

## IMPROVED ACCURACY OF INCOMPRESSIBLE APPROXIMATION OF COMPRESSIBLE EULER EQUATIONS\*

BIN CHENG†

**Abstract.** This article addresses a fundamental concern regarding the incompressible approximation of fluid motions, one of the most widely used approximations in fluid mechanics. Common belief is that its accuracy is  $O(\epsilon)$ , where  $\epsilon$  denotes the Mach number. In this article, however, we prove an  $O(\epsilon^2)$  accuracy for the incompressible approximation of the isentropic, compressible Euler equations thanks to several decoupling properties. At the initial time, the velocity field and its first time derivative are of  $O(1)$  size, but the boundary conditions can be as stringent as the solid-wall type. The fast acoustic waves are still  $O(\epsilon)$  in magnitude, since the  $O(\epsilon^2)$  error is measured in the sense of Leray projection and more physically, in time-averages. We also show when a passive scalar is transported by the flow, it is  $O(\epsilon^2)$  accurate *pointwise in time* to use incompressible approximation for the velocity field in the transport equation.

**Key words.** incompressible approximation, compressible Euler equations, convergence rates, time-averages, initial-boundary value problem, singular limits, multiscale PDEs, hyperbolic PDEs

**AMS subject classifications.** Primary, 35Q31; Secondary, 35L50, 76N15

**DOI.** 10.1137/140955173

**1. Introduction and statement of main theorem.** All fluids are compressible, which generates acoustic waves. The restoring force is the pressure gradient which results from the fluid being compressed and decompressed. The Mach number, denoted by  $\epsilon$  in our article, is defined as the typical value of the ratio of fluid speed over sound speed. In the very subsonic regime  $\epsilon \ll 1$ , incompressible (vortical) fluid motions evolve in a slower time scale than acoustic wave propagation; then, incompressible approximation is often adopted so that effectively acoustic waves are filtered out. Numerous applications and theoretical studies rely on the validity of such approximation that indeed offers more convenience and simplicity than the compressible models.

Common belief is that the incompressible approximation introduces  $O(\epsilon)$  errors. In this article, however, we prove an improved  $O(\epsilon^2)$  error estimate between the isentropic, compressible Euler equations and its incompressible counterpart, thanks to several decoupling properties. The initial data is well-prepared in the sense that its first time derivative has  $O(1)$  spatial norms, independent of the smallness of  $\epsilon$ . In a loosely equivalent way, the velocity divergence is only  $O(\epsilon)$  in spatial norms and acoustic waves have only  $O(\epsilon)$  amplitudes as well. Higher time derivatives can still grow as  $\epsilon \rightarrow 0$ . The central idea of time-averaging is repeatedly used to suppress the amplitude of acoustic waves by a factor of  $\epsilon$ . Intuitively, acoustic waves oscillate fast at temporal frequencies of  $O(\epsilon^{-1})$ , and therefore averaging them in time effectively cancels out the majority of oscillations.

We ought to point out that the nonlinear nature of fluid motions is bound to couple fast acoustic waves with the slower incompressible motions. Even when all

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\*Received by the editors January 31, 2014; accepted for publication (in revised form) August 29, 2014; published electronically December 3, 2014.

<http://www.siam.org/journals/sima/46-6/95517.html>

†Department of Mathematics, University of Surrey, Guildford GU2 7XH, United Kingdom (b.cheng@surrey.ac.uk).

acoustic waves are completely filtered out at the initial time, they are instantaneously generated from slow incompressible motions. In the atmosphere, for example, the ubiquitous acoustic waves are emitted all the time, although most are inaudible to human ears.<sup>1</sup>

To this end, time-averaging plays a crucial role to further suppress the “unwanted” contribution from acoustic waves to the incompressible dynamics. The physical relevance of time-averaging is evident from the popularity of its generalized version, time filtering. In fact, time filtering is necessary in dealing with observational and computational data when the resolution of fast acoustic waves suffers from a wide range of factors. To make even closer connection to applications, we will use the time-averaging technique to show that if a passive scalar is transported by a velocity field governed by the compressible Euler equations, then it is  $O(\epsilon^2)$  accurate *pointwise in time* to replace the velocity with its incompressible counterpart(s).

Our techniques are applicable to general bounded domains subject to the solid-wall boundary condition  $\mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$ . Several issues arise here: (1) nonlinear coupling of fast and slow dynamics does not decay or disperse in any strong sense; (2) Fourier analysis is not applicable; (3) straightforward energy estimates are not convenient for proving  $\epsilon$ -independent estimates in  $H^m$  norms for the solution. The last point is related to the fact that  $\mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$  does not hold for all spatial derivatives of  $\mathbf{v}$  and thus the boundary integrals (multiplied by  $\epsilon^{-1}$ ) resulting from the divergence theorem do not vanish. These issues will be resolved by relying on time-averaging, vorticity formulation, and the simple fact  $\partial_t^k \mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$ .

**1.1. Main results.** Upon rescaling and nondimensionalization, the isentropic, compressible Euler equations are expressed in terms of *total* density  $\rho^{\text{tot}}$  and velocity  $\mathbf{v}$ ,

$$(1.1) \quad \begin{cases} \partial_t \rho^{\text{tot}} + \nabla \cdot (\rho^{\text{tot}} \mathbf{v}) = 0, \\ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\epsilon^2} \frac{\nabla \pi(\rho^{\text{tot}})}{\rho^{\text{tot}}} = 0, \end{cases}$$

with Mach number  $\epsilon \ll 1$  bringing in acoustic waves oscillating on fast time scales. We have had the pressure law  $\pi \in C^\infty(\mathbb{R}^+)$  go through rescaling and affine transformation to satisfy

$$(1.2) \quad \pi'(1) = 1.$$

For instance, if  $\pi(\cdot)$  satisfies the  $\gamma$ -power law, then by the above assumption,  $\pi(\rho^{\text{tot}}) = (\rho^{\text{tot}})^\gamma / \gamma$ . Also, it is understood that  $|\rho^{\text{tot}} - 1| \ll 1$ , so that the pressure gradient is approximated by  $\nabla \rho^{\text{tot}}$  and the linearized acoustic waves have both phase and group velocities at order  $1/\epsilon$ , namely, the rescaled sound speed.

Without loss of generality, we only consider a connected (but not necessarily simply connected) compact spatial domain  $\Omega \subset \mathbb{R}^N$  for  $N = 2$  or  $3$ , with the “solid-wall” boundary condition

$$\mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0,$$

where  $\vec{\mathbf{n}} = \vec{\mathbf{n}}(x)$  is the outward normal to the static, smooth boundary  $\partial\Omega$ . The topology of  $\Omega$  will occasionally be a concern, e.g., in Remark 2.2.

<sup>1</sup>The sound speed is around 330 meters per second in the lower atmosphere; human’s hearing range starts at 20 Hertz. Therefore, we cannot hear wave lengths longer than 17 meters in our everyday life.

The main goal of this article is to estimate, in terms of  $\epsilon$  and initial data, the size of  $(\mathbf{v} - \tilde{\mathbf{v}})$  where  $\tilde{\mathbf{v}}$  solves the incompressible Euler equations

$$\begin{aligned} (1.3a) \quad & \partial_t \tilde{\mathbf{v}} + \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} + \nabla q = 0, \\ (1.3b) \quad & \nabla \cdot \tilde{\mathbf{v}} = 0, \\ (1.3c) \quad & \text{subject to } \tilde{\mathbf{v}} \cdot \tilde{\mathbf{n}}|_{\partial\Omega} = 0. \\ (1.3d) \quad & \text{Initial data } \tilde{\mathbf{v}}_0 \text{ satisfies (1.3b), (1.3c).} \end{aligned}$$

Here, the scalar  $q$  is an auxiliary variable, also called pressure, that enforces the incompressible condition (1.3b). Without such a term, the above system would be overdetermined.

The spatial  $H^m$  norm is defined as usual,

$$\|f(x)\|_{H^m} := \left( \sum_{|\beta| \leq m} \int_{\Omega} |\partial_x^\beta f(x)|^2 dx \right)^{1/2},$$

where multi-index  $\beta$  indicates orders of derivatives taken on each spatial dimension. Let  $H^m(\Omega)$  denote the closure space of smooth functions with finite  $H^m$  norms. Of course,  $L^2(\Omega) = H^0(\Omega)$ .

Before stating the main theorem, we clarify one technical point. The time derivatives at  $t = 0$ , denoted by  $\partial_t^k(\rho_0^{\text{tot}}, \mathbf{v}_0)$ , can be calculated without knowing the solution for  $t > 0$ . This is because, by repeatedly taking time derivatives on (1.1), one can inductively express  $\partial_t(\rho^{\text{tot}}, \mathbf{v})$ ,  $\partial_t^2(\rho^{\text{tot}}, \mathbf{v})$ ,  $\dots$ ,  $\partial_t^k(\rho^{\text{tot}}, \mathbf{v})$  solely in terms of  $(\rho^{\text{tot}}(t, x), \mathbf{v}(t, x))$  and their spatial derivatives up to the  $k$ th order evaluated at each fixed time  $t$ .

**THEOREM 1.1 (main theorem).** *Let integer  $m \geq 3$  and parameter  $\epsilon \in [0, 1/2]$ . Consider the compressible system (1.1) subject to initial data  $(\rho_0^{\text{tot}} - 1, \mathbf{v}_0) \in H^m(\Omega)$ . Assume  $(\rho_0^{\text{tot}}, \mathbf{v}_0)$  is compatible with the boundary condition, namely,  $(\partial_t^k \mathbf{v}_0) \cdot \tilde{\mathbf{n}}|_{\partial\Omega} = 0$  for  $k < m$ .*

*Let  $\tilde{\mathbf{v}}$  solve the incompressible system (1.3) subject to initial data  $\tilde{\mathbf{v}}_0 = \mathcal{P}\mathbf{v}_0$ . Here,  $\mathcal{P}$ , defined in (2.2) below, denotes the Leray projection into the incompressible velocity subspace.*

*Define*

$$E_0 := \left\| \left( \frac{\rho_0^{\text{tot}} - 1}{\epsilon}, \mathbf{v}_0 \right) \right\|_{H^m}, \quad E_{t,0} := \left\| \partial_t \left( \frac{\rho^{\text{tot}}}{\epsilon}, \mathbf{v}_0 \right) \right\|_{H^{m-1}}.$$

*Then, there exist constants  $E^*, T^*, C^*$  that only depend on  $m$ ,  $\Omega$ , and pressure law  $\pi(\cdot)$ , so that with  $E_0 \leq E^*/\epsilon$ ,*

$$(1.4) \quad \sup_{t \in [0, T^*/E_0]} \|\mathcal{P}\mathbf{v} - \tilde{\mathbf{v}}\|_{H^{m-3}} \leq C^* \epsilon^2 (E_{t,0} + E_0^2) \|\mathcal{P}\mathbf{v}_0\|_{H^m}.$$

The proof is given in section 5.

We used the clumsy notation of  $\frac{\rho^{\text{tot}} - 1}{\epsilon}$  to state the main theorem, as it will be replaced throughout the rest of this article with the density perturbation

$$(1.5) \quad \rho := \frac{\rho^{\text{tot}} - 1}{\epsilon}.$$

With this notation,

$$(1.6) \quad E_0 = \|(\rho_0, \mathbf{v}_0)\|_{H^m}, \quad E_{t,0} = \|\partial_t(\rho_0, \mathbf{v}_0)\|_{H^{m-1}},$$

so that the main theorem as well as the reformulated system (1.7) from below is invariant under the hyperbolic scaling  $\rho \rightarrow c\rho, \mathbf{v} \rightarrow c\mathbf{v}, t \rightarrow t/c, \epsilon \rightarrow \epsilon/c$  for any constant  $c > 0$ .

The original system (1.1) is then reformulated in terms of unknown pair  $(\rho, \mathbf{v})$ ,

$$(1.7a) \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = -\epsilon^{-1} \nabla \cdot \mathbf{v},$$

$$(1.7b) \quad \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + h_\epsilon(\rho) \nabla \rho = -\epsilon^{-1} \nabla \rho$$

$$(1.8) \quad \text{subject to } \mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0,$$

$$\text{where } h_\epsilon(\rho) := \left( \frac{\pi'(1 + \epsilon\rho)}{1 + \epsilon\rho} - 1 \right) \frac{1}{\epsilon}.$$

Note that by (1.2) and Taylor expansion,  $h_\epsilon(\rho) = (\pi''(1) - \pi'(1))\rho + O(\epsilon\rho^2)$ .

In a more compact form,

$$\partial_t \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} + \mathcal{N} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} = -\epsilon^{-1} \mathcal{L} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$$

with the nonlinear operator  $\mathcal{N}$  clearly defined via (1.7) and the antisymmetric operator

$$(1.9) \quad \mathcal{L} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} \nabla \cdot \mathbf{v} \\ \nabla \rho \end{pmatrix}.$$

For purely aesthetic reasons, we will use notation  $\mathcal{L} \begin{pmatrix} \rho \\ \mathbf{v} \end{pmatrix}$  and  $\mathcal{L}(\rho, \mathbf{v})$  interchangeably.

One can easily use (1.7) and Sobolev inequalities to show

$$(1.10) \quad \left| \epsilon \|\partial_t(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}} - \|\mathcal{L}(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}} \right| \leq C\epsilon \|\rho, \mathbf{v}\|_{\mathbf{H}^m} \|h_\epsilon(\rho), \mathbf{v}\|_{\mathbf{H}^m} \quad \text{for } m \geq 3,$$

where  $h_\epsilon(\rho) \sim O(\rho)$  à la Taylor expansion. Thus, having the first time derivative  $\partial_t(\rho, \mathbf{v}) \sim O(1)$  is loosely equivalent to enforcing  $\mathcal{L}(\rho, \mathbf{v}) = (\nabla \cdot \mathbf{v}, \nabla \rho) \sim O(\epsilon)$ . In other words, given  $E_0 \sim O(1)$ , preparing  $E_{t,0} \sim O(1)$  is loosely equivalent to having  $\rho_0$  to be  $O(\epsilon)$  close to constant and  $\mathbf{v}_0$  to be  $O(\epsilon)$  close to incompressibility.

*Remark 1.2.* There are two bounding factors in (1.4) that depend on initial data. Regarding the  $(E_{t,0} + E_0^2)$  factor, we note the compressible system (1.7) automatically enforces  $E_{t,0} \leq O(E_0/\epsilon + E_0^2)$ ; cf. (1.10). Then, using ill-prepared data that allow  $E_{t,0} \sim O(1/\epsilon)$  and acoustic waves of  $O(1)$  amplitudes, we would recover the  $O(\epsilon E_0)$  error estimate for ill-prepared data previously proved by Cheng in [6]. Regarding the other factor  $\|\mathcal{P}\mathbf{v}_0\|_{\mathbf{H}^m}$ , in the extreme case with purely acoustic wave or potential flow initial data,  $\mathcal{P}\mathbf{v} = 0$  is invariantly sustained by the compressible system, whereas with  $\tilde{\mathbf{v}}_0 = \mathcal{P}\mathbf{v}_0 = 0$ , the incompressible Euler system simply yields  $\tilde{\mathbf{v}} = 0$ . Then, both sides of (1.4) vanish, consistent with such well-known invariance.

