On infinite energy solutions to dissipative PDEs in unbounded domains.

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Abstract

In this thesis several problems in Partial Differential Equations in unbounded domains are studied using the techniques of uniformly local spaces and weighted energy theory.

First Coupled Burger’s equations are studied on the whole space $\mathbb{R}$ and existence of solutions in uniformly local spaces is proven in the case where the non-linearity is gradient. Moreover the uniqueness of these solutions and some additional regularity is proven.

Second the Cahn-Hilliard, and closely related Cahn-Hilliard-Oono, equations are studied on the whole space $\mathbb{R}^3$ with both polynomial and singular potentials and existence of solutions in uniformly local spaces is proven. Moreover uniqueness and additional regularity of these equations is also proven.

Third the Navier-Stokes equations are studied on the whole space $\mathbb{R}^2$ and, building on the work of Zelik who showed the existence of solutions in uniformly local spaces, the existence of a finite dimensional globally compact attractor is proven in the case where the forcing term has arbitrarily slow decay at infinity.
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1 Introduction

Differential equations were first studied by Newton due to his interest in mechanics. One of his great successes was to derive Kepler’s laws from his mathematical description of gravity. However this also led to the N-body gravitational problem, where many objects are all attracted to each other, which gives rise to systems of Ordinary Differential Equations (ODEs). For 2 bodies, say the earth and the moon, this system can be solved to give exact solutions. However for 3 or more bodies, say the earth, moon and sun, the system is not exactly solvable and the conditions under which exact solutions can be obtained is still an open problem to this day, see [74] for more information. To Leibniz is owed the current notation for calculus and important contributions to its foundation.

Throughout the eighteenth and nineteenth centuries, as calculus was applied to a wider range of disciplines, more and more differential equations were derived, which often now bear the names of their creators. Examples include the Laplace and Poisson equations, the Euler equation and the Navier-Stokes equations. These differed from the original equations as they had extension in space as well as in time and were dubbed Partial Differential Equations (PDEs). Developing an understanding of these equations and their solutions has been a crucial part of the development of science and the importance of them cannot be overstated: they have given us a view into the fundamental workings of the universe.

Throughout this era the main focus was on finding exact solutions in elementary functions. This is possible for a lot of ODEs and PDEs but it proved impossible to find a general way of solving any equation. Another related approach was to search for perturbations of known solutions, especially studying linearisations of equations around equilibria.

The qualitative study of differential equations was begun by Henri Poincaré when he studied the 3-body problem in a radically new way, see [6]. The main idea of qualitative study is to make rigorous statements about the nature of a solution without needing to express this solution explicitly. A limit cycle is a good example of this. In order to understand an equation where all solutions converge to a limit cycle it is of less importance to understand the exact trajectory a solution will take: it is much more important to have an understanding of the properties of the limit cycle itself. This is a generalisation of working near an equilibrium and shows that there are many equations where linearising around the equilibria does not capture the full behaviour of the system.

His work demonstrated that even very simple systems of ODEs can produce very complex, chaotic behaviour in their solutions, see [74]. The discovery of this new type of dynamics was a turning point in the study of differential equations: it finally ended the search for simple, explicit solutions for complex systems but opened a new avenue of research in these non-linear systems.

This complexity is compounded in the study of PDEs because they have an infinite dimensional phase space. Equations arising from the study of the weather contain turbulence,
which is easy to see in the weather itself, along with sensitive dependence on initial conditions. In fact this sensitivity is so severe that small perturbations can be amplified into large scale dynamical features which makes weather forecasting difficult and, over long time periods, impossible. Pioneering work in this area was done by Ladyzhenskaya, Temam, Foias and Prodi, see [55], [56] and [73] for summaries of this work.

Lorenz [58] then showed that even in a 3 mode Galerkin approximation of atmospheric convection, which is now named after him, there are chaotic dynamics (most pertinently sensitive dependence on initial conditions, which he called the Butterfly Effect) in such a system. The Poincaré-Bendixon theorem states that there can only be trivial dynamics in two dimensions (equilibria, limit cycles and unbounded trajectories) so these three dimensional systems with chaotic properties show this restriction is sharp.

The dynamics of the Lorenz system are, however, attracted to a compact set which turns out to be both robust and quite common in differential equations so was studied intently and given the name a global attractor. Some attractors (like an equilibria or limit cycle) have a simple structure whereas others have a fractal structure and are called strange attractors.

In general the value of an attractor is that it contains all the non-trivial dynamics of a system whereas any trajectories away from the attractor rapidly decay to it. Therefore the study of an attractor, and therefore the asymptotic dynamics of a system, can be extremely useful in understanding the behaviour of the system as a whole.

One of the chief difficulties of studying PDEs is that there is no general theorem for existence and uniqueness. The Clay Institute currently offers a Millennium Prize for a proof of uniqueness for solutions of the Navier-Stokes equation in 3 dimensions, which is a sign of the great mathematical interest in such problems.

For ODEs the Picard-Lindeloff theorem is extremely useful, though not universal, and answers the question of existence and uniqueness for many examples. This leads to a general program for the study of PDEs which will be followed in three examples in this thesis. The steps are to prove the existence and uniqueness of solutions (in an appropriate function space) and to show continuous dependence on initial conditions. If these three things are proven then the problem is said to be well posed. After this it is possible to study the asymptotic dynamics of the system, and if a global attractor exists its dimension can be measured.

When computed using an abstracted notion of dimension, of which there are many, the dimension of an attractor can be lower than that of the phase space as a whole and is often non-integer. There was great hope that this lower dimensional object could be embedded into an N-dimensional Euclidean space but there are many problems with this, the best result in this direction being the Mane projection theorem. See the Navier-Stokes section for a calculation of the dimension of an attractor using box counting (or fractal) dimension.
The study of global attractors in a bounded domain was built up over many years and the following papers give a review of that development, see [2] [14] [45] [55] [56] [61] [69] [70] [72] and [73].

When the underlying domain is unbounded (as it will be for all problems in this thesis) there are several problems which render much of the theory which was built for a bounded domain unusable. Three of these are, first, the question of infinite energy, second, the inherently infinite dimensional nature of the dynamics even when the system is dissipative and third the lack of compact embeddings in an unbounded domain.

In a discrete system the kinetic energy is $\frac{1}{2}mv^2$ for a mass $m$ and a velocity $v$. The obvious analogue of this for a continuous system is

$$\frac{1}{2} \int_\Omega \rho |u|^2 \, dx, \quad (1.1)$$

where $u$ is the velocity field and $\rho$ is the density on an appropriate domain $\Omega$.

And this works very well; it is the foundation for the $L^2$ Hilbert space and what is usually referred to as energy. However in an unbounded domain this simple approach does not work. For example consider a solution over $\mathbb{R}$ which is a constant. This solution will have infinite energy despite the fact that it is very reasonable.

To draw a distinction between solutions which have infinite energy because they have a localised singularity and those which have infinite energy because they do not decay at infinity two sets of spaces will be used. The first is the uniformly local space.

For any $1 \leq p \leq \infty$, the uniformly local Lebesgue space $L^p_b(\Omega)$ is defined as follows:

$$L^p_b(\Omega) := \left\{ u \in L^p_{loc}(\Omega) : \| u \|_{L^p_b} := \sup_{x_0 \in \Omega} \| u \|_{L^p(B_R^{x_0})} < \infty \right\}, \quad (1.2)$$

where $B^R_{x_0}$ stands for the $R$-ball in $\Omega$ centred at $x_0$.

Also its higher regularity analogues, $W^{1,p}_b$, will be used. The great advantage of this space is that it draws a clear distinction between blow-up and decay at infinity. It also has a lot of similarities to $L^\infty$ but without its limitations.

The big problem with uniformly local spaces is that it is hard to obtain estimates. The usual method of multiplying and equation by $u$ and integrating does not work because the result is a $\sup$ and an integral whose order need to be swapped and this is not possible in general.

To overcome this problem another set of spaces will be used, the weighted Sobolev spaces. For any weight function $\phi(x)$ with exponential rate of growth, the weighted Lebesgue
spaces are defined:

\[ L^p_\phi(\Omega) = \left\{ u \in L^p_{\text{loc}}(\Omega) : \|u\|_{L^p_\phi} := \left( \int_{\mathbb{R}^3} \phi(x)|u(x)|^p \, dx \right)^{1/p} < \infty \right\}. \]  

(1.3)

These weighted spaces have the extremely powerful property of being isomorphic to \( L^p \) and so all of the functional analysis which is constructed for the \( L^p \) spaces can almost immediately be applied to these weighted spaces (notably many inequalities including the interpolation inequality and some very useful embedding theorems).

Moreover if a supremum is taken with respect to all shifts,

\[ \|u\|_{L^2_b} \leq C \sup_{x_0 \in \Omega} \|u\|_{L^2_{\phi(x-x_0)}}, \]  

(1.4)

where \( \phi_{x-x_0} = \phi(x-x_0) \), then it is possible bound the \( L^2_b \) norm by the \( L^2_\phi \) norm, see, for example, [81]. This gives us the core technique in everything that follows. Multiply the equation by \( \phi u \) and then integrate and then, as a final step, take a supremum with respect to all shifts of the weight function. This allows the use of a lot of techniques which apply easily to the weighted spaces while giving final results in the uniformly local spaces.

One of the key problems with studying attractors in an unbounded domain is that they can be inherently infinite dimensional. This means they have infinitely many unstable modes, whereas one of the main attractions of attractors in a bounded domain is that they have only finitely many unstable modes. Epsilon entropy is a concept which was introduced by Kolmogorov, see [51], and is an important part of one method of measuring the dimension of an attractor. In [32] the \( \epsilon \) entropy of an attractor was estimated as follows,

\[ H_\epsilon(A|_{\Omega \cap B^R_{x_0}}) \leq C \, \text{vol}(\Omega \cap B_{x_0}^{R+k} \log \frac{1}{\epsilon}) \log \frac{1}{\epsilon}, \]  

(1.5)

where \( C \) and \( k \) are independent of \( R, \Omega \) and \( x_0 \). Moreover if the underlying domain \( \Omega = \mathbb{R}^n \) then this gives an estimate of order \( (R + \log \frac{1}{\epsilon})^n \log \frac{1}{\epsilon} \), which grows to infinity as \( R \) does.

This problem will be overcome in the third section by stipulating that the forcing term decays at infinity (arbitrarily slowly) and this will mean that the dynamics are restricted to a large ball with small tails and then the dimension of the attractor will be shown to be finite in this situation.

See [1] and [4] for early works in an unbounded domain and see [61], [25] for its continuation, with special reference to [29], [59], [81], [79], for the use of weighted energy theory.

This thesis is broken into three main sections each of which is devoted to the study of a different equation in a different dimension. The first section is on coupled Burgers’
Equations, the second on the Cahn-Hilliard equation and the third on the Navier-Stokes
equations. The unifying theme of the thesis is the use of weighted and uniformly local
spaces to investigate infinite energy solutions of PDEs in unbounded domains.

1.1 Coupled Burgers’ Equations

In this section proofs for the existence, uniqueness and regularity of coupled Burgers’
equations in a bounded and an unbounded domain will be shown. It has been designed
to be introductory so it is in two almost identical halves, one for the bounded domain
and the other for the unbounded domain, so that it is, hopefully, possible for the reader
to see the technique used in the unbounded domain more clearly.

The latter half of this section, that is the proof in an unbounded domain, or a somewhat
modified version of it, is published in [10] and some applications to the theory of shallow
water waves can be found there.

The equation
\[ u_t + \alpha u u_x = \mu u_{xx} \] (1.6)

with \( \alpha, \mu \in \mathbb{R} \) has been studied by Forsyth (1906) [39] and Bateman (1915) [7] but due
to the extensive work of Johannes Martinus Burgers (1948) [11] it is known as Burgers’
equation.

Structurally it is a heat equation with a simple non-linearity added and it has been found
to describe various phenomena such as a mathematical model of turbulence [11] (as it
can also be considered as a greatly simplified version of the Navier-Stokes equations) and
shock waves travelling in a viscous fluid [19], [77]. Moreover it has been used as a model
of non-linear acoustics, see [21].

Burgers’ equation is rare in that it is one of the few PDEs that can be solved analytically
for arbitrary initial data using the, so called, Hopf-Cole transformation [19] [46].

If the following
\[ u(x, t) = -\frac{2\mu z_x}{\alpha z}, \quad \text{with } z = z(x, t) \]
is substituted into (1.6) the result is

\[ z_t = \mu z_{xx}, \]

which is a linear heat equation.

Moreover this single Burgers’ equation can be bounded by the maximum principle, because
at any maximum \( uu_x = 0 \) and so any global maximum will exist on the boundary of the
space-time domain, in the same way as for the heat equation. This makes the analysis much simpler than in the case of coupled Burgers’ where no maximum principle applies.

Coupled Burgers’ has been used by Esipov (1995) [35] as a model of mono-dispersive and poly-dispersive sedimentation, that is the effect of gravity on particles, of types various, suspended in a fluid. He built on the work of Fletcher (1983) [38] and Barker and Grimson (1987) [5] who generated some exact, analytical, solutions to the coupled Burgers’ equations in the form of shock waves.

A conservation law is critical if the Jacobian of the flux vector, when evaluated on a constant state, has zero eigenvalues. In [10] it is shown that if a conservation law with dissipation (with some conditions on the dissipation) is critical then on a long time scale it will generate dynamics which resemble Burgers’ dynamics. If the Jacobian has a single zero eigenvalue then the dynamics will resemble single Burgers’; but if the Jacobian has multiple zero eigenvalues then the dynamics will resemble coupled Burgers’ equations.

An application to shallow water hydrodynamics is also shown where a three layer fluid is governed by a conservation law, when the different layers have different densities. This gives rise to a Jacobian with two zero eigenvalues and linearly independent eigenvectors. The dynamics can then be reduced to the case of coupled Burgers’ equations.

Results

The following coupled Burgers’ equations will be studied,

\[
\frac{\partial}{\partial t} u = \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} (f(u)), \quad x \in \Omega = \mathbb{R}, \quad u|_{t=0} = u_0
\]

for the \(N\)-component vector-valued function \(u(x, t)\), with the following conditions on the function \(f : \mathbb{R}^N \to \mathbb{R}^N\). There is a scalar-valued function \(F : \mathbb{R}^N \to \mathbb{R}\) such that

\[
\begin{cases}
1. f(u) = \nabla F(u), \\
2. f'(u) \leq C(1 + |u|), \quad \text{for all } u \in \mathbb{R}^N \\
3. F \in C^2(\mathbb{R}^N, \mathbb{R})
\end{cases}
\]

for some constant \(C\). Moreover it is assumed

\[
u_0 \in L^2_b
\]

where the space \(L^2_b\) is defined in the preliminaries section, see Definition 2.2.

Results that unique solutions to the above equations exist in the space

\[
u \in L^\infty(0, T; L^2_b), \quad \text{with } \partial_x u \in L^2_b([0, T] \times \mathbb{R})\]

will be obtained and further it will be shown that they satisfy the following estimates,
\[ \|u(t)\|_{L^2_b} \leq CT(\|u_0\|_{L^2_b}^2 + 1), \] (1.11)
\[ \int_0^T \|\partial_x u\|_{L^2_b}^2 \, dt \leq CT^2(\|u_0\|_{L^2_b} + 1). \]

Moreover, to satisfy the initial conditions, the solutions will be shown to be in
\[ C([0, T], L^2_b). \] (1.12)

If the assumptions are strengthened to
\[ u_0 \in H^1_b \] (1.13)
then the solution space can be strengthened to
\[ u \in L^\infty(0, T; H^1_b). \] (1.14)

The proof rests on the assumption that the non-linearity is gradient. This assumption is
difficult to relax as in [10] there is an example of a reduction to a coupled Burgers’ system
where the non-linearity is not gradient and the system is equivalent to a strongly damped
Boussinesq equation which exhibits finite time blow-up.

It is possible that estimate (1.11) is not sharp, as in [79] it is shown that the Navier-Stokes
equation in an infinite strip possesses an estimate that does not grow in time. However
this estimate rests on bounding the vorticity for the Navier-Stokes equation and this is
not meaningful for coupled Burgers’ equations. However coupled Burgers’ equations can
be seen as a simplification of the Navier-Stokes equation so it is possible that coupled
Burgers’ equations possess such boundedness as well.

### 1.2 Cahn-Hilliard Equation

In this section the Cahn-Hilliard equation, and the closely associated, Cahn-Hilliard-Oono
equation will be studied.

The Cahn-Hilliard equation, and many of its generalisations, are very important in the
study of phase separation in fluids, which is a significant area of study in material science.
The equation is well understood when the underlying domain is bounded. Questions
such as existence of solutions, uniqueness, regularity, dissipativity and the existence of
attractors have been well studied, even in the case of singular potentials. See [13, 22, 30,
31, 36, 41, 42, 43, 50, 26, 60, 62, 63, 65, 66, 76] (see also their references).

Studying even finite energy solutions when the underlying domain is unbounded is much
more complicated. The main tool used in a bounded domain is inverting the Laplacian,
\((-\Delta_x)^{-1}\), and using this to obtain estimates in the space \(W^{-1,2}\). However problems arise
in an unbounded domain, notably that the inverse Laplacian does not map $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ and so the tools developed are generally not applicable. This problem does not appear in a cylindrical domain, with Dirichlet boundary conditions, and a study of finite energy solutions in this case can be found in \[1, 4, 61, 29, 80\]. In this situation the case of infinite energy solutions has been studied in \[27\], see also \[8\].

In general there has not been much progress in the study of infinite energy solutions to the Cahn-Hilliard equation. There are some local results such as non-linear (diffusive) stability of relatively simple equilibria (e.g., kink-type solutions), relaxation rates to those equilibria, asymptotic expansions in a small neighbourhood of them, etc., see \[9, 52\] and references therein (also \[23\] where the viscous Cahn-Hilliard equation is studied).

Even less is established when the initial data are in the space $L^\infty(\mathbb{R}^n)$. It seems the global existence of a solution is not proven for even a cubic non-linearity $f(u) = u^3 - u$ and $f(u)$ must be linear outside of a compact set to obtain the desired result which is boundedness as time goes to infinity, see \[12\].

Much of this work is published in \[68\]

**Results**

Cahn-Hilliard equation will be studied,

$$\partial_t u = \Delta u, \quad \mu := -\Delta u + f(u) + g(x), \quad u|_{t=0} = u_0, \text{ on } \mathbb{R}^3.$$  \hspace{1cm} (1.15)

Under the assumptions that the non-linearity $f(u) = f_0(u) + \psi(u)$ satisfies:

$$\begin{align*}
1. & \quad f'_0(u) \geq 1, \quad f_0(0) = 0, \\
2. & \quad |\psi(u)| + |\psi'(u)| \leq C, \\
3. & \quad |f(u)| \leq \alpha |F(u)| + C, \quad F(u) := \int_0^u f(v) \, dv,
\end{align*}$$  \hspace{1cm} (1.16)

where $\alpha > 0$ and the external forces $g \in L^6_b(\mathbb{R}^3)$ and the initial data $u_0$ belongs to the space $\Phi_b$ defined as follows:

$$\Phi_b := \{ u \in W^{1,2}_b(\mathbb{R}^3), \quad F(u) \in L^1_b(\mathbb{R}^3) \},$$  \hspace{1cm} (1.17)

the existence of solutions which satisfy the Cahn-Hilliard equation as distributions will be shown in the following spaces

$$\begin{align*}
1. & \quad u(t) \in \Phi_b, \quad t \in [0, T]; \\
2. & \quad u(t) \in C([0, T], L^2_{\text{loc}}(\mathbb{R}^3)); \\
3. & \quad \mu \in L^2_b([0, T], W^{1,2}_b(\mathbb{R}^3)).
\end{align*}$$  \hspace{1cm} (1.18)

And those solutions will be shown to satisfy the following estimate
\[ \|u(t)\|_{W^{1,2}}^2 + \|F(u(t))\|_{L^1} + \|\nabla_x \mu\|_{L^2([0,t] \times \mathbb{R}^3)}^2 \leq \]
\[ \leq C(1 + t^4) \left( 1 + \|g\|_{L^6}^2 + \|u_0\|_{W^{1,2}}^2 + \|F(u_0)\|_{L^1} \right)^{5/2}, \]

where the constant \( C \) is independent of \( u, g \) and \( t \).

The Cahn-Hilliard-Oono equation

\[ \partial_t u = \Delta_x \mu - \lambda u, \quad \mu := -\Delta_x u + f(u) + g(x), \quad u \big|_{t=0} = u_0 \]

is also studied and, under the above conditions, solutions exist and satisfy the following estimate,

\[ \|u(t)\|_{W^{1,2}}^2 + \|F(u(t))\|_{L^1} + \|\nabla_x \mu\|_{L^2([t,t+1] \times \mathbb{R}^3)}^2 \leq \]
\[ \leq Q(\|g\|_{L^6}) + Q(\|u(0)\|_{W^{1,2}}^2 + \|F(u(0))\|_{L^1}) e^{-\sigma t}, \quad t \geq 0, \]

for some monotone increasing function \( Q \) and positive constant \( \sigma \) independent of the initial data \( u_0 \) and \( t \geq 0 \).

When the following assumption is added, that there exists a convex positive function \( \Psi \) such that

\[ \begin{align*}
1. \quad & \Psi(u) \leq C(|F(u)| + 1), \\
2. \quad & |f'(u)| \leq \Psi(u),
\end{align*} \]

then it will be shown that for both the Cahn-Hilliard and Cahn-Hilliard-Oono equations the solutions are unique and have increased regularity,

\[ u(t) \in W^{2,6}_b(\mathbb{R}^3), \]

for all \( t > 0 \).

Finally the case of singular potentials is studied, several assumptions are changed,

\[ -1 < u(t, x) < 1 \quad \text{for almost all} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \]
\[ \begin{align*}
1. \quad & f \in C^2(-1, 1), \quad f(0) = 0; \\
2. \quad & \lim_{u \to \pm 1} f(u) = \pm \infty; \\
3. \quad & \lim_{u \to \pm 1} f'(u) = +\infty.
\end{align*} \]

It will be shown that for an appropriate \( \kappa > 1 \) and \( \beta > 0 \) the following relation holds,

\[ |f(u)| \leq \beta |F(u)|^\kappa + C. \]

and, exactly as in the case of regular potentials, it is assumed that \( g \in L^6_b(\mathbb{R}^3) \).
It is then shown that solutions to the Cahn-Hilliard equation exist and satisfy the following estimate
\[
\|u(t)\|_{W^{1,2}_b}^2 + \|F(u(t))\|_{L^1_b} + \|\nabla x \mu\|_{L^2([0,t] \times \mathbb{R}^3)}^2 \leq C(1 + t^{3\kappa+1}) \left( 1 + \|g\|_{L^6_b}^2 + \|u_0\|_{W^{1,2}_b}^2 + \|F(u_0)\|_{L^1_b} \right)^{3\kappa-1/2},
\]
where \(\kappa\) is the same as in assumption (1.26) and solutions to the Cahn-Hilliard-Oono equation exist and satisfy the following estimate
\[
\|u(t)\|_{W^{1,2}_b}^2 + \|F(u(t))\|_{L^1_b} + \|\nabla x \mu\|_{L^2([t,t+1] \times \mathbb{R}^3)}^2 \leq Q(\|u_0\|_{\Phi_b})e^{-\sigma t} + Q(\|g\|_{L^6_b}),
\]
where the monotone increasing function \(Q\) and positive constant \(\sigma\) are independent of \(u_0\) and \(t\).

### 1.3 Navier-Stokes Equations

In this section the following damped Navier-Stokes equations in the whole space, \(\mathbb{R}^2\), will be studied,
\[
\begin{cases}
\partial_t u + (u, \nabla)u = \Delta u - \alpha u - \nabla p + g, \\
\nabla \cdot u = 0, \ u|_{t=0} = u_0,
\end{cases}
\]
where \(\alpha\) is a positive constant and \(g\) is the external forcing term. It will be studied under the following conditions (the exact meaning of these spaces will be made clear in the preliminaries section)
\[
u_0 \in L^2(\mathbb{R}^2), \ \text{div} \ u_0 = 0, \ g \in \dot{L}^2(\mathbb{R}^2), \ \text{div} \ g = 0, \ \text{and} \ \text{curl} \ g \in L^\infty(\mathbb{R}^2),
\]
the definition of \(\dot{L}^2\) can be found in Proposition 2.3.

Equations (5.1) model a thin layer of fluid (so thin it is considered to have zero thickness) moving over a rough underlying surface. This can be seen in geophysical models for atmospheric flow (where the thin layer is the atmosphere and the surface is the ground) and in oceanic flow (where the thin layer is the ocean and the surface its floor).

The chief advantage of reducing the thickness of the layer of fluid to zero is that the problem can be analytically attacked much more effectively than it can in 3D.

The problem of the damped Navier-Stokes (and also of the closely related damped Euler equations) has been widely studied (see [16] [17] [20] [47] [48] [49]). However these papers, in general, study the case of finite energy solutions (where the \(L^2\) norm is finite), usually in a bounded domain (with appropriate boundary conditions). The approach taken in [83] and continued in this section is to look at infinite energy solutions where the energy...
is only locally bounded (in \( L^2_b \)) which admits a much greater class of solutions, noting that \( L^2 \subset L^2_b \).

This section is essentially a continuation of the program begun by Sergey Zelik in [83]. The accomplishments in that paper are as follows

1. It shows that solutions to (1.29) are bounded given \( u_0, g \in L^2_b \), \( \text{curl } g \in L^\infty \), \( \text{div } u_0 = \text{div } g = 0 \).
2. It shows that these solutions become smoother in time.
3. It shows that these solutions are unique and continuously dependent on initial data.
4. It obtains a dissipative estimate for these solutions.
5. It proves the existence of a locally compact attractor for these solutions.

There are additional results in that paper looking at the case of \( \alpha = 0 \).

These results in [83] rest on two main ideas. The first is to estimate the non-linearity by the introduction of weighted energy spaces with parameter \( \epsilon \) and using this to obtain a family of estimates, all dissipative, parametrised by \( \epsilon \) which is dependent on the size of the initial data and the size of \( g \). For more information on this approach see [79], [82], [83].

The second is to estimate the gradient of the pressure through \( u \otimes u \) via the convolution operator utilising the decomposition of the convolution kernel, for more details on this technique see [83] and its origin, in [57].

With these two obstacles overcome the estimates become tractable and the above program is completed. In this section this program will be extended to the following, noting that \( L^2_b \subset L^2_b \) (see the preliminaries section for a definition of \( L^2_b \));

1. Use the dissipative estimate to obtain tail estimates for the solutions under the conditions in (1.30). This means that the dynamics are essentially restricted to a domain of some large radius plus some small tails outside that.
2. Use these tail estimates to show the existence of a globally compact attractor.
3. Use the smoothing estimates to show this attractor has finite fractal dimension.

The existence of the finite dimensional globally compact attractor is a consequence of the increased strictness of the conditions for \( u_0 \) and \( g \). The techniques used to prove it rely essentially on the technique built up in [83].

**Results**

Under the conditions in (1.30) the solution semi-group of equation (1.29) possesses a finite dimensional globally compact attractor in the space \( L^2_b \cap \{\text{div } u = 0\} \).
2 Preliminaries

In this section some machinery for the treatment of PDEs in an unbounded domain will be introduced, namely weighted spaces and their relationship with uniformly local spaces which will be used to obtain the key theorems relating to the equations below. Large parts of these ideas can be found in [29, 61, 79, 81].

Please note that throughout this work unimportant constants will be denoted by \( C \) and this notation will not change even when the constant itself changes, though efforts will be made to show which parameters certain constants depend on.

A ball of radius \( \epsilon \), with centre \( x_0 \), in \( \mathbb{R}^n \), is denoted
\[
B_{x_0}^\epsilon := \{ x \in \mathbb{R}^n : \| x - x_0 \| \leq \epsilon \}. \tag{2.1}
\]

**Definition 2.1** A function \( \phi \in L^\infty_{\text{loc}}(\mathbb{R}^n) \) is called a weight function with exponential rate of growth (\( \nu > 0 \)) if the conditions
\[
\phi(x) > 0 \quad \text{and} \quad \phi(x + y) \leq Ce^{\nu|x|}\phi(y), \tag{2.2}
\]
are satisfied for every \( x, y \in \mathbb{R}^n \).

Any weight function with growth rate \( \nu \) also satisfies
\[
\phi(x + y) \geq C_1 e^{-\nu|x|}\phi(y)
\]
for all \( x, y \in \mathbb{R}^n \). Important examples of weight functions which will be used, with growth rate \( \nu \), are
\[
\phi_\epsilon(x) = \frac{1}{(1 + |\epsilon x|^2)^{\gamma/2}} \quad \text{and} \quad \varphi_\epsilon(x) = e^{-\sqrt{|\epsilon x|^2 + 1}}, \tag{2.3}
\]
where \( \gamma \in \mathbb{R} \) is arbitrary and \( \epsilon < \nu \) in the second example and
\[
\theta_{R,x_0}(x) := \frac{1}{R^3 + |x - x_0|^3} \quad \text{and} \quad \rho_{R,x_0}(x) := R^{\frac{3}{2}} + |x - x_0|^\frac{3}{2}. \tag{2.4}
\]
It is worth noting here, as it becomes important when discussing initial conditions in the coupled Burger’s equations, that a simple change of variables \( y = \epsilon x \) shows that, for \( \phi_\epsilon(x) \) in (2.3) with \( \gamma = 2 \),
\[
\int_\mathbb{R} \phi_\epsilon(x) \, dx = \int_\mathbb{R} \phi_\epsilon \left( \frac{y}{\epsilon} \right) \frac{dy}{\epsilon} = \frac{\pi}{\epsilon}. \tag{2.5}
\]

Crucial for the estimates below is the fact that these functions satisfy (2.2) uniformly with respect to \( \epsilon \to 0 \). Moreover, if \( \phi(x) \) satisfies (2.2), then the shifted weight function \( \phi(x - x_0), x_0 \in \mathbb{R}^3 \), also satisfies (2.2) with the same constants \( C \) and \( \nu \).
It is not difficult to check that the function \( \varphi_\varepsilon \) in (2.3) satisfies
\[
|D_N^N \varphi_\varepsilon(x)| \leq C_N \varepsilon^N \varphi_\varepsilon(x)
\] (2.6)
for all \( N \in \mathbb{N} \) and the constant \( C_N \) is independent of \( \varepsilon \to 0 \) (here and below \( D_N^N \) stands for the collection of all partial derivatives of order \( N \) with respect to \( x \)). In addition, the first weight function of (2.3), \( \phi_\varepsilon(x) \), satisfies the improved version of (2.6)
\[
|D_N^N \phi_\varepsilon(x)| \leq C_N' \varepsilon^N |\phi_\varepsilon(x)|^{1+N/\gamma} \leq C_N' \varepsilon^N \phi_\varepsilon(x),
\] (2.7)
where \( C_N' \) is also independent of \( \varepsilon \to 0 \). Furthermore, to verify the dissipativity of the Cahn-Hilliard-Oono equation, the weight functions \( \phi_\varepsilon(t)(x) \) where the parameter \( \varepsilon = \varepsilon(t) \) depends explicitly on time will be considered. In this case,
\[
\partial_t \phi_\varepsilon(t)(x) = \frac{\varepsilon'(t)}{\varepsilon(t)} x \cdot \nabla_x \phi_\varepsilon(t)(x) = \gamma \frac{\varepsilon'(t)}{\varepsilon(t)} \phi_\varepsilon(t)(x) \frac{|\varepsilon(t)x|^2}{1+|\varepsilon(t)x|^2}
\]
and, therefore,
\[
|\partial_t \phi_\varepsilon(t)(x)| \leq \gamma \frac{|\varepsilon'(t)|}{\varepsilon(t)} \phi_\varepsilon(t)(x), \quad x \in \mathbb{R}^3.
\] (2.8)

Now the weighted and uniformly local spaces which will be used throughout this thesis are introduced, here \( \Omega \) can be \( \mathbb{R}, \mathbb{R}^2 \) and \( \mathbb{R}^3 \) as required.

