

Moduli Spaces of Transverse Deformations of Near-Horizon Geometries

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Abstract

We investigate deformations of extremal near-horizon geometries in Einstein-Maxwell-Dilaton theory, including various topological terms, and also in D=11 supergravity. By linearizing the field equations and Bianchi identities over the compact spatial cross-sections of the near-horizon geometry, we prove that the moduli associated with such deformations are constrained by elliptic systems of PDEs. The moduli space of deformations of near-horizon geometries in these theories is therefore shown to be finite dimensional.

1 Introduction

The relationship between near-horizon geometries and black hole solutions is of considerable interest. Although every extremal black hole has a well-defined near-horizon geometry, obtained by taking a certain decoupling limit of the black hole solution, there is currently no way to definitively determine if a given near-horizon geometry can be extended away from the near-horizon limit to produce a genuine black hole solution. There are also issues of uniqueness in higher dimensions. The strong uniqueness theorems established in four dimensions [1, 2, 3, 4, 5, 6, 7] break down, and there exist different black hole solutions with the same asymptotic charges, and also different black hole solutions with the same near-horizon geometries. For example, in five dimensions, the near horizon geometry $AdS_3 \times S^2$ admits two possible extensions, one to a supersymmetric black string [8], and the other to a supersymmetric black ring [9], though with different asymptotic conditions. There could also be other extensions in this case.

At the level of the near-horizon geometries, significant progress has been made in classifying such solutions, e.g. [10, 11, 12, 13, 14]. For non-supersymmetric solutions, the classifications assume the existence of sufficiently many rotational $U(1)$ isometries on the spatial cross-sections of the event horizons. For supersymmetric solutions, index theory techniques have been developed which constrain the number of supersymmetries, and which in many cases imply that supersymmetry is enhanced. The next step is to determine which near-horizon geometries actually correspond to a genuine black hole, perhaps with a given asymptotic geometry. In principle, this could be done by using the fact that in the neighbourhood of a Killing horizon, the metric can be written in Gaussian Null co-ordinates [15, 16]. There is a radial co-ordinate r , such that the horizon is located at $r = 0$, and the near-horizon solution corresponds to taking the lowest order terms in r in various components of the metric, which are assumed to be analytic in r . Determining the extension of the near-horizon solution amounts to solving the Einstein equations at higher and higher order in r . If the theory is also coupled to some matter, such as scalars or gauge fields, then these also must be appropriately expanded out order by order in r , and the various field equations solved at higher orders in r . In practice, this is extremely involved, and furthermore the Gaussian Null co-ordinate system in general will break down for sufficiently large r , so incorporating asymptotic data into the extension is difficult. It is unknown if one can systematically classify obstructions to such an extension.

In the absence of a comprehensive understanding of the theory of extensions of near-horizon geometries, we shall restrict ourselves to the first order problem. By this we mean that we shall consider the first order terms in r in the Taylor expansion of the metric, and other matter fields, and view such terms as being small perturbations of the near-horizon solution. The field equations are linearized with respect to these moduli, and we consider the issue of whether the space of moduli is finite dimensional. If the moduli space were to fail to be finite dimensional, then this would imply very little control over the possible extensions of near-horizon geometries.

In this work, we shall investigate in particular the cases of Einstein-Maxwell-Dilaton theories in any dimension, including topological terms in the special cases $D = 4, 5$; and also $D = 11$ supergravity, for which the topological term coupling is kept arbitrary. We show that the moduli space of first order radial deformations of any near-horizon solution

of such theories is finite dimensional. The method we shall use is a development of that which was first used to analyse the transverse moduli spaces of near-horizon solutions in higher dimensional Einstein gravity coupled to a cosmological constant in [17], and following that, in the case of minimal D=5 ungauged supergravity [18]. In the latter analysis, supersymmetry was used extensively in order to obtain conditions on the gauge field strength. Here, we shall not make any use of supersymmetry, and we work purely in terms of bosonic field equations.

The analysis proceeds in two steps. First, some of the metric and matter field moduli are fixed in terms of the other metric and matter field moduli, by making use of some of the Bianchi identities and field equations. Secondly, once these moduli are fixed, elliptic systems of PDEs are obtained which constrain the remaining metric and matter moduli, again by exploiting some of the Bianchi identities and field equations. Assuming that the spatial cross-sections of the near-horizon geometries are compact, this implies that the moduli space associated with the moduli appearing in these elliptic equations is finite dimensional, via standard Fredholm theory. On making use of the moduli fixing conditions, this in turn implies that the entire moduli space is finite dimensional.

The plan of this paper is as follows. In section 2, we introduce our notation, and consider in particular the metric transverse moduli. We also consider the gauge transformations which act on these moduli, and recall the proof that a gauge can be chosen for which the metric trace modulus lies in the kernel of an elliptic operator, which decouples entirely from all the other moduli [17]. In sections 3 and 4 we analyse the Einstein-Maxwell-Dilaton and D=11 supergravity theories respectively. For both theories, the matter field moduli are found, the moduli fixing conditions for metric and matter moduli are obtained, and the elliptic systems of PDEs on the remaining moduli are determined explicitly. In section 5 we present our conclusions.

2 Metric Moduli

In this section we briefly describe the metric moduli associated with radial deformations of extremal near-horizon geometries. These moduli were first considered in the analysis of transverse deformations of extremal horizons in pure gravity with a cosmological constant constructed in [17], and we summarize them here for convenience.

The metric moduli are common to all of the theories which we shall consider in the following sections. We assume that the black hole event horizon is a Killing horizon, and use Gaussian Null co-ordinates (u, r, y^A) adapted to the Killing horizon [15, 16]. The metric written in these co-ordinates is

$$ds^2 = 2du \left(dr + rh - \frac{1}{2} r^2 \Delta du \right) + ds_{\mathcal{S}}^2 \quad (2.1)$$

where $\frac{\partial}{\partial u}$ is an isometry of the geometry, which is null on the Killing horizon at $r = 0$. The metric

$$ds_{\mathcal{S}}^2 = g_{AB} dy^A dy^B \quad (2.2)$$

is the metric on spatial cross-sections of the geometry, which is u -independent and analytic in the radial co-ordinate r , and Δ , h are a scalar and 1-form on \mathcal{S} . Δ and h are also

u -independent, and analytic in r . We shall refer to g , Δ and h as the horizon (metric) data. In what follows, we shall take \mathbf{e}^i to be an orthonormal basis for \mathcal{S} . In what follows we shall find it particularly useful to choose a light-cone basis

$$\mathbf{e}^+ = du, \quad \mathbf{e}^- = dr + r h_A dy^A - \frac{1}{2} r^2 \Delta du, \quad (2.3)$$

with respect to which the spacetime metric is

$$ds^2 = 2\mathbf{e}^+ \mathbf{e}^- + ds_{\mathcal{S}}^2. \quad (2.4)$$

To obtain the moduli associated with the horizon data, we simply Taylor expand the horizon data in r ,

$$\begin{aligned} \Delta &= \overset{\circ}{\Delta}(y) + r\delta\Delta(y) + \mathcal{O}(r^2), \\ h &= \overset{\circ}{h}(y) + r\delta h(y) + \mathcal{O}(r^2), \\ g &= \overset{\circ}{g}(y) + r\delta g(y) + \mathcal{O}(r^2) \end{aligned} \quad (2.5)$$

where $\overset{\circ}{\Delta}, \overset{\circ}{h}, \overset{\circ}{g}$ are the zeroth order terms, and the metric moduli are $\delta\Delta, \delta h, \delta g$. We shall assume that $\overset{\circ}{\mathcal{S}}$, equipped with metric $\overset{\circ}{g}$, is compact.

The metric moduli admit a gauge transformation, associated with infinitesimal diffeomorphisms, assumed to be of the same order as the metric moduli, generated by the vector field [17]

$$\xi = \frac{1}{2} f \left(dr + r \overset{\circ}{h} - \frac{1}{2} r^2 \overset{\circ}{\Delta} du \right) - \frac{1}{4} r^2 \left(\overset{\circ}{\Delta} f + \mathcal{L}_{\overset{\circ}{h}} f \right) du - \frac{1}{2} r df \quad (2.6)$$

for an arbitrary smooth function f on \mathcal{S} ; with respect to which the horizon data transform as

$$\begin{aligned} \delta g_{ij} &\rightarrow \delta g_{ij} + \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j f - \overset{\circ}{h}_{(i} \overset{\circ}{\nabla}_{j)} f \\ \delta h_i &\rightarrow \delta h_i + \frac{1}{2} \overset{\circ}{\Delta} \overset{\circ}{\nabla}_i f - \frac{1}{4} (\overset{\circ}{\nabla}_i \overset{\circ}{h}_j) \overset{\circ}{\nabla}^j f - \frac{1}{4} \overset{\circ}{h}_i \overset{\circ}{h}_j \overset{\circ}{\nabla}^j f + \frac{1}{2} (\overset{\circ}{\nabla}_j \overset{\circ}{h}_i) \overset{\circ}{\nabla}^j f + \frac{1}{4} \overset{\circ}{h}_j \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}^j f \\ \delta \Delta &\rightarrow \delta \Delta + \frac{1}{2} \overset{\circ}{\nabla}^i f \left(\overset{\circ}{\nabla}_i \overset{\circ}{\Delta} - \overset{\circ}{h}_i \overset{\circ}{\Delta} \right), \end{aligned} \quad (2.7)$$

where indices i, j, \dots are with respect to the orthonormal basis $\mathbf{e}^i|_{r=0}$ on $\overset{\circ}{\mathcal{S}}$, and $\overset{\circ}{\nabla}$ denotes the Levi-Civita connection on $\overset{\circ}{\mathcal{S}}$.

Before considering the moduli space calculations of the Einstein-Maxwell-Dilaton and D=11 supergravity theories, we shall first consider the trace modulus δg_k^k . This is constrained by an elliptic PDE which decouples from the matter content, and is common to all of the theories. To see this note that under the transformation (2.7)

$$\delta g_k^k \rightarrow \delta g_k^k + \mathcal{D}f \quad (2.8)$$

where \mathcal{D} , and its adjoint \mathcal{D}^\dagger , are given by

$$\mathcal{D} \equiv \overset{\circ}{\nabla}^2 - \overset{\circ}{h} \overset{\circ}{\nabla}_i, \quad \mathcal{D}^\dagger = \overset{\circ}{\nabla}^2 + \overset{\circ}{h} \overset{\circ}{\nabla}_i + \overset{\circ}{\nabla} \overset{\circ}{h}_i. \quad (2.9)$$

We decompose δg_k^k as a sum of two terms, $\phi \in \text{Im}\mathcal{D}$, and $\phi^\perp \in (\text{Im}\mathcal{D})^\perp$ as

$$\delta g_k^k = \phi + \phi^\perp. \quad (2.10)$$

Therefore

$$\phi = \mathcal{D}(\tau), \quad \mathcal{D}^\dagger \phi^\perp = 0 \quad (2.11)$$

where τ is a smooth function. On setting $f = -\tau$ in (2.8), we have

$$\delta g_k^k = \phi^\perp \quad (2.12)$$

and hence

$$\left(\overset{\circ}{\nabla}^2 + \overset{\circ}{h} \overset{\circ}{\nabla}_i + \overset{\circ}{\nabla} \overset{\circ}{h}_i \right) \delta g_k^k = 0. \quad (2.13)$$

This is an elliptic PDE. This condition is independent of the matter content of the theory which we couple to gravity. We shall make use of this result in the analysis of the metric moduli in the following sections, in which we consider various different theories, and their associated moduli spaces. In particular, the linearized Einstein equations include a Hessian term in δg_k^k . Without the gauge fixing condition, this term would destroy the ellipticity of the associated equation. However, as the trace modulus is fixed by (2.13), the linearized Einstein equation acting on the traceless part of δg will be elliptic.