*Remark 1.3.* The local-in-time existence and uniqueness of  $\mathbf{C}([0, T], \mathbf{H}^m(\Omega))$  solution to (1.1) has been established in [19]. The compatibility condition,  $(\partial_t^k \mathbf{v}_0) \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$  for  $k < m$ , is both necessary and sufficient for  $\mathbf{H}^m$  well-posedness, although this is not the main focus of this article. On the other hand, it is crucial in our study to obtain  $\epsilon$ -independent upper bounds on  $\|\rho, \mathbf{v}\|_{\mathbf{H}^m}$  and  $\|\partial_t(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}}$ . This will be achieved in section 4.

*Remark 1.4.* Estimates on  $\|\mathcal{P}\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{H}^{m'}}$  for  $m' \in (m - 3, m)$  are obtained by interpolating between  $\|\mathcal{P}\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{H}^{m-3}} \sim O(\epsilon^2)$  from (1.4) and  $\|\mathcal{P}\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{H}^m} \sim O(1)$ .

In the final section 5, we will also prove the following corollary without relying on Leray projection. Instead, the estimate is in terms of the physically relevant time-averages.

**COROLLARY 1.5.** *Under the same hypotheses as in the main theorem, Theorem 1.1, there exist constants  $C^{**}, C^{***}$  that only depend on  $m, \Omega$  and pressure law  $\pi(\cdot)$ , so that for all times  $T \in [0, T^*/E_0]$ ,*

$$(1.11) \quad \epsilon \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{H}^{m-3}}(T) + \left\| \int_0^T \mathbf{v} - \tilde{\mathbf{v}} \right\|_{\mathbf{H}^{m-3}} \leq C^{**} \epsilon^2 (E_{t,0} + E_0^2).$$

Moreover, if two scalars  $\theta(t, x), \tilde{\theta}(t, x)$  are transported by  $\mathbf{v}, \tilde{\mathbf{v}}$ , respectively,

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta = 0,$$

$$\partial_t \tilde{\theta} + \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} = 0,$$

subject to the same initial data  $\theta_0 = \tilde{\theta}_0 \in \mathbf{H}^m(\Omega)$ , then

$$\sup_{[0, T^*/E_0]} \|\theta - \tilde{\theta}\|_{\mathbf{H}^{m-3}} \leq C^{***} \epsilon^2 (E_{t,0} + E_0^2) \|\theta_0\|_{\mathbf{H}^m}.$$

Apparently, by (1.4), the same  $O(\epsilon^2)$  accuracy holds true if we approximate the velocity by  $\mathcal{P}\mathbf{v}$ .

Note in (1.11) the gained  $\epsilon$  factor thanks to time-averaging. Also, it is  $O(\epsilon^2)$  accurate *pointwise in time* to use both  $\tilde{\mathbf{v}}$  and  $\mathcal{P}\mathbf{v}$  approximations for  $\mathbf{v}$  in the transport equation.

**1.2. Literature.** There have been numerous results regarding the singular limits of compressible Euler equations and other fluid equations in various settings, but literature on the  $O(\epsilon^2)$  accuracy is limited. To our best knowledge, such results are all represented in the form of  $O(\epsilon), O(\epsilon^2) \dots$  corrections to the incompressible approximation (e.g., [21, Theorem 3.2], [10, Theorem 3] for periodic domains under various “small divisor” conditions). Our result here confirms that the  $O(\epsilon)$  correction is in fact zero for Euler equations in general spatial domains, at least for well-prepared data. We point to two survey papers for some comprehensive lists of references: [22] with emphases on hyperbolic partial differential equations (PDEs) and homogenization in space-time and [15] with emphases on viscous fluids and weak solutions. To mention only a few earliest works in terms of well-prepared data, we refer to [8, 9, 14, 3, 13, 23]. In a closely related paper [5], the bounded derivative method is applied to numerical schemes from geophysical applications. Well-prepared conditions on initial data were later removed for problems in the whole space [26], in an exterior domain [11, 12], and in a torus [21]. These arguments more or less rely on use of Fourier analysis and/or the dispersive nature of the underlying wave equations.

Singular limit problems in a bounded spatial domain, on the other hand, remain much less studied. Even the well-posedness of hyperbolic PDEs in a bounded domain can be challenging due to the (possible) characteristic boundary. For example, Rauch elaborated in [17] that, in such settings, only estimates along tangential directions are available near the boundary. Nevertheless, Schochet [19] proved the low-Mach-number limit with solid-wall boundary condition and well-prepared initial data, but without convergence rates. In [18], Secchi proved the strong convergence of  $\mathcal{P}\mathbf{v}$  for three-dimensional Euler equations with ill-prepared initial data, again without convergence rates. Recently, Cheng proved  $O(\epsilon)$  convergence rate for ill-prepared data in [6]. The time-averaging technique used there inspired this current study; also see Cheng and Mahalov [7] for time-averaging applied to geophysical models on a sphere.

**1.3. Slow dynamics and vortical dynamics.** Later in the article, we will apply the Leray projection to the compressible system, which effectively annihilates  $\epsilon^{-1}\mathcal{L}$ . This gives a decomposition of the solution space into slow and fast subspaces and correspondingly a decomposition of the compressible system into a slow one governing the incompressible motions and a fast one governing the rapidly oscillating acoustic waves.

The slow dynamics is very closely related to the vorticity equations. Apply  $\nabla \times$  to (1.7b) so that the cancellation property  $\nabla \times \nabla = 0$  yields the equation for vorticity  $\omega := \nabla \times \mathbf{v}$

$$(1.12a) \quad \partial_t \omega + \mathbf{v} \cdot \nabla \omega + (\nabla \cdot \mathbf{v}) \omega = 0 \quad \text{in two dimensions,}$$

$$(1.12b) \quad \text{and} \quad \partial_t \omega - \omega \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \omega + (\nabla \cdot \mathbf{v}) \omega = 0 \quad \text{in three dimensions}$$

without  $O(\epsilon^{-1})$  terms contributing to  $\partial_t \omega$ . Thus, vorticity  $\omega$  evolves on a slow time scale. We have to point out that the vortical dynamics may not contain all the information of the slow dynamics if the spatial domain is not contractible. Take Remark 2.2, for example. A point vortex restricted in an annulus domain forms a steady solution of incompressible Euler equations; together with a suitable axisymmetric profile of  $\rho$ , it also solves the compressible system. However, both its vorticity and divergence are identically zero; therefore the vorticity equation, even combined with the divergence equation, does not retain all dynamical information required to solve for  $\mathbf{v}$ .

Nevertheless, the vorticity equation is widely used in practice as it has the simple structure of a transport equation which turns out to be crucial for some estimate proofs later on.

The rest of this article is organized as follows. In section 2, we define the Leray projection, prove its properties using elliptic PDE theory, and use it to extract the slow dynamics from the compressible system. Section 3 contains probably the most novelty. It explains how to use the time-averaging technique to obtain pointwise-in-time error estimates. A decoupling property particular to the compressible Euler system will allow us to gain an extra  $\epsilon$  factor, provided the data are well-prepared. Next, without concerns for boundary, the reader can skip section 4. Here, we use mixed norms to obtain  $\epsilon$ -independent bounds on the sizes of the solution and its first time derivative. The methods used here are partially similar to those of [20], but we work with both ill-prepared and well-prepared data. The final section 5 completes proofs of the main theorem, Theorem 1.1 and Corollary 1.5 and makes further comments.

*Remark 1.6.* The Sobolev inequalities used throughout this article can be summarized as follows and are proved in the appendix. Given functions  $f_1(x), f_2(x), \dots, f_j(x)$  over a two- or three-dimensional compact domain  $\Omega$ , we have estimate

$$(1.13) \quad \left\| \prod_{i=1}^j \partial_x^{\beta_i} f_i \right\|_{L^2} \lesssim \prod_{i=1}^j \|f_i\|_{H^k}$$

if one of the following conditions holds:

$$(1.14) \quad \begin{cases} |\beta_1 + \dots + \beta_j| \leq k, \text{ and } k \geq 2, \text{ or} \\ |\beta_1 + \dots + \beta_j| = k + 1, \max\{|\beta_1|, \dots, |\beta_j|\} \leq k, \text{ and } k \geq 3. \end{cases}$$

Here and below, the “similarly less than” notation  $a \lesssim b$  is understood as “ $a \leq Cb$  for a constant  $C$  solely depending on  $\Omega$ , the pressure law  $\pi(\cdot)$  and the Sobolev spaces involved.”

**2. Projections onto slow and fast subspaces.** Define  $\mathbf{X}$  to be the space of incompressible velocity fields subject to solid-wall boundary condition,

$$(2.1) \quad \mathbf{X} = \mathbf{L}^2 \text{ closure of } \left\{ \mathbf{v} \in \mathbf{C}^1(\overline{\Omega}) \mid \int_{\Omega} \mathbf{v} \cdot \nabla f = 0 \text{ for any } f \in \mathbf{H}^1(\overline{\Omega}) \right\}.$$

Define  $\mathcal{P}$  as the  $\mathbf{L}^2$ -orthogonal projection onto  $\mathbf{X}$  so that, for any  $\mathbf{v}, \mathbf{v}' \in \mathbf{L}^2(\Omega)$ ,

$$(2.2a) \quad \mathcal{P}^2 \mathbf{v} = \mathcal{P} \mathbf{v} \in \mathbf{X},$$

$$(2.2b) \quad \int_{\Omega} (\mathbf{v} - \mathcal{P} \mathbf{v}) \cdot (\mathcal{P} \mathbf{v}') = 0.$$

In fact,  $\mathcal{P}$  is the classical Leray projection subject to the solid-wall boundary condition. Then, define

$$\mathcal{Q} := 1 - \mathcal{P}.$$

These projections can be characterized conveniently by an elliptic PDE as follows.

**PROPOSITION 2.1.** *For any  $\mathbf{v} \in \mathbf{H}^k(\Omega)$  with  $k \geq 1$ , we have*

$$(2.3) \quad \mathcal{Q} \mathbf{v} = \nabla \phi \in \mathbf{H}^k(\Omega), \quad \text{where } \phi \text{ solves } \begin{cases} \Delta \phi = \nabla \cdot \mathbf{v} & \text{in } \Omega, \\ \nabla \phi \cdot \vec{\mathbf{n}} = \mathbf{v} \cdot \vec{\mathbf{n}} & \text{in } \partial \Omega. \end{cases}$$

Also,  $\phi$  is unique up to an added constant and thus  $\mathcal{Q} \mathbf{v}$  is unique.

Here and below, we always assume  $k \geq 1$  whenever the trace of an  $\mathbf{H}^k(\Omega)$  function is involved.

*Proof.* The solvability of (2.3) follows from standard elliptic PDE theory (e.g., [24, Chap. 5, Prop. 7.7]). It suffices to verify that  $(1 - \mathcal{Q})$  with  $\mathcal{Q}$  given in (2.3) equals  $\mathcal{P}$  which is uniquely defined in (2.2a), (2.2b).

Obviously, (2.3) implies  $(1 - \mathcal{Q}) \mathbf{v} \in \mathbf{X}$  for any  $\mathbf{v} \in \mathbf{H}^k(\Omega)$ . Then, for any  $\mathbf{v}^{\text{inc}} \in \mathbf{X} \cap \mathbf{H}^k(\Omega)$ , the uniqueness of  $\phi$  in (2.3), up to an added constant, implies that  $\mathcal{Q} \mathbf{v}^{\text{inc}} = 0$ . In short,  $(1 - \mathcal{Q})$  is a projection so that, for any  $\mathbf{v} \in \mathbf{H}^k$ ,  $(1 - \mathcal{Q})^2 \mathbf{v} = (1 - \mathcal{Q}) \mathbf{v} \in \mathbf{X}$ .

It remains to show (2.2b), i.e., that  $\mathcal{Q} \mathbf{v}$  is  $\mathbf{L}^2$ -orthogonal to any  $\mathbf{v}^{\text{inc}} \in \mathbf{X}$ . Since  $\mathbf{X} \cap \mathbf{C}^\infty(\overline{\Omega})$  is dense in  $\mathbf{X}$  in the  $\mathbf{L}^2$  topology, it suffices to have  $\mathbf{v}^{\text{inc}}$  as a smooth element of  $\mathbf{X}$ . Then,

$$\int_{\Omega} \mathbf{v}^{\text{inc}} \cdot (\mathcal{Q} \mathbf{v}) \, dx = \int_{\Omega} \mathbf{v}^{\text{inc}} \cdot \nabla \phi \, dx \stackrel{(a)}{=} \int_{\Omega} \nabla \cdot (\mathbf{v}^{\text{inc}} \phi) \, dx \stackrel{(b)}{=} \int_{\partial \Omega} \vec{\mathbf{n}} \cdot \mathbf{v}^{\text{inc}} \phi \, ds \stackrel{(c)}{=} 0,$$

where (a), (c) are due to the definition of  $\mathbf{v}^{\text{inc}}$  and (b) is due to the divergence theorem.  $\square$

This proposition shows that  $\mathcal{Q} \mathbf{v}$  is always a perfect gradient and therefore its curl vanishes.

$$(2.4) \quad \nabla \times (\mathcal{Q} \mathbf{v}) = 0, \quad \nabla \times (\mathcal{P} \mathbf{v}) = \nabla \times \mathbf{v}.$$

Compare them to the definitional facts,

$$\nabla \cdot (\mathcal{P} \mathbf{v}) = 0, \quad \nabla \cdot (\mathcal{Q} \mathbf{v}) = \nabla \cdot \mathbf{v}.$$

This is to say,  $\mathcal{P} \mathbf{v}$  contains all the information of the vorticity (but not necessarily vice versa!) and  $\mathcal{Q} \mathbf{v}$  contains all the information of the divergence.

From here on, we will interchangeably use  $\mathbf{v}^P$  for  $\mathcal{P} \mathbf{v}$  and  $\mathbf{v}^Q$  for  $\mathcal{Q} \mathbf{v}$ .



**2.1. Boundedness of projections and elliptic estimates.** Operators  $\mathcal{L}, \nabla \times, \mathcal{P}, \mathcal{Q}$  are all elliptic operators with nontrivial kernels, and we will employ elliptic estimates with boundary conditions to estimate them. The papers of Agmon, Douglis, and Nirenberg [1], [2] establish a “complementing boundary condition” that is necessary and sufficient for the solution operator of an  $s$ th order elliptic PDE system to be  $\mathbf{C}^k \rightarrow \mathbf{C}^{k+s}$  and  $\mathbf{H}^k \rightarrow \mathbf{H}^{k+s}$ . To treat the Euler equations (e.g., [4]), only a particular case is used: for any velocity field  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ ,

$$(2.5) \quad \|\mathbf{v}\|_{\mathbf{H}^k} \lesssim \|\nabla \cdot \mathbf{v}\|_{\mathbf{H}^{k-1}} + \|\nabla \times \mathbf{v}\|_{\mathbf{H}^{k-1}} + \|\mathbf{v}\|_{\mathbf{L}^2} \quad \text{if } \mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0 \text{ and } k \geq 1.$$

*Remark 2.2.* We point out that the  $\|\mathbf{v}\|_{\mathbf{L}^2}$  term above may not be dropped if, for example,  $\Omega$  is not contractible. Consider an annulus domain  $\Omega = \{(x, y) \mid 1 < x^2 + y^2 < 2\}$  and a point vortex  $\mathbf{v} = \nabla^\perp \ln|x^2 + y^2|$ . Then,  $\nabla \cdot \mathbf{v} = \nabla \times \mathbf{v} = 0$  and  $\mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$ , but  $\mathbf{v} \neq 0$ .

