A Four-body Model for the Breakup of Borromean Nucleus $^{22}\text{C}$

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A Borromean system is a bound 3-body system where no 2-body subsystems are bound. In nuclear physics, a nucleus that can be modelled as a Borromean system is called a Borromean nucleus; $^6$He and $^{11}$Li are good examples of this. Recent research suggests that this Borromean nature should also be exhibited by $^{22}$C, the heaviest-known carbon isotope.

In this PhD thesis, a schematic approach is taken to study reactions involving Borromean nuclei. Hyperspherical formalism (HH) and coordinate space Faddeev (CSF) method are used for creating their 3-body bound state wave functions.

We formulate the reactions of a Borromean nucleus with a stable target at incident energies ranging from tens of (MeV) to a few hundred (MeV); we adopt a 4-body reaction model to deepen our understanding of the reaction mechanism involving Borromean nuclei. The Glauber-WKB framework is used to describe these reactions, which is well-suited for these incident energies.

Introducing Watson-Migdal final state interaction, we calculate the E1 strengths for Borromean nuclei so as to elucidate their breakup mechanism and we explore the possibility of the existence of a soft dipole mode. We also calculate the differential breakup cross sections to see how the post-collision interaction can have an impact on the cross sections.

As far as $^{22}$C is concerned, it is found that the reactions are mainly focused on the forward angle region, and the contributions from the higher order terms are not significant. This implies that the non-eikonal trajectories do not play a crucial role in the reaction mechanism. Also, both E1 distributions and breakup cross sections seem to sensitive to the 2n-separation energies of the bound state wave functions, but the E1 distributions and the cross sections to $1^-$ continuum state seem not to be sensitive
to the FSIs; cross sections to $0^+$ and $2^+$ continuum states seem to be sensitive to the FSIs. Our findings does not support the view that, if an soft dipole mode exists, it is induced by the FSIs.
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CHAPTER 1

Introduction

1.1 \(^{22}\text{C} as a Borromean nucleus

In nuclear physics, the driplines mark the boundaries of the heaviest and lightest bound nuclear isotopes \([1]\). However, a nuclear isotope lying just outside the line can exist in a resonant state \([2]\). There also exist weakly bound nuclei located in the nuclear dripline regions, and the density distributions of such nuclei have long tails, which has an impact on their reaction cross sections. In fact, in the mid-1980s, the reaction cross section of \(^{11}\text{Li}\) on stable carbon target was measured and the value was around 1040 (mb) \([3]\). The value is large to the extent that it deviated from the trend of the cross sections of lighter lithium isotopes, and indicated that the rms radius of \(^{11}\text{Li}\) had to be very large. This result was interpreted as a sign of nuclear halo: the core \(^9\text{Li}\) of \(^{11}\text{Li}\) was surrounded by two correlated valence neutrons \([4]\).

Since narrow parallel momentum distributions of \(^9\text{Li}\) in the breakup of \(^{11}\text{Li}\) were observed in 1992, the concept of a halo nucleus has been developed further \([5]\). An intuitive explanation as to why a narrow parallel momentum distribution is evidence of a halo is due to the Heisenberg uncertainty principle, which states that the wave function of a valence nucleon that is localised in momentum space will be extended in configuration space. More correctly, a weakly bound nucleus with an extended wave function in configuration space will, due to the Heisenberg principle, have a localised distribution in momentum space.

From the nuclear shell perspective, there has been a growing interest in \(^{22}\text{C}\) among
the neutron rich exotic nuclei: the nucleus, which consists of 6 protons and 16 neutrons, has N=16 shell closure. Ozawa [6] indicated that the traditional magic number of N=8 disappeared for those nuclei for which \( T_z = \frac{3}{2} \) (for odd N and even Z), and magic number of N=20 disappeared at \( T_z = 4 \) (for odd N and odd Z). They showed that those nuclei with \( T_z = \frac{5}{2}, 3, \frac{7}{2}, 4 \) gave rise to a new magic number of N=16, suggesting that the appearance of the new magic number was related to neutron halo formation [6]. It has been reported that \(^{20}\text{C}\) has its first \( 2^+ \) excited state at 1588(20) (keV), and that the similarity between carbon and oxygen in terms of the behaviour of the evolution of the \( 2^+ \) energies disappeared at N=14 [7]. This suggested that the gap between \( 0d_{5/2} \) and \( 1s_{1/2} \) at N=14 disappears for the neutron-rich carbon isotopes.

In addition to the behaviour of the \( 2^+ \) excited states, the existence of a nuclear halo in \(^{22}\text{C}\) shows that the behaviour of carbon and oxygen isotopes is different near the neutron dripline. At present, the heaviest known carbon isotope is \(^{22}\text{C}\), which is considered to be a 2n-halo nucleus, where the 2n-separation energy is lower than the 1n-separation one [8].

A Borromean nucleus is a 3-body bound system which consists of core and two valence neutrons, and each of its 2-body subsystems is unbound, but can exist as a resonance state [2]. This feature of a Borromean nucleus appears in a 2n-halo nucleus. For instance, if we remove one neutron from \(^{6}\text{He}\), it becomes unbound. Likewise, we see this feature for \(^{11}\text{Li}\).

As the research of carbon isotopes goes on, growing attention has been paid to \(^{22}\text{C}\) in recent years. A recent one-proton removal experiment indicated that \(^{21}\text{C}\) is an unbound nucleus [9]. In the experiment, an upper limit of 0.070 (MeV) for the 2n-separation energy of \(^{22}\text{C}\) was extracted by using a renormalised zero-range three-body model. When it comes to the structure of \(^{22}\text{C}\), not much is known. It is thought that there exists a 2+ excited state [7], but this has not been confirmed experimentally.

The number of the confirmed halo nuclei is seven [2]: \(^{6}\text{He}, ^{11}\text{Li}, ^{11}\text{Be}, ^{14}\text{Be}, ^{15}\text{C}, ^{17}\text{B}\) and \(^{19}\text{C}\). As far as 2n-halo nuclei are concerned, it is possible that the heaviest existing carbon isotope is not the heaviest 2n-halo nucleus, and that a Borromean system can appear in heavier nuclei. Indeed, there is evidence that \(^{34}\text{Ne}\) and \(^{37}\text{Na}\) are bound, which indicates the mixing between \( 0d_{5/2} \) state and \( 0f_{7/2} \) [10]. The instability of \(^{33}\text{Ne}\) and \(^{36}\text{Na}\) has been experimentally reported [11]. It can be expected that these
1.1. **$^{22}$C AS A BORROMEAN NUCLEUS**

nuclei have two valence nucleons, and possibly the Borromean nature can appear as well.

Also, there might exist a possibility that Borromean nuclei have their tiny dipole modes, above their breakup thresholds, which emanate from the core oscillation-type motion relative to the valence neutrons [12]. For a neutron halo nucleus, the rms matter and charge radii are not identical, and so the excitation to dipole modes with low frequencies can occur [13]. The low-lying dipole excitations of $^6$He and $^{11}$Li are regarded as a more general continuum 3-body response, not a resonant mode [14].

To better understand these Borromean systems, it is important to characterize how 2n-halos and Borromean properties relate to each other, and to what extent the properties play a crucial role in forming the core+n+n structure. Thus we adopt a 3-body model (core+n+n model) and calculate various reaction observables in order to study these nuclei.

Our aim is to elucidate the properties of Borromean nuclei. To do that, after calculating their bound state wave functions, we develop a 4-body model for the reactions involving these nuclei. Accordingly, we calculate various observables, such as scattering cross sections and elastic breakup cross sections. Elastic breakup cross sections of a projectile provide us with the information about the excitation energies and potential resonances of the projectile, and we can find out about the continuum structure of it. When it comes to $^{22}$C, one of the ingredients of our 4-body model is the interaction between a stable target and $^{20}$C, which is not well known. We therefore use a folding model to calculate this interaction. Breakup S-matrix elements in which an initial state is bound and a final state is unbound will eventually be computed under several assumptions, then differential cross sections will be computed for elastic breakup reactions.

This thesis consists of eight chapters. For the remainder of this introduction chapter, we discuss the current status of $^{22}$C. Chapter 2 argues the basic theories of structure of $^{22}$C within a 3-body model, where the hyperspherical formalism to describe 3-body bound systems is introduced. In Chapter 3, we explain our basic reaction models for a 2-body system. Our framework of the reaction is based on the Glauber and WKB approximation; historically the founders of the concept of a halo nucleus used Glauber-type calculations to analyse their findings [3], because such
semi-classical approaches were appropriate at the high energies used.

In the Glauber model, it is assumed that an incident particle’s trajectory takes a straight line: an eikonal trajectory. Considering non-eikonal trajectories of a projectile, we add higher order phase shifts to the Glauber phase shift. Also, we improve the WKB by introducing a correction term which was first introduced by Rosen and Yennie [15]. We will see that, generally, the term makes the differential cross section closer to one which is calculated by the partial wave approach.

In Chapter 4, we introduce the 4-body Jacobi coordinates and develop the formalism for the reactions of a 4-body system by extending the 2-body model. Elastic scatterings and reaction cross sections which involve Borromean nuclei are considered within the 4-body framework. A folding model to estimate the interaction between $^{20}$C and a target is discussed as well.

Chapter 5 presents the E1 strength distributions of Borromean nuclei. Chapter 6 shows the elastic breakup formalism of those nuclei, and presents those numerical calculations. In Chapter 7, we consider a Watson-Migdal final state interaction (FSI), introducing this type of FSI to see how the interaction has an effect on the E1 strength and the breakup cross section. In Chapter 8, we discuss our results, and conclusions are drawn.

1.2 Review of research on $^{22}$C

Properties of $^{22}$C have been slowly investigated since the early 2000s; this nucleus is still not known well [16, 17].

The 2003 evaluation said that $^{22}$C’s 1n-separation energy and 2n-separation energy were $0.750 \pm 1.030$ (MeV) and $0.42 \pm 0.94$ (MeV), respectively [18]. Three years after that, Horiuchi and Suzuki studied the ground state structure of this nucleus in a 3-body model (core+n+n), assuming that it had the N=14 subshell closure. The fact that the rms matter radius of $^{19}$C is larger than that of $^{20}$C made this assumption more reasonable [19]. Their study indicated that the 2n-separation energy of $^{22}$C was from 0.39 to 0.57 (MeV), and also indicated that the nucleus existed almost as an s-wave 2n-halo nucleus. They found that the two valence neutrons formed spin-singlet S=0 with a probability of about 99 per cent [19], which indicated that that
1.2. REVIEW OF RESEARCH ON $^{22}\text{C}$

Figure 1.1: The rms matter radii (fm) for the He (upper left), Li (upper right), Be (lower left) and C (lower right) isotopes. For all these graphs, the vertical axis is the rms radius (fm) and the horizontal axis is the neutron number. Data are from [2], [8], [20] and [29].

S=0 components were dominant in the ground state of $^{22}\text{C}$.

In 2010, Tanaka et al measured the reaction cross sections for $^{22}\text{C} + \text{p}$ at 40 MeV/A and the cross section was $1338 \pm 274$ (mb) [20]. This experimental value was significantly larger than the theoretical value of 957 (mb) which Abu-Ibrahim et al obtained by using the Glauber optical limit approximation (OLA) [21]. It was reported that the large experimental cross section was due to the s-wave dominance of the two valence neutrons of $^{22}\text{C}$, which indicated N=14 subshell closure for the nucleus [20].

Also, the experimentally extracted rms matter radius of $^{22}\text{C}$ was $5.40 \pm 0.90$ (fm). This value remarkably deviated from the trend of the rms radii of the carbon isotopes. Figure 1.1 shows the rms radii of He, Li, Be and C isotopes. From the graph of carbon, it is apparent that the extracted radius is significantly larger than the trend. The ratio of the rms radius of $^{22}\text{C}$ to that of $^{20}\text{C}$ is about 1.81, and this value is comparable to the ratio for $^6\text{He}$ and $^{11}\text{Li}$. In terms of the rms radii, the $^4\text{He}$-to-$^6\text{He}$ ratio is about 1.71, the $^9\text{Li}$-to-$^{11}\text{Li}$ ratio is about 1.50. The $^{12}\text{Be}$-to-$^{14}\text{Be}$ ratio is about 1.20, and it is less comparable to other 2n-halo nuclei. Hence, in this regard there is similarity between $^6\text{He}$ and $^{22}\text{C}$ among the 2n-halo nuclei. Also, it may be worthwhile noting that the rms radius of $^{11}\text{Be}$ is larger than that of $^{12}\text{Be}$, and similar pattern is seen for other 2n-halo nuclei from the graph: for $^3\text{He}$ and $^4\text{He}$, $^8\text{Li}$ and $^9\text{Li}$ and $^{19}\text{C}$ and $^{20}\text{C}$.

Recent research carried out by Gaudefroy et al suggested that the 1n and 2n-separation energies are $0.3913 \pm 0.080$ (MeV) and $-0.137 \pm 0.446$ (MeV), respectively [22]. Although the uncertainty is still large, the 2n-separation energy of $^{22}\text{C}$ is less
CHAPTER 1. INTRODUCTION

<table>
<thead>
<tr>
<th>$\sqrt{\langle r_m^2 \rangle}$ (fm)</th>
<th>$S_{2n}$ (MeV)</th>
<th>approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inakura [24]</td>
<td>$\sim 3.03 - 3.34$</td>
<td>$\sim 1.0 - 5.0$</td>
</tr>
<tr>
<td>Abu-Ibrahim [21]</td>
<td>3.6-4.1</td>
<td></td>
</tr>
<tr>
<td>Horiuchi [19]</td>
<td>$\sim 3.6 - 3.7$</td>
<td>$\sim 0.39 - 0.57$</td>
</tr>
<tr>
<td>Ershov [14]</td>
<td>$\sim 0.01$</td>
<td></td>
</tr>
<tr>
<td>Kucuk [25]</td>
<td>$\sim 3.5 - 3.7$</td>
<td></td>
</tr>
<tr>
<td>Sharma [23]</td>
<td>3.133</td>
<td>5.755</td>
</tr>
<tr>
<td>Fortune [26]</td>
<td>&lt; 0.220</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.1: Theoretical values of the rms matter radius and the 2n-separation energy of $^{22}$C.

than 0.329 (MeV). Horiuchi’s conclusion that the probability of finding the two valence neutrons of $^{22}$C in the $1s_{1/2}$ was $94.1 \pm 2.6$ per cent [19] was compatible not only with the measured weak 2n-separation energy of $^{22}$C [22], but also with the measured large reaction cross section of the nucleus on a proton target [20]. This view was further supported by a neutron-removal experiment which involved a carbon target [16]. Narrow parallel momentum distribution of the residual $^{20}$C fragment was mostly due to s-wave contributions, which suggests that the valence nucleons of $^{22}$C are dominantly in $1s_{1/2}$ orbits.

As to the theoretical values of the rms radius and 2n-separation energy of $^{22}$C, those values are listed in Table 1.1. Their approaches are various: mean field approach [23, 24], stochastic variational method (SVM) [19], Hyperspherical method (HH) [14, 25] and a specific empirical formula which gives the 2n-separation energy [26].

No bound excited states of $^{22}$C has been discovered so far, probably because of the large sd gap between $1s_{1/2}$ and $0d_{3/2}$. It is thought that a 2+ excited state can exit at a higher energy level for the nucleus [7]. As for the core, $^{20}$C (an even-even nucleus), it has a bound 2+ excited state at around 1.6 (MeV): Stanoiu et al reported the excited state at 1.588(20) (MeV) [7], and Petri et al did that at 1.618(6) (MeV) [27]. Its reduced transition probability was $B(E2; 2^+_1 \rightarrow 0^+_{g.s.}) = 7.5^{+3.0}_{-1.7}\text{(stat)}^{+1.0}_{-0.4}\text{(syst)}$. 
1.2. REVIEW OF RESEARCH ON $^{22}\text{C}$

$(e^2\text{fm}^4)$ [27]. Besides, $^{20}\text{C}$ is thought to be oblate-shaped as a result of the short gap between $0d_{5/2}$ with $1s_{1/2}$ [28].
CHAPTER 2. THE 3-BODY MODEL

The 3-body model

In the present chapter, we describe a Borromean nucleus in our 3-body model. Our 3-body model enables us to regard 2n-halo nuclei, such as $^{22}$C, as 3-body systems (core+n+n), and to explore the idea that these nuclei appear Borromean in nature, by which is meant that any two-body subsystems are unbound. Borromean nuclei, including $^{22}$C, are weakly bound, and accordingly the 3-body model is suitable for describing them. To calculate the wave functions of the 3-body state, we work within the framework of hyperspherical harmonics (HH) and the solutions of coordinate space Faddeev (CSF) equations. It is often convenient to introduce the Jacobi coordinates to treat the 3-body system, and this chapter describes the coordinate system and these frameworks, then it tests its applicability to Borromean nuclei. Here we restrict ourselves to considering the cases where the core is inert and not deformed.

2.1 Jacobi coordinates

In this section, the Jacobi coordinates for a 3-body system are explained. Consider a system which consists of three particles: 1, 2 and 3, with masses $m_1$, $m_2$, $m_3$, and positions (relative to the same origin) $r_1$, $r_2$, $r_3$, respectively. In this case, we have three sets of Jacobi coordinates [31]:

$$
\mathbf{x}_i \overset{\text{def}}{=} \sqrt{\frac{m_jm_k}{m_j + m_k}} (r_k - r_j) \quad (2.1.1)
$$

$$
\mathbf{y}_i \overset{\text{def}}{=} \sqrt{\frac{m_i(m_k + m_j)}{m_1 + m_2 + m_3}} \left( r_i - \frac{m_k r_k + m_j r_j}{m_k + m_j} \right) \quad (2.1.2)
$$
2.1. JACOBI COORDINATES

and we assign the indices of (1,2,3) to $x, y$ in a way that:

$$ (i, j, k) = \begin{cases} (1, 2, 3) & \text{for coordinates set A} \\ (2, 3, 1) & \text{for coordinates set B} \\ (3, 1, 2) & \text{for coordinates set C} \end{cases} $$

(2.1.3)

Not only $x, y$, but also $\frac{m_1 r_1 + m_2 r_2 + m_3 r_3}{m_1 + m_2 + m_3}$ are the 3-body Jacobi coordinates, because, in the laboratory system, the number of independent variables $(r_1, r_2, r_3)$ for the 3-body system is nine. Indeed the latter is the centre-of-inertia of the 3-body system, and therefore, for the sake of convenience by choosing the latter as the coordinate origin, we decrease the number of the independent variables from nine to six.

With regard to the choice of coordinates set we have a degree of freedom, and therefore in this case we use set C $(x_3, y_3)$. As we need to consider the case where a 3-body system consists of a core and two valence neutrons, we adopt such a convention as particle 3 represents the core. Particle 1 and 2 are then the valence neutrons. Figure 2.1 shows these sets of coordinates: set A, B and C.

![Figure 2.1: The 3 types of 3-body unnormalised Jacobi coordinates. The system consists of valence particle 1 and 2, and a core. The X-basis is for $(x_1, y_1)$, Y-basis for $(x_2, y_2)$, T-basis for $(x_3, y_3)$.](image)

Set A is called the X-basis coordinate system, and B, C are the Y-basis and T-basis, respectively. Consequently, if we write down the Jacobi coordinates of the T-basis, they are:

$$ x = \frac{1}{\sqrt{2}} (r_2 - r_1), \quad y = \sqrt{\frac{2A_c}{A_c + 2}} \left( r_c - \frac{r_2 + r_1}{2} \right), $$

(2.1.4)
where $r_1, r_2$ are points of valence particle 1 and 2, respectively. $r_c$ is the point of the core. The subscripts of $x, y$ are omitted to avoid confusion. In addition, we need to define the unnormalised 3-body Jacobi coordinates $(X, Y)$:

$$X = r_2 - r_1, \quad Y = r_c - \frac{r_2 + r_1}{2},$$  \hspace{1cm} (2.1.5)

which are convenient to work out how the spatial configuration of a 3-body system is. Also, for later convenience, we define reduce masses for $x$ and $y$ as [32]:

$$\mu_x = \frac{m_1 + m_2}{m_1 + m_2} = m_n \times \frac{1}{2}, \quad \mu_y = \frac{m_c(m_1 + m_2)}{m_c + m_1 + m_2} = m_n \times \frac{A_c \cdot 2}{A_c + 2}. \quad (2.1.6)$$

### 2.2 3-body hyperspherical harmonics formalism

In the HH formalism, a 3-body bound wave function is calculated by solving a set of hyperradial coupled equations. These involve the hyperspherical coordinates, which are related to the 3-body (normalised) Jacobi coordinates $x, y$ as follows [33]:

$$\rho = \sqrt{x^2 + y^2} \quad (2.2.1)$$

$$\alpha = \arctan \left( \frac{x}{y} \right). \quad (2.2.2)$$

In the hyperspherical coordinates, $\rho \in [0, \infty)$ and $\alpha \in [0, \frac{\pi}{2}]$ are called the hyperradius and hyperspherical angle, respectively. Then we need to introduce the 3-body hyperspherical harmonics [32, 33]:

$$\mathcal{Y}_{KLM,L}^{\lambda_x \lambda_y}(\Omega_5) = \left[ Y_{\lambda_x} \otimes Y_{\lambda_y} \right]_{LM} N_n^{\frac{l_x + 1}{2}, \frac{l_y + 1}{2}} \left( \sin \alpha \right)^{l_x} \left( \cos \alpha \right)^{l_y} P_n^{\frac{l_x + 1}{2}, \frac{l_y + 1}{2}} \left( \cos \alpha \right) \cdot (2.2.3)$$

For this definition, we have several quantum numbers: the orbital angular momentum and its projection ($l_x, m_x$), ($l_y, m_y$) associated with the Jacobi coordinates $x, y$, respectively. $L$ is the addition of the two orbital angular momenta $l_x, l_y$, and $K$ is the hyperspherical angular momentum, which is also known as the hypermomentum [33, 35]. Here $n$ is the number of nodes of the Jacobi polynomial $P_n^{a,b}(z)$, which is expressed as [36]:

$$P_n^{a,b}(x) = \frac{(a + 1)^n}{n!} F(-n; n + a + b + 1; a + 1 + \frac{1 - x}{2}) \quad . \quad (2.2.4)$$

---

1Strictly speaking, there exist several conventions of the 3-body hyperspherical harmonics [34, 35]

2$K$ is also called the hypermoment [38, 14].
In addition, $N_n^{\alpha\beta}$ is the normalisation coefficient given by

$$N_n^{l_x+\frac{1}{2},l_y+\frac{1}{2}} = \sqrt{2(K+2)} \frac{\Gamma(n+l_x+l_y+2)\Gamma(n+1)}{\Gamma(n+l_x+\frac{3}{2})\Gamma(n+l_y+\frac{3}{2})}, \quad (2.2.5)$$

with the quantum numbers related via

$$K = 2n + l_x + l_y. \quad (2.2.6)$$

The definition also contains $\Omega_5$ which represents the set of 5 angle parameters: $\theta_x, \phi_x$, $\theta_y, \phi_y$ and the hyperspherical angle $\alpha$.

Accordingly, in the hyperspherical coordinates, a bound state wave function is written as:

$$\phi^{(3B)T}_{IMJ}(\rho, \Omega_5) = \frac{1}{(\rho)^{5/2}} \sum_{K\gamma} \chi_{K\gamma}(\rho) \left[ \mathcal{Y}^{l_xl_y}_{KL}(\Omega_5) \otimes X_S \right]_{IMJ} X_{TMT}, \quad (2.2.7)$$

$$= \frac{1}{(\rho)^{5/2}} \sum_{K\gamma} \chi_{K\gamma}(\rho) \Upsilon_{K\gamma IMJ}(\Omega_5, \sigma) \otimes X_{TMT}, \quad (2.2.8)$$

where $\gamma$ is a set of quantum numbers; $\gamma = \{l_x, l_y, L, S\}$, and $\Upsilon_{K\gamma IMJ}$ is the vector coupling of a hyperspherical harmonic $\mathcal{Y}^{l_xl_y}_{KL}$ and a spin wave function $X_{SM}$:

$$\Upsilon_{K\gamma IMJ}(\Omega_5, \sigma) = (-1)^{L-S+M_I} \sqrt{2I+1} \sum_{M_L, M_S} \begin{pmatrix} L & S & I \\ M_L & M_S & -M_I \end{pmatrix} \mathcal{Y}^{l_xl_y}_{KLM_L} X_{SM}.$$

(2.2.9)

For bound states, the hyperradial function $\chi_{K\gamma}$ satisfies the set of coupled differential equations:

$$\left[ -\frac{\hbar^2}{2m_n} \left( \frac{d^2}{d\rho^2} - \frac{(K+\frac{3}{2})(K+\frac{5}{2})}{\rho^2} \right) + V_{K\gamma, K\gamma} - E \right] \chi_{K\gamma}(\rho) = - \sum_{K'\gamma' \neq K\gamma} V_{K'\gamma', K\gamma} \chi_{K'\gamma'}, \quad (2.2.10)$$

where $m_n$ is the nucleon mass. Besides, $\{V_{K'\gamma', K\gamma}\}$ are the matrix elements of the whole interaction of the 3-body system [37]:

$$V_{K'\gamma', K\gamma} = \langle \Upsilon_{K\gamma}(\Omega_5) | V | \Upsilon_{K'\gamma'}(\Omega_5) \rangle \quad (2.2.11)$$

$$V = V_{12} + V_{1c} + V_{2c} + V_{3B}, \quad (2.2.12)$$

where $V_{3B}$ is a 3-body interaction of the system. $V_{1c}$ is the interaction between the valence particle 1 and the core of the 3-body system, and similarly $V_{2c}$ for the valence particle 2.
particle 2 and the core. \( V_{12} \) is the NN interaction between the valence particles. In our case, the absolute values of \( V_{1c}, V_{2c}, V_{12} \) decrease rapidly, because the two valence particles are neutrons. It follows that, in the asymptotic region, the hyperradial equation (2.2.10) reduces to the equation which has the second derivative term, the centrifugal term and the energy term. In the region nearest to the coordinate origin, the centrifugal term is dominant, and so the hyperradial function behaves as

\[
\chi_{K\gamma} \sim \rho^{K+5/2} \quad \text{as } \rho \to 0 , \tag{2.2.13}
\]

\[
\chi_{K\gamma} \sim \exp(-\kappa \rho) \quad \text{as } \rho \to +\infty , \tag{2.2.14}
\]

where \( \hbar \kappa = \sqrt{2m_n|E|} \) [38].

### 2.3 Coordinate space Faddeev approach

To calculate the 3-body ground state wave function, we use the Coordinate space Faddeev approach (CSF) to solve the coupled equations of motions of the 3-body system [39]:

\[
(E - H_0 - V_{2c}) \phi_1 = V_{2c}(\phi_2 + \phi_c) \tag{2.3.1}
\]

\[
(E - H_0 - V_{1c}) \phi_2 = V_{1c}(\phi_c + \phi_1) \tag{2.3.2}
\]

\[
(E - H_0 - V_{12}) \phi_c = V_{12}(\phi_1 + \phi_2) , \tag{2.3.3}
\]

where \( H_0 \) is the kinetic term of the system. Here \( \phi_1, \phi_2 \) and \( \phi_c \) are the components of the total wave function [38, 39]:

\[
\phi = \phi_1 + \phi_2 + \phi_c . \tag{2.3.4}
\]

Let us express the total Hamiltonian as

\[
H = H_0 + V_{12} + V_{2c} + V_{1c} . \tag{2.3.5}
\]

It follows that the full time-independent Schrödinger equation is written as

\[
(E - H)\phi = 0 . \tag{2.3.6}
\]

Then using the Jacobi coordinates, we expand the wave functions as [39]

\[
\phi_{\nu} = \frac{1}{x_{\nu}y_{\nu}} \sum_j R_{j,\nu}(x_{\nu}, y_{\nu}) \left[ Y_{lj}(\Omega_{x_{\nu}}) \otimes \left[ X_{sj} \otimes Y_{lj}(\Omega_{x_{\nu}}) \right] \right]_{j,M_l} \tag{2.3.7}
\]

where \( \nu = 1, 2, c . \)
Substituting this expression into (2.3.1), (2.3.2), (2.3.3), we have sets of 2-dimensional equations for $R_{j\nu}$. Then, solving these equations, we obtain the bound state wave functions.

In the 3-body system, the nn interaction can push one of the 2 valence nucleons into a forbidden state, and so we need to consider Pauli blocking. To do this, we use the PP method [33]: introducing a projection operator or using a pseudopotential, which allows us to project out an occupied states and to solve the hyperradial equations in an allowed subspace. Suppose that $|f_1\rangle$ and $|f_2\rangle$ are forbidden states for valence neutron 1 and 2, respectively. Then, the Faddeev equations which include the Pauli blocking effects are given by

\begin{align}
(E - H_0 - V_{2c}) |\phi_1\rangle &= V_{2c} (|\phi_2\rangle + |\phi_c\rangle) + F_1 |f_1\rangle, \\
(E - H_0 - V_{1c}) |\phi_2\rangle &= V_{1c} (|\phi_c\rangle + |\phi_1\rangle) + F_2 |f_2\rangle, \\
(E - H_0 - V_{12}) |\phi_c\rangle &= V_{12} (|\phi_1\rangle + |\phi_2\rangle),
\end{align}

where the functions $F_1$ and $F_2$ the functions of $y_1$ and $y_2$, respectively. The forbidden states $|f_1\rangle$ and $|f_2\rangle$ involve $x_1$ and $x_2$, respectively. Also, we have

\begin{align}
\langle \phi | f_1 \rangle = \langle \phi | f_2 \rangle = 0,
\end{align}

that is, the total wave function is orthogonal to the forbidden states. In practice, the functions $F_1$ and $F_2$ are expanded with regard to (known) spline functions $\{S_n\}_n$ [39]:

\begin{align}
F_1 &= \sum_n \lambda_{1,n} S_n(y_1), \\
F_2 &= \sum_n \lambda_{2,n} S_n(y_2).
\end{align}

Then we substitute them into (2.3.9), (2.3.10) and (2.3.11), and determine the coefficients $\lambda_{1,n}, \lambda_{2,n}$ for each $n$.

## 2.4 Bound state wave functions of $^6$He

In this section, we carry out numerical calculations for Borromean nuclei, according to the framework we present the previous sections. Since $^6$He is the lightest Borromean nucleus and its properties have been well investigated, it is worthwhile modelling it as a test case. The fact that the ground state spin-parity of the nucleus...
is the same as that of \(^{22}\)C motivates us to do this. In other words, we can investigate
the similarity between the lightest and heaviest known Borromean nuclei by applying
our model to both.

First, with regard to the 3-body HH calculations of \(^{6}\)He, we used Woods-Saxon(WS)
central potential and a spin-orbit term as the core-valence interaction [33]:
\[
V_{\text{cn}} = -v_0(r)f_{R,a}(r) + \frac{v_{\text{so}}}{r} \frac{\partial f_{R,a}(r)}{\partial r}(1 \cdot s) \tag{2.4.1}
\]
where the form factor \(f_{R,a}(r)\) is given by
\[
f_{R,a}(r) = \frac{1}{1 + e^{\frac{r-R}{a}}} \tag{2.4.2}
\]
In terms of its parametrisation, we follow that of the previous research which involved
HH [33]. We set the values of the relating parameters in the following: the WS reduced
radius is \(R = 2.0 \text{ (fm)}\), where the reduced radius is defined by \(R = r_0 \times \frac{A^{1/3}}{3} \). Here
\(r_0\) is the radial parameter of the WS potential. The diffuseness parameter is \(a = 0.7\)
(fm). For the spin-orbit term, we use \(R_{\text{so}} = 1.5, a_{\text{so}} = 0.35 \text{ (fm)}\) as its reduced radius
and diffuseness, respectively. As to the depth of the WS central potential, we set
\(v_0 = 21.5 \text{ (MeV)}\) for \(l = 2\), and \(v_0 = 0\) for \(l \geq 3\), and vary \(v_0\) for \(l = 0, 1\). Besides,
we used Gogny-Pires-De Tourreil (GPT) potential for the valence-valence interaction
whose theoretical \(1S_0\) scattering length is \(a_S = -22.12 \text{ (fm)}\) and its experimental
value is \(a_s = -23.7 \pm 0.1 \text{ (fm)}\) [40]:
\[
V_{\text{nn}} = V_C(r) + V_T(r)S_{12} + V_{LS}(r)(L \cdot S) + V_{LL}(r)L_{12} \tag{2.4.3}
\]
\[
L_{12} = (\sigma_1 \cdot \sigma_2)\mathbf{L}^2 - \frac{1}{2} \left\{ (\sigma_1 \cdot \mathbf{L})(\sigma_2 \cdot \mathbf{L}) + (\sigma_2 \cdot \mathbf{L})(\sigma_1 \cdot \mathbf{L}) \right\} \tag{2.4.4}
\]
\[
S_{12} = \frac{3(\sigma_1 \cdot r)(\sigma_2 \cdot r)}{r^2} - (\sigma_1 \cdot \sigma_2) . \tag{2.4.5}
\]
Also, we consider a HH 3-body interaction which is given by the following form [33]:
\[
V_{3\text{B,K}'}\gamma \gamma' \gamma''(\rho) = \frac{-v_{3\text{B}}\delta_{K'K} \delta_{\gamma'\gamma} \delta_{\gamma'\gamma''}}{1 + \left( \frac{\rho}{\rho_0} \right)^3} . \tag{2.4.6}
\]
In the numerical calculation on \(^{6}\)He, we used Efaddy [41], software to calculate
bound wave functions in a 3-body model.

The maximum value of the hyperangular momentum is \(K_{\text{max}}\), and here we set
\(K_{\text{max}} = 46\). In terms of the forbidden states for this case, we excluded the \(0s_{1/2}\) state.

The results of the numerical calculations are shown in Table 2.1, where several
parameter sets are shown as well. For example, for parametrisation od9 of Table 2.1,
\( v_0 = 43.6 \text{ (MeV)}, v_{so} = 40.0 \text{ (MeV)} \) and \( v_{3B} = 1.00 \text{ (MeV)} \), \( E_{3B} = -0.950 \text{ (MeV)} \) and its rms matter radius is 2.513 (fm). Also, we can see that the difference between \( dt_1 \) and \( dt_2 \) is the depth of the 3-body interaction; the potential of the 3-body interaction for \( dt_1 \) is, in depth, about six per cent smaller than that for \( dt_2 \). The resultant rms radius of \( dt_1 \) is about 0.5 per cent larger than that for \( dt_2 \).

<table>
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<tr>
<th>type</th>
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<th>( v_0 )</th>
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<th>( v_{3B} )</th>
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Table 2.1: Parametrisations of potentials for bound state wave functions of \( ^6\text{He} \). The rms matter radius and the eigen-energies of them are shown as well.

In terms of our analysis of the 2-body system (core+n), its shell model description says that the eigen-energy of \( 0s_{1/2} \) is \(-10.135 \text{ (MeV)} \), and \( 0p_{3/2}, 0p_{1/2} \) are unbound for od9. The single particle spectrum is seen in Figure 2.2, where spectra of od11 and dt1 are shown together. Figure 2.2 and Table 2.1 say that the WS central potential for od9 is, in depth, about 1.5 per cent smaller than that for od11 (and the spin-orbit potential for od9 is about 33 per cent greater than that for od11). The binding for od9, with regard to the \( 0s_{1/2} \) state, is about 3 per cent weaker than that for od11. The WS central potential for dt1 is about 3 per cent smaller than that for od11, which yields the result that the \( 0s_{1/2} \) binding for dt1 is about 6 per cent weaker than
that for od11. These results suggest that in this case the 1 per cent change of the depth of the central potential should lead to 2 per cent change of the strength of the $0s_{1/2}$ binding, provided that the change of the depth of the potential is small.

Figure 2.2: The single particle spectrum for the system of $^4$He – n, where the parametrisations of the interaction potentials are od9, od11 and dt1. Both $0p_{3/2}$ and $0p_{1/2}$ are unbound for these parametrisations.

The square modulus, $|\chi(x, y)|^2$ of the bound state wave function is shown in Figure 2.3, where we see two distinct peaks. This result is interpreted in Figure 2.4: one peak represents spatial configuration (a) where the two valence neutrons are relatively strongly correlated and located far from the core, and the other peak represents configuration (b) where the valence neutrons are less correlated and the distance between the core and the valence neutrons are shorter. It is thought that these two configurations are stationary states as the bound states of $^6$He.

Figure 2.3: The square moduli of $\chi(x, y)$ function of $^6$He (od11). The left graph shows the 3D spatial configuration of the modulus, and the right for its 2D projection colour map. Here, for convenience, unnormalised Jacobi coordinates X, Y are used.

In terms of the rms matter radius of $^6$He, as a whole, the radii listed in Table
2.4. BOUND STATE WAVE FUNCTIONS OF $^6$He

Figure 2.4: Major spatial configurations of $^6$He as a 3-body system.

2.1 are compatible with the extracted values from the experimental data: $2.45 \pm 0.10$ (fm) from $p + ^6$He scattering at around 700 (MeV/A) [2, 42], $2.46 \pm 0.09$ (fm) from $^6$He+$^{12}$C scattering at around 800 (MeV/A) [43]. In terms of the eigen-energy of $^6$He, all the listed eigen-energies in Table 2.1 are within 0.06 (MeV) of the observed value of $-0.97546(23)$ (MeV) [44]. Also, the parametrisation dt2 yields $E_{3B} = -0.9756$ (MeV), $\sqrt{\langle r_m^2 \rangle} = 2.487$ (fm) at $K_{\text{max}} = 20$; these values are consistent with the previous study which involved HH [37].

As to the convergence of the eigen-energy and rms matter radius, they are shown in Figure 2.5. As is seen in the graphs of Figure 2.5, the eigen-energies of od11 and dt1 behave in a similar way. The behaviour of the eigen-energy of dt1 is similar to that of od9, but dt1’s eigen-energy is smaller than od9’s one; the depth of the 3-body potential for od9 is about 33 per cent smaller than that for dt1, while the depth of the WS potential for od9 is about 1.3 per cent greater than that for dt1. At around...
$K_{\text{max}}$ of 20, there appears about 2 or 3 per cent change of the eigen-energy for od9 (and about 1 or 2 per cent change for dt1), while much less than 1 per cent change of the rms matter radius for both od9 and dt1. The way of convergence of the rms matter radius for od9 is similar to that for od11 and dt1.