3 Horizons in Einstein-Maxwell-Dilaton theory

In this section, we consider the moduli space associated with the Einstein-Maxwell-Dilaton theory in D dimensions. The theory has a D -dimensional metric \hat{g} , as well as Abelian gauge 2-form field strengths $F^I = dA^I$, $I = 1, \dots, N$ and uncharged scalars ϕ^a , $a = 1, \dots, M$, governed by the action:

$$S = \int d^D x \sqrt{-g} \left(R - \frac{1}{2} f_{ab}(\phi) \nabla_\mu \phi^a \nabla^\mu \phi^b - V(\phi) - \frac{1}{4} Q_{IJ}(\phi) F_{\mu\nu}^I F^{J\mu\nu} \right), \quad (3.1)$$

where the couplings f_{ab} and Q_{IJ} are functions of the scalar fields ϕ^a .

The Einstein, gauge and scalar field equations are:

$$R_{\mu\nu} - \frac{1}{2} f_{ab} \nabla_\mu \phi^a \nabla_\nu \phi^b - \frac{1}{2} Q_{IJ} F_{\mu\alpha}^I F_\nu^{J\alpha} + \frac{1}{D-2} \hat{g}_{\mu\nu} \left(\frac{1}{4} Q_{IJ} F_{\alpha\beta}^I F^{J\alpha\beta} - V \right) = 0, \quad (3.2)$$

and

$$\nabla^\nu (Q_{IJ} F_{\nu\mu}^J) = 0, \quad (3.3)$$

and

$$f_{ab}\nabla_\mu\nabla^\mu\phi^b + (\partial_b f_{ac} - \frac{1}{2}\partial_a f_{bc})\nabla_\mu\phi^b\nabla^\mu\phi^c - \partial_a V - \frac{1}{4}\partial_a Q_{IJ}F_{\mu\nu}^I F^{J\mu\nu} = 0 , \quad (3.4)$$

where $\partial_a \equiv \partial/\partial\phi^a$. The Bianchi identity is

$$dF^I = 0 . \quad (3.5)$$

We furthermore assume that $\frac{\partial}{\partial u}$ is a symmetry of the full solution, i.e.

$$\mathcal{L}_{\partial_u} A^I = 0 , \quad \mathcal{L}_{\partial_u} \phi^a = 0 . \quad (3.6)$$

The scalar fields ϕ^a depend only on r and y^A , and the field strengths F^I are written in the Gaussian Null co-ordinates as

$$F^I = \Psi^I du \wedge dr + rW_A^I du \wedge dy^A + Z_A^I dr \wedge dy^A + \frac{1}{2}\tilde{F}_{AB}^I dy^A \wedge dy^B , \quad (3.7)$$

where Ψ^I is a u -independent scalar, W^I, Z^I are u -independent 1-forms, and \tilde{F}^I is a u -independent 2-form defined on \mathcal{S} . In particular, when written in the light-cone basis, the field strengths are

$$F^I = \Psi^I \mathbf{e}^+ \wedge \mathbf{e}^- + r\mathbf{e}^+ \wedge \left(W^I - \Psi^I h + \frac{1}{2}r\Delta Z^I \right) + \mathbf{e}^- \wedge Z^I + H^I , \quad (3.8)$$

where

$$H^I = \tilde{F}^I - rh \wedge Z^I . \quad (3.9)$$

The Bianchi identity imposes the following set of conditions on the F^I components

$$\begin{aligned} \tilde{d}W^I &= 0 , & \tilde{d}\Psi^I - W^I - r\dot{W}^I &= 0 , \\ \tilde{d}Z^I - \dot{H}^I - h \wedge Z^I - r\dot{h} \wedge Z^I - rh \wedge \dot{Z}^I &= 0 , \\ \tilde{d}H^I + r\tilde{d}h \wedge Z^I - rh \wedge \tilde{d}Z^I &= 0 , \end{aligned} \quad (3.10)$$

where we denote by \tilde{d} the exterior derivative restricted to hypersurfaces $r = \text{const}$, and by $\dot{\xi}$ the Lie derivative of ξ along the vector field $\frac{\partial}{\partial r}$, i.e.

$$\dot{\xi} \equiv \mathcal{L}_{\frac{\partial}{\partial r}} \xi . \quad (3.11)$$

The decomposition of (3.2), (3.3) and (3.4) in terms of the horizon data is given in Appendix A.1.

3.1 Moduli space computation

To begin the moduli space computation, we consider the moduli associated with the scalar fields and the components of the gauge field strengths.

The scalar fields are Taylor expanded as

$$\phi^a = \overset{\circ}{\phi}^a + r\delta\phi^a + \mathcal{O}(r^2) \quad (3.12)$$

The Ψ^I , W^I and H^I components of F^I are expanded as

$$\begin{aligned} \Psi^I &= \overset{\circ}{\Psi}^I + r\delta\Psi^I + \mathcal{O}(r^2) \\ W^I &= \overset{\circ}{W}^I + r\delta W^I + \mathcal{O}(r^2) \\ H^I &= \overset{\circ}{H}^I + r\delta H^I + \mathcal{O}(r^2) \end{aligned} \quad (3.13)$$

There is however a subtlety with respect to the Z^I components. The Z^I terms appear in the F^I as $dr \wedge Z^I$, which scales linearly with r , and hence we expand Z^I as

$$Z^I = \delta Z^I + \mathcal{O}(r) \quad (3.14)$$

So, with these expansions, the scalar moduli are $\delta\phi^a$, and the gauge field moduli are $\{\delta\Psi^I, \delta W^I, \delta Z^I, \delta H^I\}$.

The above expansions are chosen to be consistent with taking the near-horizon limit. To take the near-horizon limit, we take

$$(u, r, y^A) \longrightarrow (\epsilon^{-1}u, \epsilon r, y^A), \quad \epsilon \in \mathbb{R}_{>0}. \quad (3.15)$$

On making this transformation, we note that all moduli terms (metric, scalar and gauge) are linear in ϵ . The near-horizon limit is obtained by taking $\epsilon \rightarrow 0$. Our choice of Z^I moduli expansion is also required for consistency with the $--$ component of the Einstein equation.

Having determined the moduli expansions, we start to analyse the conditions imposed on the moduli. From the Bianchi identity, we find the following conditions:

$$\tilde{d}\delta\Psi^I = 2\delta W^I, \quad (3.16)$$

$$\tilde{d}\delta Z^I - \overset{\circ}{h} \wedge \delta Z^I - \delta H^I = 0. \quad (3.17)$$

At this stage we shall specify which of the moduli are fixed by Bianchi identities and field equations. Equation (3.16) fixes δW^I in terms of $\delta\Psi^I$, and (3.17) fixes δH^I in terms of δZ^I . By using the $-$ component of the gauge field equation (A.8), we fix $\delta\Psi^I$ as follows

$$\delta\Psi^I = \overset{\circ}{\nabla}^i \delta Z_i^I - \overset{\circ}{h}^i \delta Z_i^I + \overset{\circ}{Q}^{IK} \partial_a \overset{\circ}{Q}_{KJ} \overset{\circ}{\nabla}^i \overset{\circ}{\phi}^a \delta Z_i^J - \frac{1}{2} \overset{\circ}{\Psi}^I \delta g_k^k - \overset{\circ}{Q}^{IK} \partial_a \overset{\circ}{Q}_{KJ} \overset{\circ}{\Psi}^J \delta\phi^a, \quad (3.18)$$

By using the $-i$ and the $+ -$ components of the Einstein equations, (A.5) and (A.2), we fix δh and $\delta\Delta$ respectively as follows

$$\begin{aligned} \delta h_i &= \frac{1}{2} \overset{\circ}{\nabla}_i \delta g_k^k - \frac{1}{2} \overset{\circ}{\nabla}^j \delta g_{ji} - \frac{1}{4} h_i \delta g_k^k + \frac{1}{2} \delta g_{ij} h^j \\ &+ \frac{1}{2} f_{ab} \delta\phi^a \overset{\circ}{\nabla}_i \overset{\circ}{\phi}^b + \frac{1}{2} \overset{\circ}{Q}_{IJ} (\overset{\circ}{\Psi}^I \delta Z_i^J + \delta Z^{Ij} \overset{\circ}{H}_{ij}^J), \end{aligned} \quad (3.19)$$

and

$$\begin{aligned}
\delta\Delta &= \frac{1}{3}\overset{\circ}{\nabla}_i\delta h^i + \frac{1}{12}\overset{\circ}{h}\overset{\circ}{\nabla}_i\delta g_k^k - \overset{\circ}{h}\delta h_i - \frac{1}{6}\overset{\circ}{\Delta}\delta g_k^k - \frac{1}{12}\overset{\circ}{h}_i\overset{\circ}{h}\delta g_k^k + \frac{1}{3}\delta g_{ij}\overset{\circ}{h}^i\overset{\circ}{h}^j \\
&- \frac{1}{6}\delta g_{ij}\overset{\circ}{\nabla}^i\overset{\circ}{h}^j - \frac{1}{6}\overset{\circ}{h}^j\overset{\circ}{\nabla}^i\delta g_{ij} + \frac{1}{6(D-2)}\delta\phi^a\partial_a\overset{\circ}{Q}_{IJ}(2\overset{\circ}{H}_{ij}^I\overset{\circ}{H}^{Jij} + (D-3)\overset{\circ}{\Psi}^I\overset{\circ}{\Psi}^J) \\
&+ \frac{1}{3(D-2)}\overset{\circ}{Q}_{IJ}(2\delta H_{ij}^I\overset{\circ}{H}^{Jij} - 2\overset{\circ}{H}^{I\ell_1}{}_j\overset{\circ}{H}^{J\ell_2j}\delta g_{\ell_1\ell_2} + (D-3)\overset{\circ}{\Psi}^I\delta\Psi^J) \\
&- \frac{1}{3(D-2)}\delta\phi^a\partial_a\overset{\circ}{V} + \frac{4-D}{6(D-2)}\overset{\circ}{Q}_{IJ}(\overset{\circ}{W}_i^I\delta Z^{Ji} - \overset{\circ}{\Psi}^I\overset{\circ}{h}\delta Z_i^J). \tag{3.20}
\end{aligned}$$

So, the $\delta W^I, \delta H^I, \delta\Psi^I, \delta h, \delta\Delta$ moduli are fixed in terms of the moduli $\delta Z^I, \delta g, \delta\phi^a$. We remark that the moduli $\delta\Psi^I, \delta h, \delta H^I$ are linear in $\delta Z^I, \delta g, \delta\phi^a, \overset{\circ}{\nabla}\delta Z^I, \overset{\circ}{\nabla}\delta g, \overset{\circ}{\nabla}\delta\phi^a$, whereas $\delta W^I, \delta\Delta$ involve some second order derivative terms on $\delta Z^I, \delta g, \delta\phi^a$.

We now turn to the remaining moduli $\delta Z^I, \delta g, \delta\phi^a$. We shall use the Bianchi identities and field equations to construct elliptic systems of PDEs constraining these moduli, making use of the moduli fixing conditions on $\delta W^I, \delta\Psi^I, \delta h, \delta\Delta$.

