds that $\alpha(t_2) = \alpha(t_1)$ and $\gamma(t_2) \leq \gamma(t_1)$ for any $t_1 < t_2$, $t_1, t_2 \in [a,b)$. Let $t^* = \sup\{t \in [t_1, t_2] : \alpha(t') = \alpha(t_1), \gamma(t') \leq \gamma(t_1)$ for any $t' \leq t\}$. Then we must have $t^* = t_2$. Otherwise, since $f(t^*) > g(t^*)$, Lemma 6.12 allows us to push beyond t_*, giving a contradiction. ∎

7. Appendix.

LEMMA 7.1. *Let $f : (0,T) \to \mathbb{R}$ be monotone increasing, and let $g : (0,T) \to \mathbb{R} \in BV((0,T))$, both continuous from the left and bounded, with $f(t) \geq g(t)$ for all $t \in (0,T)$. Suppose that for some constant $C > 0$ the following hold:*

(i) *$f(t') \equiv f(t)$, $f(t') > g(t')$ for any $t \in (0,T)$ with $f(t) > g(t) + \varepsilon$ and any t' with $t' - t < \frac{\varepsilon}{C}$.*

(ii) *$f(t^+) - f(t) = [g(t^+) - g(t)]^+$ for any $t \in (0,T)$.*

(iii) *$g(t_2) - g(t_1) \leq C(t_2 - t_1)$ for any $t_1 < t_2 \in [a,b)$ with $[a,b) \subset \{f > g\}$.*

Then $\partial_t f$ is concentrated on the set $\{f = g\}$, and for any Borel set $E \subset \{f = g\}$ one has $\partial_t f(E) = (\partial_t g)^+(E)$.

Before we prove this result, we prove the following lemma as preparation.

LEMMA 7.2. *Under the assumptions of the previous lemma, we have that for any $[a,b) \subset (0,T)$*

$$(\partial_t g)^+([a,b)) \leq f(b) - f(a) + C\mathcal{L}^1([a,b) \cap \{f > g\}). \tag{54}$$

Proof. First recall

$$(\partial_t g)^+([a,b)) = \sup\left\{ \sum_{i=1}^n (g(t_i) - g(t_{i-1}))^+ : a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

We fix a partition appearing on the right-hand side above. For each i, define

$$t'_{i-1} = \sup\{t \in [t_{i-1}, t_i) : [t_{i-1}, t) \cap \{f = g\} = \emptyset\},$$
$$t'_i = \inf\{t \in [t_{i-1}, t_i) : [t, t_i) \cap \{f = g\} = \emptyset\}.$$

Let D be the set of i for which $[t_{i-1}, t_i) \cap \{f = g\} \neq \emptyset$. Then for $i \in D$ one can calculate the following:

If $t_i > t_i'$, choosing $\varepsilon < t_i - t_i'$, since $[t_i' + \varepsilon, t_i) \subset \{f > g\}$, we see from point (iii) that

$$\begin{aligned} g(t_i) - g(t_{i-1}') &= g(t_i) - g(t_i' + \varepsilon) + g(t_i' + \varepsilon) - g(t_{i-1}') \\ &\leq C(t_i - t_i' - \varepsilon) + g(t_i' + \varepsilon) - g(t_{i-1}'). \end{aligned}$$

Letting $\varepsilon \to 0$, we see

$$g(t_i) - g(t_{i-1}') \leq C(t_i - t_i') + g((t_i')^+) - g(t_{i-1}').$$

On the other hand, since $[t_{i-1}, t_{i-1}') \subset \{f > g\}$, we can conclude that

$$g(t_{i-1}') - g(t_{i-1}) \leq C(t_{i-1} - t_{i-1}').$$

Combining the above calculations, we get

$$\begin{aligned} &\sum_{i=1}^{n} (g(t_i) - g(t_{i-1}))^+ = \sum_{i \in D} (g(t_i) - g(t_{i-1}))^+ + \sum_{i \notin D} (g(t_i) - g(t_{i-1}))^+ \\ &\leq \sum_{i \in D, t_i > t_i'} C(t_i - t_i') + \sum_{i \in D} C(t_{i-1}' - t_{i-1}) + \sum_{i \notin D} C(t_i - t_{i-1}) \\ &\quad + \sum_{i \in D, t_i > t_i'} (g((t_i')^+) - g(t_{i-1}'))^+ + \sum_{i \in D, t_i = t_i'} (g(t_i) - g(t_{i-1}'))^+ \\ &\leq C\mathcal{L}^1([a,b) \cap \{f > g\}) + \sum_{i \in D, t_i > t_i'} (g((t_i')^+) - g(t_{i-1}'))^+ + \sum_{i \in D, t_i = t_i'} (g(t_i) - g(t_{i-1}'))^+. \end{aligned}$$