Now, set  $\mathbf{v} = \mathbf{v}^P$  in (2.5) and use the facts that  $\nabla \cdot \mathbf{v}^P = 0, \mathbf{v}^P \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$  to obtain, without boundary condition on  $\mathbf{v}$ ,

$$(2.6) \quad \|\mathbf{v}^P\|_{\mathbf{H}^k} \lesssim \|\nabla \times \mathbf{v}^P\|_{\mathbf{H}^{k-1}} + \|\mathbf{v}^P\|_{\mathbf{L}^2} \quad \text{for } k \geq 1.$$

This gives a bound on the high norms of  $\mathbf{v}^P$  using the high norms of  $\nabla \times \mathbf{v}^P = \nabla \times \mathbf{v}$  (by (2.4)) and the  $\mathbf{L}^2$  norm of  $\mathbf{v}^P$ .

Therefore,  $\mathcal{P}, \mathcal{Q}$  are bounded operators in  $\mathbf{H}^k(\Omega)$  regardless of boundary condition,

$$(2.7) \quad \|\mathbf{v}\|_{\mathbf{H}^k} \leq \|\mathcal{P}\mathbf{v}\|_{\mathbf{H}^k} + \|\mathcal{Q}\mathbf{v}\|_{\mathbf{H}^k} \lesssim \|\mathbf{v}\|_{\mathbf{H}^k} \quad \text{for } k \geq 0.$$

The first inequality is apparently due to  $\mathcal{P} + \mathcal{Q} = 1$ . Also, for the second inequality, the case of  $k = 0$  is due to the definition of  $\mathcal{P}$  and the Pythagorean theorem.

Similar to (2.6), we can bound the norms of  $\mathbf{v}^Q$  using norms of  $\nabla \cdot \mathbf{v}^Q = \nabla \cdot \mathbf{v}$ . In fact, set  $\mathbf{v} = \mathbf{v}^Q$  in (2.5) and use (2.4) and the fact  $\mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = \mathbf{v}^Q \cdot \vec{\mathbf{n}}|_{\partial\Omega}$  to obtain

$$(2.8) \quad \|\mathbf{v}^Q\|_{\mathbf{H}^k} \lesssim \|\nabla \cdot \mathbf{v}^Q\|_{\mathbf{H}^{k-1}} + \|\mathbf{v}^Q\|_{\mathbf{L}^2} \quad \text{if } \mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0.$$

A notable feature of the above inequality is, unlike in Remark 2.2, the  $\mathbf{L}^2$  norm term above can be dropped regardless of topology of the spatial domain.

**PROPOSITION 2.3.** *Let  $k \geq 1$ . For any  $\mathbf{v} \in \mathbf{H}^k(\Omega)$ ,*

$$\|\mathbf{v}^Q\|_{\mathbf{H}^k} \lesssim \|\nabla \cdot \mathbf{v}^Q\|_{\mathbf{H}^{k-1}} = \|\nabla \cdot \mathbf{v}\|_{\mathbf{H}^{k-1}} \quad \text{if } \mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0.$$

*Proof.* Having (2.8) established, it suffices to show  $\|\mathbf{v}^Q\|_{\mathbf{L}^2} \leq C_\Omega \|\nabla \cdot \mathbf{v}\|_{\mathbf{L}^2}$ .

Take any smooth test velocity field  $\mathbf{v}'$ . By Proposition 2.1,  $\mathbf{v}' = \nabla\psi + \mathcal{P}\mathbf{v}'$ . Then, the orthogonality of  $\mathcal{P}, \mathcal{Q}$  implies  $\int_\Omega \mathbf{v}^Q \cdot \mathbf{v}' = \int_\Omega \mathbf{v}^Q \cdot \nabla\psi$ . Apply the divergence theorem and  $\mathbf{v} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$  on the right-hand side (RHS) to get  $\int_\Omega \mathbf{v}^Q \cdot \mathbf{v}' = -\int_\Omega (\nabla \cdot \mathbf{v})\psi$ . Now, since  $\Omega$  is compact, we can set  $\int_\Omega \psi = 0$  and apply the Hölder and Poincaré inequalities,

$$\int_\Omega \mathbf{v}^Q \cdot \mathbf{v}' \leq C_\Omega \|\nabla \cdot \mathbf{v}\|_{\mathbf{L}^2} \|\nabla\psi\|_{\mathbf{L}^2} \leq C_\Omega \|\nabla \cdot \mathbf{v}\|_{\mathbf{L}^2} \|\mathbf{v}'\|_{\mathbf{L}^2},$$

where the last inequality is due to  $\|\nabla\psi\|_{\mathbf{L}^2} = \|\mathcal{Q}\mathbf{v}'\|_{\mathbf{L}^2}$  and the Pythagorean theorem. This shows  $\|\mathbf{v}^Q\|_{\mathbf{L}^2} \leq C_\Omega \|\nabla \cdot \mathbf{v}\|_{\mathbf{L}^2}$ .  $\square$



**2.2. Slow and fast dynamics.** Next, we want to extract the slow dynamics from (1.7) in the form of an evolutionary system that is free of  $O(\epsilon^{-1})$  time derivatives. One could apply  $\nabla \times$  to cancel  $\epsilon^{-1}\mathcal{L}$  and get the vorticity equations (1.12a) or (1.12b), but the comments thereafter suggests that the vortical dynamics does not necessarily retain all the information of the slow dynamics. For such a reason, we will instead apply  $\mathcal{P}$  on (1.7b). The intuition is that if we define  $\mathcal{P}^\sharp(\frac{\rho}{\mathbf{v}}) := (\frac{\rho}{\mathcal{P}\mathbf{v}})$  to make  $\mathcal{L}\mathcal{P}^\sharp = 0$  and to make  $\mathcal{L}, \mathcal{P}^\sharp$  (skew-)symmetric, then hopefully they commute  $\mathcal{P}^\sharp\mathcal{L} = 0$  and therefore applying  $\mathcal{P}^\sharp$  to (1.7) will eliminate the  $\epsilon^{-1}\mathcal{L}$  term. This is easily proved using adjoint operators if  $\partial\Omega = \emptyset$  and can still be established in general provided the boundary conditions are taken care of.

PROPOSITION 2.4. For any scalar  $\rho \in H^1(\Omega)$ ,

$$\mathcal{P}(\nabla\rho) = 0.$$

*Proof.* By orthogonality of  $\mathcal{P}, \mathcal{Q}$ , we have

$$\int_{\Omega} \mathcal{P}(\nabla\rho) \cdot \mathcal{P}(\nabla\rho) = \int_{\Omega} \mathcal{P}(\nabla\rho) \cdot \nabla\rho,$$

which is zero due to  $\mathcal{P}(\nabla\rho) \in \mathbf{X}$  satisfying (2.1).  $\square$

Thus, we apply  $\mathcal{P}$  on (1.7b) to obtain the slow dynamics,

$$(2.9) \quad -\partial_t \mathbf{v}^P = \mathcal{P}(\mathbf{v} \cdot \nabla \mathbf{v}) = \mathcal{P}(\mathbf{v}^P \cdot \nabla \mathbf{v}^P) + \mathcal{P}(\mathbf{v}^P \cdot \nabla \mathbf{v}^Q + \mathbf{v}^Q \cdot \nabla \mathbf{v}^P) + \mathcal{P}(\mathbf{v}^Q \cdot \nabla \mathbf{v}^Q),$$

where we also used the fact that  $h_\epsilon(\rho)\nabla\rho$  is a perfect gradient and therefore is annihilated by  $\mathcal{P}$  according to the above proposition.

On the other hand, apply  $\mathcal{Q}$  to (1.7b) and keep (1.7a) to obtain the fast dynamics

$$(2.10a) \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = -\epsilon^{-1} \nabla \cdot \mathbf{v},$$

$$(2.10b) \quad \partial_t \mathbf{v}^Q + \mathcal{Q}(\mathbf{v} \cdot \nabla \mathbf{v}) + h_\epsilon(\rho)\nabla\rho = -\epsilon^{-1} \nabla \rho.$$

This way, the original system is decomposed into (2.10) governing the fast variables  $\rho, \mathbf{v}^Q$  with  $O(\epsilon^{-1})$  coefficients and (2.9) governing the slow variable  $\mathbf{v}^P$  whose first time derivative is  $O(1)$ . Note the density component is identically zero in the slow variable.

A key *decoupling* property is that the “fast-fast” product in (2.9) vanishes completely.

LEMMA 2.5. For any  $\mathbf{v}^Q \in H^2(\Omega)$ ,

$$\mathcal{P}(\mathbf{v}^Q \cdot \nabla \mathbf{v}^Q) = 0.$$

*Proof.* By Proposition 2.1, there exists a scalar function  $\phi$  so that  $\mathbf{v}^Q = \nabla\phi$ . Then,  $\mathbf{v}^Q \cdot \nabla \mathbf{v}^Q = (\nabla\phi) \cdot \nabla(\nabla\phi) = \frac{1}{2} \nabla |\nabla\phi|^2$ . So, by Proposition 2.4,  $\mathcal{P}(\mathbf{v}^Q \cdot \nabla \mathbf{v}^Q)$  vanishes.  $\square$

Therefore, we rewrite (2.9) as

$$(2.11) \quad -\partial_t \mathbf{v}^P = \mathcal{P}(\mathbf{v}^P \cdot \nabla \mathbf{v}^P) + \mathcal{P}\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q),$$

where bilinear operator

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) := \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_1.$$

If we set  $\mathbf{v}^Q \equiv 0$  by brutal force, (2.11) would be *formally* reduced to the incompressible Euler equations.

PROPOSITION 2.6. *Consider a velocity field  $\tilde{\mathbf{v}} \in \mathbf{C}([0, T^*] \times H^3(\Omega))$ . Then,  $\tilde{\mathbf{v}}$  solves the actual incompressible Euler equations (1.3a), (1.3b), (1.3c) if and only if it solves*

$$(2.12) \quad -\partial_t \tilde{\mathbf{v}} = \mathcal{P}(\tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}})$$

with the same initial data  $\tilde{\mathbf{v}}_0$  satisfying (1.3d).

*Proof.* “Only if.” Assume  $\tilde{\mathbf{v}}$  solves (1.3a), (1.3b), (1.3c). Apply  $\mathcal{P}$  on (1.3a). On the left-hand side (LHS), because  $\partial_t \tilde{\mathbf{v}}$  also satisfies (1.3b), (1.3c), we have  $\mathcal{P}(\partial_t \tilde{\mathbf{v}}) = \partial_t \tilde{\mathbf{v}}$  by the definition of  $\mathcal{P}$ . Also, we have  $\mathcal{P}(\nabla q) = 0$  by Proposition 2.4. Therefore, applying  $\mathcal{P}$  on (1.3a) gives us exactly (2.12).

“If.” Assume  $\tilde{\mathbf{v}}$  solves (2.12), which can be recast as

$$\partial_t \tilde{\mathbf{v}} + \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} - \mathcal{Q}(\tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}) = 0.$$

By Proposition 2.1, there exists a scalar  $\phi$  so that  $\mathcal{Q}(\tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}) = \nabla \phi$  and therefore the above equation is of the same form as (1.3a) with pressure  $q = -\phi$ . Next, since (1.3d) ensures  $\nabla \cdot \tilde{\mathbf{v}}_0 = 0$  and taking divergence on (2.12) gives  $\partial_t (\nabla \cdot \tilde{\mathbf{v}}) = 0$ , we have  $\nabla \cdot \tilde{\mathbf{v}} = 0$ , i.e., (1.3b) satisfied for all  $t \in [0, T^*]$ . Finally, restrict (2.12) on  $\partial\Omega$ , take its dot product with  $\tilde{\mathbf{n}}$ , and use the definition of  $\mathcal{P}$  to obtain  $\partial_t (\tilde{\mathbf{v}} \cdot \tilde{\mathbf{n}}) = 0$  on  $\partial\Omega$ . Since (1.3d) ensures  $\tilde{\mathbf{v}}_0 \cdot \tilde{\mathbf{n}}|_{\partial\Omega} = 0$ , we have  $\tilde{\mathbf{v}} \cdot \tilde{\mathbf{n}}|_{\partial\Omega} = 0$ , i.e., (1.3c) validated for all  $t \in [0, T^*]$ .  $\square$

Problem is, in order to estimate the difference of (2.11) and (2.12), how can we bound the “slow-fast” term  $\mathcal{P}B(\mathbf{v}^P, \mathbf{v}^Q)$  by  $O(\epsilon^2)$ ? Because of nonlinearity, the slow subspace  $\ker \mathcal{L}$  is not invariant under the coupled slow-fast dynamics. This means, even with initial data  $\mathbf{v}_0^Q = 0$  and  $\mathbf{v}_0^P \sim O(1)$ , nonlinear coupling can lead to  $\mathbf{v}^Q \sim O(\epsilon)$  in later times.

To this end, we bring to focus the key idea of this article: a generic compressible solution  $\mathbf{v}$ , without being  $O(\epsilon^2)$  pointwise in time, can still be  $O(\epsilon^2)$  in terms of its *time-averages* as long as  $\mathbf{v}_0^Q \sim O(\epsilon)$  initially. Such estimate in turn will suffice to make  $(\mathbf{v}^P - \tilde{\mathbf{v}}) \sim O(\epsilon^2)$  pointwise in time. This is the subject of the next section.

**3. Pointwise-in-time error estimates using time-averages.** In this section, we demonstrate in Lemma 3.1 and Theorem 3.3 the crucial role of time-averages in estimating  $(\mathbf{v}^P - \tilde{\mathbf{v}})$  *pointwise in time*. For brevity, throughout this section, we assume solutions  $\rho, \mathbf{v}, \tilde{\mathbf{v}} \in \mathbf{C}([0, T], H^m(\Omega))$  for integer  $m \geq 3$ .