We start with the δZ^I moduli. On taking the divergence of (3.17), we obtain

$$\begin{aligned}
\overset{\circ}{\nabla}^j\overset{\circ}{\nabla}_j\delta Z_i^I &- \overset{\circ}{\nabla}_i\overset{\circ}{\nabla}^j\delta Z_j^I - \overset{\circ}{R}_{ij}\delta Z^{Ij} - \delta Z_i^I\overset{\circ}{\nabla}_j\overset{\circ}{h}^j \\
&- \overset{\circ}{h}^j\overset{\circ}{\nabla}_j\delta Z_i^I + \delta Z^{Ij}\overset{\circ}{\nabla}_j\overset{\circ}{h}_i + \overset{\circ}{h}_i\overset{\circ}{\nabla}^j\delta Z_j^I - \overset{\circ}{\nabla}^j\delta H_{ji}^I = 0, \tag{3.21}
\end{aligned}$$

where $\overset{\circ}{R}$ denotes the Ricci tensor of $\overset{\circ}{S}$. Using the i component of the gauge field equation (A.9), we express $\overset{\circ}{\nabla}^j\delta H_{ji}^I$ as

$$\begin{aligned}
\overset{\circ}{\nabla}^j\delta H_{ji}^I &= -\overset{\circ}{\nabla}_i\overset{\circ}{\nabla}^j\delta Z_j^I + \overset{\circ}{\nabla}_i(\overset{\circ}{h}^j\delta Z_j^I) - \overset{\circ}{Q}^{IK}\partial_a\overset{\circ}{Q}_{KJ}\overset{\circ}{\nabla}_i(\overset{\circ}{\nabla}^j\overset{\circ}{\phi}^a\delta Z_j^I) \\
&+ \frac{1}{2}\overset{\circ}{\nabla}_i(\overset{\circ}{\Psi}^I\delta g_k^k) + \overset{\circ}{Q}^{IK}\partial_a\overset{\circ}{Q}_{KJ}\overset{\circ}{\nabla}_i(\overset{\circ}{\Psi}^J\delta\phi^a) + \delta g_{\ell_1\ell_2}\overset{\circ}{\nabla}^{\ell_1}\overset{\circ}{H}^{I\ell_2}{}_i + \overset{\circ}{\nabla}^j\delta g_{jk}\overset{\circ}{H}^{Ik}{}_i \\
&- \frac{1}{2}\overset{\circ}{\nabla}_j\delta g_k^k\overset{\circ}{H}^{Ij}{}_i + \frac{1}{2}\overset{\circ}{\nabla}_j\delta g^k{}_i\overset{\circ}{H}^{Ij}{}_k + \frac{1}{2}\overset{\circ}{\nabla}_i\delta g_j^k\overset{\circ}{H}^{Ij}{}_k - \frac{1}{2}\overset{\circ}{\nabla}_k\delta g_{ij}\overset{\circ}{H}^{Ijk} \\
&+ 2\overset{\circ}{h}_i\delta\Psi^I + 2\overset{\circ}{\Psi}^I\delta h_i + 2\overset{\circ}{h}^j\delta H_{ji}^I + 2\delta h^j\overset{\circ}{H}_{ji}^I - \delta\phi^b\partial_b\overset{\circ}{Q}^{IK}\partial_a\overset{\circ}{Q}_{KJ}\overset{\circ}{\nabla}^j\overset{\circ}{\phi}^a\overset{\circ}{H}_{ji}^J \\
&- \delta\phi^b\overset{\circ}{Q}^{IK}\partial_a\partial_b\overset{\circ}{Q}_{KJ}\overset{\circ}{\nabla}^j\overset{\circ}{\phi}^a\overset{\circ}{H}_{ji}^J - \overset{\circ}{Q}^{IK}\partial_a\overset{\circ}{Q}_{KJ}(\overset{\circ}{H}_{ji}^J\overset{\circ}{\nabla}^j\delta\phi^a + \delta H_{ji}^J\overset{\circ}{\nabla}^j\overset{\circ}{\phi}^a) \\
&+ \overset{\circ}{Q}^{IK}\partial_a\overset{\circ}{Q}_{KJ}\delta g_{\ell_1\ell_2}\overset{\circ}{\nabla}^{\ell_1}\overset{\circ}{\phi}^a\overset{\circ}{H}^{J\ell_2}{}_i + \overset{\circ}{Q}^{IK}\partial_a\overset{\circ}{Q}_{KJ}\delta\phi^a(\overset{\circ}{\Psi}^J\overset{\circ}{h}_i - \overset{\circ}{W}_i^J + \overset{\circ}{h}^j\overset{\circ}{H}_{ji}^J) \\
&- 2\overset{\circ}{\Delta}\delta Z_i^I - \frac{1}{2}\delta g_k^k(\overset{\circ}{W}_i^I - \overset{\circ}{\Psi}^I\overset{\circ}{h}_i) + \delta g_{ij}(\overset{\circ}{W}^{Ij} - \overset{\circ}{\Psi}^I\overset{\circ}{h}^j) \\
&+ \frac{1}{2}\delta g_k^k\overset{\circ}{h}^j\overset{\circ}{H}_{ji}^I - 2\overset{\circ}{h}^j\overset{\circ}{H}^{Ij}{}_i\delta g_{jk} - \overset{\circ}{h}_j\overset{\circ}{H}^{Ijk}\delta g_{ki}. \tag{3.22}
\end{aligned}$$

Then we substitute this expression into (3.21); the $\overset{\circ}{\nabla}_j\overset{\circ}{\nabla}^j\delta Z_i^I$ terms cancel out. Furthermore, the terms linear in $\delta H^I, \delta h$ and $\delta\Psi^I$ are rewritten using the Bianchi identity (3.17)

and (3.19), (3.18), producing terms linear in $\delta Z^I, \delta g, \delta\phi^a, \overset{\circ}{\nabla}\delta Z^I, \overset{\circ}{\nabla}\delta g, \overset{\circ}{\nabla}\delta\phi^a$. The resulting expression produces an elliptic PDE for δZ^I , with principal symbol generated by $\overset{\circ}{\nabla}^2$.

Next, we consider the scalar moduli $\delta\phi^a$. The linearized scalar field equation is

$$\begin{aligned}
& \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}^i \delta\phi^a - \delta g_{ij} \overset{\circ}{\nabla}^i \overset{\circ}{\nabla}^j \phi^a - \overset{\circ}{\nabla}^i \delta g_{ij} \overset{\circ}{\nabla}^j \phi^a + \frac{1}{2} \overset{\circ}{\nabla}^i \phi^a \overset{\circ}{\nabla}_i \delta g_k^k - \delta h^i \overset{\circ}{\nabla}_i \phi^a - \overset{\circ}{h}^i \overset{\circ}{\nabla}_i \delta\phi^a \\
& + \delta g_{ij} \overset{\circ}{h}^i \overset{\circ}{\nabla}^j \phi^a + \delta\phi^e \left[\partial_e f^{ab} (\partial_c f_{bd} - \frac{1}{2} \partial_b f_{cd}) + f^{ab} (\partial_c \partial_e f_{bd} - \frac{1}{2} \partial_b \partial_e f_{cd}) \right] \overset{\circ}{\nabla}_i \phi^c \overset{\circ}{\nabla}^i \phi^d \\
& - f^{ab} (\partial_c f_{bd} - \frac{1}{2} \partial_b f_{cd}) \delta g_{ij} \overset{\circ}{\nabla}^i \phi^c \overset{\circ}{\nabla}^j \phi^d + (2f^{ab} \partial_{(c} f_{d)b} - f^{ab} \partial_b f_{cd}) \overset{\circ}{\nabla}_i \phi^c \overset{\circ}{\nabla}^i \delta\phi^d \\
& + \frac{1}{4} \delta\phi^c (\partial_c f^{ab} \partial_b \overset{\circ}{Q}_{IJ} + f^{ab} \partial_b \partial_c \overset{\circ}{Q}_{IJ}) (2\overset{\circ}{\Psi}^I \overset{\circ}{\Psi}^J - \overset{\circ}{H}^I_{ij} \overset{\circ}{H}^{Jij}) \\
& + \frac{1}{2} f^{ab} \partial_b \overset{\circ}{Q}_{IJ} (2\overset{\circ}{\Psi}^I \delta\overset{\circ}{\Psi}^J - \overset{\circ}{H}^I_{ij} \delta H^{Jij} + \delta g_{\ell_1 \ell_2} \overset{\circ}{H}^{I\ell_1}{}_k \overset{\circ}{H}^{J\ell_2 k}) \\
& - \delta\phi^c (\partial_c f^{ab} \partial_b \overset{\circ}{V} + f^{ab} \partial_b \partial_c \overset{\circ}{V}) + \overset{\circ}{\nabla}^j \phi^a (\overset{\circ}{h}^i \delta g_{ij} - \frac{1}{2} \delta g_k^k \overset{\circ}{h}_j) \\
& + \delta\phi^a (2\overset{\circ}{\Delta} + \overset{\circ}{h}_i \overset{\circ}{h}^i - \overset{\circ}{\nabla}^i \overset{\circ}{h}_i) - \delta\phi^c f^{ab} (\partial_c f_{bd} + \partial_d f_{bc} - \partial_b f_{cd}) \overset{\circ}{h}^i \overset{\circ}{\nabla}_i \phi^d \\
& - 2\overset{\circ}{h}_i \overset{\circ}{\nabla}^i \delta\phi^a - \frac{1}{2} \delta Z^I f^{ab} \partial_b \overset{\circ}{Q}_{IJ} (\overset{\circ}{W}^{Ji} - \overset{\circ}{\Psi}^J \overset{\circ}{h}^i) = 0. \tag{3.23}
\end{aligned}$$

The terms linear in $\delta H^I, \delta h$ and $\delta\Psi^I$ are again rewritten using the Bianchi identity (3.17) and (3.19), (3.18), producing terms linear in $\delta Z^I, \delta g, \delta\phi^a, \overset{\circ}{\nabla}\delta Z^I, \overset{\circ}{\nabla}\delta g, \overset{\circ}{\nabla}\delta\phi^a$. The resulting expression produces an elliptic PDE for $\delta\phi^a$, with principal symbol generated by $\overset{\circ}{\nabla}^2$.