In the first inequality above, we used condition (iii). If $i \in D$ and $t_i > t_i'$, we will have

$$g((t_i')^+) - g(t_{i-1}') \leq f(t_i) - f(t_{i-1}').$$

Similarly, for $i \in C$ and $t_i = t_i'$, we see

$$g(t_i) - g(t_{i-1}') \leq f(t_i) - f(t_{i-1}').$$

Hence

$$\begin{aligned} \sum_{i=1}^{n} (g(t_i) - g(t_{i-1}))^+ &\leq C\mathcal{L}^1([a,b) \cap \{f > g\}) + \sum_{i \in D} (f(t_i) - f(t_{i-1}')) \\ &\leq C\mathcal{L}^1([a,b) \cap \{f > g\}) + f(b) - f(a). \end{aligned}$$

So the desired result follows. ∎

COROLLARY 7.3. *For any* $(a, b) \subset (0, T)$,

$$(\partial_t g)^+((a,b)) \leq f(b) - f(a^+) + C\mathcal{L}^1((a,b) \cap \{f > g\}). \tag{55}$$

Proof. Apply the previous lemma to $[a + \varepsilon, b)$ and send $\varepsilon \to 0$. ∎

Now we can prove Lemma 7.1 with the help of the previous lemma.

Proof. Here $\partial_t f$ is the Radon measure defined on $[0,T)$ such that $\partial_t f([a,b)) = f(b) - f(a)$ for any $[a,b) \subset [0,T)$.

First we show that $\partial_t f$ is concentrated on the set $\{f = g\}$. We show that $\partial_t(\{f > g\}) = 0$. Indeed, if $t_0 \in \{f > g\}$, let $\beta_{t_0} = \sup\{t' \geq t_0 : f(t') = f(t)\}$. From the assumption, we know $\beta_{t_0} > t_0$. Also we know $\{f > g\} = \cup_{f(t)>g(t)}[t, \beta_t)$. Each component of the set $\{f > g\}$ contains a nondegenerate interval, and hence there are only countably many components. Therefore, $\{f > g\} = \cup_i C_i$, where C_i are connected components, and has form $[a_i, b_i)$ or (a_i, b_i). It is not hard to see that f remains a constant on C_i. Otherwise, by continuity from the left, and also by point (i), one can conclude that $f(t) = g(t)$ for some $t \in C_i$—a contradiction. So $\partial_t f(C_i) = f(b_i) - f(a_i) = 0$ or $\partial_t f(C_i) = f(b_i) - f(a_i^+) = 0$.

Next, letting $E \subset \{f = g\}$ be a Borel set, we want to show $\partial_t f(E) = (\partial_t g)^+(E)$. First we show $(\partial_t g)^+(E) \leq \partial_t f(E)$. Fix $\varepsilon > 0$; then from the outer regularity of the Radon measure $\partial_t f$ and $\mathcal{L}^1$, we can find an open set U, with $E \subset U$, $\partial_t f(U - E) < \varepsilon$ and $\mathcal{L}^1(U - E) < \varepsilon$. Write $U = \cup_i(a_i, b_i)$, with (a_i, b_i) pairwise disjoint. Then we know $\sum_i \partial_t f((a_i, b_i)) \leq \partial_t E + \varepsilon$ and $\sum_i \mathcal{L}^1((a_i, b_i) - E) < \varepsilon$.