### 2.5 Bound state wave function for $^{22}\text{C}$

In this section, we present the numerical results for $^{22}\text{C}$. In our analysis, the WS central potential and a spin-orbit term are used for the interaction between the core and the valence neutron. For the interaction between the valence particles, we use the same GPT interaction [40] in the previous section. As to the 3-body interaction, the potential form is the same as the previous section, and its values depend on its parametrisation as is seen soon later.

As for the valence neutron-core interaction, we use the WS radial parameter $r_0 = 1.25$ (fm), and the diffuseness parameter $a = 0.6$ (fm), as a basic parametrisation. This parametrisation is a mixture of Horiuchi’s ones [19]. (They used four different parametrisations, and set $(r_0, a) = (1.3, 0.6)$ for the first one and $(r_0, a) = (1.25, 0.65)$ for the remaining ones [19].) They found that it was important to introduce an additional repulsive interaction for the s-waves in order to achieve unbound $1s_{1/2}$ and bound $0d_{5/2}$ simultaneously [19]. In our model, instead of introducing such a potential, we alter the depth of the interaction for the s-waves. Here we use $v_0 = 42.0$ (MeV) as the depth of the central potential for $l \geq 1$. Also, we set $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$ and $0d_{5/2}$ the forbidden states. Experimentally, it is known that the core has an excited bound state at about 1.6 (MeV) [7, 27], but we assume an inert core in this analysis.

The results of our numerical calculation are shown in Table 2.2 with which the specification of remaining parameters is shown together. As to the detail of the parametrisations, for example, parametrisations ic1-6 fix $v_0 = 33.4$ (MeV) and $v_{so} = 14.0$ (MeV), changing $v_{3B}$ from 0 to 1.1 (MeV). Parametrisations dk1-5 fix $v_{so} = 12.0$ and $v_{3B} = 0.4$, increasing $v_0$ from 33.0 to 33.8 and so on.

From our analysis of the 2-body system (core+n), for ic1, the eigen-energies are around $-18.029$, $-12.974$ and $-11.167$ (MeV), for the $0s_{1/2}$, $0p_{3/2}$ and $0p_{1/2}$ states,
2.5. BOUND STATE WAVE FUNCTION FOR $^{22}\text{C}$

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<th>$v_{3B}$ (MeV)</th>
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Table 2.2: Bound state wave functions of $^{22}\text{C}$. Parametrisations uj5-8 use $v_{so} = 16.0$ (MeV) and $v_{3B} = 0.2$ (MeV), changing $v_0$ from 34.1 to 33.3 (MeV). For hpa1, $r_0 = 1.30$ (fm), $a = 0.60$ (fm) and the depth of the central potential is the same for the all orbital angular momenta.
respectively. The eigen-energy for the $0d_{5/2}$ is $-1.443$ (MeV). Figure 2.6 shows the energy spectra of the system for hpa1, ic1 and zm3. Decreasing, by about 3 per cent, the depth of the WS central potential for the s-wave, we see that the $0s_{1/2}$ state becomes about 4 per cent more weekly bound. Compared to the results in the previous section, the depth of the central potential has a slightly weak impact on the output in this case. Also, the eigen-energies for hpa1 are around $-19.023$, $-9.848$, $-4.764$, $-0.994$ (MeV), for $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$ and $0d_{5/2}$ states, respectively. These are consistent with the previous study about the shell configurations of the system [19].

![Figure 2.6: The energy spectra of the $^{20}\text{C} - \text{n}$ system for three different parametrisations: hpa1, ic1 and zm3. For the all spectra, $0d_{5/2}$ is at the highest position in the figure, $0p_{3/2}$ is at the second highest, and the $0s_{1/2}$ is at the lowest position.](image)

The graphs of the square modulus $|\chi|^2$ are shown in Figure 2.7 where, for convenience, unnormalised 3-body Jacobi coordinates $(X, Y)$ are used. We find two peaks in the graph, and so there are two major spatial configurations for the nucleus: the valence neutrons are correlated and keeping a distance from the core (configuration (a) in Figure 2.8), and the three constituents are balanced each other to form a triangle-shaped (configuration (b) in Figure 2.8).

Convergence of the 3-body eigen-energies and rms radii of the $^{22}\text{C}$ wave functions are studied as well, and shown in Figure 2.9. Here we set $K_{\text{max}}$ = 46 for HH and 40 for CSF; calculations of the latter, in general, take longer than the former, because they consume larger memories. Therefore we restrict ourselves here to keeping the $K_{\text{max}}$ for the CSF calculations smaller than that for the HH ones. From the data

$^{3}$ The $0d_{5/2}$ state is bound by, at most, 2.93 (MeV), because this is the neutron separation energy of $^{20}\text{C}$ [48].
2.5. BOUND STATE WAVE FUNCTION FOR $^{22}$C

Figure 2.7: The square moduli of $\chi(x, y)$ function of $^{22}$C (hd3). For convenience, here the unnormalised Jacobi coordinates $X, Y$ are used.

in those graphs, it is apparent that the speed of the convergence of the eigen-energy is slower than that of the rms radius. The percentage change of the eigen-energy between $K = 40$ and $K = 46$ is 0.260 percent, while that of rms radius is 0.112 percent.

We compare the results of the present section to those of the previous section. Figure 2.10 compares the bound state wave function of $^6$He to that of $^{22}$C, and the both two graphs have two major peaks. Indeed, we find that configurations (a) and (b) in Figure 2.4 correspond to Figure 2.8’s (a) and (b), respectively. Thus in this regard, we see the similarity between $^6$He and $^{22}$C.

The difference between them is as follows. Compared to $^6$He, the bound state wave function of $^{22}$C is spatially more widespread, and its two major peaks are broader to the extent that parts of the peaks are merged together. The positions of the core are more distant from the centre-of-inertia of the two valence neutrons in the
major configurations of \(^{22}\)C. For the case of \(^6\)He, the biggest peak corresponds to the "dineutron" configuration (a) in Figure 2.4, while the most probable state for \(^{22}\)C corresponds to state (b) in Figure 2.8, where the three constituents are roughly equally spaced in a triangular configuration.

It could be argued that these differences appear because the core-n potential for \(^6\)He is significantly different from that for \(^{22}\)C: in fact, our finding is that the \(^{20}\)C-n potential plays a bigger role in the spatial configuration of \(^{22}\)C than the \(\alpha\)-n potential does in \(^6\)He.

Figure 2.11 shows the four graphs of the square moduli of \(\chi(x, y)\) for the parametrisation \(hd2\), \(ie1\), \(ic1\) and \(ic6\). Comparing the \(hd2\) to the \(ie1\), we see that the peak for the dineutron configuration of \(ie1\) is stronger than that for \(hd2\). This result seems to be natural, since the depth of the spin-orbit interaction for \(ie1\) is about 56 % shallower than that for \(hd2\). As the core-n interaction becomes weaker, the valence neutrons are more likely to keep a larger distance (in a stationary state) from the core, and
2.5. BOUND STATE WAVE FUNCTION FOR $^{22}\text{C}$

![Graph showing square moduli of the $\chi$ functions for $^{6}\text{He}$ and $^{22}\text{C}$]

Figure 2.10: The square moduli of the $\chi$ functions of $^{6}\text{He}$ and $^{22}\text{C}$. The left graph is for $^{6}\text{He}$ bound state wave function (od14) and the right for $^{22}\text{C}$ (hd3).

Accordingly the probability that a configuration involve a larger distance between the core and the center-of-inertia of the two valence neutrons becomes higher. The probability that they form a dineutron-like object becomes higher as well. Indeed, as is seen in Figure 2.11, the probability that the valence neutrons are in the shape of a dineutron-like object for ic1 is higher than that for ie1, because the WS central potential for the former is about 3% shallower in depth than that for the latter.

In terms of the 3-body force, it is not clear how this force has an effect on the spatial configuration of the system, because it works between all three objects of a system. But it could be argued that, if the 3-body force exists, the stronger the force becomes, the more strongly bound a bound 3-body system becomes, because the force is attractive. The bound state wave function for ic6 is more strongly bound than that for ic1, as the 3-body force acts for ic6 and it does not for ic1. As a result, the spatial configuration of ic6 is less widely spread than that of ic1, as can be seen from Figure 2.11. This suggests that the force evenly affects the whole configuration of the 3-body system, rather than affecting a portion of the configuration such as the probabilities that the valence neutrons form the dineutron.
Figure 2.11: The square moduli of $\chi(x,y)$ for hd2 (the upper left), ie1 (the upper right), ic1 (the lower left) and ic6 (the lower right). For convenience, the unnormalised Jacobi coordinates are used for the vertical and horizontal axes.
CHAPTER 3

2-body reaction models

In the last chapter we discussed the bound state wave functions of a 3-body system within the frameworks of our 3-body model, using HH and CSF and the 3-body Jacobi coordinates. As we wish to study the reaction processes which involve these 3-body projectiles and a structureless target, we need to build a 4-body reaction model. But before doing that, we need to consider a 2-body reaction model which involves just a projectile and a target.

The Glauber model for a 2-body reaction was developed in the late 1950s [45], and after that the model was popularised as a convenient tool to extract the sizes of nuclei [47]. The model has been used as a traditional tool to calculate cross sections for experimental nuclear data analysis [2, 3, 46]. We note that using this model for the reactions involving secondary beams of $^{11}$Li and a carbon target at around 800 MeV/A led to the discovery of a halo nucleus [3].

The Glauber model is based on the eikonal approximation, which is valid at high energies; typically it is applied for a reaction whose centre-of-mass energy is much higher than the depth of its interaction potential. For the reaction where the incident energies are from about 100 to 1000 MeV/A, the model is often used [21, 47, 48]. Although it is preferable to use the model for high energy reactions, there exists usage of Glauber-type calculations for the reactions at lower energies. For instance the reaction cross section measurement for $^{22}$C+p at 40 MeV/A was carried out, and the Glauber calculation was used to extract the rms matter radius of $^{22}$C [20].

The Glauber model was extended to take into account the effects of non-eikonal
trajectories in the early 1970s [49]. In the extended model, contributions from non-straight line trajectories were regarded as the correction to the contribution from the eikonal-trajectory. The model, which involves the WKB approximation, made it possible that cross sections of reactions at lower energies were calculated. For instance the calculations of cross sections for $^{10}$Be + $^{12}$C at 10 MeV/A and 25 MeV/A were carried out, and it turned out that the model was valid at those energies [50].

In the present chapter, we use the Glauber model with and without the non-eikonal corrections. Firstly, we need to obtain the formula for the (differential) cross section for a 2-body point elastic scattering within the Glauber-WKB framework. But before deriving the formula for the scattering amplitude, we wish to see how the WKB approximation is linked to the Glauber model. Also, we aim at improving the WKB approximation to develop our model further. One approach was offered by Good and Miller [52, 53]. A radial equation without a potential is regarded as an unperturbed equation whose solutions are known; once a potential is put into the equation, it becomes regarded as a perturbed one. They used an unperturbed equation and its perturbed one to obtain the approximated wave functions for radial equations. Rosen and Yennie generalised Good’s method, and obtained the higher order phase shift [15].

Thus in the present chapter, we discuss our 2-body reaction model under the Glauber-WKB regime. Viewing the WKB approximation from the Good-Miller (GM) perspective, we obtain the WKB phase shift.

Then expanding it in powers of $\xi U$, where $\xi = \frac{1}{\hbar v k}$ is a parameter and $U$ is an interaction potential, we obtain the lowest term and a few higher order terms of the phase shift. We see that the Glauber model is a limited case of the non-eikonal model: the Glauber phase shift is the zeroth order term of the expansion series for the WKB phase shift. Then we obtain the Rosen-Yennie phase shift to improve the WKB approximation.

### 3.1 Good-Miller’s approach to the WKB

The WKB method is one of the approximate methods to solve Schrödinger equation [94], and Good-Miller’s approach to the WKB was somewhat different to the
3.1. GOOD-MILLER’S APPROACH TO THE WKB

traditional WKB [52, 53]. Rosen and Yennie built a modified WKB approximation on the basis of Good’s research [15]. In subsequent sections we generalise our (2-body and 4-body) models by adopting Rosen-Yennie’s (RY) approach. Accordingly it could be argued that we should view the WKB method from the RY perspective, to obtain the WKB phase shift. But it could be worthwhile considering the GM method as this provides a heuristic introduction to extensions of the WKB. (Concerning how to derive the WKB phase shift from the RY perspective, we can see Appendix B.)

In this section, therefore, viewing the WKB method from the GM perspective, we derive the WKB phase shift for a 2-body reaction.

First, expanding the system’s wave function with regard to a set of spherical harmonics, we obtain the set of radial equations:

\[
\left\{ \frac{\partial^2}{\partial \rho^2} + 1 - \frac{l(l+1)}{\rho^2} - \frac{2\mu U(r)}{k^2 \hbar^2} \right\} \chi_{l,k}(\rho) = 0 ,
\]  

(3.1.1)

where \( U(r) \) is the interaction between the projectile and the target, \( \mu \) is the reduce mass and \( l \) is the angular momentum. Here \( \rho = kr \). Putting

\[
\Theta^{(GM)} = 1 - \frac{l(l+1)}{\rho^2} - \frac{2\mu U(r)}{k^2 \hbar^2} ,
\]  

(3.1.2)

we express the radial equations as

\[
\frac{\partial^2 \chi_{l,k}(\rho)}{\partial \rho^2} + \Theta^{(GM)} \chi_{l,k}(\rho) = 0 .
\]  

(3.1.3)

The Good-Miller approach is that we can use the known solutions of an unperturbed equation for the purpose of designing the solutions of its perturbed version. For example, if we consider the radial equations (3.1.3), first we have its unperturbed version \((U(r) = 0)\):

\[
\frac{\partial^2 \chi_{l,k,0}(z)}{\partial z^2} + \Theta_0^{(GM)} \chi_{l,k,0}(z) = 0 ,
\]  

(3.1.4)

where

\[
\Theta_0^{(GM)} = 1 - \frac{l(l+1)}{z^2} .
\]  

(3.1.5)

The subscript 0 of \( \chi_{l,k,0} \) signifies that it is the solution of the unperturbed equations, and \( z \) is a function of \( \rho \). Note that asymptotically \( z \to \rho \) as \( r \to +\infty \), which shows that \( \Theta_0^{(GM)} \) becomes identical with \( \Theta^{(GM)} \) at \( r = +\infty \).
Then we seek for the solution of the perturbed equations, setting \[52\]

\[
\chi_{l,k}(\rho) \propto \sqrt{\frac{z}{z'}} j_l(z) \tag{3.1.6}
\]

where \(z' = \frac{\partial z}{\partial \rho}\).

Next we set

\[
\Theta_0^{(GM)} \times (z')^2 = \Theta^{(GM)}. \tag{3.1.7}
\]

Accordingly we can take advantage of the separation of variables to carry out the integration:

\[
\int_{z_t}^{z_a} \sqrt{\Theta_0^{(GM)}} dz = \int_{\rho_t}^{\rho_a} \sqrt{\Theta^{(GM)}} d\rho, \tag{3.1.8}
\]

where the subscripts \(a\) and \(t\) of \((z_a, \rho_a)\) and \((z_t, \rho_t)\) signify that they are the values of asymptotic regions and turning points, respectively. A turning point is a point which borders a classically accessible region and inaccessible one. We see that \(\Theta_0^{(GM)}|_{z=z_t} = 0\) and \(\Theta^{(GM)}|_{\rho=\rho_t} = 0\). As to the LHS of (3.1.8), we put \(\sqrt{\frac{l(l+1)}{z}} = \cos \theta\) and set \(z_t = \sqrt{l(l+1)}\). Therefore if we define \(\theta_t\) by

\[
\cos \theta_t = \frac{\sqrt{l(l+1)}}{z_t}, \tag{3.1.9}
\]

we find that \(\theta_t = 0\). Accordingly we have

\[
\int_{z_t}^{z_a} \sqrt{\Theta_0^{(GM)}} dz = \sqrt{z_a^2 - l(l+1)} - \theta \sqrt{l(l+1)}. \tag{3.1.10}
\]

As \(z_a\) increases, we can use the approximation such that:

\[
z_a \simeq \sqrt{z_a^2 - l(l+1)}. \tag{3.1.11}
\]

Thus, by using (3.1.8), we obtain

\[
z_a \simeq \int_{\rho_t}^{\rho_a} \sqrt{\Theta^{(GM)}} d\rho + \frac{\pi}{2} \sqrt{l(l+1)}. \tag{3.1.12}
\]

Then seeing the definition (3.1.6), and by virtue of the asymptotic expression of the spherical Bessel function, we have

\[
\chi_{l,k}(\rho_a) \propto \sqrt{\frac{z_a}{z'_a}} j_l(z_a) \sim \sin \left( z_a - \frac{\pi l}{2} \right). \tag{3.1.13}
\]
3.2. EIKONAL APPROXIMATION

In general, a 2-body scattering problem provides the following condition with regard to the asymptotic region:

\[ \chi_{l,k}(\rho_a) \propto \sin \left( \rho_a - \frac{l}{2} + \delta_l \right) . \]  

(3.1.14)

Comparing the expression (3.1.13) with (3.1.14), we obtain the formula for the WKB phase shift [54]:

\[ \delta_{l}^{wkb} = \int_{\rho_{l}}^{\rho_{a}} \sqrt{\Theta^{(GM)}} d\rho + \frac{\pi \sqrt{l(l+1)}}{2} - \rho_a . \]  

(3.1.15)

Also, we replace \( l(l + 1) \) by \( (l + 1/2)^2 \). Then, for \( \rho_a \to +\infty \), we have such a relation as

\[ \int_{z_t}^{\rho_a} \sqrt{1 - \frac{(l + 1/2)^2}{z^2}} dz \simeq \sqrt{\rho_a^2 - (l + 1/2)^2} - \frac{\pi}{2} (l + 1/2) , \]  

(3.1.16)

\[ \simeq \rho_a - \frac{\pi}{2} (l + 1/2) \]  

(3.1.17)

we can rewrite the above expression for the WKB phase shift as:

\[ \delta_{l}^{wkb} = \int_{\rho_{l}}^{\rho_{a}} \sqrt{1 - \frac{(l + 1/2)^2}{\rho^2}} - \frac{2\mu U}{k^2 \hbar^2} d\rho - \int_{z_t}^{\rho_a} \sqrt{1 - \frac{(l + 1/2)^2}{z^2}} dz . \]  

(3.1.18)

Immediately we have essentially the same formula in a more familiar form: putting \( \rho = kr \) and \( z = kr_0 \), we obtain

\[ \delta_{l}^{wkb} = \int_{r_{l}}^{r_{a}} \sqrt{k^2 - \frac{(l + 1/2)^2}{r^2}} - \frac{2\mu U}{r^2 \hbar^2} dr - \int_{r_{0,l}}^{r_{0,a}} \sqrt{k^2 - \frac{(l + 1/2)^2}{r_0^2}} dr_0 , \]  

(3.1.19)

where \( r_0 \) is a just variable which links \( k \) to \( z \). Thus we obtained the expression for the WKB phase shift, from the GM perspective.

3.2 Eikonal approximation

In this section, we briefly describe the point scattering of a 2-body system within the framework of the Glauber model, and we use the eikonal approximation to derive the basic formula for the Glauber phase shift function. Suppose that a particle is moving in a potential \( U(r) \). Assume that the following conditions are met:

\[ \left| \frac{U}{E_{cm}} \right| \ll 1 \]  

(3.2.1)

\[ 1 \ll k a \]  

(3.2.2)
where $a$ is the range of the potential. The second of these conditions means that the range of the potential is much greater than the de Broglie wavelength of a projectile [55]. Also, here we assume that the following condition holds:

\[
\frac{|U|a}{\hbar v} \gg 1.
\] (3.2.3)

As we saw before, the WKB phase shift (3.1.19) contains the two turning points $r_t$ and $r_{0,t}$. Now we have used the high energy condition $\frac{2\mu|U|}{\hbar k^2} = \frac{|U|}{E_{cm}} \ll 1$, and therefore it is highly plausible that the two turning points $r_t$ and $r_{0,t}$ of (3.1.19) are close enough to each other. Thus we replace $r_t$ in (3.1.19) by $r_{0,t}$. Immediately it follows that we obtain the following expression:

\[
\chi^{\text{WKB}} = 2 \left\{ \int_{r_{0,t}}^{\infty} dr \sqrt{k^2 - \frac{(l + 1/2)^2}{r^2}} - \int_{r_{0,t}}^{\infty} dr \sqrt{k^2 - \frac{(l + 1/2)^2}{r^2}} \right\},
\] (3.2.4)

where integration is done for the classically accessible region $r > r_{0,t}$.

Here we note that, in general, the S-matrix of an elastic scattering is denoted by $S_l = e^{2i\delta_l}$ [56], where $\delta_l$ is a phase shift. It is basically related to the phase shift function $\chi(b)$ as $\chi(b) = 2\delta_l$, where an impact parameter $b$ is expressed as $b \approx \frac{(l + 1/2)}{k}$ in a high energy approximation.

From the WKB perspective, (3.2.4) can be expanded with regard to a small parameter, defined by $\xi = \frac{1}{\hbar v k}$ to obtain the phase shifts of several orders. But in this section we are interested in the lowest order approximation for (3.2.4): since we assume that the potential strength is much less than the kinetic energy of the projectile, we ignore higher order terms, retaining only the leading and first order terms:

\[
\chi^G = (-2) \int_{r_0}^{\infty} dr \frac{\mu U(r)}{\hbar^2} \left( k^2 - \frac{(l + 1/2)^2}{r^2} \right)^{-\frac{1}{2}}.
\] (3.2.5)

Now we obtain the expression of the Glauber phase shift function $\chi^G$, restricting us to the case where the potential is dependent only on the radial distance. Under the high energy condition, we have

\[
\chi^G(b) \approx -\frac{2\mu}{\hbar^2 k} \int_b^{\infty} \frac{rU(r)dr}{\sqrt{r^2 - b^2}} = -\frac{2\mu}{\hbar^2 k} \int_b^{\infty} \frac{rU(r)dr}{\sqrt{r^2 - b^2}} \]
\[= -\frac{2\mu}{\hbar^2 k} \int_0^{\infty} U(r)dz = -\frac{\mu}{\hbar^2 k} \int_{-\infty}^{\infty} U(r)dz ,
\] (3.2.6)

where the impact parameter is $b = \sqrt{r^2 - z^2}$.
3.3. DEVIATION FROM EIKONAL TRAJECTORIES

Thus, the basic formula for the Glauber phase shift function $\chi(b)$ has been derived. Since the orbital angular momentum is large, we can use the approximation that $P_l(\cos \theta) \simeq J_0(\theta l)$ [56]. Accordingly the scattering amplitude becomes

$$f(\theta) = \frac{1}{2ik} \sum (2l + 1) P_l(\cos \theta)(e^{i\chi(b)} - 1) \quad (3.2.8)$$

$$\simeq \frac{k}{i} \sum \frac{l}{k} \cdot k^{-1} J_0(l\theta)(e^{i\chi(b)} - 1) \quad (3.2.9)$$

$$= \frac{k}{i} \int bdJ_0(\theta kb)(e^{i\chi(b)} - 1). \quad (3.2.10)$$

In this derivation, the long range Coulomb interaction is not discussed for the sake of simplicity. Later we will see how to deal with such interactions in the model.

3.3 Deviation from eikonal trajectories

In a previous section, we derived the Glauber phase shift from the WKB approximation. In fact, the approximation which was used involved only the lowest order term, and in this section we show that including higher order terms allows us to make corrections to the Glauber phase shift. The basic idea is to expand the WKB phase shift in powers of the parameter $\xi U$, where $\xi = \frac{1}{\hbar wk}$, and $U$ is an interaction potential. In a high energy region, we see that $\xi U = \frac{U}{E_{cm}} \ll 1$.

To derive the non-eikonal correction terms, we first define $g(x) = \sqrt{1 + x} - 1$, and then derive its derivatives:

$$\left( \frac{\partial}{\partial x} \right)^j g(x) = \frac{(-1)^{j-1} (2j - 1)!!}{2^j (2j - 1)} (1 + x)^{\frac{1}{2} - j}. \quad (3.3.1)$$

Then, its Taylor series are:

$$g(x) = g(0) + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} (2j - 1)!!}{2^j (2j - 1)} x^j. \quad (3.3.2)$$

Using this formula, and equating $x = -2\xi U \frac{r^2}{z^2}$, we expand the WKB phase shift function as

$$\frac{\chi^{WKB}}{2} = \int_0^\infty k dz \left( \frac{z}{r} \right)^2 \left\{ \sqrt{1 - 2\xi U \frac{r^2}{z^2}} - 1 \right\} \quad (3.3.3)$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j (2j - 1)!!}{j! \cdot 2j - 1} \xi^j \int_0^\infty k dz U^j \frac{r^{2j - 2}}{z^{2j - 2}}. \quad (3.3.4)$$
The following relation holds true:
\[
\int_0^\infty \frac{1}{z^{2n}} f(z) dz = \frac{1}{(2n-1)!!} \int_0^\infty dz \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^n f(z),
\]
where \( f(z) \) is a function which vanishes at both 0 and infinity. Using the relation, we have
\[
\chi^\text{WKB} = \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} \xi^j \int_0^\infty k dz \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^{j-1} U^j r^{2(j-1)}.
\]
(3.3.6)
\[
= \sum_{m=0}^\infty \frac{(-1)^m}{(m+1)!} \xi^{(m+1)} \int_0^\infty k dz \left( \frac{1}{z} \frac{\partial}{\partial z} \right)^m U^{m+1} r^{2m}.
\]
(3.3.7)

Taking advantage of \( \frac{1}{z} \frac{\partial}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \), we obtain the same formula as Wallace [49]:
\[
\chi^\text{WKB} = \sum_{m=0}^\infty \frac{(-1)^m k}{(m+1)!} \xi^{(m+1)} \int_0^\infty dz \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^m U^{m+1} r^{2m},
\]
(3.3.8)
where the 1st order correction is the term for \( m=1 \), and the 2nd order term is for \( m=2 \), and so on. We can find that the 0-th order term (multiplied by 2) is the same as the Glauber phase shift function.

As a numerical check, we consider \( p + ^{12}\text{C} \) scatterings at several energies. Using a phenomenological Woods-Saxon potential, we compare the exact quantum mechanical method with Glauber and its 1st and 2nd corrections. By the exact quantum mechanical result, here we mean the calculation done by solving the radial equations for a 2-body reaction and calculating the scattering amplitude via a partial wave expansion. In Figure 3.1, the results are shown, where we can see that the higher order calculation is the closer to the exact calculation.
Figure 3.1: Differential cross sections for $p + ^{12}\text{C}$ scatterings at 30 (MeV)(upper left), 60 (MeV)(upper right), 120 (MeV)(lower left) and 240 (MeV)(lower right) as the ratios to Rutherford cross sections. A phenomenological optical potential is used as the interaction between them.
3.4 Rosen-Yennie correction term

So far we have treated WKB approximation as a generalisation to the eikonal approximation, and now we can improve the WKB approximation by introducing a correction term.

The term was introduced by Rosen and Yennie [15], and the idea behind the Rosen-Yennie (RY) approach is to design an approximated solution to a differential equation by using a known differential equation and its (known) solution. For example, one of the solution of the time-independent radial equation is expressed by the spherical Bessel function, and accordingly the solution of the perturbed equation can be expressed by the multiplication between the spherical Bessel function and a radius-dependent function [52].

The details can be found in Appendix B. The correction term is expressed as

\[
\frac{\chi_{RY}}{2} = -\frac{1}{24} \int_{r_0}^{\infty} \frac{dr}{r} \cdot \frac{1}{\sqrt{1 - \frac{b^2}{r^2} - 2\xi U(r)}} \cdot \left( \frac{\partial}{\partial r} r \frac{\partial}{\partial r} D(r) \right)
\]

\[
= -\frac{1}{24} \int_{0}^{\infty} \frac{dz}{r \sqrt{z^2 - 2\xi r^2 U(r)}} \cdot \left( \frac{\partial}{\partial r} r \frac{\partial}{\partial r} D(r) \right),
\]

where \( D(r) \) is defined by

\[
D(r) = \ln \left( 2 - 4\xi U(r) - 2\xi r \frac{\partial U(r)}{\partial r} \right).
\]

This term is added to (3.3.8) to improve the WKB approximation.

If we use a Woods-Saxon potential, we have

\[
\frac{\partial}{\partial r} r \frac{\partial D(r)}{\partial r} = \frac{1}{(2 - 4\xi U - 2r\xi U')^2} \cdot \{ \xi (-12U' - 20rU'' - 4r^2U''') - \xi^2 (24U'U + 40rU''U + 8r^2U'''U - 24r(U''))^2 - 4r^2U''U'' + 4r^3U'''U'' - 4r^3(U'')^2 \} ,
\]

where \( D'(r) = \frac{\partial D(r)}{\partial r} \).

As a test, elastic scatterings of \( p + ^{12}\text{C} \) at 20, 40, 80 and 120 (MeV) are considered, where the partial wave method, WKB phase shifts up to 2nd order and its RY phase shift are used. In this case, we used a Woods-Saxon potential whose parametrisation

\[
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\]
3.4. ROSEN-YENNIE CORRECTION TERM

is arbitrary; specifically $V_0 = 22.0$ (MeV), $r_0 = 1.2$ (fm), $a = 0.6$ (fm), and $W_0 = 20.0$ (MeV), $r_0i = 1.1$ (fm), $a_i = 0.6$ (fm). As is seen in Figure 3.2, the RY correction terms provide a slightly improved results especially at higher angles. This means that the RY correction term seems to give a validity to the non-eikonal approach. It turns out that for 120 MeV, the 2nd order WKB is better than its combination with the RY term at 80-100 degrees, but at 60-80 degrees the latter is better.

Figure 3.2: Elastic scatterings for $p + ^{12}C$ at 20 (MeV)(upper left), 40 (MeV)(upper right), 80 (MeV)(lower left) and 120 (MeV)(lower right). The ratios of the differential cross sections to Rutherford scattering are plotted against the scattering angle.
The previous chapter explained the formalism of the 2-body reaction model under the Glauber regime and its non-eikonal extension. Now our attention is paid to extend it to a 4-body reaction model which involves a 3-body-projectile plus a target. In this chapter, therefore we develop a formalism for a 4-body reaction. First we introduce the 4-body Jacobi coordinates to describe the 4-body system, and use the adiabatic approximation to derive the cross sections of elastic scatterings. Also we consider how to treat a long range Coulomb interaction within the Glauber-WKB framework. After that, we explain the folding model that is used to estimate the interaction between a projectile’s core and a target. Then we carry out numerical calculation of the reaction cross sections and elastic differential scattering cross sections which involve Borromean nuclei at several energies.

4.1 Adiabatic approximation

Firstly we need to introduce the 4-body Jacobi coordinates to describe our 4-body model. Suppose \( r_1, r_2 \) and \( r_c \) are the positions of valence particle 1, 2 and the core,
respectively. Then they are expressed by the Jacobi coordinates as follows:

\[ r_c = R + \sqrt{\frac{2}{A_{cA_{proj}}}} y , \]  
\[ r_2 = R - \sqrt{\frac{A_c}{2A_{proj}}} y + \sqrt{\frac{1}{2}} x , \]  
\[ r_1 = R - \sqrt{\frac{A_c}{2A_{proj}}} y - \sqrt{\frac{1}{2}} x . \]

They are expressed in Figure 4.1 as well, where a target is not rendered for convenience.

Next we treat the 4-body system as 1+3 body system: we split the Hilbert space of the whole system into two different ones. Accordingly in our 4-body model, the Hilbert space of a total 4-body system is the direct product of its two subspaces:

\[ \mathcal{H} = \mathcal{H}_{rel, cm} \times \mathcal{H}_{3B} , \]

where \( \mathcal{H}_{rel, cm} \) is the Hilbert space to which the states that represent the relative motion between a projectile and a target belong. \( \mathcal{H}_{3B} \) is the Hilbert space for a projectile (3-body system). Accordingly the 4-body scattering wave function \( \Psi^{(4B)} \) is expressed by two different wave functions, that is, the wave function \( \psi^{(+)}_{k_R} \) which represents the relative motion between the projectile and the target, and the internal wave function \( \phi^{(3B)}_{IM_I} \) of the projectile:

\[ \Psi^{(4B)}_{JM_J}(x, y, R) = \psi^{(+)}_{k_R} \otimes \phi^{(3B)}_{IM_I}(x, y) , \]

where \( k_R \) is the canonical momentum of \( R \).

Next, by definition, the 4-body wave function is a solution of the 4-body Schrödinger equation:

\[ \{ H^{(3B)} + T_R + U(x, y, R) - E^{(4B)} \} \Psi^{(4B)}_{JM_J} = 0 , \]

in which we have the sum of the interactions \( U(x, y, R) \) between each constituent of the projectile and the target, and the energy, \( E^{(4B)} \) is that of the whole system. \( T_R \) is the kinetic term which describes the relative motion between the projectile and the target. \( H^{(3B)} \) is the internal Hamiltonian of the projectile. By definition, therefore the 3-body Hamiltonian satisfies the following equation:

\[ \{ H^{(3B)} - \epsilon \} \phi^{(3B)}_{IM_I} = 0 , \]
where \( \epsilon \) is the internal energy of the projectile.

Now let us consider a situation in which the incident kinetic energy of the projectile is much larger than its internal energies: \( \epsilon \ll (E^{(4B)} - \epsilon) \). In that case, the 3-body Hamiltonian is approximately constant during the impact, and so the ground state energy \( \epsilon_0 \) of the 3-body system is substituted for the 3-body Hamiltonian: \( \{H^{(3B)} - \epsilon_0\} \phi^{(3B)}_{IJ} = 0 \). Accordingly the 4-body Schrödinger equation becomes

\[
\{T_R + U(x, y, R) + (\epsilon_0 - E^{(4B)})\} \Psi^{(4B)}_{IJ} = 0.
\] (4.1.8)

The ansatz (4.1.5) enables us to have

\[
\{T_R + U(x, y, R) + (\epsilon_0 - E^{(4B)})\} \psi^{(+)}_{kR} = 0.
\] (4.1.9)

This is the adiabatic approximation [57]. The approximation is valid if the collision time is so short that the excitation energy of the projectile cannot become large, because in such a situation the internal motion of the projectile is much slower than the relative motion between the projectile and the target. Note that the adiabatic approximation is reasonable as well when the incoming particle is weakly bound and it has one or two bound states, because in such a case the binding energy ought to be small compared to the kinetic energy for the relative motion between the projectile and the target [57, 58].

![Figure 4.1: A 4-body system consisting of the target, the projectile (core and two valence nucleons). The target is at the coordinate origin, and is not rendered for convenience. The system is expressed in the 4-body Jacobi coordinates.](image)

Using the 4-body (half-normalised) Jacobi coordinates, we compute the phase shifts. The S-matrix is written as

\[
S(b_x, b_y, b_R) = e^{i(\chi_1(b_R, b_y) + \chi_2(b_R, b_y, b_n) + \chi_3(b_R, b_y, b_n))},
\] (4.1.10)
4.2. LONG RANGE INTERACTIONS

where $\chi_{ct}$, $\chi_{vt}$ and $\chi_{vt'}$ are the phase shifts for core-target interaction, valence particle 1-target and valence particle 2-target, respectively. Here we use its shorthand notation $e^{i\chi}$ if it is convenient.

Then, with the bound 3-body states $|\phi_{\delta M_{I}'}^{(3B)}\rangle$, we write down the scattering amplitude in the 4-body Glauber model [59] :

$$f_{M'M_{I}}^{(4G)} = -ik \int_{0}^{\infty} J_{0}(qb_{R}) (\phi_{\delta M_{I}'}^{(3B)} (S-1)|\phi_{\delta M_{I}'}^{(3B)} b_{R} db_{R} . \quad (4.1.11)$$

4.2 Long range interactions

In a previous section, we used the WKB approach to build our 4-body model for the reactions which involve a Borromean projectile, under the assumption that the interaction between a projectile and a target rapidly decreases. The underlying assumption is not valid for a long-range force such as a Coulomb potential. If the interaction is a long range interaction, the S-matrix is not convergent. Accordingly in such a case, we need to introduce an effective Coulomb S-matrices to work within the Glauber-WKB framework.

To find out the effective interaction, we consider the finite size effect of a nucleus (which affects the scattering at large angles) and charge screening. To see the former, we use a crude approximation which replaces the actual charge distribution of a nucleus by a uniform charge distribution [60]. Then taking the effect of screening of charge into account, we modify a Coulomb interaction as follows:

$$V_{C} = \begin{cases} \frac{\alpha_{EM}}{2R_{C}} \left(3 - \left(\frac{r}{R_{C}}\right)^2\right) & \text{for } r < R_{C} \\ \frac{\alpha_{EM}}{r} & \text{for } R_{C} \leq r \leq a \\ 0 & \text{for } a < r \end{cases} \quad (4.2.1)$$

where $a$ is the screening radius. $R_{C}$ is the Coulomb radius of a nucleus. After that, we expand them with regard to $\frac{b}{a}$. The regularisation procedure is valid as long as the screening radius is large enough [45]. Then using the relative velocity between a projectile and a target, and introducing the Sommerfeld parameter $\eta = \frac{\alpha_{EM}}{\hbar v}$, we integrate the potential with regard to $z$. The regularisation techniques work for scatterings. But in a subsequent chapter we aim at developing our formalism for
the breakup reactions of a Borromean projectile, and in breakup the regularisation techniques are no longer valid.

Suppose the Coulomb S-matrix is $S_C$, and assume that the two valence nucleons are neutrons (not electrically charged). Then the Coulomb interaction between the core and the target is given by $U_{Br} + U_{Sc}$ where

$$U_{Br} = \frac{\alpha_{\text{EM}}}{|R - \sqrt{\frac{2}{A_{\text{proj}A_{c}y}}}|} - \frac{\alpha_{\text{EM}}}{R}, \quad (4.2.2)$$

$$U_{Sc} = \frac{\alpha_{\text{EM}}}{R}. \quad (4.2.3)$$

Here the subscripts Br and Sc signify inelastic scattering and point scattering, respectively.