**Definition 2.2** For any \( 1 \leq p \leq \infty \), the uniformly local Lebesgue space \( L^p_b(\Omega) \) is defined as follows:
\[
L^p_b(\Omega) := \left\{ u \in L^p_{\text{loc}}(\Omega) : \|u\|_{L^p_b} := \sup_{x_0 \in \Omega} \|u\|_{L^p(B^R_{x_0})} < \infty \right\},
\] (2.9)
where \( B^R_{x_0} \) stands for the \( R \)-ball in \( \Omega \) centred at \( x_0 \).

Furthermore, for any weight function \( \varphi(x) \) with exponential rate of growth, the weighted Lebesgue spaces are defined:
\[
L^p_\varphi(\Omega) = \left\{ u \in L^p_{\text{loc}}(\Omega) : \|u\|_{L^p_\varphi} := \left( \int_\Omega \varphi(x)|u(x)|^p \, dx \right)^{1/p} < \infty \right\}.
\] (2.10)
Analogously, the weighted \( (W^{l,p}_\varphi(\Omega)) \) and uniformly local \( (W^{l,p}_b(\Omega)) \) are defined as subspaces of \( \mathcal{D}'(\Omega) \) of distributions whose derivatives up to order \( l \) belong to \( L^p_\varphi(\Omega) \) or \( L^p_b(\Omega) \) respectively. That is
\[
W^{l,p}_\varphi(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) : \left( \int_\Omega \phi(x) \sum_{|\alpha| \leq l} |D^\alpha u(x)|^p \, dx \right)^{1/p} < \infty \right\}
\] (2.11)
where \( \alpha \) is the multi-index and
\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}
\]
and

\[
W^{l, p}_b(\Omega) := \left\{ u \in W^{l, p}_{\text{loc}}(\Omega) : \| u \|_{W^{l, p}_b} := \sup_{x_0 \in \Omega} \| u \|_{W^{l, p}(B_{x_0}^1)} < \infty \right\}. \quad (2.12)
\]

This works for natural \( l \) only, the weighted Sobolev spaces with fractional/negative number
of derivatives can be also defined in a standard way using interpolation/duality.

Additionally uniformly local norms can be defined with larger balls, later the following
norm will be used,

\[
\| u \|_{W^{l, p}_b(R^2)} := \sup_{x_0 \in \mathbb{R}^2} \| u \|_{W^{l, p}(B_{x_0}^R)}, \quad (2.13)
\]

and (2.22) will establish that

\[
\| u \|_{W^{l, p}_b} \leq \| u \|_{W^{l, p}_b(R^2)} \leq CR^2 \| u \|_{W^{l, p}_b}, \quad (2.14)
\]

Also Sobolev spaces, \( \dot{W}^{l, p}_b \) will be used,

\textbf{Definition 2.3} The space \( \dot{W}^{l, p}_b \) is a closed subspace of \( W^{l, p}_b \) and is defined as,

\[
\dot{W}^{l, p}_b := \left\{ u : u \in W^{l, p}_b \text{ and } \| u \|_{W^{l, p}(B_{x_0}^1)} \to 0 \text{ as } |x_0| \to \infty \right\}. \quad (2.15)
\]

\textbf{Drawing Proposition 1.3 from [32],}

\textbf{Proposition 2.4} A set \( S \subseteq \dot{W}^{l, p}_b(\mathbb{R}^n) \) is compact if and only if:

1. For every \( x_0 \in \mathbb{R}^n \), the restriction of \( S \) to \( W^{l, p}_b(B_{x_0}^R) \) is compact.

2. The set \( S \) possesses a uniform 'tail' estimate, i.e. there exists a continuous function \( D_S(z) : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{|x_0| \to \infty} D_S(|x_0|) = 0 \) and

\[
\| u \|_{W^{l, p}(B_{x_0}^1)} \leq D_S(|x_0|), \forall u \in S.
\]

see [32] for the proof.

The following proposition gives the crucial relation for estimating the uniformly local
norms of solutions using the energy estimates in weighted Sobolev spaces.

\textbf{Proposition 2.5} Let \( \phi \) be a weight function with exponential growth such that \( \| \phi \|_{L^1(\mathbb{R}^3)} < \infty \) and let \( u \in L^p_b(\mathbb{R}) \) for some \( 1 \leq p < \infty \). Then, \( u \in L^p_\phi(\mathbb{R}^3) \) and

\[
\| u \|_{L^p_\phi} \leq C \| \phi \|_{L^1}^{1/p} \| u \|_{L^p_b}, \quad (2.16)
\]
where the constant $C$ depends only on $p$ and the constants $C$ and $\nu$ in (2.2) (and is independent of the concrete choice of the functions $\phi$ and $u$). Moreover,

$$
\|u\|_{L^2_b} \leq C \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^2_{\phi(-x_0)}},
$$

(2.17)

where $C$ is also independent of the concrete choice of $u$ and $\phi$.

For the proof of this proposition, see [29] or [81].

The machinery will mainly use estimate (2.17) in the situation where $\phi = \phi_\varepsilon$ is one of the special weight functions of (2.3) and $\varepsilon > 0$ is a small parameter. In this case, for $\gamma > 3$ (which is required in 3 dimensions to ensure the weight function is integrable in the whole space), $\|\phi_\varepsilon\|_{L^1} \sim \varepsilon^{-3}$ and (2.16) reads

$$
\|v\|_{L^p_b} \leq C\varepsilon^{-3/p} \|v\|_{L^p_{\phi_\varepsilon}},
$$

(2.18)

where the constant $C > 0$ is independent of $\varepsilon \to 0$.

The proper spaces for functions of time with values in some uniformly local space will be needed, say $u : [0, T] \to W^{l,p}_b(\Omega)$. With a slight abuse of notations, the following space is denoted, $L^q_b([0, T], W^{l,p}_b(\Omega))$ the subspace of distributions defined by the following norm:

$$
\|u\|_{L^q_b([0, T], W^{l,p}_b(\Omega))} := \sup_{(t,x_0) \in [0, T-1] \times \Omega} \|u\|_{L^q((t,t+1), W^{l,p}_b(B^{1}_{x_0}))}.
$$

(2.19)

Note that a more standard definition of $L^q_b([0, T], W^{l,p}_b(\Omega))$ would be via the following norm:

$$
\|u\|_{L^q_b([0, T], W^{l,p}_b(\Omega))} := \sup_{t \in [0, T-1]} \left( \int_{t}^{t+1} \|u(t)\|_{W^{l,p}_b}^q dt \right)^{1/q}.
$$

(2.20)

which differs from (2.19) by the changed order of supremum over $x_0 \in \Omega$ and integral in time and is slightly stronger than (2.19). The main reason to use (2.19) instead of (2.20) is that the first norm can be estimated through the associated weighted space analogously to (2.17) which is essential since all estimates in uniformly local spaces are usually obtained with the help of the associated weighted estimates. Thus, the first norm gives the natural and useful generalization of the space $L^q([0, T], W^{l,p}(\Omega))$ to the uniformly local case and the second norm (2.20) requires more delicate additional arguments to be properly estimated and is of only limited interest.

Now two lemmas concerning multiplying weight functions. Lemma 2.5 from [83],

**Lemma 2.6** Let $\theta_{x_0}(x)$ be the weight function defined via (2.4). Then the following estimate holds:

$$
\int_{x \in \mathbb{R}^2} \theta_{x_0}(x) \theta_{y_0}(x) \, dx \leq C \theta_{x_0}(y_0)
$$

(2.21)

where $C$ is independent of $x_0, y_0 \in \mathbb{R}^2$. 

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Proposition 2.7 Let \( \theta \) be a weight function of exponential growth rate such that

\[
\int_{x \in \mathbb{R}^2} \theta \, dx < \infty.
\]

Then, for every \( u \in W^{1,p}_b(\mathbb{R}^2) \) and every \( \kappa \geq 1 \),

\[
\|u\|_{W^{1,p}(B_{\kappa x_0})}^p \leq C \int_{\mathbb{R}^2} \theta(y-x_0) \|u\|_{W^{1,p}(B_{y})}^p \, dy,
\]

(2.22)

The scaled analogue of (2.22) reads,

\[
\|u\|_{W^{1,p}(B_{\kappa x_0})}^p \leq C \kappa R \int_{\mathbb{R}^2} \theta_{R,x_0}(y) \|u\|_{W^{1,p}(B_{y})}^p \, dy.
\]

(2.23)

For the proof see [83].

Lemma 2.8 Let \( \theta_{x_0}(x) \) and \( \rho_{x_0}(x) \) be defined,

\[
\theta_{x_0}(x) := \frac{1}{1 + |x-x_0|^3}, \quad \rho_{x_0}(x) := 1 + |x-x_0|^\frac{3}{2}.
\]

(2.24)

Then the following estimate holds:

\[
\int_{x \in \mathbb{R}^2} \theta_{x_0}(x) \rho_{y_0}(x) \, dx \leq C \rho_{x_0}(y_0)
\]

(2.25)

where \( C \) is independent of \( x_0, y_0 \in \mathbb{R}^2 \).

Proof

The basic triangle inequality gives,

\[
|x - y_0| \leq |x - x_0| + |x_0 - y_0|.
\]

(2.26)

From this the next inequality is evident, the square roots being valid because all terms are positive and greater than 1,

\[
1 + |x - y_0|^\frac{3}{2} \leq 2(1 + |x - y_0|)^\frac{3}{2} \leq 2(1 + |x - x_0| + |x_0 - y_0| + |x - x_0||x_0 - y_0|)^\frac{3}{2} = 2(1 + |x - x_0|)^\frac{3}{2}(1 + |x_0 - y_0|)^\frac{3}{2}.
\]

(2.27)

Now for the integrand,
\[
\theta_{x_0}(x) \rho_{y_0}(x) = \frac{1}{1 + |x - x_0|^3} (1 + |y - y_0|) \frac{1}{2}
\]
(2.28)

\[
\leq \frac{2}{1 + |x - x_0|^3} (1 + |x - x_0|) \frac{1}{2} \rho_{x_0}(y_0)
\]

And integrating over \( x \in \mathbb{R}^2 \) the result is obtained. \( \square \)

**Lemma 2.9** Let \( \theta_{x_0}(x) \) and \( \rho_{x_0}(x) \) be weights defined via (2.4). Then the following estimate holds:

\[
\int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \rho_{R,y_0}(x) \, dx \leq CR^{-1} \rho_{R,x_0}(y_0)
\]
(2.29)

where \( C \) is independent of \( x_0, y_0 \in \mathbb{R}^2 \).

**Proof**

Lemma 2.8 gives the result for the case where \( R = 1 \) and this can be extended to the case of arbitrary \( R \) by scaling. First note

\[
\rho_{R,x_0}(x) = R^{\frac{1}{2}} + |x - x_0| \frac{1}{2} = R^{\frac{1}{2}} \rho_{1,x_0} \left( \frac{x}{R} \right),
\]
(2.30)

\[
\theta_{R,x_0}(x) = \frac{1}{R^{3} + |x - x_0|^3} = R^{-3} \theta_{1,x_0} \left( \frac{x}{R} \right).
\]

Then if \( R y = x \) and \( R^2 dy = dx \) using (2.25) it can be seen that

\[
\int_{\mathbb{R}^2} \theta_{R,x_0}(x) \rho_{R,y_0}(x) \, dx = \int_{\mathbb{R}^2} R^{-3} \theta_{1,x_0} \left( \frac{x}{R} \right) R^{\frac{1}{2}} \rho_{1,y_0} \left( \frac{x}{R} \right) \, dx = \]
(2.31)
\[
= \int_{\mathbb{R}^2} R^{-3} \theta_{1,x_0} \left( y \right) R^{\frac{1}{2}} \rho_{1,y_0} \left( y \right) R^2 \, dy \leq \]
\[
\leq CR^{-\frac{1}{2}} \rho_{1,x_0} \left( \frac{y_0}{R} \right) = CR^{-1} \rho_{R,x_0}(y_0).
\]

Further it is clear that, with \( R y = x \) and \( R^2 dy = dx \)

\[
\int_{\mathbb{R}^2} \theta_{R,x_0}(x) \, dx = R^{-3} \int_{\mathbb{R}^2} \theta_{1,x_0} \left( \frac{x}{R} \right) \, dx = \]
(2.32)
\[
= R^{-3} R^2 \int_{\mathbb{R}^2} \theta_{1,y_0} \left( y \right) \, dy = CR^{-1}.
\]

Further define two cut-off functions and an associated space,
Definition 2.10 For any set \( \Phi \),

\[
\eta_{R,x_0}(x) := \begin{cases} 
1 & : x \in B^R_{x_0} \\
0 & : x \notin B^{2R}_{x_0}
\end{cases}
\]
and \( \chi_{R,x_0}(x) := 1 - \eta_{R,x_0}(x) \).

(2.33)

such that \( \eta_{R,x_0} \) is continuous and

\[
|\nabla \eta_{R,x_0}| \leq \frac{1}{R} \eta_{2R,x_0},
\]

(2.34)

holds pointwise. The cut-off space is defined as subsets of \( L_{\text{loc}}^p(\mathbb{R}^2) \) such that the following semi-norm is finite,

\[
\|u\|^2_{L^2_{\eta_{R,x_0}}} := \int_{\mathbb{R}^2} \eta_{R,x_0}|u|^2 \, dx < \infty.
\]

(2.35)

Norms of the type \( \int_{x_0 \in \Omega} \phi(x_0) \|u\|^p_{L^p(B^R_{x_0})} \, dx_0 \) will be used frequently, where \( \phi \) is a weight function of exponential growth rate. The following Proposition 2.4 from [83] establishes that these norms are equivalent for different values of \( R \).

Proposition 2.11 Let \( \phi \) be a weight function of exponential growth rate and let \( 1 \leq p \leq \infty, \, R > 0, \, \Omega = \mathbb{R}^n \). Then,

\[
C_1 \int_{x_0 \in \Omega} \phi(x_0) \|u\|^p_{L^p(B^R_{x_0})} \, dx_0 \leq \|u\|^p_{L^p(\Omega)} \leq C_2 \int_{x_0 \in \Omega} \phi(x_0) \|u\|^p_{L^p(\Omega)} \, dx_0,
\]

(2.36)

where the constants \( C_i \) depend on \( R \) but are independent of \( u \) and the concrete choice of the weight \( \phi \).

For the proof see [83] or [28] and references therein. Moreover the following proposition establishes two inequalities in the case when different values of \( R \) are chosen.

Proposition 2.12 Let \( \theta_{R,x_0} \) be a weight function of exponential growth rate defined by (2.4) and let \( 1 \leq p \leq \infty, \, R > 0, \, \Omega = \mathbb{R}^n, \, l > 1 \) and \( \kappa > 1 \). Then,

\[
\|u\|^2_{H^l(B^R_{x_0})} \leq \int_{\Omega} \theta_{R,x_0}(x) \|u\|^2_{H^l(\Omega)} \, dx, \text{ and}
\]

(2.37)

\[
\int_{\Omega} \theta_{R,x_0}(x) \|u\|^p_{L^2(B^R_{x_0})} \, dx \leq C_\kappa \int_{\Omega} \theta_{R,x_0}(x) \|u\|^p_{L^2(\Omega)} \, dx
\]

(2.38)

where \( C_\kappa \) depends on \( \kappa \) but not on \( R \) or \( \phi \).

For the proof of this see [83], equations (2.19) and (2.21).

Now some lemmas to aid in the estimation process.
Lemma 2.13 \( L^\infty([0,T],L^2) \cap L^2([0,T],H^1) \) is embedded in \( L^4([0,T],L^4) \) in a one dimensional bounded domain and the following estimate holds

\[
\|u\|_{L^4([0,T],L^4)} \leq \|u\|_{L^\infty([0,T],L^2)}^{\frac{1}{2}} \|u\|_{L^2([0,T],H^1)}^{\frac{1}{2}}.
\] (2.39)

**Proof**
First establish that \( \|u\|_{L^\infty}^2 \leq \|u\|_{L^2} \|u\|_{H^1} \) by the following estimate,

\[
u^2(x) = \frac{1}{2} \int_0^x \partial_x (u^2) \, dx = \int_0^x u \partial_x u \, dx \leq \|u\|_{L^2} \|\partial_x u\|_{L^2} \leq \|u\|_{L^2} \|u\|_{H^1}.
\] (2.40)

And then use this for the following estimate,

\[
\int_0^T \|u(t)\|_{L^4}^4 \, dt \leq \int_0^T \|u(t)\|_{L^2}^2 \|u(t)\|_{H^1}^2 \, dt \leq \|u\|_{L^\infty([0,T],L^2)}^2 \|u\|_{L^2([0,T],H^1)}^2,
\] (2.41)

which establishes the result.

\( \square \)

Lemma 2.14 The space

\[
H^1([0,T];H^{-1}) \cap L^2([0,T];H^1)
\] (2.42)

is compactly embedded in \( L^{6-\delta}([0,T];H^{6-\delta}) \), for a small \( \delta > 0 \), in a one dimensional bounded domain and the following estimate holds

\[
\|u\|_{L^{6-\delta}([0,T];H^{6-\delta})} \leq C \|u\|_{H^\frac{1}{2}([0,T];H^\frac{1}{2})} \leq C \|u\|_{H^1([0,T];H^{-1}) \cap L^2([0,T];H^1)}
\] (2.43)

**Proof**
Let \( \Omega \) denote the bounded domain. By the interpolation inequality it is possible to obtain

\[
\|u\|_{H^\alpha([0,T],H^{1-2\alpha})} \leq C \|u\|_{H^{1/2}([0,T],H^{1/2})}, \quad 0 \leq \alpha \leq 1
\] (2.44)

and noting the maximum regularity that can be obtained in both space and time is at \( \alpha = \frac{1}{3} \) the Sobolev embedding theorem can be used to show that \( H^{\frac{1}{2}}([0,T],H^{\frac{1}{2}}) \) is compactly embedded in \( L^{6-\delta}([0,T] \times \Omega) \) for any \( \delta > 0 \) giving the above result. \( \square \)

Lemma 2.15 [Krasnoselskii] Let \( f(u) : \mathbb{R}^N \to \mathbb{R}^N \) be a function on \( N \)-dimensional Euclidean space which is continuous in \( u \). Let \( p \geq 1 \) and \( u \in L^{2p}([0,T];L^{2p}) \), \( |f(x,u)| \leq C(1 + |u|^2) \) and \( F(u)(x) := f(u(x)) \). Then \( F \) is a bounded and continuous map from \( L^{2p}([0,T];L^{2p}) \) to \( L^p([0,T];L^p) \).

For the proof see [24].
Lemma 2.16 If

\[ u \in L^2([0, T], H^1_0) \text{ and } \partial_t u \in L^2([0, T], H^{-1}) \]

on a one dimensional bounded domain then \( u \in C([0, T], L^2) \) and

\[ \|u\|_{C([0, T], L^2)} \leq C\|\partial_t u\|_{L^2([0, T], H^{-1})}^{\frac{1}{2}} \cdot \|u\|_{L^2([0, T], H^1)}^{\frac{1}{2}} \]

**Proof**

First extend \( u \) to the interval \([-T, T]\) by reflection, that is \( u(-t) = u(t) \), calling this new function \( \hat{u} \). Then mollify \( \hat{u} \) to obtain a sequence \( u_n \) of continuous functions with values in \( H^1_0 \) which converge to \( \hat{u} \).

Then take a cut-off function, \( \psi \) such that

\[
\begin{aligned}
1. \psi(t) &= 1 \text{ for } t \geq 0 \\
2. 0 < \psi(t) < 1 \text{ for } -\frac{T}{2} < t < 0 \\
3. \psi(t) &= 0 \text{ for } t \leq -\frac{T}{2} \\
4. \psi'(t) \text{ bounded.}
\end{aligned}
\]

Now consider

\[ \frac{d}{dt}(\psi\|v\|_{L^2}^2) = 2\psi(\partial_t v, v)_{L^2} + \psi'\|v\|_{L^2}^2. \] (2.45)

Taking \( v = u_n - u_m \), integrating, using Hölder’s inequality and taking a supremum, obtain

\[ \sup_{t \in [0, T]} \|u_n(t) - u_m(t)\|_{L^2}^2 \leq \int_{-T}^{T} \left| 2\psi(\partial_t u_n - \partial_t u_m, u_n - u_m)_{L^2} + \psi'\|u_n(t) - u_m(t)\|_{L^2}^2 \right| dt \leq \]

\[ \leq 2\|\partial_t u_n - \partial_t u_m\|_{L^2([-T, T], H^{-1})} \cdot \|u_n - u_m\|_{L^2([-T, T], H^1)} > \int_{-T}^{T} \psi'\|u_n(t) - u_m(t)\|_{L^2}^2 dt \]

which is a Cauchy sequence in a complete space, therefore \( u \in C([0, T], L^2) \). \qed

Lemma 2.17 Given a weight function \( \psi \in C(\mathbb{R}^n, \mathbb{R}) \) such that \( 0 < \psi < 1 \) and \( |\partial_x \psi| \leq \psi \), and if \( v \) is defined as \( v = \psi^\frac{1}{2} u \), then,

\[ C\|u\|_{H^1_0}^2 \leq \|v\|_{H^1}^2 \leq C\|u\|_{H^1_0}^2. \]

**Proof**
\[ \|u\|^2_{H^1_\psi} = \int \psi(|u|^2 + |\partial_x u|^2) \, dx = \int |\psi^{\frac{1}{2}} u|^2 + |\psi^{\frac{1}{2}} \partial_x u|^2 \, dx \leq (2.47) \]

\[ \leq \int |\psi^{\frac{1}{2}} u|^2 + |\psi^{\frac{1}{2}} \partial_x u|^2 + |\partial_x \psi^{\frac{1}{2}} u|^2 \, dx \leq \int |\psi^{\frac{1}{2}} u|^2 + |\partial_x (\psi^{\frac{1}{2}} u)|^2 \, dx = \|v\|^2_{H^1_\psi} \]

and

\[ \|v\|^2_{H^1_\psi} = \int |\psi^{\frac{1}{2}} u|^2 + |\partial_x (\psi^{\frac{1}{2}} u)|^2 \, dx \leq C \int \psi(|u|^2 + |\partial_x u|^2) \, dx = C\|u\|^2_{H^1_\psi} \]

\[ \leq C \int \psi(|u|^2 + |\partial_x u|^2) \, dx = C\|u\|^2_{H^1_\psi} \]

as

\[ (\partial_x \psi^{\frac{1}{2}})^2 = \left( \frac{1}{2} \psi^{-\frac{1}{2}} \partial_x \psi \right)^2 \leq \left( \frac{1}{2} \psi^{\frac{1}{2}} \right)^2 \leq C\psi. \]

\[ \square \]

**Lemma 2.18** Given a weight function \( \phi \) and a function \( u \in H^2(\mathbb{R}) \) it is true that

\[ \| \partial_x u \|^2_{L^2_\phi} \leq C \| u \|^2_{L^2_\phi} + C \| u \|_{L^2_\phi} \| \partial_{xx} u \|_{L^2_\phi}, \]

(2.49)

where \( C \) depends only on \( \phi \).

**Proof**

Using integration by parts,

\[ \| \partial_x u \|^2_{L^2_\phi} = \int \phi \partial_x u \partial_x u \, dx \]

\[ = -\int \phi \partial_{xx} u u \, dx - \int \phi_x \partial_x u u \, dx \]

\[ = -\int \phi \partial_{xx} u u \, dx + \frac{1}{2} \int \phi_{xx} u^2 \, dx \]

\[ \leq C\|u\|_{L^2_\phi} \| \partial_{xx} u \|_{L^2_\phi} + C\|u\|^2_{L^2_\phi}. \]

\[ \square \]
3 Coupled Burgers’ Equations

In an effort to elucidate the ideas behind the analysis of the equation in an unbounded domain, and because this is a good route from which to approach the problem, an analysis of coupled Burgers’ equations in a bounded domain will be presented first and then, afterwards, the process will be repeated for the same equation in an unbounded domain.

3.1 Coupled Burgers’ equations in a bounded domain

3.1.1 The first energy estimate

In this section the first energy estimate for the class of coupled Burgers’ equations is derived,

\[ \partial_t u = \partial_{xx} u + \partial_x (f(u)), \quad x \in \Omega = [0, \pi], \quad u|_{t=0} = u_0 \]  

for the \(N\)-component vector-valued function \(u(x, t)\), with the following conditions on the function \(f : \mathbb{R}^N \to \mathbb{R}^N\). There is a scalar-valued function \(F : \mathbb{R}^N \to \mathbb{R}\) such that

\[
\begin{align*}
1. & \quad f(u) = \nabla F(u), \\
2. & \quad f'(u) \leq C(1 + |u|), \quad \text{for all } u \in \mathbb{R}^N \\
3. & \quad F \in C^2(\mathbb{R}^N, \mathbb{R})
\end{align*}
\]  

for some constant \(C\). Moreover it is assumed

\[ u|_{t=0} = u_0 \in L^2 \text{ and } u|_{\partial \Omega} = 0 \]  

noting that the analysis would go through the same for periodic boundary conditions, which are not studied here.

**Definition 3.1** Define a weak solution as a function, \(u\), such that

\[ u \in L^\infty([0, T], L^2) \cap L^2([0, T], H^1_0) \quad \text{and} \quad \partial_x u \in L^2([0, T], L^2), \]  

and \(u\) satisfies (3.1) as a distribution, that is \(u\) satisfies

\[ (u, \partial_t \psi)_{L^2} = (\partial_x u, \partial_x \psi)_{L^2} + (f(u), \partial_x \psi)_{L^2}, \]  

for any vector \(\psi \in C_0^\infty([0, T] \times \mathbb{R}^2)\). Where \(C_0^\infty\) is infinitely differentiable functions in space and time with compact support. Moreover

\[ u \in C([0, T], L^2), \]  

and the initial conditions are understood as an identity in this space.

**Theorem 3.2** Let assumptions (3.2) and (3.3) hold. For any \(t > 0\) any solution to equation (3.1) in the sense of (3.5) satisfies the following estimates,
\[ \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_x u(\tau)\|_{L^2}^2 \, d\tau = \|u(0)\|_{L^2}^2 \]  
(3.7)

\[ \|u(t)\|_{L^2}^2 \leq e^{-2\lambda t} \|u(0)\|_{L^2}^2 \]  
(3.13)

where \( \lambda \) is the smallest eigenvalue of the Laplacian with Dirichlet boundary conditions, which in this case is 1.

**Proof**

Take the equation (3.1), multiply by \( u \) and integrate in space to obtain.

\[ \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 = (\partial_x (f(u)), u)_{L^2} \]  
(3.8)

now integrate the right hand side by parts to get

\[ (\partial_x (f(u)), u)_{L^2} = -(f(u), \partial_x u)_{L^2} = -(\partial_x (F(u)), 1)_{L^2} = 0 \]

by the fundamental theorem of calculus as the right hand side is a total differential of a function with zero boundary conditions (note this also goes through in the periodic case as the boundary conditions will cancel each other, as they will in all the other integrations by parts.)

This gives

\[ \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 = 0. \]  
(3.9)

So, integrating in time, the following is arrived at,

\[ \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_x u(\tau)\|_{L^2}^2 \, d\tau = \|u(0)\|_{L^2}^2 \]  
(3.10)

or more succinctly, utilising the initial conditions (3.3), derive,

\[ u \in L^\infty([0, T], L^2) \cap L^2([0, T], H_0^1). \]  
(3.11)

Moreover it is possible to return to (3.9) and utilise Poincaré’s inequality to show that

\[ \frac{d}{dt} \|u\|_{L^2}^2 + 2\lambda \|u\|_{L^2}^2 \leq 0 \]  
(3.12)

and using Grönwall’s inequality, derive

\[ \|u(t)\|_{L^2}^2 \leq e^{-2\lambda t} \|u(0)\|_{L^2}^2 \]  
(3.13)

so all solutions decay exponentially, where \( \lambda \) is the first eigenvalue of the Laplacian. \( \square \)
3.1.2 Galerkin proof of existence

This section contains a proof of existence of solutions to (3.1), using the method of Galerkin.

**Theorem 3.3** Let assumptions (3.2) and (3.3) hold. Equation (3.1) possess at least one solution in the sense of (3.5) for any given initial data.

**Proof**

The projector, $P_n$, is defined as

$$P_n u = \sum_{j=1}^{n} (u, e_j)_{L^2} e_j$$

(3.14)

where the set $e_j$ is the eigenvectors of the Laplacian in one dimension with corresponding eigenvalues $\lambda_j$, parametrised by $j$, which is the Fourier basis for $L^2[0, \pi]$. Explicitly

$$e_j = \sqrt{\frac{2}{\pi}} \sin(j x), \quad \lambda_j = -j^2.$$  

(3.15)

Now apply the projector $P_n$ to (3.1) to get

$$\frac{\partial}{\partial t} \sum_{i=1}^{n} u_i e_i = \frac{\partial^2}{\partial x^2} \sum_{i=1}^{n} u_i e_i + \sum_{i=1}^{n} \left( \frac{\partial}{\partial x} (f(u)), e_i \right)_{L^2} e_i,$$

(3.16)

where $u_i(t)$ are the Fourier coefficients of $u$. The system (3.16) is not solvable because the unknown function $u$ still appears in the non-linearity. For this reason truncate $u$, using the approximating sequence

$$f \left( \sum_{j=1}^{n} u_j e_j \right)$$

(3.17)

and integrate by parts in the final term to get,

$$\frac{\partial}{\partial t} \sum_{i=1}^{n} u_i e_i = \frac{\partial^2}{\partial x^2} \sum_{i=1}^{n} u_i e_i - \sum_{i=1}^{n} \left( f \left( \sum_{j=1}^{n} u_j e_j \right), e_i \right)_{L^2} e_i.$$  

(3.18)

These unknown $u_i$’s are now no longer the Fourier coefficients but it will be shown that they converge to them as $n \to \infty$. Next take an inner product of (3.18) with some fixed $e_k$ to get

$$\frac{du_k}{dt} = \lambda_k u_k + \left( f \left( \sum_{j=1}^{n} u_j e_j \right), e_k \right)_{L^2}.$$

(3.19)

Now take a dot product of (3.19) with $u_k$, for each $k$, and then take a sum of these equations over $k$ setting $u_n = \sum_{k=1}^{n} u_k e_k$. This gives,
\[ \frac{d}{dt} \left( \sum_{k=1}^{n} |u_k|^2 \right) - \sum_{k=1}^{n} \lambda_k |u_k|^2 = \left( f(u_n), \sum_{k=1}^{n} u_k e_k \right), \]  

(3.20)

which can be rewritten, using Parseval’s equality and an integration by parts in the non-linearity as,

\[ \frac{d}{dt} \|u_n\|_{L^2}^2 + \|\partial_x u_n\|_{L^2}^2 = (\partial_x f(u_n), u_n). \]  

(3.21)

For any \( n \) (3.20) is a coupled system of first order ODEs (with initial conditions \( P_n u_0 \)) and by the Picard-Lindelöf theorem each has a unique local solution. From (3.21) it is simple to show, by following exactly the method of Theorem 3.2, that these solutions are global in time and bounded by the estimate

\[ \|u_n(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_x u_n(\tau)\|_{L^2}^2 \, d\tau = \|u_n(0)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \]  

(3.22)

for any \( n \), where \( u_n(0) = P_n(u_0) \), in the space

\[ L^\infty([0, T], L^2) \cap L^2([0, T], H^1_0). \]  

(3.23)

From (3.22) it is clear the sequence \( u_n \) is uniformly bounded in \( L^2([0, T], H^1_0) \). Then, since the space \( L^2([0, T], H^1_0) \) is reflexive, by the Banach-Alaoglu theorem, the sequence \( u_n \) is weakly sequentially pre-compact in \( L^2([0, T], H^1_0) \). Moreover, again due to (3.22), the sequence is also bounded in \( L^\infty([0, T], L^2) \), which is the dual space to the separable Banach space \( L^1([0, T], L^2) \). Again by the Banach-Alaoglu theorem, the sequence \( u_n \) is weak-star sequentially pre-compact in \( L^\infty([0, T], L^2) \).