Define the time-averaging (indeed, integrating) operator

$$\overline{\mathbf{v}}(T, x) := \int_0^T \mathbf{v}(t, x) dt.$$

First, estimate the slow-fast product  $B(\mathbf{v}^P, \mathbf{v}^Q)$  of (2.11), which is the extra term compared to the incompressible system (2.12). By the product rule,  $B(\mathbf{v}^P, \mathbf{v}^Q) = \partial_t B(\mathbf{v}^P, \overline{\mathbf{v}^Q}) - B(\partial_t \mathbf{v}^P, \overline{\mathbf{v}^Q})$ , so we apply time-averaging,

$$\begin{aligned} \overline{B(\mathbf{v}^P, \mathbf{v}^Q)}(t) &= B(\mathbf{v}^P, \overline{\mathbf{v}^Q}) \Big|_0^t - \int_0^t B(\partial_t \mathbf{v}^P, \overline{\mathbf{v}^Q}) \\ &= B(\mathbf{v}^P, \overline{\mathbf{v}^Q}) \Big|_0^t + \int_0^t B(\mathcal{P}(\mathbf{v}^P \cdot \nabla \mathbf{v}^P) + \mathcal{P}B(\mathbf{v}^P, \mathbf{v}^Q), \overline{\mathbf{v}^Q}), \end{aligned}$$

where  $\partial_t \mathbf{v}^P$  was replaced via the slow dynamics (2.11). Apply the bounds of  $\mathcal{P}$ ,  $\mathcal{Q}$  as in (2.7) and the Sobolev inequalities (1.13) to obtain

$$(3.1) \quad \sup_{[0,T]} \|\overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathbf{H}^{m-2}} \lesssim \sup_{[0,T]} \left\{ \|\overline{\mathbf{v}^Q}\|_{\mathbf{H}^m} \|\mathbf{v}^P\|_{\mathbf{H}^m} (1 + T \|\mathbf{v}\|_{\mathbf{H}^m}) \right\}.$$

Here, the  $\|\overline{\mathbf{v}^Q}\|_{\mathbf{H}^m}$  factor measures the size of time-averaged fast variable, for which we will prove a crucial  $O(\epsilon^2)$  upper bound. In fact, by Proposition 2.3,

$$\|\overline{\mathbf{v}^Q}\|_{\mathbf{H}^m} \lesssim \|\overline{\nabla \cdot \mathbf{v}}\|_{\mathbf{H}^{m-1}} \lesssim \|\nabla(\overline{\nabla \cdot \mathbf{v}})\|_{\mathbf{H}^{m-2}},$$

where the last estimate is due to the Poincaré inequality and the zero spatial mean of  $\nabla \cdot \mathbf{v}$ . Now, replace  $\nabla(\overline{\nabla \cdot \mathbf{v}})$  on the RHS using the continuity equation (2.10a),

$$(3.2) \quad \|\overline{\mathbf{v}^Q}\|_{\mathbf{H}^m}(T) \lesssim \epsilon \left\| \nabla(\rho(T, \cdot) - \rho_0) + \nabla \int_0^T \nabla \cdot (\rho \mathbf{v}) \right\|_{\mathbf{H}^{m-2}}.$$

Already, we have gained an  $\epsilon$  factor in the bound of  $\overline{\mathbf{v}^Q}$ . But this is not enough for  $O(\epsilon^2)$ . A key *decoupling property* here is that the quadratic terms in the RHS contain no “slow-slow” product which would have made an  $O(\|\mathbf{v}^P\|^2) \sim O(1)$  contribution. Instead, everything in the RHS of (3.2) has a factor from the fast variables  $\rho, \mathbf{v}^Q$ , or more precisely  $(\nabla \cdot \mathbf{v}^Q, \nabla \rho) = \mathcal{L}(\rho, \mathbf{v})$ . Thus, combine (3.2) and Sobolev inequalities (1.13) to get

$$(3.3) \quad \|\overline{\mathbf{v}^Q}\|_{\mathbf{H}^m}(T) \lesssim \epsilon \sup_{[0,T]} \left\{ \|\mathcal{L}(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}} (1 + T \|\rho, \mathbf{v}\|_{\mathbf{H}^m}) \right\}.$$

Plug it into (3.1) to prove the following lemma.

LEMMA 3.1. *Let integer  $m \geq 3$ . Suppose the compressible Euler equations (1.7) admit solution  $(\rho, \mathbf{v}) \in \mathbf{C}([0, T], \mathbf{H}^m(\Omega))$ . Then, there exists a constant  $C$  solely depending on  $m, \Omega$  so that*

$$\sup_{[0,T]} \|\overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathbf{H}^{m-2}} \leq C \epsilon \sup_{[0,T]} \left\{ \|\mathcal{L}(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}} \|\mathbf{v}^P\|_{\mathbf{H}^m} (1 + T \|\rho, \mathbf{v}\|_{\mathbf{H}^m})^2 \right\}.$$

Meanwhile, estimate (1.10) relates  $\mathcal{L}(\rho, \mathbf{v}) \sim O(\epsilon)$  to  $\partial_t(\rho, \mathbf{v}) \sim O(1)$ . Thus, we will show in section 4, by preparing initial data so that  $\|\partial_t(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}} \sim O(1)$  at  $t = 0$ , it will remain  $O(1)$  for finite times, making  $\mathcal{L}(\rho, \mathbf{v}) \sim O(\epsilon)$ , hence making the RHS in the above lemma  $O(\epsilon^2)$ .

Having bounded the time-average of the “extra term”  $\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)$  in the compressible system (2.11), we move on to show how it helps us to estimate  $(\mathbf{v}^P - \tilde{\mathbf{v}})$ . Before the main result in Theorem 3.3, we prove the following technical lemma showing that error propagation in a bilinear time-dependent system heavily relies on the time-averages of its coefficients.

LEMMA 3.2. *Let  $\mathbf{b}(\cdot, \cdot)$  be a bilinear operator (made precise below). Consider time-dependent systems*

$$(3.4) \quad \partial_t u_i + \mathbf{b}(v_i, u_i) = \gamma_i, \quad i = 1, 2, \quad \text{subject to the same initial data}$$

with  $v_i \in \mathbf{C}([0, T], \mathcal{B})$ ,  $u_i \in \mathbf{C}([0, T], \mathcal{B}')$ , and  $\overline{\gamma_1 - \gamma_2} \in \mathbf{C}([0, T], \mathcal{B}'')$  for some Banach spaces  $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ .

Let  $\mathcal{H} \supset \mathcal{B}', \mathcal{B}''$  be a (real) Hilbert space so that  $b(\cdot, \cdot)$  is bounded as  $\mathcal{B} \times \mathcal{B}' \mapsto \mathcal{H}$ , and also bounded as  $\mathcal{B} \times \mathcal{B}'' \mapsto \mathcal{H}$ . Furthermore, assume

$$(3.5) \quad \langle u, b(v, u) \rangle_{\mathcal{H}} \leq C \|v\|_{\mathcal{B}} \|u\|_{\mathcal{H}}^2 \quad \text{for } v \in \mathcal{B}, u \in \mathcal{B}' \cap \mathcal{B}'',$$

with some constant  $C$ . Then, with  $M := \sup_{[0, T]} \{ \|v_1\|_{\mathcal{B}} \}$ , we have

$$\sup_{[0, T]} \|u_1 - u_2\|_{\mathcal{H}} \leq \sup_{[0, T]} \|\overline{\gamma_1 - \gamma_2}\|_{\mathcal{H}} + \frac{e^{CMT} - 1}{CM} \sup_{[0, T]} \left\{ \|b(v_1, \overline{\gamma_1 - \gamma_2}) + b(v_1 - v_2, u_2)\|_{\mathcal{H}} \right\}.$$

The significance of this result is that  $|u_1 - u_2|$  measured pointwise in time is affected by  $\gamma_i$  only via time-average  $(\overline{\gamma_1 - \gamma_2})$ , not the pointwise-in-time values of  $|\gamma_1 - \gamma_2|$ .

A similar effect comes from  $(\overline{v_1 - v_2})$  as well, but we will neither prove nor use it in this article.

*Proof* (Lemma 3.2). Set  $i = 1, 2$  in (3.4) and subtract them to get

$$\partial_t(u_1 - u_2) + b(v_1, u_1 - u_2) + b(v_1 - v_2, u_2) = \gamma_1 - \gamma_2.$$

Then, define the time integral of the RHS

$$\xi := \overline{(\gamma_1 - \gamma_2)},$$

and replace the RHS of the previous equation with  $\partial_t \xi$ , recasting it into

$$\partial_t(u_1 - u_2 - \xi) + b(v_1, u_1 - u_2 - \xi) = -b(v_1, \xi) - b(v_1 - v_2, u_2).$$

The boundedness hypotheses on  $b(\cdot, \cdot)$  guarantee every bilinear term above is in  $\mathbf{C}([0, T], \mathcal{H})$  and therefore so is  $\partial_t(u_1 - u_2 - \xi)$ . This allows us to take the  $\mathcal{H}$  inner product of this equation with  $(u_1 - u_2 - \xi)$ . Then, apply (3.5) to get

$$\frac{d}{dt} \|u_1 - u_2 - \xi\|_{\mathcal{H}} \leq C \|v_1\|_{\mathcal{B}} \|u_1 - u_2 - \xi\|_{\mathcal{H}} + \|b(v_1, \overline{\gamma_1 - \gamma_2}) + b(v_1 - v_2, u_2)\|_{\mathcal{H}}.$$

Finally, for fixed  $T$ , relax  $\|v_1\|_{\mathcal{B}}$  to  $M$  and also relax the last term to its maximum over  $[0, T]$  to arrive at a differential inequality with constant coefficients. This is easily solved to confirm the desired conclusion.  $\square$

We also state a convenient fact due to  $\mathbf{H}^2 \subset \mathbf{L}^\infty$  on the two- or three-dimensional domain,

$$(3.6) \quad \text{for } k \geq 0, \quad \|fg\|_{\mathbf{H}^k} \lesssim \|f\|_{\mathbf{H}^{k+2}} \|g\|_{\mathbf{H}^k}.$$

**THEOREM 3.3** (time-averaging estimates). *Consider the incompressible Euler equations (2.12) and the slow dynamics (2.11), i.e.,*

$$(3.7) \quad \partial_t \tilde{\mathbf{v}} + \mathcal{P}(\tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}}) = 0,$$

$$(3.8) \quad \partial_t \mathbf{v}^P + \mathcal{P}(\mathbf{v}^P \cdot \nabla \mathbf{v}^P) = -\mathcal{P}\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q),$$

both of which are subject to the solid-wall boundary condition and the same initial data  $\tilde{\mathbf{v}}_0 = \mathbf{v}_0^P$ . Suppose  $(\tilde{\mathbf{v}}, \rho, \mathbf{v} = \mathbf{v}^P + \mathbf{v}^Q) \in \mathbf{C}([0, T^b], \mathbf{H}^m(\Omega))$  for  $m \geq 3$ . Then, there exists constants  $D_1, D_2$  only dependent on  $m, \Omega$  so that

$$\sup_{[0, T]} \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathbf{H}^{m-3}} \leq \epsilon D_1 \sup_{[0, T]} \left\{ \|\mathcal{L}(\rho, \mathbf{v})\|_{\mathbf{H}^{m-1}} \|\mathbf{v}^P\|_{\mathbf{H}^m} \right\}$$

for  $T \in [0, T^b] \cap [0, D_2 / \sup_{[0, T^b]} \|(\rho, \mathbf{v}, \tilde{\mathbf{v}})\|_{\mathbf{H}^m}]$ .

*Proof.* Recall we defined  $\mathbf{X}$  in (2.1) as the space of incompressible velocity fields subject to solid-wall boundary condition. To fit the notation of Lemma 3.2, set

$$\begin{aligned} u_1 = v_1 &= \tilde{\mathbf{v}}, & \gamma_1 &= 0, \\ u_2 = v_2 &= \mathbf{v}^P, & \gamma_2 &= -\mathcal{P}\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q), \\ \mathcal{B} = \mathcal{B}' &= \mathbf{H}^m \cap \mathbf{X}, & \mathcal{B}'' &= \mathbf{H}^{m-2} \cap \mathbf{X}, \end{aligned}$$

and in particular set

$$\mathcal{H} = \mathbf{H}^{m-3} \cap \mathbf{X}.$$

We then endow  $\mathcal{H}$  with the following inner product:

$$(3.9) \quad \langle \mathbf{v}, \mathbf{v}' \rangle_{\mathcal{H}} := \begin{cases} \langle \mathbf{v}, \mathbf{v}' \rangle_{\mathbf{L}^2} & \text{if } m = 3, \\ \langle \mathbf{v}, \mathbf{v}' \rangle_{\mathbf{L}^2} + \langle \nabla \times \mathbf{v}, \nabla \times \mathbf{v}' \rangle_{\mathbf{H}^{m-4}} & \text{if } m \geq 4. \end{cases}$$

The induced  $\mathcal{H}$  norm, by the virtue of (2.6), is then equivalent to the  $\mathbf{H}^{m-3}$  norm

$$(3.10) \quad \|\mathbf{v}\|_{\mathcal{H}} \lesssim \|\mathbf{v}\|_{\mathbf{H}^{m-3}} \lesssim \|\mathbf{v}\|_{\mathcal{H}}.$$

By (2.7), we see  $\mathcal{P}$  is a bounded operator over  $\mathcal{H}$  as well as bounded over  $\mathbf{H}^{m-3}$ .

Set the bilinear operator

$$(3.11) \quad \mathbf{b}(\mathbf{v}, \mathbf{v}') = \mathcal{P}(\mathbf{v} \cdot \nabla \mathbf{v}')$$

so that, by (3.10) and (3.6) ( $k = m - 3$ ), all the boundedness assumptions on  $\mathbf{b}(\cdot, \cdot)$  in Lemma 3.2 are satisfied. This leaves us only (3.5) to validate, i.e., we need to show, with  $m \geq 3$ ,

$$(3.12) \quad \langle \mathbf{v}', \mathcal{P}(\mathbf{v} \cdot \nabla \mathbf{v}') \rangle_{\mathcal{H}} \lesssim \|\mathbf{v}\|_{\mathbf{H}^m} \|\mathbf{v}'\|_{\mathbf{H}^{m-3}}^2 \quad \text{for } \mathbf{v} \in \mathbf{H}^m \cap \mathbf{X}, \mathbf{v}' \in \mathbf{H}^{m-2} \cap \mathbf{X}.$$

Here, we used the  $\mathbf{H}^{m-3}$  norm instead of the  $\mathcal{H}$  norm in the RHS, thanks to (3.10). Indeed, by definition (3.9), we estimate the  $\mathbf{L}^2$  component and the higher derivative component, respectively, in the above inequality. The  $\mathbf{L}^2$  component is simply zero as we can use the  $\mathbf{L}^2$ -orthogonality of  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathbf{v}' \in \mathbf{X} = \text{image } \mathcal{P}$  to get

$$(3.13) \quad \langle \mathbf{v}', \mathcal{P}(\mathbf{v} \cdot \nabla \mathbf{v}') \rangle_{\mathbf{L}^2} = \langle \mathbf{v}', \mathbf{v} \cdot \nabla \mathbf{v}' \rangle_{\mathbf{L}^2} = \int_{\Omega} \frac{1}{2} \mathbf{v} \cdot \nabla |\mathbf{v}'|^2 = 0.$$

Here, the last equality is due to the divergence theorem and  $\mathbf{v} \in \mathbf{X} \cap \mathbf{H}^m$ . Also note from (3.12) that  $\mathbf{v}' \in \mathbf{H}^{m-2}$ , causing no regularity problem.