Lastly, we consider the metric moduli δg . The linearized ij component of the Einstein equations is

$$\overset{\circ}{\nabla}^2 \delta g_{ij} - \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \delta g_k^k - (\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j \overset{\circ}{\nabla}_\ell) \delta g^\ell{}_i - (\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_i - \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_\ell) \delta g^\ell{}_j = \mathcal{A}_{ij}, \tag{3.24}$$

where

$$\begin{aligned}
\mathcal{A}_{ij} = & (\overset{\circ}{\nabla}_i \delta g^k_j + \overset{\circ}{\nabla}_j \delta g^k_i - \overset{\circ}{\nabla}^k \delta g_{ij}) \overset{\circ}{h}_k - \overset{\circ}{\nabla}_{(i} (\overset{\circ}{h}_{j)} \delta g_k^k) + 2 \overset{\circ}{\nabla}_{(i} (\delta g_{j)k} \overset{\circ}{h}^k) \\
& + 2 \overset{\circ}{\nabla}_{(i} (\overset{\circ}{\nabla}_{j)} \overset{\circ}{\phi}^b \overset{\circ}{f}_{ab} \delta \phi^a) + 2 \overset{\circ}{\nabla}_{(i} (\delta Z_j^J \overset{\circ}{Q}_{IJ} \overset{\circ}{\Psi}^I) + 2 \overset{\circ}{\nabla}_{(i} (\overset{\circ}{H}_{j)k}^J \overset{\circ}{Q}_{IJ} \delta Z^I k) \\
& - 8 \overset{\circ}{h}_{(i} \delta h_{j)} + 2(-\overset{\circ}{\Delta} + \frac{1}{2} \overset{\circ}{\nabla}_k \overset{\circ}{h}^k - \overset{\circ}{h}_k \overset{\circ}{h}^k) \delta g_{ij} \\
& - 2 \delta g_{(i}^k \overset{\circ}{\nabla}_{|k|} \overset{\circ}{h}_{j)} - 2 \overset{\circ}{h}^k \overset{\circ}{\nabla}_{(i} \delta g_{j)k} + 2 \overset{\circ}{h}^k \overset{\circ}{\nabla}_k \delta g_{ij} - 2 \overset{\circ}{h}_{(i} \overset{\circ}{\nabla}^k \delta g_{j)k} \\
& + 4 \overset{\circ}{h}_k \overset{\circ}{h}_{(i} \delta g_{j)}^k + 2 \overset{\circ}{h}_{(i} \overset{\circ}{\nabla}_{j)} \delta g_k^k + \delta g_k^k (\overset{\circ}{\nabla}_{(i} \overset{\circ}{h}_{j)} - \overset{\circ}{h}_i \overset{\circ}{h}_j) - \delta \phi^c \partial_c \overset{\circ}{f}_{ab} \overset{\circ}{\nabla}_i \overset{\circ}{\phi}^a \overset{\circ}{\nabla}_j \overset{\circ}{\phi}^b \\
& - 2 \overset{\circ}{f}_{ab} \overset{\circ}{\nabla}_i \delta \phi^a \overset{\circ}{\nabla}_j \overset{\circ}{\phi}^b - \delta \phi^a \partial_a \overset{\circ}{Q}_{IJ} \overset{\circ}{H}_{ik}^I \overset{\circ}{H}_j^J + \overset{\circ}{Q}_{IJ} \delta g_{kl} \overset{\circ}{H}_i^I \overset{\circ}{H}_j^J + 2 \overset{\circ}{Q}_{IJ} \delta H_{k(i}^I \overset{\circ}{H}_{j)}^J \\
& - \frac{2}{D-2} \overset{\circ}{g}_{ij} \delta \phi^a \partial_a \overset{\circ}{V} + \frac{1}{2(D-2)} \overset{\circ}{g}_{ij} \delta \phi^a \partial_a \overset{\circ}{Q}_{IJ} (\overset{\circ}{H}_{kl}^I \overset{\circ}{H}^{Jkl} - 2 \overset{\circ}{\Psi}^I \overset{\circ}{\Psi}^J) \\
& + \frac{1}{2(D-2)} \delta g_{ij} \overset{\circ}{Q}_{IJ} (\overset{\circ}{H}_{kl}^I \overset{\circ}{H}^{Jkl} - 2 \overset{\circ}{\Psi}^I \overset{\circ}{\Psi}^J) - \frac{2}{D-2} \delta g_{ij} \overset{\circ}{V} \\
& + \frac{1}{D-2} \overset{\circ}{g}_{ij} \overset{\circ}{Q}_{IJ} (\delta H_{kl}^I \overset{\circ}{H}^{Jkl} - \delta g_{mn} \overset{\circ}{H}^{Im} \overset{\circ}{H}^{Jnk} - 2 \delta \Psi^I \overset{\circ}{\Psi}^J) + 2 \delta \phi^a \overset{\circ}{f}_{ab} \overset{\circ}{h}_{(i} \overset{\circ}{\nabla}_{j)} \overset{\circ}{\phi}^b \\
& + \frac{2}{D-2} \overset{\circ}{g}_{ij} \overset{\circ}{Q}_{IJ} \delta Z_\ell^I (\overset{\circ}{W}^{J\ell} - \overset{\circ}{\Psi}^J \overset{\circ}{h}^\ell) - 2 \overset{\circ}{Q}_{IJ} (\overset{\circ}{W}_{(i} \delta Z_{j)}^J - \overset{\circ}{\Psi}^I \delta Z_{(j}^J \overset{\circ}{h}_{i)}), \tag{3.25}
\end{aligned}$$

which is a linear expression in $\delta Z^I, \delta \phi^a, \delta g, \overset{\circ}{\nabla} \delta Z^I, \overset{\circ}{\nabla} \delta \phi^a, \overset{\circ}{\nabla} \delta g$. Furthermore, in (3.24), terms of the form $(\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j \overset{\circ}{\nabla}_\ell) \delta g^\ell_i$ can be rewritten as terms linear in δg and the Riemann tensor $\overset{\circ}{R}$, hence can be incorporated into the algebraic term on the RHS. The trace term δg_k^k is fixed by the elliptic condition (2.13), so (3.24) is an elliptic set of PDEs for the traceless part of δg , with principal symbol generated by $\overset{\circ}{\nabla}^2$.

Taken together the conditions (3.21), (3.23), (3.24) and (2.13) constitute elliptic PDEs on the moduli $\delta Z^I, \delta \phi^a, \delta g$. The remaining moduli $\{\delta W^I, \delta H^I, \delta \Psi^I, \delta h, \delta \Delta\}$ are fixed in terms of $\{\delta Z^I, \delta \phi^a, \delta g\}$ by (3.16), (3.17), (3.18), (3.19) and (3.20). The moduli space is therefore finite dimensional.

3.2 Including Topological Terms

In this section we shall add a topological term to the Einstein-Maxwell-Dilaton action (3.1), and analyse explicitly in four and five dimensions any change to the analysis reported in section 3.1. Such cases are of particular relevance in the context of four and five-dimensional supergravity theories. We shall show in all cases that the addition of the topological terms does not affect the principal symbol of the systems of PDEs. Therefore we prove that the finiteness of the moduli space is also true in the presence of a topological term in the action.

3.2.1 Topological Terms in $D = 4$

In $D = 4$, we shall consider the following topological term

$$S_{top} = \int t_{IJ}(\phi) F^I \wedge F^J , \quad (3.26)$$

where $t_{IJ}(\phi)$ are functions of the scalar fields. The gauge field equation, in the presence of topological term, becomes

$$\nabla^\nu(Q_{IJ}(\phi)F_{\nu\mu}^J) + \epsilon_{\mu\nu\rho\sigma}F^{J\rho\sigma}\nabla^\nu t_{IJ}(\phi) = 0 . \quad (3.27)$$

and the scalar field equation becomes

$$\begin{aligned} f_{ab}\nabla_\mu\nabla^\mu\phi^b + (\partial_b f_{ac} - \frac{1}{2}\partial_a f_{bc})\nabla_\mu\phi^b\nabla^\mu\phi^c - \partial_a V \\ - \frac{1}{4}\partial_a Q_{IJ}F_{\mu\nu}^I F^{J\mu\nu} + \frac{1}{4}F_{\mu\nu}^I F_{\rho\sigma}^J \epsilon^{\mu\nu\rho\sigma}\partial_a t_{IJ} = 0 , \end{aligned} \quad (3.28)$$

The changes provided by the topological term relevant to the moduli space computation are here presented. The $-$ component of the gauge field equation, which enters into the moduli fixing for $\delta\Psi^I$, is modified by:

$$\epsilon^{+-ij}\left(\partial_a t_{IJ}\dot{\phi}^a(H_{ij}^J + 2rh_i Z_j^J) - 2Z_j^J\partial_a t_{IJ}\tilde{\nabla}_i\phi^a\right) , \quad (3.29)$$

which is linear on the moduli $\delta\phi^a$ and δZ^I . The i component of the gauge field equation, which is used to construct the elliptic system for δZ^I , is modified by:

$$2\epsilon^{+-ij}\left(\Psi^J\partial_a t_{IJ}\tilde{\nabla}_j\phi^a - rW_j^J\partial_a t_{IJ}\dot{\phi}^a\right) , \quad (3.30)$$

which when linearized includes terms linear in δZ^I , $\delta\Psi^I$, $\delta\phi^a$. On eliminating the $\delta\Psi^I$ term, the extra terms can be rewritten as terms linear in δZ^I , δg , $\delta\phi^a$, $\overset{\circ}{\nabla}\delta Z^I$, which do not affect the ellipticity of the resulting system.

The scalar field equation is modified by:

$$\epsilon^{+-ij}\partial_a t_{IJ}\left(\Psi^I H_{ij}^J + 2rZ_j^J(\Psi^I h_i - W_i^I)\right) , \quad (3.31)$$

and includes terms linear in δZ^I , $\delta\Psi^I$ and δH^I . On eliminating the $\delta\Psi^I$ and δH^I terms, again the extra terms can be rewritten as terms linear in δZ^I , δg , $\delta\phi^a$, $\overset{\circ}{\nabla}\delta Z^I$, which do not affect the ellipticity of the resulting system.

3.2.2 Topological Terms in $D = 5$

In $D = 5$, we shall consider the topological term

$$S_{top} = \int C_{IJK}A^I \wedge F^J \wedge F^K , \quad (3.32)$$

where C_{IJK} are constants. The gauge field equation, in the presence of the topological term, becomes

$$\nabla^\mu(Q_{IJ}(\phi)F_{\mu\nu}^J) - \frac{1}{4}C_{IJK}\epsilon_{\nu\mu\rho\sigma\lambda}F^{J\mu\rho}F^{K\sigma\lambda} = 0 . \quad (3.33)$$

We show here the relevant changes to the equations of motion. The $-$ component of the gauge field equation, entering into the $\delta\Psi^I$ moduli fixing condition, is modified by:

$$-C_{IJK}\epsilon^{+-ijk}Z_i^J H_{jk}^K , \quad (3.34)$$

which provides a linear contribution in δZ^I . The i component of the gauge field equation, which enters into the construction of the elliptic system for δZ^I , is modified by:

$$C_{IJK}\epsilon^{+-ijk}\left(-\Psi^J H_{jk}^K + 2r(W_j^J - \Psi^J h_j)Z_k^K\right) , \quad (3.35)$$

which when linearized includes terms linear in δZ^I , $\delta\Psi^I$. On eliminating the $\delta\Psi^I$ term, the extra terms can be rewritten as terms linear in δZ^I , δg , $\delta\phi^a$, $\overset{\circ}{\nabla}\delta Z^I$, which do not affect the ellipticity of the resulting system.

4 D=11 Supergravity

The bosonic field content of D=11 supergravity is the D=11 metric \hat{g} , and a 4-form F , $F = dC$. The analysis of the moduli of transverse deformations of extremal horizons in $D = 11$ supergravity proceeds in a rather similar fashion to that of the Einstein-Maxwell-Dilaton theory. We shall again assume that the Killing vector $\frac{\partial}{\partial u}$ is a symmetry of the full solution, i.e.

$$\mathcal{L}_{\frac{\partial}{\partial u}}F = 0 , \quad (4.1)$$

We decompose F in Gaussian Null co-ordinates as

$$F = du \wedge dr \wedge \Psi + r du \wedge W + dr \wedge Z + X \quad (4.2)$$

where Ψ is a u -independent 2-form, W, Z are u -independent 3-forms, and X is a u -independent 4-form on \mathcal{S} , which are all assumed to be analytic in r .

The Bianchi identity $dF = 0$ then decomposes as

$$\tilde{d}\Psi - W - r\dot{W} = 0, \quad \tilde{d}W = 0, \quad \tilde{d}Z - \dot{X} = 0, \quad \tilde{d}X = 0 . \quad (4.3)$$

The gauge field equations are given by

$$\nabla^\nu F_{\nu\lambda_1\lambda_2\lambda_3} = \frac{q}{(4!)^2}\epsilon_{\lambda_1\lambda_2\lambda_3}{}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8}F_{\mu_1\mu_2\mu_3\mu_4}F_{\mu_5\mu_6\mu_7\mu_8} \quad (4.4)$$

where q is a constant. Here we have included a topological term in the action proportional to $qF \wedge F \wedge C$. On imposing supersymmetry, the value of q is fixed by requiring consistency

of the gauge field equations with the integrability conditions of the gravitino Killing spinor equations. However, here our analysis is purely in the bosonic sector, so the value of q is kept arbitrary.