Accordingly we see that its phase shift function is expressed as

$$\chi_C = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} U_{Br} dz - \frac{1}{\hbar v} \int_{-\infty}^{\infty} U_{Sc} dz, \quad (4.2.4)$$

where we use just a 0th order (Glauber) framework to obtain the functions. Then we define the first term and the second of the RHS in (4.2.4) by $\chi_{Br}$ and $\chi_{Sc}$, respectively. The latter describes the point Coulomb scattering between a projectile and a target. Using a cutoff procedure [45], we see that the phase shift contains the cutoff-dependent part. But the part does not physically contribute to the cross section, and therefore $\chi_{Sc}$ becomes [61, 62]

$$\chi_{Sc} = 2\eta \ln k_{BR}. \quad (4.2.5)$$

With regard to $\chi_{Br}$, we have

$$\chi_{Br} = -\eta \int_{-\infty}^{\infty} \frac{1}{|b_R - \sqrt{\frac{2}{A_{\text{proj}A_{c}}}} b_y|^2 + (z - \sqrt{\frac{2}{A_{\text{proj}A_{c}}} z_y})^2} dz + \eta \int_{-\infty}^{\infty} \frac{1}{R} dz. \quad (4.2.6)$$

In terms of the first term of the RHS of the equation, we substitute $z' = z - \sqrt{\frac{2}{A_{\text{proj}A_{c}}} z_y}$ into the equation. Immediately we have

$$\chi_{Br} = -\eta \int_{-\infty}^{\infty} \frac{1}{|b_R - \sqrt{\frac{2}{A_{\text{proj}A_{c}}} b_y}|^2 + z'^2} dz' + \eta \int_{-\infty}^{\infty} \frac{1}{R} dz'. \quad (4.2.7)$$
4.2. LONG RANGE INTERACTIONS

Accordingly we have

$$\chi_{Br} = -2\eta \left[ \ln \left( \sqrt{z^2 + \left| b_R - \sqrt{\frac{2}{A_{proj} A_c}} b_y \right|^2} + z \right) - \ln \left( \sqrt{z^2 + b_R^2} + z \right) \right]_0^\infty,$$  \hspace{1cm} (4.2.8)

and it follows that

$$\chi_{Br} = -2\eta (-1) \left[ \ln \sqrt{b_R - \left| \sqrt{\frac{2}{A_{proj} A_c}} b_y \right| - \ln b_R} \right].$$  \hspace{1cm} (4.2.9)

Then we can express it as

$$\chi_{Br}(b_R, b_y) = \eta \ln \left( 1 - \frac{2\hat{b}_R}{b_R} \cdot \sqrt{\frac{2}{A_{proj} A_c}} b_y + \frac{2}{A_{proj} A_c} \cdot \frac{b_y^2}{b_R^2} \right),$$  \hspace{1cm} (4.2.10)

where $\hat{b}_R = \frac{b_R}{m}$. Assuming that $\sqrt{\frac{2}{A_{proj} A_c}} b_y$ is small, we approximate (4.2.10) by

$$\chi_{Br}(b_R, b_y) \simeq -\eta \frac{2\hat{b}_R}{b_R} \cdot \sqrt{\frac{2}{A_{proj} A_c}} b_y,$$  \hspace{1cm} (4.2.11)

$$= -Q \hat{b}_R \cdot b_y,$$  \hspace{1cm} (4.2.12)

where we define the following quantity [63]:

$$Q = \frac{2\eta}{b_R} \sqrt{\frac{2}{A_c A_{proj}}}. \hspace{1cm} (4.2.13)$$

Therefore the Coulomb S-matrix $S_C$ can be expressed as

$$S_C = e^{i\chi_C} = e^{i\chi_{Br}} e^{i\chi_{Sc}}.$$  \hspace{1cm} (4.2.14)

Writing down $e^{i\chi_{Br}}$ as

$$e^{i\chi_{Br}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \chi_{Br}^n,$$  \hspace{1cm} (4.2.15)

we replace its first order term by the first order term of the perturbation theory:

Then we replace the S-matrix as follows:

$$S_C \rightarrow e^{-ib_R \cdot b_y} + iQ \hat{b}_R - iQ \xi \left\{ K_1(\xi) \hat{b}_R \cdot b_y + iK_0(\xi) z_y \right\},$$  \hspace{1cm} (4.2.16)

where $\xi = \frac{(E_{ex} - E_{gs})b_R}{\hbar v}$ is the adiabacity parameter [63]. The whole S-matrix is expressed as the multiplication of the S-matirx and the nuclear S-matrix.
4.3 Folding potentials

In a previous section, we adopted the adiabatic approximation to build a formalism for reactions involving Borromean nuclei. But along with the 3-body projectile wave function, the other ingredients is to calculate the projectile constituent-target interactions and, in the case of $^{22}$C, we need to discuss the interaction between the projectile core and the target.

Our approach is to fold the sum of interactions between the constituent nucleons of the core and the target with the density of the core, and subsequently derive a potential of the total interaction between the two nuclei (core + target). To do this, we consider the density of the core, and integrate it with the potential which describes the (average) interaction between each constituent of the core and the target.

Firstly, we assume a simple Gaussian distribution for the density of the $^{20}$C core to make our calculations easier. In this case, the Gaussian density, $\rho_g = \rho_0 e^{-r^2/r_0^2}$, is characterised by its radial parameter $r_0$, along with its normalisation factor $\rho_0 = \frac{1}{\pi^{3/2}r_0^3}$. The familiar formula, $\sqrt{\langle r^2 \rangle} = \sqrt{3}r_0$, for the rms radius, allows us to determine the radial parameter.

Secondly, we test to what extent the Gaussian density works for $^{12}$C($^{12}$C, $^{12}$C)$^{12}$C and $^{12}$C($^{16}$O, $^{16}$O)$^{12}$C elastic scatterings. In these calculations, we fold the potential for $p + ^{12}$C over the density of $^{12}$C and $^{16}$O, respectively, whose radial parameters are derived from their rms radii. If this works, we will extrapolate the use of the Gaussian density to $^{20}$C.

With the folding density and a Woods-Saxon potential, folding potentials are constructed as follows:

$$U_f(r) = A_{\text{proj}} \times \int \rho_g(r')U_{\text{WS}}(|r - r'|)dr' , \quad (4.3.1)$$

$$U_{\text{WS}} = \frac{-V_0}{1 + \exp \left( \frac{r - r_0 \times A^{1/3}}{a} \right)} + \frac{-iW_0}{1 + \exp \left( \frac{r - r_0 \times A^{1/3}}{a} \right)} , \quad (4.3.2)$$

where $A_{\text{proj}}$ is the mass number of a projectile, and $r$ refers to the radial distance from the target to the point at which the value of the potential is evaluated, provided that the target sits at the coordinate origin. As to the parametrisations of the potential, we use the parameters which are fitted to the experimental data of $p + ^{12}$C scatterings.
4.3. FOLDING POTENTIALS

Here we do not use a spin-orbit potential, since both $^{12}\text{C}$ and $^{16}\text{O}$ are spinless particles. Generally, the spin-dependent terms do not affect cross sections [64].

Then we calculate the differential elastic scattering cross sections for calculations for $^{12}\text{C} + ^{12}\text{C}$ at 1016 (MeV), and for $^{16}\text{O} + ^{12}\text{C}$ at 1503 (MeV). Figure 4.2 compares the cross sections to the experimental data, where we see that the folding model calculations agree with the experimental data.

Figure 4.2: The differential scattering cross sections (as the ratio to the Rutherford cross sections) plotted as a function of the scattering angle. The left graph is for $^{12}\text{C} + ^{12}\text{C}$ at 1016 (MeV), and the right for $^{16}\text{O} + ^{12}\text{C}$ at 1503 (MeV). The experimental data are from [65, 66].

Seeing that the folding model works for $^{12}\text{C} + ^{12}\text{C}$ and $^{16}\text{O} + ^{12}\text{C}$, we need to move on to the case for $^{20}\text{C}$. At this stage we extrapolate the usage of the Gaussian folding density of $^{20}\text{C}$ to the nucleus. We wish to see to what extent this density is realistic. To see this, we compare the Gaussian density to Kucuk-Tostevin (KT) density. The KT density is the density of $^{20}\text{C}$ which is derived from the Hartree-Fock calculation. Figure 4.3 presents the Gaussian density and Kucuk-Tostevin (KT) density for $^{20}\text{C}$.

Due to the fact that there is a difference in the way of normalisation between the two densities, we multiply the Gaussian density by a factor of $A = 20$ [67]. The rms matter radius of the former is 2.913 (fm), and the latter 2.913 (fm). From the data in the graphs of Figure 4.3, we can see that KT density is smaller than the Gaussian near the coordinate origin; the Gaussian density is about 65 per cent larger than the KT density at the coordinate origin. This difference stems from the fact that KT density includes the effect of Pauli blocking, while the Gaussian does not. As the radial distance increases, the Gaussian density more rapidly decreases than KT does; the Gaussian density is smaller than KT density by a factor of about 1000 at a radial
Figure 4.3: Folding densities plotted against the radial distance. The vertical axis is naturally scaled in the left graph, and logarithmically scaled in the right graph. Due to the difference in the way of normalisation, we multiply the Gaussian density by a factor of $A = 20$.

distance of 10 (fm).

Now, in order to obtain the folding potentials for $^{20}\text{C} + {}^{12}\text{C}$, we fold the potentials for $p + {}^{12}\text{C}$ over the folding densities. Table 4.1 presents the parametrisations of the potential for $p + {}^{12}\text{C}$.

Then we show the resultant folding potentials. For example, Figure 4.4 presents the folding potentials for $^{20}\text{C} + {}^{12}\text{C}$ at (300 MeV/A). The upper left graph of Figure 4.4 is their real parts, and the upper right is the logarithm of the moduli of them. The lower left is for their imaginary parts, and the lower left is for the logarithm of the moduli of them. As can be seen from those graphs, the both real and imaginary parts of the Gaussian folding potential are little different from those of KT folding potential, as far as a short-radial-distance region is concerned. The former starts to deviate from the latter as the radial distance goes beyond 10 (fm), but it seems that the moduli of the both are small in the region to the extent that they do not contribute to the cross sections so much.
Figure 4.4: The folding potentials, for $^{20}\text{C}+^{12}\text{C}$ at 300 (MeV/A), shown as a function of the radial distance. The upper left graph is their real parts, and the upper right is for the logarithm of the moduli of them. The lower left is for the imaginary parts, and the lower right is for the logarithm of the moduli of them. RI2 is used for the parametrisation.
<table>
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<th>parameters for WS</th>
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<td>RI2 84.666 (1016 MeV)</td>
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</tr>
<tr>
<td>RI2 93.937 (1503 MeV)</td>
<td>$(22.0, 1.1, 0.55) (11.0, 1.20, 0.65)$</td>
</tr>
<tr>
<td>EW1 84.666 (1016 MeV)</td>
<td>$(18.0, 1.1, 0.70) (8.0, 1.20, 0.70)$</td>
</tr>
<tr>
<td>EW1 93.937 (1503 MeV)</td>
<td>$(18.0, 1.1, 0.70) (8.0, 1.20, 0.70)$</td>
</tr>
<tr>
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<td>$(29.3, 1.1, 0.55) (11.9, 1.20, 0.65)$</td>
</tr>
<tr>
<td>EW1 99.1</td>
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</tr>
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<td>RI2 122</td>
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</tr>
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Table 4.1: The parametrisations of the Woods-Saxon central potentials at various energies. Here the Coulomb radius is set $1.2 \times 12^{1/3}$ (fm).
4.4 Folding potentials fitted into Woods-Saxon potentials

In the previous section, we treated a folding model to construct the potentials for $^{20}\text{C} + ^{12}\text{C}$. In practice, if we use a folding model to describe the interaction between a projectile and a target, the folding potential is not so much useful for when we work on the higher order terms which are related to non-eikonal trajectories. Therefore we fit the folding potential into the Woods-Saxon central potential. Table 4.2 represents the parametrisations for the reaction $^{20}\text{C} + ^{12}\text{C}$ at different laboratory energies (per nucleon).

Then, the fitted Woods-Saxon potentials are compared to the original folding potentials. For example, Figure 4.5 represents the moduli of the real and imaginary parts of the folded and fitted potentials for $^{20}\text{C} + ^{12}\text{C}$ at 99.1 (MeV/A). The moduli of the potential are plotted as a function of the radial distance. The vertical axis is scaled naturally or logarithmically. The same kind of graphs for 300 (MeV/A) are presented in Figure 4.6.

As can be seen from those graphs, we can find that the effective range of the folding potentials seems to be around 10 (fm). Generally, it seems that the Gaussian folding potentials are well fitted, but actually the folding potentials start to deviate from the fitted potentials at around the radial distance of 7.5 (fm). The fitted ones have as slightly longer tail, and accordingly the resultant reaction cross sections which involve the fitted ones tend to be larger than those involve the folding potentials.
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<td>RI2</td>
<td>250</td>
</tr>
<tr>
<td></td>
<td>(293.8, 2.650, 0.5296, 1.047), (151.8, 2.841, 0.5678, 1.114)</td>
</tr>
<tr>
<td>HE1</td>
<td>290</td>
</tr>
<tr>
<td></td>
<td>(266.9, 2.650, 0.5296, 1.047), (169.2, 2.841, 0.5678, 1.114)</td>
</tr>
<tr>
<td>RI2</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>(341.5, 2.650, 0.5296, 1.047), (171.6, 2.841, 0.5678, 1.114)</td>
</tr>
<tr>
<td>HE1</td>
<td>305</td>
</tr>
<tr>
<td></td>
<td>(258.6, 2.650, 0.5296, 1.047), (164.6, 2.841, 0.5678, 1.114)</td>
</tr>
</tbody>
</table>

Table 4.2: The parametrisations for $^{20}$C + $^{12}$C at several energies. These were derived by fitting the folding potentials (with the Gaussian folding density) into a Woods-Saxon type potential. Here the Woods-Saxon reduced radius is given as $R_{WS} = r_0 \times (20^{1/3} + 12^{1/3})$. 
Figure 4.5: The moduli of the real and imaginary parts of the folding potentials and those of fitted potentials, for $^{20}\text{C} + ^{12}\text{C}$ at 99.1 (MeV/A) plotted against the radial distance. The folding is done with the Gaussian density. The vertical axes of the left graph and the right are scaled naturally and logarithmically, respectively. The parametrisations are EW1.

Figure 4.6: The moduli of the real and imaginary parts of the folding potentials and those of fitted potentials, for $^{20}\text{C} + ^{12}\text{C}$ at 300 (MeV/A) plotted against the radial distance. The folding is done with the Gaussian density. The vertical axes of the left graph and the right are scaled naturally and logarithmically, respectively. The parametrisations are EW1.
4.5 Reaction cross sections

In this section we derive the formula for the reaction cross section of a 3-body projectile within the Glauber-WKB 4-body framework. Here we write the S-matrix for the reaction by a shorthand notation $e^{i\chi}$ as seen in (4.1.10). Then we expand the S-matrix with regard to the spherical harmonics:

$$e^{i\chi(R_b,y_b,x_b)} = \sum_{\lambda M\lambda_x \lambda_y} \Gamma_{\lambda M\lambda}(\Omega_x,\Omega_y) \Delta_{\lambda_x \lambda_y}^{\lambda M\lambda}(R_b,\rho,\alpha),$$  \hspace{1cm} (4.5.1)

where $\Gamma_{\lambda M\lambda}(\Omega_x,\Omega_y) = [Y_{\lambda_x} \otimes Y_{\lambda_y}]_{\lambda M\lambda}$, and $\Delta_{\lambda_x \lambda_y}^{\lambda M\lambda}$ is given by

$$\Delta_{\lambda_x \lambda_y}^{\lambda M\lambda}(R_b,\rho,\alpha) = \int \Gamma_{\lambda M\lambda}(\Omega_x,\Omega_y) e^{i\chi(R_b,y_b,x_b)} d\Omega_x d\Omega_y.$$  \hspace{1cm} (4.5.2)

The S-matrix elements for the reaction are the matrix elements of the S-matrix:

$$S_{J'\lambda' M'}(R_b) = \langle \phi^{(3B)}_{J'\lambda'} | e^{i\chi(R_b,y_b,x_b)} | \phi^{(3B)}_{J M\lambda} \rangle.$$  \hspace{1cm} (4.5.3)

Then the matrix elements are expressed as

$$S_{J'\lambda' M'}(R_b) = \sum_{\lambda M\lambda_x \lambda_y} \sum_{\lambda' \lambda} \langle \gamma' J' | \Gamma_{\lambda M\lambda} | \gamma J \rangle \tilde{\Delta}_{\xi \xi'}^{\lambda M\lambda_x \lambda_y}(R_b),$$  \hspace{1cm} (4.5.4)

where we write a set of quantum number as $\xi = \{\gamma, K\}$, and $\tilde{\Delta}_{\xi \xi'}^{\lambda M\lambda_x \lambda_y}(R_b)$ is defined by

$$\tilde{\Delta}_{\xi \xi'}^{\lambda M\lambda_x \lambda_y}(R_b) = \int_{-1}^{1} A_{\xi \xi'}^{L_x L_y} n^L_{\lambda_x \lambda_y}(\alpha) \int_{0}^{\infty} K' \gamma' (\rho) \Delta_{\lambda_x \lambda_y}^{\lambda M\lambda}(R_b,\rho,\alpha) \chi_{K'_\gamma}(\rho) d\rho \frac{\sin(2\alpha)}{8} d\cos(2\alpha).$$  \hspace{1cm} (4.5.5)

and $A_{\xi \xi'}^{L_x L_y} n^L_{\lambda_x \lambda_y}(\alpha)$ is expressed as

$$A_{\xi \xi'}^{L_x L_y} n^L_{\lambda_x \lambda_y}(\alpha) = (\sin \alpha)^{L_x' + L_y'} (\cos \alpha)^{L_x + L_y} N_{n'}^{L_x' + \frac{1}{2}, L_y'} N_{n}^{L_x + \frac{1}{2}, L_y + \frac{1}{2}} \times P_{n'}^{L_x' + \frac{1}{2}, L_y'} \cos(2\alpha) P_{n}^{L_x + \frac{1}{2}, L_y + \frac{1}{2}} (\cos 2\alpha).$$  \hspace{1cm} (4.5.6)

The matrix elements of $\Gamma_{\lambda M\lambda}$ can be written as

$$\langle \gamma' J' | \Gamma_{\lambda M\lambda} | \gamma J \rangle = (-1)^{J' - M'} \left( \begin{array}{cc} J' & \lambda \\ -M' & M \end{array} \right) \langle \gamma' J' | \Gamma_{\lambda} | \gamma J \rangle,$$  \hspace{1cm} (4.5.7)
where $\langle \gamma' J' || \Gamma_\lambda || \gamma J \rangle$ is the reduced matrix element. The reduced element is expressed as

$$
\langle \gamma' J' || \Gamma_\lambda || \gamma J \rangle = (-1)^{J' + J + \lambda + L_x + L_y} \sqrt{(2J + 1)(2J' + 1)} \delta_{S'S} \delta_{L'L} \delta_{\lambda\lambda} \times (2L_x + 1)(2L + 1)(2\lambda + 1) \times \frac{1}{4\pi} \sqrt{(2L'_x + 1)(2L_x + 1)(2\lambda_x + 1)(2\lambda_y + 1)}
$$

$$
\times \begin{pmatrix} L'_x & L'_y & L' \\ \lambda_x & \lambda_y & \lambda \end{pmatrix} \begin{pmatrix} L_x & L_x & 0 \\ \lambda_x & \lambda_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L'_y & L'_y & L'_y \\ \lambda_y & \lambda_y & \lambda_y \\ 0 & 0 & 0 \end{pmatrix}
$$

Then the reaction cross section is given by

$$
\sigma_{\text{reac}} = 2\pi \int_0^{R_b} R_b dR_b \left( 1 - |S_{J'M'}(R_b)|^2 \right)
$$

Figure 4.7 shows the reaction cross sections for $^{22}\text{C} + ^{12}\text{C}$ at various incident energies. In the graph, we use both the Gaussian density and KT density as the folding density, and compared to each other. The two cross sections are close altogether, but the Gaussian density yields a slightly larger cross sections. For reference, Horiuchi’s theoretical values calculated under the Optical Limit Approximation (OLA) are presented as well [48]. It turns out that their values are larger than ours at the incident energies below 150 (MeV).
Figure 4.7: Reaction cross sections for $^{22}\text{C} + ^{12}\text{C}$ at various incident energies. Both Gaussian and KT folding densities are used to calculate the cross sections. ic4 is used for the bound wave function, and EW1 is used for the parametrisation of the core-target interaction. For reference, Horiuchi’s theoretical values are presented [48].

4.6 Borromean nucleus elastic scatterings

In previous sections, we presented our 4-body reaction model, and built the formalism to calculate differential cross sections for a 4-body system. In this section, we carry out numerical calculations for Borromean nuclei. First we do it for $^6\text{He}$, and next for $^{22}\text{C}$.

4.6.1 $^6\text{He}$ scattering

Firstly we consider the elastic scattering of $^6\text{He} + ^{12}\text{C}$ at 38.3 (MeV/A). The Glauber-WKB 4-body model enables us to calculate differential cross sections for the reaction.

For valence-target interaction, we use the Woods-Saxon central potential whose the parametrisation is: a depth of 38.6 (MeV), a radial parameter of 1.15 (fm) and a diffuseness of 0.73 (fm); for its imaginary part, the depth, the radial parameter and diffuseness are 7.15 (MeV), 1.25 (fm) and 0.44 (fm), respectively. This parametrisation was used by Fannon [68]. Here we omit the spin-orbit force. This parametrisation is actually for $p + ^{12}\text{C}$ at 40 (MeV/A), but we substitute it for $n + ^{12}\text{C}$. As to the
core-target interaction, we adopt Goldberg’s parametrisation [69] in which the Woods-Saxon reduced radius was given by $R = r_0 \times A^{1/3}$. Specifically, we set: $V_0 = 108.0$ (MeV), $R = 2.782$ (fm) and $a = 0.760$ (fm) for the depth, the reduced radius and the diffuseness for the real part. We use: $W_0 = 16.99$ (MeV), $R_i = 4.226$ and $a_i = 0.468$ (fm) for the depth, reduced radius and the diffuseness of the imaginary part.

The graph of the cross sections is seen in Figure 4.8, where the cross sections as the ratio to the Rutherford one are plotted against the scattering angle. Here we set the maximum of hypermoment 38. In the graph, the theoretical calculations are compared to the experimental data; the experimental data are from [70]. From the graph, it is apparent that the main difference between the Glauber and the higher order calculations is seen at angles of 10 to 25 degrees. We can see that the Glauber calculation has less absorption than higher order calculations. The higher order terms take into account the non-eikonal trajectories of the incident particle, and so the present result suggests that the Glauber calculation overestimates the cross section at higher angles. Since this reaction is at 38.3 (MeV/A), the non-eikonal contribution is important. Accordingly it is thought that the higher order phase shift plays a role in suppressing the cross section at higher angels.

4.6.2 $^{22}$C scattering

Now we need to study the known heaviest Carbon isotope by considering its...
scattering. Figure 4.9 presents the differential scattering cross sections as the ratio to the differential Rutherford cross section $d\sigma_R$ for the reaction $^{22}\text{C} + ^{12}\text{C}$ at 240 and 290 (MeV/A) as these energies allow us to use the semiclassical approximations.

![Figure 4.9](image)

Figure 4.9: The differential cross section as the ratio to $d\sigma_R$ plotted against the scattering angle for the reaction $^{22}\text{C} + ^{12}\text{C}$ at 240 (the left graph) and 290 (the right graph) (MeV/A). hd1 is used as the bound wave function.

At these incident energies, as can be seen from the graphs, the scattering is observed mostly at forward angles; the scattering cross sections tend to begin to rapidly decrease at around an scattering angle of 5 degrees. On top of that, the graphs indicate that the Glauber calculation is little different to the higher order and RY calculations.

When the incident energy is 240 (MeV/A), the centre-of-mass energy for the core-target system is around 1800 (MeV) and wave number is about 25 (1/fm). Accordingly a previous section tells us that the condition for the high energy condition is met:

$$\frac{|U|}{E_{\text{cm}}} \approx \frac{271}{1800} \ll 1.$$  \hspace{1cm} (4.6.1)

From a previous section, we can estimate that the effective range $a_e$ of the potential seems to be 10 (fm), and so the high energy condition is obviously satisfied: $1 \ll ka_e$.

Also, as to the WKB approximation, we have

$$\frac{|U|a_e}{\hbar v} \approx \frac{271 \times 10}{140} \gg 1 ,$$  \hspace{1cm} (4.6.2)

and so in this case the WKB condition is satisfied as well. For these reasons, we find that it is understandable that the Glauber calculation and the higher order calculations yield similar values. Figure 4.10 shows the graphs of the same reaction
Figure 4.10: The differential cross section as the ratio to $d\sigma_R$ plotted against the scattering angle for the reaction $^{22}\text{C} + ^{12}\text{C}$ at 134 (the left graph) and 180 (the right graph) (MeV/A). hd1 is used as the bound wave function.

except that the incident energies are 134 and 180 (MeV/A). We can see that the similar argument holds true at these energies as well.
In a nuclear reaction process, a charged projectile interacts with a charged target via electromagnetic and nuclear forces. Investigations of electromagnetic effects may help us to understand their importance in the reaction process compared to the strong nuclear interaction. If the reaction involves a breakup event, the breakup can occur either directly or in a sequential way.

In non-sequential breakup, the incident particle of a reaction disintegrates without going through any other physical processes [72]. This notion was supported in the early 1980s: the results of the scattering experiment for $^7$Li on $^{120}$Sn suggested that the Coulomb interaction excited the projectile directly from its ground state into the continuum of $\alpha + ^3$H [73]. This excitation had been known as a rapid non-sequential process or a direct breakup mode [74]. In fact, the theoretical differential cross section which is based on the direct breakup model deviates from experimental data as the scattering angle increases. But this does not mean that this mechanism is incorrect, because the final state interactions (FSIs) between the target and the breakup fragments contribute to bringing the cross section down at higher angles in a rapid non-sequential breakup process [75]. (In contrast, in an indirect process, for instance for the reaction $^9$Be + $^{120}$Sn, the projectile disintegrates at around 290 (fm) into $^8$Be + n. The FSIs are therefore less important in that case.)

In indirect breakup, the incident particle passes through an intermediate state, and then breaks up [72], and this was relevant to a Borromean nucleus. In the late 1980s, it had been pointed out that a weakly-bound nucleus had a soft dipole mode
(also called a soft giant dipole mode [12]) because of electromagnetic excitation [4]. The notion of the soft mode was extended by considering the relative motion between the valence neutrons and the core, and interpreted as a collective oscillatory motion whose amplitude is large, but whose frequency is low. As the restoring force is weak, due to the low density of the neutron halo, they oscillate at a low frequency. The logical consequence of the extended notion was that if this soft giant dipole mode is observed, then the nucleus has a halo state in its ground state [12].

It was expected that Borromean nuclei would have soft dipole modes and, in 1989, an enhanced electromagnetic dissociation (EMD) cross section for $^{11}$Li at 800 MeV/A was reported, where an interaction cross section $\sigma^{(EMD)}_1$ of about 1720 (mb) was observed [76]; that value was about 20 times larger than that for the reaction $^{12}$C + $^{208}$Pb. The experimental result offered a possibility of the existence of soft dipole resonance (SDR) for this nucleus [76].

Several years after of this experiment, with measurement of the decay energy spectrum of $^{11}$Li, it was reported that the SDR calculation agreed with their spectrum (which other models, such as a dineutron cluster model, overestimated at lower decay energies), but the lifetime of the resonance was shorter than expected [77]. Their finding was that the derived width $\Gamma_0 = 0.80$ (MeV) led to a lifetime of $\tau = 250$ (fm/c) for the resonance; about one-fifth of the oscillation period for an excited energy of 1 (MeV) [77]. In addition, the longitudinal component of the velocity difference between the $^9$Li core and the valence neutrons deviated from zero, which suggested that the core was accelerated by the Coulomb interaction between the core and the lead target. It turned out that the breakup event happened within 30 (fm) of the lead target, which placed a constraint on the lifetime of the soft dipole state of $^{11}$Li: the figure was about 85 (fm/c), or about 6.6 % of the oscillation period [77]. In another dissociation experiment of $^{11}$Li on a lead target, the derived E1 strength distribution (as a function of decay energy) suggested that the resonance appeared at the decay energy $E_d = 0.7$ (MeV) and the width $\Gamma = 0.8$ (MeV), which was well reproduced by the SDR model. However, the velocity difference between the core and the valence neutrons indicated that the mean lifetime of the excited state of $^{11}$Li was about 50 (fm/c) (and, accordingly, the width was about 4.0 (MeV)), much shorter than the SDR theory prediction [78]. On the whole, the results of these two
experiments were in favour of the view that the breakup was rather non-sequential [77, 78]. Also, Coulomb breakup of the one-neutron halo nucleus $^{11}$Be yielded an E1 distribution that was consistent with the direct breakup mechanism [79]. The claim of the existence of the soft dipole mode was treated with scepticism [80]. Nonetheless, in 2015 it was reported that the soft dipole mode of $^{11}$Li was observed at 1.03 ± 0.03 (MeV) with a width of 0.51 ± 0.11 [81].

When it comes to $^{22}$C, although there are no experimental data on E1 strength, it has been studied theoretically [24] and the possibility of the soft dipole mode was explored [13]. More extensive research on this is required to elucidate the structure of this nucleus. Thus, in this chapter we consider the E1 transition strength for a halo nucleus by deriving the formula for it and calculating it for Borromean nuclei.

### 5.1 Basic formulation of E1 response functions

In this section, we derive the formula of the E1 strength distribution within the 3-body model which we have used so far. Since this chapter treats 3-body continuum wave functions, we now introduce the wave numbers $k'_x, k'_y$ for the 3-body Jacobi coordinates. (For further detail, see Appendix A.) Then, using these wave numbers, we define excitation energy $E_{ex}$ (measured from the breakup threshold) by [32]:

$$E_{ex} = E'_x + E'_y = \frac{\hbar^2 k'^2_x}{2\mu_x} + \frac{\hbar^2 k'^2_y}{2\mu_y}, \quad (5.1.1)$$

where energies $E'_x$ and $E'_y$ for the coordinates are defined by [32]:

$$E'_x = \frac{\hbar^2 k'^2_x}{2\mu_x} = \frac{\hbar^2}{2m_n} \cdot \left(\frac{2}{1}\right), \quad E'_y = \frac{\hbar^2 k'^2_y}{2m_n} \cdot \left(\frac{2 + A_c}{2A_c}\right). \quad (5.1.2)$$

Then suppose that the hyperspherical coordinates are related to the Jacobi ones as [32]:

$$|x| = \rho \sin \alpha \quad |y| = \rho \cos \alpha \quad (5.1.3)$$

$$|k_x| \sqrt{\frac{m_n}{\mu_x}} = k \sin \alpha_k \quad |k_y| \sqrt{\frac{m_n}{\mu_y}} = k \cos \alpha_k. \quad (5.1.4)$$

Using $E_{ex} = \frac{\hbar^2 k'^2}{2m_n}$, we have

$$E'_x = E_{ex}(\sin \alpha_k)^2, \quad E'_y = E_{ex}(\cos \alpha_k)^2. \quad (5.1.5)$$
5.1. BASIC FORMULATION OF E1 RESPONSE FUNCTIONS

Then we denote a 3-body bound state and continuum wave functions by $\phi_{\text{IM}_J}(x, y)$ and $\phi_{k'_xk'_yJ'M_J}(x, y)$, respectively. The E1 strength distribution of a process is given by

$$\frac{d\mathcal{B}}{dE_{\text{ex}}} = \frac{1}{2J + 1} \left( \frac{A_{\text{proj}}}{A_c} \right)^3 \sum_{M_\mu M_J} \int dk'_x dk'_y \delta(E_{\text{ex}} - E'_x - E'_y) \times \left| \langle \phi_{k'_xk'_yJ'M_J}(x, y) | \hat{M}_\mu^{(E1)} | \phi_{\text{IM}_J}(x, y) \rangle \right|^2,$$  

(5.1.6)

where $\hat{M}_\mu^{(E1)}$ is the E1 transition operator. The factor $\left( \frac{A_{\text{proj}}}{A_c} \right)^3$ comes from the normalisation of a bound wave function.

Since the electric multipole moment of order $\lambda$ and its projection $\mu$ is expressed, with the charge distribution $\rho$, as [82, 83]

$$Q(E; \lambda, \mu) = \int r^\lambda Y_{\lambda\mu}(\Omega) \rho(r) dr .$$  

(5.1.7)

Here we are using the normalised Jacobi coordinates, and so the distance between the core and the center-of-inertia of the two valence neutrons is $\sqrt{\frac{A_c+2}{2A_c}} |y|$. Accordingly we have the relation of $r = \frac{2}{A_{\text{proj}}+2} \cdot \sqrt{\frac{A_c+2}{2A_c}} |y|$, which yields the expression for the E1 transition operator. The operator is given as [62]

$$\hat{M}_\mu^{(E1)}(y) = eZ_c \sqrt{\frac{2}{A_c A_{\text{proj}}}} |y| Y_{1\mu}(\Omega') .$$  

(5.1.8)

Since in this case the two valence neutrons are electrically neutral, in the normalised 3-body Jacobi coordinates, only the core’s charge is considered (and accordingly $eZ_c$ appears in (5.1.8).)

After that, we define a matrix element $\mathcal{M}_\mu^{\Xi}$ by

$$\langle \phi_{k'_xk'_yJ'M_J}(x, y) | \hat{M}_\mu^{(E1)} | \phi_{\text{IM}_J}(x, y) \rangle = \sum_{K'L'M_{J'}} (-1)^{L'-S'+M_{J'}} \frac{j'}{M_{L'} M_{S'} - M_{J'}} \times \mathcal{Y}_{K'L'}^{L'L'}(\Omega_{k'5}) \mathcal{M}_\mu^{\Xi},$$  

(5.1.9)

where $\phi_{k'_xk'_yJ'M_J}(x, y)$ denotes the continuum wave function, $\phi_{\text{IM}_J}(x, y)$ the bound wave function, and $\mathcal{M}_\mu^{\Xi}$ contains the Kronecker deltas $\delta_{SS'}\delta_{M_{S'}M_S}$ from the norm between $|S, M_S\rangle$ and $|S', M_{S'}\rangle$. In our notation, $j' = \sqrt{2J + 1}$, and the superscript $\Xi$ denotes a set of quantum numbers; in this case it denotes

$$\Xi = \{ J', M_{J'}, K', L', M_{L'}, S', L'_{\gamma}, L'_{\delta}, J, M_J \} .$$  

(5.1.10)
Here we note that the dimensions of $\mathcal{M}^E_\mu$ are (MeV$^{1/2}$ · fm$^{9/2}$), which is the same as
\[
\langle \phi^{(3B)}_{k,L,M,J}(x,y) | \mathcal{M}^{(E1)}_\mu | \phi^{(3B)}_{J,M,J}(x,y) \rangle.
\]

Then, if we use plane waves for the continuum of the projectile, we need to express them in hyperspherical coordinates. Indeed, the synthesis of a 3-body plane wave with a spin part is expanded in terms of hyperspherical harmonics as [33] or [84]
\[
\frac{1}{(2\pi)^3} \left[ e^{i \mathbf{k}_r \cdot \mathbf{x} + i \mathbf{k}_p \cdot \mathbf{y}} \otimes | S' \rangle \right]_{J',M,J'} = \frac{1}{(k\rho)^{5/2}} \sum_{K'L'J'M'} \mathcal{Y}^{L'L'}_{K'J'}(\Omega_{k\rho}) \mathcal{Y}^{J'M'}_M(\Omega S') \hat{M}^{(E1)}_\mu(y) \hat{J} \times \sum_{M_{S'}} (-1)^{L'-S'+M_{J'}} \mathcal{Y}^{L'L'}_{K'J'}(\Omega S') | S'M_{S'} \rangle.
\]
(5.1.11)

while we express the 3-body bound wave function as
\[
\phi^{(3B)}_{J,M} = \sum_{KL,L,L,S,L,M,L,S} \left[ \phi^{(3B)}_{KL} \otimes | S \rangle \right]_{J,M}.
\]
(5.1.12)

Here we add the factor of $(2\pi)^{-3}$ to the plane wave function. We will see the explanation concerning this factor in Chapter 6.