For the higher derivatives of (3.12), it is needed only for  $m \geq 4$ . Upon taking curl of (3.11) and using  $\nabla \times (\mathcal{P}\mathbf{v}) = \nabla \times \mathbf{v}$  given in (2.4), one has

$$\nabla \times \mathcal{P}(\mathbf{v} \cdot \nabla \mathbf{v}') = \mathbf{v} \cdot \nabla (\nabla \times \mathbf{v}') + \mathbf{b}_p(\nabla \mathbf{v}, \nabla \mathbf{v}')$$

for some bilinear *polynomial*  $\mathbf{b}_p(\cdot, \cdot)$ . By (3.6) ( $k = m - 4$ ),

$$\|\mathbf{b}_p(\nabla \mathbf{v}, \nabla \mathbf{v}')\|_{\mathbf{H}^{m-4}} \lesssim \|\mathbf{v}\|_{\mathbf{H}^m} \|\mathbf{v}'\|_{\mathbf{H}^{m-3}}.$$

Also by (3.6) ( $k = 0$ ),

$$\sum_{j=0}^{m-5} \left\| \nabla^j [\mathbf{v} \cdot \nabla (\nabla \times \mathbf{v}')] \right\|_{\mathbf{L}^2} + \left\| \nabla^{m-4} [\mathbf{v} \cdot \nabla (\nabla \times \mathbf{v}')] - \mathbf{v} \cdot \nabla (\nabla^{m-4} \nabla \times \mathbf{v}') \right\|_{\mathbf{L}^2} \lesssim \|\mathbf{v}\|_{\mathbf{H}^m} \|\mathbf{v}'\|_{\mathbf{H}^{m-3}}.$$

Combining the above three lines of calculations, we obtain

$$\langle \nabla \times \mathbf{v}', \nabla \times \mathcal{P}(\mathbf{v} \cdot \nabla \mathbf{v}') \rangle_{\mathbf{H}^{m-4}} \lesssim \|\mathbf{v}\|_{\mathbf{H}^m} \|\mathbf{v}'\|_{\mathbf{H}^{m-3}}^2 + \langle \nabla^{m-4} \nabla \times \mathbf{v}', \mathbf{v} \cdot \nabla (\nabla^{m-4} \nabla \times \mathbf{v}') \rangle_{L^2}.$$

But the last term is simply zero by the divergence theorem and  $\mathbf{v} \in \mathbf{X} \cap \mathbf{H}^m$ . Therefore, (3.12), hence (3.5), is confirmed.

Having validated all assumptions of Lemma 3.2, we arrive at, with  $M := \sup_{[0,T]} \{\|\tilde{\mathbf{v}}\|_{\mathbf{H}^m}\}$ ,

$$(3.14) \quad \sup_{[0,T]} \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathcal{H}} \leq \sup_{[0,T]} \|\overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathcal{H}} + \frac{e^{CMT} - 1}{CM} \sup_{[0,T]} \left\{ \|\mathbf{b}(\tilde{\mathbf{v}}, \overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}) + \mathbf{b}(\tilde{\mathbf{v}} - \mathbf{v}^P, \mathbf{v}^P)\|_{\mathcal{H}} \right\}.$$

To estimate the second last bilinear term, apply (3.6) ( $k = m - 3$ ),

$$\|\mathbf{b}(\tilde{\mathbf{v}}, \overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)})\|_{\mathcal{H}} \lesssim \|\tilde{\mathbf{v}} \cdot \nabla \overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathbf{H}^{m-3}} \lesssim \|\tilde{\mathbf{v}}\|_{\mathbf{H}^m} \|\overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathbf{H}^{m-2}}.$$

Comparing the last  $\mathbf{H}^{m-2}$  norm with Lemma 3.1, we see why we had to choose the  $\mathbf{H}^{m-3}$  norm in this current theorem.

To estimate the last bilinear term of (3.14), apply (3.6) (with  $k = m - 3$  but  $f = \nabla \mathbf{v}^P, g = \tilde{\mathbf{v}} - \mathbf{v}^P$ )

$$\|\mathbf{b}(\tilde{\mathbf{v}} - \mathbf{v}^P, \mathbf{v}^P)\|_{\mathcal{H}} \lesssim \|(\tilde{\mathbf{v}} - \mathbf{v}^P) \cdot \nabla \mathbf{v}^P\|_{\mathbf{H}^{m-3}} \lesssim \|\mathbf{v}^P\|_{\mathbf{H}^m} \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathbf{H}^{m-3}}.$$

Note it is crucial to only use the  $\mathbf{H}^{m-3}$  norm of  $(\tilde{\mathbf{v}} - \mathbf{v}^P)$  because it is needed to bring closure to the  $\mathbf{H}^{m-3}$  estimate started from the LHS of (3.14).

Substitute the above two estimates into (3.14), use the equivalence (3.10), and rearrange to obtain, with  $M = \sup_{[0,T]} \{\|\tilde{\mathbf{v}}\|_{\mathbf{H}^m}\}$ ,  $M^P := \sup_{[0,T]} \{\|\mathbf{v}^P\|_{\mathbf{H}^m}\}$ ,

$$(3.15) \quad \left(1 - c_3 \frac{e^{CMT} - 1}{CM} M^P\right) \sup_{[0,T]} \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathbf{H}^{m-3}} \leq (c_1 + c_2(e^{CMT} - 1)) \sup_{[0,T]} \|\overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathbf{H}^{m-2}}$$

for some universal positive constants  $c_1, c_2, c_3, C$ . If

$$(3.16) \quad MT, M^P T \leq D_2,$$

for a constant  $D_2$ , then by the mean value theorem,

$$\frac{e^{CMT} - 1}{CM} M^P = \frac{e^{CMT} - 1}{CMT} T M^P \leq e^{CD_2} D_2.$$

Thus, we can choose  $D_2$  so that the coefficient in the LHS of (3.15) is no less than 1/2, yielding

$$\sup_{[0,T]} \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathbf{H}^{m-3}} \leq 2 (c_1 + c_2(e^{CD_2} - 1)) \sup_{[0,T]} \|\overline{\mathbf{B}(\mathbf{v}^P, \mathbf{v}^Q)}\|_{\mathbf{H}^{m-2}}.$$

Combining it with Lemma 3.1 and (3.16), we conclude the proof.  $\square$

**4. Estimates independent of  $\epsilon$ .** What does it mean to have estimates independent of  $\epsilon$ ? Basically, we want calculations to be invariant under the hyperbolic scaling  $\rho \rightarrow c\rho, \mathbf{v} \rightarrow c\mathbf{v}, t \rightarrow t/c, \epsilon \rightarrow \epsilon/c$  for any constant  $c > 0$ . In Theorem 4.6, for example, we will show  $H^m$  solutions exist at least for time interval of order  $1/\|(\rho_0, \mathbf{v}_0)\|_{H^m}$ . During this time interval, the solution’s  $H^m$  norm is at most inflated by a constant and, very importantly, the  $H^{m-1}$  norm of its first time derivative is also only inflated by a constant. The latter estimate can be loosely stated as “what starts well-prepared stays well-prepared.” Recall that having  $\partial_t(\rho, \mathbf{v}) \sim O(1)$  is equivalent to having  $(\nabla \cdot \mathbf{v}, \nabla \rho) \sim O(\epsilon)$  as suggested in (1.10)

The main difficulty is, even though  $\mathcal{L}$  is skew-self-adjoint, namely,  $\int_{\Omega} (\frac{\rho}{\mathbf{v}}) \cdot \mathcal{L}(\frac{\rho}{\mathbf{v}}) dx = 0$  for  $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$ , it is in general not the case for the spatial derivatives, namely,

$$\int_{\Omega} \partial_x^\beta \left( \frac{\rho}{\mathbf{v}} \right) \cdot \partial_x^\beta \mathcal{L} \left( \frac{\rho}{\mathbf{v}} \right) dx \neq 0 \quad \text{if } \beta \neq 0 \text{ and } \partial\Omega \neq \emptyset.$$

This would’ve introduced  $O(\epsilon^{-1})$  terms in the energy estimate. In addition, a very sophisticated mollification procedure would’ve been needed in estimating the highest spatial derivatives [17].

On the other hand, we can recruit higher time derivatives since  $\partial\Omega$  is static and

$$(4.1) \quad \int_{\Omega} \partial_t^k \left( \frac{\rho}{\mathbf{v}} \right) \cdot \partial_t^k \mathcal{L} \left( \frac{\rho}{\mathbf{v}} \right) dx = 0 \quad \text{if } \partial_t^k \left( \frac{\rho}{\mathbf{v}} \right) \in H^1(\Omega) \text{ and } \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

If one considers  $[0, T] \times \partial\Omega$  as the lateral boundary of the time-space domain, then the  $\partial_t$  derivatives are precisely taken in the tangential directions, resonating with the argument in [17] that, near a characteristic boundary, tangential and normal derivatives are estimated differently.

We first rescale the original system into an equivalent one without explicit dependence on  $\epsilon$ . At the end of this section, we scale it back to the original formulation for which the essential results will still be independent of  $\epsilon$ , so long as hyperbolic scaling is respected in the estimates.

To this end, we introduce

$$(4.2) \quad V := (\check{\rho}, \check{\mathbf{v}}) := (\epsilon\rho, \epsilon\mathbf{v}), \quad \tau := t/\epsilon,$$

and rewrite the Euler equations (1.7) for  $(\rho, \mathbf{v})$  into a system for  $V$ ,

$$(4.3) \quad \partial_\tau V + \check{\mathbf{v}} \cdot \nabla V = -\sigma(\check{\rho})\mathcal{L}(V), \quad \check{\mathbf{v}} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

where the diagonal matrix

$$\sigma(\check{\rho}) := \text{diag} \left\{ 1 + \check{\rho}, \frac{\pi'(1 + \check{\rho})}{1 + \check{\rho}}, \dots, \frac{\pi'(1 + \check{\rho})}{1 + \check{\rho}} \right\}.$$

This rescaled,  $\epsilon$ -free system will be the main subject of this section.

Since by the scaling assumption (1.2) we have  $\sigma(0) = I$  and  $\sigma(\check{\rho})$  is  $\mathbf{C}^\infty$ , there exists a constant  $R_\pi$  only depending on  $\pi(\cdot)$  such that

$$(4.4) \quad \text{for } |\check{\rho}| \leq R_\pi, \quad \text{diagonal matrix } \sigma(\check{\rho}) \in [1/2, 2].$$

For example, if  $\pi(\cdot)$  satisfies the  $\gamma$ -power law, then one must have  $\pi'(1 + \check{\rho}) = (1 + \check{\rho})^{\gamma-1}$ . For the physically relevant case  $\gamma \in [1, 2)$ , it is easy to calculate  $R_\pi = \min\{\frac{1}{2}, 1 - 2^{\frac{1}{1-\gamma}}\}$ .



We introduce a mixed norm for  $V(\tau, x)$  at any fixed time  $\tau$ ,

$$\|V\|_m(\tau) := \left( \sum_{k=0}^m \|\partial_\tau^k V\|_{H^{m-k}(\Omega)}^2(\tau) \right)^{1/2}.$$

It encapsulates all mixed space-time derivatives up to order  $m$ .

For brevity, we make the a priori assumption throughout this section that

$$(4.5) \quad \text{integer } m \geq 3, \quad V \in \mathbf{C}([0, \tau^*], H^m(\Omega)), \quad \|V\|_m(\tau) \leq 1, \quad \text{and} \quad |\check{\rho}(\tau, x)| \leq R_\pi$$

for all  $\tau \in [0, \tau^*]$ , unless specified otherwise. The last two inequalities are not stringent at all because by the scaling of  $V$  in (4.2), (4.3), one expects  $\|V\|_m$  and  $|\check{\rho}|$  to be  $O(\epsilon)$  as long as the original unknown satisfies scaling  $\|(r, \mathbf{v})\|_{H^m} \sim O(1)$ .

**4.1. Mixed norms.** First, we establish some basic facts of  $\|\cdot\|$  as direct consequences of Sobolev inequalities. Given functions  $f_1(\tau, x), f_2(\tau, x), \dots, f_j(\tau, x)$  and a product of mixed mixed derivatives  $(\partial_{\tau,x}^{\beta_1} f_1) (\partial_{\tau,x}^{\beta_2} f_2) \dots (\partial_{\tau,x}^{\beta_j} f_j)$ , we have

$$(4.6) \quad \left\| \prod_{i=1}^j \partial_{\tau,x}^{\beta_i} f_i \right\|_{L^2} \lesssim \prod_{i=1}^j \|f_i\|_k$$

if one of the following conditions holds:

$$\begin{cases} |\beta_1 + \dots + \beta_j| \leq k, \quad \text{and} \quad k \geq 2, \quad \text{or} \\ |\beta_1 + \dots + \beta_j| = k + 1, \quad \max\{|\beta_1|, \dots, |\beta_j|\} \leq k, \quad \text{and} \quad k \geq 3. \end{cases}$$

The proof is exactly the same as the one given in the appendix where only spatial derivatives are involved.

Estimates on  $\sigma(\check{\rho})$  and its matrix inverse  $\sigma^{-1}(\check{\rho})$  will also be needed. By (4.4),  $\check{\rho}$ -derivatives of  $\sigma$  and  $\sigma^{-1}$  can be bounded by constants only depending on pressure law  $\pi(\cdot)$  and the order of derivatives, i.e.,

$$(4.7) \quad \left| \frac{d^k}{d\check{\rho}^k} \sigma(\check{\rho}) \right| + \left| \frac{d^k}{d\check{\rho}^k} \sigma^{-1}(\check{\rho}) \right| \lesssim 1 \quad \text{for} \quad |\check{\rho}| \leq R_\pi.$$

Also, by the mean value theorem and  $\sigma(0) = I$ ,

$$(4.8) \quad |\sigma(\check{\rho}) - I| + |\sigma^{-1}(\check{\rho}) - I| \lesssim |\check{\rho}| \quad \text{for} \quad |\check{\rho}| \leq R_\pi.$$

Now, inductively apply the chain rule and product rule to obtain, for multi-index  $\beta$ ,

$$\begin{aligned} \partial_{\tau,x}^\beta \sigma(\check{\rho}) &= \text{linear combination of } (\partial_{\tau,x}^{\beta_1} \check{\rho}) (\partial_{\tau,x}^{\beta_2} \check{\rho}) \dots (\partial_{\tau,x}^{\beta_j} \check{\rho}) \frac{d^j}{d\check{\rho}^j} \sigma(\check{\rho}) \\ &\text{over all integers } j \in [1, |\beta|] \text{ and multi-indices satisfying } \beta_1 + \dots + \beta_j = \beta. \end{aligned}$$

Therefore, by (4.6), (4.7), and the assumptions  $\|V\|_m \leq 1, m \geq 3$  set in (4.5),

$$\|\partial_{\tau,x}^\beta \sigma(\check{\rho})\|_{L^2} \lesssim \|\check{\rho}\|_m \quad \text{for } 1 \leq |\beta| \leq m.$$

Obviously, this estimate works for matrix inverse  $\sigma^{-1}$  as well. Sum such estimates for all  $|\beta|$  from 1 to  $m$ , and use (4.8) for the  $|\beta| = 0$  case to arrive at

$$(4.9) \quad \|\sigma(\check{\rho}) - I\|_m + \|\sigma^{-1}(\check{\rho}) - I\|_m \lesssim \|\check{\rho}\|_m.$$

A similar estimate works for  $\partial_\tau \sigma$ . In fact, applying (4.6) to  $\partial_\tau \sigma = (\frac{d}{d\bar{\rho}} \sigma(\check{\rho}) - \frac{d}{d\bar{\rho}} \sigma(0)) \partial_\tau \check{\rho} + \frac{d}{d\bar{\rho}} \sigma(0) \partial_\tau \check{\rho}$ , and noting that  $\frac{d}{d\bar{\rho}} \sigma(\check{\rho}) - \frac{d}{d\bar{\rho}} \sigma(0)$  can be bounded in a way similar to (4.9), namely,  $\|\frac{d}{d\bar{\rho}} \sigma(\check{\rho}) - \frac{d}{d\bar{\rho}} \sigma(0)\|_{m-1} \lesssim \|\check{\rho}\|_{m-1} \leq 1$ , we have

$$(4.10) \quad \|\partial_\tau \sigma(\check{\rho})\|_{m-1} + \|\partial_\tau \sigma^{-1}(\check{\rho})\|_{m-1} \lesssim \|\partial_\tau \check{\rho}\|_{m-1}.$$