Therefore there exists a subsequence (relabelled for convenience \( u_n \)) which is convergent weakly in \( L^2([0, T], H^1_0) \) and weak star in \( L^\infty([0, T], L^2) \) to some function \( u \) which is in the intersection of these spaces. Moreover using a standard result for weak convergences \( u \) is bounded as

\[ \|u\|_{L^2([0, T], H^1_0)}^2 \leq \liminf_{n \to \infty} \|u_n\|_{L^2([0, T], H^1_0)}^2 \leq \|u_0\|_{L^2}^2, \]  

(3.24)

and likewise

\[ \|u\|_{L^\infty([0, T], L^2)}^2 \leq \liminf_{n \to \infty} \|u_n\|_{L^\infty([0, T], L^2)}^2 \leq \|u_0\|_{L^2}^2. \]  

(3.25)

To verify that \( u \) is a solution to the equation it must be checked that \( u \) satisfies functional equality (3.5) for any test function \( \psi \in C_0^\infty \). Let such a function \( \psi \) now be fixed, because \( u_n \) is a solution to (3.18) it is true that

\[ (u_n, \partial_t \psi)_{L^2} - (\partial_x u_n, \partial_x \psi)_{L^2} = (f(u_n), \partial_x P_n \psi)_{L^2}. \]  

(3.26)

Passing to the limit \( n \to \infty \) in the linear terms is straightforward because weak convergence in \( L^2([0, T], H^1_0) \) is established. For the non-linear term recall \( u_n \) is bounded in

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2.16 this sequence converges weakly in $C_u$ and, as these are both Hilbert spaces, $u$ already established that the sequence $u_n$ is bounded in $L^2([0, T], L^2)$ and moreover $\partial_{xx}u_n$ is clearly bounded in this space. From (3.18) derive that $\partial_t u_n \in L^2([0, T], H^{-1})$ and so $u_n \in H^1([0, T], H^{-1})$. Lemma 2.14 applies and it is shown that $u_n$ is compactly embedded in $L^{6-\delta}([0, T], L^{6-\delta})$ for any $\delta > 0$. Applying the theorem of Krasnoselsky (Lemma 2.15) $f(u_n) \to f(u)$ strongly in $L^2([0, T], L^2)$ and so passing to the limit in the non-linear term is accomplished.

It remains to verify that the initial conditions are satisfied in the sense of (3.6). It is already established that the sequence $u_n$ is bounded in $L^2([0, T], H^1) \cap H^1([0, T], H^{-1})$ and, as these are both Hilbert spaces, $u_n$ converges weakly to some limit $u$. By Lemma 2.16 this sequence converges weakly in $C([0, T], L^2)$. The functional $(\delta(t)\phi, \cdot)_{L^2}$, where $\delta(t)$ is the Dirac delta distribution and $\phi \in L^2$ is arbitrary, is a linear functional on the space $C([0, T], L^2)$. Therefore, by the definition of weak convergence,

$$(u_n(0), \phi)_{L^2} = (\delta(t)\phi, u_n)_{L^2} \to (\delta(t)\phi, u_0)_{L^2} = (u(0), \phi)_{L^2}. \tag{3.27}$$

Because $u_n(0) = P_n u_0$, where $u_0$ is the desired initial data, it is clear that $u_n(0)$ converges strongly to $u_0$ as $n \to \infty$ and therefore from (3.27) it can be concluded that $u(0) = u_0$ as desired. This completes the proof.

3.1.3 Uniqueness

**Theorem 3.4** Let assumptions (3.2) and (3.3) hold. Any solution to (3.1) in the sense of (3.5) is unique with respect to the initial data.

**Proof**

To obtain uniqueness take the difference of two equations (with solutions $u$ and $v$ respectively)

$$\partial_t w - \partial_{xx} w = \partial_x f(u) - \partial_x f(v), \tag{3.28}$$

where $w = u - v$. Take the dot product of this equation with $w$ and integrate the result over $\Omega$, integrate by parts in the non-linear term and use Cauchy-Schwarz’ and Young’s inequalities to obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2 = (\partial_x (f(u) - f(v)), w)_{L^2} \leq \frac{1}{2} \|f(u) - f(v)\|_{L^2}^2 + \frac{1}{2} \|\partial_x w\|_{L^2}^2 \tag{3.29}$$

Now use a simple mean value idea,

$$\text{if } \psi(s) = f(su + (1-s)v) \text{ then } \frac{d}{ds} \psi(s) = f'(su + (1-s)v)(u - v), \tag{3.30}$$

which, when integrated gives

$$\psi(1) - \psi(0) = f(u) - f(v) = \int_0^1 f'(su + (1-s)v)ds(u - v), \tag{3.31}$$
so this leads to

\[ \| f(u) - f(v) \|_{L^2}^2 = \int_{\Omega} \int_{0}^{1} f'(su + (1-s)v) \, ds \, w \cdot \int_{0}^{1} f'(su + (1-s)v) \, ds \, w \, dx \]  

(3.32)

and if the maximum growth rate of \( f' \) is used, based on (3.2), use Hölder’s inequality, note that \( \| u \|_{L^2}^2 \leq C \) by (3.11), and again use the interpolation \( \| u \|_{L^\infty}^2 \leq \| u \|_{L^2} \| u \|_{H^1} \) and come finally to

\[ \int_{\Omega} \int_{0}^{1} f'(su + (1-s)v) \, ds \, w \cdot \int_{0}^{1} f'(su + (1-s)v) \, ds \, w \, dx \]  

(3.33)

\[ \leq C \int_{\Omega} (1 + \| u \|^2 + \| v \|^2) w \cdot w \, dx \]

\[ \leq C \| 1 + u + v \|_{L^2}^2 \cdot \| w \|_{L^\infty}^2 \leq C \| w \|_{L^2}^2 \cdot \| w \|_{H^1} \leq C \| w \|_{L^2}^2 + \frac{1}{2} \| \partial_x w \|_{L^2}^2 \]

and, as above with the regularity estimate, a final estimate remains,

\[ \frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 + \frac{1}{2} \| \partial_x w \|_{L^2}^2 \leq C \| w \|_{L^2}^2 \]  

(3.34)

which leads to the conclusion that

\[ \| u(t) - v(t) \|_{L^2}^2 \leq C \| u(0) - v(0) \|_{L^2}^2 \cdot e^{Ct} \]  

(3.35)

and that therefore the solutions are unique. \( \square \)

### 3.1.4 Regularity

In this section a regularity estimate will be done to show the regularity of the solutions whose existence, in the sense of (3.4), was proven above.

**Theorem 3.5** Let assumptions (3.2) and (3.3) hold and let \( u(0) \in H^1_0 \). A solution to (3.1) in the sense of (3.4) is in \( L^\infty([0,T];H^1_0) \) and satisfies estimate

\[ \| \partial_x u(t) \|_{L^2}^2 \leq e^{-\beta t} \| \partial_x u(0) \|_{L^2}^2 + C_\beta e^{-2\lambda t} \| u(0) \|_{L^2}^2, \]  

(3.36)

where \( \lambda \) is the first eigenvalue of the Dirichlet Laplacian on \([0,\pi]\), \( \beta \) is arbitrary and \( C_\beta \) depends on \( \beta \).

**Proof**

Taking the equation (3.1), multiplying by \( -\partial_x u \) and integrating in space establishes

\[ \frac{1}{2} \frac{d}{dt} \| \partial_x u \|_{L^2}^2 + \| \partial_{xx} u \|_{L^2}^2 = - (\partial_x(f(u)), \partial_{xx} u)_{L^2} \]  

(3.37)

Now all that needs doing is to estimate the right hand side in terms of the left hand side. Use Cauchy-Schwartz’ and Young’s inequalities and then use (3.2) to bound the non-linearity to get
\[-(\partial_x(f(u)), \partial_{xx}u)_{L^2} \leq \frac{1}{2} \|\partial_x(f(u))\|_{L^2}^2 + \frac{1}{2} \|\partial_{xx}u\|_{L^2}^2 \leq C\|(1 + |u|)\partial_x u\|_{L^2}^2 + \frac{1}{2} \|\partial_{xx}u\|_{L^2}^2.\]  \hspace{1cm} (3.38)

Now move the latter term to the left hand side of the estimate and cancel it there. With the remaining term estimate it thus,

\[C\|(1 + |u|)\partial_x u\|_{L^2}^2 \leq C\|(u \cdot \partial_x u)^2\|_{L^1} + C\|\partial_x u\|_{L^2}^2 \leq C\|u\|_{L^2}^2 \|\partial_x u\|_{L^\infty}^2 + C\|\partial_x u\|_{L^2}^2.\]  \hspace{1cm} (3.39)

If \(\beta\|\partial_x u\|_{L^2}^2\) is added to both sides the following is arrived at,

\[\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2}^2 + \|\partial_{xx}u\|_{L^2}^2 + \beta \|\partial_x u\|_{L^2}^2 \leq C\|u\|_{L^2}^2 \|\partial_x u\|_{L^\infty}^2 + (C + \beta) \|\partial_x u\|_{L^2}^2.\]  \hspace{1cm} (3.40)

Now \(H^\alpha\) is embedded in \(W^{1,\infty}\) if \(\alpha > \frac{3}{2}\) and \(H^{\frac{3}{2}}\) can be interpolated between \(H^2\) and \(L^2\) with interpolation constant \(\theta = \frac{1}{4}\) to give,

\[\|\partial_x u\|_{L^\infty}^2 \leq \|u\|_{H^{\frac{3}{2}}}^2 \leq \|u\|_{L^2}^2 \|\partial_{xx}u\|_{L^2}^2.\]  \hspace{1cm} (3.41)

Moreover the second term on the right hand side of (3.40) can be interpolated between \(L^2\) and \(H^2\), which results in,

\[\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2}^2 + \|\partial_{xx}u\|_{L^2}^2 + \beta \|\partial_x u\|_{L^2}^2 \leq C\|u\|_{L^2}^2 \|\partial_{xx}u\|_{L^2}^2 + C_\beta \|u\|_{L^2} \|\partial_{xx}u\|_{L^2},\]  \hspace{1cm} (3.42)

where \(C_\beta\) depends on \(\beta\). Now use Young’s inequality on the first term with exponents 4 and \(\frac{4}{3}\) and on the second term with exponents both 2 to get,

\[\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2}^2 + \|\partial_{xx}u\|_{L^2}^2 + \beta \|\partial_x u\|_{L^2}^2 \leq C\|u\|_{L^2}^4 + \frac{1}{4} \|\partial_{xx}u\|_{L^2}^2 + C_\beta \|u\|_{L^2}^2.\]  \hspace{1cm} (3.43)

Cancelling the \(\partial_{xx}u\) terms and integrating in time over the domain \([t, t + 1]\) to get

\[\|\partial_x u(t + 1)\|_{L^2}^2 \leq e^{-\beta t} \|\partial_x u(t)\|_{L^2}^2 + C_\beta \int_t^{t+1} \|u\|_{L^2}^2 + \|u\|_{L^2}^2 \ dt.\]  \hspace{1cm} (3.44)

which can be combined with (3.13) to give the final result,

\[\|\partial_x u(t)\|_{L^2}^2 \leq e^{-\beta t} \|\partial_x u(0)\|_{L^2}^2 + C_\beta e^{-2L} \|u(0)\|_{L^2}^2.\]  \hspace{1cm} (3.45)

Therefore this gives, when combined with Theorem 3.2, \(u \in L^\infty([0, T], H^1)\) so long as the initial conditions are smooth enough.

The above theorems show that under the assumptions above the solutions to coupled Burgers’ equations in a bounded domain exist, are unique and become regular in a very short time interval and that they decay exponentially in time. This is very similar to the heat equation in a bounded domain.
3.2 Coupled Burgers’ equations in an unbounded domain

3.2.1 The first energy estimate

In this section, following the same program as that for a bounded domain, the first energy estimate for the class of coupled Burgers’ equations will be derived,

\[ \partial_t u = \partial_{xx} u + \partial_x (f(u)), \quad x \in \Omega = \mathbb{R}, \quad u|_{t=0} = u_0 \]  

(3.46)

for the \(N\)-component vector-valued function \(u(x,t)\), with the following conditions on the function \(f : \mathbb{R}^N \to \mathbb{R}^N\). There is a scalar-valued function \(F : \mathbb{R}^N \to \mathbb{R}\) such that

\[ \begin{aligned}
1. & \quad f(u) = \nabla F(u), \\
2. & \quad f'(u) \leq C(1 + |u|), \text{ for all } u \in \mathbb{R}^N \\
3. & \quad F \in C^2(\mathbb{R}^N, \mathbb{R})
\end{aligned} \]  

(3.47)

for some constant \(C\). Moreover it is assumed

\[ u_0 \in L^2_b \]  

(3.48)

where the space \(L^2_b\) is defined above in Definition 2.2.

**Definition 3.6** A weak solution of (3.46) is a function, \(u\), such that

\[ u \in L^\infty([0,T],L^2_b) \quad \text{and} \quad \partial_x u \in L^2_b([0,T] \times \mathbb{R}), \]  

(3.49)

and \(u\) satisfies (3.46) as a distribution. That is, \(u\) satisfies

\[ (u, \partial_t \psi)_{L^2} = (\partial_x u, \partial_x \psi)_{L^2} + (f(u), \partial_x \psi)_{L^2}, \]  

(3.50)

for any vector \(\psi \in C^\infty_0([0,T] \times \mathbb{R})\), that is infinitely differentiable functions in space and time with compact support. Moreover

\[ u \in C([0,T],L^2_{\text{loc}}) \text{ and } u(\cdot,0) = u_0. \]  

(3.51)

**Theorem 3.7** Let (3.47) and (3.48) hold. For any \(t > 0\) any solution to equation (3.46) satisfies the following estimates,

\[ \|u(t)\|_{L^2_b} \leq CT(\|u_0\|_{L^2_b} + 1), \]  

(3.52)

for some constant \(C\) and

\[ \int_0^T \|\partial_x u\|_{L^2_b}^2 \, dt \leq CT^2(\|u_0\|_{L^2_b} + 1), \]  

(3.53)

where \(\phi\) is defined in (2.3).
Remark. Because of the type of non-linearity and the assumptions which are made it can be shown that any solution grows at most linearly, which bounds the solution against finite time blow-up and is enough to show global existence, but it cannot be shown using this technique that the solution is bounded as $t \to \infty$.

Proof

To show that the solution is bounded in the uniformly local space $L^2_b$ it is enough to show it is bounded in an appropriate weighted space $L^2_\phi$ where the supremum of the weighted norm is taken with the weight centred at all points on the line, see Proposition 2.5.

Now make use of a scalar-valued weight function $\phi(x)$, as defined above in (2.3) with $\gamma = 2$, with the properties,

$$|\partial_x \phi| \leq \epsilon \phi^2 \leq \epsilon \phi \quad \text{and} \quad \int_\mathbb{R} \phi \, dx = \frac{C}{\epsilon} < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}} \phi(x) \leq 1. \quad (3.54)$$

Take the dot product of $\phi u$ with equation (3.46) and integrate in space,

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2_\phi}^2 + \|\partial_x u\|_{L^2_\phi}^2 + (\partial_x u, u \partial_x \phi)_{L^2} = (\partial_x (f(u)), \phi u)_{L^2}, \quad (3.55)$$

To estimate the right hand side note that

$$\partial_x (F(u) \phi) = f(u) \phi \partial_x u + F(u) \partial_x \phi, \quad (3.56)$$

and the fact that $F(u(x)) \phi(x) \to 0$ as $x \to \pm \infty$ (in the sense of distributions), to simplify the estimate (3.55),

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2_\phi}^2 + \|\partial_x u\|_{L^2_\phi}^2 + (\partial_x u, u \partial_x \phi)_{L^2} = (F(u) - uf(u), \partial_x \phi)_{L^2} := \mathcal{R}. \quad (3.57)$$

Now using that $F(u)$ and $f(u) \cdot u$ are at most cubic (from 3.47) it is possible to bound $\mathcal{R}$,

$$\mathcal{R} \leq C(|u|^3 + 1, |\partial_x \phi|)_{L^2} \quad (3.58)$$

For the third term on the left-hand side of (3.57), use Hölder’s inequality and (3.54)

$$|(\partial_x u, u \partial_x \phi)_{L^2}| \leq \|\partial_x u\|_{L^2_\phi} \epsilon \|u\|_{L^2_\phi} \leq \|\partial_x u\|_{L^2_\phi}^2 + \frac{1}{2} \epsilon^2 \|u\|_{L^2_\phi}^2. \quad (3.59)$$

The estimate (3.57) is now

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2_\phi}^2 + \frac{1}{2} \|\partial_x u\|_{L^2_\phi}^2 - \frac{1}{2} \epsilon^2 \|u\|_{L^2_\phi}^2 \leq \mathcal{R}. \quad (3.60)$$

\(^1\)as $|f'(u)| \leq C(1 + |u|)$ implies $|f(u)| \leq C(1 + |u|^2)$ and $|F(u)| \leq C(1 + |u|^3)$ using Young’s inequality repeatedly.
To estimate the non-linearity define \( v = \phi^\frac{1}{2} u \), assume \( \epsilon < 1 \) and argue

\[
\int \partial_x \phi ||u||^3 \, dx \leq \int \phi^\frac{3}{2} |u|^3 \, dx = \int \phi^\frac{3}{2} u \, dx = \int \phi^\frac{3}{2} \, dx = ||v||^3_{L^3}. \tag{3.61}
\]

Since \( H^\frac{5}{2} \) is embedded in \( L^3 \) in one dimension it is possible to obtain \( ||v||_{L^3} \leq ||v||_{H^\frac{1}{6}} \) and then interpolate between \( H^1 \) and \( L^2 \) with exponent \( \frac{1}{6} \) to get

\[
||v||_{H^\frac{1}{6}}^3 \leq C(||v||_{H^1}^\frac{5}{6}, ||v||_{L^2}^\frac{5}{6})^3 \tag{3.62}
\]

then, using Lemma 2.17, to get

\[
\int \partial_x \phi ||u||^3 \, dx \leq C(||u||_{H^\frac{5}{6}}^\frac{5}{2}, ||u||_{L^2}^\frac{5}{2}). \tag{3.63}
\]

Now break up the \( H^\frac{5}{6} \) norm into two pieces and use Young's inequality, (3.2) and (3.54) to find that the right hand side of (3.60) becomes

\[
\mathbb{R} \leq C(||u||^3 + 1, ||\partial_x \phi||_{L^2}) \tag{3.64}
\]

\[
\leq C \epsilon \left(||u||_{H^1}^\frac{1}{2}, ||u||_{L^2}^\frac{5}{2}\right) + C \epsilon \int \phi(x) \, dx
\]

\[
\leq C \epsilon \left(||u||_{L^2}^\frac{5}{2} + ||\partial_x u||_{L^2}^\frac{1}{2}\right) ||u||_{L^2}^\frac{5}{2} + C
\]

\[
\leq C \epsilon ||u||_{L^2}^3 + \frac{1}{4} ||\partial_x u||_{L^2}^2 + \frac{3}{4} \epsilon^2 ||u||_{L^3}^\frac{10}{3} + C.
\]

Cancelling the \( ||\partial_x u||_{L^2} \) term on the right with the one on the left leads to the refinement of (3.60)

\[
\frac{d}{dt} ||u||_{L^2}^2 + \frac{1}{4} ||\partial_x u||_{L^2}^2 \leq \epsilon^2 ||u||_{L^2}^2 + C \epsilon ||u||_{L^2}^3 + \frac{3}{4} \epsilon^2 ||u||_{L^3}^\frac{10}{3} + C, \tag{3.65}
\]

noting the use of Young's inequality, with exponents \( \frac{3}{2} \) and 3, on \( \epsilon^2 ||u||_{L^2}^2 \cdot 1 \leq \frac{2}{3} \epsilon^3 ||u||_{L^2}^3 + \frac{1}{3} \) to reduce this term to some small cubic component and a constant and again with exponents \( \frac{10}{9} \) and 10 to obtain \( C \epsilon^{-\frac{1}{20}} \epsilon^{\frac{21}{20}} ||u||_{L^2}^3 \leq C \epsilon^\frac{7}{6} ||u||_{L^3}^\frac{10}{3} + C \epsilon^{-\frac{1}{2}}. \) This leads to a final estimate

\[
\frac{d}{dt} ||u||_{L^2}^2 + \frac{1}{4} ||\partial_x u||_{L^2}^2 \leq C \epsilon^\frac{7}{6} ||u||_{L^3}^\frac{10}{3} + C \epsilon^{-\frac{1}{2}} + C. \tag{3.66}
\]

The final steps are to show that \( ||u||_{L^2} \) is bounded on any finite time interval, \( 0 \leq t \leq T \); these will be to first turn to the initial conditions, and then recast the problem as an ODE and then use two changes of variables to show that \( ||u||_{L^2} \) is bounded on any finite time interval.

With the initial condition \( u_0 \in L^2_b \) it follows that \( ||u_0||_{L^2_b}^2 = \sup_x \int_{x}^{x+1} |u_0|^2 \, dx = C. \)
Estimating the norm in $L^2_\phi$ to get
\[
\|u_0\|_{L^2_\phi} = \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} \phi(y) |u_0|^2 \, dy \leq \sum_{n \in \mathbb{Z}} \sup_{y \in [n,n+1]} \phi(y) \|u_0\|_{L^2_{[n,n+1]}}^2 
\]
\[
\leq \|u_0\|_{L^2_\phi}^2 \sum_{n \in \mathbb{Z}} \sup_{y \in [n,n+1]} \phi(y). 
\]

The weight function satisfies $\phi(x + y) \leq C e^x \phi(y)$ by definition so this leads to
\[
\|u_0\|_{L^2_\phi} \leq \|u_0\|_{L^2_\phi}^2 \sum_{n \in \mathbb{Z}} \phi(n) \leq C \|u_0\|_{L^2_\phi}^2 \int_{\mathbb{R}} \phi(y) \, dy,
\]
which by condition (3.54) ($\int_{\mathbb{R}} \phi(y) \, dy = \frac{C}{\epsilon} < \infty$), yields
\[
\|u_0\|_{L^2_\phi}^2 \leq \frac{C}{\epsilon} \|u_0\|_{L^2_\phi}^2 \leq \frac{I}{\epsilon}
\]
for $I = C \|u_0\|_{L^2_\phi}^2$. Noting that the $\|\partial_x u\|$ term can be neglected because it is positive on the left of (3.66) it is now possible to recast the problem as an ODE, that is
\[
Y' = \epsilon^{\frac{3}{2}} Y^{\frac{5}{2}} + C \epsilon^{-\frac{1}{2}} + C \quad \text{with} \quad Y(0) = \frac{I}{\epsilon},
\]
and
\[
\|u(t)\|_{L^2_\phi}^2 \leq Y(t) \quad \text{for} \quad t \in [0, T].
\]

Now rescale space such that $Z = \epsilon Y$ to get initial data $Z(0) = I$ and the equation
\[
Z' \leq \epsilon^{\frac{1}{2}} Z^{\frac{5}{2}} + C \epsilon^{\frac{1}{4}} + C \epsilon.
\]

Now if time is rescaled such that $\tau = \epsilon^{\frac{1}{2}} t$, $Z(\tau)|_{\tau=0} = I$ is obtained and
\[
Z_\tau \leq Z^{\frac{5}{2}} + C + C \epsilon^{\frac{1}{4}}.
\]

Now choosing $\epsilon$ to be sufficiently small it is possible to conclude that $Z(\tau) \leq 2C(I + 1)$ if $\tau \leq \beta = \beta(I)$.

Descaling time leads to $Z(t) \leq 2C(I + 1)$ if $t \leq \beta \epsilon^{-\frac{1}{2}}$ so take $\epsilon \cong \frac{\beta^2}{T^2}$ (as $\epsilon$ has not been fixed for the entire estimate) which leads to $Z(t) \leq C(I + 1)$ if $t \in [0, T]$. So $Y(t) \leq CT^2(I + 1)$ in this interval and this means $\|u(t)\|_{L^2_{\phi_*}} \leq CT^2(I + 1)$ for $t \in [0, T]$.

Now if $\phi(x)$ satisfies (3.54) then $\phi(x - x_0)$ satisfies (3.54) for any arbitrary $x_0 \in \mathbb{R}$. So if a supremum over $\mathbb{R}$ is taken, (recalling that $I = C \|u_0\|_{L^2_\phi}$),
\[
\|u(t)\|_{L^2_\phi} \leq \sup_{x_0 \in \mathbb{R}} \|u\|_{L^2_{\phi_*(x-x_0)}} \leq CT(\|u_0\|_{L^2_\phi} + 1),
\]
is obtained which shows the solution is bounded on all finite time intervals and that the asymptotic growth rate is at most linear in $T$.  

Moving on to establish the second part of the theorem, integrating (3.66) in time,
\[ \int_0^T \| \partial_x u \|^2_{L^2_\phi} dt \leq \int_0^T C \epsilon^{\frac{7}{6}} \| u \|_{L^2_\phi}^{\frac{10}{3}} + \epsilon^{-\frac{1}{2}} + C dt. \] (3.75)
Equation (3.74) fixes the growth rate of \( \| u \|_{L^2_\phi} \) and \( \epsilon \) is fixed just before this, \( \epsilon \approx \frac{\beta}{T^2} \), and so these are combined to obtain
\[ \int_0^T \| \partial_x u \|^2_{L^2_\phi} dt \leq C \int_0^T (\| u_0 \|_{L^2_\phi} + 1) dt \leq C T^2 (\| u_0 \|_{L^2_\phi} + 1). \] (3.76)

3.2.2 Proof of Existence

It was proven above, in Theorem 3.3, that for any bounded, one-dimensional, domain, a solution to (3.46) exists. Using this and estimate (3.74) existence of solutions in an unbounded domain can now be proven.

Theorem 3.8 Equation (3.46) under conditions (3.47) and (3.48) possesses a solution in the sense of definition 3.6.

Proof
Let \( \Omega_N = [-N, N] \subset \mathbb{R} \), for \( N = 1, 2... \) and let \( \nu_N(x) \in C_0^\infty(\mathbb{R}) \) be a sequence of functions such that \( \nu_N(x) = 1 \) if \( x \in [-N+1, N-1] \) and \( \nu_N(x) = 0 \) if \( x \notin [-N, N] \) and \( \| \nu_N \|_{C^2} \leq C \).

Now let \( u_N \) be the solution to,
\[ \partial_t u_N = \partial_{xx} u_N + \partial_x (f(u_N)), \quad u_N|_{t=0} = \nu_N u_0, \quad u_N|_{\partial \Omega_N} = 0. \] (3.77)

Theorem 3.3 shows that such a solution, \( u_N \), exists and Theorem 3.4 proves it is unique and the solution is in the space
\[ L^\infty([0, T], L^2[-N, N]) \cap L^2([0, T], H^1_0[-N, N]) \] (3.78)
with
\[ \partial_x u \in L^2([0, T], L^2[-N, N]). \] (3.79)

Estimates (3.52) and (3.53) are valid for each \( u_N \) as the steps of the proof can be repeated word by word for all \( N \) to give that on any domain \([K, K+1]\) for \( |K| < N-1 \),
\[ \| u_N \|_{L^\infty([0, T], L^2[K, K+1])} \leq C(T, \| u_0 \|_{L^2_\phi}), \quad \| u_N \|_{L^2([0, T], H^1_0[K, K+1])} \leq C(T, \| u_0 \|_{L^2_\phi}), \] (3.80)
where \( C \) is independent \( N \). The desired solution \( u \) will be constructed as the limit of the sequence \( u_N \).

From (3.80) it is clear the sequence \( u_N \) is uniformly bounded in \( L^2([0, T], H^1[-M, M]) \) for every fixed \( M \in \mathbb{N} \). Then, since the space \( L^2([0, T], H^1[-M, M]) \) is reflexive, by
the Banach-Alaoglu theorem, the sequence $u_N$ is weakly sequentially pre-compact in $L^2([0, T], H^1[-M, M])$ for every $M \in \mathbb{N}$. Moreover, again due to (3.80), the sequence is also bounded in $L^\infty([0, T], L^2[-M, M])$, which is the dual space to the separable Banach space $L^1([0, T], L^2[-M, M])$. Again by the Banach-Alaoglu theorem, the sequence $u_N$ is weak-star sequentially pre-compact in $L^\infty([0, T], L^2[-M, M])$.

Therefore Cantor’s diagonalisation procedure can be used to show there exists a subsequence of $u_N$ (which will for simplicity be labelled $u_N$) which is convergent weakly in $L^2([0, T], H^1[-M, M])$ and weak-star in $L^\infty([0, T], L^2[-M, M])$ for every $M \in \mathbb{N}$ to some function,

$$u \in L^\infty([0, T], L^2_{\text{loc}}(\mathbb{R})) \cap L^2([0, T], H^1_{\text{loc}}(\mathbb{R})).$$

(3.81)

Now it must be checked that $u$ belongs to the uniformly local spaces stated in definition 3.6. Using a standard property of weak star convergence and (3.80) it is clear that,

$$\|u\|_{L^\infty([0,T],L^2([K,K+1])))} \leq \lim\inf_{N \to \infty} \|u_N\|_{L^\infty([0,T],L^2([K,K+1)))} \leq C(T, \|u_0\|_{L^2_b}),$$

(3.82)

for every fixed $K \in \mathbb{R}$. Taking the supremum over $K \in \mathbb{R}$, it can be seen that $u \in L^\infty([0, T], L^2_b(\mathbb{R}))$ and that,

$$\|u\|_{L^\infty([0,T],L^2_b(\mathbb{R})))} \leq C(T, \|u_0\|_{L^2_b}).$$

(3.83)

Moreover the fact that $\partial_x u \in L^2_b([0, T] \times \mathbb{R})$ can be established analogously. Thus the regularity requirements of definition 3.6 are satisfied and it remains to check this constructed $u$ satisfies the equation in the sense of distributions.