Finally, we have the Einstein equations:

$$R_{\mu\nu} = \frac{1}{12} F_{\mu\lambda_1\lambda_2\lambda_3} F_{\nu}{}^{\lambda_1\lambda_2\lambda_3} - \frac{1}{144} \hat{g}_{\mu\nu} F_{\lambda_1\lambda_2\lambda_3\lambda_4} F^{\lambda_1\lambda_2\lambda_3\lambda_4} . \quad (4.5)$$

The decomposition of (4.4) and (4.5) in terms of the horizon data is provided in Appendix A.2.

4.1 Moduli space computation

In this section we shall prove the finiteness of the black hole moduli space in $D = 11$ supergravity by using the same procedure adopted for the Einstein-Maxwell-Dilaton theory. We must again identify the moduli associated with the gauge fields, in this case the 4-form F . We shall adopt the following expansions for Ψ , W and X via

$$\begin{aligned} \Psi &= \overset{\circ}{\Psi} + r\delta\Psi + \mathcal{O}(r^2) \\ W &= \overset{\circ}{W} + r\delta W + \mathcal{O}(r^2) \\ X &= \overset{\circ}{X} + r\delta X + \mathcal{O}(r^2) . \end{aligned} \quad (4.6)$$

For the Z moduli, we note that as Z appears in F as $dr \wedge Z$, which scales linearly with r , we shall expand out Z as

$$Z = \delta Z + \mathcal{O}(r) \quad (4.7)$$

This expansion is consistent with requiring that under the transformation

$$(u, r, y^A) \longrightarrow (\epsilon^{-1}u, \epsilon r, y^A) , \quad \epsilon \in \mathbb{R}_{>0} . \quad (4.8)$$

all moduli terms are linear in ϵ , as well as being required for consistency with the -- Einstein equation. The gauge theory moduli are therefore $\delta\Psi, \delta W, \delta X, \delta Z$.

As done previously, we shall use the Bianchi identities and field equations to fix some of the metric and gauge field moduli. The Bianchi identity provides the following conditions:

$$\tilde{d}\delta\Psi = 2\delta W , \quad (4.9)$$

$$\tilde{d}\delta Z = \delta X , \quad (4.10)$$

which we use to fix δW and δX , respectively, in terms of $\delta\Psi$ and δZ . By using the $-k_1 k_2$ component of the gauge field equations (4.4), we further fix $\delta\Psi$ as

$$\begin{aligned} \delta\Psi_{k_1 k_2} &= \overset{\circ}{\nabla}{}^\ell \delta Z_{\ell k_1 k_2} - \overset{\circ}{h}{}^\ell Z_{\ell k_1 k_2} + \delta g_{k_1}{}^\ell \overset{\circ}{\Psi}_{\ell k_2} - \delta g_{k_2}{}^\ell \overset{\circ}{\Psi}_{\ell k_1} \\ &\quad - \frac{1}{2} \delta g_\ell{}^\ell \Psi_{k_1 k_2} - \frac{q}{72} \epsilon_{k_1 k_2}{}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7} \delta Z_{\ell_1 \ell_2 \ell_3} \overset{\circ}{X}_{\ell_4 \ell_5 \ell_6 \ell_7} . \end{aligned} \quad (4.11)$$

Also, by using the $-i$ and $+-$ components of the Einstein equations (A.18) and (A.16), we fix δh and $\delta\Delta$ respectively as follows

$$\begin{aligned}\delta h_i &= \frac{1}{2}\overset{\circ}{\nabla}_i\delta g_k{}^k - \frac{1}{2}\overset{\circ}{\nabla}^j\delta g_{ji} + \frac{1}{2}\delta g_{ij}\overset{\circ}{h}{}^j - \frac{1}{4}\overset{\circ}{h}_i\delta g_k{}^k \\ &+ \frac{1}{4}\overset{\circ}{\Psi}_{\ell_1\ell_2}\delta Z_i{}^{\ell_1\ell_2} + \frac{1}{12}\delta Z_{\ell_1\ell_2\ell_3}\overset{\circ}{X}_i{}^{\ell_1\ell_2\ell_3},\end{aligned}\quad (4.12)$$

and

$$\begin{aligned}\delta\Delta &= \frac{1}{3}\overset{\circ}{\nabla}_i\delta h^i + \frac{1}{12}\overset{\circ}{h}{}^i\overset{\circ}{\nabla}_i\delta g_k{}^k - \overset{\circ}{h}{}^i\delta h_i - \frac{1}{6}\overset{\circ}{\Delta}\delta g_k{}^k - \frac{1}{12}\overset{\circ}{h}_i\overset{\circ}{h}{}^i\delta g_k{}^k + \frac{1}{3}\delta g_{ij}\overset{\circ}{h}{}^i\overset{\circ}{h}{}^j \\ &- \frac{1}{6}\delta g_{ij}\overset{\circ}{\nabla}{}^{i\circ j}h - \frac{1}{6}\overset{\circ}{h}{}^j\overset{\circ}{\nabla}{}^{\circ i}\delta g_{ij} + \frac{1}{9}\delta\Psi_{\ell_1\ell_2}\overset{\circ}{\Psi}{}^{\ell_1\ell_2} - \frac{1}{9}\overset{\circ}{\Psi}{}^{\ell_1}{}_m\overset{\circ}{\Psi}{}^{\ell_2m}\delta g_{\ell_1\ell_2} \\ &- \frac{1}{108}(\overset{\circ}{W} - \overset{\circ}{h} \wedge \overset{\circ}{\Psi})_{\ell_1\ell_2\ell_3}\delta Z^{\ell_1\ell_2\ell_3} + \frac{1}{216}\delta X_{\ell_1\ell_2\ell_3\ell_4}\overset{\circ}{X}{}^{\ell_1\ell_2\ell_3\ell_4} \\ &- \frac{1}{108}\overset{\circ}{X}{}^m{}_{\ell_1\ell_2\ell_3}\overset{\circ}{X}{}^{n\ell_1\ell_2\ell_3}\delta g_{mn} - \frac{1}{216}(\overset{\circ}{h} \wedge \delta Z)_{\ell_1\ell_2\ell_3\ell_4}\overset{\circ}{X}{}^{\ell_1\ell_2\ell_3\ell_4}.\end{aligned}\quad (4.13)$$

We remark that the expressions for $\delta\Psi$, δX and δh are linear in δZ , δg , $\overset{\circ}{\nabla}\delta Z$, $\overset{\circ}{\nabla}\delta g$, whereas the expressions for δW and $\delta\Delta$ involve some second order derivatives acting on δg , δZ . The remaining (unfixed) moduli are δg and δZ .

We shall now proceed to find elliptic systems of PDEs constraining δZ and δg , starting with δZ . By taking the divergence of the Bianchi identity (4.10), we obtain

$$\overset{\circ}{\nabla}{}^i\overset{\circ}{\nabla}_i\delta Z_{k_1k_2k_3} - 3\overset{\circ}{\nabla}_{[k_1}\overset{\circ}{\nabla}{}^{\ell}\delta Z_{k_2k_3]\ell} - 3\overset{\circ}{R}_{\ell[k_1}\delta Z^{\ell}_{k_2k_3]} - \overset{\circ}{\nabla}{}^{\ell}\delta X_{\ell k_1k_2k_3} = 0, \quad (4.14)$$

and by using the $k_1k_2k_3$ component of the gauge field equation (A.14), we express the

divergence of δX as follows

$$\begin{aligned}
\overset{\circ}{\nabla}^{\ell} \delta X_{\ell k_1 k_2 k_3} &= -3 \overset{\circ}{\nabla}_{[k_1} \overset{\circ}{\nabla}^{\ell} \delta Z_{k_2 k_3] \ell} + 3 \overset{\circ}{\nabla}_{[k_1} (\overset{\circ}{h}^{\ell} \delta Z_{k_2 k_3] \ell}) + 6 \overset{\circ}{\nabla}_{[k_1} (\delta g_{k_2}^{\ell} \overset{\circ}{\Psi}_{k_3] \ell}) \\
&+ \frac{3}{2} \overset{\circ}{\nabla}_{[k_1} (\overset{\circ}{\Psi}_{k_2 k_3] \ell} \delta g_{\ell}^{\ell}) + \frac{q}{72} \epsilon_{[k_1 k_2}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7} \overset{\circ}{\nabla}_{k_3]} (\delta Z_{\ell_1 \ell_2 \ell_3} \overset{\circ}{X}_{\ell_4 \ell_5 \ell_6 \ell_7}) \\
&+ \frac{3}{2} (\overset{\circ}{\nabla}_m \delta g_{[k_1}^{\ell} + \overset{\circ}{\nabla}_{[k_1} \delta g_{|m] \ell} - \overset{\circ}{\nabla}^{\ell} \delta g_{m[k_1]}) \overset{\circ}{X}_{k_2 k_3] \ell}^m + \delta g_{ij} \overset{\circ}{\nabla}^i \overset{\circ}{X}_{k_1 k_2 k_3}^j \\
&+ \overset{\circ}{\nabla}^m \delta g_{m \ell} \overset{\circ}{X}_{k_1 k_2 k_3}^{\ell} - \frac{1}{2} \overset{\circ}{\nabla}^{\ell} \delta g_m^m \overset{\circ}{X}_{\ell k_1 k_2 k_3} + \overset{\circ}{\nabla}^{\ell} (\overset{\circ}{h} \wedge \delta Z)_{\ell k_1 k_2 k_3} \\
&- 2 \overset{\circ}{\Delta} \delta Z_{k_1 k_2 k_3} - \delta g_{ij} \overset{\circ}{h}^i \overset{\circ}{X}_{k_1 k_2 k_3}^j + 2 \delta h^{\ell} \overset{\circ}{X}_{\ell k_1 k_2 k_3} + \overset{\circ}{h}^{\ell} \delta X_{\ell k_1 k_2 k_3} \\
&+ 2 (\delta h \wedge \overset{\circ}{\Psi})_{k_1 k_2 k_3} + 2 (\overset{\circ}{h} \wedge \delta \Psi)_{k_1 k_2 k_3} + 3 \delta g_{[k_1}^{\ell} (\overset{\circ}{W} - \overset{\circ}{h} \wedge \overset{\circ}{\Psi})_{k_2 k_3] \ell} \\
&+ \overset{\circ}{h}^{\ell} \delta X_{\ell k_1 k_2 k_3} - 2 \overset{\circ}{h}^{\ell} (\overset{\circ}{h} \wedge \delta Z)_{\ell k_1 k_2 k_3} - \frac{1}{2} \delta g_{\ell}^{\ell} (\overset{\circ}{W} - \overset{\circ}{h} \wedge \overset{\circ}{\Psi})_{k_1 k_2 k_3} \\
&+ \frac{1}{2} (-2 \overset{\circ}{h}^{\ell} \delta g_{\ell}^q + \overset{\circ}{h}^q \delta g_{\ell}^{\ell}) \overset{\circ}{X}_{q k_1 k_2 k_3} + 3 \overset{\circ}{h}^q \delta g_{[k_1}^{\ell} \overset{\circ}{X}_{k_2 k_3] \ell q} \\
&- \frac{q}{24} \epsilon_{k_1 k_2 k_3}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} (\delta \Psi_{\ell_1 \ell_2} \overset{\circ}{X}_{\ell_3 \ell_4 \ell_5 \ell_6} + \overset{\circ}{\Psi}_{\ell_1 \ell_2} \delta X_{\ell_3 \ell_4 \ell_5 \ell_6} - \overset{\circ}{\Psi}_{\ell_1 \ell_2} (\overset{\circ}{h} \wedge \delta Z)_{\ell_3 \ell_4 \ell_5 \ell_6}) \\
&+ \frac{q}{12} \delta g_{\ell_1 m} \epsilon_{k_1 k_2 k_3}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} \overset{\circ}{\Psi}_{\ell_2}^m \overset{\circ}{X}_{\ell_3 \ell_4 \ell_5 \ell_6} + \frac{q}{6} \delta g_{\ell_3 m} \epsilon_{k_1 k_2 k_3}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} \overset{\circ}{\Psi}_{\ell_1 \ell_2} \overset{\circ}{X}_{\ell_4 \ell_5 \ell_6}^m \\
&+ \frac{q}{18} \epsilon_{k_1 k_2 k_3}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} (\overset{\circ}{W} - \overset{\circ}{h} \wedge \overset{\circ}{\Psi})_{\ell_1 \ell_2 \ell_3} \delta Z_{\ell_4 \ell_5 \ell_6} . \tag{4.15}
\end{aligned}$$

On substituting this expression back into (4.14), the $\overset{\circ}{\nabla}_{[k_1} \overset{\circ}{\nabla}^{\ell} \delta Z_{k_2 k_3] \ell}$ terms cancel out. Furthermore, there are a number of terms which are linear in δX , δh , $\delta \Psi$ which are eliminated on using the Bianchi identity (4.10), together with (4.12) and (4.11), producing terms which are linear in δZ , δg , $\overset{\circ}{\nabla} \delta Z$, $\overset{\circ}{\nabla} \delta g$. The resulting PDEs are an elliptic system for δZ , with principal symbol generated by $\overset{\circ}{\nabla}^2$.