Now, thanks to the  $\epsilon$ -free formulation of (4.3), combine (4.9), (4.10) with Sobolev inequalities (4.6) to inductively estimate  $\partial_\tau V, \partial_\tau^2 V, \dots, \partial_\tau^m V$  and obtain, under the a priori assumption (4.5)

$$(4.11) \quad \|\partial_\tau V\|_{m-1} \lesssim \|\partial_\tau V\|_{\mathbf{H}^{m-1}} \leq \|V\|_m \lesssim \|V\|_{\mathbf{H}^m}.$$

**4.2. Vorticity estimates.** Define  $\check{\omega} := \nabla \times \check{\mathbf{v}}$ . Take  $\nabla \times$  of the momentum equation in (4.3),

$$(4.12) \quad \partial_\tau \check{\omega} + \mathbf{b}^{vor}(\check{\mathbf{v}}, \check{\omega}) = 0,$$

$$\text{where} \quad \mathbf{b}^{vor}(\mathbf{v}, \omega) = \begin{cases} \mathbf{v} \cdot \nabla \omega + (\nabla \cdot \mathbf{v}) \omega & \text{in two dimensions} \\ \mathbf{v} \cdot \nabla \omega + (\nabla \cdot \mathbf{v}) \omega - \omega \cdot \nabla \mathbf{v} & \text{in three dimensions} \end{cases}$$

It is then an exercise of energy estimates with Sobolev inequalities to show (e.g., [25, section 17.3]), with any  $\check{\mathbf{v}}$  satisfying (4.5) and  $\mathbf{v} \cdot \check{\mathbf{n}}|_{\partial\Omega} = 0$ ,

$$(4.13) \quad \|\check{\omega}\|_{\mathbf{H}^{m-1}}^2 \Big|_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} \|\check{\omega}\|_{\mathbf{H}^{m-1}}^2 \|\nabla_x \check{\mathbf{v}}\|_{\mathbf{H}^{m-1}} \quad \text{for } 0 \leq \tau_1 < \tau_2 \leq \tau^*.$$

Note that due to  $\check{\omega} \in \mathbf{H}^{m-1}(\Omega)$ , before taking the  $(m-1)$ th spatial derivatives of (4.12), one should mollify  $\check{\omega}$  by first extending it outside  $\partial\Omega$ , then convolving it with a smooth kernel and restricting it back to  $\Omega$  — cf. [25, p. 489]. Velocity  $\check{\mathbf{v}}$  needs not to be mollified because it is assumed to be  $\mathbf{H}^m(\Omega)$ . It also means  $\check{\mathbf{v}}|_{\partial\Omega}$  is a defined function, allowing us to utilize the  $\check{\mathbf{v}} \cdot \check{\mathbf{n}}|_{\partial\Omega} = 0$  condition and divergence theorem to show such identities as

$$\int_{\Omega} (\check{\mathbf{v}} \cdot \nabla f) f \, dx = -\frac{1}{2} \int_{\Omega} (\nabla \cdot \check{\mathbf{v}}) |f|^2 \, dx \quad \text{for } f \in \mathbf{H}^1(\Omega).$$

Next, recall (2.5) which gives a bound for  $V$  using  $\mathcal{L}(V) = (\nabla \cdot \check{\mathbf{v}}, \nabla \check{\rho})$  and  $\nabla \times \check{\mathbf{v}}$ ,

$$(4.14) \quad \|V\|_{\mathbf{H}^k} \lesssim \|\mathcal{L}(V)\|_{\mathbf{H}^{k-1}} + \|\nabla \times \check{\mathbf{v}}\|_{\mathbf{H}^{k-1}} + \|V\|_{\mathbf{L}^2} \quad \text{if } \check{\mathbf{v}} \cdot \check{\mathbf{n}}|_{\partial\Omega} = 0 \text{ and } k \geq 1.$$

So it remains to estimate  $\mathcal{L}(V)$ . In light of (4.1) and the fact that  $\mathcal{L}(V)$  and  $\partial_\tau V$  are connected via (4.3), we now move on to estimate  $\|\partial_t^k V\|_{\mathbf{L}^2}$  for  $k = 0, 1, \dots, m$ .

**4.3. Diagnostic estimates and recurrence.** We now connect mixed norms of  $\mathcal{L}(V)$  and  $\partial_\tau V$  via rescaled system (4.3) and its time derivatives at any fixed time. We call such estimates “diagnostic” as opposed to “prognostic” estimates such as (4.13) for  $\check{\omega}$  in the form of time-dependent integral and differential inequalities. Diagnostic estimates do *not* rely on evolutionary properties of (4.3) and will not involve Gronwall-type inequalities. They instead come from algebraic manipulation of (4.3) at a fixed time  $\tau$ , typically using the product rule and Sobolev inequalities.

Elliptic estimate (4.14) and the fact that  $\partial_\tau^k \check{\mathbf{v}} \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$  for  $k \in [0, m - 1]$  imply

$$\|\partial_\tau^k V\|_{\mathbf{H}^{m-k}} \lesssim \|\mathcal{L}(\partial_\tau^k V)\|_{\mathbf{H}^{m-k-1}} + \|\nabla \times \partial_\tau^k \check{\mathbf{v}}\|_{\mathbf{H}^{m-k-1}} + \|\partial_\tau^k V\|_{\mathbf{L}^2}.$$

On the other hand, take  $\partial_\tau^k$  derivative on (4.3) to get, for  $k \in [0, m - 1]$ ,

$$-\mathcal{L}(\partial_\tau^k V) = \partial_\tau^{k+1} V + \partial_\tau^k (\check{\mathbf{v}} \cdot \nabla V) + \partial_\tau^k ((\sigma - I)\mathcal{L}(V)).$$

Combine these two to obtain a recursive inequality, for every  $k \in [0, m - 1]$ ,

$$(4.15) \quad \|\partial_\tau^k V\|_{\mathbf{H}^{m-k}} \lesssim \|\partial_\tau^{k+1} V\|_{\mathbf{H}^{m-k-1}} + \|\partial_\tau^k (\check{\mathbf{v}} \cdot \nabla V, (\sigma - I)\mathcal{L}(V), \check{\omega})\|_{\mathbf{H}^{m-k-1}} + \|\partial_\tau^k V\|_{\mathbf{L}^2}.$$

At one end of this recursive chain is  $\|V\|_{\mathbf{H}^m}$ , the desirable norm, and at the other end is  $\|\partial_\tau^m V\|_{\mathbf{H}^0}$ , which will be estimated prognostically using energy method in the next subsection.

Now, exclude  $k = 0$  and connect from  $k = 1$  to  $m - 1$  in (4.15), using the definition of  $\|\cdot\|$ , and replace  $\partial_\tau \check{\omega}$  with the bilinear term from (4.12) to get

$$\|\partial_\tau V\|_{\mathbf{H}^{m-1}} \lesssim \|\partial_\tau (\check{\mathbf{v}} \cdot \nabla V, (\sigma - I)\mathcal{L}(V))\|_{m-2} + \|\mathbf{b}^{vor}(\check{\mathbf{v}}, \check{\omega})\|_{m-2} + \sum_{k=1}^m \|\partial_\tau^k V\|_{\mathbf{L}^2}.$$

Combine it with Sobolev inequalities (4.6), bounds on  $(\sigma - I)$  in (4.9), and the equivalence of  $\|\cdot\|_m$  and  $\|\cdot\|_{\mathbf{H}^m}$  in (4.11) to reach the following lemma.

LEMMA 4.1 (diagnostic estimates on first time derivative). *Consider  $V(\tau, x)$ , a solution of (4.3) in the a priori setting of (4.5). Then,*

$$\|\partial_\tau V\|_{\mathbf{H}^{m-1}} \lesssim \|V\|_{\mathbf{H}^m}^2 + \sum_{k=1}^m \|\partial_\tau^k V\|_{\mathbf{L}^2}.$$

The extra quadratic term will cause no trouble, due to the scaling argument below (4.5).

For the  $k = 0$  case of (4.15), upon applying the same (4.6), (4.9), (4.11), one obtains  $\|V\|_{\mathbf{H}^m} \lesssim \|\partial_\tau V\|_{\mathbf{H}^{m-1}} + \|\check{\omega}\|_{\mathbf{H}^{m-1}} + \|V\|_{\mathbf{H}^m}^2 + \|V\|_{\mathbf{L}^2}$ . Combine it with Lemma 4.1 to get

$$2d_1 \|V\|_{\mathbf{H}^m} \leq \|\check{\omega}\|_{\mathbf{H}^{m-1}} + \|V\|_{\mathbf{H}^m}^2 + \sum_{k=0}^m \|\partial_\tau^k V\|_{\mathbf{L}^2}$$

for some constant  $d_1 > 0$ . Then, relax one of the  $\|V\|_{\mathbf{H}^m}$  factors on the RHS to  $d_1$  and absorb the associated quadratic term into the LHS, proving the next lemma.

LEMMA 4.2 (diagnostic estimates). *Consider  $V(\tau, x)$ , a solution of (4.3) in the a priori setting of (4.5). If furthermore  $\|V\|_{\mathbf{H}^m} \leq d_1$  for some constant  $d_1$  solely depending on  $m, \Omega$  and pressure law  $\pi(\cdot)$ , then*

$$d_1 \|V\|_{\mathbf{H}^m} \leq \|\check{\omega}\|_{\mathbf{H}^{m-1}} + \sum_{k=0}^m \|\partial_\tau^k V\|_{\mathbf{L}^2}.$$

**4.4. Prognostic estimates of  $\|\partial_\tau^k V\|_{L^2}$ .** Take the  $\partial_\tau^k$  derivative of (4.3) and single out the highest derivatives,

$$(4.16) \quad R^{(k)} := -\sum_{j=1}^k \binom{k}{j} (\partial_\tau^j \check{\nu}) \cdot \nabla \partial_\tau^{k-j} V + \binom{k}{j} (\partial_\tau^j \sigma) \mathcal{L}(\partial_\tau^{k-j} V) = \partial_\tau(\partial_\tau^k V) + \check{\nu} \cdot \nabla \partial_\tau^k V + \sigma \mathcal{L}(\partial_\tau^k V).$$

PROPOSITION 4.3 (diagnostic estimates of commutator). *Given a solution  $V(\tau, x)$  to (4.3) satisfying a priori assumption (4.5). Then, for all  $k \in [0, m]$ ,*

$$\|R^{(k)}\|_{L^2} \lesssim \|\partial_\tau V\|_{H^{m-1}} \|V\|_{H^m}.$$

*Proof.* The case  $k = 0$  is trivial, so we consider  $k \in [1, m]$ . Notice that in the definition of  $R^{(k)}$  in (4.16), every product contains a factor as  $\partial_\tau \sigma$ ,  $\partial_\tau \check{\nu}$  or their higher derivative, and another factor as  $\nabla V$  or its higher derivative. Therefore, by Sobolev inequalities (4.6) with  $k = m - 1$ ,

$$\|R^{(k)}\|_{L^2} \lesssim \|(\partial_\tau \sigma, \partial_\tau \check{\nu})\|_{m-1} \|\partial_x V\|_{m-1}.$$

Combine it with (4.10) and (4.11) to conclude the proof.  $\square$

Before obtaining estimates of  $\|\partial_t^k V\|_{L^2}$  in the next lemma, we carry out some calculation for any  $W$  with the same number of components as  $V$  and with regularity  $W \in H^1(\Omega)$ ,  $\partial_\tau W \in L^2(\Omega)$ .

$$\begin{aligned} & 2 \int_\Omega (\sigma^{-1} W) \cdot (\partial_\tau W + \check{\nu} \cdot \nabla W) \\ \stackrel{(a)}{=} & \int_\Omega \partial_\tau (\sigma^{-1} W \cdot W) + \int_\Omega \check{\nu} \cdot \nabla (\sigma^{-1} W \cdot W) - \left( \int_\Omega (\partial_\tau \sigma^{-1}) W \cdot W + \int_\Omega (\check{\nu} \cdot \nabla \sigma^{-1}) W \cdot W \right) \\ \stackrel{(b)}{=} & \int_\Omega \partial_\tau (\sigma^{-1} W \cdot W) + \int_\Omega \check{\nu} \cdot \nabla (\sigma^{-1} W \cdot W) - \int_\Omega (\partial_\tau \check{\rho} + \check{\nu} \cdot \nabla \check{\rho}) \left( \frac{d}{d\check{\rho}} \sigma^{-1} W \cdot W \right) \\ \stackrel{(c)}{=} & \int_\Omega \partial_\tau (\sigma^{-1} W \cdot W) - \int_\Omega (\nabla \cdot \check{\nu}) (\sigma^{-1} W \cdot W) - \int_\Omega (\partial_\tau \check{\rho} + \check{\nu} \cdot \nabla \check{\rho}) \left( \frac{d}{d\check{\rho}} \sigma^{-1} W \cdot W \right) \\ \stackrel{(d)}{=} & \frac{d}{d\tau} \int_\Omega \sigma^{-1} W \cdot W - \int_\Omega \nabla \cdot \check{\nu} \left( \sigma^{-1} - \sigma_{1,1} \frac{d}{d\check{\rho}} \sigma^{-1} \right) W \cdot W. \end{aligned}$$

Here, (a) is by the product rule and the fact that  $\sigma$  is diagonal, (b) is by the chain rule, (c) is by the divergence theorem and  $\check{\nu} \cdot \vec{n}|_{\partial\Omega} = 0$ , and (d) is a simple substitution via the mass equation of (4.3) with  $\sigma_{1,1}$  denoting the first entry of matrix  $\sigma$ .

Next, add and subtract a  $2 \int_\Omega (\sigma^{-1} W) \cdot (\sigma \mathcal{L}(W)) = 2 \int_\Omega W \cdot \mathcal{L}(W)$  term and rearrange it,

$$\begin{aligned} \frac{d}{d\tau} \int_\Omega \sigma^{-1} W \cdot W &= -2 \int_\Omega W \cdot \mathcal{L}(W) + \int_\Omega \nabla \cdot \check{\nu} \left( \sigma^{-1} - \sigma_{1,1} \frac{d}{d\check{\rho}} \sigma^{-1} \right) W \cdot W \\ &\quad + 2 \int_\Omega (\sigma^{-1} W) \cdot (\partial_\tau W + \check{\nu} \cdot \nabla W + \sigma \mathcal{L}(W)). \end{aligned}$$

Use (4.7) to bound the maxima of  $\sigma, \sigma^{-1}, \partial_\rho \sigma^{-1}$ , and also use  $|\nabla \cdot \check{\nu}|_{L^\infty} \lesssim \|V\|_{H^m}$  to arrive at, for  $W \in H^1(\Omega)$  and  $\partial_\tau W \in L^2(\Omega)$ ,

$$(4.17) \quad \left| \frac{d}{d\tau} \|W\|_{L^2}^2 \right| \lesssim \left| \int_\Omega W \cdot \mathcal{L}(W) \right| + \|V\|_{H^m} \|W\|_{L^2}^2 + \|\partial_\tau W + \check{\nu} \cdot \nabla W + \sigma \mathcal{L}(W)\|_{L^2} \|W\|_{L^2}.$$

Here,

$$\|W\|_{L^2_\sigma} := \left( \int_\Omega \sigma^{-1} W \cdot W dx \right).$$

LEMMA 4.4 (prognostic estimates). *Consider  $V(\tau, x)$  a solution to (4.3) satisfying a priori assumption (4.5). Then, for all  $k \in [0, m]$  and  $0 \leq \tau_1 < \tau_2 \leq \tau^*$ ,*

$$\|\partial_\tau^k V\|_{L^2_\sigma}^2 \Big|_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} \left( \|\partial_\tau^k V\|_{L^2_\sigma} + \|\partial_\tau V\|_{H^{m-1}} \right) \|\partial_\tau^k V\|_{L^2_\sigma} \|V\|_{H^m}.$$

Note  $k = m$  is included here. Also, for  $k \geq 1$ , both sides are in some sense quadratic in  $\partial_\tau V$ , which will eventually yield desirable bound on the inflation of  $\|\partial_\tau V\|_{H^{m-1}}$ .