To establish that $u$ is a distributional solution it needs be verified that the functional equality in definition 3.6 holds for any smooth test function $\psi \in C^\infty_0([0, T] \times \mathbb{R})$ with compact support. Let now an arbitrary function $\psi$ be fixed and let $M$ be fixed such that the support of $\psi$ is in the segment $[-M, M]$. Then, since $u_N$ is a distributional solution of the auxiliary problem (3.77), for sufficiently large $N$, it is true that,

$$(u_N, \partial_t \psi)_{L^2} = (\partial_x u_N, \partial_x \psi)_{L^2} + (f(u_N), \partial_x \psi)_{L^2}.$$

(3.84)

It only remains to pass to the limit $N \to \infty$ in this equality. Passing to the limit in the linear terms is straightforward since the weak convergence $u_N \to u$ in the space $L^2([0, T], H^1[-M, M])$ is established, so only the non-linear term requires work. To this end recall that the sequence $u_N$ is uniformly bounded in the space

$$L^\infty([0, T], L^2[-M, M]) \cap L^2([0, T], H^1[-M, M]).$$

(3.85)

Lemma 2.13 gives that $u_N$ is in $L^4([0, T], L^4[-M, M])$ and so, using (3.2),

$$\partial_x f(u_N) \in L^2([0, T], H^{-1}[-M, M]).$$

(3.86)

Also $\partial_{xx} u_N$ is also in this space and so, from (3.77), $\partial_t u_N \in L^2([0, T], H^{-1}[-M, M])$ so $u_N \in H^1([0, T], H^{-1}[-M, M])$. This space is compactly embedded in the Banach space.
\( L^{6-\delta}([0, T] \times [-M, M]) \) by Lemma 2.14 and therefore it can be concluded that \( u_N \to u \) strongly in \( L^{6-\delta}([0, T] \times [-M, M]) \) for any \( \delta > 0 \). Then by the theorem of Krasnoselsky (Lemma 2.15), \( f(u_N) \to f(u) \) strongly in \( L^2([0, T], L^2[-M, M]) \) and so passing to the limit in the non-linear term is straightforward. Thus it has been verified that the functional equality in definition 3.6 holds for any test function \( \psi \) and therefore \( u \) is a distributional solution to the main equation.

It remains to verify that the constructed solution \( u \) satisfies the initial conditions in the sense of definition 3.6. Fix a cutoff function \( \kappa \in C_0^\infty \) such that \( \kappa = 1 \) on \([L_1 + 1, L_2 - 1]\) and \( \kappa = 0 \) outside \([L_1, L_2]\) with \( L_2 > L_1 + 2 \). It is clear that \( \kappa u_N \in L^2([0, T], H_0^1[L_1, L_2]) \) by (3.80) and, by the same logic as the paragraph above, \( \kappa \partial_t u_N \in L^2([0, T], H^{-1}) \). Now \( \kappa u_N \) satisfies the conditions of Lemma 2.16 and so \( u_N \in C([0, T], L^2[L_1, L_2]) \). Passing to the limit \( N \to \infty \) and varying \( L \) and \( M \) demonstrates that, analogously with the case in a bounded domain, \( u \in C([0, T], L^2_{\text{loc}}(\mathbb{R})) \) and that \( u(x, 0) = u_N(x, 0) = u_0 \) on any \([L_1, L_2]\).

Therefore \( u \) is a solution to the equation in the sense of definition 3.6.

\[ \square \]

### 3.2.3 Uniqueness

**Theorem 3.9** Let (3.47) and (3.48) hold, then the weak solutions to (3.46) in the sense of definition 3.6 are unique.

**Proof**

To obtain uniqueness start by taking the difference of two equations (with solutions \( u \) and \( v \) respectively)

\[
\partial_t w - \partial_{xx} w = \partial_x f(u) - \partial_x f(v),
\]

where \( w = u - v \). Take the dot product of this equation with \( \phi w \), where \( \phi \) is defined in (2.3), and integrate the result over \( \mathbb{R} \) and use \( \partial_x \phi \leq \epsilon \phi \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \| w \|_{L_\phi^2}^2 + \frac{1}{2} \| \partial_x w \|_{L_\phi^2}^2 - \epsilon^2 \| w \|_{L_\phi^2}^2 = -(f(u) - f(v), \partial_x (\phi w))_{L^2} \leq (3.87)
\]

\[
\leq -(f(u) - f(v), \phi \partial_x w + \epsilon \phi w)_{L^2} \leq \frac{1}{2} \| f(u) - f(v) \|_{L_\phi^2}^2 + \frac{1}{4} \| \partial_x w \|_{L_\phi^2}^2 + \epsilon^2 \| w \|_{L_\phi^2}^2.
\]

All of these terms are reasonably benign, apart from \( \| f(u) - f(v) \|_{L_\phi^2}^2 \) which will be estimated using a mean value type theorem (see (3.31)), through which

\[
\| f(u) - f(v) \|_{L_\phi^2}^2 \leq \int_0^1 \int f'(su + (1 - s)v) ds \cdot \int_0^1 f'(su + (1 - s)v) ds \cdot w \phi \ dx, \tag{3.89}
\]

can be obtained. Now use assumption (3.47), \( f'(u) \leq C(1 + |u|) \), and take the maximum value of the inner integral to arrive at

\[
\| f(u) - f(v) \|_{L_\phi^2}^2 \leq C \int_\mathbb{R} (1 + |u|^2 + |v|^2) w \cdot w \phi \ dx, \tag{3.90}
\]

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for which the identity from Proposition 2.11 can be used,

$$
\int_{\mathbb{R}} |u| \phi \, dx \leq C \int_{\mathbb{R}} |\phi(s)| |u|_{L^1([s,s+1])} \, ds,
$$

(3.91)
to obtain

$$
\|f(u) - f(v)\|_{L^2_\phi}^2 \leq C \int_{\mathbb{R}} \|(1 + |u|^2 + |v|^2)w \cdot w\|_{L^1([s,s+1])} |\phi(s)| \, ds \leq C \int_{\mathbb{R}} \|(1 + |u|^2 + |v|^2)\|_{L^1([s,s+1])} |w|_{L^2([s,s+1])}^2 |\phi(s)| \, ds,
$$

(3.92)
and as \(\|u\|_{L^2([s,s+1])}^2 \leq \|u\|_{L^2_\phi}^2 \leq C\) from (3.52) the following is obtained, on which it is possible to use interpolation \(\|w\|_{L^2([s,s+1])} \leq \|w\|_{L^2([0,T])}\|w\|_{H^1([s,s+1])}\) to get

$$
\|f(u) - f(v)\|_{L^2_\phi}^2 \leq C \int_{\mathbb{R}} |\phi(s)| |w|_{L^2([s,s+1])}^2 \, ds \leq C \int_{\mathbb{R}} |\phi(s)| |w|_{L^2([s,s+1])} \|w\|_{H^1([s,s+1])} \, ds \leq C \left( \int_{\mathbb{R}} |\phi(s)| |w|_{L^2([s,s+1])}^2 \, ds \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |\phi(s)| |w|_{H^1([s,s+1])}^2 \, ds \right)^{\frac{1}{2}} \leq C \|w\|_{L^2_\phi} \|w\|_{H^1} \leq C \|w\|_{L^2_\phi}^2 + \frac{1}{4} \|\partial_x w\|_{L^2_\phi}^2,
$$

(3.93)
which, cancelling the \(\|\partial_x w\|_{L^2_\phi}^2\) terms with the one on the left of (3.88) and dropping that term as it is strictly positive, leads to a final estimate

$$
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2_\phi}^2 \leq (C + 2\epsilon^2) \|w\|_{L^2_\phi}^2.
$$

(3.94)
The final step is to centre the weight function at all points \(x_0\) along the line and take the supremum over \(x_0\), giving uniqueness in uniformly local spaces. \(\square\)

### 3.2.4 Regularity

Having proven existence of solutions in \(L^\infty([0,T],L^2_b)\) the following regularity result will now be proven. First assume

$$
u_0 \in H^1_b(\mathbb{R}).
$$

(3.95)

**Theorem 3.10 (Regularity)** Let assumptions (3.47), (3.48) and (3.95) hold, and let \(u\) be a weak solution of (3.46) such that

$$
\|u\|_{L^\infty([0,T],L^2_b)} \leq K,
$$

(3.96)
then \(u \in L^\infty([0,T],H^1_b)\), \(\partial_{xx} u \in L^2_b([0,T] \times \mathbb{R})\) and

$$
\|u\|_{L^\infty([0,T],H^1_b)} + \|\partial_{xx} u\|_{L^2_b([0,T] \times \mathbb{R})} \leq C(\|u_0\|_{H^1_b} + K^5 + 1),
$$

(3.97)
where \(C\) is a constant independent of \(u, T\) and \(u_0\).
Proof  
Take the dot product of equation (3.46) with \(-\partial_x(\phi \partial_x u)\) and integrating in space where \(\phi\) is a weight function satisfying (3.54). This gives
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2_\phi}^2 + \|\partial_{xx} u\|_{L^2_\phi}^2 + (\partial_x(\phi) \partial_x u, \partial_{xx} u)_{L^2} = -(\partial_x f(u), \partial_x(\phi \partial_x u))_{L^2}.
\] (3.98)

Now using the assumption (3.54)
\[
|\langle \partial_x(\phi) \partial_x u, \partial_{xx} u \rangle_{L^2}| \leq \frac{1}{4} \|\partial_x u\|_{L^2_\phi}^2 + \frac{\epsilon^2}{4} \|\partial_x u\|_{L^2_\phi}^2, \tag{3.99}
\]
is obtained and again using (3.54) to obtain on the right
\[
|\langle \partial_x f(u), \partial_x(\phi \partial_x u) \rangle_{L^2}| \leq \frac{3}{2} \|\partial_x f(u)\|_{L^2_\phi}^2 + \frac{\epsilon}{8} \|\partial_x u\|_{L^2_\phi}^2, \tag{3.100}
\]
The estimate is now in a good form apart from the \(\partial_x f(u)\) term. Using the assumptions (3.47) on \(f'\) and repeating exactly steps (3.90) to (3.93) with \(w\) replaced by \(\partial_x u\) it can be argued that,
\[
\frac{3}{2} \|\partial_x f(u)\|_{L^2_\phi}^2 \leq C \|1 + |u|\cdot \partial_x u\|_{L^2_\phi}^2 \tag{3.101}
\]
\[
\leq C \|u\|_{L^2_\phi}^2 \|\partial_x u\|_{L^2_\phi} \|\partial_x u\|_{H^1_\phi} \leq C (K^2 + 1)^2 \|\partial_x u\|_{L^2_\phi}^2 + \frac{1}{4} \|\partial_{xx} u\|_{L^2_\phi}^2.
\]
Which leads to
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2_\phi}^2 + \frac{1}{4} \|\partial_{xx} u\|_{L^2_\phi}^2 \leq C (K^2 + 1)^2 \|\partial_x u\|_{L^2_\phi}^2. \tag{3.102}
\]
Finally using the weighted interpolation inequality from Lemma 2.18, and Young’s inequality,
\[
\|\partial_x u\|_{L^2_\phi}^2 \leq C \|u\|_{L^2_\phi}^2 + C \|u\|_{L^2_\phi} \|\partial_{xx} u\|_{L^2_\phi}, \tag{3.103}
\]
to obtain
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u\|_{L^2_\phi}^2 + \frac{1}{8} \|\partial_{xx} u\|_{L^2_\phi}^2 \leq C (K^4 + 1)^2 \|u\|_{L^2_\phi}^2 \leq C (K^5 + 1)^2. \tag{3.104}
\]
Grönwall’s inequality can now be used to arrive at the desired estimate,
\[
\|\partial_x u(t)\|_{L^2_\phi}^2 + \int_0^T \|\partial_{xx} u(s)\|_{L^2_\phi}^2 \, ds \leq C (\|u_0\|_{H^1_\phi} + K^5 + 1)^2. \tag{3.105}
\]
This estimate remains true with respect to all shifts, \(s \in \mathbb{R}\), which shift the weight function, \(\phi_s(x) = \phi(x - s)\) and so if a supremum is taken of both sides the desired result is obtained in uniformly local spaces.

\[ \square \]
4 Cahn-Hilliard Equation

4.1 The key estimate and global existence

In this section the first energy estimate for the solutions of the Cahn-Hilliard equation

\[ \partial_t u = \Delta x \mu, \quad \mu := -\Delta x u + f(u) + g(x), \quad u\big|_{t=0} = u_0 \] (4.1)

is obtained. This allows the existence of global (in time) solutions in uniformly local spaces to be proven with uniqueness and regularity studied afterwards.

Here \( u = u(t,x) \) and \( \mu = \mu(t,x) \) are an unknown solution and chemical potential respectively, \( \Delta x \) is a Laplacian with respect to \( x \), \( f \) is the given non-linearity and \( g \) is the external forcing term.

Assume that the non-linearity \( f(u) = f_0(u) + \psi(u) \) satisfies these conditions,

\[
\begin{align*}
1. & \quad f_0 \in C^1, \quad f_0'(0) \geq 1, \quad f_0(0) = 0, \\
2. & \quad |\psi(u)| + |\psi'(u)| \leq C, \\
3. & \quad |f(u)| \leq \alpha |F(u)| + C, \quad F(u) := \int_0^u f(v) \, dv,
\end{align*}
\]

where \( \alpha > 0 \). These assumptions are satisfied by all polynomials of odd order, even order are disallowed because \( f_0' \geq 1 \), and even some potentials of exponential growth rate are allowed. Because of the first and second assumptions of (4.2), it is true that

\[ F(u) \geq \beta |u|^2 - C, \quad u \in \mathbb{R} \] (4.3)

for some positive constants \( \beta \) and \( C \). This is true because if \( f_0' \geq 1 \) then \( f \geq u + C \) and so \( F(u) \geq \beta u^2 + C \) and an important corollary of this is that if \( \|u\|_{L^2} \leq \|F(u)\|_{L^1} \).

The following assumption is also made on the external forcing term,

\[ g \in L_b^6(\mathbb{R}^3). \] (4.4)

Moreover assume that,

\[ u\big|_{t=0} = u_0 \in \Phi_b := \{ u \in W^{1,2}_b(\mathbb{R}^3), \quad F(u) \in L^1_b(\mathbb{R}^3) \}. \] (4.5)

A function \( u \) is a solution of (4.1) on the time interval \( t \in [0,T] \) if

\[
\begin{align*}
1. & \quad u(t) \in \Phi_b, \quad t \in [0,T]; \\
2. & \quad u(t) \in C([0,T], L^2_{b, loc}(\mathbb{R}^3)); \\
3. & \quad u \in L^2_{b}([0,T], W^{1,2}_b(\mathbb{R}^3))
\end{align*}
\]

and equation (4.1) is satisfied in the sense of distributions. Note that the third assumption of (4.6), together with Lemmas 4.2 and 4.3 (see below) for the semi-linear equation

\[ -\Delta x u + f(u) = \mu - g \]
in the uniformly local space \( L^6_b(\mathbb{R}^3) \), imply that
\[ u \in L^2_b([0, T], W^{2,6}_b(\mathbb{R}^3)), \quad f(u) \in L^2_b([0, T], L^6_b(\mathbb{R}^3)). \]

This now leads to the main result of this section.

**Theorem 4.1** Let assumptions (4.2) and (4.5) hold. Then, for every \( T > 0 \), equation (4.1) possesses at least one solution in the sense of (4.6) which satisfies the following estimate:

\[
\|u(t)\|_{W^{1,2}_b}^2 + \|F(u(t))\|_{L^1_b}^2 + \|\nabla_x \mu\|_{L^2_b([0,T] \times \mathbb{R}^3)}^2 \leq C(1 + t^4) \left( 1 + \|g\|_{L^6_b}^2 + \|u_0\|_{W^{1,2}_b}^2 + \|F(u_0)\|_{L^1_b} \right)^{5/2},
\]

where the constant \( C \) is independent of \( u, g \) and \( t \).

**Proof**

Below only the formal derivation of the key a priori estimate (4.7) will be given. The existence of solutions can be then deduced using the approximation of the infinite energy data \((u_0, g)\) by the finite energy functions \((u_n^0, g_n)\) (for which the existence and regularity of a solution is immediate and the derivation of (4.7) is justified) and passing to the limit in a local topology. Since these arguments are standard, see e.g., [29, 27, 80], they are omitted and this thesis will concentrate on the derivation of the key estimate.

Weight functions with polynomial growth rate will be utilised,

\[
\phi(x) := \frac{1}{(1 + |x|^2)^{5/2}}, \quad \phi_{\varepsilon,x_0}(x) := \phi(\varepsilon(x - x_0)), \quad \varepsilon > 0, \quad x_0 \in \mathbb{R}^3.
\]

which corresponds to (2.3) with \( \gamma = 5 \) and gives \( \phi_\varepsilon \in L^1(\mathbb{R}^3) \). These weights satisfy estimate (2.7) uniformly also with respect to \( x_0 \in \mathbb{R}^3 \). Now multiply the first equation of (4.1) by \( \phi_\varepsilon \mu = \phi_{\varepsilon,x_0} \mu \) and integrate over the domain, where \( x_0 \in \mathbb{R}^3 \) is parameter. This gives,

\[
(\partial_t u, \phi_\varepsilon \mu) = (\Delta_x \mu, \phi_\varepsilon \mu).
\]

The term on the right hand side can be integrated by parts twice to give,

\[
(\Delta_x \mu, \phi_\varepsilon \mu) = - (\phi_\varepsilon, |\nabla_x \mu|^2) + \frac{1}{2} (\Delta_x \phi_\varepsilon, |\mu|^2).
\]

The term on the left hand side can be expanded and the first term can be integrated by parts to give,

\[
(\partial_t u \phi_\varepsilon, -\Delta_x u + f(u) + g) = \frac{d}{dt} \left( (F(u), \phi_\varepsilon) + \frac{1}{2} (|\nabla_x u|^2, \phi_\varepsilon) + (g, \phi_\varepsilon u) \right) + (\partial_t u, \nabla_x \phi_\varepsilon \nabla_x u).
\]
This last term can be converted using (4.1) and integrated by parts as follows,
\[ (\partial_t u, \nabla_x \phi \nabla_x u) = (\Delta_x \mu, \nabla_x \phi \nabla_x u) = -(\nabla_x \mu, \nabla_x (\nabla_x \phi \nabla_x u)) \] (4.12)
which yields in total,
\[ \frac{d}{dt} \left( (F(u), \phi) + \frac{1}{2}(|\nabla_x u|^2, \phi) + (g, \phi u) \right) + (|\nabla_x \mu|^2, \phi) = \]
\[ = \frac{1}{2}(|\mu|^2, \Delta_x \phi) + (\nabla_x \mu, \nabla_x (\nabla_x \phi \cdot \nabla_x u)). \]

The next step is to prove two lemmas which will be used on left hand side.

**Lemma 4.2** Let assumptions (4.2) and (4.5) hold. Then the following inequality holds:
\[ (\phi^3_{\epsilon}, |f(u)|^6) \leq C_1\|
abla_x (\phi^{1/2}_{\epsilon} \mu)\|_{L^2}^6 + C_2\epsilon^{-3}(1 + \|g\|_{L^6}^6), \] (4.14)
where the constants \(C_i\) are independent of \(\epsilon\).

**Proof**
Rewrite the equation for \(\mu\) as follows
\[ -\Delta_x u + f_0(u) = \mu - \psi(u) - g \] (4.15)
and multiply it by \(\phi^3_{\epsilon} f_0(u) |f_0(u)|^4\) and integrate. For the first term, using integration by parts, it is true that,
\[ (-\Delta_x u, \phi^3_{\epsilon} f_0(u) |f_0(u)|^4) = (\nabla_x u, \nabla_x \phi^3_{\epsilon} f_0(u) |f_0(u)|^4 + \phi^3_{\epsilon} \nabla_x (f_0(u) |f_0(u)|^4)) = \]
\[ = (\nabla_x u, \nabla_x \phi^3_{\epsilon} f_0(u) |f_0(u)|^4 + (\phi^3_{\epsilon} |f_0(u)|^4 f_0(u), |\nabla_x u|^2) \]
Using that \(f_0(u) \geq 0\) the second term on the right hand side is strictly positive and so can be neglected. Integrate by parts again in the first term and again neglect the strictly positive term to obtain in total,
\[ (\phi^3_{\epsilon}, |f_0(u)|^6) \leq \|(-\Delta_x \phi^3_{\epsilon}, F_0(u))\| + (\phi^3_{\epsilon} (\mu + \psi(u) - g), f_0(u) |f_0(u)|^4), \] (4.17)
where \(F_0(u) := \int_0^u f_0(u) |f_0(u)|^4 du\). Due to the monotonicity of \(f_0\) it is true that
\[ |F_0(u)| \leq |u| |f_0(u)|^5 \leq |f_0(u)|^6 \]
and, therefore, because of (2.7), the first term in the right-hand side of (4.17) is absorbed by the left-hand side if \(\epsilon\) is small enough. Then, estimating the second term in the right-hand side by Hölder’s inequality and recalling that \(\psi\) is bounded, obtain
\[ (\phi^3_{\epsilon}, |f_0(u)|^6) \leq C(\phi^3_{\epsilon}, |\mu|^6) + C(|g|^6 + 1, \phi^3_{\epsilon}). \] (4.18)

For the second term on the right hand side split the domain into cubes, \(\square_{i,j,k} = [i, i + 1] \times [j, j + 1] \times [k, k + 1]\) and estimate as follows,
(\|g\|^{6}, \phi^{3}_{\varepsilon}) = \sum_{i,j,k} \int_{\mathbb{R}^{3}} |g|^{6}\phi^{3}_{\varepsilon} \, dx \leq \sum_{i,j,k} \sup_{\mathbb{R}^{3}} \phi^{3}_{\varepsilon} \int_{\mathbb{R}^{3}} |g|^{6} \, dx \leq \|g\|_{L^{b}_{6}}^{6} \sum_{i,j,k} \sup_{\mathbb{R}^{3}} \phi^{3}_{\varepsilon} \leq C\|g\|_{L^{6}}^{6} \int_{\mathbb{R}^{3}} \phi^{3}_{\varepsilon} \, dx = C\varepsilon^{-3}\|g\|_{L^{6}}^{6}.

Now using estimate (2.18) with \( p = 1 \) together with the Sobolev inequality (see (7.18))

\[ \| (\phi^{\frac{3}{2}}_{\varepsilon} \mu) \|_{L^{6}} \leq C \| \nabla_{x} (\phi^{\frac{1}{2}}_{\varepsilon} \mu) \|_{L^{2}}, \]

(4.20)

the following

\[ (\phi^{3}_{\varepsilon}, |f_{0}(u)|^{6}) \leq C\| \nabla_{x} (\phi^{1/2}_{\varepsilon} \mu) \|_{L^{2}}^{6} + C\varepsilon^{-3}(\|g\|_{L^{6}}^{6} + 1) \]

is obtained which implies (4.14) and finishes the proof of the lemma.

\[ \square \]

**Lemma 4.3**

Let assumptions (4.2) and (4.5) hold. Then, the following estimate is valid:

\[ (\phi_{\varepsilon}, |D^{2}_{x}u|^{2}) \leq C(\phi_{\varepsilon}, |\nabla_{x}u|^{2}) + C(\phi_{\varepsilon}, |\nabla_{x}u|^{2}) + C(\phi_{\varepsilon}, |g|^{2}), \]

(4.21)

where the constant \( C \) is independent of \( \varepsilon \) and \( D^{2}_{x}u \) means the collection of all second derivatives of \( u \) with respect to \( x \).

**Proof**

First split the left hand side of (4.21) as follows,

\[ (\phi_{\varepsilon}, |D^{2}_{x}u|^{2}) = (\phi_{\varepsilon}, |\Delta_{x}u|^{2}) + \sum_{i \neq j} (\phi_{\varepsilon} \partial^{2}_{x_{i}x_{j}}u, \partial^{2}_{x_{j}x_{i}}u). \]

(4.22)

Multiply (4.15) by \(-\nabla_{x} (\phi_{\varepsilon} \nabla_{x}u)\) and integrate by parts to get

\[ (\phi_{\varepsilon}, |\Delta_{x}u|^{2}) + (\nabla_{x} \phi_{\varepsilon}, \Delta_{x}u \nabla_{x}u) + (f_{0}' \phi_{\varepsilon}, |\nabla_{x}u|^{2}) \leq \]

\[ \leq (\nabla_{x}(\mu - \psi(u) - g), \phi_{\varepsilon} \nabla_{x}u) \]

(4.23)

Now use the monotonicity of \( f_{0}(u) \) together with (2.7), Hölder’s and Young’s inequalities, to show that

\[ (\phi_{\varepsilon}, |\Delta_{x}u|^{2}) \leq C(\phi_{\varepsilon}, |\nabla_{x}u|^{2}) + C(\phi_{\varepsilon}, |\nabla_{x}u|^{2}) + C(\phi_{\varepsilon}, |g|^{2}). \]

(4.24)

Now only estimating the mixed second derivatives of \( u \) based on (4.24) is required. To do this, note that

\[ (\phi_{\varepsilon}, |\Delta_{x}u|^{2}) = \sum_{i,j} (\phi_{\varepsilon} \partial^{2}_{x_{i}x_{j}}u, \partial^{2}_{x_{j}x_{i}}u) \]

and, for \( i \neq j \), integrate by parts repeatedly to get,

\[ (\phi_{\varepsilon} \partial^{2}_{x_{i}x_{j}}u, \partial^{2}_{x_{j}x_{i}}u) = -(\partial_{x_{i}} \phi_{\varepsilon}, \partial_{x_{j}} \partial_{x_{i}}u, \partial_{x_{j}} \partial_{x_{i}}u) - (\partial_{x_{i}} \phi_{\varepsilon}, \partial_{x_{j}} \partial_{x_{i}}u, \partial_{x_{j}} \partial_{x_{i}}u) = (\phi_{\varepsilon}, |\partial^{2}_{x_{i}x_{j}}u|^{2}) + \]

\[ + (\partial_{x_{i}} \phi_{\varepsilon}, \partial_{x_{j}} \partial_{x_{i}}u, \partial_{x_{j}} \partial_{x_{i}}u) + (\partial_{x_{i}} \phi_{\varepsilon}, \partial_{x_{j}} \partial_{x_{i}}u, \partial_{x_{j}} \partial_{x_{i}}u) + (\partial^{2}_{x_{i}x_{j}} \phi_{\varepsilon}, \partial_{x_{j}} \partial_{x_{i}}u, \partial_{x_{j}} \partial_{x_{i}}u) = \]

\[ = (\phi_{\varepsilon}, |\partial^{2}_{x_{i}x_{j}}u|^{2}) + (\partial^{2}_{x_{i}x_{j}} \phi_{\varepsilon}, \partial_{x_{j}} \partial_{x_{i}}u, \partial_{x_{j}} \partial_{x_{i}}u) - \frac{1}{2} (\partial^{2}_{x_{i}x_{j}} \phi_{\varepsilon}, |\partial_{x_{j}} \partial_{x_{i}}u|^{2}) - \frac{1}{2} (\partial^{2}_{x_{j}x_{i}} \phi_{\varepsilon}, |\partial_{x_{j}} \partial_{x_{i}}u|^{2}) \]

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which together with (4.24) and (2.7) imply (4.21) and finishes the proof of the lemma. □

Now the first energy estimate will be derived. Note that due to assumptions (4.2) on the non-linearity \( f(u) \), there exists a constant \( C \) (independent of \( \varepsilon \)) such that the function

\[
E_{\phi_\epsilon}(t) := (F(u(t)), \phi_\epsilon) + \frac{1}{2}(|\nabla_x u(t)|^2, \phi_\epsilon) + C\varepsilon^{-3}(\|g\|^2_{L^2_\varepsilon} + 1)
\]

(4.26)
satisfies the inequalities

\[
(|F(u(t))|, \phi_\epsilon) + \frac{1}{2}(|\nabla_x u(t)|^2, \phi_\epsilon) \leq E_{\phi_\epsilon}(t) \leq \|g\|^2_{L^2_\varepsilon} + 1
\]

(4.27)

where the constant \( C' \) is independent of \( \varepsilon \). The following estimate can be introduced

\[
\|\nabla_x (\phi_\varepsilon^2 \mu)\|^2_{L^2} = \|\phi_\varepsilon^1 \nabla_x \mu + \mu \nabla_x \phi_\varepsilon^2\|^2_{L^2} \leq 2(\phi_\varepsilon, |\nabla_x \mu|^2) + 2((\nabla_x \phi_\varepsilon^3)^2, |\mu|^2) = 2(\phi_\varepsilon, |\nabla_x \mu|^2) + (\phi_\varepsilon^{-1}(\nabla_x \phi_\varepsilon)^2, |\mu|^2)
\]

(4.28)

and in the form

\[
0 \leq -\frac{1}{4} \|\nabla_x (\phi_\varepsilon^{1/2} \mu)\|^2_{L^2} + \frac{1}{4}(\phi_\varepsilon^{-1}(\nabla_x \phi_\varepsilon)^2, |\mu|^2) + \frac{1}{2}(\phi_\varepsilon, |\nabla_x \mu|^2),
\]

it can be added to the right hand side of (4.13) while the inequality remains true. Equation (4.13) then becomes

\[
\frac{d}{dt} E_{\phi_\epsilon}(t) + \frac{1}{2}(|\nabla_x \mu|^2, \phi_\epsilon) + \frac{1}{4} \|\nabla_x (\phi_\varepsilon^{1/2} \mu)\|^2_{L^2} \leq \frac{C(\phi_\varepsilon^{-1}(\nabla_x \phi_\varepsilon)^2 + \Delta_x \phi_\varepsilon, |\mu|^2) + \frac{1}{4}(\phi_\varepsilon, |\nabla_x \mu|^2) + C(|D_x^2 \phi_\varepsilon|^2 |\nabla_x u|^2, \phi_\varepsilon^{-1}) + (|D_x^2 u|^2 |\nabla_x \phi_\varepsilon|^2, \phi_\varepsilon^{-1}).
\]

(4.29)

Cancel the second term on the right hand side of the left, expand \( \mu = -\Delta_x u + f(u) + g \) in the first term of the right-hand side and use (2.7) together with (2.18) to get the following,

\[
\phi_\varepsilon^{-1}(\nabla_x \phi_\varepsilon)^2 + \Delta_x \phi_\varepsilon \leq \phi_\varepsilon^{-1}(\varepsilon \phi_\varepsilon^0) + \varepsilon^2 \phi_\varepsilon^2
\]

(4.30)

\[
\phi_\varepsilon^{-1} D_x^2 \phi_\varepsilon \leq \varepsilon^2 \phi_\varepsilon^2
\]

\[
\phi_\varepsilon^{-1} |\nabla_x \phi_\varepsilon|^2 \leq \varepsilon^2 \phi_\varepsilon^2
\]

and use them to obtain

\[
\frac{d}{dt} E_{\phi_\epsilon}(t) + \frac{1}{4}(|\nabla_x \mu|^2, \phi_\epsilon) + \frac{1}{4} \|\nabla_x (\phi_\varepsilon^{1/2} \mu)\|^2_{L^2} \leq \frac{C(\varepsilon^2 \phi_\varepsilon^{7/5}, |\Delta_x u|^2 + |f(u)|^2 + |g|^2) + C(\varepsilon^2 \phi_\varepsilon^{7/5}, |\nabla_x u|^2) + C(|D_x^2 u|^2, \varepsilon^2 \phi_\varepsilon^{7/5})}{C(\varepsilon^2 \phi_\varepsilon^{7/5}, |\Delta_x u|^2 + |f(u)|^2 + |g|^2) + C(\varepsilon^2 \phi_\varepsilon^{7/5}, |\nabla_x u|^2) + C(|D_x^2 u|^2, \varepsilon^2 \phi_\varepsilon^{7/5})}
\]

(4.31)
Now recalling that $\phi_\varepsilon^p \leq \phi_\varepsilon$ if $p > 1$ by (4.8) and using (4.19) arrive at,
\[
\leq C\varepsilon^2(\phi_\varepsilon^{7/5}, |f(u)|^2) + C\varepsilon^2(\phi_\varepsilon, |\nabla_x u|^2) + C\varepsilon^{-1}g^2_{L^2} + C\varepsilon^2(\phi_\varepsilon, |D^2_x u|^2). \tag{4.32}
\]
Using Lemmas 4.2 and 4.3 to estimate the last term on the right the following is obtained,
\[
\frac{d}{dt} E_{\phi_\varepsilon}(t) + \beta \left( (|\nabla_x \mu|^2, \phi_\varepsilon) + \|\nabla_x (\phi_\varepsilon^{1/2} \mu)\|^2_{L^2} + (\phi_\varepsilon^3, |f(u)|^6)^{1/4} \right) \leq C\varepsilon^2(\phi_\varepsilon, |\nabla_x u|^2) + C\varepsilon^2(\phi_\varepsilon^{7/5}, |f(u)|^2) + C\varepsilon^{-3}(g^5_{L^6} + 1), \tag{4.33}
\]
where $\beta$ is some positive constant independent of $\varepsilon \to 0$.