From the definition of the matrix element (5.1.9), we see the specific expression of the element to be
\[
\mathcal{M}^E_\mu = \int dx dy \frac{1}{(k\rho)^{5/2}} \sqrt{k\rho} J_{K'J}(k\rho) \mathcal{Y}^{L'L'}_{K'J'}(\Omega_{k\rho}) \hat{M}^{(E1)}_\mu(y) \hat{J} \times \sum_{KL,M_{S},L,L,S} \phi^{(3B)}_{KL}(x,y) \left( -1 \right)^{L'-S'+M_{J'}} \begin{pmatrix} L & S & J \\ M_L & M_S & -M_J \end{pmatrix} \delta_{S'S} \delta_{M\rho'M_S}.
\]
(5.1.13)

Summing over $S$ and $M_S$ yields
\[
\mathcal{M}^E_\mu = \int dx dy \frac{1}{(k\rho)^{5/2}} \sqrt{k\rho} J_{K'J}(k\rho) \mathcal{Y}^{L'L'}_{K'J'}(\Omega_{k\rho}) \hat{M}^{(E1)}_\mu(y) \hat{J} \times \sum_{KL,M_{L},L,L} \phi^{(3B)}_{KL}(x,y) \left( -1 \right)^{L'-S'+M_{J'}} \begin{pmatrix} L & S' & J \\ M_L & M_{S'} & -M_J \end{pmatrix}.
\]
(5.1.14)
5.1. BASIC FORMULATION OF E1 RESPONSE FUNCTIONS

Considering this, we can now derive a general formula for E1 transition distribution. Note that the phase factors of $k_x'$, $k_y'$ are written as

$$dK_k' = \mu_x \sqrt{2\mu_x E_x'} \frac{1}{\hbar^3} dE_x' d\Omega_k' \tag{5.1.15}$$

$$dK_y' = \mu_y \sqrt{2\mu_y E_y'} \frac{1}{\hbar^3} dE_y' d\Omega_k' \tag{5.1.16}$$

Then we substitute for $dK_x'$, $dK_y'$ in (5.1.6) to express the E1 strength distribution as:

$$
\frac{dB}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^{\frac{3}{2}} \sum_{M_{j'}M_{j\mu}} \mu_x \sqrt{2\mu_x} \frac{1}{\hbar^3} \int \sqrt{E_{ex} - E_y'} \sqrt{E_{ex} - E_x'} d\Omega_k' dK_y' \\
\times |\langle \phi_{k_y'k_y'}^{(3B)}(x,y)| M_{\mu}^{(E1)} |\phi_{jM_j}^{(3B)}(x,y)\rangle|^2 \\
\times \frac{\mu_y \sqrt{2\mu_x E_x'}}{\hbar^3} dE_x' d\Omega_k' dE_y' d\Omega_k' |\phi_{k_y'k_y'}^{(3B)}| \langle x,y | M_{\mu}^{(E1)} |\phi_{jM_j}(x,y)\rangle |^2 \\
= \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^{\frac{3}{2}} \sum_{M_{j'}M_{j\mu}} \mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y} \frac{1}{\hbar^3} \\
\times \int \sqrt{(E_{ex} - E_y')E_x'|d\Omega_k'|dE_y'|d\Omega_k'|\langle \phi_{k_y'k_y'}^{(3B)}| (x,y)| M_{\mu}^{(E1)} |\phi_{jM_j}(x,y)\rangle |^2} \\
\times \frac{\mu_y \sqrt{2\mu_x E_x'}}{\hbar^3} dE_x'|d\Omega_k'|dE_y'|d\Omega_k'.
\tag{5.1.17}
$$

For convenience, we collect coefficients and phases as

$$K_{L''L'S'J'}^{M_LM_{L''}M_{L'}} = (-1)^{(L''-S'\pm M_{L''}+L'-S'\pm M_{L'})} (2J' + 1) \\
\times \begin{pmatrix} L'' & S' & J' \\ M_{L''} & M_{S'} & -M_{J'} \end{pmatrix} \begin{pmatrix} L' & S' & J' \\ M_{L'} & M_{S'} & -M_{J'} \end{pmatrix}, \tag{5.1.18}
$$

hence,

$$\frac{dB}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^{\frac{3}{2}} \sum_{M_{j'}M_{j\mu}} \mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y} \frac{1}{\hbar^3} \\
\times \int \sqrt{(E_{ex} - E_y')E_x'|d\Omega_k'|dE_y'|d\Omega_k'| \\
\times \sum_{K''L''M_Lr'L''r'} K_{L''L'S'J'}^{M_L} K_{L''L'}^{M_L} \gamma_{K''L''}^{(3B)}(\Omega_{k''}) \gamma_{K''L'}^{(3B)}(\Omega_{k''}) M_{\mu}^{(E1)} M_{\mu}^{(E1)}, \tag{5.1.19}
$$
where the quantum numbers which are signified by double prime appear due to taking the square modulus of the matrix element (5.1.9). For notational convenience, we define $\Xi''$ to represent

$$\Xi'' = \{J', M_{J'}, K'', L'', M_{L''}, S', L''_x, L''_y, J, M_J\}. \quad (5.1.20)$$

Then the integration with regard to $\Omega_{K'_x}, \Omega_{K'_y}, E_y'$ yields

$$\frac{dE}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^3 \sum_{M_{J'}M_{J''}} \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{\hbar^3} 2E_{ex}^2 \times \sum_{K''L''L''_xL''_y} K''L''L''_xL''_y \sum_{M_{L''}M_{S''}} \delta_{K'K''} \delta_{L'L''} \delta_{M_{L''}M_{S''}} \delta_{L''_xL''_y} \mathcal{M}_\mu^{\Xi''} \mathcal{M}_\mu^{\Xi''}.$$ 

Taking the summation over $(K'', L'', M_{L''}, L''_x, L''_y)$, we have

$$\frac{dE}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^3 \sum_{M_{J'}M_{J''}} \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{\hbar^3} 2E_{ex}^2 \times \sum_{K'L'M_{L'}M_{S'}} (2J' + 1) \left( \begin{array}{ccc} L' & S' & J' \\ M_{L'} & M_{S'} & -M_{J'} \end{array} \right) \times \mathcal{M}_\mu^{\Xi''} \mathcal{M}_\mu^{\Xi''}. \quad (5.1.21)$$

Immediately, the basic formula for E1 transition strength becomes:

$$\frac{dE}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^3 \sum_{M_{J'}M_{J''}} \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{\hbar^3} 2E_{ex}^2 \times \sum_{K'L'M_{L'}M_{S'}} (2J' + 1) \left( \begin{array}{ccc} L' & S' & J' \\ M_{L'} & M_{S'} & -M_{J'} \end{array} \right) \mathcal{M}_\mu^{\Xi''} \mathcal{M}_\mu^{\Xi''}. \quad (5.1.22)$$

Thus we have derived the formula. In terms of the summation over $K'$, only odd $K'$ contribute to the E1 strength distribution due to the parity constraint. The parity of $\mathcal{Y}_{K'L'}^{L''L''_xL''_y}$ is $(-1)^{K'}$ [34]. In addition, since a Borromean nucleus consists of two valence nucleons plus its core, we express the factorisation of reduce masses for the system in the 3-body Jacobi coordinates as

$$\mu_x\mu_y = \frac{m_1m_2m_c}{m_1 + m_2 + m_c} = \frac{A_c}{A_{proj}} m_n^2. \quad (5.1.23)$$
5.2. AN EXPRESSION FOR \( \mathcal{M}_\mu^\Xi \)

By virtue of this, we define the E1 phase factor by

\[
\rho_{\text{ex}}^{(E1)}(\mu_x, \mu_y, E_{\text{ex}}) = \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{\hbar^2} \frac{2E_{\text{ex}}^2}{A_{\text{proj}}} \left( \frac{A_{\text{proj}}}{A_c} \right)^{3/2} = \frac{4E_{\text{ex}}^2}{\hbar^6m_n^3},
\]

(5.1.24)

where \( \rho_{\text{ex}}^{(E1)}(\mu_x, \mu_y, E_{\text{ex}}) \) is seen in the formula (5.1.22) and has dimensions (MeV\(^{-1}\) · fm\(^{-6}\)). Accordingly the dimensions of E1 strength distribution are (fm\(^3\)), but here we use (e\(^2\) · fm\(^2\) · MeV\(^{-1}\)).

5.2 An expression for \( \mathcal{M}_\mu^\Xi \)

In this section, we express explicitly the reduced matrix elements of the E1 transition operator. In the previous section, we expressed the \( \mathcal{M}_\mu^\Xi \) with a plane wave. Now we focus on having an explicit expression for it. Note that the initial total angular momentum of \(^{22}\)C is zero \((J = 0^+)\). Then, recalling that the orbital part of the bound wave function is

\[
\phi_{KL}^{(3B)L_\alpha L_\gamma}(x, y) = \sum_{KLL_\alpha L_\gamma} \frac{1}{\rho_5^{1/2}} \chi_{KL}^{L_\alpha L_\gamma}(\Omega_5),
\]

(5.2.1)

we write (5.1.14) as

\[
\mathcal{M}_\mu^\Xi = iK\epsilon Z_e \sqrt{\frac{2}{A_{\text{proj}}A_c}} \sum_{KLL_\alpha L_\gamma} \int_0^\infty \frac{J_{K'2}(k\rho)}{(k\rho)^{5/2}} \sqrt{k\rho \rho_5^{1/2}} \chi_{KL}^{L_\alpha L_\gamma}(\rho) \rho_5^5 d\rho
\]

\[
\times \int \chi_{KL'}^{L_\alpha L_\gamma}(\Omega_5) \cos \alpha \chi_{KL}^{L_\alpha L_\gamma}(\Omega_5) \sin(2\alpha) \frac{d\Omega_x d\Omega_y}{8}
\]

\[
\times (-1)^{L-M_L} \frac{1}{L} \delta_{LS} \delta_{M_L-M_S} \delta_{S'M_S}.
\]

(5.2.2)

Using the definition of HH, we have

\[
\mathcal{M}_\mu^\Xi = iK\epsilon Z_e \sqrt{\frac{2}{A_{\text{proj}}A_c}} \sum_{KLL_\alpha L_\gamma} \sum_{M_L,M_L'} \int_0^\infty \frac{J_{K'2}(k\rho)}{(k\rho)^{5/2}} \sqrt{k\rho \rho_5^{1/2}} \chi_{KL}^{L_\alpha L_\gamma}(\rho) \rho_5^5 d\rho
\]

\[
\times C_{L'2L_\alpha L_\gamma} \int Y_{L_x,M_{L_x}}^*(\Omega_x) Y_{L_y,M_{L_y}}(\Omega_y) d\Omega_x \int Y_{L'_x,M_{L'_x}}^*(\Omega_y) Y_{L_y,M_{L_y}}(\Omega_y) d\Omega_y
\]

\[
\times (-1)^{L-M_L} \frac{1}{L} (-1)^{L'-L_y+M_{L_y}+M_{L'_y}} \begin{pmatrix} L'_{x} & L_y & L \\ M_{L_x} & M_{L_y} & -M_{L'_y} \end{pmatrix}
\]

\[
\times (-1)^{L_x-L_{\alpha}+M_L} \begin{pmatrix} L_x & L_y & L \\ M_{L_x} & M_{L_y} & -M_L \end{pmatrix} \delta_{LS} \delta_{M_L-M_S} \delta_{S'S'M_S}.
\]

(5.2.3)
where \( C_{\nu L'\mu' n L_n, \mu n L_y} \) is defined by

\[
C_{\nu L'\mu' n L_n, \mu n L_y} = \int_{-1}^{1} N_{\nu n}^{L'\mu' + \frac{1}{2} L_y' + \frac{1}{2}} P_{\nu n}^{L'\mu' + \frac{1}{2} L_y' + \frac{1}{2}} \times (\cos \alpha) \\
\times N_{\nu n}^{L_n + \frac{1}{2} L_y + \frac{1}{2}} P_{\nu n}^{L_n + \frac{1}{2} L_y + \frac{1}{2}} (\sin \alpha)^{L'\mu' + L_n} (\cos \alpha)^{L_y' + L_y} \sin 2\alpha \frac{d\cos 2\alpha}{8} .
\]

(5.2.4)

This coefficient \( C_{\nu L'\mu' n L_n, \mu n L_y} \) becomes a constant because it involves the integration with regard to \( \cos 2\alpha \).

Next, we seek to obtain the explicit expression for \( M^\Xi_\mu \), which involves integration over \( \Omega_y \). We make use of the following formula regarding spherical harmonics to obtain

\[
\int \langle K L_L L_y, M_L M_y \rangle d\Omega_y = \int (-1)^{\mu L_y} Y_{L_y, M_y} Y_{1 \mu} Y_{L_y} \langle K L_L, M_L \rangle d\Omega_y \\
= (-1)^{\mu L_y} \sqrt{\frac{3}{4\pi}} \hat{L}_y \hat{L}_y \\
\times \left( \begin{array}{ccc}
L_y & 1 & L_y \\
M_{L} & \mu & M_{L_y}
\end{array} \right) \left( \begin{array}{ccc}
L_y & 1 & L_y \\
0 & 0 & 0
\end{array} \right).
\]

(5.2.6)

To deal with remaining parts of the \( M^\Xi_\mu \) is not trivial. As to the detail of the derivation of \( M^\Xi_\mu \) see Appendix C.2. Eventually, we reach the following expression:

\[
M^\Xi_\mu = \sum_{K L L_y} (-1)^{L'_y + \mu + 1} \left( \begin{array}{ccc}
L' & 1 & L \\
M_{L'} & \mu & M_{L}
\end{array} \right) \tilde{M}^{K L L_L L_y'}_{K L L_L L_y} \delta_{L S} \delta_{M_L - M_S} \delta_{S' S} \delta_{M_{L_y} M_S} ,
\]

(5.2.7)

where the reduced matrix element \( \tilde{M}^{K L L_L L_y'}_{K L L_L L_y} \) is defined by

\[
\tilde{M}^{K L L_L L_y'}_{K L L_L L_y} = i^{K'} e Z c \sqrt{\frac{2}{A_{proj} A_c}} \left( \begin{array}{ccc}
L' & 1 & L \\
L_y & L_x' & L_y'
\end{array} \right) \left( \begin{array}{ccc}
L_y & 1 & L_y \\
0 & 0 & 0
\end{array} \right) \\
\times C_{L'_y L' \mu' n L_n L_y} \hat{L}_y \hat{L}_y' \sqrt{\frac{3}{4\pi}} \int_0^\infty \frac{J_{K'+2}(k\rho)}{k^2} \rho^{3/2} \chi_{K KL} (\rho) d\rho .
\]

(5.2.8)

Here we call this the reduced matrix element, because the reduced matrix elements of a tensor operator are defined by \[85\]

\[
\langle \gamma' j' m' | T(k, q) | \gamma j m \rangle = (-1)^{j' - m'} \left( \begin{array}{ccc}
j' & k & j \\
-j' & m' & q
\end{array} \right) \langle \gamma' j' | T(k) | \gamma j \rangle .
\]

(5.2.9)
5.3 Numerical calculation

In this section, our numerical calculations of the E1 strength distribution are shown.

5.3.1 E1 strength for $^6$He

It is useful to consider first the electric dipole strength distribution where experimental data are available [86]. On the theoretical side, the E1 transition strengths were calculated within CSF and HH frameworks by Cobis and Danilin, respectively [87, 33]. Figure 5.1 compares our theoretical values with these early theoretical results and the experimental data. Our calculation involves HH 3-body bound state wave functions with plane wave continuum wave functions. We chose 42 and 41 as the maximum initial and final hypermoments, respectively.

![Figure 5.1](image.png)

Figure 5.1: The E1 strength distributions of $^6$He as a function of an excitation energy. Our calculation is compared with Danilin’s theoretical value and Cobis’s, and experimental data as well. Our calculation involves od13 for the bound wave function. The experimental data are from [86].

As is seen from the graph, Danilin’s HH calculation generally agrees with the experimental data except that it overestimates the E1 strength at the low-lying energies. The calculation involving CSF seems to overestimate in the low-lying region.
and underestimate at higher excitation energies. These HH and CSF calculations have the relatively sharp peaks at around 1.2 (MeV). On the whole, our calculation underestimates the experimental data, and is smaller than the values which CSF and HH provide, although the CSF calculation provides smaller values than ours at those energies higher about 3.1 (MeV). This implies that, at least at a low-lying region, a more realistic continuum wave function can provide a larger and sharper peak than a plane wave continuum wave function. (In the next subsection, we see a result similar to this.) Corrections to our plane wave continuum wave functions will be explored in Chapter 7.

5.3.2 E1 strength for $^{22}_C$

When it comes to $^{22}_C$, so far there have been no experimental measurements of its E1 strength distribution. In terms of the theoretical approaches to address this issue, HH and self-consistent Random Phase Approximation (RPA) calculations have been carried out by Ershov and Inakura, respectively [13, 24].

Figure 5.2 shows our theoretical E1 strength distributions of this nucleus, plotted against the excitation energy. The E1 strengths calculated by Ershov are shown

![Figure 5.2: The E1 strength distributions of $^{22}_C$ plotted against the excitation energy. For our calculations, ic1 and ic4 are used for the bound state wave functions. Our calculations do not involve a final state interaction.](image_url)
5.3. NUMERICAL CALCULATION

together in the graph. Our E1 distributions involve bound state wave functions whose parametrisations are ic1 and ic4. Basically these bound state wave functions are different from Ershov’s ones, but the 2n-separation energies of the bound state wave functions (ic1 and ic4) are close to those of Ershov. As far as the peak positions of the distributions are concerned, it seems that our results are comparable to Ershov’s calculations. Comparing their results to ours, we find that theirs have generally higher and sharper peaks than ours. It can be argued that the value of their E1 distribution is generally larger than that of ours as they use more realistic continuum wave functions. This is comparable to the findings in the previous subsection for $^6$He where, at a low-lying region, more realistic continuum wave functions gave higher and sharper peaks. Thus in this regard we have seen the similarity between $^6$He and $^{22}$C.

Also, in Figure 5.3, the E1 strength distributions are plotted against the excitation energy, for several different 2n-separation energies. From the graph, we see that the peak position of the distribution is shifted to higher energies (and the peak becomes broader and smaller) as the 2n-separation energy increases.

Figure 5.3: E1 strength distributions of $^{22}$C involving five different bound state wave functions. In this case, we used ic1-4 and ic6 for the bound state wave functions. Our calculations do not involve a final state interaction.
In the previous chapter, we discussed EMD (also called Coulomb breakup), where heavy targets are used such as lead. In fact, contribution to a breakup cross section comes from not only Coulomb part but also nuclear part, which raises the issue that we need to separate the former contribution to the latter [88]. To do this, we need to build a realistic model to describe a nuclear breakup reaction, and therefore we need to extend our discussion into nuclear breakup reactions which typically involve a beryllium or carbon target.

Basically breakup reactions are classified according to a reaction mechanism: elastic breakup and inelastic breakup [89]. The former is a process where the projectile is broken into constituent parts, and the target remains in its ground state, and the nucleon tends to continue to move in a forward direction [90, 91].

In the present chapter, we discuss elastic breakup reactions of Borromean nuclei on a stable nucleus. This helps us to study the continuum structure of a projectile, since a halo nucleus often tends to break apart after its collision with a target. In terms of the breakup reactions involving Borromean nuclei, several 4-body models were built.

The DWBA-type 4-body model included a recoil effect and took into account of the fact that halo nuclei are spatially extended. This model also treated the Coulomb breakup and nuclear breakup on the same footing [32, 90]. The continuum discretized coupled channel (CDCC) 4-body method was a quantum mechanical approach so that the method was valid at those energies which do not permit semiclassical and
adiabatic approximations. The method was called 4-body CDCC [92].

An eikonal-type 4-body model involved adiabatic approximations and the eikonal approximation [62, 93]. This model has been used to study breakup reactions involving $^6$He and $^{11}$Li.

When it comes to $^{22}$C, the interaction between its core and carbon target has not been investigated well. For this reason, we adopt our folding model in our analysis to estimate the interaction between the core and the target as was done in Chapter 4. The Glauber-WKB 4-body model which we develop can have advantage that it is valid at lower energies and higher scattering angles, compared to a usual eikonal-type model.

Thus, in the present chapter, within the 4-body Glauber-WKB model, we formulate the elastic breakup for Borromean nuclei and calculate breakup cross sections for Borromean nuclei.

Here we note that thanks to a cluster model, we can write the total Hilbert space $\mathcal{H}_{\text{tot}}$ as the product of two subspaces: the Hilbert space $\mathcal{H}_{4B,\text{cm}}$ for the system involving a projectile and a target and the Hilbert space $\mathcal{H}_{3B}$ for the projectile’s internal system. That is,

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{4B,\text{cm}} \otimes \mathcal{H}_{3B}.$$  \hspace{1cm} (6.0.1)

This cluster approach is adopted by others as well [32], and with this approach, we analyse the breakup reactions of Borromean nuclei such as $^{22}$C.

### 6.1 Formalism for breakup

In this section, we derive formulae to calculate the differential breakup cross section. Adopting the Glauber-WKB approach which was used in a previous chapter, we develop a formalism to calculate the observable.

#### 6.1.1 Derivation of the 5-folded differential cross section

In this subsection, we address the calculation of the transition matrix in order...
to derive the expression of a 5-folded differential cross section. Suppose that, for a
4-body system, a transition occurs from an initial state $\alpha$ to a final one $\alpha'$. Here
we denote the wave numbers of the valence particle 1, 2, the core and the target by
$k'_1, k'_2, k'_c$ and $k'_t$, respectively\(^1\).

The exclusive cross section\(^2\) for this process is expressed as [94]:

$$d\sigma_{\alpha'\alpha} = \frac{(2\pi)^4}{v} |T_h|^2 \delta(E_f - E_i) \delta(p_f - p_i) \prod_{\nu=1,2,c,t} dk'_\nu , \quad (6.1.1)$$

where $v$ is the relative velocity of the colliding system, and

$$p_f = \hbar (k'_1 + k'_2 + k'_c + k'_t) , \quad E_f = E'_1 + E'_2 + E'_c + E'_t . \quad (6.1.2)$$

Now, for the above formula (6.1.1), we derive the explicit expression for the part
involving the integration with regard to momenta. To see this, we begin by using the
following relation:

$$\frac{\partial(k_f, k'_r, k'_y, k'_z)}{\partial(k'_1, k'_2, k'_c, k'_t)} = 1 , \quad (6.1.3)$$

which enables us to change the Cartesian coordinates into the 4-body Jacobi coordinates. The relation (6.1.3) can be obtained by a slightly lengthy but straightforward
calculation. Next, in the centre-of-mass frame of the 4-body system, we see that
$p_{t,cm} = p_{t,cm} = 0$, which yields (6.1.5):

$$\int |T_h|^2 \delta(E_f - E_i) \prod_{\nu=1,2,c,t} dp'_\nu = \int |T_h|^2 \delta(E_f - E_i) \delta(p_f - p_i) \prod_{\nu=x,y,R,f} dp'_\nu$$

$$= \int |T_h|^2 \delta(E_f - E_i) \prod_{\nu=x,y,R} dp'_\nu . \quad (6.1.4)$$

Then we express $\prod_{\nu=x,y,R} dp'_\nu$ by the momentum spherical coordinates and energies:

$$\prod_{\nu=x,y,R} dp'_\nu = \prod_{\nu=x,y,R} p'_\nu d\Omega'_{k_\nu} = \prod_{\nu=x,y,R} \mu_\nu p'_\nu dE'_\nu d\Omega'_{k_\nu} . \quad (6.1.6)$$

\(^1\)In Goldberger and Watson’s book, the initial momentum of a system and final momenta are
denoted by $P_\alpha$ and \{$k_j\}_j$, respectively [94].

\(^2\)In fact, there exist several ways of standardising a differential breakup cross section [51, 95].
Those conventions differ by a factor of constant, and those differences are absorbed by other parts
of differences such as the way of normalisation of a wave function [32].
Here the reduced masses \( \{ \mu_j \}_{j=x,y,R} \) associated with the Jacobi coordinates are introduced:

\[
\begin{align*}
\mu_R &= \frac{m_t(m_c + m_2 + m_1)}{m_t + m_c + m_2 + m_1} = m_n \frac{A_{\text{targ}}(A_c + 2)}{A_{\text{targ}} + (A_c + 2)}, \\
\mu_y &= \frac{m_c(m_2 + m_1)}{m_c + m_2 + m_1} = m_n \frac{A_c \cdot 2}{A_c + 2}, \\
\mu_x &= \frac{m_1m_2}{m_2 + m_1} = \frac{m_n}{2},
\end{align*}
\]

(6.1.7) (6.1.8)

where we implicitly assume that the projectile consists of a core and two valence neutrons. Then, since the energy conservation \( E_f = E_i \) must hold in the total centre-of-mass frame, we can integrate the integrand of (6.1.5) with regard to \( E_R \).

\[
\int |T_{fi}|^2 \delta(E_f - E_i) \prod_{\nu=x,y,R} dp'_\nu = \int |T_{fi}|^2 \mu_R p'_R d\Omega'_R \prod_{\nu=x,y} \mu_\nu p'_\nu dE'_\nu d\Omega'_k \nu.
\]

(6.1.9)

Then dividing the above equation by \( d\Omega'_k x d\Omega'_k y dE'_y dE'_x \), we have the following 5-fold differential cross section:

\[
\frac{d^5 \sigma}{dE'_y dE'_x d\Omega'_k x d\Omega'_k y} = \sum_{\alpha_i} \left( \frac{2\pi}{v} \right)^4 \rho(p'_y, p'_x) |T_{fi}|^2,
\]

(6.1.10)

where the phase space factor \( \rho(p'_y, p'_x) \) is given by

\[
\rho(p'_y, p'_x) = \mu_R \mu_y \mu_x p'_R p'_y p'_x.
\]

(6.1.11)

The phase space factor is not a function of \( p'_R \), since the momentum is determined by the integration with regard to \( E'_R \). The factor is now written as

\[
\rho(p'_y, p'_x) = 2 \mu_x^{3/2} \mu_y^{3/2} \sqrt{E'_x E'_y \mu_R p'_R} = 2 \mu_x^{3/2} \mu_y^{3/2} \sqrt{(E_{ex} - E'_y) E'_y \mu_R p'_R}.
\]

(6.1.12)

because the excitation energy is \( E_{ex} = E'_x + E'_y = \frac{p^2}{2m_x} + \frac{p'^2}{2m_y} \) (See [32]). To sum up, in this subsection we have derived the explicit expression of the phase space factor for the exclusive cross section. In a later subsection, we derive the formula for the exclusive cross section for the breakup reaction.

### 6.1.2 Calculation of the transition matrix

In this subsection, we adopt a T-matrix approach and introduce a breakup transition matrix and breakup S matrix (elements), and deal with them within our 4-body...
model. First, we denote the final wave number vector of the whole system by \( \mathbf{k}'_R \). Its solid angular variables are \( \Omega_{\mathbf{k}_R} = (\theta_R, \phi_R) \), and this characterises the direction of final momentum of (the centre-of-mass of) the incident particle in the centre of mass frame. During the process, the momentum transferred to the projectile is

\[
\mathbf{q} = \mathbf{k}'_R - \mathbf{k}_R .
\]

Then the transition matrix element is expressed as

\[
T_t(\mathbf{k}'_x, \mathbf{k}'_y) = \left( \frac{A_{\text{proj}}}{A_c} \right)^{\frac{3}{2}} i \hbar v \int d\mathbf{R}_b e^{i\mathbf{q}\cdot\mathbf{R}_b} t(\mathbf{k}'_x, \mathbf{k}'_y, \mathbf{R}_b) ,
\]

where \( t(\mathbf{k}'_x, \mathbf{k}'_y, \mathbf{R}_b) \) is a breakup matrix element for the whole system, and the factor \( \left( \frac{A_{\text{proj}}}{A_c} \right)^{\frac{3}{2}} \) comes from the normalisation of a bound wave function\(^3\). As to the breakup matrix, it is written as

\[
t(\mathbf{k}'_x, \mathbf{k}'_y, \mathbf{R}_b) = \langle \phi^{(3B)}_{\mathbf{k}_x', \mathbf{k}_y', J M_{J'}}(\mathbf{x}, \mathbf{y}) | (e^{i\chi (R_b, y_b, x_b)} - 1) | \phi^{(3B)}_{J M_j}(\mathbf{x}, \mathbf{y}) \rangle
\]

\[
= \langle \phi^{(3B)}_{\mathbf{k}_x', \mathbf{k}_y', J M_{J'}}(\mathbf{x}, \mathbf{y}) | \sum_{\lambda_x, \lambda_y, \lambda_{M_J}} \Gamma_{\lambda x \lambda y \lambda M_J}(\Omega_x, \Omega_y) \Delta^{\lambda M_J}_{\lambda_x \lambda_y}(R_b, \rho, \alpha) | \phi^{(3B)}_{J M_j}(\mathbf{x}, \mathbf{y}) \rangle ,
\]

where \( \Gamma_{\lambda x \lambda y \lambda M_J}(\Omega_x, \Omega_y) \equiv [Y_{\lambda x} \otimes Y_{\lambda y}]_{\lambda M_J} \), and

\[
\Delta^{\lambda M_J}_{\lambda_x \lambda_y}(R_b, \rho, \alpha) = \int \Gamma_{\lambda x \lambda y \lambda M_J} e^{i\chi (R_b, y_b, x_b)} d\Omega_x d\Omega_y .
\]

Note that, from (4.1.10), \( e^{i\chi (R_b, y_b, x_b)} = e^{i(\chi_R + \chi_{y'z'} + \chi_{x't'})} \).

In considering elastic breakup reactions, we implicitly assume that the momentum transfer is small. Accordingly the momentum transfer becomes

\[
q \approx 2k_R \sin \left( \frac{\theta}{2} \right) , \quad k'_R \simeq k_R .
\]

Then the continuum wave function is expanded as [33]:

\[
\phi^{(3B)}_{\mathbf{k}_x', \mathbf{k}_y', J M_{J'}}(\mathbf{x}, \mathbf{y}) = \frac{1}{(k' \rho)^{\frac{3}{2}}} \sum_{\xi', \xi_0} \chi_{\xi', \xi_0}(k' \rho) \left[ \mathcal{Y}_{K_{L_x'} L_{L_y'}}(\Omega_5) \otimes X_{S'} \right]_{J' M_{J'}}
\]

\[
\times \sum_{M_{L_x'} M_{L_y'}} \langle L_{L_x'} S_{L_x'} M_{L_x'} M_{L_y'} | J' M_{J'} \rangle \mathcal{Y}_{K_{L_x'} L_{L_y'} M_{L_x'} M_{L_y'}}(\Omega_5) ,
\]

where \( \xi = \{ K, L, S, L_x, L_y \} \) is a set of quantum numbers. The quantum numbers which are involved with wave numbers of the continuum wave function are signified

\[\text{\footnotesize \footnote{The six-dimensional integration is with regard to } x \text{ and } y.}\]
6.1. FORMALISM FOR BREAKUP

by subscript k. For example, \( L'_{xk}, L'_{yk} \) are the angular momentum quantum numbers which are linked to \( \Omega_{5k} \).

As was seen in (5.1.11) of a previous chapter, if we use a plane wave as a continuum wave function and expand it with regard to HH, then the Bessel function involved is:

\[
\chi^{\pi'}_{\xi', \xi_k}(\rho) = i^{K'} J_{K'+2}(k' \rho) \sqrt{k' \rho} .
\]  

(6.1.20)

Then the breakup t matrix element is expressed as

\[
t(k'_x, k'_y, R_b) = \sum_{J'M_J} \sum_{\xi' \xi \lambda \lambda_M} \langle L'_k S'_k M'_k M'_k | J' M_J \rangle \Delta^{\lambda M_{L_x} \lambda_{y}}_{\xi' \xi}(R_b) ,
\]  

(6.1.21)

where \( \Delta^{\lambda M_{L_x} \lambda_{y}}_{\xi' \xi}(R_b) \) is written as

\[
\Delta^{\lambda M_{L_x} \lambda_{y}}_{\xi' \xi}(R_b) = \int_{-1}^{1} d \cos(2\alpha) \frac{1}{8} (\sin 2\alpha) L'_{x}^{J'} + \frac{1}{2} L'_{y}^{J'} + \frac{1}{2} L_{x}^{J} + \frac{1}{2} L_{y}^{J} + \frac{1}{2}
\]

\[
\times \sin(2\alpha) P_{L'_{x}^{J'} + \frac{1}{2}, L'_{y}^{J'} + \frac{1}{2}} (\cos 2\alpha)
\]

\[
\times (\sin \alpha)^{L_{x}^{J} + L'_{y}^{J'}} (\cos \alpha)^{L_{y}^{J} + L'_{y}^{J'}} \int_{0}^{\infty} \frac{1}{k' \rho} \chi^{\pi'}_{\xi' \xi_k}(\rho) \Delta^{\lambda M_{L_x} \lambda_{y}}_{\xi' \xi}(R_b, \rho, \alpha) \chi^{\pi}_{\xi}(\rho) \, d\rho .
\]  

(6.1.22)

As for the reduced matrix elements in (6.1.21), we derive the expression for them as follows [85]:

\[
\langle \xi' J' || \Gamma_{\lambda} || \xi J \rangle = (-1)^{L' + S + J + \lambda} \sqrt{(2J + 1)(2J' + 1)} \delta_{S,S'} \begin{pmatrix} L' & J' & S \\ J & L & \lambda \end{pmatrix}
\]

\[
\times \langle L'_{x}^{J'} L'_{y}^{J'} || \Gamma_{\lambda} || L_{x}^{J} L_{y}^{J} \rangle .
\]  

(6.1.23)

Since the rank \( \lambda \) spherical tensor operator \( \Gamma_{\lambda M_{\lambda}} \) is constructed by tensor multiplication of \( Y_{\lambda_{x}} \) and \( Y_{\lambda_{y}} \), the reduced matrix elements are:

\[
\langle L'_{x}^{J'} L'_{y}^{J'} || \Gamma_{\lambda} || L_{x}^{J} L_{y}^{J} \rangle = \langle L'_{x} || Y_{\lambda_{x}} || L_{x} \rangle \langle L'_{y} || Y_{\lambda_{y}} || L_{y} \rangle
\]

\[
\times \sqrt{(2L' + 1)(2L + 1)} \begin{pmatrix} L_{x}^{J} & L_{y}^{J} & \lambda_{x} \\ L'_{x} & L'_{y} & \lambda_{y} \end{pmatrix} .
\]  

(6.1.24)
Then we use the following relation:

\[
\langle l' || Y_k || l \rangle = (-1)^l' \sqrt{\frac{(2l' + 1)(2k + 1)(2l + 1)}{4\pi}} \begin{pmatrix} l' & k & l \\ 0 & 0 & 0 \end{pmatrix},
\]

and therefore the reduced matrix elements are written as:

\[
\langle \xi' J' || \Gamma \lambda || \xi J \rangle = (-1)^{L' + S + J + \lambda + L_x + L_y} \sqrt{(2J + 1)(2J' + 1)(2\lambda + 1)} \delta_{S' S} \begin{pmatrix} L' & J' & S \\ J & L & \lambda \end{pmatrix}
\times \sqrt{(2L' + 1)(2L + 1)} \frac{1}{4\pi} \sqrt{(2L' x + 1)(2L_x + 1)(2\lambda + 1)} \sqrt{(2L' y + 1)(2L_y + 1)(2\lambda + 1)}.
\]

Thus the transition matrix element is written as

\[
T_{\xi}(k', x', k, y') = \left( \frac{A_{0}}{A} \right)^{\frac{3}{2}} i\hbar \sum_{J,M,J',M'} \sum_{L_x,L_y,M_{L_x},M_{L_y}} \langle L_{k,M}, J,M || J', M,J' \rangle \chi^{L_{k'M}} \chi^{L_{k'y}} (\Omega_{5k})
\times \sum_{\lambda M_{\lambda}} (-1)^{J'-M_{J'}} \begin{pmatrix} J' & \lambda & J \\ -M_{J'} & M_{\lambda} & M_{J} \end{pmatrix}
\times \int dR_{b} e^{i\mathbf{q} \cdot \mathbf{R}_{b}} \sum_{\xi \xi_{\lambda x} \lambda_{y}} \langle \xi' || \Gamma \lambda || \xi J \rangle \Delta^{\xi \xi_{\lambda x} \lambda_{y}} (R_{b}) \).
\]

**6.1.3 Coherent wave functions**

In the previous subsection, we derived a specific form of the T matrix element. As to the reaction models of Borromean nuclei, so far we have discussed them mainly under a semiclassical regime: we built the Glauber-WKB 4-body model, and its Rosen-Yennie extension, using the adiabatic approximation. In the model, basically the 4-body wave function \( \Psi_{JM_{J}}^{(4B)}(x, y, R) \) is expressed as

\[
\Psi_{JM_{J}}^{(4B)}(x, y, R) = \psi_{kR}^{(+)} \otimes \phi_{JM_{J}}^{(3B)}(x, y),
\]

where \( \psi_{kR}^{(+)} = e^{iK_{R}} \omega(x, y, b_{R}) = e^{iK_{R}} e^{-\frac{i}{\hbar} \int Udz} \). But we see that the modulating function \( \omega \) does not depend on the azimuthal angle \( \phi \) of \( R \).
Thus we make the wave function explicitly dependent on the angle by keeping a consistency for the phase of different wave functions differing by a rotation. Putting a rotation factor $e^{-iφ_R J_z}$ and a traditional eikonal wave function together yields a coherent wave function [58]. Here the $J_z$ is the $z$ component of the total angular momentum of the projectile. The rotation operator acts on the projectile wave function as:

$$e^{iφ_R J_z} \psi_{kR}^{(+)} = e^{iφ_R M_J} \psi_{kR}^{(+)} , \quad (6.1.29)$$

Here $M_J = M_{L_k} + M_S$ is the projection of the total angular momentum.

Taking this into account, we express the transition matrix element as

$$T_{\bar{n}}(k', k) = \left( \frac{A_{\text{proj}}}{A_c} \right)^{\frac{3}{4}} i\hbar v \sum_{J'M,J'k, M'k, M'_{L_k}, M'_{S_k}} \langle L'_k | S'_k | M'_{L'_k} M'_{S'_k} | J'M_J \rangle \gamma_{M'_{L'_k} M'_{S'_k}}^{L_k S_k} (Ω_{5k})$$

$$\times \sum_{\lambda \lambda' M_\lambda} (-1)^{J'-M_J} \left( \begin{array}{ccc} J' & λ & J \\ -M_J & -M_J & M_J \end{array} \right) \times \int d\mathbf{R}_b e^{i\mathbf{q}\cdot\mathbf{R}_b} \sum_{ξ' ξ λ_ξ λ_μ} \langle ξ' J'|||Γ_λ M_\lambda||ξ J⟩ \tilde{Γ}_{ξ' ξ}^{λ M_\lambda λ_ξ λ_μ}(R_b)e^{-iφ_R (M_{JP} - M_J)} . \quad (6.1.30)$$

Then we integrate it with regard to $φ_R$:

$$T_{\bar{n}} = \left( \frac{A_{\text{proj}}}{A_c} \right)^{\frac{3}{4}} i\hbar v \sum_{J'M,J'k, M'k, M'_{L_k}, M'_{S_k}} \langle L'_k | S'_k | M'_{L'_k} M'_{S'_k} | J'M_J \rangle \gamma_{M'_{L'_k} M'_{S'_k}}^{L_k S_k} (Ω_{5k})$$

$$\times \sum_{\lambda \lambda' M_\lambda} (-1)^{J'-M_J} \left( \begin{array}{ccc} J' & λ & J \\ -M_J & -M_J & M_J \end{array} \right) \times 2π i^{M_\lambda} \int R_b dR_b J_{|M_\lambda|}(qR_b) \sum_{ξ' ξ λ_ξ λ_μ} \langle ξ' J'|||Γ_λ M_\lambda||ξ J⟩ \tilde{Γ}_{ξ' ξ}^{λ M_\lambda λ_ξ λ_μ} , \quad (6.1.31)$$

where $M_\lambda = M_{JP} - M_J$. Here we used the formula:

$$\int_0^{2π} dφ e^{iφ_R cos φ} e^{iNφ} = 2π i^N J_N(qR_b) , \quad (6.1.32)$$

for $N = 0, 1, 2, \ldots$ Now for convenience, expressing the integration-related terms as

$$U_{\lambda M_\lambda}(q) = 2π i^{M_\lambda} \int R_b dR_b J_{|M_\lambda|}(qR_b) \sum_{ξ' ξ λ_ξ λ_μ} \langle ξ' J'|||Γ_λ M_\lambda||ξ J⟩ \tilde{Γ}_{ξ' ξ}^{λ M_\lambda λ_ξ λ_μ} , \quad (6.1.33)$$
we have the following expression for the T-matrix element:

\[
T_{J'M'k'}^{S'M'_k} = \left( \frac{A_{\text{proj}}}{A_c} \right)^3 \frac{i \hbar v}{M_{J'M'k'}M_{S'M'_k}} \sum_{M_{J'M'k'}M_{S'M'_k}} \langle L_k' S_k'M_{L_k'}M_{S_k'} | J'M_{J'} \rangle \gamma^{L_k' L_k}_{M_{L_k}' M_{L_k}}(\Omega_{5k}) \\
\times \sum_{\lambda M_{\lambda}} (-1)^{J' - M_{J'}} \left( \begin{array}{ccc}
J' & \lambda & J \\
-M_{J'} & M_{\lambda} & M_j
\end{array} \right) U_{JM_{J} M_{J'}}^{J'M'}(q) .
\]  

(6.1.34)

### 6.1.4 Differential cross sections for a breakup reaction

In a previous subsection, using the T matrix approach, we derived the expression for the 5-folded differential breakup cross section. We now show the derivation of the exclusive cross section. The cross section is expressed as:

\[
d five \sigma_{kx} d \Omega'_{kx} d \Omega_{kx} d \Omega_{kx} d \Omega_{kx} d \Omega_{kx} = \frac{(2\pi)^4}{\hbar^2} \rho \sqrt{E_y(E_{ex} - E_y)} (2J + 1) \sum_{J'S'_k M_{S'_k}} \left| T_{J'S'_k M_{S'_k}}^{S'M'_k} \right|^2 ,
\]

(6.1.35)

where \( J \) and \( J' \) are the initial and final total angular momentum, respectively. Here \( \alpha \) is the quantum number for the internal degrees of freedom of the core of a Borromean nucleus. The reduced phase space factor is

\[
\rho = \frac{2\mu_R p_R}{\hbar^3} \left( \frac{\mu}{\mu_R} \right)^{3/2} = \frac{2\mu_R p_R}{\hbar^3} \cdot \frac{m_n^2}{\hbar^6} \left( \frac{A_c}{A_{\text{proj}}} \right)^{3/2} ,
\]

(6.1.36)

and the factor \( \left( \frac{A_c}{A_{\text{proj}}} \right)^{3/2} \) is cancelled out by \( \left( \frac{A_{\text{proj}}}{A_c} \right)^{3/2} \) which comes from the normalisation of a bound state wave function.