*Proof.* First, restrict the value of  $k \in [0, m - 1]$  so that  $\partial_\tau^k V \in H^1(\Omega)$  and  $\partial_\tau^k \check{\nu} \cdot \vec{n}|_{\partial\Omega} = 0$  is well-defined, allowing us to apply the divergence theorem to have

$$(4.18) \quad \int_\Omega \partial_\tau^k V \cdot \mathcal{L}(\partial_\tau^k V) = 0 \quad \text{if } k \in [0, m - 1].$$

Thus, we set  $W = \partial_\tau^k V$  in (4.17) where we also apply (4.18) to cancel out the first term in the RHS and use Proposition 4.3 to estimate the last term to prove the lemma for  $k \in [0, m - 1]$ .

Estimating the highest derivative  $\partial_\tau^m V$  requires more care because it is merely in  $L^2(\Omega)$ . Thus, (4.17) and (4.18) are not directly applicable here. One remedy is to apply mollification in time, namely, a time filter, to increase time regularity. Here instead, we demonstrate the closely related time-averaging technique. Let  $W$  in (4.17) be the time-average of  $\partial_\tau^m V$ , i.e., with small  $\delta > 0$ ,

$$W = W^\delta := \frac{1}{\delta} \int_\tau^{\tau+\delta} \partial_\tau^m V = \frac{1}{\delta} \partial_\tau^{m-1} V \Big|_\tau^{\tau+\delta}.$$

Then,  $W^\delta \in H^1(\Omega)$ ,  $\partial_\tau W^\delta \in L^2(\Omega)$ , so that (4.17) holds and also the divergence theorem applies,

$$(4.19) \quad \int_\Omega W^\delta \cdot \mathcal{L}(W^\delta) = 0.$$

It remains to estimate the first factor of the last term in (4.17). At fixed time  $\tau$ , we have

$$(4.20) \quad \begin{aligned} & \delta \cdot \left( \partial_\tau W^\delta + \check{\nu} \cdot \nabla W^\delta + \sigma \mathcal{L}(W^\delta) \right) \\ &= \left( \partial_\tau \partial_\tau^{m-1} V + \check{\nu} \cdot \nabla \partial_\tau^{m-1} V + \sigma \mathcal{L}(\partial_\tau^{m-1} V) \right) \Big|_\tau^{\tau+\delta} \\ & - \left( \check{\nu} \Big|_\tau^{\tau+\delta} \right) \cdot \nabla \partial_\tau^{m-1} V(\tau + \delta, \cdot) - \left( \sigma \Big|_\tau^{\tau+\delta} \right) \mathcal{L}(\partial_\tau^{m-1} V(\tau + \delta, \cdot)). \end{aligned}$$

In the RHS, the first term is indeed  $\int_\tau^{\tau+\delta} \partial_\tau R^{(m-1)}$ . By the same reasoning as used in Proposition 4.3, we find  $\|\partial_\tau R^{(m-1)}\|_{L^2} \lesssim \|\partial_\tau V\|_{H^{m-1}} \|V\|_{H^m}$ . The second term has a

factor  $\check{v}|_{\tau}^{\tau+\delta} = \int_{\tau}^{\tau+\delta} \partial_{\tau} \check{v}$  and we know  $|\partial_{\tau} \check{v}|_{L^{\infty}} \lesssim \|\partial_{\tau} \check{v}\|_{H^{m-1}}$ . Similarly, the last term has a factor  $\sigma|_{\tau}^{\tau+\delta} = \int_{\tau}^{\tau+\delta} \partial_{\tau} \sigma$  and we know  $|\partial_{\tau} \sigma|_{L^{\infty}} \leq |\sigma'|_{L^{\infty}} |\partial_{\tau} \check{\rho}|_{L^{\infty}} \lesssim \|\partial_{\tau} \check{\rho}\|_{H^{m-1}}$ . Finally, notice that  $\|\nabla \partial_{\tau}^{m-1} V\|_{L^2} + \|\mathcal{L}(\partial_{\tau}^{m-1} V)\|_{L^2} \lesssim \|V\|_{H^m}$  by (4.11). Apply all these estimates in the  $L^2$  norm of (4.20) to obtain

$$\delta \|\partial_{\tau} W^{\delta} + \check{v} \cdot \nabla W^{\delta} + \mathcal{L}(W^{\delta})\|_{L^2} \lesssim \int_{\tau}^{\tau+\delta} \|\partial_{\tau} V\|_{H^{m-1}} \max_{[\tau, \tau+\delta]} \|V\|_{H^m}.$$

Now, apply this estimate to  $\int_{\tau_1}^{\tau_2-\delta}$  of (4.17) with  $W = W^{\delta}$ , apply (4.19), and pass the limit as  $\delta \rightarrow 0+$  to prove the lemma for  $k = m$ . Note by  $V \in \mathbf{C}([0, \tau^*], H^m)$ , we have  $\partial_{\tau}^k V \in \mathbf{C}([0, \tau^*], H^{m-k})$ , and thus  $\lim_{\delta \rightarrow 0} \|W^{\delta} - \partial_{\tau}^m V\|_{L^2}(\tau) = 0$ .  $\square$

**4.5. Estimates of  $\|(\rho, \mathbf{v})\|_{H^m}$  and  $\|\partial_t(\rho, \mathbf{v})\|_{H^{m-1}}$ .** We will still mostly work with  $V = \epsilon(r, \mathbf{v})$ , only reconnecting with  $\mathbf{v}$  and  $r \approx \rho$  in Theorem 4.6 near the end of this section. The goal to keep in mind is the existence time in terms of  $\tau$  at the same order of  $1/\|V_0\|_{H^m}$  with the  $\|V\|_{H^m}$  norm only inflated by a constant. We also need to see similar inflation of  $\|\partial_{\tau} V\|_{H^{m-1}}$  but will tolerate some additional term that is quadratic in  $\|V\|_{H^m}$ .

LEMMA 4.5 (estimates on  $V$  and  $\partial_{\tau} V$ ). *Consider a solution of (4.3),  $V \in \mathbf{C}([0, \tau^*], H^m(\Omega))$  with  $m \geq 3$ . Then, there exist positive constants  $\tau^{\sharp}, C_v, C_1, C_2, C_3$  that solely depend on  $m, \Omega$  and pressure law  $\pi(\cdot)$  so that if  $\|V\|_{\mathbf{C}([0, \tau^*], H^m)} \leq C_v$ , then for times  $\tau \in [0, \tau^*] \cap [0, \tau^{\sharp}/\|V_0\|_{H^m}]$ ,*

$$(4.21) \quad \|V\|_{H^m} \leq C_1 \|V_0\|_{H^m},$$

$$(4.22) \quad \|\partial_{\tau} V\|_{H^{m-1}} \leq C_2 \|\partial_{\tau} V_0\|_{H^{m-1}} + C_3 \|V_0\|_{H^m}^2.$$

Notice, under the hyperbolic rescaling (4.2), one has  $\|V\|_{H^m} = \epsilon \|(r, \mathbf{v})\|_{H^m}$  and  $\|\partial_{\tau} V\|_{H^m} = \epsilon^2 \|\partial_t(r, \mathbf{v})\|_{H^m}$  and such substitution would not change the structure of (4.21), (4.22).

*Proof.* We pick constant  $C_v$  so that  $\|V\|_{\mathbf{C}([0, \tau^*], H^m)} \leq C_v$  implies a priori assumption (4.5) and  $\|V\|_{H^m} \leq d_1$  as required by Lemma 4.2.

Introduce the shorthand notation

$$\begin{aligned} F(\tau) &:= \|V\|_{H^m}(\tau), & f(\tau) &:= \|\partial_{\tau} V\|_{H^{m-1}}(\tau), \\ \Phi(\tau) &:= \left( \|\check{\omega}\|_{H^{m-1}}^2 + \|V\|_{L^2_{\check{x}}}^2 + \phi^2(\tau) \right)^{1/2}, & \phi(\tau) &:= \left( \sum_{k=1}^m \|\partial_{\tau}^k V\|_{L^2_{\check{x}}}^2 \right)^{1/2}, \end{aligned}$$

where lowercase  $f, \phi$  involve at least one  $\partial_{\tau}$  derivatives and our eventual goal is to estimate  $F, f$ .

By definition and  $|\sigma| \in [1/2, 2]$ , we have  $\phi \leq \sqrt{2} \|\partial_{\tau} V\|_{m-1}$  and  $\Phi \leq \sqrt{2} \|V\|_m$ . Combine it with (4.11) to get

$$(4.23) \quad \Phi \lesssim F, \quad \phi \lesssim f.$$

Meanwhile, we have been gathering diagnostic estimates in Lemmas 4.1 and 4.2, i.e.,

$$(4.24) \quad F \lesssim \Phi, \quad f \lesssim F^2 + \phi,$$

and prognostic estimates in (4.13) and Lemma 4.4 which sum up to

$$(4.25) \quad \int_{\tau_1}^{\tau_2} \Phi^2 \lesssim \int_{\tau_1}^{\tau_2} (\Phi + f)\Phi F, \quad \phi^2 \Big|_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} (\phi + f)\phi F.$$

(i) Estimate of  $F(\tau)$ . In the first part of (4.25), relax  $f$  to  $F$  à la (4.11) and relax  $F$  to  $\Phi$  à la (4.24) to obtain  $\Phi^2|_{\tau_1}^{\tau_2} \leq 2c_1 \int_{\tau_1}^{\tau_2} \Phi^3$  for some constant  $c_1$ . Thus,  $\Phi^2(\tau) \leq \Phi^2(0) + 2c_1 \int_0^\tau \Phi^3$  and by the continuity of  $\Phi(\tau)$  and the comparison principle,

$$\Phi \leq \tilde{\Phi} \text{ solving } \frac{d}{d\tau}(\tilde{\Phi})^2 = 2c_1(\tilde{\Phi})^3, \quad \tilde{\Phi}(0) = \Phi(0),$$

$$\implies \Phi(\tau) \leq \tilde{\Phi}(\tau) = \frac{\Phi(0)}{1 - c_1\Phi(0)\tau},$$

as long as the RHS is bounded. Thus

$$1 - c_1\Phi(0)\tau \geq 1/2 \implies \Phi(\tau) \leq 2\Phi(0).$$

Then, by the equivalence of  $F, \Phi$  as in the first parts of (4.23), (4.24), we proved (4.21) as well as the  $\tau$  interval prescribed above it.

(ii) Estimate of  $f(\tau)$ . Combine the second parts of (4.24), (4.25), and relax  $F(\tau)$  to  $F(0)$  à la (4.21),

$$\phi^2|_{\tau_1}^{\tau_2} \leq 2c_2 \int_{\tau_1}^{\tau_2} (\phi + F^2(0)) \cdot \phi F(0).$$

By the continuity of  $\phi(\tau)$  and the comparison principle,

$$\phi \leq \tilde{\phi} \text{ solving } \frac{d}{d\tau}(\tilde{\phi})^2 = 2c_2(\tilde{\phi} + F^2(0)) \cdot \tilde{\phi} F(0), \quad \tilde{\phi}(0) = \phi(0)$$

$$\implies \phi(\tau) \leq \tilde{\phi}(\tau) = -F^2(0) + e^{c_2 F(0)\tau}(\phi(0) + F^2(0)).$$

Combine it with the  $\tau$  interval above (4.21) and the second parts of (4.23), (4.24) to prove (4.22).  $\square$

This lemma leads to the final theorem of this section.

**THEOREM 4.6 (uniform estimates).** *Under the same hypotheses as the main theorem, Theorem 1.1, with  $E_0, E_{t,0}$  equivalently given in (1.6), namely,  $E_0 = \|(\rho_0, \mathbf{v}_0)\|_{H^m}$ ,  $E_{t,0} = \|\partial_t(\rho_0, \mathbf{v}_0)\|_{H^{m-1}}$ , there exist constants  $E^*, T^\sharp, C^\sharp, C_t^\sharp, C_{\mathcal{L}}^\sharp$  that solely depend on  $m, \Omega$ , and pressure law  $\pi(\cdot)$  so that*

$$(4.26a) \quad E_0 \leq E^*/\epsilon \text{ implies there exists a unique } \mathbf{C}^1 \text{ solution for } t \in [0, T^\sharp/E_0].$$

More precisely,

$$(4.26b) \quad \|(\rho, \mathbf{v})\|_{\mathbf{C}([0, T^\sharp/E_0], H^m)} \leq C^\sharp E_0,$$

$$(4.26c) \quad \|\partial_t(\rho, \mathbf{v})\|_{\mathbf{C}([0, T^\sharp/E_0], H^{m-1})} \leq C_t^\sharp (E_{t,0} + E_0^2),$$

$$(4.26d) \quad \|\mathcal{L}(\rho, \mathbf{v})\|_{\mathbf{C}([0, T^\sharp/E_0], H^{m-1})} \leq C_{\mathcal{L}}^\sharp \epsilon (E_{t,0} + E_0^2).$$

*Proof.* The short time existence of classical solutions is established in [19], so we only prove the estimates here. The continuation method is always at our disposal, since the compatibility condition  $(\partial_t^k \mathbf{v}_0) \cdot \vec{n}|_{\partial\Omega} = (\partial_\tau^k \check{\mathbf{v}}_0) \cdot \vec{n}|_{\partial\Omega} = 0$  is invariant under hyperbolic rescaling (4.2).