It only remains to estimate the second term in the right-hand side of (4.33). Using Hölder’s inequality with exponents $\frac{5}{4}$ and 5, and then using Young’s inequality with exponents $\frac{5}{2}$ and $\frac{5}{3}$, obtain
\[
\begin{align*}
\varepsilon^2(\phi_\varepsilon^{7/5}, |f(u)|^2) & = \varepsilon^2([\phi_\varepsilon, |f(u)|]^{4/5}, [\phi_\varepsilon^{1/2}, |f(u)|]^{6/5}) = \\
& = \varepsilon^2(\|\phi_\varepsilon f(u)\|^{\frac{5}{4}}(\phi_\varepsilon^{\frac{1}{2}} f(u))^{\frac{5}{2}} L^1 \leq \varepsilon^2(\|\phi_\varepsilon f(u)\|^{\frac{5}{4}}(\phi_\varepsilon^{\frac{1}{2}} f(u))^{\frac{5}{2}} L^5 = \\
& = \varepsilon^2(\phi_\varepsilon, |f(u)|)^{4/5}(\phi_\varepsilon^3, |f(u)|^6)^{1/5} \leq C\varepsilon^5(\phi_\varepsilon, |f(u)|)^2 + \beta(\phi_\varepsilon^3, |f(u)|^6)^{1/3}.
\end{align*}
\]
The second term just derived can be cancelled on the left. Then, using the third assumption of (4.2) and (4.27), the following is arrived at
\[
\frac{d}{dt} E_{\phi_\varepsilon}(t) + \beta \|\nabla x \mu\|^2_{H^2} \leq C\varepsilon^2 E_{\phi_\varepsilon}(t) + C\varepsilon^5|E_{\phi_\varepsilon}(t)|^2 + C\varepsilon^{-3}(g^2_{L^6} + 1). \tag{4.35}
\]
It is claimed that (4.35) is sufficient to derive the key estimate (4.7) and finish the proof of the theorem. Indeed, due to (2.18),
\[
E_{\phi_\varepsilon}(0) \leq C\varepsilon^{-3}(\|u_0\|^2_{W^{1,2}} + \|F(u_0)\|_{L^1} + g^2_{L^6} + 1) \tag{4.36}
\]
and, using in addition that $\varepsilon^2 y \leq \varepsilon^5 y^2 + \varepsilon^{-1}$ and absorbing the $\varepsilon^{-1}$ into the $\varepsilon^{-3}$ term, it can be seen that the function $V_\varepsilon(t) := \varepsilon^3 E_{\phi_\varepsilon}(t)$ solves the inequality
\[
\frac{d}{dt} V_\varepsilon + \beta \varepsilon^3 \|\nabla x \mu\|^2_{L^2} \leq \varepsilon^2 V_\varepsilon^2 + C(\|g\|^2_{L^6} + 1), \quad V_\varepsilon(0) \leq C(1 + \|g\|^2_{L^6} + \|u_0\|_{\Phi_b}), \tag{4.37}
\]
where the constant $C$ is independent of $\varepsilon$ and
\[
\|u_0\|_{\Phi_b} := \|u_0\|^2_{W^{1,2}} + \|F(u_0)\|_{L^1}. \tag{4.38}
\]
Let $T > 0$ now be fixed arbitrarily and consider inequality (4.37) on the time interval $t \in [0, T]$ only. Assume, in addition, that the $\varepsilon > 0$ is chosen in such a way that the inequality
\[
\varepsilon^2 V_\varepsilon^2(t) \leq C(1 + \|g\|^2_{L^6} + \|u_0\|_{\Phi_b}) \tag{4.39}
\]
is satisfied for all $t \in [0, T]$. Then, from (4.37), conclude that
\[
V_\varepsilon(t) \leq 2C(T + 1)(1 + \|g\|^2_{L^6} + \|u_0\|_{\Phi_b}), \quad t \in [0, T]. \tag{4.40}
\]
Thus, in order to satisfy (4.39), fix $\varepsilon = \varepsilon(T, u_0, g)$ as follows
\[
\varepsilon := \frac{1}{2(T + 1)[C(1 + \|g\|_{L^6}^2 + \|u_0\|_{\Phi_b})]^{1/2}}.
\] (4.41)
Then inequality (4.40) will be indeed satisfied for all $t \in [0, T]$ (with $\varepsilon$ fixed by (4.41)) which gives
\[
E_{\phi_\varepsilon}(T) \leq \varepsilon^{-3}V_\varepsilon(T) \leq C(T + 1)^4(1 + \|g\|_{L^6}^2 + \|u_0\|_{\Phi_b})^{3/2}. \tag{4.42}
\]
Now recall that $\phi_\varepsilon(x) = \phi_{\varepsilon, x_0}$ depends on the parameter $x_0 \in \mathbb{R}^3$ and estimate (4.42) is uniform with respect to this parameter. Thus, taking the supremum with respect to this parameter and using (2.17), the following is obtained,
\[
\|u(T)\|_{W^{1,2}}^2 + \|F(u(T))\|_{L^6} \leq 2 \sup_{x_0 \in \mathbb{R}^3} E_{\phi_\varepsilon, x_0}(T) \leq C(T + 1)^4(1 + \|g\|_{L^6}^2 + \|u_0\|_{\Phi_b})^{3/2}. \tag{4.43}
\]
Estimate (4.43) together with (4.37) (which is needed for estimating the gradient of $\mu$) imply (4.7) and finishes the proof of the theorem. □

**Corollary 4.4** Under the assumptions of Theorem 4.1, the solution $u$ possesses the following additional regularity:
\[
\|\partial_t u\|_{L^2([0, T], W^{1,2}(B_{x_0}))}^2 + \|\mu\|_{L^2([0, T], L^6(B_{x_0}))}^2 + \|f(u)\|_{L^2([0, T], L^6(B_{x_0}))}^2 + \|u\|_{W^{2,6}(B_{x_0})}^2 \leq C(T + 1)^4(1 + \|g\|_{L^6}^2 + \|u_0\|_{\Phi_b})^{3/2}, \tag{4.44}
\]
where the constant $C$ is independent of $x_0 \in \mathbb{R}^3$, $T$ and $u_0$.

Indeed, the estimate of the first term in the left-hand side follows from the identity $\partial_t u = \Delta_x \mu$ and estimate (4.7), the estimate for the $L^6$-norm of $\mu$ follows from the presence of the term $\|\nabla_x (\phi_\varepsilon^{1/2} \mu)\|_{L^2}^2$ in the left-hand side of (4.29) and Sobolev embedding. Finally, the estimate for two last terms in the left-hand side is a corollary of the corresponding estimate for $\mu$ and the $L^6$-maximal regularity for the semi-linear equation (4.15).

### 4.2 Uniqueness

The aim of this section is to verify that the solution $u$ of the Cahn-Hilliard equation (4.1) constructed in Theorem 4.1 is unique. To this end, more assumptions are set on the non-linearity $f$. It is assumed that there exists a convex positive function $\Psi$ such that
\[
\begin{align*}
1. \quad & \Psi(u) \leq C(|F(u)| + 1), \\
2. \quad & |f'(u)| \leq \Psi(u). \tag{4.45}
\end{align*}
\]
Note that the conditions (4.2) and (4.45) on the non-linearity $f$ do not look very restrictive and are satisfied, for example, by any polynomial of odd order with positive highest coefficient, even order is disallowed because $f_0' \geq 1$, and even by some exponentially growing potentials.

The main result of this section is the following theorem.
**Theorem 4.5** Let assumptions (4.2), (4.5) and (4.45) hold and let $u_1, u_2$ be two solutions of problem (4.1) satisfying (4.6). Then, the following estimate holds:

$$
\|u_1(t) - u_2(t)\|_{W^{1,2}_0} \leq C_T \|u_1(0) - u_2(0)\|_{W^{1,2}_0},
$$

(4.46)

where the constant $C_T$ depends only on $T$ and on the $(4.6)$-norms of the solutions $u_1$ and $u_2$.

**Proof**

Let $v(t) := u_1(t) - u_2(t)$. Then, this function solves the equation

$$
\partial_t v = \Delta_x(-\Delta_xv + l(t)v), \quad v|_{t=0} = v_0, \quad l(t) := \int_0^1 f'(su_1 + (1-s)u_2) \, ds
$$

(4.47)

which is rewritten in the following equivalent form (adapted to the $H^{-1}$-energy estimates):

$$
(-\Delta_x + 1)^{-1}\partial_t v = \Delta_xv - l(t)v - (-\Delta_x + 1)^{-1}(\Delta_xv - l(t)v).
$$

(4.48)

Now let $\varphi(x) := e^{-\sqrt{\epsilon|x|^2} + 1}$ be the exponential weight function (see (2.3)) and let $\varphi_\epsilon(x) = \varphi_\epsilon(x - x_0)$. Moreover let $w := (-\Delta_x + 1)^{-1}v$. Then this function satisfies estimate (2.6) uniformly also with respect to $x_0 \in \mathbb{R}^3$. Multiply (4.48) by the following function:

$$
-\nabla_x(\varphi_\epsilon \nabla_x((-\Delta_x + 1)^{-1}v)) + \varphi_\epsilon(-\Delta_x + 1)^{-1}v = \varphi_\epsilon v - \nabla_x \varphi_\epsilon \cdot \nabla_x((-\Delta_x + 1)^{-1}v)
$$

which is equivalent to

$$
-\nabla_x(\varphi_\epsilon, \nabla_xw) + \varphi_\epsilon w = \varphi_\epsilon v - \nabla_x \varphi_\epsilon \cdot \nabla_xw,
$$

and integrate over $x$. On the left, with one integration by parts, this becomes

$$
(\partial_t w, -\nabla_x(\varphi_\epsilon, \nabla_xw) + \varphi_\epsilon w) = \frac{1}{2} \frac{d}{dt} (|\nabla_x w|^2 + |w|^2, \varphi_\epsilon),
$$

(4.49)

and on the right this becomes, again with some terms integrated by parts,

$$
(\Delta_x v - l(t)v - (-\Delta_x + 1)^{-1}(\Delta_xv - l(t)v), \varphi_\epsilon v - \nabla_x \varphi_\epsilon \cdot \nabla_x w) =
$$

$$
= -(|\nabla_x v|^2, \varphi_\epsilon) + (-\Delta_x \varphi_\epsilon, |v|^2) - (\Delta_x v, \nabla_x \varphi_\epsilon \cdot \nabla_x w) +
$$

$$
- (l(t)v, \varphi_\epsilon v) + (l(t)v, \nabla_x \varphi_\epsilon \cdot \nabla_x w) +
$$

$$
+((-\Delta_x + 1)^{-1}(\Delta_xv - l(t)v), \varphi_\epsilon v) - ((-\Delta_x + 1)^{-1}(\Delta_xv - l(t)v), \nabla_x \varphi_\epsilon \cdot \nabla_x w).
$$

(4.50)

Combining these the following is obtained

$$
\frac{1}{2} \frac{d}{dt} (|\nabla_x w|^2 + |w|^2, \varphi_\epsilon) + (|\nabla_x v|^2, \varphi_\epsilon) + (l(t)v, \varphi_\epsilon v) = (-\Delta_x \varphi_\epsilon, |v|^2) -
$$

$$
-(\Delta_x v, \nabla_x \varphi_\epsilon \cdot \nabla_x w) + (l(t)v, \nabla_x \varphi_\epsilon \cdot \nabla_x w) +
$$

$$
+((-\Delta_x + 1)^{-1}(\Delta_xv - l(t)v), \varphi_\epsilon v) - ((-\Delta_x + 1)^{-1}(\Delta_xv - l(t)v), \nabla_x \varphi_\epsilon \cdot \nabla_x w).
$$

(4.51)
Using now the weighted maximal regularity for the equation \(-\Delta_x w + w = v\), for sufficiently small \(\varepsilon\). Parameter \(\varepsilon\) must be sufficiently small in order to exclude the kernel of the Laplacian on an unbounded domain. It is true that

\[
\|w\|_{W^{s+2,2}_\varepsilon} \sim \|v\|_{W^{s+2}_\varepsilon}, \quad s \in \mathbb{R},
\]

(4.52)

where the equivalence constants depend only on \(s\), see \([29, 61]\). Therefore, integrating by parts and using (2.6) together with (4.52) and Cauchy-Schwartz inequality, three terms are transformed as follows

\[
-(\Delta_x v, \nabla \varphi \cdot \nabla_x w) + (-(\Delta_x + 1)^{-1})\Delta_x v, \varphi \cdot v) +
\]

(4.53)

and the rest are carried unchanged and the following is obtained,

\[
\frac{d}{dt}\|w\|_{W^{2,2}_\varepsilon}^2 + \|v\|_{W^{1,2}_\varepsilon}^2 + (|l(t)|v, \varphi \varepsilon v) \leq C\|v\|_{L^2_\varepsilon}^2 +
\]

(4.54)

\[
+ (|l(t)| \cdot |v|, \varphi \varepsilon |\nabla_x w| + |(\Delta_x + 1)^{-1}(\varphi \varepsilon v)| + |(\Delta_x + 1)^{-1}(\nabla_x \varphi \varepsilon \cdot \nabla_x w)|.
\]

Let now

\[
h := \varphi^{-1}_\varepsilon(\varphi \varepsilon |\nabla_x w| + |(\Delta_x + 1)^{-1}(\varphi \varepsilon v)| + |(\Delta_x + 1)^{-1}(\nabla_x \varphi \varepsilon \cdot \nabla_x w)|).
\]

Then, applying the Cauchy-Schwartz inequality together with the weighted maximal regularity for the Laplacian, it can be concluded that

\[
(|l(t)|, h) \leq (|l(t)|, \varphi \varepsilon ^2) + (|l(t)|, \varphi \varepsilon h^2) \leq
\]

(4.55)

\[
\leq (|l(t)|v, \varphi \varepsilon v) + \int_{\mathbb{R}^3} \varphi \varepsilon \|l(t)|h^2\|\mathcal{L}^1(B_{1/2}^0) \, dx_0 \leq
\]

\[
\leq (|l(t)|v, \varphi \varepsilon v) + \|l(t)\|_{L^1} \int_{\mathbb{R}^3} \varphi \varepsilon \|h\|_{L^\infty(B_{1/2}^0)}^2 \, dx_0 \leq
\]

\[
\leq (|l(t)|v, \varphi \varepsilon v) + C\|l(t)\|_{L^1} \int_{\mathbb{R}^3} \|h\|_{W^{3/2,2}_{\varphi \varepsilon}}^2 \, dx_0 \leq
\]

\[
\leq (|l(t)|v, \varphi \varepsilon v) + C\|l(t)\|_{L^1} \|h\|_{W^{3/2,2}_{\varphi \varepsilon}}^2 \leq (|l(t)|v, \varphi \varepsilon v) + C\|l(t)\|_{L^1} \|v\|_{W^{3/2,2}_{\varphi \varepsilon}}^2.
\]

Now estimate the \(L^1\)-norm of \(l(t)\) using assumptions (4.45). Namely,

\[
\|l(t)\|_{L^1} \leq \int_0^1 \|f'(su_1 + (1 - s)u_2)\|_{L^1} \, ds \leq
\]

(4.56)

\[
\leq \int_0^1 \|\Psi(su_1 + (1 - s)u_2)\|_{L^1} \, ds \leq \int_0^1 \|s\Psi(u_1) + (1 - s)\Psi(u_2)\|_{L^1} \, ds \leq
\]

\[
\leq \|\Psi(u_1)\|_{L^1} + \|\Psi(u_2)\|_{L^1} \leq C(\|F(u_1)\|_{L^1} + \|F(u_2)\|_{L^1} + 1).
\]
Thus, \( \|l(t)\|_{L^1_b} \leq C_T = C_T(u_1, u_2) \) and using (4.55), (4.54) is rewritten as follows

\[
\frac{d}{dt} \|w\|_{W^{1,2}_{\varphi}}^2 + \|v\|_{W^{1,2}_{\varphi}}^2 \leq C_T\|v\|_{W^{3/4,2}}^2.
\] (4.57)

Now interpolating the \( W^{3/4,2} \)-norm between the \( W^{1,2} \) and \( W^{-1,2} \)-norms, and arriving at

\[
\frac{d}{dt} \|w\|_{W^{1,2}_{\varphi=x_0}}^2 \leq C_T\|w\|_{W^{1,2}_{\varphi=x_0}}^2,
\] (4.58)

where the constant \( C_T \) grows polynomially in \( T \) (according to (4.7)). Applying Grönwall’s inequality to this relation, taking the supremum over \( x_0 \in \mathbb{R}^3 \) and using (2.17), (4.46) appears and the proof of the theorem is finished. \( \square \)

This next corollary of the proved theorem shows that the constructed solution \( u \) is somewhat smoother, with a stronger regularity result proven in the next section.

**Corollary 4.6** Let assumptions (4.2), (4.5) and (4.45) hold and let, in addition, \( u_0 \) be smooth enough to guarantee that \( \partial_t u(0) \in W^{-1,2}_b(\mathbb{R}^3) \). Then, \( \partial_t u(t) \in W^{-1,2}_b \) for all \( t \geq 0 \) and its norm grows at most polynomially in time. If \( \partial_t u(0) \notin W^{-1,2}_b(\mathbb{R}^3) \), then nevertheless \( \partial_t u(t) \in W^{-1,2}_b(\mathbb{R}^3) \) for all \( t > 0 \) and the following estimate holds:

\[
\|\partial_t u(t)\|_{W^{-1,2}_b}^2 \leq C t^{-1}(1 + t^N)Q(1 + \|g\|_{L^2_b}^2 + \|u_0\|_{\varphi_b})
\] (4.59)

for some monotone function \( Q \) and constants \( C \) and \( N \) independent of \( u_0 \) and \( t \).

**Proof**

Indeed, differentiating equation (4.1) in time and denoting \( v(t) := \partial_t u(t) \), it can be seen that \( v \) solves the equation

\[
\partial_t v = \Delta_x (-\Delta_x v + f'(u)v), \quad v|_{t=0} = \partial_t u(0)
\] (4.60)

which is almost identical to equation (4.47). Therefore, denoting \( w(t) := (-\Delta_x + 1)^{-1}v(t) \) and arguing exactly as in the proof of the theorem, the following is derived,

\[
\frac{d}{dt} \|\partial_t u(t)\|_{W^{-1,2}_b}^2 \leq C(t)\|\partial_t u(t)\|_{W^{-1,2}_b}^2,
\] (4.61)

where \( C(t) \) grows polynomially in time, for the same reason as (4.58). This estimate, together with (4.44) proves both assertions of the corollary. \( \square \)

### 4.3 Regularity

In this section an improved regularity result will be proven under the assumption of additional smoothness of the initial data.

**Theorem 4.7** Let assumptions (4.2), (4.5) and (4.45) hold and let, in addition, \( u_0 \in W^{2,6}_b(\mathbb{R}^3) \). Then, \( u(t) \in W^{2,6}_b(\mathbb{R}^3) \) for all \( t \geq 0 \) and its norm grows at most polynomially in time. If \( u(0) \notin W^{2,6}_b(\mathbb{R}^3) \), then nevertheless \( u(t) \in W^{2,6}_b(\mathbb{R}^3) \) for all \( t > 0 \).
Proof
Rewriting the Cahn-Hilliard equation in the form
\[ \mu = -(-\Delta x + 1)^{-1} \partial_t u(t) + (-\Delta x + 1)^{-1} \mu \]
and using the maximal regularity for the Laplacian in the uniformly local spaces, it can be seen that
\[ \|\mu(t)\|_{W_b^{1,2}} \leq C\|\partial_t u(t)\|_{W_b^{-1,2}} + C\|\mu(t)\|_{W_b^{-1,2}}. \] (4.62)
Moreover, according to Theorem 4.1 and using (4.2),
\[ \|\mu(t)\|_{W_b^{-2,2}} \leq C\|(-\Delta x + 1)^{-1}(\Delta_x u(t) - f(u(t)) + g)\|_{L_b^2} \leq C\|\mu(t)\|_{W_b^{1,2}} + \|g\|_{L_b^2} + \|f(u(t))\|_{L_b^1} \leq C(\|u(t)\|_{\Phi_b} + \|g\|_{L_b^6}) \leq C(t, u_0, g). \] (4.63)
Thus, due to (4.62), (4.63) and interpolation,
\[ \|\mu(t)\|_{L_b^6} \leq C\|\mu(t)\|_{W_b^{1,2}} \leq C\|\partial_t u(t)\|_{W_b^{-1,2}} + \|u(t)\|_{\Phi_b} + \|g\|_{L_b^6} \leq C\|\partial_t u(t)\|_{W_b^{-1,2}} + C(t, u_0, g), \] (4.64)
where the constant \( C(t, u_0, g) \) grows polynomially in time.

Estimate (4.64) together with Corollary 4.6 give the assertion of Theorem 4.7 for the \( L_b^6 \)-norm of \( \mu \). In order to obtain the analogous assertions for the \( W_b^{2,6} \)-norm of \( u \), the \( L_b^6 \)-maximal regularity theorem for the semi-linear equation (4.15) is applied which gives
\[ \|u(t)\|_{W_b^{2,6}} \leq C(1 + \|\mu(t)\|_{L_b^6}) \leq C_1(\|\partial_t u(t)\|_{W_b^{-1,2}} + \|u(t)\|_{\Phi_b} + \|g\|_{L_b^6}), \] (4.65)
Thus, the theorem is proved. \( \Box \)

The proved regularity is more than enough to initialize the standard bootstrapping process and to verify that the smoothness of \( u(t) \) is restricted only by the smoothness of \( f \) and \( g \). In particular, if both of them are \( C^\infty \)-smooth the solution will be \( C^\infty \)-smooth as well. If they are, in addition, real analytic, one has the real analytic in \( x \) solution \( u(t, x) \) as well (for \( t > 0 \)).

4.4 Dissipative estimate for the Cahn-Hilliard-Oono equation

In this section the above developed techniques are applied to the so-called Cahn-Hilliard-Oono equation
\[ \partial_t u = \Delta_x u - \lambda u, \quad \mu := -\Delta_x u + f(u) + g(x), \quad u|_{t=0} = u_0 \] (4.66)
which differs from the classical Cahn-Hilliard equation by the presence of an extra term \( \lambda u \) where the constant \( \lambda > 0 \). The extra dissipative term has been initially introduced to model the long-range non-local interactions (see [67] and also [64] for further details) and essentially simplifies the analysis of the long-time behaviour of the Cahn-Hilliard equations in unbounded domains and as will be seen, guarantees the dissipativity of the equation in the uniformly local spaces. To be more precise, the following theorem can be considered as the main result of the section.
Theorem 4.8 Let the assumptions of Theorem 4.1 hold. Then the Cahn-Hilliard-Oono equation (4.66) possesses at least one global in time solution in the sense of (4.6) (for all \( T > 0 \)) which satisfies the following estimate:

\[
\begin{align*}
\|u(t)\|_{W_x}^2 + \|F(u(t))\|_{L_b} + \|\nabla_x \mu\|_{L_b}^2 \leq & \quad (4.67) \\
\leq & \quad Q(\|g\|_{L_b}) + Q(\|u(0)\|_{W_x}^2 + \|F(u(0))\|_{L_b}) e^{-\sigma t}, \quad t \geq 0,
\end{align*}
\]

for some monotone increasing function \( Q \) and positive constant \( \sigma \) independent of the initial data \( u_0 \) and \( t \geq 0 \).

Proof

The proof of this theorem is analogous to Theorem 4.1, but the presence of the dissipative term \( \lambda u \) produces the extra term \( \lambda V \) in the left-hand side of (4.37) with some positive \( \lambda_0 \) independent of \( \varepsilon \) and this gives global existence and the dissipative estimate for \( V \) if \( \varepsilon = \varepsilon(u_0, g) \) is small enough. However, this is still not enough to deduce the dissipative estimate for \( u(t) \) since the parameter \( \varepsilon \) still depends on the initial data \( u_0 \). To overcome this difficulty, consider the time-dependent parameter \( \varepsilon = \varepsilon(t) \). To be more precise, let \( \phi_{\varepsilon,x_0}(x) \) be the same as in (4.8). Then, due to (2.8),

\[
|\partial_t \phi_{\varepsilon,x_0}(x)| \leq C_t[\phi_{\varepsilon,x_0}(x)], \quad C_t := 5 \cdot \frac{|\varepsilon'(t)|}{\varepsilon(t)}.
\]

(4.68)

Multiply equation (4.66) by \( \phi \mu = \phi_{\varepsilon(t),x_0} \mu \), where the function \( \varepsilon(t) \) will be specified below, and integrate over \( x \). Then, analogously to (4.13),

\[
\frac{d}{dt} \left( (F(u), \phi_{\varepsilon}) + \frac{1}{2} \left| \nabla_x u \right|^2, \phi_{\varepsilon} \right) + \left| \nabla_x \mu \right|^2, \phi_{\varepsilon} \right) = \quad (4.69)
\]

\[
= \frac{1}{2} \left( |\mu|^2, \Delta \phi_{\varepsilon} \right) + \left( \nabla_x \mu, \nabla_x \left( \nabla_x \phi_{\varepsilon} \cdot \nabla_x u \right) \right) - \lambda(\phi_{\varepsilon}, |\nabla u|^2 + f(u)u + gu) + (\partial_t \phi_{\varepsilon}, F(u) + \frac{1}{2} |\nabla u|^2 + gu),
\]

where the extra two terms in the right-hand side are due to the extra term \( \lambda u \) and the dependence of \( \phi_{\varepsilon} \) on time. Note that the assumptions (4.2) imply that

\[
F(u) \leq f(u)u + C
\]

for some constant \( C \) and, therefore,

\[
- \lambda(\phi_{\varepsilon}, |\nabla u|^2 + f(u)u + gu) \leq - \lambda E_{\phi_{\varepsilon}}(t) + C\varepsilon^{-3}(1 + \|g\|_{L_b}^2),
\]

where \( E_{\phi_{\varepsilon}}(t) \) is defined by (4.26). Furthermore, using (4.68) together with (4.27),

\[
(\partial_t \phi_{\varepsilon}, F(u) + \frac{1}{2} |\nabla u|^2 + gu) \leq 2C_t E_{\phi_{\varepsilon}} + C(C_t + 1)\varepsilon^{-3}(1 + \|g\|_{L_b}^2).
\]

Thus, the extra terms are estimated as follows

\[
- \lambda(\phi_{\varepsilon}, |\nabla u|^2 + f(u)u + gu) + (\partial_t \phi_{\varepsilon}, F(u) + \frac{1}{2} |\nabla u|^2 + gu) \leq \quad (4.70)
\]

\[
\leq - \lambda/2 E_{\phi_{\varepsilon}}(t) + C\varepsilon^{-3}(1 + \|g\|_{L_b}^2)
\]

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if the parameter $\varepsilon(t)$ satisfies the following extra condition:

$$C_t = 5 \frac{|\varepsilon'(t)|}{\varepsilon(t)} \leq \frac{\lambda}{2}$$  \hspace{1cm} (4.71)

which is assumed to be satisfied from now on. The rest of the terms in (4.69) can be estimated using Lemmas 4.2 and 4.3 exactly as in the proof of Theorem 4.1 which gives the following dissipative analogue of (4.35):

$$\frac{d}{dt} E_{\phi_\varepsilon}(t) + \frac{\lambda}{2} E_{\phi_\varepsilon}(t) + \beta \| \nabla_x \mu \|^2_{L^2_{\phi_\varepsilon}} \leq C \varepsilon^2 E_{\phi_\varepsilon}(t) + C \varepsilon^3 [E_{\phi_\varepsilon}(t)]^2 + C \varepsilon^{-3}(\|g\|^2_{L^6_b} + 1).$$  \hspace{1cm} (4.72)

Leaving (4.36) unchanged and again using that $\varepsilon^2 y \leq \varepsilon^5 y^2 + \varepsilon^{-1}$, it can be seen that the function $V_{\varepsilon}(t) := \varepsilon^3 E_{\phi_\varepsilon}(t)$ solves the dissipative analogue of inequality (4.37):

$$\frac{d}{dt} V_{\varepsilon} + \frac{\lambda}{2} V_{\varepsilon} + \beta \varepsilon^3 \| \nabla_x \mu \|^2_{L^2_{\phi_\varepsilon}} \leq \varepsilon^2 V_{\varepsilon}^2 + C(\|g\|^2_{L^6_b} + 1),$$

where the constant $C$ is independent of $\varepsilon$ and $\|u_0\|_{\phi_b}$ is defined by (4.38).