Then we move on to the formula for the cross section. For more detail of the derivation, see Appendix D. Then, we see that the differential cross section is expressed by

\[
d two \sigma_{kx} d \Omega_{kx} d \Omega_{kx} d \Omega_{kx} d \Omega_{kx} d \Omega_{kx} = (2\pi)^4 \left( \frac{m_n^2}{\hbar^4} \right)^{3/2} \frac{k_{RR}}{2(2J + 1)} \sum_{J'S'_k M_{S'_k}} 2E_{ex}^2 \left| U_{JM_{J} M_{J'}}^{J'M'} \right|^2 .
\]

(6.1.37)

Then we obtain the inclusive cross section; we use the following formula

\[
\int_0^\infty J_\mu(a_0 t) J_\mu(a_1 t) \, dt = \frac{1}{a_0} \delta(a_0 - a_1) , \quad 0 < a_0 < a_1 ,
\]

(6.1.38)
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in order to calculate $\int d\Omega_{kR}|U_{\lambda M,\lambda}^{J J}|^2$. Then using the relation $qdq = k^2 R \sin \theta d\theta$, we obtain:

$$\int d\Omega_{kR}|U_{\lambda M,\lambda}^{J J}|^2 = \frac{2\pi}{k_R} \int_0^\infty R_b dR_b \int_0^\infty R'_b dR'_b (2\pi)^2 \int_0^\infty J_{|\lambda M|}(R_b q) J_{|\lambda M|}(R'_b q) qdq \frac{1}{k_R} \Lambda_{\lambda M,\lambda}^{J J}$$

$$= 2\pi \int_0^\infty R_b dR_b (2\pi)^2 \frac{1}{k_R} \Lambda_{\lambda M,\lambda}^{J J}$$

where

$$\Lambda_{\lambda M,\lambda}^{J J}(R_b) = \sum_{\xi,\xi'} \langle \xi', J'||\Gamma_{\lambda M,\lambda}\xi\rangle \Delta_{\xi,\xi'} \lambda M, \lambda M \rangle \langle \xi, J'| \Lambda_{\lambda M,\lambda}^{J J} \rangle.$$

(6.1.39)

Then we use the approximation $k_R \approx k'_R$. As a consequence, we obtain the inclusive cross section:

$$\frac{d\sigma}{dE_{ex}} = (2\pi)^6 \left( \frac{m_n}{\hbar^2} \right)^{\frac{3}{2}} \frac{(2\pi)^2}{(2J + 1)} \sum_{J', \xi_{\lambda M,\lambda}} \frac{1}{2\lambda + 1} \frac{2}{E_{ex}^2} \int_0^\infty R_b dR_b \Lambda_{\lambda M,\lambda}^{J J}.$$

(6.1.41)

Note that the factor $(2\pi)^6$ in the RHS of (6.1.41) is absorbed and disappear if we use the conventional continuum wave function $\frac{1}{(2\pi)^3} \phi_{k_x, k_y}(x, y)$ [33].

6.2 Results of the calculations

In this section we present the numerical results obtained with the formalism described in the previous section.

6.2.1 $^6$He breakup

Here we consider $^6$He + $^{208}$Pb elastic breakup at 70 (MeV/A). Here we use the WS central potential for $\alpha - ^{208}$Pb potential; $V_0 = 79.5$ and $W_0 = 36.5$ (MeV) for the depths, $a_r = 0.893$ and $a_i = 0.846$ (fm) for the diffusenesses, and $R_r = 7.51$ and $R_i = 7.59$ (fm) for the WS reduced radii [96]. For $n - ^{208}$Pb potential, we use the same sort of potential whose parametrisation is: $V_0 = 28.0$ and $W_0 = 6.49$ (MeV) for the depths, $a = 0.647$ (fm) for the both real and imaginary diffusenesses, and $R = 7.37$ (fm) for the both real and imaginary WS reduced radii [97].
We note that, concerning the cross section for $J' = 0^+$ continuum state, the T matrix elements and cross sections are not convergent as long as we use a plane wave as the continuum state of a Borromean nucleus. Therefore it may be necessary to use the smoothing factor for the S-matrix elements of $0^+$. In terms of $1^-$ and $2^+$, their T matrix elements can be avoided diverging, because the angular part of the elements vanishes as the impact parameter increases. (By contrast, for $0^+$, it inevitably involves the integration $\int Y^*_{00}Y_{00}d\Omega$ which cannot vanish.)

![Graphs showing differential breakup cross sections for $^6\text{He} + ^{208}\text{Pb}$ at 70 (MeV/A).](image)

Figure 6.1: The differential breakup cross sections for $^6\text{He} + ^{208}\text{Pb}$ at 70 (MeV/A). The upper left graph is for $0^+$, the upper right for $1^-$, and the lower graph for $2^+$. The cross sections are evaluated at $E = 1.0$ (MeV). For our calculation, od14 is used.

Then we compare our calculations to the calculation of early research which was done by Baye [62]. Figure 6.1 presents the differential breakup cross sections, as a function of the scattering angle, at $E = 1.0$ (MeV). Our calculations involve the PP method in trimming the forbidden states of the Borromean system, while they adopted the supersymmetric (SUSY) approach to do this [62]. Also, for $0^+$, our calculation involves the smoothing factor, which may enlarge the difference between our calculation and theirs. For $1^-$, the trends of the angular distributions of their calculations are similar to those of our calculations, but their calculations provide larger cross sections. For $2^+$, the value of their cross sections are relatively closer to
that of ours, but the shapes of the two curves differ.

Also, for the purpose of seeing how two different bound state wave functions provide different results, we used two different bound state wave functions: od14 and od15 whose 2n-separation energies are $S_{2n} = 0.976$ and $S_{2n} = 0.966$ (MeV), respectively. Our finding was that the resultant cross sections seemed not so much to be sensitive to this difference.

Then we move on to numerical calculation with regard to nuclear breakup. Here we consider $^6$He + $^{12}$C at 240 (MeV/A).

Figure 6.2: The energy distribution of breakup cross section for $^6$He + $^{12}$C at 240 (MeV/A). The upper left graph is for $0^+$, the upper right for $1^-$, and the lower graph for $2^+$. For our calculation, od14 is used for the bound state wave function.

In terms of the core-target potential we use a WS central optical potential for $^{12}$C + $^{12}$C at 200 (MeV/A) whose reduced radial parameter is adjusted for the alpha particle [98, 99]. Specifically, we find that for the real part of the WS potential the depth is 18.5 (MeV), the reduced radius is $1.0 \times \frac{4^{1/3}+12^{1/3}}{12^{1/3}+12^{1/3}} = 0.8467$ (fm) and the diffuseness is 0.75 (fm). Likewise for the imaginary part, the depth, the reduced radius and the diffuseness are 100.7 (MeV), 0.5097 (fm) and 0.819 (fm), respectively.

As to the neutron-target potential, we use a global nucleon-nucleus WS optical model potential (OMP) [97]. For the real part, the depth is 18.920 (MeV), the reduced radius is 2.5797 (fm) and the diffuseness is 0.67602 (fm). For the imaginary part, we
see that the depth, the reduced radius and the diffuseness are, 11.391 (MeV), 2.5797 (fm) and 0.67602 (fm), respectively.

Our finding was that at this energy, the difference in the cross section between different WKB orders was negligible. It is plausible that at this incident energy the eikonal approximation is a good approximation.

Then we compare our calculation to CDCC calculations [99], which are shown in Figure 6.2. As can be seen from the graph, for $0^+$, in a low-lying region the CDCC gives a larger cross section than our calculation, but at higher energies our cross section becomes larger than theirs. For both $1^+$ and $2^+$, the cross section for CDCC’s is larger than that for our model. For $1^-$, both the CDCC and our model have peaks in the low-lying region, but the peak value of the CDCC calculation is larger than that of our model. Also, from the graph, it is apparent that the CDCC calculation for $2^+$ yield a sharp peak at around 1 (MeV) which our model does not produce. It can be argued that a plane wave as a continuum state does not provide the resonance.

Also, Figure 6.3 compares our calculation to the experimental data, where the CDCC calculation is shown together. From the graph, we can see that although our model does not produce the $2^+$ resonance, our model is in line with the experimental data at higher energies; the CDCC calculation underestimates the cross section at higher energies. We do not have an intuitive answer to the question why our model

Figure 6.3: Breakup cross sections for $^6\text{He} + ^{12}\text{C}$ at 240 (MeV/A). The cross sections are totals of those for $0^+$, $1^-$ and $2^+$. For our calculation, the bound state wave function of od14 is used. The experimental data are from [86].

which uses plane waves as continuum states of this nucleus is closer to the experimental data at higher energies. In the next chapter, we see how the Watson-Migdal final
state interaction affect the cross section. It can be argued that if the introduction of the final state interaction to our model makes things worse, a plane wave is more suitable for describing the reality as far as higher energies are concerned.

### 6.2.2 $^{22}$C breakup

Now we move on to breakup of $^{22}$C; specifically we consider $^{22}$C+$^{12}$C. In Chapter 4, in relation to the core-target interaction, we compared the Kucuk-Tostevin (KT) density of $^{20}$C to the Gaussian density. In fact, we tested both densities to calculate the cross sections, and our finding was that this difference is not significant in this case. The breakup cross section for this reaction is not sensitive to the folding density used to produce the core-target potential.

Firstly, we calculate exclusive breakup cross sections for $^{22}$C+$^{12}$C at 100 (MeV/A). Figure 6.4 compares the angular distributions of the cross sections that involve a few different bound state wave functions. The differential cross sections are evaluated at $E = 1.0$ (MeV). As can be seen from the graphs in Figure 6.4, in general, the cross section becomes larger as the 2n-separation energy of $^{22}$C becomes larger. We also find that the shapes of the curves of these cross sections are similar.

In terms of their inclusive cross sections, we show them in Figure 6.5, where we also consider how the higher order terms of the WKB expansion have an impact on the cross sections. The graphs in Figure 6.5 tell us that for both $0^+$ and $1^-$, there is no significant difference in the cross section between the different WKB orders, while for $2^+$, the higher order terms slightly have an effect on the cross section. As to our result that the non-eikonal effect is more sensitive to the $2^+$ than to the $0^+$ and $1^-$, it can be argued that for the nuclear breakup reaction the non-eikonal effect can be more important as the transfer of angular momentum becomes larger. As a whole, for this reaction the effects of the higher order terms are not significant.

Also, from these graphs, we see that the cross sections for $0^+$ are much larger than those for $1^-$ and $2^+$. We do not have a clear explanation why the cross sections for $0^+$ are much larger, but it might be possible that in effect the $0^+$ is dominant in the continuum state of this nucleus. Also, in these graphs we see a relatively sharp
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Figure 6.4: The angular distributions of the breakup cross sections for $^{22}$C + $^{12}$C at 100 (MeV/A). The upper left graph is for $0^+$, the upper right for $1^-$, and the lower graph for $2^+$. The cross sections are evaluated at $E = 1.0$ (MeV). ic1, ic3 and ic6 are used for the bound state wave functions.

Then we see how the difference of bound state wave functions affects the inclusive cross section. Figure 6.6 compares the cross sections which involve a few different bound state wave functions. The 2n-separation energies of ic1, ic3 and ic6 are 0.214, 0.316 and 0.456 (MeV), respectively. Accordingly, it seems that in general, the peak position is shifted to lower energies (and the strength of the peak becomes stronger) as the 2n-separation energy decreases. In particular, for $1^-$ this result is consistent with our finding for E1 transition of $^{22}$C.

Next we see how differences in the projectile-target interaction affects the cross section. Figure 6.7 shows the angular distributions of the cross sections at 300 (MeV/A)$^4$ involving two different parametrisations RI2 and HE1. These parameters were shown

\footnote{As to HE1 at 300 (MeV/A), we substitute HE1 of 305 (MeV/A) for the parametrisation as the two energies are close to each other.}
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Figure 6.5: Breakup cross sections as a function of energy for $^{22}$C + $^{12}$C at 100 (MeV/A). The upper left graph is for $0^+$ continuum state, the upper right for $1^-$, and the lower graph for $2^+$, respectively. ic6 is used for the bound state wave function.

in Chapter 4). The cross sections are evaluated at $E_{ex} = 1$ (MeV). As can be seen from the graphs of Figure 6.7, in general RI2 gives larger cross sections in a higher angle region. It can be argued that because the depth of the real part of the potential of RI2 is deeper than that of HE1.
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Figure 6.6: The cross sections for $^{22}\text{C} + ^{12}\text{C}$ at 100 (MeV/A), where the upper left graph is for $0^+$, the upper right for $1^-$, and the lower graph for $2^+$. For the bound state wave functions, ic1, ic3 and ic6 are used.

Figure 6.7: Differential breakup cross sections plotted as a function of the scattering angle for $^{22}\text{C} + ^{12}\text{C}$ at 300 (MeV/A). The cross sections are evaluated at $E_{\text{ex}} = 1$ (MeV). The upper left graph is for $0^+$, the upper right for $1^-$, and the lower graph for $2^+$. The calculations with RI2 are compared to those with HE1. ic6 is used for the bound state wave function.
7.1 The Watson-Migdal approach

In a nuclear reaction, a final state interaction (FSI) can take place between two or more of the fragments produced during the reaction and can often contribute to observables, such as cross sections \([100]\). FSIs are distinguished from the primary mechanisms, which are the interactions that produce the particular reaction outcomes, and take place when the interacting particles are in their post-collision state (or final state) and spatially within the interacting region \([100]\).

Originally, FSIs were studied to describe pi meson production following nucleon-nucleon collisions at energies near the meson production threshold, where final state nuclear forces significantly affect the spectrum of the produced mesons \([100]\). In such a case, the relative velocity of two nucleons is slow in their final state \([101]\).

The FSI model was applied to neutron-neutron (nn) FSIs, and later extended to proton-proton (pp) FSIs. It was later applied to nucleon-deuteron charge exchange reactions where it led to a significant enhancement in the cross sections. In \((d, nn)\) charge exchange, FSIs produced a strong and sharp peak at around 0.2 (MeV) with a width of 0.4 (MeV), while in \((d, pp)\) they led to a weaker and broader energy spectrum with a peak at around 0.6 (MeV) and a width of about 1.2 (MeV) \([102]\).

Our approach in this work is to follow the method of Watson, where the FSI was derived from the dispersion relation \([94]\). To gain the insight of the FSI, we firstly...
consider a simple 2-body scattering problem:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R_{nl}(r)}{\partial r} \right) + \left( \kappa^2 - \frac{l(l+1)}{r^2} - \frac{2\mu v(r)}{\hbar^2} \right) R_{nl}(r) = 0 , \tag{7.1.1} \]

where \( v(r) \) is the interaction between the two particles, and \( \kappa \) is the wave number. Setting \( R_{nl} = u_{\kappa l} r \), we see as usual:

\[ \frac{\partial^2 u_{\kappa l}(r)}{\partial r^2} + \left( \kappa^2 - \frac{l(l+1)}{r^2} - \frac{2\mu v(r)}{\hbar^2} \right) u_{\kappa l}(r) = 0 . \tag{7.1.2} \]

We assume that the FSI is non-negligible only for s-waves, and set the boundary conditions as

\[ u_{s0}(0) = 0 , \quad \frac{\partial u_{s0}(0)}{\partial r} = c_0 , \tag{7.1.3} \]

where \( c_0 \) is a constant. We focus on the equation for the s-wave:

\[ \frac{\partial^2 u_{s0}(r)}{\partial r^2} + \left( \kappa^2 - \frac{2\mu v(r)}{\hbar^2} \right) u_{s0}(r) = 0 , \tag{7.1.4} \]

where the solutions are expressed by the Jost function \( f(\kappa, r) \). Its boundary condition is

\[ \lim_{r \to +\infty} e^{i\kappa r} f(\kappa, r) = 1 . \tag{7.1.5} \]

Note that \( f(\kappa, r) \) and \( f(-\kappa, r) \) are independent solutions of (7.1.4); indeed the Wronskian of these is non-zero:

\[ W[f(\kappa, r), f(-\kappa, r)] = f(\kappa, r) \frac{\partial f(-\kappa, r)}{\partial r} - f(-\kappa, r) \frac{\partial f(\kappa, r)}{\partial r} = 2i\kappa . \tag{7.1.6} \]

Also, if \( \kappa \) is real, the differential equation (7.1.4) is real, and therefore we see from the boundary condition (7.1.5) that [94]:

\[ f^*(-\kappa, r) = f(\kappa, r) . \tag{7.1.7} \]

Secondly we use the notation:

\[ f(\kappa) \equiv f(\kappa, 0) , \tag{7.1.8} \]

and it follows from (7.1.7) that

\[ |f(-\kappa)| = |f(\kappa)| . \tag{7.1.9} \]
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Using \( f(\kappa) \) and the boundary conditions of \( u_{\kappa 0} \), we can rewrite \( u_{\kappa 0}(r) \) as

\[
\frac{1}{c_0} u_{\kappa 0}(r) = \frac{1}{2i\kappa} \left\{ f(\kappa)f(-\kappa, r) - f(-\kappa)f(\kappa, r) \right\} . \tag{7.1.10}
\]

Also, using this and (7.1.6) we express \( f(\kappa) \) as

\[
f(\kappa) = W \left[ f(\kappa, r), \frac{1}{c_0} u_{\kappa 0}(r) \right] . \tag{7.1.11}
\]

By using this expression, we consider the asymptotic form of \(-f(-\kappa)e^{-i\kappa r} + f(\kappa)e^{i\kappa r}\) as \( r \to +\infty \): after a simple calculation, we see that

\[
\frac{1}{c_0} u_{\kappa 0}(r \to +\infty) \to \frac{f(-\kappa)}{2i\kappa} \left[ -e^{-i\kappa r} + \frac{f(\kappa)}{f(-\kappa)} e^{i\kappa r} \right] . \tag{7.1.12}
\]

Accordingly, we express the S-matrix in terms of \( f(\kappa) \):

\[
S_0(\kappa) = e^{2i\delta_0(\kappa)} = \frac{f(\kappa)}{f(-\kappa)} , \tag{7.1.13}
\]

and so we have \( S_0(-\kappa) = S_0^{-1}(\kappa) \). Therefore we have \( e^{2i\delta_0(-\kappa)} = e^{-2i\delta_0(\kappa)} \), that is

\[
\delta_0(-\kappa) = -\delta_0(\kappa) . \tag{7.1.14}
\]

Writing \( f(\kappa) = |f(\kappa)|e^{i\text{Arg}(f(\kappa))} \), and using (7.1.13) and (7.1.7), we see:

\[
e^{2i\delta_0} = S_0(\kappa) = \frac{f(\kappa)}{f(-\kappa)} = \frac{f(\kappa)}{f(\kappa)^*} = \frac{|f(\kappa)|e^{i\text{Arg}(f(\kappa))}}{|f(\kappa)|e^{-i\text{Arg}(f(\kappa))}} = e^{2i\text{Arg}(f(\kappa))} , \tag{7.1.15}
\]

and therefore we find that \( \text{Arg}(f(\kappa)) = \delta_0(\kappa) \). Thus we have

\[
f(\kappa) = |f(\kappa)|e^{i\delta_0} . \tag{7.1.16}
\]

Using (7.1.12), (7.1.16) and the fact that the asymptotic form of \( u_{\kappa 0} \) is \( \sqrt{\frac{2}{\pi}} \sin (\kappa r + \delta_0) \), we see

\[
c_0 = \left( \sqrt{\frac{\pi}{2}} \frac{|f(-\kappa)|}{\kappa} \right)^{-1} . \tag{7.1.17}
\]

Now we use the effective range approximation to obtain a specific form of the function \( f(\kappa) \):

\[
\kappa \cot \delta_0 = \frac{1}{a} + \frac{r_e}{2\kappa^2} , \tag{7.1.18}
\]

where the parameters \( a \) and \( r_e \) are the scattering length and the effective range, respectively. Then we have:

\[
S_0(\kappa) = e^{2i\delta_0} = \frac{i + \cot \delta_0}{i - \cot \delta_0} = \frac{\frac{1}{a} + \frac{r_e \kappa^2}{2} + i\kappa}{\frac{1}{a} + \frac{r_e \kappa^2}{2} - i\kappa} . \tag{7.1.19}
\]
We express the S-matrix as

\[ S_0(\kappa) = \frac{1}{a} + \frac{r_e\kappa^2}{2} + i\kappa = \frac{(\kappa + iA)(\kappa - iB)}{(\kappa - iA)(\kappa + iB)}, \quad (7.1.20) \]

where \( A \) and \( B \) are given as

\[ \frac{1}{A - B} = \frac{r_e}{2}, \quad \frac{r_eAB}{2} = \frac{1}{a}. \quad (7.1.21) \]

Then we use \( S_0 = \frac{f(\kappa)}{f(-\kappa)} \) and (7.1.7) to obtain \( f(\kappa) = \frac{\kappa + iB}{\kappa - iA} \), and we write this with the scattering length and the effective range:

\[ f(\kappa) = \frac{\kappa - iB}{\kappa - iA} = \frac{\kappa^2 + B^2}{\frac{r_e^2}{2}} \frac{1}{a} + \frac{r_e\kappa^2}{2} - i\kappa. \quad (7.1.22) \]

From (7.1.21), since the effective range is positive and the scattering length is negative, we have \( A > 0 \) and \( B < 0 \). Then we have two values for \( B \):

\[ B = -\frac{1}{r_e} \pm \sqrt{\frac{1}{r_e^2} + \frac{2}{ar_e}}, \] and we choose the larger value for it so that in a very low-lying region, our FSI model becomes closer to Watson’s original FSI model\(^1\).

Thus we see that

\[ f(\kappa) = \frac{\kappa - iB}{\kappa - iA} = \frac{\kappa^2 + B^2}{\frac{r_e^2}{2}} \frac{1}{a} + \frac{r_e\kappa^2}{2} - i\kappa. \quad (7.1.23) \]

We now define \( \mathcal{F} \) as

\[ \mathcal{F} \equiv \left[ \begin{array}{c} u_{\kappa 0} \\ \sqrt{\frac{2}{\pi}} \sin (\kappa r) \end{array} \right]_{r=0} = \frac{c_0}{\kappa \sqrt{\frac{2}{\pi}}} = \frac{1}{f(-\kappa)}, \quad (7.1.24) \]

for s-waves, while \( \mathcal{F} \) is unity for non-s-waves (as we restrict ourselves to the s-wave FSI). \( |\mathcal{F}|^2 \) is called the enhancement factor [94]. It follows from (7.1.24) that the enhancement factor is the ratio of the probability of finding the interacting two particles at the zero distance \( |u_{\kappa 0}|^2 \), to that for no interaction between the particles \( \sqrt{\frac{2}{\pi}} \sin (\kappa r)|^2 \) [94]. Thus our approach to incorporating the FSI is to include the enhancement factor within a T-matrix element.

With regard to the application of FSI to Borromean nuclei, the model was used for calculating the differential Coulomb breakup cross section of \(^{11}\)Li, where it was found that the nn FSI was an important factor in reproducing the nn relative energy spectrum for the cross section [103].

\(^1\)Strictly speaking, the FSI model which is derived from the dispersion theory is different to Watson’s original FSI model which was proposed in 1952.
7.2. WATSON-MIGDAL FSI FOR E1 STRENGTH DISTRIBUTION

We expect that the FSIs affect reactions involving $^6\text{He}$ and $^{22}\text{C}$; as for $^{22}\text{C}$, how the FSIs affect the cross sections has not so far been investigated. In the rest of this chapter, therefore, we develop the Watson-Migdal FSI within the 4-body Glauber-WKB framework, in the subsequent sections. We also calculate the E1 transition strength and breakup cross sections with this FSI.

7.2 Watson-Migdal FSI for E1 strength distribution

The SDR of a Borromean nucleus is interpreted as an oscillation-like motion with a low frequency between its core and the valence neutrons [12, 76]. To seek for the possibility of the existence of the SDR for $^6\text{He}$ and $^{22}\text{C}$, we introduce an FSI for a Borromean nucleus. Here from the view of the efficiency of numerical calculation, we focus on an FSI which appears between the core and a (post-collision) dineutron-like object that, if exists, is thought to be formed during a short-time period. We also assume that the FSI is a Watson-Migdal-type interaction.

Therefore in this section, we formulate the E1 transition strength within the framework of the Watson-Migdal model and see how the strength distribution is affected by FSIs. To incorporate the Watson-Migdal type FSI features into our model, we now modify our basic formulation for E1 transition strength (5.1.22). For the sake of convenience, we begin with the expression (5.1.19):

\[
\frac{dB}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^3 \sum_{M_LM_J} \frac{\mu_x \sqrt{2} \mu_x \mu_y \sqrt{2} \mu_y}{\hbar^3} \int \sqrt{(E_{ex} - E_{y})E_{y}d\Omega_k dE_y d\Omega_{k_y}} \times \sum_{K,L'J'LM_J'Jx} K^{L',J'L,Jx}_{L',L,Jx} \frac{S_{L'}(\Omega_{k_y})S_{L}(\Omega_{k_y})M_{\mu}^L M_{\mu}^{L}}{K^{L',J'L,Jx}_{L',L,Jx}} \cdot (7.2.1)
\]

Note that in this section, again for the sake of convenience, we introduce a new superscript $C, C''$ which is defined by

\[
C = \{ J', M_{J'}, L', M_{L'}, S', L'_y, L'_x, J, M_J \} \quad (7.2.2)
\]
\[
C'' = \{ J', M_{J'}, L'', M_{L''}, S', L''_y, L''_x, J, M_J \}, \quad (7.2.3)
\]

where $C$ and $C''$ are subsets of $\Xi$ and $\Xi''$ defined in (5.1.10) and (5.1.20), but excluding the hypermoment labels $K'$ and $K''$. In this chapter, we use $C$ and $C''$ instead of $\Xi$. 

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and $\Xi''$ because we do not integrate with regard to $\cos (2\alpha_k)$ and do not sum over $K''$ until we introduce the Watson-Migdal FSI.

Accordingly we rewrite (5.1.19) as

$$\frac{dB}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^{\frac{3}{2}} \sum_{M'\mu, M_\mu} \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{h^3} \int_{-1}^{1} \sqrt{(E_{ex} - E_y) E_y} d\Omega_{k_x} dE_y d\Omega_{k_y}$$

$$\times \sum_{K''L''M''} \mathcal{M}_{K''L''M''} \mathcal{M}_{K''L''M''}^{K'''} \mathcal{M}_{K''L''M''}^{K''C'} \mathcal{M}_{K''L''M''}^{K''C'} , \quad (7.2.4)$$

where

$$R_{L''L''S'''J''}^{K''L''S'''J''} = (-1)^{(L''-S'+M'+L'-S'+M')} (2J' + 1)$$

$$\times \left( \begin{array}{ccc} L'' & S' & J' \\ M_{L''} & M_{S'} & -M_{J'} \end{array} \right) \left( \begin{array}{ccc} L' & S' & J' \\ M_{L'} & M_{S'} & -M_{J'} \end{array} \right). \quad (7.2.5)$$

Then, by integrating with regard to $\Omega_{k_x}$, $\Omega_{k_y}$ and summing over the indices of spherical harmonics, we have

$$\frac{dB}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^{\frac{3}{2}} \sum_{M'\mu, M_\mu} \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{h^3} \int_{-1}^{1} \sqrt{(E_{ex} - E_y) E_y} E_{ex} d(\cos 2\alpha_k)$$

$$\times \sum_{K''L''S'''J''} \mathcal{M}_{K''L''S'''J''}^{K''C'} \delta_{L''L'} \delta_{M_{L''}M_{L'}} \delta_{L''L''} \delta_{L''L''} \mathcal{M}_{K''L''S'''J''}^{K''C'}$$

$$\times J_{L_{ax}, L_{ay}}^{L''L''} (\cos 2\alpha_k) \mathcal{M}_{K''C'}^{K''C'} (\sin \alpha_k)^{2L_x} (\cos \alpha_k)^{2L_y} , \quad (7.2.6)$$

where $J_{L_{ax}, L_{ay}}^{L''L''} (\cos 2\alpha_k)$ is defined by

$$J_{L_{ax}, L_{ay}}^{L''L''} (\cos 2\alpha_k) = N_{L_{ax}}^{L''L''} \frac{1}{2} L_{ay}^{L''L''} \frac{1}{2} N_{L_{ay}}^{L''L''} \frac{1}{2} L_{ax}^{L''L''} \frac{1}{2} P_{L_{ax}}^{L''L''} \frac{1}{2} L_{ay}^{L''L''} \frac{1}{2} P_{L_{ay}}^{L''L''} \frac{1}{2} (\cos 2\alpha_k) \left( \begin{array}{c} \frac{1}{2} \cos \alpha_k \end{array} \right) . \quad (7.2.7)$$

Then, we carry out the summation over the $L''$, $M_{L''}$, $L''$ and $L''$. The Kronecker deltas in (7.2.6) enables us to have a slightly simpler form:

$$\frac{dB}{dE_{ex}} = \frac{1}{2J + 1} \left( \frac{A_{proj}}{A_c} \right)^{\frac{3}{2}} \sum_{M'\mu, M_\mu} \frac{\mu_x \sqrt{2\mu_x} \mu_y \sqrt{2\mu_y}}{h^3} \int_{-1}^{1} \sqrt{(E_{ex} - E_y) E_y} E_{ex} d(\cos 2\alpha_k)$$

$$\times \sum_{K'K''L''S'''J''} R_{L''L''S'''J''}^{K''L''S'''J''} (\cos 2\alpha_k) \mathcal{M}_{K''C'}^{K''C'} (\sin \alpha_k)^{2L_x} (\cos \alpha_k)^{2L_y} . \quad (7.2.8)$$

From (5.1.5), we have the relation $E_y' = E_{ex}(\cos \alpha_k)^2$. We therefore express $E_y'$ in terms of $E_{ex}$ and $\alpha_k$. Since $\alpha_k$ ranges from 0 to $\frac{\pi}{2}$, both $\sin \alpha_k$ and $\cos \alpha_k$ are positive.
Accordingly, \( \sqrt{(E_{\text{ex}} - E'_y)}E'_y \) becomes \( E_{\text{ex}} \sin \alpha_k \cos \alpha_k \), and immediately we have

\[
\frac{d \mathcal{B}}{dE_{\text{ex}}} = \frac{1}{2J + 1} \left( \frac{A_{\text{proj}}}{A_c} \right)^1 \sum_{M_{1\mu} M_{1\mu}} \frac{\mu_x \sqrt{2} \mu_x \mu_y \sqrt{2} \mu_y}{\hbar^3} \times \frac{E_{\text{ex}}^2}{2} \int_{-1}^1 d(\cos 2\alpha_k) \sum_{K'' K'''} \left( \sin \alpha_k \right)^{2L_z + 1} \left( \cos \alpha_k \right)^{2L_y + 1} \times R_{L/L' S'/J'}^{M_{1\mu} M_{1\mu}'} n_L' n_{L''} (\cos 2\alpha_k) \mathcal{M}_{\mu}^{K'' C} \mathcal{M}_{\mu}^{K' C} .
\]

(7.2.9)

Now we introduce the Watson-Migdal FSI. Thus we multiply (7.2.9) by the enhancement factor \( |\mathcal{F}|^2 \), which yields

\[
\frac{d \mathcal{B}}{dE_{\text{ex}}} = \frac{1}{2J + 1} \left( \frac{A_{\text{proj}}}{A_c} \right)^1 \sum_{M_{1\mu} M_{1\mu}} \frac{\mu_x \sqrt{2} \mu_x \mu_y \sqrt{2} \mu_y}{\hbar^3} \times \frac{E_{\text{ex}}^2}{2} \int_{-1}^1 d(\cos 2\alpha_k) \sum_{K'' K'''} \left| \mathcal{F}(\cos 2\alpha_k) \right|^2 \times (\sin \alpha_k)^{2L_z + 1} (\cos \alpha_k)^{2L_y + 1} R_{L/L' S'/J'}^{M_{1\mu} M_{1\mu}'} n_L' n_{L''} (\cos 2\alpha_k) \mathcal{M}_{\mu}^{K'' C} \mathcal{M}_{\mu}^{K' C} ,
\]

(7.2.10)

where the square root of the enhancement factor is written as:

\[
\mathcal{F}(\cos 2\alpha_k) = \frac{r_{2\alpha}}{2} \left\{ k_y^2 + \left( \frac{-1}{r_{c}} + \sqrt{\frac{1}{r_{c}^2} + \frac{2}{ar_{c}}} \right)^2 \right\} .
\]

(7.2.11)

Then we use \( E'_y = \frac{\hbar^2 k_y^2}{2\mu_y} = E_{\text{ex}} (\cos \alpha_k)^2 = \frac{\mu_y E_{\text{ex}} (1 + \cos 2\alpha_k)}{\hbar^2} \) to obtain:

\[
\mathcal{F}(\cos 2\alpha_k) = \frac{r_{2\alpha}}{2} \left\{ \frac{\mu_y E_{\text{ex}} (1 + \cos 2\alpha_k)}{\hbar^2} + \left( \frac{-1}{r_{c}} + \sqrt{\frac{1}{r_{c}^2} + \frac{2}{ar_{c}}} \right)^2 \right\} .
\]

(7.2.12)

Then we proceed to express (7.2.10) as follows:

\[
\frac{d \mathcal{B}}{dE_{\text{ex}}} = \frac{1}{2J + 1} \left( \frac{A_{\text{proj}}}{A_c} \right)^1 \sum_{M_{1\mu} M_{1\mu}} \frac{\mu_x \sqrt{2} \mu_x \mu_y \sqrt{2} \mu_y}{\hbar^3} \times \frac{E_{\text{ex}}^2}{2} \int_{-1}^1 d(\cos 2\alpha_k) \sum_{K'' K'''} \left| \mathcal{F}(\cos 2\alpha_k) \right|^2 \times \frac{1}{2L_z + 1} (1 - \cos 2\alpha_k)^{L_z + 1/2} (1 + \cos 2\alpha_k)^{L_y + 1/2} \times \left( \begin{array}{ccc} L' & S' & J' \\ M_{1\mu} & M_{1\mu} & -M_{1\mu} \end{array} \right)^2 J' L_z L_y n'' \left( \cos 2\alpha_k \right) \mathcal{M}_{\mu}^{K'' C} \mathcal{M}_{\mu}^{K' C} ,
\]

(7.2.13)
and accordingly

\[
\frac{dB}{dE_{\text{ex}}} = \frac{1}{2J+1} \left( \frac{A_{\text{proj}}}{A_c} \right)^2 \sum_{M_J M_{J'} L'M_{S'} L'_S} \frac{\mu_x \sqrt{2} \mu_x}{\hbar^3} \frac{\mu_y \sqrt{2} \mu_y}{\hbar^3} \left( \frac{L' S' J'}{M_{L'} M_{S'} - M_{J'}} \right)^2 \tilde{j}^2
\]

\[
\times \frac{E_{\text{ex}}^2}{2L_x' + L_y' + 2} \int_{-1}^{1} d(\cos 2\alpha_k) |F(\cos 2\alpha_k)|^2
\]

\[
\times (1 - \cos 2\alpha_k)^{L_x' + 1/2} (1 + \cos 2\alpha_k)^{L_y' + 1/2} J_{L_x' L_y'}^{L_x L_y} (\cos 2\alpha_k) \mathcal{M}_{\mu'}^{K'\mu} \mathcal{M}_{\mu}^{K'\mu}.
\]

(7.2.14)

Then, recalling that we have the formula for the phase space factor (5.1.24):

\[
\rho_{\text{ex}}^{(E1)}(\mu_x, \mu_y, E_{\text{ex}}) = \frac{\mu_x \sqrt{2} \mu_x}{\hbar^3} \frac{\mu_y \sqrt{2} \mu_y}{\hbar^3} 2E_{\text{ex}}^2 \left( \frac{A_{\text{proj}}}{A_c} \right)^2 = \frac{4E_{\text{ex}}^2}{\hbar^6 m^3_n},
\]

(7.2.15)

we see that the expression of the E1 transition strength with the FSI is:

\[
\frac{dB}{dE_{\text{ex}}} = \rho_{\text{ex}}^{(E1)} \sum_{M_{J'} M_{J''} L'M_{S'}} \sum_{L'M_{S'} L'_S} \left( \frac{L' S' J'}{M_{L'} M_{S'} - M_{J'}} \right)^2 \tilde{j}^2 \frac{1}{2L_x' + L_y' + 3} \int_{-1}^{1} d(\cos 2\alpha_k) |F(\cos 2\alpha_k)|^2
\]

\[
\times (1 - \cos 2\alpha_k)^{L_x' + 1/2} (1 + \cos 2\alpha_k)^{L_y' + 1/2} J_{L_x' L_y'}^{L_x L_y} (\cos 2\alpha_k) \mathcal{M}_{\mu'}^{K'\mu} \mathcal{M}_{\mu}^{K'\mu}.
\]

(7.2.16)

### 7.2.1 E1 strength for \(^6\text{He}\)

We firstly calculate the E1 strength for \(^6\text{He}\) with FSI in order to compare with experimental data. Figure 7.1 compares several different E1 distributions. For the purpose of comparison, No FSI calculation is shown together. From the left graphs, we can see that for \(r_e = 1.5\) (fm), the enhancement effect becomes bigger and the peak of the distribution is shifter to lower energies as the scattering length becomes longer. In the right graph, the effective range of a post-collision core-dineutron interaction is set 2.5 (fm); we can see that the longer the scattering length becomes, the more strongly enhanced the E1 transition is. But for the same scattering length, the enhancement is more significant for \(r_e = 1.5\) (fm) than that for \(r_e = 2.5\) (fm).