Therefore, by applying hyperbolic rescaling (4.2) to the target conclusions (4.26), it suffices to show there exist universal constants  $e^*, \tau^\sharp, c_0, c_1, c_2$ , suitably chosen so that

$$(4.27a) \quad \|V_0\|_{H^m} \leq e^* \text{ implies there exists a unique } \mathbf{C}^1 \text{ solution for } \tau \in [0, \tau^\sharp/\|V_0\|_{H^m}]$$



and that

$$(4.27b) \quad \|V\|_{\mathbf{C}([0, \tau^\sharp/\|V_0\|_{\mathbb{H}^m}], \mathbb{H}^m)} \leq c_0 \|V_0\|_{\mathbb{H}^m},$$

$$(4.27c) \quad \|\partial_\tau V\|_{\mathbf{C}([0, \tau^\sharp/\|V_0\|_{\mathbb{H}^m}], \mathbb{H}^{m-1})} \leq c_1 (\|\partial_\tau V_0\|_{\mathbb{H}^{m-1}} + \|V_0\|_{\mathbb{H}^m}^2),$$

$$(4.27d) \quad \|\mathcal{L}(V)\|_{\mathbf{C}([0, \tau^\sharp/\|V_0\|_{\mathbb{H}^m}], \mathbb{H}^{m-1})} \leq c_2 (\|\partial_\tau V_0\|_{\mathbb{H}^{m-1}} + \|V_0\|_{\mathbb{H}^m}^2).$$

Indeed, choose  $e^* = C_v/C_1$  with  $C_1 > 1$  and  $C_v$  used in Lemma 4.5. Then, by the continuity argument and Lemma 4.5, the a priori assumption  $\|V\|_{\mathbb{H}^m} \leq C_v$  as well as (4.27b), (4.27c) remains true in the time interval given in (4.27a). Finally, (4.27d) is by a simple deduction from (4.27b), (4.27c), the  $\epsilon$ -free formulation (4.3), Sobolev inequalities, and bounds of  $\sigma$  in (4.9).  $\square$

**5. Proof of the main theorem and concluding remarks.** Now we prove the main theorem, Theorem 1.1, using the time-averaging estimates in Theorem 3.3 and the  $\epsilon$ -independent estimates in Theorem 4.6.

*Proof* (main theorem, Theorem 1.1). First, regarding the time interval of validity, by (4.26b) of Theorem 4.6, replace  $T^b$  with  $T^\sharp/E_0$  in the last line of Theorem 3.3 to get

$$(5.1) \quad T \in [0, T^\sharp/E_0] \cap [0, D_2/\sup_{[0, T^\sharp/E_0]} \|(\rho, \mathbf{v}, \tilde{\mathbf{v}})\|_{\mathbb{H}^m}].$$

By estimate (4.26b) again, and by a similar estimate well known to be true for  $\tilde{\mathbf{v}}$  (e.g., [25, Chap. 17, Thm. 3.2]), we further shorten the second time interval to  $[0, D_2/(C^\sharp E_0)]$ . Therefore, take  $T^* := \min\{T^\sharp, D_2/C^\sharp\}$  to make both Theorems 3.3 and 4.6 valid for  $T \in [0, T^*/E_0]$ .

On this time interval, Theorem 3.3 guarantees

$$\sup_{[0, T]} \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathbb{H}^{m-3}} \lesssim \epsilon \sup_{[0, T]} \|\mathcal{L}(\rho, \mathbf{v})\|_{\mathbb{H}^{m-1}} \sup_{[0, T]} \|\mathbf{v}^P\|_{\mathbb{H}^m}.$$

One  $\epsilon$  factor is in place, and (4.26d) of Theorem 4.6 guarantees another  $\epsilon$  factor from  $\mathcal{L}(\rho, \mathbf{v})$ ,

$$(5.2) \quad \sup_{t \in [0, T]} \|\mathbf{v}^P - \tilde{\mathbf{v}}\|_{\mathbb{H}^{m-3}} \lesssim \epsilon^2 (E_{t,0} + E_0^2) \sup_{[0, T]} \|\mathbf{v}^P\|_{\mathbb{H}^m}.$$

So, our last job is to bound  $\|\mathbf{v}^P\|_{\mathbb{H}^m}$  in terms of  $\|\mathbf{v}_0^P\|_{\mathbb{H}^m}$ . Scale vorticity estimate (4.13) back to variables  $\omega, t$  so that  $\|\omega\|_{\mathbb{H}^{m-1}}^2|_{t_1}^{t_2} \lesssim \int_{t_1}^{t_2} \|\mathbf{v}\|_{\mathbb{H}^m} \|\omega\|_{\mathbb{H}^{m-1}}^2$ . Apply energy method to (2.11), noting  $\mathbf{v}^P \cdot \vec{\mathbf{n}}|_{\partial\Omega} = \mathbf{v}^Q \cdot \vec{\mathbf{n}}|_{\partial\Omega} = 0$ , to get  $\|\mathbf{v}^P\|_{\mathbb{L}^2}^2|_{t_1}^{t_2} \lesssim \int_{t_1}^{t_2} \|\mathbf{v}\|_{\mathbb{H}^m} \|\mathbf{v}^P\|_{\mathbb{L}^2}^2$ . Combine these two Gronwall inequalities with elliptic estimate (2.6) to obtain  $\|\mathbf{v}^P\|_{\mathbb{H}^m}(T) \leq \|\mathbf{v}_0^P\|_{\mathbb{H}^m} e^{CT \sup_{[0, T]} \|\mathbf{v}\|_{\mathbb{H}^m}}$ . Then, the exponent can be relaxed to a constant due to (5.1). The proof is complete!  $\square$

The proof of Corollary 1.5 is as follows. First, regarding  $\|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbb{H}^{m-3}}$ , combine (4.26d) of Theorem 4.6 with elliptic estimates from Proposition 2.3 to obtain  $\|\mathbf{v}^Q\|_{\mathbb{H}^m} \lesssim \epsilon(E_{t,0} + E_0^2)$ . By the  $O(\epsilon^2)$  estimate of  $\|\mathbf{v}^P - \tilde{\mathbf{v}}\|_{\mathbb{H}^{m-3}}$  from the main theorem, this is more than enough to prove the  $O(\epsilon)$  estimate of  $\|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbb{H}^{m-3}}$ .

Second, regarding  $\int_0^T \mathbf{v} - \tilde{\mathbf{v}} = \int_0^T \mathbf{v}^Q + \int_0^T \mathbf{v}^P - \tilde{\mathbf{v}}$ , one starts with estimate (3.3) of  $\overline{\mathbf{v}^Q}$  and estimate (4.26d) of  $\mathcal{L}(\rho, \mathbf{v})$  to obtain

$$(5.3) \quad \left\| \int_0^T \mathbf{v}^Q \right\|_{\mathbb{H}^{m-1}} \lesssim \epsilon^2 (E_{t,0} + E_0^2) (1 + T \sup_{[0, T]} \|(\rho, \mathbf{v})\|_{\mathbb{H}^m}).$$

Combine it with (5.2), (5.1) to complete the proof of (1.11).

Last, regarding the transport equations in Corollary 1.5, we rewrite them as

$$\begin{aligned} \partial_t \tilde{\theta} + \tilde{\mathbf{v}} \cdot \nabla \tilde{\theta} &= 0, \\ \partial_t \theta + \mathbf{v}^P \cdot \nabla \theta &= -\mathbf{v}^Q \cdot \nabla \theta, \end{aligned}$$

and fit it into the notation of Lemma 3.2,

$$\begin{aligned} v_1 &= \tilde{\mathbf{v}}, & u_1 &= \tilde{\theta}, & \gamma_1 &= 0, \\ v_2 &= \mathbf{v}^P, & u_2 &= \theta, & \gamma_2 &= -\mathbf{v}^Q \cdot \nabla \theta, \\ \mathcal{B} &= \mathbf{H}^m \cap \mathcal{X}, & \mathcal{B}' &= \mathbf{H}^m, & \mathcal{B}'' &= \mathbf{H}^{m-1} \end{aligned}$$

with

$$b(v, u) = v \cdot \nabla u, \quad \mathcal{H} = \mathbf{H}^{m-3}.$$

Then, simple application of Sobolev inequalities and the fact that  $\tilde{\mathbf{v}} \in \mathcal{X}$  can validate all assumptions of Lemma 3.2, including (3.5). Therefore, we have, with  $M := \sup_{[0, T]} \{\|\tilde{\mathbf{v}}\|_{\mathbf{H}^m}\}$ ,

$$\begin{aligned} (5.4) \quad & \sup_{[0, T]} \|\tilde{\theta} - \theta\|_{\mathbf{H}^{m-3}} - \sup_{[0, T]} \|\overline{\mathbf{v}^Q \cdot \nabla \theta}\|_{\mathbf{H}^{m-3}} \\ & \leq \frac{e^{CMT} - 1}{CM} \sup_{[0, T]} \left\{ \|b(\tilde{\mathbf{v}}, \overline{\mathbf{v}^Q \cdot \nabla \theta}) + b(\tilde{\mathbf{v}} - \mathbf{v}^P, \theta)\|_{\mathbf{H}^{m-3}} \right\} \\ & \lesssim T \|\tilde{\mathbf{v}}\|_{\mathbf{H}^m} \|\overline{\mathbf{v}^Q \cdot \nabla \theta}\|_{\mathbf{H}^{m-2}} + T \|\tilde{\mathbf{v}} - \mathbf{v}^P\|_{\mathbf{H}^{m-3}} \|\theta\|_{\mathbf{H}^m}, \end{aligned}$$

where the last estimate is due to (5.1) and Sobolev inequalities.

To estimate the time-average  $\overline{\mathbf{v}^Q \cdot \nabla \theta}$ , we perform integrating by parts and use the transport equation of  $\theta$  itself,

$$\overline{\mathbf{v}^Q \cdot \nabla \theta} = \overline{\mathbf{v}^Q \cdot \nabla \theta} \Big|_0^T - \int_0^T \overline{\mathbf{v}^Q \cdot \nabla \partial_t \theta} = \overline{\mathbf{v}^Q \cdot \nabla \theta} \Big|_0^T + \int_0^T \overline{\mathbf{v}^Q \cdot \nabla (\mathbf{v} \cdot \nabla \theta)}.$$

Take its  $\mathbf{H}^{m-2}$  norm, and apply (5.3) and Sobolev inequalities to get

$$(5.5) \quad \|\overline{\mathbf{v}^Q \cdot \nabla \theta}\|_{\mathbf{H}^{m-2}} \lesssim \epsilon^2 (E_{t,0} + E_0^2) \sup_{[0, T]} \|\theta\|_{\mathbf{H}^m} (1 + T \sup_{[0, T]} \|\mathbf{v}\|_{\mathbf{H}^m}).$$

Finally, the estimation of  $\|\theta\|_{\mathbf{H}^m}$  follows the standard energy method together with the same mollification for proving vorticity estimate (4.13),

$$\|\theta\|_{\mathbf{H}^m}^2 \Big|_{t_1}^{t_2} \lesssim \int_{t_1}^{t_2} \|\mathbf{v}\|_{\mathbf{H}^m} \|\theta\|_{\mathbf{H}^m}^2 \implies \|\theta\|_{\mathbf{H}^m}(T) \leq \|\theta_0\|_{\mathbf{H}^m} e^{CT \sup_{[0, T]} \|\mathbf{v}\|_{\mathbf{H}^m}}.$$

Therefore, in (5.4), apply this estimate together with (5.5), (5.2), and (5.1) to finish the proof of the very last inequality of Corollary 1.5.

For future studies, we comment on the possibilities of sharpening the error estimates for practical use such as numerical analysis, because the incompressible approximation is ubiquitously important. One aspect is to get some good bounds on the inequality constants  $C^*, T^*$ , etc. This can benefit from using optimal constants in the Sobolev inequalities, making all  $\lesssim$  relations explicitly  $\leq$  relations. In addition, for the easier case  $\partial\Omega = \emptyset$ , one can drastically reduce the steps of the energy

method in section 4, potentially reducing constants as well. Another aspect is to utilize dispersive and/or dissipative mechanisms which the current article does not reply on. It will be very interesting to see what role they can play when combined with time-averaging.

Furthermore, we note that it is easy to extend our techniques to domains living in two- and three-dimensional Riemannian manifolds, in which case two major analytical tools remain valid: Stokes' theorem as a generalization of the divergence theorem and Sobolev inequalities. Also, calculations carried out in this article mostly rely on a handful of coordinate-independent operators, i.e.,  $\nabla, \nabla \cdot, \nabla \times, \mathbf{v} \cdot \nabla, \Delta$ . Then, our results and techniques can be applied to interesting areas such as geophysical fluid dynamics on a sphere and relativistic fluid dynamics.

**Appendix.** We prove the Sobolev (type) inequality as described in Remark 1.6, that is, on a smooth, compact domain in two or three dimensions,

$$(5.6) \quad \left\| \prod_{i=1}^j \partial_x^{\beta_i} f_i \right\|_{L^2} \lesssim \prod_{i=1}^j \|f_i\|_{H^k},$$

if one of the following conditions holds:

$$\begin{cases} |\beta_1 + \cdots + \beta_j| \leq k, \quad \text{and } k \geq 2, \quad \text{or} \\ |\beta_1 + \cdots + \beta_j| = k + 1, \quad \max\{|\beta_1|, \dots, |\beta_j|\} \leq k, \quad \text{and } k \geq 3. \end{cases}$$

For convenience, assume  $k \geq |\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_j|$ . First, it is an easy exercise of proof by contradiction (for example) to show that, under the above hypotheses, we must have

$$(5.7) \quad |\beta_3| \leq k - 2,$$

and one of the following scenarios must occur:

$$(5.8) \quad \begin{cases} |\beta_2| \leq k - 2 \quad \text{or} \\ |\beta_1| = |\beta_2| = k - 1. \end{cases}$$

Next, by Hölder's inequality,

$$\left\| \prod_{i=1}^j \partial_x^{\beta_i} f_i \right\|_{L^2} \leq \left\| \partial_x^{\beta_1} f_1 \partial_x^{\beta_2} f_2 \right\|_{L^2} \prod_{i=3}^j \left\| \partial_x^{\beta_i} f_i \right\|_{L^\infty}.$$

Then, by (5.7) and Sobolev embedding  $H^2 \subset L^\infty$ , we can relax all the above  $L^\infty$  norms to  $H^k$  norms and therefore the proof of (5.6) is reduced to proving

$$(5.9) \quad \left\| \partial_x^{\beta_1} f_1 \partial_x^{\beta_2} f_2 \right\|_{L^2} \lesssim \|f_1\|_{H^k} \|f_2\|_{H^k} \quad \text{if one of (5.8) holds.}$$

Under the first scenario of (5.8), we simply apply as before the same combination of Hölder's inequality and Sobolev embedding  $H^2 \subset L^\infty$  to finish the proof of (5.9). Under the second scenario of (5.8), apply a different version of Hölder's inequality

$$\left\| \partial_x^{\beta_1} f_1 \partial_x^{\beta_2} f_2 \right\|_{L^2} \leq \left\| \partial_x^{\beta_1} f_1 \right\|_{L^4} \left\| \partial_x^{\beta_2} f_2 \right\|_{L^4}$$

and a different version of Sobolev embedding  $H^1 \subset L^4$  to finish the proof of (5.9).

**Acknowledgments.** The author would like to thank I. Gallagher for very insightful discussions while both were participating in the Mathematics for the Fluid Earth program at the Isaac Newton Institute, autumn 2013. The author thanks INI for its hospitality and is very grateful to the organizers of this program for offering a visiting fellowship. The author is also thankful for stimulating discussions with and expertise from H. Beirão da Veiga, M. Cullen, S. Jin, M. Oliver, I. Roulstone, P. Secchi, S. Schochet, E. Tadmor, M. Tang, C. Xie, B. Wingate, S. Zelik, C. Zeng, and the anonymous referee(s). The author gratefully acknowledges administrative support by the Department of Mathematics of the University of Surrey, which particularly expedited the accomplishment of this work.

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