It is claimed that inequality (4.73) is enough to deduce the desired dissipative estimate (4.67) and finish the proof of the theorem. Indeed, restyling it as

$$\frac{d}{dt} V_{\varepsilon} + \frac{\lambda}{4} V_{\varepsilon} \leq V_{\varepsilon} \left( \varepsilon^2 V_{\varepsilon} - \frac{\lambda}{4} \right) + C_g,$$  \hspace{1cm} (4.74)

where $C_g = C(1 + \|g\|^2_{L^6_b})$, it can be seen that, under the assumption

$$\varepsilon^2(t) V_{\varepsilon}(t) \leq \frac{\lambda}{4}, \quad t \geq 0,$$  \hspace{1cm} (4.75)

the first term in the right-hand side of (4.74) will be negative (and, therefore, can be omitted) and $V_{\varepsilon}(t)$ will satisfy the estimate

$$V_{\varepsilon}(t) \leq \frac{4C_g}{\lambda} + V_{\varepsilon}(0) e^{\frac{\lambda}{4} t}, \quad t \geq 0.$$  \hspace{1cm} (4.76)

Using this observation, it is not difficult to show that both estimates (4.76) and (4.75) will be satisfied if the parameter $\varepsilon_0(t)$ is chosen in such way that

$$\varepsilon^2(t) \left( \frac{4C_g}{\lambda} + V_{\varepsilon}(0) e^{\frac{\lambda}{4} t} \right) \leq \frac{\lambda}{4}.$$

Thus, the function $\varepsilon(t) \ll 1$ is fixed to satisfy the two inequalities (4.71) and (4.77). In particular, take

$$\varepsilon(t) = \varepsilon_0 \left( \frac{\lambda/4}{\frac{4C_g}{\lambda} + V_{\varepsilon}(0) e^{-\sigma t}} \right)^{\frac{1}{2}},$$  \hspace{1cm} (4.78)
where $\varepsilon_0 > 0$ and $\sigma > 0$ are proper small constants. Indeed, condition (4.77) will be satisfied if $\varepsilon_0 \leq 1$ and $\sigma \leq \lambda/4$. In order to check (4.71), note that

$$
\left| \frac{\varepsilon'(t)}{\varepsilon(t)} \right| = \left| \frac{d}{dt} \log \varepsilon(t) \right| = \frac{1}{2} \cdot \frac{V_\varepsilon(0)\sigma e^{-\sigma t}}{\frac{4\varepsilon_0}{\lambda} + V_\varepsilon(0)e^{-\sigma t}} \leq \frac{1}{2}\sigma
$$

(4.79)

and (4.71) will also be satisfied if $\sigma \leq \lambda/5$. Thus, for that choice of $\varepsilon(t)$ estimate (4.76) is satisfied and, therefore,

$$
E_{\Phi_{\varepsilon(t), x_0}}(t) \leq \varepsilon(t)^{-3}V_\varepsilon(t) \leq C \left( C_g + V_\varepsilon(0)e^{-\sigma t} \right)^{\frac{3}{2}} \left( C_g + V_\varepsilon(0)e^{-\sigma t} \right)
$$

(4.80)

uniformly with respect to $x_0 \in \mathbb{R}^3$. Taking the supremum with respect to $x_0 \in \mathbb{R}^3$ from both sides of (4.80) and using (2.17), the following is arrived at

$$
\|u(t)\|_{W^{1,2}_b}^2 + \|F(u(t))\|_{L^1_b} \leq Q(\|g\|_{L^6_b}) + Q(\|u(0)\|_{W^{1,2}_b}^2 + \|F(u(0))\|_{L^1_b})e^{-\sigma t}
$$

(4.81)

for the properly chosen monotone function $Q$ and positive constant $\sigma$. Estimate (4.81) together with (4.73) (which is needed for estimating the gradient of $\mu$) implies (4.67) and finishes the proof of the theorem.

Now the uniqueness and further regularity of solutions for the case of the Cahn-Hilliard-Oono equation are discussed.

**Proposition 4.9** Let the assumptions of Theorem 4.8 hold and, in addition, (4.45) be satisfied. Then the solution $u(t)$ constructed in Theorem 4.8 is unique and, for every two solutions $u_1(t)$ and $u_2(t)$ of the Cahn-Hilliard-Oono equation, estimate (4.46) holds.

Indeed, the presence of the extra term $\lambda \mu$ in (4.66) does not make any essential difference for the uniqueness proof which repeats almost word by word the proof of Theorem 4.5 and by this reason is omitted.

The following corollary is the dissipative analogue of Corollary 4.6.

**Corollary 4.10** Let the assumptions of Proposition 4.9 hold and let, in addition, the initial data $u_0$ be such that $\partial_t u(0) \in W_b^{-1,2}(\mathbb{R}^3)$. Then, $\partial_t u(t) \in W_b^{-1,2}(\mathbb{R}^3)$ for all $t > 0$ and the analogue of dissipative estimate (4.67) is valid:

$$
\|\partial_t u(t)\|_{W^{-1,2}_b} \leq Q(\|\partial_t u(0)\|_{W^{-1,2}_b} + \|u(0)\|_{\Phi_b})e^{-\gamma t} + Q(\|g\|_{L^6_b})
$$

(4.82)

for proper monotone function $Q$ and positive constant $\gamma$. Moreover, if $\partial_t u(0) \not\in W_b^{-1,2}(\mathbb{R}^3)$ then, nevertheless, $\partial_t u(t) \in W_b^{-1,2}(\mathbb{R}^3)$ and the following estimate holds:

$$
\|\partial_t u(t)\|_{W^{-1,2}_b} \leq C t^{-1/2}Q(\|u(0)\|_{\Phi_b} + \|g\|_{L^6_b}), \quad t \in (0, 1]
$$

(4.83)

for some monotone increasing function $Q$ and positive $C$.  

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Indeed, arguing exactly as in Corollary 4.6, the following estimate is obtained
\[
\frac{d}{dt}\|\partial_t u(t)\|_{W_{x,t}^{r,1,2}}^2 \leq Q(\|u(t)\|_{\Phi_b})\|\partial_t u(t)\|_{W_{x,t}^{r,1,2}}^2, \tag{4.84}
\]
where \(Q(z) = C(1+z)^8\), see (4.56) and (4.57). Multiply this inequality by \(t\) and integrate in time to get
\[
t\|\partial_t u(t)\|_{W_{x,t}^{r,1,2}}^2 \leq (Q(\|u(t)\|_{\Phi_b}) + 1) \int_0^t \|\nabla_x u(t)\|_{L_{x,t}^2}^2 \, dt, \quad t \in (0,1].
\]
Taking the supremum over all shifts \(x_0 \in \mathbb{R}^3\) and using (2.17) together with (4.67),
\[
\|\partial_t u(t)\|_{W_{x,t}^{r,1,2}}^2 \leq t^{-1}Q(\|u(0)\|_{\Phi_b} + \|g\|_{L_b^1})
\]
for some new monotone function \(Q\). Thus, (4.83) is verified. In addition, the last estimate gives that
\[
\|\partial_t u(t+1)\|_{W_{x,t}^{r,1,2}} \leq Q(\|u(t)\|_{W_{x,t}^{r,1,2}} + \|g\|_{L_b^1})
\]
which together with (4.67) proves also the dissipative estimate (4.82) for \(t \geq 1\). Finally, estimate (4.82) on the finite time interval \(t \in [0,1]\) follows directly from Grönwall’s inequality applied to (4.84) and the corollary is proved. \(\square\)

Furthermore, the dissipative analogue of Corollary 4.7 also holds.

**Corollary 4.11** Let the assumptions of Proposition 4.9 hold and let, in addition \(u_0 \in W_{b}^{2,6} (\mathbb{R}^3)\). Then, \(u(t) \in W_{b}^{2,6} (\mathbb{R}^3)\) for all \(t > 0\) and the analogue of (4.82) holds. If \(u_0 \notin W_{b}^{2,6} (\mathbb{R}^3)\) then, nevertheless, \(u(t) \in W_{b}^{2,6} (\mathbb{R}^3)\) for all \(t > 0\) and the analogue of smoothing property (4.83) also holds.

As is noted above, the verified \(W^{2,6}\)-regularity of solutions (Corollary 4.7) allows one to obtain further smoothness of solutions (restricted only by the regularity of \(g\) and \(f\)) by standard bootstrapping arguments. In the case of the Cahn-Hilliard-Oono equation, the obtained estimates for the higher norms will be also dissipative.

Note also that the proved dissipative estimate (4.67) together with the smoothing properties established in Corollaries 4.10 and 4.11 allow one to define the dissipative solution semi-group in the phase space \(\Phi_b\)
\[
S(t) : \Phi_b \rightarrow \Phi_b, \quad S(t)u_0 = u(t), \quad \Phi_b := \{u_0 \in W_{b}^{1,2} (\mathbb{R}^3), \, F(u_0) \in L_b^1 (\mathbb{R}^3)\} \tag{4.85}
\]
and verify that this semi-group possesses an absorbing set bounded in \(W_{b}^{2,6} (\mathbb{R}^3)\). This, together with the Lipschitz continuity (4.46) allows one, in turn, to establish the existence of the so-called locally compact global attractor \(A \subset W_{b}^{2,6} (\mathbb{R}^3)\) (see [61] for more details) for the solution semi-group (4.85) associated with the Cahn-Hilliard-Oono equation. After that one can also study the upper and lower bounds for its Kolmogorov’s \(\varepsilon\)-entropy, etc. These things are possible (when the key dissipative estimate is obtained, of course, see [2, 3, 61, 73, 80, 82, 79, 81] and references therein); and are outside the scope of this thesis.
4.5 Cahn-Hilliard equation with singular potentials

In the previous sections, the case when the non-linearity is regular $f \in C^1(\mathbb{R})$ has been considered. In this section, the case of the so-called singular potentials is considered where the non-linearity $f$ is defined on the interval $(-1, 1)$ only and has singularities at $u = \pm 1$, a situation which is currently of great interest, see [18, 27, 31] and references therein. The typical example here is the so-called logarithmic potential

$$f(u) = \log \frac{1 + u}{1 - u} - \alpha u$$

(4.86)

or the polynomial singularity

$$f(u) = \frac{u}{(1 - u^2)^\gamma} - \alpha u,$$

(4.87)

where $l > 0$.

In this case, it is additionally assumed that the solution $u(t, x)$ is always in between minus and plus one:

$$-1 < u(t, x) < 1 \text{ for almost all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

(4.88)

and therefore $f(u(t, x))$ has sense.

Following [27] (see also [18, 22, 63] and references therein), assume that the non-linearity $f$ satisfies

$$\begin{align*}
1. & \quad f \in C^2(-1, 1), \quad f(0) = 0; \\
2. & \quad \lim_{u \to \pm 1} f(u) = \pm \infty; \\
3. & \quad \lim_{u \to \pm 1} f'(u) = +\infty
\end{align*}$$

(4.89)

and, exactly as in the case of regular potentials, assume that $g \in L_6^b(\mathbb{R}^3)$.

However, in contrast to the case of bounded or cylindrical domains, assumptions (4.89) look insufficient to derive the key a priori estimate (at least using the method developed above). Indeed, the third assumption of (4.2) which connects the growth rate of $f(u)$ and its antiderivative $F(u)$ has been essential in the derivation of that estimate. But this assumption is clearly wrong for the case of singular potentials where $f(u)$ is growing faster than $F(u)$ as $u \to \pm 1$. In particular, for the case of non-linearity (4.86) as well as non-linearity (4.87) with $l < 1$, the potential $F(u)$ is bounded near $u = \pm 1$, so $f(u)$ cannot be reasonably estimated through $F(u)$ near the singularities and, by this reason, this technique is unable to treat these cases. But if $l > 1$, the non-linearity (4.87) satisfies

$$|f(u)| \leq \beta |F(u)|^\kappa + C$$

(4.90)

for some positive $\beta$ and $C$ and some $\kappa \in (1, \infty)$ (for (4.87), the equivalent is $\kappa = 1 + \frac{1}{l-1}$).

As shown in the next theorem this assumption is sufficient in order to obtain the analogues of theorems 4.1 and 4.8 for the case of singular potentials.
Theorem 4.12 Let the assumptions (4.89) and (4.90) hold and \( g \in L_b^6(\mathbb{R}^3) \). Then, for every \( u_0 \in \Phi_b \), the Cahn-Hilliard equation (4.1) possesses at least one global solution \( u(t) \), \( t \geq 0 \), (in the sense of (4.6) plus the extra assumption (4.88)) which satisfies the following analogue of (4.7):

\[
\|u(t)\|^2_{W_b^1,2} + \|F(u(t))\|_{L_b^1} + \|\nabla_x \mu\|^2_{L_b^2([0,t] \times \mathbb{R}^3)} \leq C(1 + t^{3\kappa+1}) \left( 1 + \|g\|^2_{L_b^6} + \|u_0\|^2_{W_b^1,2} + \|F(u_0)\|_{L_b^1} \right)^{3\kappa-1/2},
\]

where \( \kappa \) is the same as in assumption (4.90).

Proof
As in the proof of Theorem 4.1, only the formal derivation of the key estimate (4.91) is done and the existence of a solution can be then obtained in a standard way, see [29, 61]. The derivation of this estimate is also similar to what has been done in the proof of Theorem 4.2. However, the weight function (4.8) is no longer appropriate and more general weights should be used, namely \( \phi_{\varepsilon}(x) \) defined in (2.3) with the parameter

\[
\gamma = 3 + \frac{2}{2\kappa - 1},
\]

where \( \kappa \) is the same as in assumption (4.90).

Indeed, multiplying equation (4.1) by \( \phi_{\varepsilon} \mu(t) = \phi_{\varepsilon,x_0}(x)\mu(t) \) where \( \phi_{\varepsilon,x_0}(x) = \phi_{\varepsilon}(x - x_0) \) and \( \phi_{\varepsilon} \) is defined by (2.3) (with the parameter \( \varepsilon \) being specified below), and arguing exactly as in the proof of Theorem 4.1 (as it is not difficult to see, Lemmas 4.2 and 4.3 remain true for the singular potentials, because the only condition they require is the monotonicity of \( f_0' \), so no difference so far), the following analogue of estimate (4.33) is obtained:

\[
\frac{d}{dt} E_{\phi_{\varepsilon}}(t) + \beta \left( \|\nabla_x \mu\|^2, \phi_{\varepsilon} \right) + \|\nabla_x (\phi_{\varepsilon}^{1/2} \mu)\|^2_{L_2} + (\phi_{\varepsilon}^{3/2}, |f(u)|^6) \leq C \varepsilon^2 (\phi_{\varepsilon}, |\nabla_x u|^2) + C \varepsilon^2 (\phi_{\varepsilon}^{1+2/\gamma}, |f(u)|^2) + C \varepsilon^{-3} (\|g\|^6_{L_b^6} + 1),
\]

where the weighted energy \( E_{\phi_{\varepsilon}} \) is defined by (4.26) and satisfies (4.27) (the exponent 7/5 in the second term of the right-hand side of (4.33) is now replaced by \( 1 + 2/\gamma \) due to the choice of a different weight function, see (2.7)).

However, in order to estimate the second term in the right-hand side of (4.93), modify (4.34) interpolating between \( L_{\phi_{\varepsilon}}^{1/\kappa} \) and \( L_{\phi_{\varepsilon}^2}^6 \) (instead of \( L_{\phi_{\varepsilon}}^1 \) and \( L_{\phi_{\varepsilon}^2}^6 \)). Next derive from (4.92) that

\[
1 + \frac{2}{\gamma} = \frac{4\kappa}{6\kappa - 1} + 3 \cdot \frac{2\kappa - 1}{6\kappa - 1}.
\]

Now with the Hölder’s inequality (with exponents \( \frac{6k-1}{4k} \) and \( \frac{6k-1}{2k-1} \)) and Young’s inequality
(with exponents $\frac{6k-1}{2}$ and $\frac{6k-1}{6k-3}$), it can be seen that

$$
C\varepsilon^2(\phi^1+2/\gamma, |f(u)|^2) = C\varepsilon^2(\|\phi\|_{\varepsilon}/\|f(u)\|^{1/\gamma})^\frac{4k}{6k-1}, [\phi^3, |f(u)|^6]^{\frac{2k-1}{6k-4}} = (4.94)
$$

$$
= C\varepsilon^2\|\phi\|_{\varepsilon}/\|f(u)\|^{1/\gamma})^\frac{4k}{6k-1}, [\phi^3, |f(u)|^6]^{\frac{2k-1}{6k-4}} \leq
$$

$$
\leq C\varepsilon^2(\|\phi\|_{\varepsilon}, |f(u)|^\frac{1}{\gamma})^\frac{4k}{6k-1}, [\phi^3, |f(u)|^6]^{\frac{2k-1}{6k-4}} =
$$

$$
= C\varepsilon^2(\|\phi\|_{\varepsilon}, |f(u)|^\frac{1}{\gamma})^\frac{4k}{6k-1}, [\phi^3, |f(u)|^6]^{\frac{2k-1}{6k-4}} =
$$

$$
= C\left(\varepsilon^{6k-1}(\|\phi\|_{\varepsilon}, |f(u)|^\frac{1}{\gamma})^{2\kappa}\right)^\frac{2}{6k-1}(\|\phi^3, |f(u)|^6\|^{1/3})^{\frac{6k-3}{6k-4}} \leq
$$

Inserting this estimate into the right-hand side of (4.93) and using (4.90) and (4.27), the following analogue of inequality (4.35) is arrived at

$$
\frac{d}{dt}E_{\phi_\varepsilon}(t) + \beta\|\nabla x\|_{L^2_{\phi_\varepsilon}}^2 \leq C\varepsilon^2E_{\phi_\varepsilon}(t) + C\varepsilon^{6k-1}[E_{\phi_\varepsilon}(t)]^{2\kappa} + C\varepsilon^{-3}(\|g\|_{L^6_{\phi_\varepsilon}}^2 + 1). \quad (4.95)
$$

As in the proof of Theorem 4.1, this inequality implies the desired estimate (4.91). Indeed, introducing $V_\varepsilon(t) := \varepsilon^3E_{\phi_\varepsilon}(t)$, multiplying by $\varepsilon^3$ and eliminating the first term in the right-hand side via Young’s inequality,

$$
\frac{d}{dt}V_\varepsilon + \beta\varepsilon^3\|\nabla x\|_{L^2_{\phi_\varepsilon}}^2 \leq \varepsilon^2V_\varepsilon^{2\kappa} + C(\|g\|_{L^6_{\phi_\varepsilon}}^2 + 1), \quad V_\varepsilon(0) \leq C(1 + \|g\|_{L^6_{\phi_\varepsilon}}^2 + \|u_0\|_{\phi_\varepsilon}). \quad (4.96)
$$

As in the proof of Theorem 4.1, if

$$
\varepsilon^2V_\varepsilon^{2\kappa} \leq C(1 + \|g\|_{L^6_{\phi_\varepsilon}}^2 + \|u_0\|_{\phi_\varepsilon}) \quad (4.97)
$$

then it can be concluded from (4.96) that

$$
V_\varepsilon(t) \leq 2C(T + 1)(1 + \|g\|_{L^6_{\phi_\varepsilon}}^2 + \|u_0\|_{\phi_\varepsilon}), \quad t \in [0, T]. \quad (4.98)
$$

Therefore if $\varepsilon = \varepsilon(T, u_0, g)$ is fixed by

$$
\varepsilon := \frac{1}{[2(T + 1)]^\kappa[C(1 + \|g\|_{L^6_{\phi_\varepsilon}}^2 + \|u_0\|_{\phi_\varepsilon})]^{\kappa-1/2}}. \quad (4.99)
$$

then (4.97) will be satisfied and so will (4.98). Thus,

$$
E_{\phi_{\varepsilon,x_0}}(T) \leq \varepsilon^{-3}V_\varepsilon(T) \leq C(T + 1)^{3\kappa+1}(1 + \|g\|_{L^6_{\phi_\varepsilon}}^2 + \|u_0\|_{\phi_\varepsilon})^{3\kappa-1/2}
$$

and the desired estimate (4.91) follows now by applying the supremum over $x_0 \in \mathbb{R}^2$ and using (2.17). Theorem 4.12 is proved.

The next theorem gives the analogue of Theorem 4.12 for the Cahn-Hilliard-Oono equation with singular potentials.
Theorem 4.13 Let the assumptions (4.89) and (4.90) hold and \( g \in L^6_b(\mathbb{R}^3) \). Then, for every \( u_0 \in \Phi_b \), the Cahn-Hilliard-Oono equation (4.66) possesses at least one global solution \( u(t), t \geq 0 \), (in the sense of (4.6) plus the extra assumption (4.88)) which satisfies the following analogue of (4.43):

\[
\|u(t)\|_{W^{1,2}_b}^2 + \|F(u(t))\|_{L^1_b} + \|\nabla x \mu\|_{L^2_b([t,t+1] \times \mathbb{R}^3)}^2 \leq Q(\|u_0\|_{\Phi_b})e^{-\sigma t} + Q(\|g\|_{L^6_b}), \tag{4.100}
\]

where the monotone increasing function \( Q \) and positive constant \( \sigma \) are independent of \( u_0 \) and \( t \).

The proof of this theorem repeats almost word by word the proof of Theorem 4.8. The only difference is that \( \gamma = 3 + \frac{2}{\kappa-1} \) should be used instead of \( \gamma = 5 \) in the definition of the weight function \( \phi_{\varepsilon(t)}(x) \) and use the refined interpolation inequality (4.94) instead of (4.34). For this reason, it is not presented here.

The uniqueness Theorem 4.5 can also be extended to the singular case. However, this requires the control of the derivative \( f'(u) \) through \( f(u) \) or \( F(u) \) and assumptions (4.45) are again not compatible with singular potentials and must be modified. For instance, if it is assumed that

\[
|f'(u)| \leq [\Psi(u)]^{\kappa_1}, \quad \Psi(u) \leq C_1 f(u) + C_2, \quad \kappa_1 < 8/5 \tag{4.101}
\]

for some convex function \( \Psi \) and positive \( C_1 \) and \( C_2 \), then arguing as Theorem 3.4 in [27], uniqueness may be established as well as the further regularity of a solution and it may be verified, in particular, that the solution \( u \) becomes separated from any singularities for positive times (\( \|u(t)\|_{L^\infty} \leq 1 - \delta \), for some \( \delta > 0 \)). After that the further investigation of the problem can be constructed exactly as for the case of regular potentials.

Note also that condition (4.101) is stronger than (4.90) which is needed for the global existence of a solution. In particular, for the non-linearities (4.87), \( k_1 > 5/3 \) is needed in (4.101) (instead of \( k_1 > 1 \)).
5 Navier-Stokes Equations

In this section the following damped Navier-Stokes equations in the whole space, $\mathbb{R}^2$, are studied,
\[
\begin{aligned}
\partial_t u + (u, \nabla)u &= \Delta u - \alpha u - \nabla p + g, \\
\nabla \cdot u &= 0, \quad u|_{t=0} = u_0,
\end{aligned}
\]
(5.1)

where $\alpha$ is a positive constant and $g$ is the external forcing term. This equation will be studied under the following conditions
\[
u_0 \in L^2_b(\mathbb{R}^2), \quad \text{div} \, u_0 = 0, \quad g \in \dot{L}^2_b(\mathbb{R}^2), \quad \text{div} \, g = 0, \quad \text{and} \quad \text{curl} \, g \in L^\infty(\mathbb{R}^2).
\]
(5.2)

Under these conditions the following result will be established, which is the central result of this section.

**Theorem 5.1** Under the conditions in (5.2) the solution semi-group of equation (5.1) possesses a finite dimensional globally compact attractor in the space $L^2_b \cap \{\text{div} \, u = 0\}$.

The issue of excluding the pressure is dealt with in an elegant way in [83], building on a technique first developed in [57]. The basic idea is to take the divergence of (5.1), assume the external force $g$ is divergence free and then obtain the following equation,
\[
-\Delta p = \text{div}((u, \nabla)u) = \text{div}((\text{div}(u \otimes u))) \quad (5.3)
\]
where $u \otimes u$ is the tensor product of $u$ with itself. This equation can then be solved using a standard Green’s Function approach. The integration kernel for the Laplacian in two dimensions is $K_{2d}(x) := -\frac{1}{2\pi} \log |x|$ which yields,
\[
p(y) = \int_{\mathbb{R}^2} K_{2d}(y - x) \text{div}((u \otimes u)) \, dx.
\]
(5.4)

This expression can now be integrated by parts twice, only two terms will be shown here, the others follow similarly,
\[
\partial_{x_1} \partial_{x_1} K_{2d}(x) = \partial_{x_1} \partial_{x_1} \left( \frac{1}{2\pi} \log(x_1^2 + x_2^2) \right) = \left( \frac{1}{4\pi} \log \left( \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} \right) \right) = \\
= \partial_{x_1} \partial_{x_1} \left( \frac{1}{4\pi} \log \left( \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} \right) \right) = \partial_{x_1} \left( \frac{2}{4\pi} \log \left( \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} \right) \right) = \\
= \frac{1}{4\pi} \left( \frac{2}{x_1^2 + x_2^2} - \frac{4x_1^2}{(x_1^2 + x_2^2)^2} \right) = \frac{1}{2\pi} \frac{|x|^2 - 2x_1^2}{|x|^4},
\]
and
\[
\partial_{x_1} \partial_{x_2} K_{2d}(x) = \partial_{x_1} \partial_{x_2} \left( \frac{1}{4\pi} \log(x_1^2 + x_2^2) \right) = \partial_{x_2} \left( \frac{1}{4\pi} \frac{2x_1}{(x_1^2 + x_2^2)} \right) = \frac{1}{4\pi} \frac{-4x_1x_2}{(x_1^2 + x_2^2)^2} = \frac{1}{2\pi} \frac{-2x_1x_2}{|x|^4}.
\]

This result can be expressed for all four terms as

\[
p(y) := KH := \int_{\mathbb{R}^2} \sum_{ij} K_{ij}(y-x) u_i(x) u_j(x) dx,
\]

where

\[
H = u \otimes u \quad \text{and} \quad K_{ij}(x) := \frac{1}{2\pi} \frac{|x|^2 \delta_{ij} - 2x_i x_j}{|x|^4};
\]

this expression being derived in section 3 of [83].

However it is not possible to obtain useful estimates for \(p(y)\) as the pressure can be unbounded on an unbounded domain so one more step is needed which is to bound only the gradient of pressure rather than the pressure itself.

So using (5.7) an operator is constructed which is denoted \(\nabla P(u \otimes u)\) such that

\[
\nabla P(u \otimes u) : [L_b^p(\mathbb{R}^2)]^2 \rightarrow [W_{-1,2}^1(\mathbb{R}^2)]^2.
\]

This leads to the following lemma, which is Lemma 3.5 from [83].

**Lemma 5.2** Let the exponents \(1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, H \in [L_b^p(\mathbb{R}^2)]^4\) and \(v \in [W^{1,q}(\mathbb{R}^2)]^2\) be divergence free. Then the following estimate holds,

\[
\left| \langle \nabla P(H), \eta_{R,x_0} v \rangle \right| \leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|H\|_{L^p(B_R^p)} dx \|\eta_{R,x_0}^{\frac{1}{2}} v\|_{L^q}
\]

where \(C\) is independent of \(R\) and \(x_0\), \(\eta_{R,x_0}\) is the cutoff function defined in (2.33) and \(\theta_{R,x_0}\) is defined by (2.4).

See [83] for the proof.

This then leads on to the definition of the weak solution used here, which is the same as Definition 3.2 from [83].

**Definition 5.3** Let the external forces \(g \in L_b^2(\mathbb{R}^2)\) be divergence free. A function \(u(t,x)\) is a weak solution of problem (5.1) if
\[ u \in L^\infty([0, T], L^2_b(\mathbb{R}^2)), \; \nabla u \in L^2_b([0, T] \times \mathbb{R}^2) \]  \hspace{1cm} (5.11)

and the equation is satisfied in the sense of distributions with \( \nabla p = \nabla P(u \otimes u) \) defined in (5.7) and (5.9).

Note that according to the embedding theorem, \( u \in L^4_b([0, T] \times \mathbb{R}^2) \) and \( u \otimes u \in L^2_b([0, T] \times \mathbb{R}^2) \) and due to (5.9), \( \nabla p \in L^2_b([0, T], W_\infty^{-1,2}(\mathbb{R}^2)) \) which means equation (5.1) can be understood as an equality in this space.

See [83] for more details.

### 5.1 Estimates for all time

In this section estimates for the solutions to (5.1) will be derived. These will include a vorticity estimate, a dissipative estimate and a tail estimate which are sufficient to prove the existence of a globally compact attractor in the next section.

Proposition 5.2 from [83] gives the following for weak solutions of (5.1), it is usual parabolic smoothing for solutions which exist locally in time,

**Proposition 5.4**  Let \( u_0, g \in L^2_b \) be divergence free. Then the local weak solution \( u(t) \) becomes smoother, \( u(t) \in W^{1,2}_b(\mathbb{R}^2) \) and the following estimate holds:

\[
\|u\|_{L^\infty([t, T], W^{1,2}_b)} + \|u\|_{L^2_b([t, T], W^{2,2}_b)} \leq C T^{-1/2} Q(\|g\|_{L^2_b} + \|u_0\|_{L^2_b}),
\]

(5.12)

where \( 0 < T(\|u_0\|_{L^2_b}, \|g\|_{L^2_b}) \ll 1 \) and the monotone increasing function \( Q \) is independent of the concrete choice of \( u(t) \).

Once the global existence of solutions is proven it is clear that solutions will satisfy this regularity estimate for all time.

Applying the curl operator to (5.1) and denoting \( w := \text{curl } u \) the following is obtained,

\[
\partial_t w + (u, \nabla)w = \Delta w - \alpha w + \text{curl } g, \; w|_{t=0} = \text{curl } u_0.
\]

(5.13)

As this is a heat equation with a transport term it has a maximum/comparison principle from which the following estimate is derived (which is Proposition 5.3 in [83]).

**Proposition 5.5**  Let \( u_0, g \in L^2_b \) be divergence free, let \( \text{curl } g \in L^\infty(\mathbb{R}^2) \) and let \( \text{curl } u_0 \in L^\infty(\mathbb{R}^2) \). The the following estimate holds for \( w := \text{curl } u \),

\[
\|w(t)\|_{L^\infty} \leq \|\text{curl } u_0\|_{L^\infty} e^{-\alpha t} + \frac{1}{\alpha} \|\text{curl } g\|_{L^\infty}.
\]

(5.14)

Moreover the restriction on the initial conditions of \( u \) can be removed using the classical smoothing estimates for the heat equation. Proposition 5.4 from [83] states;
Proposition 5.6 Let \( u_0, g \in L^2_b \) be divergence free and \( \text{curl } g \in L^\infty(\mathbb{R}^2) \). Then for any weak solution \( u(t) \) of problem (5.1), \( \text{curl } u := w(t) \in L^\infty(\mathbb{R}^2) \) for all \( t \in (0,T] \) and the following estimate holds
\[
\|w(t)\|_{L^\infty(\mathbb{R}^2)} \leq t^{-N}Q(\|u_0\|_{L^2_b}) + Q(\|g\|_{L^2_b} + \|\text{curl } g\|_{L^\infty}) \tag{5.15}
\]
for some positive \( N \) and monotone function \( Q \) which are independent of \( t, u_0 \) and \( g \).

Therefore with \( u_0 \in L^2_b \) by assumption this proposition can be used to show \( \|w(1)\|_{L^\infty} \) is bounded. Time can then be shifted to \( \hat{t} = t - 1 \) so that \( \|w(\hat{t} = 0)\|_{L^\infty} \) is bounded. Then \( \hat{t} \) can be relabelled \( t \) and (5.14) can be used without loss of generality.

The next step is to establish a dissipative estimate for solutions to (5.1). This is Theorem 6.1 from [83].

Proposition 5.7 Let \( u_0, g \in L^2_b \) be divergence free and let \( \text{curl } g \in L^\infty \) then the unique weak solution \( u(t) \) of (5.1) exists globally in time and the following estimate holds
\[
\|u(t)\|_{L^2_b} \leq G(\|u_0\|_{L^2_b}) e^{-\beta t} + G(\|g\|_{L^2_b} + \|\text{curl } g\|_{L^\infty}) \tag{5.16}
\]
where \( \beta \) is a positive constant and \( G \) is a monotone function and both are independent of \( t \) and \( u_0 \).

Proof

In this proof only the key energy estimate will be derived. The existence of solutions can then be shown by a variety of well known methods, for example the Galerkin method, see [83] for more details.

By Proposition 5.6 it can be assumed that \( \text{curl } u_0 \in L^\infty(\mathbb{R}^2) \) and the vorticity estimate may be used from \( t = 0 \).