We also compare our calculation with those of Cobis and Danilin, which is shown in Figure 7.2. As is seen in the graph, it seems that our FSI model yields theoretical
Figure 7.1: E1 strength distributions for $^6$He with several different FSI parameters, plotted against the excitation energy. No FSI calculation and experimental data are shown together. For our calculation, od14 is used as the bound state wave function. Experimental data are from [86].

values which are closer to the experimental data than their models do. Cobis and Danilin produced relatively sharp peaks at around 1.0 and 1.3, respectively; both of the peaks deviate from the experimental data. Our FSI model is in line with the experimental data provided that the scattering length is -25 (fm) and the effective range is 1.5 (fm). If the scattering length is kept -25 (fm) and the effective range climbs to 2.5 (fm), our FSI model underestimates the E1 strength, but the trend of the E1 distribution is still similar to that of the experimental data. These results suggest that the post-collisional core-dineutron interaction plays a vital role in the EMD of $^6$He.

Also, we carry out calculations of E1 transitions for numerical checking purposes; the enhancement factor heads toward unity as the energy increases. The No FSI calculation and the FSI calculation should agree with each other in the high energy limit. In the left graph of Figure 7.3, difference between the curve for FSI and that for No FSI decreases as the energy increases. The right graph of Figure 7.3 shows that a very long scattering length and a large effective range lead to a situation where the enhancement factor becomes close to unity. Accordingly we see that the resultant E1 transition involving the FSI becomes close to that of the No FSI calculation. This is regarded as another numerical check.
CHAPTER 7. FINAL STATE INTERACTIONS

Figure 7.2: E1 strength distributions for $^6$He. For our calculation, od14 is used for the bound state wave function. Experimental data are from [86].

Figure 7.3: E1 transition strength distributions of $^6$He with and without the FSI.

7.2.2 E1 strength for $^{22}$C

As far as $^{22}$C is concerned, its E1 strengths are shown in Figure 7.4, where several different scattering length parameters are used, and No FSI calculations are shown together. As can be seen from the graph, the FSIs have almost no effect on the E1 strength in the low-lying region, and have tiny effects on that in higher energies. For example, for the effective range being kept fixed 2.5 (fm), if the modulus of scattering length increases from 15 (fm) to 55 (fm), the resultant E1 strength does little change.

The effects of the FSI on the E1 strengths for $^{22}$C can appear as the effective range becomes smaller. In Figure 7.5 a few E1 distributions are shown, where the effective range is set 0.5 (fm), and No FSI calculation is shown together. From the graphs,
we can see that the enhancement effects of the FSI appear in a higher energy region, but the effects are not significant.

It is argued that if the SDR exists in $^{22}$C, basically the time scale of its post-collisional core-dineutron interaction may be shorter than that for $^{11}$Li as the core of $^{22}$C is heavier (and the relative velocity between the dineutron and the core is larger) than that of $^{11}$Li. Considering the fact that the peak position of the $^{11}$Li SDR is at around 1.0 (MeV) and the FWHM is about 0.5 (MeV), we do not genuinely believe that a $^{22}$C SDR can appear at about 6.0 (MeV) with a much larger FWHM (even if the bump is a resonance). Thus we interpret the small bumps in Figure 7.5 as something that differs from the SDR.

Also, we see how differently the FSI affects the E1 strengths for other $^{22}$C bound state wave functions. The E1 strengths involving two different bound state wave functions.
functions are shown in Figure 7.6, where No FSI calculations and Ershov’s E1 distributions are shown together. From the graph, it is apparent that for both ic1 and ic3,

![Graph showing E1 strength distributions for 22C.](image)

the differences between the No FSI calculations and the FSI calculations are negligible; a peak-shift phenomenon does not occur if the FSI is switched on. This suggests that, in general, the post-collisional core-dineutron interaction is insensitive to the E1 transition strengths for this nucleus.

We also compare our results to Ershov’s ones. In their model, core-n interaction is a Woods-Saxon-type potential, and the Gaussian potential is used for nn interaction [13]. As can be seen in Figure 7.6, the peak positions of our E1 strengths do not significantly differ from theirs. Also, as was seen before, there is little difference in peak positions between their E1 strength and our E1 distribution of No FSI calculation. These things suggest that the FSIs are not so sensitive to the peak position of an SDR for this nucleus, and that the 20C-dineutron interaction does not play an important role in creating a soft dipole resonance.

These findings are different to those about 6He where the FSI made a significant effect on the E1 distribution, and also changed the peak position of the E1 distribution. It is thought that, the post-collisional core-dineutron interaction has a much more significant impact on the E1 strength in 6He than in 22C. Thus it seems that as far as a post-collisional core-dineutron interaction is concerned, there is no similarity between 6He and 22C.

In relation to this, we note that the valence particles of 6He are in the 0p3/2 state,
and those of $^{22}\text{C}$ are mainly in the $1s_{1/2}$ state; for $^6\text{He}$ the p-wave component occupies more than 80 percent and the s-wave component is less than 7 percent, while for $^{22}\text{C}$ the s-wave does about 80 percent. Thus we argue that it is possible that this difference can have an impact on the sensitivity.

Taken all these things into consideration, we put forward two different theories: the first theory is that the peaks which appear in our E1 distributions have something to do with the SDR for $^{22}\text{C}$; in this regard, the SDR mechanism is not completely incompatible with our results, but the FSI does not play a key role in forming an SDR. The second theory is that, provided that the FSI plays a vital role in forming an SDR state, the peaks in our E1 distributions are not linked to an SDR-type resonance, which can imply that there exists no SDR state for this Borromean nucleus. Although our discussion is not still conclusive, if the second theory is true, it will support the view that the EMD of $^{22}\text{C}$ can be a direct breakup. Here, we regard the second theory as a hypothesis, and argue that the first theory is closer to our current conclusion about an SDR of $^{22}\text{C}$.

### 7.3 Elastic breakup with Watson-Migdal FSI

In the previous section, we introduced Watson-Migdal final state interactions in our calculation of E1 strength distributions. We now incorporate these FSIs into our 4-body breakup model. The main difference between the formulae with and without FSIs is that the former contains the enhancement factor $|\mathcal{F}|^2$, and is integrated numerically with regard to $\cos(2\alpha_k)$.

As to the derivation of the formula, see Appendix D. In the Appendix page, we see that the expression (D.3.9) for the exclusive breakup cross section is:

$$
\frac{d^2\sigma}{d\Omega'_{kR}dE_{ex}} = (2\pi)^4 2k_R \frac{k'_R}{k \hbar} \frac{m_n^3}{\hbar^6} \cdot \frac{1}{2J + 1} \int \frac{E_{ex}^2}{2} d(\cos 2\alpha_k) \sum_{K_{kR}K_{kR}'} \left( \sin \alpha_k \right)^{2L'_{z_k}+1} \left( \cos \alpha_k \right)^{2L_{z_k}'+1} J_{L_{z_k}L_{z_k}'} L_{z_k} L_{z_k}' n_k n'_k (\cos 2\alpha_k) \sum_{\lambda M_{\lambda}} \frac{1}{2\lambda + 1} |U_{\lambda M_{\lambda}}| \left| ^{2} \right. $$

(7.3.1)
CHAPTER 7. FINAL STATE INTERACTIONS

Here the square modulus of $U_{\lambda M_\lambda}^{J'}(q, \cos 2\alpha_k)$ is given as:

$$|U_{\lambda M_\lambda}^{J'}|^2 = |2\pi|^2 \int R_0 dR_0 J_{|M_\lambda|}(qR_0) \int R_0' dR_0' J_{|M_\lambda|}(qR_0') \sum_{\xi,\lambda_\alpha,\lambda_\beta,\lambda_\gamma,\xi'} |F|^2 D^{J'J\xi}_{\lambda M_\lambda,\lambda_\alpha,\lambda_\beta,\lambda_\gamma}(R_b) D^{J\xi'}_{\lambda M_\lambda,\lambda_\alpha,\lambda_\beta,\lambda_\gamma}(R_b') \ , \ (7.3.2)$$

where the subscript $\xi$ is a set of quantum numbers: $\xi = \{K, L, S, L_y, L_x\}$, and $D^{J'J\xi}_{\lambda M_\lambda,\lambda_\alpha,\lambda_\beta,\lambda_\gamma}(R_b)$ is defined by

$$D^{J'J\xi}_{\lambda M_\lambda,\lambda_\alpha,\lambda_\beta,\lambda_\gamma}(R_b) = \langle \xi'J' || \Gamma_{\lambda M_\lambda} || \xi J \rangle \Delta^\lambda_{\xi'\xi} \lambda_\alpha \lambda_\beta \lambda_\gamma \ . \ (7.3.3)$$

As was done in Chapter 6, for a 4-body breakup reaction, we now move on to formulate the inclusive cross section which involves the FSI. Its derivation is done in Appendix D, where we can see that the expression (D.3.14) for the inclusive cross section is:

$$\frac{d\sigma}{dE_{\text{ex}}} = (2\pi)^6 \cdot \frac{m_3^3}{\hbar^6} \cdot \frac{2\pi}{2J + 1} E_{\text{ex}}^2 \int d(\cos 2\alpha_k) \sum_{K'_{x_k}'; y_k} (\sin \alpha_k)^{2L'_{x_k} + 1} (\cos \alpha_k)^{2L'_{y_k} + 1} J_{L'_{x_k}L'_{y_k}n_k}^L (\cos 2\alpha_k)$$

$$\times \sum_{\lambda M_\lambda} \frac{1}{2\lambda + 1} \int R_0 dR_0 \sum_{\xi,\lambda_\alpha,\lambda_\beta,\lambda_\gamma,\xi'} |F|^2 D^{J'J\xi}_{\lambda M_\lambda,\lambda_\alpha,\lambda_\beta,\lambda_\gamma}(R_b) D^{J\xi'}_{\lambda M_\lambda,\lambda_\alpha,\lambda_\beta,\lambda_\gamma}(R_b') \ . \ (7.3.4)$$

As is seen from the result (6.1.41) in Chapter 6, the factor $(2\pi)^6$ on the RHS of (7.3.4) is absorbed and disappears if we use the conventional continuum wave function $\phi^{(3B)}_{k_x,k_y}(x,y)$ [33].

### 7.3.1 Calculation for $^6\text{He}$ elastic breakup with FSI

Now we move on to numerical calculations of the inclusive cross section. First we consider $^6\text{He} + ^{12}\text{C}$ at 240 (MeV/A), and see how the FSIs have an impact on the breakup cross section. Figure 7.7 presents some cross sections that involve a few different inputs of FSI parameters. No FSI calculations and the CDCC calculations are shown together in the graphs. From the graphs, it seems that in general once the FSI is switched on, the cross section is increased. The biggest enhancement effect is seen for $0^+$, making a peak at around 0.8 (MeV). For $1^-$, it seems that the peaks of the cross sections become stronger and sharper as the scattering length becomes
Figure 7.7: Comparison of breakup cross sections which involve several different scattering lengths, for $^6$He + $^{12}$C at 240 (MeV/A). The upper left graph is for $0^+$, the upper right for $1^-$ and the lower graph for $2^+$. For our calculation, od14 is used for the bound state wave function.

longer. Also, we found that the enhancement effect became weaker as the effective range became longer, although here we do not show the graph. These findings are consistent with the previous results of E1 transition strengths for $^6$He. Similarly for $2^+$, the cross section is more easily enhanced as the effective range decreases, and the cross section becomes larger as the scattering length becomes longer. It seems that the peak positions of the cross sections seem to be located at a higher excitation energy than those for $1^-$. 

The sums of the cross sections for $0^+$, $1^-$ and $2^+$ are shown in Figure 7.8, where the CDCC calculation and the experimental data are shown together. As is seen from the graph, the introduction of the FSI worsens the cross section, especially at higher energies. For our calculations, basically the contribution to the cross sections from $0^+$ becomes large when the FSI is switched on, which causes our calculations to overestimate the cross section at higher energies. Indeed, for $0^+$, its cross section is sensitive to the FSI parameters, and the enhancement effect seems to be large even for short scattering length. As we discussed in the previous Chapter, it can be argued
that a plane wave as a continuum state is more suitable for describing the reality as far as higher energies are concerned.

In this calculation, the FSI in our calculation does not produce the $2^+$ resonance, creating a peak at around 0.8 (MeV) and changing the values of the cross sections, which suggests that the FSI does not involve the $2^+$ resonance. This implies the conclusion that the $2^+$ resonance is not related to the post-collision alpha-dineutron interaction.

### 7.3.2 Calculation for $^{22}$C elastic breakup with FSI

We now consider how the FSIs have an effect on the inclusive breakup cross section involving $^{22}$C. As to the $^{22}$C+$^{12}$C at 100 (MeV/A), we plot the breakup cross sections against energy in Figure 7.9, where No FSI calculations are shown together.

As is seen in the graphs, for $0^+$ continuum states, the enhancement effect of the FSIs are larger, compared to other states. This is similar to our findings for $^6$He+$^{12}$C at (240 MeV/A). For $1^-$, we see that the effect of the FSI on the cross section is negligible, and that is consistent with our findings about the E1 transition strengths of $^{22}$C. For $2^+$, it seems that the peak becomes stronger as the scattering
length becomes longer and the effective range becomes shorter. Also, we see that the peak of $2^+$ is shifted to higher energies once the FSI is switched on. We do not have

![Breakup cross sections](image)

Figure 7.9: Breakup cross sections for $^{22}\text{C} + ^{12}\text{C}$ at 100 (MeV/A). The cross sections are for $0^+$ (the upper left graph), $1^-$ (the upper right), and $2^+$ (the lower). ic6 is used for the bound state wave function.

a clear explanation why the FSI has almost no impact on the cross section for $1^-$, although this result itself is consistent with our findings for E1 transition for $^{22}\text{C}$.

For the case of nuclear breakup of $^6\text{He}$, we saw that the FSI had a non-negligible impact on the cross section for $1^-$. Thus, in this regard we can see a difference between $^6\text{He}$ and $^{22}\text{C}$.

For both EMD and nuclear breakup, we have a similar result that cross sections to $1^-$ are insensitive to the FSIs. This suggests that it is possible that we can deny that the difference in reaction process between EMD and nuclear breakup is crucial for the post-collision states of $^{22}\text{C}$, as far as $1^-$ is concerned.

However, it could be argued that the post-collision state for nuclear breakup is more complicated than that for EMD: Compared to EMD, nuclear breakup allows a projectile to breakup at a spatial point closer to its target because of the nuclear attractive interaction (and weaker Coulomb force), and so its post-collision state is affected more by the FSI between the core and the target. If we include the core-
target FSI for this reaction, the peak of the cross section for $1^-$ might be shifted to a higher or lower energy region. Also, it is possible that the introduction of the core-target FSI can change the sensitivity of the $0^+$ cross section to the FSI.

Thus, in order to see more detailed post-collision state for this reaction, we need to include the core-target FSI.

We next check how different initial bound state wave functions affect the cross section under the FSI regime. Figure 7.10 shows the cross sections involving different bound wave functions, where the FSI is kept fixed: $a = -15$ (fm) and $r_e = 4.0$ (fm). As is seen in the graph, the larger the $2n$-separation energy becomes, the smaller the peak value becomes and the higher the energy it is shifted to. This result is consistent with our findings for E1 transition for $^{22}$C.

We also calculate the angular distributions of the breakup cross section. Figure 7.11 shows the distributions which are evaluated at $E = 1.0$ (MeV). For the purpose of comparison, No FSI calculation are presented as well. From the graphs, it is apparent that the FSIs have a much more significant effect on the cross sections for $0^+$ than for other states. When it comes to $1^-$, the FSIs have no impact on the cross section,
while for $2^+$ the FSIs enhance the cross section. These findings are in line with the

Figure 7.11: The angular distributions of the breakup cross section for $^{22}\text{C} + ^{12}\text{C}$ at 100 (MeV/A). The upper left graph is for $0^+$, the upper right for $1^-$, and the lower graph for $2^+$. The cross sections are evaluated at $E = 1.0$ (MeV). ic6 is used for the bound state wave functions.

results for the inclusive cross section.
In this work, we adopted a 3-body model (core+n+n) to study Borromean nuclei, focusing in particular on $^{22}\text{C}$, and developed the 4-body Glauber-WKB model to study reactions involving these nuclei. Using the Jacobi coordinates and HH/CSF, we calculated the 3-body bound state wave functions for $^6\text{He}$ and $^{22}\text{C}$, and studied both the similarities and differences between them. Both had two clear peaks in their probability density distributions, but the peaks for $^6\text{He}$ were sharper than those for $^{22}\text{C}$. For $^6\text{He}$, the highest probability corresponded to "dineutron" configuration, while for $^{22}\text{C}$, the highest corresponded to a "triangular" configuration.

For $^{22}\text{C}$, we tested how the 3-body force has an effect on the spatial configuration of the 3-body wave function, and found that broadly, the force affects the entire wave function. (This is not against our intuition.) However, our results suggested that the $^{20}\text{C}$-n potential was much more important than the 3-body force concerning the spatial configuration of the nucleus.

With regard to our reaction models, we began by considering 2-body scattering and investigated the WKB approximation from Good-Miller’s perspective, and we used the Glauber-WKB model to describe the reaction. Then we extended the validity of the model by adopting Wallace’s approach to include non-eikonal corrections; we used the WKB approximation and expanded the WKB phase shift with regard to a parameter, whereby the Glauber phase shift is the lowest order term in the expansion. The higher order phase shifts corresponded to the non-eikonal trajectories of the projectiles, so they should reflect a more accurate model than the eikonal one. In
addition, we aimed at adding Rosen-Yennie correction for the purpose of improving the WKB calculations, and we saw that, on the whole, the non-eikonal phase shifts and the RY correction provided us with scattering cross sections closer to the results that the partial wave calculations yielded.

We then generalised our reaction models to a 4-body system which involved a (3-body Borromean) projectile and a 1-body target. We applied the adiabatic approximation to the reaction under the assumption that the collision time is short so that the excitation energy of the projectile remains smaller than the center-of-mass energy of the system. Indeed, the adiabatic approximation was reasonable for such a nucleus, which is weakly bound and has one bound excited state, because such a nucleus has a binding energy that is much smaller than the kinetic energy of the 3-body projectile. The spatially extended halos have a localised momentum distribution, and the slow internal motion of the projectile means that it has a small excitation energy. Our numerical calculations of scattering cross sections for $^6$He+$^{12}$C at 38.3 (MeV/A) indicated that the Glauber calculation underestimated the cross section, and that the higher order calculations made things slightly worse at higher scattering angles.

As for reactions of $^{22}$C+$^{12}$C at around 100 to 300 (MeV/A), we needed to estimate the interaction between the core and the target, and so we used a folding model. Our calculations for the reactions showed that the scattering cross sections were mainly focused in the forward direction, and the cross sections started to decrease continuously at around a scattering angle of 5 degrees. Unlike the case of $^6$He, the higher order calculations gave cross sections that were similar to those from the Glauber calculation. This was mainly because the absolute values of the depths of the potential were small to the extent that the high energy condition was satisfied easily. This result suggests that the non-eikonal trajectories do not play a crucial role at these energies.

We then studied the continuum excitation mechanism of Borromean nuclei by calculating their E1 transition strengths. For $^6$He, both Cobis and Danilin overestimated the E1 transition strength at lower energies, and Cobis underestimated it in a higher energy region. Danilin’s calculation involving more realistic continuum wave function gave a larger strength than our No FSI calculation did. This suggested that more realistic continuum wave function provides a larger strength in a low-lying region.
For $^{22}$C, our theoretical values were, on the whole, smaller than Ershov’s theoretical values which used a more realistic continuum wave function for the final state.

So far we had been using plane waves for the 3-body final states, and to correct for this we introduced the final state interaction.

For $^6$He, the E1 distribution was enhanced and its peak was shifted to lower energies once the Watson-Migdal FSI was introduced. It seemed that our FSI model was better than Cobis’s model and Danilin’s model in the sense that the trends of our E1 distributions are similar to the trend of the experimental data. This suggested that the post-collisional core-dineutron interaction played a crucial role in the EMD of $^6$He. In contrast to this, the E1 distributions for $^{22}$C were not significantly affected by the FSI. It could be argued that the peak seen in the E1 distribution for $^{22}$C was related to the soft dipole resonance (SDR), but this resonance was not induced by the FSI.

We calculated the breakup cross sections for $^6$He + $^{12}$C at 240 (MeV/A) within our 4-body Glauber-WKB model, and compared them to CDCC calculations. The CDCC calculations reproduced the well-known $2^+$ resonance, while our model did not. But our model provided the cross section which was closer to the experimental data than CDCC at higher energies. Once the Watson-Migdal FSI was introduced, our theoretical cross sections increased for $2^+$ but the sharp peak did not appear. For $1^-$, the cross section was enhanced and its peak was strengthened by the FSI. Also, the cross section for $0^+$ was more sensitive to the FSIs than the other continuum states. The introduction of the FSI made things worse as far as higher energies are concerned.

Then we calculated the differential breakup cross sections for the reaction $^{22}$C+$^{12}$C at about 100 (MeV/A). The exclusive breakup cross sections were evaluated at low excitation energies. In our 4-body model, estimating the interaction between the core and the target, we used a single folding model to describe it. Testing two different folding density (a Gaussian density and KT density), we saw that, at forward scattering angles, the two cross sections were similar. This meant that the breakup cross section was not so sensitive to the behaviour of the density near the coordinate origin.

We also calculated the inclusive cross section for the same reaction at 100 (MeV/A).
Our findings suggested that for $1^-$, the effect of the higher order terms of the WKB expansion on the cross section was subtle, while for $2^+$, the non-eikonal effect was slightly more significant. We do not have a decisive explanation why there appeared the difference between $1^-$ and $2^+$. In terms of $1^-$, we saw that our FSI model did not have a significant impact on the cross section, which was consistent with our finding about E1 strength for $^{22}$C. The cross section for $2^+$ had a relatively broad peak at around $E = 2.0$ (MeV); it was enhanced and its peak was shifted to higher energies once the FSI was switched on. The cross section to $0^+$ showed a strong sensitivity to the FSIs. Also, our calculation indicated that the peak position was shifted to the lower energy region, and the peak became strengthened, as the 2n-separation energy of the bound wave function decreases.

As for future work, it will be desirable to include core excitation and deformation in the calculation, and to see how these impact on the E1 transition strength distributions and differential breakup cross sections involving Borromean nuclei. It is also worthwhile exploring further where the 3-body force comes from. It is of course important to obtain continuum wave functions by solving the hyperradial equations, and to explore other resonance states. Here, since we only used plane wave final state wave functions, we compensated for this by using a Watson-Migdal type FSI; it would be useful to developing the model further. For instance, we neglected the FSI between the valence neutrons. Thus, considering other kinds of FSIs such as these between projectile constituents and target may be important. In the present research, we used phenomenological values for the scattering length and the effective range, but we will need to estimate realistic values for those parameters by analysing a 2-body system.
APPENDIX A

Jacobi coordinates

This supplementary chapter is to provide a detailed description of Jacobi coordinates. There exist unnormalised and normalised Jacobi coordinates for a N-body system. First of all, in terms of the unnormalised Jacobi coordinates, let us define a transformation of coordinates as [105]:

\[
T_j :
\begin{cases}
  u_j = r_j - R_{j-1} \\
  R_j = \frac{1}{M_j} (m_j r_j + M_{j-1} R_{j-1})
\end{cases}
\]  

(A.0.1)

\[
(j = 2, 3, \ldots, N)
\]

where \( R_j \) is regarded as the center-of-mass of the subsystem which consists of particle 1, 2, ..., j. \( R_1 \) is equal to \( r_1 \). \( M_j \) is the total mass of the subsystem:

\[
M_1 = m_1, \quad M_j = M_{j-1} + m_j \quad (j = 2, 3, \ldots, N).
\]

(A.0.3)

(A.0.4)

Next we consider its inverse transformation \( T_j^{-1} \):

\[
T_j^{-1} = \begin{cases}
  r_j = \frac{M_{j-1}}{M_j} u_j + R_j \\
  R_{j-1} = \left( -\frac{m_j}{M_j} \right) u_j + R_j
\end{cases}
\]

(A.0.5)

Then consider their momenta; let \( Q_j \) be the corresponding change of variables.

\[
Q_j = \begin{cases}
  v_j = \frac{M_{j-1}}{M_j} p_j - \frac{m_j}{M_j} P_{j-1} \\
  P_j = p_j + P_{j-1}
\end{cases}
\]

(A.0.6)
where $P_j = p_j$. Here the capital bold letter $P_j$ is the total linear momentum of the subsystem of particle 1, 2, ..., $j$. The inverse of $Q_j$ is

$$Q^{-1}_j = \begin{cases} p_j = v_j + \left( \frac{m_j}{M_j} \right) P_j \\ P_{j-1} = -v_j + \left( \frac{M_{j-1}}{M_j} \right) P_j \end{cases}. \quad (A.0.7)$$

Thus the N-body Jacobi coordinates are $(R_N, u_2, u_3, \ldots, u_N, P_N, v_2, v_3, \ldots, v_N)$.

**3-body Jacobi coordinates:** Then we write down the 3-body Jacobi coordinates, according to the definition that we just introduced. The coordinates are expressed as follows:

$$u_2 = r_2 - r_1, \quad u_3 = r_3 - \frac{m_2 r_2 + m_1 r_1}{m_1 + m_2}, \quad (A.0.8)$$

$$R_3 = \frac{m_3 r_3 + m_2 r_2 + m_1 r_1}{m_1 + m_2 + m_3}, \quad (A.0.9)$$

and

$$v_2 = \frac{m_1 m_2}{m_1 + m_2} \left( \frac{p_2}{m_2} - \frac{p_1}{m_1} \right), \quad v_3 = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3} \left( \frac{p_3}{m_3} - \frac{p_1 + p_2}{m_1 + m_2} \right), \quad (A.0.10)$$

$$P_3 = p_1 + p_2 + p_3. \quad (A.0.11)$$

**3-body normalised Jacobi coordinates:** Then we consider the 3-body system which consists of particle 1 and 2, and the core, whose mass numbers are $A_1, A_2, A_c$, respectively. For the system the normalised 3-body Jacobi coordinates $(R_3, y, x, P_3, \hbar \kappa_x, \hbar \kappa_y)$ are [32]:

$$x = \sqrt{\frac{A_1 A_2}{A_1 + A_2}} (r_2 - r_1), \quad y = \sqrt{\frac{(A_1 + A_2)A_c}{A_1 + A_2 + A_c}} \left( r_c - \frac{A_2 r_2 + A_1 r_1}{A_1 + A_2} \right), \quad (A.0.12)$$

$$R_3 = \frac{A_c r_c + A_2 r_2 + A_1 r_1}{A_1 + A_2 + A_c}, \quad (A.0.13)$$

and

$$\kappa_x = \sqrt{\frac{A_1 A_2}{A_1 + A_2}} \left( \frac{p_2}{m_2} - \frac{p_1}{m_1} \right), \quad \kappa_y = \sqrt{\frac{(A_1 + A_2)A_c}{A_1 + A_2 + A_c}} \left( \frac{p_c}{m_c} - \frac{p_1 + p_2}{m_1 + m_2} \right), \quad (A.0.14)$$

$$P_3 = p_1 + p_2 + p_3. \quad (A.0.15)$$

We assume that $m_1 = m_n A_1, m_2 = m_n A_2, m_c = m_n A_c$, where $m_n$ is the nucleon mass.
APPENDIX A. JACOBI COORDINATES

As to the relation between the coordinates and hyperspherical coordinates is [32]

\[ \rho = \sqrt{x^2 + y^2}, \quad \alpha = \arctan \left( \frac{x}{y} \right) \]  \quad (A.0.16)

\[ \kappa \ast \hbar = \hbar \sqrt{\kappa_x^2 + \kappa_y^2} = \sqrt{2m_n|E_\kappa|}, \quad \alpha_\kappa = \arctan \left( \frac{\kappa_x}{\kappa_y} \right). \]  \quad (A.0.17)

Then we note that the conventions of the unnormalised momentum Jacobi coordinates \( k_x, k_y \) differ between authors. For example, in Danilin’s convention [33],

\[ k_x = \kappa_x \]  \quad (A.0.18)

\[ k_y = \kappa_y. \]  \quad (A.0.19)

As to Ershov’s convention [32], they are:

\[ k_x = \sqrt{\frac{A_1 A_2 - \kappa_x}{A_1 + A_2}} \kappa_x, \quad k_y = \sqrt{\frac{(A_1 + A_2) A_c}{A_1 + A_2 + A_c}} \kappa_y. \]  \quad (A.0.20)

For the unnormalised momentum Jacobi coordinates, we take Ershov’s convention.

Then we just see how the differentials in the Cartesian coordinates transform into those in the Jacobi coordinates.

\[ dr_{12} dr_{12c} = \frac{\partial (r_{12}, r_{12c})}{\partial (x, y)} dx dy. \]  \quad (A.0.21)

The Jacobian matrices are written as

\[ \frac{\partial r_{12}}{\partial x} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad \frac{\partial r_{12}}{\partial y} = 0 \otimes 1_3 \]  \quad (A.0.22)

\[ \frac{\partial r_{12c}}{\partial x} = 0 \otimes 1_3, \quad \frac{\partial r_{12c}}{\partial y} = \begin{pmatrix} \frac{A_{\text{proj}}}{2A_c} & 0 & 0 \\ 0 & \frac{A_{\text{proj}}}{2A_c} & 0 \\ 0 & 0 & \frac{A_{\text{proj}}}{2A_c} \end{pmatrix} \]  \quad (A.0.23)

and so it turns out that the Jacobian is \( (\frac{A_{\text{proj}}}{A_c})^{3/2} \). Thus we have

\[ dr_{12} dr_{12c} = \left( \frac{A_{\text{proj}}}{A_c} \right)^{3/4} dx dy. \]  \quad (A.0.24)

As to the relation between the Jacobi coordinates and the hyperspherical coordinates, using \( dx dy = \rho d\rho d\alpha \), we find that

\[ dx dy = x^2 dx d\Omega_x y^2 dy d\Omega_y \]  \quad (A.0.25)

\[ = \rho^5 d\rho (\sin \alpha \cos \alpha)^2 d\alpha d\Omega_x d\Omega_y. \]  \quad (A.0.26)
Using the equation (A.0.24) in the previous section, we have:

\[
dr_{12}dr_{12} = \left( \frac{A_{\text{proj}}}{A_c} \right)^{\frac{3}{2}} \rho^5 d\rho (\sin \alpha \cos \alpha)^2 d\alpha d\Omega_x d\Omega_y .
\] (A.0.27)

Also, for the momentum Jacobi coordinates, we have [32]:

\[
dk_x dk_y = \left( \frac{\mu_x \mu_y}{m_n^2} \right)^{3/2} k^5 (\sin \alpha_k \cos \alpha_k)^2 dk d\alpha_k d\Omega_{kx} d\Omega_{ky}
\] (A.0.28)

\[
= \left( \frac{A_c}{A_{\text{proj}}} \right)^{3/2} k^5 (\sin \alpha_k \cos \alpha_k)^2 dk d\alpha_k d\Omega_{kx} d\Omega_{ky} .
\] (A.0.29)

Therefore, if we define their excited energies by

\[
E'_x = \frac{\hbar^2 k_x'^2}{2\mu_x}, \quad E'_y = \frac{\hbar^2 k_y'^2}{2\mu_y},
\] (A.0.30)

then we have

\[
dk'_x dk'_y = \left( \frac{2\mu_x 2\mu_y}{\hbar^4} \right)^{\frac{3}{2}} \sqrt{E'_x E'_y} dE'_x dE'_y d\Omega_{k'x} d\Omega_{k'y}
\] (A.0.31)

\[
= \left( \frac{4m_n^2}{\hbar^4} \cdot \frac{A_c}{A_{\text{proj}}} \right)^{3/2} \sqrt{E'_x E'_y} dE'_x dE'_y d\Omega_{k'x} d\Omega_{k'y} .
\] (A.0.32)

**4-body normalised Jacobi coordinates:** The normalised 4-body Jacobi coordinates \(x, y, R\) are defined by

\[
x = \frac{1}{\sqrt{2}} r_{12} = \frac{1}{\sqrt{2}} (r_2 - r_1) ,
\] (A.0.33)

\[
y = \sqrt{\frac{2A_c}{A_c + 2}} r_{12c} = \sqrt{\frac{2A_c}{A_c + 2}} \left( r_c - \frac{1}{2}(r_1 + r_2) \right) ,
\] (A.0.34)

\[
R = \frac{r_1 + r_2 + A_c r_c}{A_{\text{proj}}} ,
\] (A.0.35)

where the position of the target is chosen as the coordinate origin. Then, as to each particle’s position, they are expressed as follows:

\[
r_c = R + \sqrt{\frac{2}{A_{\text{proj}} A_c}} y
\] (A.0.36)

\[
r_2 = R - \sqrt{\frac{A_c}{2A_{\text{proj}}}} y + \frac{1}{\sqrt{2}} x
\] (A.0.37)

\[
r_1 = R - \sqrt{\frac{A_c}{2A_{\text{proj}}}} y - \frac{1}{\sqrt{2}} x .
\] (A.0.38)

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As to their b-components, immediately we have:

\[ b_c = b_R + \sqrt{\frac{2}{A_{proj} A_c}} b_y \]  \hspace{1cm} (A.0.39)

\[ b_2 = b_R - \sqrt{\frac{A_c}{2 A_{proj}}} b_y + \frac{1}{\sqrt{2}} b_x \]  \hspace{1cm} (A.0.40)

\[ b_1 = b_R - \sqrt{\frac{A_c}{2 A_{proj}}} b_y - \frac{1}{\sqrt{2}} b_x \]  \hspace{1cm} (A.0.41)
Rosen-Yennie extension

In this supplementary chapter, we derive the formula for the Rosen-Yennie phase shift. First we present the important aspects of the Rosen-Yennie approach. Next, we revisit the WKB approximation from the RY perspective and derive the WKB phase shift. Then we obtain the expression for the RY phase shift.

B.1 Essential ingredients for the RY approach

In this section, we present important ingredients of the RY method. GM method involved a perturbed equation and its unperturbed equation. RY did as well; the RY approach was approximating the solution of the perturbed equation by both the solution and its derivative of the unperturbed equation.

As usual, firstly we write the radial equation as:

\[ \left\{ \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} - \frac{2\mu U(r)}{\hbar^2} + k^2 \right\} \chi_{l,k} = 0 \Leftrightarrow \frac{\partial^2 \chi_{l,k}}{\partial \rho^2} + \Theta(\rho) \chi_{l,k} = 0, \quad (B.1.1) \]

where \( \Theta(\rho) = 1 - \frac{l(l+1)}{\rho^2} - \frac{U(r)}{E_{cm}} \), and \( \rho = kr \). Here we define \( \rho_t \) as the classical turning point.