Next, we consider the metric moduli δg . The linearized ij component of the Einstein equations is

$$\overset{\circ}{\nabla}^2 \delta g_{ij} - \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \delta g_k^k - (\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j \overset{\circ}{\nabla}_\ell) \delta g_{\ell}^i - (\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_i - \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_\ell) \delta g_{\ell}^j = \mathcal{C}_{ij} , \tag{4.16}$$

where

$$\begin{aligned}
\mathcal{C}_{ij} = & (\overset{\circ}{\nabla}_i \delta g^k_j + \overset{\circ}{\nabla}_j \delta g^k_i - \overset{\circ}{\nabla}^k \delta g_{ij}) \overset{\circ}{h}_k - \overset{\circ}{\nabla}_{(i} (\overset{\circ}{h}_{j)} \delta g_k^k) + 2 \overset{\circ}{\nabla}_{(i} (\delta g_{j)k} \overset{\circ}{h}^k) \\
& + 2 \overset{\circ}{\nabla}_{(i} (\overset{\circ}{\nabla}_{j)} \overset{\circ}{\phi}^b \overset{\circ}{f}_{ab} \delta \phi^a) + 2 \overset{\circ}{\nabla}_{(i} (\delta Z_j^J \overset{\circ}{Q}_{IJ} \overset{\circ}{\Psi}^I) + 2 \overset{\circ}{\nabla}_{(i} (\overset{\circ}{H}_{j)k}^J \overset{\circ}{Q}_{IJ} \delta Z^{Ik}) \\
& - 8 \overset{\circ}{h}_{(i} \delta h_{j)} + 2(-\overset{\circ}{\Delta} + \frac{1}{2} \overset{\circ}{\nabla}_k \overset{\circ}{h}^k - \overset{\circ}{h}_k \overset{\circ}{h}^k) \delta g_{ij} \\
& - 2 \delta g_{(i}^k \overset{\circ}{\nabla}_{|k|} \overset{\circ}{h}_{j)} - 2 \overset{\circ}{h}^k \overset{\circ}{\nabla}_{(i} \delta g_{j)k} + 2 \overset{\circ}{h}^k \overset{\circ}{\nabla}_k \delta g_{ij} - 2 \overset{\circ}{h}_{(i} \overset{\circ}{\nabla}^k \delta g_{j)k} \\
& + 4 \overset{\circ}{h}_k \overset{\circ}{h}_{(i} \delta g_{j)}^k + 2 \overset{\circ}{h}_{(i} \overset{\circ}{\nabla}_{j)} \delta g_k^k + \delta g_k^k (\overset{\circ}{\nabla}_{(i} \overset{\circ}{h}_{j)} - \overset{\circ}{h}_i \overset{\circ}{h}_j) - 2 \delta \Psi_{\ell(i} \overset{\circ}{\Psi}_{j)}^{\ell} \\
& - \overset{\circ}{\Psi}_i^m \overset{\circ}{\Psi}_j^n \delta g_{mn} + (\overset{\circ}{h} \wedge \overset{\circ}{\Psi} - \overset{\circ}{W})_{\ell_1 \ell_2 (i} \delta Z_j)^{\ell_1 \ell_2} + \frac{1}{3} \delta X_{\ell_1 \ell_2 \ell_3 (i} \overset{\circ}{X}_{j)}^{\ell_1 \ell_2 \ell_3} \\
& + \frac{1}{2} \overset{\circ}{X}_{i \ell_1 \ell_2}^m \overset{\circ}{X}_j^{\ell_1 \ell_2 n} \delta g_{mn} - \frac{1}{3} (\overset{\circ}{h} \wedge \delta Z)_{\ell_1 \ell_2 \ell_3 (i} \overset{\circ}{X}_{j)}^{\ell_1 \ell_2 \ell_3} - \frac{1}{3} \overset{\circ}{g}_{ij} \delta \Psi_{\ell_1 \ell_2} \overset{\circ}{\Psi}^{\ell_1 \ell_2} - \frac{1}{6} \delta g_{ij} \overset{\circ}{\Psi}_{mn} \overset{\circ}{\Psi}^{mn} \\
& + \frac{1}{3} \overset{\circ}{g}_{ij} \overset{\circ}{\Psi}_m^{\ell} \overset{\circ}{\Psi}^{mq} \delta g_{\ell q} + \frac{1}{72} \delta g_{ij} \overset{\circ}{X}_{\ell_1 \ell_2 \ell_3 \ell_4} \overset{\circ}{X}^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{36} \overset{\circ}{g}_{ij} \delta X_{\ell_1 \ell_2 \ell_3 \ell_4} \overset{\circ}{X}^{\ell_1 \ell_2 \ell_3 \ell_4} \\
& - \frac{1}{18} \overset{\circ}{g}_{ij} \overset{\circ}{X}^m_{\ell_1 \ell_2 \ell_3} \overset{\circ}{X}^{n \ell_1 \ell_2 \ell_3} \delta g_{mn} + \frac{1}{9} \overset{\circ}{g}_{ij} (\overset{\circ}{W} - \overset{\circ}{h} \wedge \overset{\circ}{\Psi})_{\ell_1 \ell_2 \ell_3} \delta Z^{\ell_1 \ell_2 \ell_3} \\
& - \frac{1}{36} \overset{\circ}{g}_{ij} (\overset{\circ}{h} \wedge \delta Z)_{\ell_1 \ell_2 \ell_3 \ell_4} \overset{\circ}{X}^{\ell_1 \ell_2 \ell_3 \ell_4}, \tag{4.17}
\end{aligned}$$

which is linear in $\delta Z, \delta g, \overset{\circ}{\nabla} \delta Z, \overset{\circ}{\nabla} \delta g$. Furthermore, in (4.16), terms of the form $(\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j \overset{\circ}{\nabla}_\ell) \delta g^{\ell}_i$ can be rewritten as terms linear in δg and the Riemann tensor $\overset{\circ}{R}$, hence can be incorporated into the algebraic term on the RHS. The trace term δg_k^k is again fixed by the elliptic condition (2.13), so (4.16) is an elliptic set of PDEs for the traceless part of δg , with principal symbol generated by $\overset{\circ}{\nabla}^2$.

Taken together, the conditions (4.14) and (4.16) and (2.13) constitute elliptic PDEs on the moduli $\{\delta Z, \delta g\}$, with the remaining moduli $\{\delta W, \delta \Psi, \delta X, \delta h, \delta \Delta\}$ fixed in terms of $\{\delta Z, \delta g\}$ by (4.9), (4.10), (4.11), (4.12) and (4.13). The moduli space is therefore finite dimensional.

5 Conclusions

We have proven that the moduli space of transverse deformations of extremal event horizons in Einstein-Maxwell-Dilaton theory in any dimension, and including topological terms in four and five dimensions, is finite dimensional. We have also demonstrated the same result in eleven-dimensional supergravity, with an arbitrary coupling of the topological term. The treatment of the gauge field moduli in both these theories is very similar. We remark that some of the Einstein-Maxwell-Dilaton theories could be obtained by dimensional reduction of pure gravity in higher dimensions. In such cases, the finite dimensionality of the moduli spaces would be inherited from the higher dimensional calculation via the result of [17]. However, the case of D=11 supergravity cannot be obtained from such a reduction.

There are a number of further theories for which it would be interesting to investigate the moduli space of the transverse deformations, such as gravity coupled to a non-abelian gauge theory. Horizons in such theories have been considered in [19]. Higher derivative theories, for example α' -corrected heterotic theory, could also be considered. Supersymmetric event horizons in this theory were analyzed in [20], and it is unclear how the higher derivative terms would alter the elliptic systems. Beyond proving that the moduli space of deformations is finite-dimensional, the next step is to actually count the moduli, or at least to obtain further conditions on the number of moduli. Work in this direction is in progress.

Appendix A The field equations

In this appendix we list the various components of the field equations, expressed in terms of the horizon data. We denote by $\tilde{\nabla}$ the Levi-Civita connection on \mathcal{S} , restricted to $r = \text{const.}$

A.1 Einstein-Maxwell-Dilaton theory

The components of the Einstein equation (3.2) are:

The $++$ component:

$$\begin{aligned} R_{++} - \frac{1}{2}r^2 Q_{IJ}(W_i^I W^{Ji} - 2W_i^I \Psi^J h^i + \Psi^I \Psi^J h_i h^i) \\ - \frac{1}{4}r^3 \Delta Q_{IJ}(2W_i^I Z^{Ji} - 2\Psi^I Z_i^J h^i) - \frac{1}{8}r^4 \Delta^2(f_{ab} \dot{\phi}^a \dot{\phi}^b + Q_{IJ} Z_i^I Z^{Ji}) = 0. \end{aligned} \quad (\text{A.1})$$

The $+ -$ component:

$$\begin{aligned} R_{+-} + \frac{1}{2(D-2)} Q_{IJ}(2H_{ij}^I H^{Jij} + (D-3)\Psi^I \Psi^J) - \frac{1}{D-2} V \\ + r \frac{4-D}{2(D-2)} Q_{IJ}(W_i^I Z^{Ji} - \Psi^I Z_i^J h^i) \\ + r^2 \left(\frac{4-D}{4(D-2)} \Delta^2 Q_{IJ} Z_i^I Z^{Ji} - \frac{1}{4} \Delta f_{ab} \dot{\phi}^a \dot{\phi}^b \right) = 0. \end{aligned} \quad (\text{A.2})$$

The $--$ component:

$$R_{--} - \frac{1}{2} f_{ab} \dot{\phi}^a \dot{\phi}^b - \frac{1}{2} Q_{IJ} Z_i^I Z^{Ji} = 0. \quad (\text{A.3})$$