For this proof the parameter \( R \) will depend on time and so the cutoff and weight functions, which are defined in (2.33) and (2.4) respectively, differentiate in time as follows.
\[
\left| \frac{\partial}{\partial t} \eta_{R(t),x_0}(x) \right| \leq C \frac{|R'(t)|}{R(t)} \eta_{2R(t),x_0}(x), \tag{5.17}
\]
and
\[
\left| \frac{\partial}{\partial t} \theta_{R(t),x_0}(x) \right| \leq C \frac{|R'(t)|}{R(t)} \theta_{R(t),x_0}(x). \tag{5.18}
\]

Now multiply the first equation of (5.1) by \( \eta_{R,x_0} u \), where \( \eta_{R,x_0} \) is defined in (2.33), and integrate over \( \mathbb{R}^2 \). Going term by term, using (2.34) and integrating by parts, construct the following estimates,
\[(\partial_t u, \eta_{R,x_0} u) = \frac{1}{2} \frac{d}{dt} \|u\|_{L^2_x}^2 - \frac{1}{2} \left( |u|^2, \partial_t \eta_{R(t),x_0} \right), \quad (5.19)\]

\[(u, \nabla u)u, \eta_{R,x_0} u) = \int_{R^2} \sum_{i,j=1,2} u_i \partial_i u_j \eta_{R,x_0} u_j \ dx \leq - \frac{1}{2} \int_{R^2} \sum_{i,j=1,2} u_i u_j \partial_i \eta_{R,x_0} \ dx \leq CR^{-1} \|u\|_{L^3_x(B_{2R}^1)}^3, \]

\[-\Delta u, \eta_{R,x_0} u) = (\eta_{R,x_0}, |\nabla u|^2) + (\nabla u, \partial_\eta_{R,x_0} u) = \]

\[(\eta_{R,x_0}, |\nabla u|^2) - \frac{1}{2} (\Delta \eta_{R,x_0}, u) \geq \|\nabla (\eta_{R,x_0} u)\|_{L^2}^2 - CR^{-2} \|u\|_{L^2_x(B_{2R}^1)}^2, \]

\[-C \|u\|_{L^2_x}^2 - (\nabla P(u), \eta_{R,x_0} u), \]

\[(g, \eta_{R,x_0} u) \leq C \|g\|_{L^2_x}^2 + \frac{\alpha}{2} \|u\|_{L^2_x}^2. \]

Now noting that,

\[|(|u|^2, \partial_t \eta_{R(t),x_0})| \leq C \frac{|R(t)|}{R(t)} \|u\|_{L^2_x(B_{2R}^1)}^2, \quad (5.20)\]

this yields in totality,

\[\frac{1}{2} \frac{d}{dt} \|u\|_{L^2_x}^2 + \alpha \|u\|_{L^2_x}^2 + \|\nabla (\eta_{R,x_0} u)\|_{L^2}^2 - CR^{-1} \|u\|_{L^3_x(B_{2R}^1)} \leq \]

\[\leq C \|g\|_{L^2_x}^2 + CR^{-2} \|u\|_{L^2_x(B_{2R}^1)}^3 + CR^{-1} \|u\|_{L^3_x(B_{2R}^1)}^3 + |(\nabla P(u), \eta_{R,x_0} u)|. \quad (5.21)\]

Estimate (5.10) can now be used, together with Hölder and Young’s inequalities, with exponents 3 and \(\frac{3}{2}\), to estimate the pressure term as follows,

\[|\nabla P(u), \eta_{R,x_0} u)| \leq C \int_{R^2} \theta_{R,x_0}(x) \|u \otimes u\|_{L^3_x(B_{2R}^1)} \ dx \cdot \|u\|_{L^3_x(B_{2R}^1)} \leq \]

\[\leq C \left( \int_{R^2} \theta_{R,x_0}(x) \|u \otimes u\|_{L^3_x(B_{2R}^1)} \ dx \right)^{\frac{1}{3}} \left( \int_{R^2} \theta_{R,x_0}(x) \|u\|_{L^3_x(B_{2R}^1)}^3 \ dx \right)^{\frac{2}{3}} \cdot \|u\|_{L^3_x(B_{2R}^1)} \leq \]

\[\leq C \left( \int_{R^2} \theta_{R,x_0}(x) \|u\|_{L^3_x(B_{2R}^1)}^3 \ dx \right)^{\frac{2}{3}} \cdot \|u\|_{L^3_x(B_{2R}^1)} \leq \]

\[\leq C \int_{R^2} \theta_{R,x_0}(x) \|u\|_{L^3_x(B_{2R}^1)}^3 \ dx + CR^{-1} \|u\|_{L^3_x(B_{2R}^1)}^3, \]

where all constants are independent of \(R \gg 1\). This gives in total,
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2_{R,x_0}}^2 + \alpha \|u\|_{L^2_{R,x_0}}^2 + \|\nabla (\eta_{R,x_0} u)\|_{L^2}^2 - C \frac{|R'(t)|}{R(t)} \|u\|_{L^2_{\partial(B^2_R(t))}}^2 \leq (5.23)
\]

\[
\leq C \|g\|_{L^2_{R,x_0}}^2 + CR^{-2} \|u\|_{L^2(B^2_R)}^2 + CR^{-1} \|u\|_{L^3(B^2_{R,x_0})}^3 + C \int_{R^2} \theta_{R,x_0}(x) \|u\|_{L^3(B)}^3 \, dx.
\]

Now denote

\[
Z_{R,x_0}(u(t)) := \int_{y_0 \in \mathbb{R}^2} \theta_{R,x_0}(y_0) \|u(t)\|_{L^2_{R,x_0}}^2 \, dy_0. 
\]

Inequality (5.23) can be multiplied by \(\theta_{R,x_0}\) and integrated over \(\mathbb{R}^2\). Note that

\[
\int_{x_0 \in \mathbb{R}^2} \theta_{R(t),0}(x_0) \frac{d}{dt} \|u\|_{L^2_{\theta(t),x_0}}^2 \, dx_0 = 
\]

\[
\frac{d}{dt} Z_{R(t),0}(u) - \int_{x_0 \in \mathbb{R}^2} \partial_t \theta_{R(t),0}(x_0) \|u\|_{L^2_{\theta(t),x_0}}^2 \, dx_0,
\]

and that this second term can be estimated using (5.18) via

\[
\left| \int_{x_0 \in \mathbb{R}^2} \partial_t \theta_{R(t),0}(x_0) \|u\|_{L^2_{\theta(t),x_0}}^2 \, dx_0 \right| \leq C \frac{R'(t)}{R(t)} Z_{R(t),0}(u). 
\]

Therefore using (2.38) the following can be obtained,

\[
\frac{d}{dt} Z_{R,x_0}(u) + \beta Z_{R,x_0}(u) + 
\]

\[
\left( \beta - K_1 \frac{|R'(t)|}{R(t)} \right) Z_{R,x_0}(u) + 2 \beta \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|_{W^1,2(B_R)}^2 \, dx \leq 
\]

\[
\leq C Z_{R,x_0}(g) + CR^{-1} \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|_{L^3(B_r)}^3 \, dx,
\]

where the positive constants \(C\) and \(\beta\) are independent of \(R\). Three lemmas from [83] are now required to estimate the \(L^3\) term.

**Lemma 5.8** Let the vector field \(u \in [W^{1,2}_{0}(B^2_{x_0})]^2 \) be such that \(\text{div } u, \text{curl } u \in L^\infty(B^2_{x_0})\). Then

\[
\|u\|_{L^3(B^2_{x_0})} \leq C \|u\|_{L^2(B^2_{x_0})}^{\frac{1}{2}} \left( \|\text{curl } u\|_{L^\infty(B^2_{x_0})} + \|\text{div } u\|_{L^\infty(B^2_{x_0})} \right)^{\frac{1}{2}}, \quad (5.28)
\]

where the constant \(C\) is independent of \(R\) and \(x_0\). Moreover, for any \(2 < p < \infty\),

\[
\|u\|_{L^\infty(B^2_{x_0})} \leq C \|u\|_{L^2(B^2_{x_0})}^{\frac{1}{p}} \left( \|\text{curl } u\|_{L^p(B^2_{x_0})} + \|\text{div } u\|_{L^p(B^2_{x_0})} \right)^{1-\frac{1}{p}}, \quad (5.29)
\]

where \(C\) may depend on \(p\), but is independent of \(R\) and \(x_0 \in \mathbb{R}^2\).
For the proof see [83], Appendix 2. Moreover, similarly, the case of additional regularity can be treated.

**Lemma 5.9** Let the vector field \( u \in [W^{1,2}_0(B^R_0)]^2 \) be such that \( \text{div} \, u, \text{curl} \, u \in L^\infty(B^R_0) \). Then

\[
\|u\|_{W^{1,4}(B^R_0)} \leq C \left( \|\text{curl} \, u\|_{L^4(B^R_0)} + \|\text{div} \, u\|_{L^4(B^R_0)} \right)
\]  

(5.30)

where the constant \( C \) is independent of \( R \).

For the proof see [83], Lemma 5.6 and Appendix 2. Additionally bringing another lemma from [83].

**Lemma 5.10** Let the assumptions of Proposition 5.11 hold, the following estimate is valid,

\[
\int_{\mathbb{R}^2} \theta_{R,x_0}(x)\|u\|^3_{L^3(B^R_0)} \, dx \leq \quad (5.31)
\]

\[
\leq CR^3 \left( R^{-1}\|u\|_{L^2_{\eta,R}} + \|w\|_{L^\infty} \right) \int_{\mathbb{R}^2} \theta_{R,x_0}(x)\|u\|^2_{W^{1,2}(B^R_0)} \, dx,
\]

where \( \theta_{R,x_0} \) is defined in (2.4), \( L^2_{\eta,R} \) is defined in (2.13), and \( C \) is independent of \( R \) and \( x_0 \).

**Proof**

Using (5.28) and the cut-off functions (2.33) estimate the \( L^3 \) norm as follows:

\[
\|u\|^3_{L^3(B^R_0)} \leq \|w_{R,x_0}\|^3_{L^2(B^R_0)} \leq \]

\[
\leq C \|u\|^3_{L^2(B^R_0)} (\|\text{curl}(\eta_{R,x_0}u)\|_{L^\infty} + \|\text{div}(\eta_{R,x_0}u)\|_{L^\infty})^{\frac{3}{2}} \leq \]

\[
\leq C \|u\|^3_{L^2(B^R_0)} (CR^{-1}\|u\|_{L^\infty(B^R_0)} + C\|w\|_{L^\infty(B^R_0)})^{\frac{3}{2}} \leq \]

\[
\leq C \|u\|^3_{L^2(B^R_0)} \|w\|_{L^\infty}^{\frac{3}{2}} + CR^{-\frac{3}{2}}\|u\|^3_{L^2(B^R_0)} \|u\|_{L^\infty(B^R_0)},
\]

where the constant \( C \) is independent of \( R \) and \( x_0 \). Analogously, using estimate (5.29), with \( p = 4 \), and the fact that \( W^{1,2} \) is embedded in \( L^4 \), obtain

\[
\|u\|_{L^\infty(B^R_0)} \leq \|\eta_{2R,x_0}u\|_{L^\infty} \leq C \|u\|_{L^2(B^R_0)}^{\frac{1}{4}} (\|w\|_{L^4(B^R_0)} + R^{-1}\|u\|_{L^4(B^R_0)})^{\frac{3}{4}} \leq \]

\[
\leq C \|u\|^\frac{3}{4}_{W^{1,2}(B^R_0)} (\|w\|_{L^\infty}^{\frac{1}{4}}\|u\|_{L^2(B^R_0)}^{\frac{1}{2}} + R^{-\frac{3}{2}}\|u\|_{L^2(B^R_0)}) \leq \]

\[
\leq CR^{\frac{3}{2}}\|w\|_{L^\infty} + C\|u\|_{W^{1,2}(B^R_0)},
\]

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where the constant $C$ is independent of $R$ and $x_0$. Inserting this estimate into the right hand side of (5.32) arrive at

$$
\|u\|_{L^3(B^R_{0_0})}^3 \leq C\|u\|_{L^2(B^R_{40})}^2 \|w\|_{L^\infty} + CR^{-\frac{1}{2}} \|u\|_{L^2(B^R_{40})} \|u\|_{W^{1,2}(B^R_{40})} \leq (5.34)
$$

where the constant $C$ is independent of $R$ and $x_0$ and recalling that $L^2_{b,R}$ is defined in (2.13). Finally multiply through by $\theta_{R,x_0}$, integrate over $\mathbb{R}^2$ and use (2.38) to establish the result.

Returning to the proof of the proposition insert (5.31) into (5.27) and arrive at the following.

$$
\frac{d}{dt} Z_{R,x_0}(u(t)) + \beta Z_{R,x_0}(u(t)) + \left( \beta - K_1 \frac{|R'(t)|}{|R(t)|} \right) Z_{R,x_0}(u) + (2\beta - KR^{-\frac{1}{2}}(R^{-1}\|u\|_{L^2_{R,R}} + \|w\|_{L^\infty})) \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|_{W^{1,2}(B^R_{40})}^2 \, dx \leq CZ_{R,x_0}(g),
$$

where the positive constants $C, K$ and $\beta$ are independent of $R$ and $x_0$ and

$$
Z_{R,x_0}(u(t)) := \int_{y_0 \in \mathbb{R}^2} \theta_{R,x_0}(y_0) \|u(t)\|_{L^2_{b,R,x_0}}^2 \, dy_0
$$

where $\theta$ is defined in (2.4).

If the following two assumptions are true, firstly

$$
\frac{|R'(t)|}{R(t)} \leq \frac{\beta}{K_1},
$$

and

$$
KR^{-\frac{1}{2}}(R^{-1}\|u\|_{L^2_{R,R}} + \|w\|_{L^\infty}) \leq 2\beta,
$$

then the third and fourth terms of (5.35) can be dropped and the following estimate can be obtained using Grönwall’s inequality,

$$
Z_{R,x_0}(u(t)) \leq C\left(Z_{R,x_0}(u(0))e^{-\beta t} + Z_{R,x_0}(g)\right),
$$

where $C$ is a constant. Using Proposition 2.11 this leads to

$$
R(t)^{-1}\|u\|_{L^2_{b,R}(t)} \leq C(\|u_0\|_{L^2_{b}} e^{-\beta t} + \|g\|_{L^2_{b}}),
$$

where $L^2_{b,R}$ is defined in (2.13). This can then be combined with (5.14) to give,
\[ R(t)^{-1} \| u(t) \|_{L^2_{b,R(t)}} + \| w(t) \|_{L^\infty} \leq C(\| u_0 \|_{L^2_b} + \| \text{curl} u_0 \|_{L^\infty}) e^{-\beta t} + \| g \|_{L^2_b} + \| \text{curl} g \|_{L^\infty} ) \]

(5.41)

If the function
\[ t \to \sup_{x_0 \in \mathbb{R}^2} Z_{R,x_0}(u(t)) \]

is continuous then it can be concluded that (5.37) and (5.38) are satisfied if \( R(t) \) is taken as
\[ R(t) = \frac{\beta^2 C^2}{K^2} \left( (\| u_0 \|_{L^2_b} + \| \text{curl} u_0 \|_{L^\infty}) e^{-\gamma t} + \| g \|_{L^2_b} + \| \text{curl} g \|_{L^\infty} \right)^2, \]

(5.43)

which gives
\[ \left| \frac{R'(t)}{R(t)} \right| = \frac{\beta^2 C^2}{K^2} \left( (\| u_0 \|_{L^2_b} + \| \text{curl} u_0 \|_{L^\infty}) e^{-\gamma t} + \| g \|_{L^2_b} + \| \text{curl} g \|_{L^\infty} \right)^2 \leq 2 \gamma, \]

(5.44)

where \( \gamma = \beta \min\{\frac{1}{2}, \frac{1}{K_1}\} \) is chosen such that (5.37) is satisfied.

A priori function (5.42) is not known to be continuous. However if \( g \) and \( u_0 \) have compact support then it can be proven to be continuous. By approximating the infinite energy initial data and forcing term by a sequence of finite energy functions with compact support the continuity of (5.42) can be proven. For more details on this method see [83]. Therefore, as all the assumptions made thus far are justified, equation (5.40) is justified and the following estimate holds,

\[ \| u(t) \|_{L^2_b} \leq \| u(t) \|_{L^2_{b,R(t)}} \leq \]

(5.45)

\[ \leq C \left( (\| u_0 \|_{L^2_b} + \| \text{curl} u_0 \|_{L^\infty}) e^{-\gamma t} + \| g \|_{L^2_b} + \| \text{curl} g \|_{L^\infty} \right)^2 \left( \| u_0 \|_{L^2_b} e^{-\gamma t} + \| g \|_{L^2_b} \right) \]

giving the desired estimate and finishing the proof.

\[ \square \]

The next step is to establish a tail estimate for the solutions for large times, this will be utilised in proving the existence of the global attractor. The following interpolation inequality will be used,

\[ \| u \|_{L^p_b} \leq C\| u \|_{L^2_b}^{\frac{2}{p}}\| u \|_{H^1_b}^{1-\frac{2}{p}} \]

(5.46)

where \( 2 \leq p < \infty \). It can be shown that if \( \| u \|_{L^2_b(B_{|x_0|})} \rightarrow 0 \) as \( |x_0| \rightarrow \infty \) then likewise \( \| u \|_{L^2_b(B_{|x_0|})} \rightarrow 0 \) for any solution \( u(t) \) whose \( H^1_b \) norm is bounded. The first step is to establish the following result for \( L^2_b \).
Proposition 5.11 If $u_0 \in L^2_b$, $g \in \dot{L}^2_b$, $\text{curl } g \in L^\infty$ and $u(t)$ is a weak solution of (5.1) with $u_0$ and $g$ divergence free then, for any $\tau > 0$, there exists a time $T$ and a radius $R$ such that,

$$\|u\|_{L^\infty([t,t+1],L^2_b(\mathbb{R}^2 \setminus B_R^0))} \leq \tau$$

when $t > T$, where $T$ and $R$ depend only on $g$ and $\|u_0\|_{L^2_b}$.

Proof

Begin from (5.39). Split the second term into two pieces, one for a large ball centred at the origin of radius $Q$ (which will be determined later) and another outside that.

$$Z_{R,x_0}(g) = \int_{y_0 \in B_0^Q} \frac{1}{R^3 + |y_0 - x_0|^3} \|g\|^2_{L^2_{R,y_0}} \, dy_0 + \int_{y_0 \not\in B_0^Q} \frac{1}{R^3 + |y_0 - x_0|^3} \|g\|^2_{L^2_{R,y_0}} \, dy_0$$

(5.48)

For the first term note that

$$\|g\|^2_{L^2_{R,y_0}} \leq R \|g\|^2_{L^2_b} \leq CR$$

(5.49)

for some constant $C$. For the second term note that if $g \in \dot{L}^2_b$ (which is a stricter condition than in the proof above) then for any $\delta > 0$ there exists some sufficiently large radius (which will be taken as $Q$) such that

$$\|g\|^2_{L^2_{R,y_0}} < \delta \text{ when } |y_0| > Q.$$  

(5.50)

This gives an estimate

$$Z_{R,x_0}(g) \leq CR \int_{y_0 \in B_0^Q} \frac{1}{R^3 + |y_0 - x_0|^3} \, dy_0 + \int_{y_0 \not\in B_0^Q} \frac{\delta}{R^3 + |y_0 - x_0|^3} \, dy_0,$$

(5.51)

for which there exists a constant $X$ such that $Z_{R,x_0}(g) \leq \frac{\epsilon}{\tau}$ when $|x_0| > X$ for any $\epsilon > 0$.

Returning to (5.39) it is clear that given a sufficiently large $X$ and time $t$ that

$$Z_{R,x_0}(u(t)) \leq \epsilon$$

(5.52)

for any $\epsilon > 0$. Finally using (2.23) and taking a supremum over the domain a tail estimate for $L^2_b$ is obtained.

It is now possible to extend this further to $L^d_b$.

Proposition 5.12 If $u_0 \in L^2_b$, $g \in \dot{L}^2_b$, $\text{curl } g \in L^\infty$ and $u(t)$ is a weak solution of (5.1) with $u_0$ and $g$ divergence free then, for any $\tau > 0$, there exists a time $T$ and a radius $R$ such that,

$$\|u\|_{L^\infty([t,t+1],L^d_b(\mathbb{R}^2 \setminus B_R^0))} \leq \tau$$

(5.53)

when $t > T$.  

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Proof

From (5.46),

\[ \|u\|_{L^4_b} \leq C\|u\|_{L^2_b}^{\frac{1}{2}}\|u\|_{H^1_b}^{\frac{3}{2}}. \]  
(5.54)

Proposition 5.11 bounds the \(L^2_b\) norm and (5.12) bounds the \(H^1_b\) norm giving the result. □

5.2 Existence of the Globally Compact Attractor

The solution semi-group is denoted as

\[ S_h u_0 := u(h), \quad S_h : H_b \rightarrow H_b, \quad H_b := \{u_0 \in [L^2_b(\mathbb{R}^2)]^2, \text{div } u_0 = 0\} \]  
(5.55)

where \(u(h)\) is the unique global weak solution of (5.1), which is proven to exist in [83], Theorems 4.1 and 4.2, for some time \(h > 0\).

A set \(A\) is an attractor for the semi-group \(S_h\) if it satisfies either of the following two definitions from [32]. The first defines a locally compact attractor, the second defines a globally compact attractor. Note that both of these objects are more generally known as "global attractors", see the introduction for more details. They are global because their attracting basin is the whole space. What is at issue in these proofs is the topology in which this attraction takes place. If that topology is local then the attractor is a "locally compact global attractor" which will be called a "locally compact attractor", if that topology is global then the attractor is a "globally compact global attractor", which will be called a "globally compact attractor".

**Definition 5.13** A set \(A \subset H_b\) is the locally compact attractor of solution semi-group \(S_h\) if:

1. The set \(A\) is bounded in \(H_b\) and compact in \(H_{\text{loc}} := \{u_0 \in [L^2_{\text{loc}}(\mathbb{R}^2)]^2, \text{div } u_0 = 0\}\).
2. The set \(A\) is strictly invariant, i.e. \(S_h A = A\) for all \(h > 0\).
3. \(A\) is an attracting set for the semi-group \(S_h\), i.e. for every neighbourhood \(O(A)\) of the attractor in the space \(H_{\text{loc}}\) and every bounded subset \(B \subset H_b\), there exists \(T = T(O, B) \geq 0\) such that

\[ S_h B \subset O(A) \text{ for } h \geq T. \]

**Definition 5.14** A set \(A \subset H_b\) is the globally compact attractor of solution semi-group \(S_h\) if:

1. The set \(A\) is compact in \(H_b\).
2. The set \(A\) is strictly invariant, i.e. \(S_h A = A\) for all \(h > 0\).
3. A is an attracting set for the semi-group $S_h$, i.e. for every neighbourhood $O(A)$ of the attractor in the space $H_b$ and every bounded subset $B \subset H_b$, there exists $T = T(O,B) \geq 0$ such that

$$S_hB \subset O(A) \text{ for } h \geq T.$$  

(see e.g. [14] and [15] for details).

The following corollary, which is Corollary 6.3 in [83], proves the existence of a locally compact attractor.

**Corollary 5.15** Let the assumptions of Proposition 5.7 hold. Then the associated solution semi-group possesses a locally compact attractor $A$, which is generated by all bounded solutions of (5.1) defined for all $t \in \mathbb{R}$:

$$A = K|_{t=0},$$

where $K \subset L^\infty(\mathbb{R},H_b)$, and $H_b := \{u_0 \in [L^2_b]^2, \text{div } u_0 = 0\}$ is the set of all solutions of (5.1) which are defined for all $t \in \mathbb{R}$ and bounded.

For the proof see [83].

This corollary defines the attractor $A$ as the set of all solutions which are bounded for all time, $t \in \mathbb{R}$. Therefore any result which requires a ”sufficiently large time”, such as, for example, Proposition 5.12 can be assumed to hold at $t = 0$ as an infinite amount of time has already passed by this point. For example if $t = N$ is required for the result to hold then $u_0$ can be defined as

$$u_0 := S_N u(-N), \quad \text{(5.56)}$$

and therefore the result holds for $u_0$.

Now it is time to go on to prove the existence of a globally compact attractor.

**Proposition 5.16** Under the conditions in (5.2), the solution semi-group of (5.1) possesses a globally compact attractor $A$ in the topology $\hat{L}^2_b$.

**Proof**

It must be proven that the set of solutions of (5.1) is attracted to the set $A$ and that this set is compact, invariant and attracting.

By Proposition 2.4 a set in $W^{\frac{3}{2},p}_b$ is compact if it is compact when restricted to a ball of finite radius and is arbitrarily small outside this ball. Proposition 5.4 shows the set of solutions is in $H^1_b$ after a short time, which means on the attractor all solutions have this regularity. The space $H^1$ is compactly embedded in $L^2$ on any ball of finite radius. Moreover Proposition 5.12 proves the solutions are arbitrarily small outside a ball of sufficiently large radius $R$. Therefore for any set an $R$ can be chosen such that all solutions
in that set are arbitrarily small outside this large ball. So, on the set \( A \) the conditions of Proposition 2.4 are satisfied.

Proposition 5.7 gives invariance.

Therefore the attractor is compact and invariant.

Finally, arguing as in [32], assume \( \mathbb{A} \) is not an attracting set for \( \mathbb{S}_h \) in the uniform topology of \( H_b \). Therefore there exists a sequence of solutions \( u_n(t) \) belonging to some subset \( B \subset H_b \) and a sequence \( t_n \to \infty \) such that

\[
\text{dist}_{H_b}(u_n(t_n), \mathbb{A}) \geq \beta > 0. \tag{5.57}
\]

Since \( \mathbb{A} \) is a locally compact attractor, then there exists a \( u_0 \in \mathbb{A} \) such that

\[
\|u_n(t_n) - u_0\|_{L^2_t(B^R_0)} \to 0, \text{ for every } R > 0. \tag{5.58}
\]

This combined with Proposition 5.12 contradicts (5.57) and so \( \mathbb{A} \) is an attracting set in the uniform topology of \( H_b \).

Therefore \( \mathbb{A} \) is compact, invariant and attracting and so is the desired globally compact attractor. \( \square \)

### 5.3 Dimension of the Globally Compact Attractor

This section is built on work in [32] and references therein.

**Definition 5.17** For a set \( A \) and any \( \epsilon > 0 \) let \( N_\epsilon(A) \) be the minimal number of balls of radius \( \epsilon \) required to cover the set \( A \).

**Definition 5.18** If a set \( A \) is compact, for any \( \epsilon > 0 \), it can be covered by a finite number of balls of radius \( \epsilon \). The fractal dimension of \( A \), \( D_f \), is defined as

\[
D_f(A) = \lim_{\epsilon \to 0} \frac{\ln N_\epsilon(A)}{\ln(1/\epsilon)}.
\]

This next proposition is a simplified version of Theorem 4.1 from [32] and is reproduced here for convenience.

**Proposition 5.19** Consider any two Banach spaces \( \Phi \) and \( \Psi \) where \( \Psi \) is compact in \( \Phi \) and consider a semi-group evolution \( S_t : \Phi \to \Phi \) with \( S := S_1 \). If a set \( A \subset \Phi \) is compact and invariant under \( S \), that is

\[
S(A) = A,
\]

and for any \( u_1, u_2 \in A \), the following estimates hold:

\[
S(u_1) - S(u_2) = L(u_1, u_2) + K(u_1, u_2), \tag{5.59}
\]
\[ \|L(u_1, u_2)\|_\Phi \leq \kappa \|u_1 - u_2\|_\Phi, \]

and

\[ \|K(u_1, u_2)\|_\Psi \leq C \|u_1 - u_2\|_\Phi, \]

with \( C \) and \( \kappa \) constants with \( \kappa < \frac{1}{2} \) then \( D_f(A) < \infty \).

**Proof**

\( A \) is compact in \( \Phi \) so it can be covered by a single ball of radius \( R \) and centre \( u_0 \). This will be denoted \( A \subset B(R, \Phi, u_0) \). From (5.59), for any \( u \in A \) it is true that,

\[ S(u) - S(u_0) = L(u, u_0) + K(u, u_0). \tag{5.60} \]

Therefore, using the estimates for \( L \) and \( K \),

\[ S(B(R, \Phi, u_0)) \subset B(\kappa R, \Phi, 0) \oplus B(CR, \Psi, Su_0), \tag{5.61} \]

where \( \oplus \) is understood here as the sum of two sets. For any \( \epsilon > 0 \) it is possible to cover any compact set with balls of radius \( \epsilon \). It has been assumed that \( \Psi \) is compact in \( \Phi \). Therefore, for any \( \epsilon > 0 \), \( B(CR, \Psi, Su_0) \) can be covered by a finite number of balls denoted \( B(\epsilon R, \Phi, u_i) \). This gives

\[ S(B(R, \Phi, u_0)) \subset B(\kappa R, \Phi, 0) \oplus \bigcup_i B(\epsilon R, \Phi, u_i). \tag{5.62} \]

However the \( u_i \)'s may not themselves be on the attractor. If, for any \( i \), \( B(\kappa R, \Phi, 0) \oplus B(\epsilon R, \Phi, u_i) \) contains a point on the attractor, labelled \( w_i \), then,

\[ B(\kappa R, \Phi, 0) \oplus B(\epsilon R, \Phi, u_i) \subset B(2\kappa R, \Phi, 0) \oplus B(2\epsilon R, \Phi, w_i), \tag{5.63} \]

and in total for the entire set,

\[ S(B(R, \Phi, u_0)) \subset B(2\kappa R, \Phi, 0) \oplus \bigcup_j B(2\epsilon R, \Phi, w_i), \tag{5.64} \]

with the \( w_i \)'s definitely on the attractor.

To ensure this new net is constructed from balls smaller than the original net (which was made of a single ball of size \( R \)) the following condition must be satisfied,

\[ 2R(\kappa + \epsilon) < R, \text{ which implies } \kappa < \frac{1}{2}, \tag{5.65} \]

recalling that \( \epsilon \) can be chosen to be arbitrarily small.

Now let \( \gamma := 2(\kappa + \epsilon) < 1 \). The number of balls required to construct this new covering is

\[ N_\gamma B(R, \Phi, u_0) \cap A = N_\gamma B(R, \Phi, 0) \cap A = \]

\[ = N_\gamma B(1, \Phi, 0) \cap A := N < \infty. \]

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Repeating this argument for each ball of radius \( \gamma R \) in \( \Phi \) it is possible to cover \( A \) with \( N_{\gamma R} = N^3 \) balls and, repeating it again \( p \) times it is possible to cover \( A \) with \( N_{\gamma^p R} = N^p \) balls. Applying the definition above it can be deduced that,

\[
D_f(B) \leq \lim_{p \to \infty} \frac{\ln N^p}{\ln \left( \frac{1}{\gamma^p} \frac{1}{R} \right)} = \lim_{p \to \infty} \frac{\ln N}{\ln \left( \frac{1}{\gamma R} \right)} = \frac{\ln N}{\ln \gamma} < \infty.
\]

(5.67)

\( \square \)

### 5.4 Estimates on the attractor

In this section additional estimates will be established for solutions on the attractor. In the next section, using these estimates the conditions of Proposition 5.19 will be shown to hold and the final result will be established.

This next proposition will gather the results obtained for the regularity of solutions on the attractor.

**Proposition 5.20** Solutions to (5.1) are attracted to the attractor \( A \). Solutions on the attractor satisfy the following estimates,

\[
\| u \|_{W^{1,4}_b} \leq C \left( \| \text{curl } u \|_{L^4_b} + \| \text{div } u \|_{L^4_b} \right) \leq C \| \text{curl } g \|_{L^\infty},
\]

(5.68)

by Lemma 5.9 combined with Proposition 5.5,

\[
\| u \|_{L^\infty([t,t+1],W^{1,4}_b)} + \| u \|_{L^2([t,t+1],L^2_b)} \leq C,
\]

(5.69)

by Proposition 5.4 and, for any \( \tau > 0 \), there exists a sufficiently large radius \( R \) such that,

\[
\| u \|_{L^\infty([t,t+1],L^4_b(\mathbb{R}^2 \setminus B^R_0))} \leq \tau,
\]

(5.70)

by Proposition 5.12.