Then from the previous corollary, we know

$$
\begin{aligned}
(\partial_t g)^+(E) \leq \sum_i (\partial_t g)^+((a_i, b_i)) &\leq \sum_i (f(b_i) - f(a_i^+)) + \sum_i C\mathcal{L}^1((a_i, b_i) \cap \{f > g\}) \\
&\leq \partial_t f(E) + \varepsilon + C \sum_i \mathcal{L}^1((a_i, b_i) - E) \leq \partial_t f(E) + 2\varepsilon.
\end{aligned}
$$

Since ε is arbitrary, it follows that $(\partial_t g)^+(E) \leq \partial_t f(E)$.

Now to prove the reverse inequality, Again we choose a cover $E \subset U$, with $U = \cup_i(a_i, b_i)$, (a_i, b_i) pairwise disjoint, such that $(\partial_t g)^+(E) \geq \sum_i (\partial_t g)^+((a_i, b_i)) - \varepsilon$.

We can assume $(a_i, b_i) \cap E \neq \emptyset$ for each i. Denote

$$
\begin{aligned}
a_i' &= \sup\{t \in (a_i, b_i) : (a_i, t) \cap \{f = g\} = \emptyset\}, \\
b_i' &= \inf\{t \in (a_i, b_i) : [t, b_i) \cap \{f = g\} = \emptyset\}.
\end{aligned}
$$

For those i with $b_i' < b_i$, we can decrease b_i, so that $f(b_i) - f((b_i')^+) + |g(b_i) - g((b_i')^+)| < \varepsilon 2^{-i}$. From the left continuity of f and g, one has $f(b_i') = g(b_i')$. Now if $b_i = b_i'$, then

$$
(\partial_t g)^+((a_i, b_i)) \geq g(b_i) - g(a_i^+) \geq f(b_i) - f(a_i^+).
$$

If $b_i > b_i'$, then

$$
\begin{aligned}
(\partial_t g)^+((a_i, b_i)) &\geq (g((b_i')^+) - g(b_i'))^+ + (g(b_i') - g(a_i^+))^+ \\
&\geq f((b_i')^+) - f(b_i') + f(b_i') - f(a_i^+) \geq f(b_i) - f(a_i^+) - \varepsilon 2^{-i}.
\end{aligned}
$$

Summing up, we get

$$
\sum_i (\partial_t g)^+((a_i, b_i)) \geq \sum_i (f(b_i) - f(a_i^+)) - \varepsilon \geq \partial_t f(E) - \varepsilon.
$$

The proof is complete. ∎

Acknowledgments. The algorithm in this paper was developed at an Environmental Modelling in Industry study group (September 21–24, 2015) at the Isaac Newton Institute in Cambridge. Jingrui Cheng wishes to thank his advisor, Mikhail Feldman, for helpful discussions and suggestions.

REFERENCES

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and the Wasserstein spaces of probability measures*, Lectures in Mathematics, ETH Zurich, Birkhäuser, 2005.

[2] O. Bokhove, B. Cheng, M. Cullen, A. Dedner, G. Esler, J. Norbury, M. R. Turner, and J. Vanneste, *Convection in a Single Column: Modelling, Algorithm and Analysis*, Met Office Solutions Report, http://www.turing-gateway.cam.ac.uk/documents/mfsg-sept2015/, 2016.

[3] D. Goldman, *Weak Lagrangian Solutions to a One Dimensional Model of Moist Semi-geostrophic Equations*, Master's Thesis, University of Toronto, 2008.

[4] M. W. Holt, *Semi-geostrophic moist frontogenesis in a Lagrangian model*, Dyn. Atmos. Ocean., 14 (1990), pp. 463–481.

[5] A. M. Lock and J. Norbury, *A column model of moist convection: Some exact equilibrium solutions*, Quart. J. Roy. Meteorol. Soc., 137 (2011), pp. 979–991, https://doi.org/10.1002/qj.820.

[6] Met Office, *Forecaster's Reference Book*, 2nd ed., Met O. 1012, Bracknell, UK, 1993.

[7] G. J. Shutts, *The adjustment of a rotating, stratified fluid subject to localized sources of mass*, Q. J. R. Meteorol. Soc., 120 (1994), pp. 361–386.

[8] G. J. Shutts and M. E. B. Gray, *A numerical modelling study of the geostrophic adjustment process following deep convection*, Quart. J. Roy. Meteorol. Soc., 120 (1994), pp. 1145–1178.