Secondly, we introduce a new variable \( t = t(\rho) \) which is a function of \( \rho \), and consider the equations that have the unperturbed solutions \( \chi_{l,k,0} \) to (B.1.1): the subscript 0 of the \( \chi_{l,k,0} \) menas that this is a solution of the unperturbed equations:

\[ \frac{\partial^2 \chi_{l,k,0}}{\partial t^2} + \Theta_0 \chi_{l,k,0} = 0, \quad (B.1.2) \]
where $\Theta_0(t) = 1 - \frac{l(l + 1)}{t^2}$. As the definition of the introduced variable, we use the following relation:

$$\Theta_0 \left( \frac{\partial t}{\partial \rho} \right)^2 = \Theta.$$  \hspace{1cm} (B.1.3)

Then we introduce two new functions $\alpha(\rho)$, $\beta(\rho)$, and both of the two are functions of $\rho$. Using these functions, we express the solution to (B.1.1) as:

$$\chi_{l,k}(\rho) = \alpha \chi_{l,k,0} + \beta \frac{\partial \chi_{l,k,0}}{\partial t}.$$  \hspace{1cm} (B.1.4)

Immediately we have its first derivative with regard to $\rho$:

$$\chi'_{l,k}(\rho) = \alpha' \chi_{l,k,0} + \alpha \frac{\partial \chi_{l,k,0}}{\partial t} t' + \beta' \frac{\partial \chi_{l,k,0}}{\partial t} + \beta \frac{\partial^2 \chi_{l,k,0}}{\partial t^2},$$  \hspace{1cm} (B.1.5)

where $t' = \frac{\partial t}{\partial \rho}$, $\alpha' = \frac{\partial \alpha}{\partial \rho}$ and $\beta' = \frac{\partial \beta}{\partial \rho}$. Here we used (B.1.2) to obtain (B.1.5). And then we have the second derivative of $\chi_{l,k}(\rho)$ in a straightforward manner:

$$\chi''_{l,k}(\rho) = \left\{ \alpha'' - 2 \beta' t' \Theta_0 - \alpha (t')^2 \Theta_0 - \beta t'' \Theta_0 - \beta (t')^2 \frac{\partial \Theta_0}{\partial t} \right\} \chi_{l,k,0}$$

$$+ \left\{ 2 \alpha t' - \beta (t')^2 \Theta_0 + \alpha t'' + \beta'' \right\} \frac{\partial \chi_{l,k,0}}{\partial t}.$$  \hspace{1cm} (B.1.6)

Using the definition of $t$ (B.1.3), we have

$$\chi''_{l,k}(\rho) = \left\{ \alpha'' - 2 \beta' t' \Theta_0 - \alpha \Theta - \beta t'' \Theta_0 - \beta (t')^2 \frac{\partial \Theta_0}{\partial t} \right\} \chi_{l,k,0}$$

$$+ \left\{ 2 \alpha t' - \beta \Theta + \alpha t'' + \beta'' \right\} \frac{\partial \chi_{l,k,0}}{\partial t}.$$  \hspace{1cm} (B.1.7)

Then we use (B.1.4) to have

$$\chi''_{l,k}(\rho) + \Theta \chi_{l,k}(\rho) = \left\{ \alpha'' - 2 \beta' t' \Theta_0 - \beta t'' \Theta_0 - \beta (t')^2 \frac{\partial \Theta_0}{\partial t} \right\} \chi_{l,k,0}$$

$$+ \left\{ 2 \alpha t' + \alpha t'' + \beta'' \right\} \frac{\partial \chi_{l,k,0}}{\partial t}.$$  \hspace{1cm} (B.1.8)

The LHS of this equation is zero because the perturbed equation (B.1.1) holds. Accordingly the RHS is zero as well. Note that in this section, for the sake of convenience, let us denote the differentiation with regard to $\rho$ by prime. Now we use these $\chi_{l,k}(\rho)$, $\chi''_{l,k}(\rho)$ in (B.1.1), and we deal with the coefficients of $\chi_{l,k,0}$, $\frac{\partial \chi_{l,k,0}}{\partial t}$ independently, which yields the following differential equations:

$$\alpha'' - 2 \beta' t' \Theta_0 - \beta t'' \Theta_0 - \beta (t')^2 \frac{\partial \Theta_0}{\partial t} = 0$$  \hspace{1cm} (B.1.9)

$$(2 \alpha t' + \alpha t'' + \beta'') = 0$$  \hspace{1cm} (B.1.10)
These equations are reexpressed as

\[ \alpha'' - 2\sqrt{\Theta_0 t'} \frac{\partial}{\partial \rho} \left( \beta \sqrt{\Theta_0 t'} \right) = 0 \]  \hspace{1cm} (B.1.11)

\[ \beta'' + 2\sqrt{t'} \frac{\partial}{\partial \rho} \left( \alpha \sqrt{t'} \right) = 0 \]  \hspace{1cm} (B.1.12)

Thirdly, we expand these \( \alpha, \beta \) in powers of \( \hbar^2 \), and it follows that

\[ \alpha = \alpha_0 + \alpha_2 \hbar^2 + \alpha_4 \hbar^4 + \cdots \]  \hspace{1cm} (B.1.13)

\[ \beta = \beta_1 \hbar + \beta_3 \hbar^3 + \beta_5 \hbar^5 + \cdots . \]  \hspace{1cm} (B.1.14)

Then we substitute these into (B.1.11) and (B.1.12), and retain the lowest orders for each of them. It turns out that

\[ \alpha''_0 - 2\sqrt{\Theta_0 t'} \frac{\partial}{\partial \rho} \left( \beta_1 \hbar \sqrt{\Theta_0 t'} \right) = 0 \]  \hspace{1cm} (B.1.15)

\[ 2\sqrt{t'} \frac{\partial}{\partial \rho} \left( \alpha_0 \sqrt{t'} \right) = 0 \]  \hspace{1cm} (B.1.16)

Accordingly, we have the following two relations:

\[ \beta_1 \hbar \frac{1}{2\sqrt{\Theta_0 t'}} \int_{\rho_0}^{\rho} \frac{\alpha''_0}{\sqrt{\Theta_0 t'}} d\rho \]  \hspace{1cm} (B.1.17)

\[ \alpha_0 \sqrt{t'} = \text{const} \]  \hspace{1cm} (B.1.18)

As for the constant of the RHS in (B.1.21), we set it unity, because \( \chi_{l,k} \) should get close to the unperturbed solution \( \chi_{l,k,0} \) in the asymptotic region. Thus we should set

\[ \alpha_0 = \sqrt{\frac{1}{t'}} . \]  \hspace{1cm} (B.1.19)

Therefore we find that

\[ \lim_{\rho \to \infty} \alpha_0 = 1. \]  \hspace{1cm} (B.1.20)

Immediately, using (B.1.3), we have

\[ \alpha_0 = \frac{1}{\sqrt{t'}} = \left( \frac{\Theta_0}{\Theta} \right)^{1/4} , \]  \hspace{1cm} (B.1.21)

where \( t' = \frac{\partial t}{\partial \rho} \).

Now we aim at expressing (B.1.17) in another form. To do this, we do integral by parts to see

\[ \int_{\rho_0}^{\infty} \frac{\alpha''_0}{\sqrt{\Theta_0 t'}} d\rho = \int_{\rho_0}^{\infty} \frac{\alpha''_0 \alpha_0}{\sqrt{\Theta_0}} d\rho \]

\[ = I_1 - I_2 + I_3 , \]  \hspace{1cm} (B.1.22)
where

\[ I_1 = \left[ \frac{\alpha_0' \alpha_0}{\sqrt{\Theta_0}} \right]_{\rho_t}^{\infty} \] \tag{B.1.23}

\[ I_2 = \int_{\rho_t}^{\infty} \frac{\alpha_0' \alpha_0}{\sqrt{\Theta_0}} d\rho \] \tag{B.1.24}

Then we evaluate \( \alpha_0' \alpha_0' \): since \( \alpha_0^2 = \frac{\Theta_0}{\Theta_0'} \), we differentiate the both sides with regard to \( \rho \), and after that, we use (B.1.21). This yields:

\[ \alpha_0' = \frac{1}{4} \left( \frac{1}{\Theta_0} \frac{\partial \Theta_0}{\partial t} - \Theta' \sqrt{\Theta_0} \Theta'^{\frac{3}{2}} \right), \] \tag{B.1.25}

and therefore

\[ I_1 = \frac{1}{4} \left[ \frac{1}{\Theta_0^{\frac{3}{2}}} \frac{\partial \Theta_0}{\partial t} \right]_{t_t}^{\infty} - \frac{1}{4} \left[ \frac{\Theta'}{\Theta_0^{\frac{3}{2}}} \right]_{\rho_t}^{\infty} \] \tag{B.1.26}

\[ I_3 = \frac{1}{8} \int_{\rho_t}^{\infty} \left( \frac{(\Theta_0')^2 t'}{\Theta_0^{\frac{5}{2}}} - \frac{\Theta' \Theta_0''}{\Theta_0^{\frac{3}{2}}} \right) d\rho \] \tag{B.1.27}

where \( t_t = t(\rho_t) \).

Before calculating \( I_2 \), we see

\[ \frac{t''}{t'} = \frac{1}{2} \left( \frac{1}{\Theta} \frac{\partial \Theta}{\partial t} - \Theta' \frac{\partial \Theta}{\partial t} \right) \] \tag{B.1.28}

\[ \alpha_0' = -\frac{1}{2} (t')^{-\frac{3}{2}} t'' \] \tag{B.1.29}

Thus we have the expression for \( I_2 \):

\[ I_2 = \frac{1}{16} \int_{\rho_t}^{\infty} \left\{ \frac{(\Theta')^2}{\Theta_0^{\frac{5}{2}}} - 2 \frac{\Theta'}{\Theta_0^{\frac{3}{2}}} \frac{\partial \Theta_0}{\partial t} + \frac{t'}{\Theta_0^{\frac{5}{2}}} \left( \frac{\partial \Theta_0}{\partial t} \right)^2 \right\} d\rho \] \tag{B.1.30}

As a result, we obtain the following expression:

\[ \int_{\rho_t}^{\infty} \frac{\alpha_0''}{\sqrt{\Theta_0}} d\rho = \frac{1}{16} \left[ \int_{t_t}^{\infty} \frac{1}{\Theta_0^{\frac{5}{2}}} \left( \frac{\partial \Theta_0}{\partial t} \right)^2 dt - \int_{\rho_t}^{\infty} (\Theta')^2 \frac{\Theta_0''}{\Theta_0^{\frac{5}{2}}} d\rho \right] + \frac{1}{4} \left[ \frac{1}{\Theta_0^{\frac{3}{2}}} \frac{\partial \Theta_0}{\partial t} \right]_{t_t}^{\infty} - \frac{1}{4} \left[ \frac{\Theta'}{\Theta_0^{\frac{3}{2}}} \right]_{\rho_t}^{\infty}. \] \tag{B.1.31}

We see that the following relation holds true:

\[ \frac{1}{12} \int_{t_t}^{\infty} \frac{1}{\sqrt{\Theta_0}} \frac{d}{dt} \left( \frac{\Theta_0''}{\partial \Theta_0} \right) dt = \frac{1}{12} \left[ \frac{1}{\sqrt{\Theta_0}} \frac{\partial \Theta_0}{\partial t} \right]_{t_t}^{\infty} + \frac{1}{24} \left[ \Theta_0^{-\frac{3}{2}} \frac{\partial \Theta_0}{\partial t} \right]_{t_t}^{\infty} + \frac{1}{16} \int_{t_t}^{\infty} \Theta_0^{-\frac{5}{2}} \left( \frac{\partial \Theta_0}{\partial t} \right)^2 dt, \] \tag{B.1.32}
where the relation (B.1.32) is obtained by repeating integration by parts. Therefore, by using (B.1.32), we obtain:

\[
\int_{\rho_t}^{\infty} \frac{a''_0}{\sqrt{\Theta_0}} \sqrt{\Theta_0} \, d\rho = \frac{1}{12} \int_{\rho_t}^{\infty} \frac{d}{dt} \left( \frac{\partial^2 \Theta_0}{\partial \rho^2} \right) \, dt - \frac{1}{12} \int_{\rho_t}^{\infty} \frac{1}{\sqrt{\Theta}} \, d\rho \left( \frac{\Theta''}{\Theta'} \right) \, d\rho \\
+ \frac{4}{25} \left\{ \left[ \Theta^{-3/2} \frac{\partial \Theta_0}{\partial t} \right]_t^{\infty} - [\Theta^{-3/2} \Theta']_t^{\infty} \right\} - \frac{1}{12} \left\{ \left[ \frac{1}{\sqrt{\Theta}} \frac{\partial \Theta_0}{\partial t} \right]_t^{\infty} - \left[ \frac{1}{\sqrt{\Theta}} \Theta' \right]_t^{\infty} \right\}
\] (B.1.33)

Also we need to mention that we set \( \lim_{t \to \rho_t} \Theta_0 = 0 \), and this leads to

\[
\Theta_0(t_t) = 1 - \frac{l(l+1)}{t_t^2} = 0
\]  
\[
\Rightarrow t_t = \sqrt{l(l+1)} .
\] (B.1.34)

Under this condition, we see that \( t' \in [t_t, \infty) \) is real and finite. And accordingly we have

\[
\Theta_0(t) = 1 - \frac{t^2}{t^2_t}
\] (B.1.35)

Further, we now obtain the expression for \( \lim_{t \to t_t} t' \). To see this, we use L'Hôpital’s rule:

\[
\lim_{t \to t_t} t' = \lim_{\rho \to \rho_t} \sqrt{\Theta} \Theta_0
\] (B.1.36)

\[
= \lim_{\rho \to \rho_t} \frac{1}{2} \Theta^{-1/2} \Theta' + \frac{1}{2} \Theta^{-1/2} \frac{\partial \Theta_0}{\partial t} t',
\] (B.1.37)

and it follows that, for \( t = t_t \)

\[
\frac{t_t \Theta'_t}{2} \frac{1}{(t')^2} = t'.
\] (B.1.38)

Hence we obtain

\[
\lim_{t \to t_t} t' = \left( \frac{1}{2} t_t \Theta'_t \right)^{1/3}.
\] (B.1.39)

where we define \( \Theta'_t \) by

\[
\lim_{\rho \to \rho_t} \Theta' \overset{\text{def}}{=} \Theta'_t.
\] (B.1.40)
Moreover, we need to see the behaviours of $\Theta_0$ and $\Theta$ near $t = t_t, \rho = \rho_t$, respectively. For $t \approx t_t, \rho \approx \rho_t$, we have
\begin{align}
\Theta &= (t')^2 \Theta_0 \\
&\approx \left(\frac{t_t \Theta'_t}{2}\right)^{2/3} \Theta_0 .
\end{align}

Accordingly making use of the fact that $\Theta_0$ is very small near the region, we have
\begin{align}
\Theta' &\approx \left(\frac{t_t \Theta'_t}{2}\right)^{2/3} \frac{\partial \Theta_0}{\partial t} t' \\
&\approx \left(\frac{t_t \Theta'_t}{2}\right)^{3/3} \frac{\partial \Theta_0}{\partial t} \\
\text{and}
\Theta'' &\approx \left(\frac{t_t \Theta'_t}{2}\right)^{4/3} \frac{\partial^2 \Theta_0}{\partial t^2} .
\end{align}

Therefore it follows that, for example
\begin{align}
\frac{1}{\Theta_0^{3/2}} \frac{\partial \Theta_0}{\partial t} - \frac{\Theta'}{\Theta^{3/2}} &\approx \frac{1}{\Theta_0^{3/2}} \frac{\partial \Theta_0}{\partial t} - \left(\frac{t_t \Theta'_t}{2}\right)^{3/3} \frac{\partial \Theta_0}{\partial t} \\
&\to 0 ,
\end{align}

for $\rho \to \rho_t$. Similarly, as $\rho \to \rho_t$, we have
\begin{align}
\frac{\partial^2 \Theta_0}{\partial t^2} / \Theta_0^{1/2} - \Theta'' / \Theta^{1/2} &\approx \frac{\partial^2 \Theta_0}{\partial t^2} / \Theta_0^{1/2} - \frac{\partial^2 \Theta_0}{\partial t^2} / \Theta_0^{1/2} \\
&\to 0 .
\end{align}

Thus, it turns out that the integrated part in (B.1.33) vanishes, and we have
\begin{align}
\int_{\rho_t}^{\infty} \frac{\alpha''_0}{\sqrt{\Theta_0'}} d\rho &= \frac{1}{12} \int_{t_t}^{\infty} \frac{1}{\sqrt{\Theta_0}} \frac{d}{dt} \left(\frac{\partial^2 \Theta_0}{\partial t^2} / \Theta_0^{1/2}\right) dt - \frac{1}{12} \int_{\rho_t}^{\infty} \frac{1}{\sqrt{\Theta}} \frac{d}{d\rho} \left(\Theta'' / \Theta\right) d\rho .
\end{align}

Since $\lim_{\rho \to \infty} \Theta_0 t' = 1$, we substitute these things into (B.1.17):
\begin{align}
2h\beta_1(\infty) &= \lim_{\rho \to \infty} \frac{1}{\sqrt{\Theta_0 t'}} \int_{\rho_t}^{\rho} \frac{\alpha''_0}{\sqrt{\Theta_0'}} d\rho \\
&= \frac{1}{12} \int_{t_t}^{\infty} \frac{1}{\sqrt{\Theta_0}} \frac{d}{dt} \left(\Theta'' / \Theta\right) dt - \frac{1}{12} \int_{\rho_t}^{\infty} \frac{1}{\sqrt{\Theta}} \frac{d}{d\rho} \left(\Theta'' / \Theta\right) d\rho .
\end{align}

The $h\beta_1$ is the first order correction to the WKB approximation.
B.2 WKB from the RY perspective

In this supplementary section, we view the WKB method from the RY perspective and obtain the WKB phase shift, making use of the equations which were obtained in the previous section. To do this, we set $\rho = e^\lambda$ and rewrite the radial wave function of the radial equation (B.1.1) as

$$\chi_{l,k}(\rho) = e^{\frac{\lambda}{2}} u_{l,k}(\lambda),$$

(B.2.1)

where $\lambda \in (-\infty, \infty)$. Accordingly the radial perturbed $u_{l,k}(\lambda)$ and unperturbed solutions $u_{l,k,0}(\Lambda)$ satisfy the following equations:

$$\frac{\partial^2 u_{l,k}}{\partial \lambda^2} + \Theta^{(RY)} u_{l,k} = 0,$$

(B.2.2)

$$\frac{\partial^2 u_{l,k,0}}{\partial \Lambda^2} + \Theta_0^{(RY)} u_{l,k,0} = 0,$$

(B.2.3)

where

$$\Theta^{(RY)} = e^{2\lambda} \left(1 - \frac{2\mu U}{\hbar^2 k^2}\right) - (l + 1/2)^2,$$

(B.2.4)

$$\Theta_0^{(RY)} = e^{2\Lambda} - (l + 1/2)^2.$$  

(B.2.5)

Here $\Lambda$ is the function of $\lambda$: $\Lambda = \Lambda(\lambda)$.

Then we use the same procedure as the previous section:

$$u_{l,k} = \alpha^{(RY)} u_{l,k,0} + \beta^{(RY)} \frac{\partial u_{l,k,0}}{\partial \Lambda},$$

(B.2.6)

$$\Theta^{(RY)} \left(\frac{\partial \Lambda}{\partial \lambda}\right)^2 = \Theta^{(RY)}.$$  

(B.2.7)

In a similar fashion, we have

$$\alpha_0^{(RY)} = \sqrt{\frac{1}{\frac{\partial \Lambda}{\partial \lambda}}},$$

(B.2.8)

$$\hbar \beta_1^{(RY)}(\lambda) = \frac{1}{2\sqrt{\Theta_0^{(RY)} \frac{\partial \Lambda}{\partial \lambda}}} \int_{\lambda_a}^{\lambda_t} \int_{\lambda_a}^{\lambda_t} \frac{\partial^2 \alpha_0^{(RY)}}{\partial \lambda^2} \frac{\partial \Lambda}{\partial \lambda} d\lambda,$$

(B.2.9)

where the subscripts $a$ and $t$ of $\lambda_a$ and $\lambda_t$ signify that they are the values of the asymptotic region and the turning point, respectively. Thus we obtain the following equation:

$$\int_{\Lambda_a}^{\Lambda_t} \sqrt{\Theta_0^{(RY)}} d\Lambda = \int_{\lambda_a}^{\lambda_t} \sqrt{\Theta^{(RY)}} d\lambda.$$  

(B.2.10)
and we reexpress its RHS, again using \( \rho = e^\lambda \), as:

\[
\int_{\lambda_i}^{\lambda_f} \sqrt{\Theta^{(RY)}} d\lambda = \int_{\rho_i}^{\rho_f} \sqrt{1 - \frac{2\mu U}{\hbar^2 k^2} - \frac{(l + 1/2)^2}{\rho^2}} d\rho .
\]  

(B.2.11)

Likewise for the LHS, putting \( \rho_0 = e^\Lambda \), we have

\[
\int_{\Lambda_i}^{\Lambda_f} \sqrt{\Theta_0^{(RY)}} d\Lambda = \int_{\rho_{0,i}}^{\rho_{0,f}} \sqrt{1 - \frac{(l + 1/2)^2}{\rho_0^2}} d\rho_0 .
\]  

(B.2.12)

Then we put \( \cos \theta = \frac{(l + \frac{1}{2})}{\rho_0} \), and set \( \rho_{0,t} = (l + \frac{1}{2}) \). Then we have

\[
\int_{\Lambda_i}^{\Lambda_f} \sqrt{\Theta_0^{(RY)}} d\Lambda = \sqrt{\rho_{0,a} - (l + 1/2)^2 - \theta(l + 1/2)} .
\]  

(B.2.13)

For large \( \rho_{0,a} \), we have

\[
\int_{\Lambda_i}^{\Lambda_f} \sqrt{\Theta_0^{(RY)}} d\Lambda \simeq \rho_{0,a} - \frac{\pi}{2} \left( l + \frac{1}{2} \right) .
\]  

(B.2.14)

Substituting (B.2.10) and (B.2.11) into (B.2.14) yields

\[
\int_{\rho_i}^{\rho_f} \sqrt{1 - \frac{2\mu U}{\hbar^2 k^2} - \frac{(l + 1/2)^2}{\rho^2}} d\rho \simeq \rho_{0,a} - \frac{\pi}{2} \left( l + \frac{1}{2} \right) .
\]  

(B.2.15)

\( \chi_{l,k}(\rho) \) is the solution of the radial equation (the perturbed equation) (B.1.1), and we approximated the solution by the solution of the unperturbed equation. Therefore we can have the following relation:

\[
\chi_{l,k}(\rho) \propto \rho_0 j_l(\rho_0) .
\]  

(B.2.16)

As we used the relation (B.2.1) and \( \rho_0 = e^\Lambda \), for asymptotic regions we have

\[
u_{l,k}(\lambda_a) \propto \sqrt{e^\Lambda} j_l(e^\Lambda) ,
\]  

(B.2.17)

and so we have

\[
u_{l,k}(\lambda_a) \propto \sin \left( e^\Lambda_a - \frac{\pi l}{2} \right) = \sin \left( \rho_{0,a} - \frac{\pi l}{2} \right) .
\]  

(B.2.18)

Then we use (B.2.15) to see that its asymptotic expression should become

\[
u_{l,k}(\lambda_a) \propto \sin \left( \int_{\rho_i}^{\rho_f} \sqrt{1 - \frac{2\mu U}{\hbar^2 k^2} - \frac{(l + 1/2)^2}{\rho^2}} d\rho + \frac{\pi}{2} \left( l + \frac{1}{2} \right) - \frac{\pi l}{2} \right) .
\]  

(B.2.19)
B.3. RY PHASE SHIFT

As usual, the asymptotic expression for the solution of the radial equation is

\[ u_{l,k}(\lambda_a) \propto \sin \left( e^{\lambda_a} - \frac{\pi l}{2} + \delta_l \right) = \sin \left( \rho_a - \frac{\pi l}{2} + \delta_l \right). \]  

(B.2.20)

Comparing (B.2.19) with (B.2.20), we have

\[ \delta^{\text{wkb}}_l = \int_{\rho_t}^{\rho_a} \sqrt{1 - \frac{2\mu U}{\hbar^2 k^2} - \frac{(l + 1/2)^2}{\rho^2}} d\rho - \int_{\rho_0}^{\rho_t} \sqrt{1 - \frac{(l + 1/2)^2}{\rho^2}} d\rho_0. \]  

(B.2.21)

Thus we obtain the WKB phase shift from the RY perspective:

\[ \delta^{\text{wkb}}_l = \int_{\rho_t}^{\rho_a} \sqrt{1 - \frac{2\mu U}{\hbar^2 k^2} - \frac{(l + 1/2)^2}{\rho^2}} d\rho - \int_{\rho_0}^{\rho_t} \sqrt{1 - \frac{(l + 1/2)^2}{\rho^2}} d\rho_0. \]  

(B.2.22)

B.3 RY phase shift

In the previous section we obtained the lowest order phase shift \( \delta^{\text{wkb}}_l \). In this supplementary section, we derive the formula for the RY phase shift. In a previous section, we obtained a general expression (B.1.52) for the RY term. From (B.2.4) and (B.2.5), we have

\[ \Theta^{(\text{RY})}(\lambda) = e^{2\Lambda} \left( 1 - \frac{2\mu U}{\hbar^2 k^2} \right) - (l + 1/2)^2, \]  

(B.3.1)

\[ \Theta_0^{(\text{RY})}(\Lambda) = e^{2\Lambda} - (l + 1/2)^2, \]  

(B.3.2)

Immediately we see that the first and second derivatives of \( \Theta_0^{(\text{RY})} \) are:

\[ \frac{\partial \Theta_0^{(\text{RY})}}{\partial \Lambda} = 2e^{2\Lambda}, \quad \frac{\partial^2 \Theta_0^{(\text{RY})}}{\partial \Lambda^2} = 4e^{2\Lambda}. \]  

(B.3.3)

Thus we substitute them into (B.1.52), which yields

\[ 2\hbar \beta_1^{(\text{RY})} = -\frac{1}{12} \int_{\lambda_0}^{\lambda_t} \frac{1}{\sqrt{\Theta^{(\text{RY})}}} \frac{\partial}{\partial \lambda} \left( \frac{\partial^2 \Theta^{(\text{RY})}}{\partial \lambda^2} \right) d\lambda. \]  

(B.3.4)

Thus we derived the RY phase shift.

Now we see that \( \hbar \beta_1^{(\text{RY})} \) is the RY phase shift. Indeed, in the previous section, we set

\[ u_{l,k} = \alpha_{l,k,0}^{(\text{RY})} u_{l,k,0} + \beta^{(\text{RY})} \frac{\partial u_{l,k,0}}{\partial \Lambda}, \]  

(B.3.5)

\[ \simeq \alpha_0^{(\text{RY})} u_{l,k,0} + \hbar \beta_1^{(\text{RY})} \frac{\partial u_{l,k,0}}{\partial \Lambda}. \]  

(B.3.6)
APPENDIX B. ROSEN-YENNIE EXTENSION

Its asymptotic expression should be

\[ u_{l,k} \propto \alpha_0^{(RY)} \sin \left( \Lambda - \frac{\pi l}{2} \right) + \hbar \beta_1^{(RY)} \cos \left( \Lambda - \frac{\pi l}{2} \right), \]  

(B.3.7)

\[ = \sqrt{\left( \alpha_0^{(RY)} \right)^2 + (\hbar \beta_1^{(RY)})^2} \sin \left( \Lambda - \frac{\pi l}{2} + \delta_l^{(RY)} \right), \]  

(B.3.8)

where

\[ \sin \delta_l^{(RY)} = \frac{\hbar \beta_1^{(RY)}}{\sqrt{\left( \alpha_0^{(RY)} \right)^2 + (\hbar \beta_1^{(RY)})^2}}. \]  

(B.3.9)

Since \( \alpha_0^{(RY)} \) should go to unity, and \( \hbar \beta_1^{(RY)} \) is small, approximately we have

\[ \delta_l^{(RY)} \simeq \sin \delta_l^{(RY)} \simeq \hbar \beta_1^{(RY)}. \]  

(B.3.10)

Thus we found that \( \hbar \beta_1^{(RY)} \) is the RY phase shift.
C.1 C coefficients for E1 transition distribution

In this supplementary section, we express the coefficient $C_{L'x'\nu'Lx\nu}$ explicitly. To see this, we use the Jacobi polynomials which involve HH. For the bound wave function of a Borromean nucleus, the Jacobi polynomials are expressed as [36, 106]:

$$ P_{n+\frac{1}{2},Lx+\frac{1}{2}}^{Lx+1,\frac{1}{2}}(\cos 2\alpha) = \frac{1}{2^n} \sum_{\nu=0}^{n} \frac{\Gamma(n+Lx+3/2)\Gamma(n+Ly+3/2)(\cos 2\alpha - 1)^{n-\nu}(\cos 2\alpha + 1)^{\nu} \Gamma(n-Ly + \nu + 3/2)}{\Gamma(n+1)\Gamma(n+Lx-\nu + 3/2)\Gamma(n-\nu + 1)\Gamma(Ly + \nu + 3/2)}, $$

and similarly for those related to the continuum wave function:

$$ P_{n'+\frac{1}{2},Lx'+\frac{1}{2}}^{L'x'+1,\frac{1}{2}}(\cos 2\alpha) = \frac{1}{2^{n'}} \sum_{\nu'=0}^{n'} \frac{\Gamma(n'+L'_x+3/2)\Gamma(n'+L'_y+3/2)(\cos 2\alpha - 1)^{n'-\nu'}(\cos 2\alpha + 1)^{\nu'} \Gamma(n-L'_y + \nu' + 3/2)}{\Gamma(n'+1)\Gamma(n'+L'_x-\nu' + 3/2)\Gamma(n'-\nu' + 1)\Gamma(L'_y + \nu' + 3/2)}, $$

The coefficient involves the above 2 Jacobi polynomials and $\cos \alpha$, $\sin \alpha$, and therefore
the integration part \( \tilde{C}^{\nu' \nu}_{L_x, L_y, n' L_{x_y}} \) of (5.2.4) becomes

\[
\tilde{C}^{\nu' \nu}_{L_x, L_y, n' L_{x_y}} = \int_0^{\frac{\pi}{2}} (\sin \alpha)^{2(n' - \nu')} (\cos \alpha)^{2\nu'} (\cos \alpha)^{2(n - \nu)} (\cos \alpha)^{2\nu} \times (\sin \alpha)^{L_x' + 1 + \nu} (\cos \alpha)^{L_y' + 1 + \nu} (\sin \alpha)^2 (\cos \alpha)^2 d\alpha.
\]

Then we use the beta function to express it as:

\[
\tilde{C}^{\nu' \nu}_{L_x, L_y, n' L_{x_y}} = \int_0^{\frac{\pi}{2}} (\cos \alpha)^{L_x' + 1 + \nu + 3 + 2\nu' + 2\nu} (\sin \alpha)^{L_y' + 1 + \nu + 2(n' - \nu') + 2(n - \nu)} d\alpha
\]

\[
= \frac{1}{2} B \left( \frac{L_y' + L_{y_x} + 4 + 2\nu' + 2\nu}{2}, \frac{L_x' + L_{x_y} + 3 + 2(n' - \nu') + 2(n - \nu)}{2} \right)
\]

\[
= \frac{\Gamma \left( \frac{L_y' + L_{y_x} + 4 + 2\nu' + 2\nu}{2} \right)}{\Gamma \left( \frac{L_x' + L_{x_y} + 3 + 2(n' - \nu') + 2(n - \nu)}{2} \right)} \times \frac{\Gamma \left( \frac{L_x' + L_{x_y} + 7 + 2n' + 2n + L_x' + L_{x_y}}{2} \right)}{\Gamma \left( \frac{L_y' + L_{y_x} + 7 + 2n' + 2n + L_x' + L_{x_y}}{2} \right)}.
\]

Thus the coefficient \( C_{L_x, L_y, n' L_{x_y}} \) is rewritten as

\[
C_{L_x, L_y, n' L_{x_y}} = N_n^{L_x + \frac{1}{2}, L_y + \frac{1}{2}, n + \frac{1}{2}, L_{x_y} + \frac{1}{2}}
\]

\[
\times \left( \sum_{\nu = 0}^{n'} \frac{\Gamma(n' + L_x' + 3/2)\Gamma(n' + L_y' + 3/2)(-1)^{n' - \nu'}}{\Gamma(n' + 1)\Gamma(n' + L_x' - \nu' + 3/2)\Gamma(n' - \nu' + 1)\Gamma(L_y' + \nu' + 3/2)} \right)
\]

\[
\times \left( \sum_{\nu = 0}^{n} \frac{\Gamma(n + L_x + 3/2)\Gamma(n + L_y + 3/2)(-1)^{n - \nu}}{\Gamma(n + 1)\Gamma(n + L_x - \nu + 3/2)\Gamma(n - \nu + 1)\Gamma(L_y + \nu + 3/2)} \right)
\]

\[
\times \tilde{C}^{\nu' \nu}_{L_x, L_y, n' L_{x_y}}.
\]

(C.1.2)

and we can see that the coefficient \( C_{L_x, L_y, n' L_{x_y}} \) is a constant. The normalisation constant \( N_n^{L_x + \frac{1}{2}, L_y + \frac{1}{2}} \) of the Jacobi polynomial is written as

\[
N_n^{L_x + \frac{1}{2}, L_y + \frac{1}{2}} = \sqrt{2(K + 2)} \sqrt{\frac{\Gamma(n + L_x + L_y + 2)\Gamma(n + 1)}{\Gamma(n + L_x + 3/2)\Gamma(n + L_y + 3/2)}}.
\]

(C.1.3)

C.2 Derivation of \( \mathcal{M}_\mu^\Xi \)

In this supplementary section, we derive \( \mathcal{M}_\mu^\Xi \). Begin by the following expression
C.2. DERIVATION OF $\mathcal{M}_\mu \tilde{\xi}$

for the matrix element $\mathcal{M}_\mu \tilde{\xi}$:

$$
\mathcal{M}_\mu \tilde{\xi} = i^{K'} e Z e \sqrt{2 \frac{2}{A_{\text{proj}} A_c}} \sum_{k L L_y M_{L_x} M_{L_y} M_{L_z}} \sum_{L_y} \int_0^\infty \frac{J_{K' + 2}(k \rho)}{(k \rho)^{5/2}} \frac{L_{L_y} \Lambda_y}{\rho^{5/2}} \rho^5 d \rho
\times C_{L_y L'_y n' L_y n} \\
\times (-1)^{M_{L_y}} \sqrt{2L + 1} \frac{1}{4\pi} \left( \begin{array}{ccc} L_y' & 1 & L_y \\ -M_{L_y}' & \mu & M_{L_y} \end{array} \right) \left( \begin{array}{ccc} L_x' & L_y' & L' \\ M_{L_x}' & M_{L_y}' & -M_{L_y}' \end{array} \right) \\
\times (-1)^{L_y - L_y' + M_{L_y} \sqrt{2L + 1} + 1} \left( \begin{array}{ccc} L_x' & L_y' & L' \\ M_{L_x}' & M_{L_y}' & -M_{L_y}' \end{array} \right) \\
\times (-1)^{L_y - L_y' + M_{L_y}} \delta_{LS} \delta_{M_L - M_{L_y}} \delta_{S's'} \delta_{M_{S'}M_S}.
$$

(C.2.1)

As to some of the 3j-symbols of the above equation, we arrange them as

$$
\left( \begin{array}{ccc} L_y' & 1 & L_y \\ -M_{L_y}' & \mu & M_{L_y} \end{array} \right) \left( \begin{array}{ccc} L_x' & L_y' & L' \\ M_{L_x}' & M_{L_y}' & -M_{L_y}' \end{array} \right) \left( \begin{array}{ccc} L_x' & L_y' & L \\ M_{L_x}' & M_{L_y} & -M_{L_y} \end{array} \right) \\
= (-1)^{L_y' + 1 + L_y + L_y' + L_x + L_y} \left( \begin{array}{ccc} L' & L_x' & L_y' \\ -M_{L_y'} & M_{L_x}' & M_{L_y} \end{array} \right) \left( \begin{array}{ccc} L_y' & 1 & L_y' \\ M_{L_y} & \mu & -M_{L_y}' \end{array} \right) \left( \begin{array}{ccc} L_y & L_x & L \\ M_{L_y} & M_{L_x}' & -M_{L_y} \end{array} \right).
$$

(C.2.2)

This relation implies that $M_{L_y} + M_{L_y'} = M_L$. And it follows that, as to the $(-1)$ factor, its exponent becomes

$$
The \text{exponent} = (L - L_y - L_y' + M_{L'} + M_{L_y'}) + (L + 1 + L_y' + L_x') \quad \text{(C.2.3)}
= -L_y + 1 + L_x' + M_{L'} + M_{L_y'}
M_{L_y} + M_{L_y'} = M_L \rightarrow = -L_y + 1 + L_x' + M_{L'} - M_L + (M_{L_y'} + M_{L_x'} + M_{L_y})
= L_y' + 1 + M_{L'} - M_L
+ (-L_y - L_y' + L_x' + M_{L_y} + M_{L_y'} + M_{L_y'}) \quad \text{(C.2.4)}
$$
Then, we use the following relation about Wigner symbols:

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
m_1 & m_2 & m_3
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
j_4 & j_5 & j_6
\end{pmatrix} = \sum_{m_4 m_5 m_6} P \times \begin{pmatrix}
  j_1 & j_5 & j_6 \\
m_1 & m_5 & -m_6
\end{pmatrix}
\times \begin{pmatrix}
  j_4 & j_2 & j_6 \\
-m_4 & m_2 & m_6
\end{pmatrix} \begin{pmatrix}
  j_4 & j_5 & j_3 \\
m_4 & -m_5 & m_3
\end{pmatrix}.
\]
\hspace{1cm} (C.2.5)

where \( P = (-1)^{j_4 + j_5 + j_6 + m_4 + m_5 + m_6} \). By virtue of this formula, we can have

\[
\begin{pmatrix}
  L' & 1 & L \\
L_y & L'_y & L'_y
\end{pmatrix}
\begin{pmatrix}
  L' & 1 & L \\
-M_{L'} & \mu & M_L
\end{pmatrix} = \sum_{M_{L'_y} - M_{L_y}} (-1)^{-L_y - L'_y + M_{L'_y} + M_{L_y} + M_{L'_y}}
\times \begin{pmatrix}
  L' & L'_y & L'_y \\
-M_{L'} & M_{L'_y} & M_{L'_y}
\end{pmatrix}
\times \begin{pmatrix}
  L_y & 1 & L'_y \\
M_{L_y} & \mu & -M_{L'_y}
\end{pmatrix} \begin{pmatrix}
  L_y & L'_x & L \\
M_{L_y} & M_{L'_x} & -M_L
\end{pmatrix}.
\]
\hspace{1cm} (C.2.6)

Eventually, we have

\[
M_{\mu}^e = i^{K_x} e Z e \sqrt{\frac{2}{A_{\text{proj}} A_e}} \sum_{KMLy} \int_0^\infty \frac{J_{K'_x+2}(k\rho)}{(k\rho)^{5/2}} \sqrt{k_1^x c_{KLMLy}} \rho^5 d\rho
\times C_{L'_x L'_y L_x L_y n}
\times \sqrt{\frac{(2L'_y + 1)(2L_y + 1)}{4\pi}} \begin{pmatrix}
L'_y & 1 & L_y \\
0 & 0 & 0
\end{pmatrix}
\times \frac{1}{\sqrt{2L + 1}} \times \sqrt{2L + 1} \times (\hat{1}_{LZ} \delta_{M_L - M_L'} \delta_{S'} S' \delta_{M_{S'} M_S})
\times (-1)^{L'_y - M_{L_L} + M_{L'_L} + 1} \begin{pmatrix}
L' & 1 & L \\
-M_{L'} & \mu & M_L
\end{pmatrix} \begin{pmatrix}
L' & 1 & L \\
L_y & L'_x & L'_y
\end{pmatrix}.
\]
\hspace{1cm} (C.2.7)
C.2. DERIVATION OF $\mathcal{M}_\mu^\Xi$

That is, we obtain the expression for the $\mathcal{M}_\mu^\Xi$:

$$
\mathcal{M}_\mu^\Xi = i^{K'} e Z e \sqrt{\frac{2}{A_{\text{proj}} A}} \sum_{KLM L_y} (-1)^{L'_y + \mu + 1} 
\times \left( \begin{array}{ccc}
L' & 1 & L \\
-M_{L'} & \mu & M_L \\
- \end{array} \right) \left( \begin{array}{ccc}
L' & 1 & L \\
L_y & L'_x & L'_y \\
0 & 0 & 0 \\
\end{array} \right) 
\times C_{L_x L'_x, n' L_x L_y} \sqrt{\frac{(2L'_y + 1)3(2L_y + 1)(2L' + 1)}{4\pi}} 
\times \delta_{LS} \delta_{M_{L'} M_S} \delta_{S' S} \delta_{M_{S'} M_S} 
\times \int_0^\infty \frac{J_{K'}(k \rho)}{(k \rho)^{5/2}} \sqrt{k \rho \rho^L_{KL}} \frac{\chi_{KL}}{\rho^{5/2}} \rho^5 d\rho . 
$$

(C.2.8)
In this supplementary chapter, we derive the formula for a breakup cross section for a 4-body system, and provide a formula which is relevant to the cross section. Also we build a 4-body breakup reaction model by using FSI.