The $+i$ component:

$$\begin{aligned} R_{+i} + \frac{1}{2} r Q_{IJ}(\Psi^I W_i^J - \Psi^I \Psi^J h_i - H_{ij}^I W^{Jj} + H_{ij}^I \Psi^J h^j) \\ - \frac{1}{4} r^2 \Delta(f_{ab} \dot{\phi}^a \tilde{\nabla}_i \dot{\phi}^b + Q_{IJ} H_{ij}^I Z^{Jj} - Q_{IJ} \Psi^I Z_i^J) + \frac{1}{4} r^3 \Delta h_i f_{ab} \dot{\phi}^a \dot{\phi}^b = 0. \end{aligned} \quad (\text{A.4})$$

The $-i$ component:

$$R_{-i} - \frac{1}{2}f_{ab}\dot{\phi}^a\tilde{\nabla}_i\phi^b - \frac{1}{2}Q_{IJ}(\Psi^I Z_i^J + Z^{Ij}H_{ij}^J) + \frac{1}{2}rh_i f_{ab}\dot{\phi}^a\dot{\phi}^b = 0 . \quad (\text{A.5})$$

The ij component:

$$\begin{aligned} R_{ij} &- \frac{1}{2}f_{ab}\tilde{\nabla}_i\phi^a\tilde{\nabla}_j\phi^b - \frac{1}{2}Q_{IJ}H_{ik}^I H_j^{Jk} - \frac{1}{D-2}g_{ij}V \\ &+ \frac{1}{4(D-2)}g_{ij}Q_{IJ}(H_{k\ell}^I H^{Jk\ell} - 2\Psi^I\Psi^J) \\ &+ r\left(f_{ab}h_{(i}\dot{\phi}^a\tilde{\nabla}_{j)}\phi^b + \frac{1}{D-2}g_{ij}Q_{IJ}Z_\ell^I(W^{J\ell} - \Psi^J h^\ell) - Q_{IJ}(W_{(i}^I Z_{j)}^J - \Psi^I Z_{(j}^J h_{i)})\right) \\ &+ r^2\left(-\frac{1}{2}h_i h_j f_{ab}\dot{\phi}^a\dot{\phi}^b - \frac{1}{2}\Delta Q_{IJ}Z_i^I Z_j^J + \frac{1}{2(D-2)}\Delta g_{ij}Q_{IJ}Z_\ell^I Z^{J\ell}\right) = 0 . \end{aligned} \quad (\text{A.6})$$

The components of the gauge field equation (3.3) are:

The $+$ component:

$$\begin{aligned} &-\tilde{\nabla}^i W_i^I + h^i W_i^I + \Psi^I \tilde{\nabla}^i h_i - h^i h_i \Psi^I - \frac{1}{2}\tilde{d}h_{ij}H^{Iij} \\ &Q^{IK}\partial_a Q_{KJ}(\Psi^J h_i \tilde{\nabla}^i \phi^a - W_i^J \tilde{\nabla}^i \phi^a) + h_i \tilde{\nabla}^i \Psi^I \\ &+ r\left(Q^{IK}\partial_a Q_{KJ}(\dot{\phi}^a h^i W_i^J - h^2 \Psi^J \dot{\phi}^a - \frac{1}{2}\Delta Z_i^J \tilde{\nabla}^i \phi^a - \frac{1}{2}\Delta \dot{\phi}^a \Psi^J)\right. \\ &\left.- \frac{1}{2}\Delta \dot{\Psi}^I + h^i \dot{W}_i^I + \frac{3}{2}\Delta h^i Z_i^I - Z_i^I \tilde{\nabla}^i \Delta - \frac{1}{2}\Delta \tilde{\nabla}^i Z_i^I - h^2 \dot{\Psi}^I - \Psi^I h^i \dot{h}_i\right. \\ &\left.+ (\frac{1}{2}\dot{g}_k{}^k h_j - h^i \dot{g}_{ij})(W^{Ij} - \Psi^I h^j) - \frac{1}{4}\dot{g}_k{}^k \Delta \Psi^I + h_i \dot{h}_j H^{Iij}\right) \\ &+ \frac{r^2}{2}\left(Q^{IK}\partial_a Q_{KJ}\dot{\phi}^a \Delta h^i Z_i^J + (\frac{1}{2}\dot{g}_k{}^k h_j - h^i \dot{g}_{ij})\Delta Z^{Ij}\right. \\ &\left.+ 2\dot{\Delta} h^i Z_i^I + \Delta h^i \dot{Z}_i^I - \Delta \dot{h}^i Z_i^I\right) = 0 . \end{aligned} \quad (\text{A.7})$$

The $-$ component:

$$\begin{aligned} &\dot{\Psi}^I - \tilde{\nabla}^i Z_i^I + h^i Z_i^I - Q^{IK}\partial_a Q_{KJ}\tilde{\nabla}^i \phi^a Z_i^J + \frac{1}{2}\dot{g}_k{}^k \Psi^I + Q^{IK}\partial_a Q_{KJ}\dot{\phi}^a \Psi^J \\ &+ r\left(Q^{IK}\partial_a Q_{KJ}\dot{\phi}^a h^i Z_i^J + h^i \dot{Z}_i^I + \dot{h}^i Z_i^I + (\frac{1}{2}\dot{g}_k{}^k h_j - h^i \dot{g}_{ij})Z^{Ij}\right) = 0 . \end{aligned} \quad (\text{A.8})$$

The i component:

$$\begin{aligned}
& W_i^I + \tilde{\nabla}^j H_{ji}^I - \Psi^I h_i - h^j H_{ji}^I + Q^{IK} \partial_a Q_{KJ} \tilde{\nabla}^j \phi^a H_{ji}^J \\
& + r \left(Q^{IK} \partial_a Q_{KJ} (\dot{\phi}^a W_i^J - \Psi^J \dot{\phi}^a h_i - \dot{\phi}^a h^j H_{ji}^J) + 2\Delta Z_i^I \right. \\
& + \dot{W}_i^I - \dot{\Psi}^I h_i - \Psi^I \dot{h}_i - h^j \dot{H}_{ji}^I - \dot{h}^j H_{ji}^I + \frac{1}{2} \dot{g}_k{}^k (W_i^I - \Psi^I h_i) \\
& \left. - \frac{1}{2} \dot{g}_k{}^k h^j H_{ji}^I + h^j \dot{g}_{jk} H^{Ik}{}_i + h_j H^{Ijk} \dot{g}_{ki} - \dot{g}_{ij} (W^{Ij} - \Psi^I h^j) \right) \\
& + r^2 \left(Q^{IK} \partial_a Q_{KJ} \Delta \dot{\phi}^a Z_i^J + \dot{\Delta} Z_i^I + \Delta \dot{Z}_i^I + \frac{1}{2} \dot{g}_k{}^k \Delta Z_i^I - \dot{g}_{ij} \Delta Z^{Ij} \right) = 0. \quad (\text{A.9})
\end{aligned}$$

The scalar field equation (3.4) reduces in terms of the horizon data to:

$$\begin{aligned}
& \tilde{\nabla}_i \tilde{\nabla}^i \phi^a - h^i \tilde{\nabla}_i \phi^a + f^{ab} (\partial_c f_{bd} - \frac{1}{2} \partial_b f_{cd}) \tilde{\nabla}_i \phi^c \tilde{\nabla}^i \phi^d \\
& + \frac{1}{4} f^{ab} \partial_b Q_{IJ} (2\Psi^I \Psi^J - H_{ij}^I H^{Jij}) - f^{ab} \partial_b V \\
& + r \left(\tilde{\nabla}^j \phi^a (h^i \dot{g}_{ij} - \frac{1}{2} \dot{g}_k{}^k h_j) + \dot{\phi}^a (2\Delta + h_i h^i) - \dot{\phi}^a \tilde{\nabla}^i h_i - 2h_i \tilde{\nabla}^i \dot{\phi}^a \right. \\
& \left. - f^{ab} (\partial_c f_{bd} + \partial_d f_{bc} - \partial_b f_{cd}) \dot{\phi}^c \tilde{\nabla}_i \phi^d h^i - \frac{1}{2} f^{ab} \partial_b Q_{IJ} (Z_i^I W^{Ji} - h^i Z_i^I \Psi^J) \right) \\
& + r^2 \left(-h^j \dot{\phi}^a (h^i \dot{g}_{ij} - \frac{1}{2} \dot{g}_k{}^k h_j) + (\Delta + h^i h_i) \ddot{\phi}^a + \dot{\Delta} \dot{\phi}^a + 2h_i \dot{h}^i \dot{\phi}^a + \frac{3}{4} \dot{g}_k{}^k \Delta \dot{\phi}^a \right. \\
& \left. + f^{ab} (\partial_c f_{bd} - \frac{1}{2} \partial_b f_{cd}) \dot{\phi}^c \dot{\phi}^d (\Delta + h^i h_i) - \frac{1}{4} f^{ab} \partial_b Q_{IJ} \Delta Z_i^I Z^{Ji} \right) = 0. \quad (\text{A.10})
\end{aligned}$$

A.2 D=11 Supergravity

The gauge field equation (4.4) decomposes into the following components.

The $+ - k$ component:

$$\begin{aligned}
& \tilde{\nabla}^\ell \Psi_{\ell k} - r h^\ell \dot{\Psi}_{\ell k} - \frac{1}{2} r (\tilde{d}h)^{mn} Z_{mnk} + r h^\ell \dot{g}_\ell{}^q \Psi_{qk} - \frac{1}{2} r h^q \Psi_{qk} \dot{g}_m{}^m + r h^\ell \dot{g}_k{}^q \Psi_{\ell q} \\
& = \frac{q}{576} \epsilon_k{}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8} X_{\ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_5 \ell_6 \ell_7 \ell_8} - \frac{qr}{72} \epsilon_k{}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8} h_{\ell_1} Z_{\ell_2 \ell_3 \ell_4} X_{\ell_5 \ell_6 \ell_7 \ell_8}. \quad (\text{A.11})
\end{aligned}$$

The $+k_1 k_2$ component:

$$\begin{aligned}
& \tilde{\nabla}^\ell (W - h \wedge \Psi)_{\ell k_1 k_2} + 2r \Delta h^\ell Z_{\ell k_1 k_2} + h^\ell (W - h \wedge \Psi)_{\ell k_1 k_2} \\
& + r h^\ell (\dot{W} - \dot{h} \wedge \Psi - h \wedge \dot{\Psi})_{\ell k_1 k_2} + \frac{1}{2} (-\tilde{d}h^{mn} + r h^m \dot{h}^n) X_{mnk_1 k_2} \\
& + \frac{1}{2} r \tilde{d}h^{mn} (h \wedge Z)_{mnk_1 k_2} + \frac{1}{2} r (-2h^\ell \dot{g}_\ell{}^q + h^q \dot{g}_m{}^m) (W - h \wedge \Psi)_{qk_1 k_2} \\
& - r h^\ell \dot{g}_{k_1}{}^q (W - h \wedge \Psi)_{\ell q k_2} + r h^\ell \dot{g}_{k_2}{}^q (W - h \wedge \Psi)_{\ell q k_1} \\
& = \frac{q}{72} \epsilon_{k_1 k_2}{}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7} (W - h \wedge \Psi)_{\ell_1 \ell_2 \ell_3} X_{\ell_4 \ell_5 \ell_6 \ell_7} - \frac{qr}{18} \epsilon_{k_1 k_2}{}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7} h_{\ell_1} Z_{\ell_2 \ell_3 \ell_4} W_{\ell_5 \ell_6 \ell_7}. \quad (\text{A.12})
\end{aligned}$$