In terms of the notation, here we denote a set of quantum numbers \( \{K, L, L_y, L_x\} \) by \( \eta \). Also, for notational convenience, we denote a set of quantum numbers \( \{K, L, S, L_y, L_x\} \) by \( \xi \).

\section{Integration in the momentum Jacobi coordinates}

In this appendix section, we give a proof of the formula (D.1.5). Starting by the definition of the wave number in the 3-body Jacobi coordinates which is seen in Ershov’s paper (eq. A4 and A6 in Ref. [32]), we write it as

\[ k_x = \sqrt{\frac{\mu_x}{m_n}} k \sin \alpha_k, \quad k_y = \sqrt{\frac{\mu_y}{m_n}} k \cos \alpha_k \tag{D.1.1} \]

and then we have

\[ E_y = \frac{\hbar^2 k^2}{2\mu_y} = \frac{\hbar^2 k^2}{2m_n} (\cos \alpha_k)^2 = E_{ex} (\cos \alpha_k)^2. \tag{D.1.2} \]

It follows that

\[ dE_y = \frac{E_{ex}}{2} d(\cos 2\alpha_k). \tag{D.1.3} \]
Now we prove the following formula:

\[ F_1 = \int_0^{E_{\text{ex}}} dE_y \sqrt{E_y(E_{\text{ex}} - E_y)} \int d\Omega_{x\gamma} \Omega_{y\gamma} \gamma_{K'L'}^{L_x L_y} (\Omega_{k\delta}) \gamma_{KL}^{L_x L_y} (\Omega_{k\delta}) \]  
\[ = 2E_{\text{ex}}^2 \delta_{K'K} \delta_{L'L} \delta_{Mx,My} \delta_{L_x L_y} \delta_{L'_x L'_y} . \]  
(D.1.4)

To see this, let us compute \( F_1 \); write down the integration:

\[ F_1 = \int_0^{E_{\text{ex}}} dE_y \sqrt{E_y(E_{\text{ex}} - E_y)} \int d\Omega_{x\gamma} d\Omega_{y\gamma} \gamma_{K'L'}^{L_x L_y} (\Omega_{k\delta}) \gamma_{KL}^{L_x L_y} (\Omega_{k\delta}) . \]  
(D.1.5)

Then the integration with regard to \( \Omega_{x\gamma}, \Omega_{y\gamma} \) yields:

\[ F_1 = \int_0^{E_{\text{ex}}} dE_y \sqrt{E_y(E_{\text{ex}} - E_y)} (\sin \alpha_k)^{2L_x} (\cos \alpha_k)^{2L_y} \sum_{M_{L_x},M_{L_y},M_{x,y}} \hat{L} \hat{L}' (-1)^{M_L + M_{L'}} \]
\[ \times \left( \begin{array}{ccc} L_x & L_y & L \\ M_{L_x} & M_{L_y} & -M_L \end{array} \right) \left( \begin{array}{ccc} L'_x & L'_y & L' \\ M'_{L_x} & M'_{L_y} & -M_{L'} \end{array} \right) \]
\[ \times J_{L_x L_y n}^{L'_x L'_y n'} \delta_{L_x L_x} \delta_{L_y L_y} \delta_{M_{L_x} M_{L_y}} \delta_{M_{L_x} M_{L_y}} , \]  
(D.1.7)

where the expression \( J_{L_x L_y n}^{L'_x L'_y n'} (\cos 2\alpha_k) \) is

\[ J_{L_x L_y n}^{L'_x L'_y n'} (\cos 2\alpha_k) = N_{n'}^{L'_x + \frac{1}{2}, L'_y + \frac{1}{2}} N_{n}^{L_x + \frac{1}{2}, L_y + \frac{1}{2}} P_{n'}^{L'_x + \frac{1}{2}, L'_y + \frac{1}{2}} (\cos 2\alpha_k) P_{n}^{L_x + \frac{1}{2}, L_y + \frac{1}{2}} (\cos 2\alpha_k) . \]  
(D.1.8)

Then it is summed over \( M_{L_x}, M_{L_y} \), and we have

\[ F_1 = \int_0^{E_{\text{ex}}} dE_y \sqrt{E_y(E_{\text{ex}} - E_y)} (\sin \alpha_k)^{2L_x} (\cos \alpha_k)^{2L_y} \sum_{M_{L_x},M_{L_y}} \hat{L} \hat{L}' (-1)^{M_L + M_{L'}} \]
\[ \times \left( \begin{array}{ccc} L_x & L_y & L \\ M_{L_x} & M_{L_y} & -M_L \end{array} \right) \left( \begin{array}{ccc} L'_x & L'_y & L' \\ M'_{L_x} & M'_{L_y} & -M_{L'} \end{array} \right) \]
\[ \times J_{L_x L_y n}^{L'_x L'_y n'} \delta_{L_x L_x} \delta_{L'_y L'_y} \]  
(D.1.9)

Making use of the fact that \( L'_x = L_x \) and \( L'_y = L_y \) (because of the Kronecker deltas), we have

\[ F_1 = \int_0^{E_{\text{ex}}} dE_y \sqrt{E_y(E_{\text{ex}} - E_y)} (\sin \alpha_k)^{2L_x} (\cos \alpha_k)^{2L_y} \sum_{M_{L_x},M_{L_y}} \hat{L} \hat{L}' (-1)^{M_L + M_{L'}} \]
\[ \times \left( \begin{array}{ccc} L_x & L_y & L \\ M_{L_x} & M_{L_y} & -M_L \end{array} \right) \left( \begin{array}{ccc} L_x & L_y & L' \\ M_{L_x} & M_{L_y} & -M_{L'} \end{array} \right) \]
\[ \times J_{L_x L_y n}^{L'_x L'_y n'} \delta_{L'_x L_x} \delta_{L'_y L_y} \]  
(D.1.10)
APPENDIX D. BREAKUP CROSS SECTION

We deal with the 3j-symbols:

\[
F_1 = \int_0^{E_{\text{ex}}} dE_y \sqrt{E_y(E_{\text{ex}} - E_y)} (\sin \alpha_k) \cos^{2L_y} (\cos \alpha_k) J_{L_z}^{L_x} L_g n L_{n'} \delta_L^{L_z L_x} \delta_L^{L_g L_{n'}} \delta_M^{L_z L_x} \delta_M^{L_g L_{n'}} .
\]

(D.1.11)

Then, using (D.1.2), we have

\[
F_1 = \int_{-1}^{1} \frac{E_{\text{ex}}^4}{2} d(\cos 2\alpha_k) \sqrt{1 + \cos 2\alpha_k} \left( \frac{E_{\text{ex}} - E_{\text{ex}}}{2} \right)
\times (\sin \alpha_k) \cos^{2L_y} (\cos \alpha_k) J_{L_z}^{L_x} L_g n L_{n'} \delta_L^{L_z L_x} \delta_L^{L_g L_{n'}} \delta_M^{L_z L_x} \delta_M^{L_g L_{n'}} .
\]

(D.1.12)

\[
= \frac{E_{\text{ex}}^2}{2^{L_x + L_y + 2}} \int_{-1}^{1} \frac{d(\cos 2\alpha_k)}{2} (1 - \cos 2\alpha_k)^{L_x + \frac{1}{2}} (1 + \cos 2\alpha_k)^{L_y + \frac{1}{2}}
\times J_{L_z}^{L_x} L_g n L_{n'} \delta_L^{L_z L_x} \delta_L^{L_g L_{n'}} \delta_M^{L_z L_x} \delta_M^{L_g L_{n'}} .
\]

(D.1.13)

Then we use the orthogonality of the Jacobi polynomials: for \(-1 < \alpha, \beta\)

\[
\int_{-1}^{1} \omega(x) P_n^{\alpha, \beta}(x) P_{n'}^{\alpha, \beta}(x) dx = \delta_{nn'} h_n ,
\]

(D.1.14)

where \(\omega(x) = (1 - x)^\alpha (1 + x)^\beta\), and

\[
h_n = \frac{2^{\alpha + \beta + 1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{\Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)} .
\]

(D.1.15)

Thus using the orthogonality relation, we can proceed to

\[
F_1 = \frac{E_{\text{ex}}^2}{2^{L_x + L_y + 2}} \times \frac{2^{L_x + L_y + 2}}{2n + L_x + L_y + 2} \frac{\Gamma(n + L_x + 3/2) \Gamma(n + L_y + 3/2)}{\Gamma(n + 1) \Gamma(n + L_x + L_y + 2)}
\times \delta_{n'n'} N_n^{L_x + \frac{1}{2} L_y + \frac{1}{2}} N_{n'}^{L_x + \frac{1}{2} L_y + \frac{1}{2}} \delta_L^{L_z L_x} \delta_L^{L_g L_{n'}} \delta_M^{L_z L_x} \delta_M^{L_g L_{n'}} .
\]

(D.1.16)

\[
= 2E_{\text{ex}}^2 \delta_{K'K} \delta_L^{L_z L_x} \delta_L^{L_g L_{n'}} \delta_M^{L_z L_x} \delta_M^{L_g L_{n'}} .
\]

(D.1.17)

where we used the relation \(K = L_x + L_y + 2n\): if \(n' = n\), \(L'_y = L_y\) and \(L'_x = L_x\), then \(K' = K\). Thus we proved the formula (D.1.5). Note that the coefficient 2 in the RHS of (D.1.17) comes from the square of the normalisation factor \(N_n^{L_x + 1/2 L_y + 1/2}\). The normalisation factor is written as

\[
N_n^{L_x + \frac{1}{2} L_y + \frac{1}{2}} = \sqrt{2(K + 2)} \frac{\Gamma(n + L_x + L_y + 2) \Gamma(n + 1)}{\Gamma(n + L_x + 3/2) \Gamma(n + L_y + 3/2)} ,
\]

(D.1.18)

and let us note that this normalisation factor is for the normalisation of HH. Thus the formula in ([32] eq.18) is proved. Also we see that (D.1.5) suggests that the following useful formula is true as well:

\[
\int_{-1}^{1} d(\cos 2\alpha) (\sin \alpha)^{2L_x + 1} (\cos \alpha)^{2L_y + 1} J_{L_z}^{L_x} L_g n (\cos 2\alpha) = 4\delta_{n'n'} .
\]

(D.1.19)
D.2. Derivation of breakup cross section

Here we also note that the definition of HH is from [32] (eq. A8 and A10):

\[
\psi^{IaIy}_K (\alpha) = N_n^{Ia} \frac{1}{2} L_y^{Ia} \frac{1}{2} (\sin \alpha)^{Ia} (\cos \alpha)^{Iy} P_n^{Ia} \frac{1}{2} L_y^{Iy} \frac{1}{2} (\cos (2\alpha)) .
\]

Here we also note that the definition of HH is from [32] (eq. A8 and A10):

\[
Y^{IaIy}_{KLp} (\Omega_5) = \psi^{IaIy}_K (\alpha) \left[ Y_{Ia} (\Omega_x) \otimes Y_{Iy} (\Omega_y) \right]_{LM_L} \tag{D.2.20}
\]

This supplementary section presents how to derive the expression for the differential breakup cross section for a 4-body system without the final state interaction. For the purpose of deriving the double differential inclusive cross section, we integrate the 5-folded cross section with regard to \(E'_y, \Omega'_{\text{ex}}, \Omega'_{\text{gy}}\):

\[
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \int d\Omega_{5k} |T_{J'}^{5k} S_k^{LM}_k|^2 = |i\hbar|^2 \int d\Omega_{5k} \sum_{M_{J'}, M_{L_{J'}}, M_{x_{J'}}, M_{y_{J'}}, M_{z_{J'}}} (-1)^{L_{J'} + L_{y'} + M_{x_{J'}} + M_{y'}}
\]

\[
\times \hat{J'} \hat{J} Y_{K'L_{J'}M_{L_{J'}}}^{J_{L_{J'}}L_y y_{J'}} (\Omega_{5k}) Y_{K''L_{J'}M_{L_{J'}}}^{J_{L_{J'}}L_x x_{J'}} (\Omega_{5k})
\]

\[
\times \left( \begin{array}{ccc}
L_k'' & S_k'' & J' \\
M_{L_k''} & M_{S_k''} & -M_{J'}
\end{array} \right) \left( \begin{array}{ccc}
L_k' & S_k' & J' \\
M_{L_k'} & M_{S_k'} & -M_{J'}
\end{array} \right) \times \sum_{\lambda \lambda' \lambda_{M_{J'}} \lambda_{M_{J'}}} \lambda 
\]

\[
\times (-1)^{J' - M_{J'}} \left( \begin{array}{ccc}
J' & \lambda & J \\
-M_{J'} & M_{\lambda} & M_{J}
\end{array} \right) \right) U_{\lambda \lambda_{M_{J'}}}^{J', J'_{M_{J'}}} U_{\lambda_{M_{J'}}}^{J', J} ,
\]

(D.2.1)

where we denote \(\sqrt{E_y' (E_{\text{ex}} - E'_y)} dE'_y d\Omega'_{\text{ex}} d\Omega'_{\text{gy}}\) by \(d\Omega_{5k}\). As to \(U_{\lambda M_{\lambda_{J}}}^{J, J'}\), we write this as

\[
U_{\lambda M_{\lambda_{J}}}^{J, J'} (q) = 2 \pi \int |M_{\lambda_{J}}| R_{bl} d\Omega_{5k} (q R_{bl}) \sum \langle \xi' J' || \Gamma_{\lambda M_{\lambda_{J}}} || \xi J \rangle \Delta_{\xi' \xi}^{\lambda M_{\lambda_{J}}} ,
\]

(D.2.2)

where \(\Gamma_{\lambda M_{\lambda_{J}}} = [Y_{\lambda x} \otimes Y_{\lambda y}]_{\lambda M_{\lambda_{J}}}\). Here \(\Delta_{\xi' \xi}^{\lambda M_{\lambda_{J}}} \) is given as

\[
\Delta_{\xi' \xi}^{\lambda M_{\lambda_{J}}} = \int_{-1}^1 d \cos (2\alpha) \frac{1}{8} (\sin 2\alpha) N_n^{L_x} \frac{1}{2} L_x^{L_x} \frac{1}{2} P_n^{L_y} \frac{1}{2} L_y^{L_y} \frac{1}{2} N_n^{L_y} \frac{1}{2} L_y^{L_y} \frac{1}{2} P_n^{L_x} \frac{1}{2} L_x^{L_x} \frac{1}{2} \int_0^{+\infty} \frac{1}{k_2''} \chi_{\xi' \xi}^{\lambda M_{\lambda_{J}}} p \Delta_{\lambda \lambda_{J}}^{\lambda M_{\lambda_{J}}} dp ,
\]

(D.2.3)

where

\[
\Delta_{\lambda \lambda_{J}}^{\lambda M_{\lambda_{J}}} = \int \Gamma_{\lambda M_{\lambda_{J}}} \chi_{\lambda \lambda_{J}}^{\chi_{\xi' \xi} \lambda \lambda_{J}} d\Omega_x d\Omega_y .
\]

(D.2.4)
Next we consider the integration of the above equation with regard to \( \Omega_{5k} \). The related formula (D.1.5) has been already demonstrated. That is,

\[
\int_{0}^{E_{\text{ex}}} dE_{\text{y}} \sqrt{E_{\text{y}}(E_{\text{ex}} - E_{\text{y}})} \int d\Omega_{kz} d\Omega_{k\gamma} \gamma^*(\Omega'_{45})\gamma(\Omega'_{45}) = 2E_{\text{ex}}^2 \delta_{K'K} \delta_{L'L} \delta_{M_L'M_L} \delta_{L'_L'L_L} \delta_{L'_L'L_L}.
\]

\[ \text{(D.2.5)} \]

This formula enables us to integrate with regard to \( \Omega_{5k} \), and take its summation over \( K'', L''_k, M_{L''_k} \), and \( L''_y, L''_y' \):

\[
\left( \frac{A_{\text{proj}}}{A_{\text{c}}} \right)^{-3/2} \int d\Omega_{5k} |T_{J'}^{S_{L''_k}M_{L''_k}}|^2 = \frac{|i\hbar v|^2}{2E_{\text{ex}}^2} \sum_{M_J'} \sum_{M_{L''_k}} (-1)^{M_J' + M_{L''_k}} \times \hat{J}^* \hat{J}^T \left( \begin{array}{ccc}
L_k & S_{L''_k} & J' \\
M_{L''_k} & M_{L''_k} & -M_{J'}
\end{array} \right) \left( \begin{array}{ccc}
L_k & S_{L''_k} & J' \\
M_{L''_k} & M_{L''_k} & -M_{J'}
\end{array} \right) \\
\times \sum_{\lambda \lambda' M_{\lambda} M_{\lambda'}} (-1)^{J' - M_{J'}} \left( \begin{array}{ccc}
J' & \lambda & J \\
-M_{J'} & M_{\lambda'} & M_{\lambda}
\end{array} \right) \times (-1)^{J' - M_{J'}} \left( \begin{array}{ccc}
J' & \lambda & J \\
-M_{J'} & M_{\lambda} & M_{\lambda}
\end{array} \right) U_{J' \lambda \lambda'} U_{\lambda \lambda' J'}.
\]

\[ \text{(D.2.6)} \]

Then we find that, thanks to the Wigner 3-j’s property, \( L_k' + S_k = M_{J'} \) and \( L_k' + S_{L''_k} = M_{J'}' \), and so we have \( M_{J'}' = M_{J'} \):

\[
\left( \frac{A_{\text{proj}}}{A_{\text{c}}} \right)^{-3/2} \int d\Omega_{5k} |T_{J'}^{S_{L''_k}M_{L''_k}}|^2 = \frac{|i\hbar v|^2}{2E_{\text{ex}}^2} \sum_{M_{J'}} \sum_{M_{L''_k}} (-1)^{M_{J'} + M_{L''_k}} \times \hat{J}^* \hat{J}^T \left( \begin{array}{ccc}
L_k & S_{L''_k} & J' \\
M_{L''_k} & M_{L''_k} & -M_{J'}
\end{array} \right) \left( \begin{array}{ccc}
L_k & S_{L''_k} & J' \\
M_{L''_k} & M_{L''_k} & -M_{J'}
\end{array} \right) \\
\times \sum_{\lambda \lambda' M_{\lambda} M_{\lambda'}} (-1)^{J' - M_{J'}' \lambda} \left( \begin{array}{ccc}
J' & \lambda & J \\
-M_{J'} & M_{\lambda'} & M_{\lambda}
\end{array} \right) \times (-1)^{J' - M_{J'}'} \left( \begin{array}{ccc}
J' & \lambda & J \\
-M_{J'} & M_{\lambda} & M_{\lambda}
\end{array} \right) U_{J' \lambda \lambda'} U_{\lambda \lambda' J'}.
\]

\[ \text{(D.2.7)} \]

After rearranging the Wigner 3-j symbols which involve \( \lambda, \lambda' \), we take the summation
over $M_J$ and $M'_J$:

$$
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \int d\Omega_{5k} |T_{J'k}^{S'kM'k}|^2 = |i\hbar v|^2 2E_{\text{ex}}^2 \sum_{M_J M'_J} \sum_{M_k M'_k} (-1)^{M_J+M'_J} \\
\times \sum_{\lambda M_k \lambda M'_k} \frac{\delta_{\lambda M_k \lambda M'_k}}{(2\lambda'+1)(2\lambda+1)} U_{\lambda M_k}^{J'J} U_{\lambda M_k}^{J'J} .
$$

(D.2.8)

Accordingly the summation over $\lambda', M_{\lambda'}$ yields:

$$
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \int d\Omega_{5k} |T_{J'k}^{S'kM'k}|^2 = |i\hbar v|^2 2E_{\text{ex}}^2 \sum_{M_J M'_J} \sum_{M_k M'_k} (-1)^{M_J+M'_J} \\
\times \sum_{\lambda M_k \lambda M'_k} \frac{1}{2\lambda+1} U_{\lambda M_k}^{J'J} U_{\lambda M_k}^{J'J} .
$$

(D.2.9)

Then we take the summation over $M_{L_k}', M_{S_k}'$:

$$
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \sum_{M_{S_k}} \int d\Omega_{5k} |T_{J'k}^{S'kM'k}|^2 = |i\hbar v|^2 2E_{\text{ex}}^2 \sum_{M_{S_k}} \sum_{M_{S_k} M_{L_k} M'_{L_k}} (-1)^{M_{S_k}+M'_{L_k}} \\
\times \sum_{\lambda M_k \lambda M'_k} \frac{1}{2\lambda+1} U_{\lambda M_k}^{J'J} U_{\lambda M_k}^{J'J} .
$$

(D.2.10)

and it follows that

$$
\sum_{M_{S_k}} \int d\Omega_{5k} |T_{J'k}^{S'kM'k}|^2 = \left( \frac{A_{\text{proj}}}{A_c} \right)^{3/2} |i\hbar v|^2 2E_{\text{ex}}^2 \sum_{M_{S_k}} \sum_{\lambda M_k \lambda M'_k} \frac{1}{2\lambda+1} U_{\lambda M_k}^{J'J} U_{\lambda M_k}^{J'J} .
$$

(D.2.11)

As a result, by using the formula (D.2.11), we can write the exclusive differential breakup cross section as follows:

$$
d^3\sigma = \int_0^{E_{\text{ex}}} dE' \int d\Omega' \int d\Omega' \int d\Omega' \int d\Omega' \int d\Omega' R_{\text{ex}} \frac{d^5\sigma}{d\Omega'} = (2\pi)^4 \frac{\tilde{\rho}}{2J+1} \int d\Omega_{5k} \sum_{\alpha'J'S'kM'S'k} |T_{J'k}^{S'kM'k}|^2 ,
$$

(D.2.12)
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where the reduced phase shift factor is

\[ \tilde{\rho} = \frac{2 \mu R p_R'}{h^3} \cdot \frac{m_n^3}{h^6} \left( \frac{A_c}{A_{\text{proj}}} \right)^{\frac{3}{2}}. \]  

\[ \text{(D.2.13)} \]

We proceed to calculate to have

\[ \frac{d^2 \sigma}{d \Omega_{kR}^\prime dE_{ex}} = \left( \frac{A_c}{A_{\text{proj}}} \right)^{\frac{3}{2}} \left( \frac{2 \pi}{h^3} \right)^{\frac{3}{2}} \frac{\mu R k_R^2}{h^2} \cdot \left( \frac{m_n^2}{h^4} \right) \frac{1}{2J+1} \sum_{\alpha, J', M_{j', S_k'}} \int d\Omega_{5k} |T_{j'} S_k' M_{j', S_k'}|^2, \]

\[ = \frac{(2\pi)^4}{h v} \frac{\mu R k_R^2}{h^2} \cdot \left( \frac{m_n^2}{h^4} \right) \frac{1}{2J+1} |i h v|^2 \sum_{\alpha, J', \lambda M_{\lambda A}} \frac{1}{2\lambda + 1} 2 E_{ex}^2 |U_{\lambda M_{\lambda A}}^{J, J'}|^2. \]

\[ \text{(D.2.14)} \]

Then we obtain the expression for the differential breakup cross section:

\[ \frac{d^2 \sigma}{d \Omega_{kR}^\prime dE_{ex}} = (2\pi)^4 \left( \frac{m_n^2}{h^4} \right)^{3/2} \frac{k_R k_R'}{2(2J+1)} \sum_{\alpha, J', \lambda M_{\lambda A}} \frac{1}{2\lambda + 1} 2 E_{ex}^2 |U_{\lambda M_{\lambda A}}^{J, J'}|^2. \]

\[ \text{(D.2.15)} \]

D.3 Watson-Migdal type FSI

In this supplementary section, we derive the formula for the differential breakup cross section which involves the Watson-Migdal type final state interaction.

We begin by considering the square modulus of the T-matrix. We integrate this with regard to \( \Omega_{5k} \):

\[ \left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \int d\Omega_{5k} |T_{j'} S_k' M_{j', S_k'}|^2 = |i h v|^2 \int d\Omega_{5k} \sum_{\lambda', \lambda M_{\lambda A}} \sum_{J', M_{j', S_k'}} \sum_{M_{j', S_k'}} (-1)^{L_{\lambda}^\prime Q + L_k + M_{j', S_k'}} \]

\[ \times \tilde{J}' \tilde{Y}_{\lambda' M_{\lambda A} j'} M_{j', S_k'} \tilde{Y}_{\lambda' M_{\lambda A} k} (\Omega_{5k}) \tilde{Y}_{\lambda' M_{\lambda A} k} (\Omega_{5k}) \]

\[ \times \left( \begin{array}{ccc} L_k' & S_k' & J' \\ M_{L_k'} & M_{S_k'} & -M_{J'} \end{array} \right) \left( \begin{array}{ccc} L_k' & S_k' & J' \\ M_{L_k'} & M_{S_k'} & -M_{J'} \end{array} \right) \]

\[ \times \sum_{ \lambda M_{\lambda A} M_{\lambda A} } (-1)^{J' - M_{j'}} \left( \begin{array}{ccc} J' & \lambda & J \\ -M_{j'} & \lambda & J \end{array} \right) \]

\[ \times (-1)^{J' - M_{j'}} \left( \begin{array}{ccc} J' & \lambda & J \\ -M_{j'} & \lambda & J \end{array} \right) \]

\[ \times (-1)^{J' - M_{j'}} U_{\lambda M_{\lambda A}}^{J', J'} U_{\lambda M_{\lambda A}}^{J, J'}, \]

\[ \text{(D.3.1)} \]

where, again, we denote \( \sqrt{E_y' (E_{ex} - E_y') dE_y' d\Omega_{kx} d\Omega_{ky}} \) by \( d\Omega_{5k} \).
Integration with regard to \( d\Omega_{kx} d\Omega_{ky} \) gives Kronecker deltas in terms of \( L_k, M_{L_k} \) and \( L_{2k}, L_{yk}, \) and so we see that

\[
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \int d\Omega_{5k} |T_{J'} J' L' S'_k\rangle |M_{L_k} S_k\rangle^2 = |i\hbar v|^2 \int \sqrt{E_g'(E_{\text{ex}} - E_g')} dE_g' \sum_{M_{L_k}} \sum_{\eta_{L_k}} \left( -1 \right)^{M_{L'} + M_{J'}} \\
\times \hat{J}' \hat{J}' (\sin \alpha_k)^{2L_{s_k}} (\cos \alpha_k)^{2L'_{s_k}} J' \left( \begin{array}{ccc}
J' & S'_k & J' \\
M'_{L_k} & M'_{S_k} & -M_J \\
\end{array} \right) \\
\times \sum_{\lambda \lambda' M_{\lambda} M_{\lambda'}} \left( -1 \right)^{J' - M_{J'}} \left( \begin{array}{ccc}
J' & \lambda & J \\
M_{J'} & M_{\lambda} & M_{J} \\
\end{array} \right) U_{\lambda \lambda M_{\lambda}} U_{\lambda' \lambda M_{\lambda'}}.
\]

(D.3.2)

Then the property of the 3-j symbol involving \( L'_{k}, S'_{k}, J' \) enables us to see that \( M'_{J'} = M_{J'} \). We take the summation over \( M_{J}, M'_{J'} \), and after that we take the summation over \( \lambda', M_{\lambda'} \) to see that

\[
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \int d\Omega_{5k} |T_{J'} J' L' S'_k\rangle |M_{L_k} S_k\rangle^2 = |i\hbar v|^2 \int \sqrt{E_g'(E_{\text{ex}} - E_g')} dE_g' \sum_{M_{L_k}} \sum_{\eta_{L_k}} \left( -1 \right)^{M_{L'} + M_{J'}} \\
\times \hat{J}' \hat{J}' (\sin \alpha_k)^{2L_{s_k}} (\cos \alpha_k)^{2L'_{s_k}} J' \left( \begin{array}{ccc}
J' & S'_k & J' \\
M'_{L_k} & M'_{S_k} & -M_J \\
\end{array} \right) \\
\times \sum_{\lambda M_{\lambda}} \frac{1}{2\lambda + 1} U_{\lambda M_{\lambda}} U_{\lambda M_{\lambda}'}.
\]

(D.3.3)

Then the summation over \( M'_{L_k}, M'_{S_k} \) yields:

\[
\left( \frac{A_{\text{proj}}}{A_c} \right)^{-3/2} \sum_{M_{S_k}} \int d\Omega_{5k} |T_{J'} J' L' S'_k\rangle |M_{L_k} S_k\rangle^2 = |i\hbar v|^2 \int \sqrt{E_g'(E_{\text{ex}} - E_g')} dE_g' \sum_{M_{L_k}} \sum_{\eta_{L_k}} \left( -1 \right)^{M_{L'} + M_{J'}} \\
\times (\sin \alpha_k)^{2L_{s_k}} (\cos \alpha_k)^{2L'_{s_k}} J' \left( \begin{array}{ccc}
J' & S'_k & J' \\
M'_{L_k} & M'_{S_k} & -M_J \\
\end{array} \right) \\
\times \sum_{\lambda M_{\lambda}} \frac{1}{2\lambda + 1} U_{\lambda M_{\lambda}} U_{\lambda M_{\lambda}'}.
\]

(D.3.4)
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The square modulus of $U_{\lambda M\lambda}^{J\ell J}(q)$ is replaced by that of $U_{\lambda M\lambda}^{J\ell J}(q, \cos 2\alpha_k)$:

$$|U_{\lambda M\lambda}^{J\ell J}|^2 = [2\pi]^2 \int R_d dR_b J_{|M\lambda|}(q R_b) \int R''_d dR''_b J_{|M\lambda|}(q R''_b)$$

$$\times \sum_{\xi' \xi \lambda_b \xi' \xi' \lambda' \lambda' \lambda''} |F|^2 D_{\lambda M\lambda \lambda\xi}^{J\ell J\xi'} \langle R_b \rangle D_{\lambda M\lambda \lambda' \lambda''}^{J\ell J\xi''} \langle R''_b \rangle,$$  \hspace{1cm} (D.3.5)

where

$$D_{\lambda M\lambda \lambda\xi}^{J\ell J\xi'} \langle R_b \rangle = \langle \xi' J' || \Gamma_{\lambda M\lambda} || \xi J \rangle \Delta_{\xi \xi}^{\lambda M\lambda \lambda\xi}.$$  \hspace{1cm} (D.3.6)

Accordingly we have:

$$\left( \frac{A_{\text{proj}}}{A_k} \right)^{-3/2} \sum_{M\xi'_{k'}} \int d\Omega_{5k} |T_{\lambda M\lambda}^{\xi\xi'}|^{2} = \left| \frac{E_{\text{ex}}}{2} \right|^2 \int d\Omega_{5k} \sum_{J M\xi'_{k'}} |T_{\lambda M\lambda}^{\xi\xi'}|^{2},$$

$$\times (\sin \alpha_k)^2 L_{b, k}^{L_{b, k}^{L_{b, k}}+1} \cos \angle \langle \xi, \xi' \rangle \sum_{\lambda M\lambda} \frac{1}{2 \lambda + 1} |U_{\lambda M\lambda}^{J\ell J}|^2.$$  \hspace{1cm} (D.3.7)

Then we proceed to have the expression for the exclusive differential cross section with the FSI:

$$\frac{d^2 \sigma}{d\Omega_{5k} dE_{\text{ex}}} = \frac{(2\pi)^4}{\hbar^3} \frac{m^3}{h^3} \left( \frac{A_{\text{proj}}}{A_k} \right)^{\frac{3}{2}} \frac{1}{2 J + 1} \int d\Omega_{5k} \sum_{J M\xi'_{k'}} |T_{\lambda M\lambda}^{\xi\xi'}|^{2},$$

$$\times (\sin \alpha_k)^2 L_{b, k}^{L_{b, k}^{L_{b, k}}+1} \cos \angle \langle \xi, \xi' \rangle \sum_{\lambda M\lambda} \frac{1}{2 \lambda + 1} |U_{\lambda M\lambda}^{J\ell J}|^2.$$  \hspace{1cm} (D.3.8)

and therefore we have:

$$\frac{d^2 \sigma}{d\Omega_{5k} dE_{\text{ex}}} = \frac{(2\pi)^4}{\hbar^3} \frac{m^3}{h^3} \left( \frac{A_{\text{proj}}}{A_k} \right)^{\frac{3}{2}} \frac{1}{2 J + 1} \int d\Omega_{5k} \sum_{K''_{k''} K'_{k'}} |T_{\lambda M\lambda}^{\xi\xi'}|^{2},$$

$$\times (\sin \alpha_k)^2 L_{b, k}^{L_{b, k}^{L_{b, k}}+1} \cos \angle \langle \xi, \xi' \rangle \sum_{\lambda M\lambda} \frac{1}{2 \lambda + 1} |U_{\lambda M\lambda}^{J\ell J}|^2.$$  \hspace{1cm} (D.3.9)

Then we obtain the expression of the inclusive cross section with the FSI. In advance of obtaining the expression, we need to calculate the integration of $|U_{\lambda M\lambda}^{J\ell J}|^2$ with regard to $\Omega''_{k R}$:

$$\int d\Omega''_{k R} |U_{\lambda M\lambda}^{J\ell J}|^2 = [2\pi]^2 \int R_d dR_b J_{|M\lambda|}(q R_b) \int R''_d dR''_b J_{|M\lambda|}(q R''_b) 2\pi \int \frac{1}{k_R} dq d\Omega$$

$$\times \sum_{\xi' \xi \lambda_b \xi' \xi' \lambda' \lambda' \lambda''} |F|^2 D_{\lambda M\lambda \lambda\xi}^{J\ell J\xi'} \langle R_b \rangle D_{\lambda M\lambda \lambda' \lambda''}^{J\ell J\xi''} \langle R''_b \rangle.$$  \hspace{1cm} (D.3.10)
Using the relation $qdq = k_R^2 \sin \theta d\theta$ and the following formula

$$\int_0^\infty J_\mu(a_0 \xi_0) J_\mu(a_1 \xi_1) d\xi_0 = \frac{1}{a_0} \delta (a_0 - a_1) \ , \ \ 0 < a_0 < a_1 \ , \quad (D.3.11)$$

we see:

$$\int d\Omega_{kR} |U_{LM\lambda}^I| = |2\pi|^2 \int R_0 dR_0 \int R_0' dR_0' \frac{1}{k_R^3} \delta (R_0' - R_0) \times \sum_{\xi \lambda \alpha} |\mathcal{F}|^2 D_{\lambda \lambda \lambda \lambda}^{J'J} \xi \xi' \alpha \beta (R_0) D_{\lambda \lambda \lambda \lambda}^{J''J''} \xi \xi' \alpha \beta (R_0) \ ,$$

$$= |2\pi|^2 \int R_0 dR_0 \frac{1}{k_R^3} \sum_{\xi \lambda \alpha} |\mathcal{F}|^2 D_{\lambda \lambda \lambda \lambda}^{J'J} \xi \xi' \alpha \beta (R_0) D_{\lambda \lambda \lambda \lambda}^{J''J''} \xi \xi' \alpha \beta (R_0) \ . \quad (D.3.12)$$

Then we use the approximation $k_R \approx k'_R$. As a consequence, we obtain the inclusive cross section:

$$\frac{d\sigma}{dE_{\text{ex}}} = (2\pi)^4 2k_R k'_R \cdot \frac{m_3^3}{h^6} \cdot \frac{1}{2J + 1} \int \frac{E_{\text{ex}}^2}{2} d(\cos \alpha_k) \sum_{K_k' K_k''} \times (\sin \alpha_k)^{2L'_k + 1} (\cos \alpha_k) \sum_{L'' L''} L'_{L_k} L'_{L_k} n_k' n_k'' \sum_{\lambda M} \sum_{\lambda M} \frac{1}{2\lambda + 1} \int d\Omega_{kR} |U_{LM\lambda}^{J'J} |^2 ,$$

$$= (2\pi)^4 2k_R k'_R \cdot \frac{m_3^3}{h^6} \cdot \frac{1}{2J + 1} \int \frac{E_{\text{ex}}^2}{2} d(\cos \alpha_k) \sum_{K_k' K_k''} \times (\sin \alpha_k)^{2L'_k + 1} (\cos \alpha_k) \sum_{L'' L''} L'_{L_k} L'_{L_k} n_k' n_k'' \sum_{\lambda M} \sum_{\lambda M} \frac{1}{2\lambda + 1} \times |\mathcal{F}|^2 D_{\lambda \lambda \lambda \lambda}^{J'J} \xi \xi' \alpha \beta (R_0) D_{\lambda \lambda \lambda \lambda}^{J''J''} \xi \xi' \alpha \beta (R_0) . \quad (D.3.13)$$

Thus we derive the formula for the inclusive cross section:

$$\frac{d\sigma}{dE_{\text{ex}}} = (2\pi)^6 \cdot \frac{m_3^3}{h^6} \cdot \frac{2\pi}{2J + 1} E_{\text{ex}} \int d(\cos \alpha_k) \sum_{K_k' K_k''} (\sin \alpha_k)^{2L'_k + 1} (\cos \alpha_k)^{2L'_k + 1} \times J_{L'_{L_k} L'_{L_k}} n_k' n_k'' \sum_{\lambda M} \sum_{\lambda M} \frac{1}{2\lambda + 1} \int R_0 dR_0 \times \sum_{\xi \lambda \alpha} |\mathcal{F}|^2 D_{\lambda \lambda \lambda \lambda}^{J'J} \xi \xi' \alpha \beta (R_0) D_{\lambda \lambda \lambda \lambda}^{J''J''} \xi \xi' \alpha \beta (R_0) . \quad (D.3.14)$$
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