The $-k_1k_2$ component:

$$\begin{aligned}\tilde{\nabla}^\ell Z_{\ell k_1 k_2} &= \dot{\Psi}_{k_1 k_2} + h^\ell Z_{\ell k_1 k_2} - \dot{g}_{k_1}^\ell \Psi_{\ell k_2} + \dot{g}_{k_2}^\ell \Psi_{\ell k_1} + \frac{1}{2} \dot{g}_m^m \Psi_{k_1 k_2} \\ &+ \frac{q}{72} \epsilon_{k_1 k_2}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7} Z_{\ell_1 \ell_2 \ell_3} X_{\ell_4 \ell_5 \ell_6 \ell_7}\end{aligned}\quad (\text{A.13})$$

and the $k_1k_2k_3$ component:

$$\begin{aligned}&\tilde{\nabla}^\ell (X - rh \wedge Z)_{\ell k_1 k_2 k_3} + 2r \Delta Z_{k_1 k_2 k_3} - (h^\ell + r \dot{h}^\ell) X_{\ell k_1 k_2 k_3} + (W - h \wedge \Psi)_{k_1 k_2 k_3} \\ &+ r(\dot{W} - \dot{h} \wedge \Psi - h \wedge \dot{\Psi})_{k_1 k_2 k_3} - 3r \dot{g}_{[k_1}^\ell (W - h \wedge \Psi)_{k_2 k_3] \ell} - r h^\ell \dot{X}_{\ell k_1 k_2 k_3} \\ &\quad + 2r h^\ell (h \wedge Z)_{\ell k_1 k_2 k_3} + \frac{1}{2} r \dot{g}_m^m (W - h \wedge \Psi)_{k_1 k_2 k_3} \\ &\quad - \frac{1}{2} r (-2h^\ell \dot{g}_\ell^q + h^q \dot{g}_m^m) X_{q k_1 k_2 k_3} - 3r h^q \dot{g}_{[k_1}^\ell X_{k_2 k_3] \ell q} \\ &= -\frac{q}{24} \epsilon_{k_1 k_2 k_3}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} \Psi_{\ell_1 \ell_2} (X - rh \wedge Z)_{\ell_3 \ell_4 \ell_5 \ell_6} + \frac{qr}{18} \epsilon_{k_1 k_2 k_3}^{\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6} (W - h \wedge \Psi)_{\ell_1 \ell_2 \ell_3} Z_{\ell_4 \ell_5 \ell_6} .\end{aligned}\quad (\text{A.14})$$

It should be noted that in (A.11)-(A.14), we have suppressed the appearance of terms of the form \dot{Z} , and also $\dot{\Delta}Z, \dot{h}Z, \dot{g}Z$, because as we explain in section 4.1, Z is linear in the moduli, and hence these terms are suppressed in the moduli space calculation. Furthermore, (A.12) has been simplified by making use of (A.13) to eliminate the $\tilde{\nabla}^\ell Z_{\ell k_1 k_2}$ term from (A.12).

The Einstein equation (4.5) decomposes into the following components:

The $++$ component:

$$R_{++} = \frac{1}{12} r^2 (W - h \wedge \Psi)_{\ell_1 \ell_2 \ell_3} (W - h \wedge \Psi + r \Delta Z)^{\ell_1 \ell_2 \ell_3} . \quad (\text{A.15})$$

The $+ -$ component:

$$\begin{aligned}R_{+-} &= -\frac{1}{6} \Psi_{\ell_1 \ell_2} \Psi^{\ell_1 \ell_2} + \frac{1}{36} r (W - h \wedge \Psi)_{\ell_1 \ell_2 \ell_3} Z^{\ell_1 \ell_2 \ell_3} \\ &- \frac{1}{144} X_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{72} r (h \wedge Z)_{\ell_1 \ell_2 \ell_3 \ell_4} X^{\ell_1 \ell_2 \ell_3 \ell_4} .\end{aligned}\quad (\text{A.16})$$

The $+i$ component:

$$\begin{aligned}R_{+i} &= -\frac{1}{4} r \Psi_{\ell_1 \ell_2} (W - h \wedge \Psi + \frac{1}{2} r \Delta Z)_i^{\ell_1 \ell_2} - \frac{1}{12} r^2 (W - h \wedge \Psi)_{\ell_1 \ell_2 \ell_3} (h \wedge Z)_i^{\ell_1 \ell_2 \ell_3} \\ &+ \frac{1}{12} r X_i^{\ell_1 \ell_2 \ell_3} (W - h \wedge \Psi + \frac{1}{2} r \Delta Z)_{\ell_1 \ell_2 \ell_3} .\end{aligned}\quad (\text{A.17})$$

The $-i$ component:

$$R_{-i} = \frac{1}{4} \Psi_{\ell_1 \ell_2} Z_i^{\ell_1 \ell_2} + \frac{1}{12} Z_{\ell_1 \ell_2 \ell_3} X_i^{\ell_1 \ell_2 \ell_3} . \quad (\text{A.18})$$

The ij component:

$$\begin{aligned}
R_{ij} &= -\frac{1}{2}\Psi_{il}\Psi_j{}^\ell + \frac{1}{2}r(W - h \wedge \Psi + \frac{1}{2}r\Delta Z)_{\ell_1\ell_2(i}Z_j)^{\ell_1\ell_2} + \frac{1}{12}X_{i\ell_1\ell_2\ell_3}X_j{}^{\ell_1\ell_2\ell_3} \\
&+ \frac{1}{6}r(h \wedge Z)_{\ell_1\ell_2\ell_3(i}X_j)^{\ell_1\ell_2\ell_3} + \frac{1}{12}g_{ij}\Psi_{\ell_1\ell_2}\Psi^{\ell_1\ell_2} - \frac{1}{144}g_{ij}X_{\ell_1\ell_2\ell_3\ell_4}X^{\ell_1\ell_2\ell_3\ell_4} \\
&- \frac{1}{18}rg_{ij}(W - h \wedge \Psi)_{\ell_1\ell_2\ell_3}Z^{\ell_1\ell_2\ell_3} + \frac{1}{72}rg_{ij}(h \wedge Z)_{\ell_1\ell_2\ell_3\ell_4}X^{\ell_1\ell_2\ell_3\ell_4} . \quad (\text{A.19})
\end{aligned}$$

Appendix B Ricci Tensor

The Ricci tensor components in the frame basis (2.3) are

$$\begin{aligned}
R_{++} &= r^2\left(\frac{1}{2}\tilde{\nabla}_i\tilde{\nabla}^i\Delta - \frac{3}{2}h^i\tilde{\nabla}_i\Delta - \frac{1}{2}\Delta\tilde{\nabla}_ih^i + \Delta h^ih_i + \frac{1}{4}\tilde{d}h_{ij}\tilde{d}h^{ij}\right) \\
&+ r^3\left(-h^i\tilde{\nabla}_i\dot{\Delta} - \frac{1}{2}\dot{\Delta}\tilde{\nabla}_ih^i + \frac{1}{2}\dot{h}^i\tilde{\nabla}_i\Delta + \frac{1}{2}\Delta\tilde{\nabla}_i\dot{h}^i + 2\dot{\Delta}h^ih_i - \Delta h^i\dot{h}_i\right. \\
&+ \left.\frac{1}{4}\Delta\dot{g}_k{}^kh_ih^i - \frac{1}{2}\Delta\dot{g}_{ij}h^ih^j - \frac{1}{4}\dot{g}_k{}^kh_i\tilde{\nabla}^i\Delta + \frac{1}{2}\dot{g}_{ij}h^i\tilde{\nabla}^j\Delta - h^i\dot{h}^j\tilde{d}h_{ij}\right) + \mathcal{O}(2) , \quad (\text{B.1})
\end{aligned}$$

$$\begin{aligned}
R_{+-} &= \frac{1}{2}\tilde{\nabla}_ih^i - \Delta - \frac{1}{2}h^ih_i \\
&+ r\left(\frac{1}{2}\tilde{\nabla}_i\dot{h}^i - \frac{1}{2}\Delta\dot{g}_k{}^k - \frac{1}{4}\dot{g}_k{}^kh_ih^i + \frac{1}{2}\dot{g}_{ij}h^ih^j - 2\dot{\Delta} - 2h^i\dot{h}_i\right) + \mathcal{O}(2) , \quad (\text{B.2})
\end{aligned}$$

$$R_{--} = \mathcal{O}(2) , \quad (\text{B.3})$$

$$\begin{aligned}
R_{+i} &= r\left(\frac{1}{2}\tilde{\nabla}^k\tilde{d}h_{ik} + h^j\tilde{d}h_{ji} + \Delta h_i - \tilde{\nabla}_i\Delta\right) \\
&+ r^2\left(-\frac{1}{2}\Delta\dot{h}_i + \frac{1}{2}\dot{h}_i\tilde{\nabla}_jh^j + h^j\tilde{\nabla}_j\dot{h}_i - \frac{1}{2}h_i\tilde{\nabla}_j\dot{h}^j - \frac{1}{2}\tilde{\nabla}_i(h_j\dot{h}^j)\right. \\
&+ 2\dot{\Delta}h_i - \frac{1}{2}\tilde{\nabla}_i\dot{\Delta} + 3h^jh_{[i}\dot{h}_{j]} - \frac{3}{4}\Delta\dot{g}_{ij}h^j + \frac{1}{2}\dot{g}_{ij}\tilde{\nabla}^j\Delta + \frac{1}{4}\Delta\tilde{\nabla}^j\dot{g}_{ij} \\
&- \left.\frac{1}{4}\tilde{\nabla}_i(\Delta\dot{g}_k{}^k) + \frac{3}{8}\Delta h_i\dot{g}_k{}^k + \frac{1}{2}\dot{g}_i{}^j\tilde{d}h_{jk}h^k + \frac{1}{4}\dot{g}_k{}^kh^j\tilde{d}h_{ji} + \frac{1}{2}\tilde{d}h_i{}^j\dot{g}_{jk}h^k\right) + \mathcal{O}(2) , \quad (\text{B.4})
\end{aligned}$$

$$R_{-i} = \dot{h}_i + \frac{1}{2}\tilde{\nabla}^j\dot{g}_{ji} - \frac{1}{2}\tilde{\nabla}_i\dot{g}_k{}^k + \frac{1}{4}h_i\dot{g}_k{}^k - \frac{1}{2}\dot{g}_{ij}h^j + \mathcal{O}(2) , \quad (\text{B.5})$$

$$\begin{aligned}
R_{ij} &= \tilde{\mathcal{R}}_{ij} + \tilde{\nabla}_{(i} h_{j)} - \frac{1}{2} h_i h_j \\
&+ r \left(\tilde{\nabla}_{(i} \dot{h}_{j)} - 3 h_{(i} \dot{h}_{j)} + \left(-\Delta + \frac{1}{2} \tilde{\nabla}_k h^k - h_k h^k \right) \dot{g}_{ij} - \dot{g}_{(i}{}^k \tilde{\nabla}_{|k|} h_{j)} \right. \\
&- h^k \tilde{\nabla}_{(i} \dot{g}_{j)k} + h^k \tilde{\nabla}_k \dot{g}_{ij} - h_{(i} \tilde{\nabla}^k \dot{g}_{j)k} + 2 h_k h_{(i} \dot{g}_{j)}^k + h_{(i} \tilde{\nabla}_{j)} \dot{g}_k{}^k \\
&\left. + \frac{1}{2} \dot{g}_k{}^k (\tilde{\nabla}_{(i} h_{j)} - h_i h_j) \right) + \mathcal{O}(2) .
\end{aligned} \tag{B.6}$$

Here $\mathcal{O}(2)$ consists of terms linear in $\ddot{h}, \ddot{\Delta}, \ddot{g}$, and terms quadratic in $\dot{h}, \dot{\Delta}, \dot{g}$, which play no role in the moduli space calculations.

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Data Management:

No data beyond those presented and cited in this work are needed to validate this study.

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