The next step is to decompose the equation into two parts. To do this take the difference between the equations for \( u_1 \) and \( u_2 \) and defining \( v = u_1 - u_2 \) to get

\[
\partial_t v + (u_1 \cdot \nabla)v + (v \cdot \nabla)u_2 = \Delta v + \nabla P - \alpha v
\]

(5.71)

\[
\text{div } v = 0, \quad v|_{t=0} = u_1^0 - u_2^0.
\]

Now use the identity \((u \cdot \nabla)u = \nabla \cdot (u \otimes u)\) and cutoff functions \( \chi_{Q,x_1} + \eta_{Q,x_1} = 1 \), defined in (2.33) for some sufficiently large \( Q \) which will be chosen later (just after equation (5.91)) and also any \( x_1 \) which is on the attractor and define \( v = w + z \) to split (5.71) into two,

\[
\partial_t w + \nabla \cdot (\chi_{Q,x_1} (u_1 \otimes w + w \otimes u_2)) = \Delta w + \nabla P_w - \alpha w
\]

(5.72)

\[
\text{div } w = 0, \quad w|_{t=0} = v|_{t=0},
\]

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and

\[
\partial_t z + \nabla \cdot (\eta_{Q,x_1} (u_1 \otimes w + w \otimes u_2)) + \nabla \cdot (u_1 \otimes z + z \otimes u_2) = \Delta z + \nabla P_z - \alpha z \tag{5.73}
\]

\[
\text{div } z = 0, \quad z|_{t=0} = 0,
\]

where \(\nabla P_w\) and \(\nabla P_z\) are unknown pressure terms sufficient to ensure the divergence free conditions are maintained.

Now it is important to obtain estimates for the solutions to these decomposed equations.

**Proposition 5.21** Under the conditions in (5.2) all solutions to equation (5.71) satisfy the following estimate.

\[
\|w(t)\|_{L^2_{t,b}}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_t^{t+1} \|\nabla w\|_{L^2_{\eta,\theta R,x_0}}^2 \, dt \leq C e^{-Ct} \|w(0)\|_{L^2_{t,b}}^2. \tag{5.74}
\]

**Proof**

First to estimate (5.72) multiply by cutoff function \(\eta_{R,x_0} w\) defined in (2.33), integrate over \(\mathbb{R}^2\) and use (2.34) to obtain,

\[
\begin{align*}
(\partial_t w, \eta_{R,x_0} w) &= \frac{1}{2} \frac{d}{dt} \|w\|_{L^2_{\eta,\theta R,x_0}}^2, \\
(\nabla \cdot (\chi_{Q,x_1} (u_1 \otimes w + w \otimes u_2)) \cdot \eta_{R,x_0} w), \\
(\Delta w, \eta_{R,x_0} w) &= - (\nabla w, \nabla (\eta_{R,x_0} w)) = - \|\nabla w\|_{L^2_{\eta,\theta R,x_0}}^2 \leq (\nabla P_w, \eta_{R,x_0} w), \\
(\alpha w, \eta_{R,x_0} w) &= \alpha \|w\|_{L^2_{\theta R,x_0}}^2,
\end{align*}
\]

which, when combined, give

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2_{\eta,\theta R,x_0}}^2 + \alpha \|w\|_{L^2_{\eta,\theta R,x_0}}^2 + \|\nabla w\|_{L^2_{\eta,\theta R,x_0}}^2 \leq \frac{1}{2R} \|w\|_{L^2_{\eta,\theta R,x_0}}^2 + \\
+ \frac{1}{2R} \|\nabla w\|_{L^2_{\eta,\theta R,x_0}}^2 + (\nabla P_w, \eta_{R,x_0} w) - (\nabla \cdot (\chi_{Q,x_1} (u_1 \otimes w + w \otimes u_2)) \cdot \eta_{R,x_0} w).
\]

Estimating the non-linearity first integrate by parts and use (2.34) to obtain

\[
\begin{align*}
(\chi_{Q,x_1} (u_1 \otimes w + w \otimes u_2), \nabla (\eta_{R,x_0} w)) \leq \\
(\chi_{Q,x_1} | (u_1 \otimes w + w \otimes u_2), \frac{1}{R} |\eta_{\theta R,x_0} w| + |\eta_{R,x_0} \nabla w|).
\end{align*}
\]
The method for estimating this term is splitting the domain into squares, \( \Box_{i,j} = [i, i + 1] \times [j, j + 1] \), using Hölder’s inequality and then summing over \( \mathbb{Z}^2 \) to return to the whole domain. The estimate for the most difficult term will be shown and the others follow similarly. Thus

\[
(\chi_{Q,x_1} \big| (u_1 \otimes w), |\eta_{R,x_0} \nabla w|) \leq \sum_{i,j \in \mathbb{Z}} \chi_{Q,x_1} |(u_1 \otimes w), |\eta_{R,x_0} \nabla w| \ dx \leq 
\]

\[
\sum_{i,j \in \mathbb{Z}} ||\chi_{Q,x_1} u_1||_{L^4(\Box_{i,j})} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \leq 
\]

\[
\sup_{i,j \in \mathbb{Z}} ||\chi_{Q,x_1} u_1||_{L^4(\Box_{i,j})} \sum_{i,j \in \mathbb{Z}} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})}.
\]

Using Ladyzhenskaya’s inequality on the \( L^4 \) term inside the sum and noting that the \( L^4 \) term outside the bracket is equivalent to \( L^4_b \) the following is obtained

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \sum_{i,j \in \mathbb{Z}} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \leq \quad (5.79)
\]

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \sum_{i,j \in \mathbb{Z}} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \times
\]

\[
\left( ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} + ||\nabla \eta_{R,x_0}||_{L^2(\Box_{i,j})} w\right)^\frac{1}{2} ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})}
\]

and using (2.34) this can be rearranged as,

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \sum_{i,j \in \mathbb{Z}} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} + \quad (5.80)
\]

\[
+ ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})}
\]

Now use Young’s inequality on the first term with powers 4 and \( \frac{4}{3} \) and on the second term with both powers 2 to get

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \sum_{i,j \in \mathbb{Z}} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} + ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} + ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \leq \quad (5.81)
\]

which, noting that \( ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \leq ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \), reduces to

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \sum_{i,j \in \mathbb{Z}} ||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} + ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} \leq \quad (5.82)
\]

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \left(||w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})} + ||\nabla w\eta_{R,x_0}^\frac{1}{2}||_{L^2(\Box_{i,j})}\right) \leq 
\]

\[
C ||\chi_{Q,x_1} u_1||_{L^4_b} \left(||w||_{L^2_R,x_0}^2 + ||\nabla w||_{L^2_R,x_0}^2\right) 
\]
For now set
\[ \tau = \| \chi_{Q,x_1} u_1 \|_{L^4_b} \]  
(5.83)
and once the other terms have been estimated in the same way the following is obtained for the non-linearity,
\[ C_\tau \| w \|_{L^2_{R,R^0}}^2 + C_\tau \| \nabla w \|_{L^2_{R,R^0}}^2 = \]  
(5.84)
Applying Lemma 5.2 to the pressure term the following is deduced,
\[
(\nabla P_w, \eta_{R,x_0} w) = (\nabla P(\chi_{Q,x_1} (u_1 \otimes w + w \otimes u_2), \eta_{R,x_0} w) \leq 
\leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| \chi_{Q,x_1} (u_1 \otimes w + w \otimes u_2) \|_{L^2(B_R^0)} dx \| \eta_{R,x_0}^2 w \|_{L^2} \leq 
\leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \left( \| \chi_{Q,x_1} u_1 \otimes w \|_{L^2(B_R^0)} + 
+ \| \chi_{Q,x_1} w \otimes u_2 \|_{L^2(B_R^0)} \right) dx \| \eta_{R,x_0}^2 w \|_{L^2} \leq 
\leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \left( \| \chi_{Q,x_1} u_1 \|_{L^4(B_R^0)} \| w \|_{L^4(B_R^0)} + 
+ \| \chi_{Q,x_1} u_2 \|_{L^4(B_R^0)} \| w \|_{L^4(B_R^0)} \right) dx \| \eta_{R,x_0}^2 w \|_{L^2}.
\]
Utilising (5.83) it can be argued the above is less than or equal to
\[
C_\tau \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| w \|_{L^4(B_R^0)} dx \| \eta_{R,x_0}^2 w \|_{L^2} \]  
(5.85)
and using Hölder’s and Young’s inequality and (2.32) the following is arrived at
\[
\leq \left( C_\tau^2 \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| w \|_{L^4(B_R^0)} dx \right)^2 + \tau \| w \|_{L^2_{R,R^0}}^2 \leq 
\leq \left( \int_{\mathbb{R}^2} \theta_{R,x_0} dx \right) C_\tau \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| w \|_{L^4(B_R^0)}^2 dx + \tau \| w \|_{L^2_{R,R^0}}^2 \leq 
\leq C R^{-1} \tau \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| w \|_{L^2(B_R^0)}^2 dx + \tau \| w \|_{L^2_{R,R^0}}^2 .
\]
Using Ladyzhenskaya’s inequality and then Young’s inequality to get
\[
\leq C R^{-1} \tau \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \left( \| w \|_{L^2(B_R^0)}^2 + \| \nabla w \|_{L^2(B_R^0)}^2 \right) dx + \tau \| w \|_{L^2_{R,R^0}}^2 \]  
(5.86)
Now return to (5.76) using these estimates and obtain
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2_{L^2_{\rho R,x_0}} + \alpha \| w \|^2_{L^2_{\rho R,x_0}} + \| \nabla w \|^2_{L^2_{\rho R,x_0}} \leq \frac{1}{2R} \| w \|^2_{L^2_{\rho_{2R,x_0}}} + \frac{1}{2R} \| \nabla w \|^2_{L^2_{\rho_{2R,x_0}}} + \tag{5.89}
\]
\[
C \tau \| w \|^2_{L^2_{\rho_{2R,x_0}}} + C \tau \| \nabla w \|^2_{L^2_{\rho_{2R,x_0}}} + CR^{-1} \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| w \|^2_{L^2(B^R_x)} \, dx +
\]
\[
+ C \tau \| w \|^2_{L^2_{\rho_{2R,x_0}}} + CR^{-1} \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \| \nabla w \|^2_{L^2(B^R_x)} \, dx.
\]

Now multiply the whole expression by \( \theta_{R,\cdot}(x_0) \) and integrate over \( x_0 \in \mathbb{R}^2 \) and recall (2.21). Introducing, exactly as above,

\[
Z_{R,y_0}(w) = \int_{\mathbb{R}^2} \theta_{R,y_0}(x_0) \| w \|^2_{L^2_{\rho_{2R,x_0}}} \, dx_0,
\]

(5.90)
to get

\[
\frac{1}{2} \frac{d}{dt} Z_{R,y_0}(w) + \alpha Z_{R,y_0}(w) + Z_{R,y_0}(\nabla w) \leq
\]
\[
C \tau Z_{R,y_0}(w) + R^{-1} \left( C \tau + \frac{1}{2} \right) Z_{2R,y_0}(w) + R^{-1} \left( C \tau + \frac{1}{2} \right) Z_{2R,y_0}(\nabla w).
\]

By (5.70) if \( Q \) is chosen to be large enough then \( \tau \) can be made arbitrarily small. With \( \tau \) small enough and \( R \) big enough it is possible to use (2.12) to cancel all the terms on the right with those on the left. This then, for large enough time, leads on to a dissipative estimate for \( Z_{R,y_0} \),

\[
Z_{R,y_0}(w(t)) \leq e^{-Ct} Z_{R,y_0}(w(0)).
\]

(5.92)

Using (2.37) and then taking a supremum over all values \( x_0 \in \mathbb{R}^2 \) a dissipative estimate for \( w \) can be obtained, namely

\[
\| w(t) \|^2_{L^2_b} + \sup_{x_0 \in \mathbb{R}^2} \int_t^{t+1} \| \nabla w \|^2_{L^2_{\rho R,x_0}} \, dt \leq Ce^{-Ct} \| w(0) \|^2_{L^2_b},
\]

(5.93)
finishing the proof. \( \square \)

**Proposition 5.22** Under the conditions of (5.2) the solutions to equation (5.73) satisfy the following estimate

\[
\| z(t) \|^2_{L^2_{\rho R,x_0}} + \int_0^t \| \nabla z(t) \|^2_{L^2_{\rho R,x_0}} \, dt \leq Ce^{Ct} \| v(0) \|^2_{L^2_b},
\]

(5.94)
where \( \rho_{R,x_0} \) is defined in (2.4) and \( C \) is independent of \( R \) and \( x_0 \).

**Proof**
Multiply the whole of equation (5.73) by \( \eta_{R,x_0} z \) and integrate over the domain to obtain, exactly as in (5.75),

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The non-linear term is essentially the same, yielding

$$\frac{1}{2} \frac{d}{dt} \|z\|^2_{L_{qR,x_0}} + \alpha \|z\|^2_{L_{qR,x_0}} + \|\nabla z\|^2_{L_{qR,x_0}} \leq 0$$  \hspace{1cm} (5.95)$$

$$\leq \frac{1}{2R} \|z\|^2_{L_{qR,x_0}} + \frac{1}{2R} \|\nabla z\|^2_{L_{qR,x_0}} + (\nabla \cdot \eta_{Q,x_1}(u_1 \otimes w + w \otimes u_2), \eta_{R,x_0} z) +$$

$$+ (\nabla \cdot (u_1 \otimes z + z \otimes u_2), \eta_{R,x_0} z) + (\nabla P, \eta_{R,x_0} z).$$

Estimating first the non-linear term involving \( w \) integrate by parts and use Hölder’s inequality to get,

$$\nabla \cdot (\eta_{Q,x_1} u_1 \otimes w + w \otimes u_2), \eta_{R,x_0} z) \leq$$

$$\leq \|\eta_{Q,x_1} u_1\|_{L^\infty} \|\eta_{Q,x_1} w\|_{L_{qR,x_0}} \left( \|\nabla z\|_{L_{qR,x_0}}^2 + \frac{1}{R} \|z\|_{L_{qR,x_0}}^2 \right) +$$

$$+ \|\eta_{Q,x_1} u_2\|_{L^\infty} \|\eta_{Q,x_1} w\|_{L_{qR,x_0}} \left( \|\nabla z\|_{L_{qR,x_0}}^2 + \frac{1}{R} \|z\|_{L_{qR,x_0}}^2 \right) \leq$$

$$\leq C \|\eta_{Q,x_1} w\|_{L_{qR,x_0}}^2 + C_1 \|\nabla z\|_{L_{qR,x_0}}^2 + \frac{C}{R} \|z\|_{L_{qR,x_0}}^2,$$

where \( C_1 \) is small as estimates for \( u_1, u_2 \) in the \( L^\infty \) norm are already obtained in (5.68) \((W^{1,4} \text{ is embedded in } L^\infty)\), and \( Q \) is chosen just after (5.91). The result for the second non-linear term is essentially the same, yielding

$$\nabla \cdot (u_1 \otimes z + z \otimes u_2), \eta_{R,x_0} z) \leq C_1 \|\nabla z\|_{L_{qR,x_0}}^2 + \frac{C}{R} \|z\|_{L_{qR,x_0}}^2.$$

For the pressure again recall (5.10) and obtain

$$\nabla P, \eta_{R,x_0} z) \leq$$

$$\leq C \int_{R^2} \theta_{R,x_0}(x) \|\eta_{Q,x_1} \ (u_1 \otimes w + w \otimes u_2) + (u_1 \otimes z + z \otimes u_2) \|_{L^2(B_R^p)} \ dx \ \|z\|_{L_{qR,x_0}}^2.$$

Again because estimates for \( u_1 \) and \( u_2 \) in the \( L^\infty \) norm have been obtained in (5.68) this reduces to

$$\leq C \int_{R^2} \theta_{R,x_0}(x) \left( \|\eta_{Q,x_1} w\|_{L^2(B_R^p)}^2 + \|z\|_{L^2(B_R^p)}^2 \right) \ dx + C \|z\|_{L_{qR,x_0}}^2,$$

and plugging this in to (5.95) this leaves the following estimate,

$$\frac{1}{2} \frac{d}{dt} \|z\|^2_{L_{qR,x_0}} + \alpha \|z\|^2_{L_{qR,x_0}} + \|\nabla z\|^2_{L_{qR,x_0}} \leq$$

$$\leq C \|z\|_{L_{qR,x_0}}^2 + C_2 \|\nabla z\|_{L_{qR,x_0}}^2 + C \|\eta_{Q,x_1} w\|_{L_{qR,x_0}}^2 +$$

$$+ C \int_{R^2} \theta_{R,x_0}(x) \|\eta_{Q,x_1} w\|_{L^2(B_R^p)}^2 + \|z\|_{L^2(B_R^p)}^2 \ dx.$$
where $C_2$ is small. Multiply the whole equation by $\rho_{R,y_0}(x_0)$ which is defined in (2.4), utilise (2.29) and integrate over $x_0 \in \mathbb{R}^2$. Introduce

$$Y_{R,y_0}(w) = \int_{\mathbb{R}^2} \rho_{R,y_0}(x_0)\|w\|_{L^2_{\rho_{R,x_0}}}^2 \, dx_0.$$  

(5.101)

It is possible to use (2.36) which gives

$$Y_{2R,y_0} \leq CY_{R,y_0}$$  

(5.102)

so it is possible to cancel all terms except four to obtain

$$\frac{d}{dt}Y_{R,y_0}(z) + Y_{R,y_0}(\nabla z) \leq CY_{R,y_0}(z) + CY_{R,y_0}(\eta_{Q,x_1} w)$$  

(5.103)

which yields

$$\|z(t)\|_{L^2_{\rho_{R,x_0}}}^2 + \int_0^t \|\nabla z(t)\|_{L^2_{\rho_{R,x_0}}}^2 \, dt \leq e^{Ct} \|z(0)\|_{L^2_{\rho_{R,x_0}}}^2 + e^{Ct} \int_0^t e^{-Cs} \|\eta_{Q,x_1} w(s)\|_{L^2_{\rho_{R,x_0}}}^2 \, ds.$$  

(5.104)

Considering (5.93) it is possible to argue that the integral on the right hand side is bounded by a constant proportional to $\|v(0)\|_{L^2_b}$ and $Q$, also considering the initial conditions for $z$, which are $z(0) = 0$ as defined in (5.73), to arrive at

$$\|z(t)\|_{L^2_{\rho_{R,x_0}}}^2 + \int_0^t \|\nabla z(t)\|_{L^2_{\rho_{R,x_0}}}^2 \, dt \leq C e^{Ct} \|v(0)\|_{L^2_b}^2,$$  

(5.105)

finishing the proof. \hfill \Box

**Proposition 5.23** Under the conditions of (5.2) the solutions to equation (5.73) satisfy the following estimate

$$\|\nabla z(t)\|_{L^2} \leq C e^{Ct} \|w_0\|_{L^2_b}^2,$$  

(5.106)

where $C$ is independent of $t$.

**Proof**

Multiply equation (5.73) by $-\Delta z$ and integrate over $\mathbb{R}^2$, simplifying a few terms, using (2.34) and integration by parts, obtain the following;

$$(\partial_t z, -\Delta z) = \frac{1}{2} \frac{d}{dt} \|\nabla z\|_{L^2}^2,$$  

(5.107)

$$(\Delta z, -\Delta z) = -\|\Delta z\|_{L^2}^2,$$  

(5.108)

and
\[ (-\alpha z, -\Delta z) = -\alpha \|\nabla z\|_{L^2}^2. \]  
\text{(5.109)}

This yields in total
\[ \frac{1}{2} \frac{d}{dt} \|\nabla z\|_{L^2}^2 + \alpha \|\nabla z\|_{L^2}^2 + \|\Delta z\|_{L^2}^2 = \]
\text{(5.110)}
\[ = (\nabla \cdot \eta_{Q,x_1}(u_1 \otimes w + w \otimes u_2), \Delta z) + 
\quad + (\nabla \cdot (u_1 \otimes z + z \otimes u_2), \Delta z) - (\nabla P_z, \Delta z). \]

The pressure term cancels because \( z \) is divergence free,
\[ (\nabla P_z, \Delta z) = \]
\text{(5.111)}
\[ = (P_z, \text{div}(\Delta z)) = (P_z, \Delta(\text{div} z)) = 0. \]

The four non-linear terms are estimated using the usual estimates for the Navier-Stokes non-linearity and (2.34). That is,
\[ (\nabla \cdot \eta_{Q,x_1}(u_1 \otimes w + w \otimes u_2), \Delta z) \leq \]
\text{(5.112)}
\[ \leq \left( \frac{1}{Q} |\eta_{2Q,0}(u_1 \otimes w + w \otimes u_2)| + |\eta_{Q,x_1} \nabla \cdot (u_1 \otimes w + w \otimes u_2)|, |\Delta z| \right) \leq 
\quad \frac{1}{Q} \|u_1\|_L^\infty \|\eta_{2Q,0} w\|_{L^2} \|\Delta z\|_{L^2} + \|\nabla u_1\|_{L^4} \|\eta_{Q,x_1} w\|_{L^4} \|\Delta z\|_{L^2} + 
\quad + \frac{1}{Q} \|\eta_{2Q,0} w\|_{L^2} \|u_2\|_{L^\infty} \|\Delta z\|_{L^2} + \|\eta_{Q,x_1} \nabla w\|_{L^2} \|u_2\|_{L^\infty} \|\Delta z\|_{L^2} \leq 
\quad \leq C(\|u_1\|^{\frac{2}{3}}_{W^{1,4}} + \|u_2\|^{\frac{2}{3}}_{W^{1,4}})(\|\eta_{2Q,0} w\|_{H^1}^2 + \frac{1}{4} \|\Delta z\|_{L^2}^2)
\]

and, using Ladyzhenskaya’s inequality for the \( L^4 \) term,
\[ (\nabla \cdot (u_1 \otimes z + z \otimes u_2), \Delta z) \leq \]
\text{(5.113)}
\[ \leq \|\nabla u_1\|_{L^4} \|z\|_{L^4} \|\Delta z\|_{L^2} + \|\nabla z\|_{L^2} \|u_2\|_{L^\infty} \|\Delta z\|_{L^2} \leq 
\quad \leq C(\|z\|_{L^2} \|u_1\|^{\frac{2}{3}}_{W^{1,4}} + \|u_2\|^{\frac{2}{3}}_{W^{1,4}})(\|\nabla z\|_{L^2}^2 + \frac{1}{4} \|\Delta z\|_{L^2}^2)
\]

Putting all this together yields,
\[ \frac{1}{2} \frac{d}{dt} \|\nabla z\|_{L^2}^2 + \alpha \|\nabla z\|_{L^2}^2 + \frac{1}{2} \|\Delta z\|_{L^2}^2 \leq \]
\text{(5.114)}
\[ \leq C(\|z\|_{L^2} \|u_1\|^{\frac{2}{3}}_{W^{1,4}} + \|u_2\|^{\frac{2}{3}}_{W^{1,4}})(\|\nabla z\|_{L^2}^2 + C(\|u_1\|^{\frac{2}{3}}_{W^{1,4}} + \|u_2\|^{\frac{2}{3}}_{W^{1,4}})(\|\eta_{2Q,0} w\|_{H^1}^2).
\]

Dropping the positive terms on the left this expression can be recast as,
\[
\frac{d}{dt} \| \nabla z(t) \|_{L^2} \leq C(\| z \|_{L^2} \| u_1 \|_{W^{1,4}}^2 + \| u_2 \|_{W^{1,4}}^2) \| \nabla z \|_{L^2} + \\
+ C(\| u_1 \|_{W^{1,4}}^2 + \| u_2 \|_{W^{1,4}}^2) \| \eta_{2Q0w} \|_{H^1}^2.
\]

(5.115)

Estimate (5.74) bounds the second term,
\[
\frac{d}{dt} \| \nabla z(t) \|_{L^2} \leq C(\| z \|_{L^2} \| u_1 \|_{W^{1,4}}^2 + \| u_2 \|_{W^{1,4}}^2) \| \nabla z \|_{L^2} + \\
+ C(\| u_1 \|_{W^{1,4}}^2 + \| u_2 \|_{W^{1,4}}^2) 2Qe^{-Ct} \| w_0 \|_{L^2}^2.
\]

(5.116)

The size of \( \| z \|_{L^2} \) is bounded by Proposition 5.22 and (5.68) bounds the norms of \( u \), therefore, integrating in time and noting that \( \| \nabla z(0) \|_{L^2} = 0 \) the result is established,
\[
\| \nabla z(t) \|_{L^2} \leq Ce^{Ct} \| w_0 \|_{L^2}^2.
\]

(5.117)

\(\Box\)

5.5 Final result

Before the final proof a simple embedding lemma is required.

**Lemma 5.24** In the whole space \( H^1 \cap L^2_{\rho_{R,0}} \) is compactly embedded in \( L^2_b \), where \( \rho_{R,0} \) is defined in (2.4).

**Proof**

In a bounded domain the result follows from the Sobolev embedding theorem, see Lemma 7.5, (in a bounded domain \( L^2_b \) is equivalent to \( L^2 \) and \( H^1 \) is compactly embedded in \( L^2 \)).

In an unbounded domain take a set \( A \subset H^1 \cap L^2_{\rho_{R,0}} \).

For any \( \epsilon > 0 \) there exists a large radius \( \hat{R} \) such that all \( u \in A \) have \( \| u \|_{L^2(B^\hat{R} \setminus B^\hat{R}_0)} \leq \frac{\epsilon}{2} \) because \( u \in L^2_{\rho_{R,0}} \).

Because \( H^1 \) is compactly embedded in \( L^2(B^\hat{R}) \) for any \( \epsilon > 0 \) the set \( \{ u_{\hat{R}_0}, u \in A \} \) can be covered by finitely many balls of radius \( \frac{\epsilon}{2} \) in \( L^2 \) such that, for any \( u \in A \), \( \| u - u_i \|_{L^2(B^\hat{R}_0)} \leq \frac{\epsilon}{2} \), where \( u_i \) is the centre of the ball which contains \( u \).

Therefore, for any \( \epsilon > 0 \) a finite set of balls can be constructed such that for any \( u \in A \), there exists a \( u_i \) such that,
\[
\| u - u_i \|_{L^2} = \| u - u_i \|_{L^2(B^\hat{R})} + \| u \|_{L^2(B^\hat{R}_0)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

(5.118)

Therefore \( A \) is covered by a finite set of balls of radius \( \epsilon \) in \( L^2_b \). Moreover \( L^2_b \) is complete and so by the Hausdorff criterion \( A \) is compact in \( L^2_b \) and therefore the embedding is compact.

\(\Box\)
Theorem 5.25 Under the conditions in (5.2) the solution semi-group of equation (5.1) possesses a finite dimensional globally compact attractor in $L^2_b \cap \{\text{div } u = 0\}$.

Proof
Recall first that $u_1 - u_2 = v = w + z$ for any two solutions, $u_1$ and $u_2$ of (5.1).

Proposition 5.16 proves the existence of the globally compact attractor.

Label

$$L(u_1, u_2) := w(t), \text{ and } K(u_1, u_2) := z(t)$$

and Propositions 5.21, 5.22 and 5.23 provide the following estimates

$$\|w(t)\|_{L^2_b}^2 \leq C_1 e^{-C_2 t} \|w(0)\|_{L^2_b}^2 = C_1 e^{-C_2 t} \|u_1(0) - u_2(0)\|_{L^2_b}^2,$$  \hspace{1cm} (5.120)

and

$$\|z(t)\|_{L^2_{pr,0}}^2 + \|\nabla z\|_{L^2} \leq C_3 e^{C_4 t} \|u_1(0) - u_2(0)\|_{L^2_b}^2,$$  \hspace{1cm} (5.121)

noting that all four $C$’s are different constants in these estimates. Fix $t$ such that

$$C_1 e^{-C_2 t} \leq \frac{1}{2}$$  \hspace{1cm} (5.122)

and this gives a value for $C_3 e^{C_4 t}$ which is finite. By Lemma 5.24 the space $\Psi = H^1 \cap L^2_{pr,0}$ is compactly embedded in $\Phi = L^2_b$. With this choice of spaces and the above estimates the conditions of Proposition 5.19 are satisfied proving the attractor has finite dimension. □
6 Further Work

In this section some possible directions for further work will be discussed.

The key open problem which relates to all three of the equations above in the whole spaces in which they are considered is whether the solutions remain bounded without the presence of the additional damping term.

In the first section Coupled Burgers’ equations were discussed and the result obtained was that the uniformly local energy norm grows linearly in time. However it is unclear as to whether this result is sharp. In [10] an example where the non-linearity is not gradient is presented and the resulting equation has finite time blow-up. Moreover in the scalar case the equation has a maximum-principle and so global bounds are immediate. It seems likely, though it is not studied here, that the addition of a dissipation term such as $-\alpha u, \alpha > 0$ to the equation would result in a dissipative energy estimate. So the key open problem from a mathematical perspective is whether such a term is necessary for global bounds in time and what happens as such a term tends to zero. From a physical perspective such a dissipative equation has not yet generated much interest and consequently is not well studied.

In the second section the Cahn-Hilliard equation was studied and the estimate obtained was that the uniformly local energy estimate grew as $t$ to the fourth power (or even higher in the case of singular potentials). It is again unknown whether this is sharp. The Cahn-Hilliard-Oono equation was shown above to have a dissipative estimate so the addition of an arbitrarily small dissipation term is enough to ensure the decay of the energy estimate. From one direction it is possible that as $\lambda \to 0$ the estimate will converge to a global bound in time however this is yet to be established and may well not be true. Some progress in this direction was made in [12] where an $L^\infty$ bound for $u$ is obtained under the conditions that the non-linearity is a constant outside a sufficiently large ball. Of course this is insufficient to treat the problems of greatest physical interest, those with singular potentials, but gives some hope that a better estimate might be achievable.

In the third section the Navier-Stokes equations were studied using mainly the results from [83] where again this key problem arises. With the presence of an additional damping term a dissipative estimate is obtained and without such a term the best estimate is growth with the fifth power of time, this itself being an improvement on the previous exponential result. Again what happens in the limit as the dissipative term tends to zero is unknown. Some solid progress in this direction has been made in [40] for $[0, L] \times \mathbb{R}$ for some $L > 0$ with periodic boundary conditions. In that paper a global bound for the solution is shown to exist and, rather more significantly, the vorticity is shown to decay to zero as $t \to \infty$ showing the convergence to a laminar regime equivalent to the heat equation in an appropriate Galilean frame. It is an interesting open question as to whether this result can be extended to the whole space $\mathbb{R}^2$. 
7 Appendix: Inequalities and Embeddings

Here is a list of the inequalities and embeddings used throughout the thesis. A nice review of these inequalities can be found in Appendix B.2 of [37].

Young’s inequality

Lemma 7.1 For any two positive real numbers $a$ and $b$,
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.
\] (7.1)

Proof
The log function is concave and if $t$ is chosen such that $t = \frac{1}{p}$ and $(1-t) = \frac{1}{q}$ it is true that,
\[
\log(ta^p + (1-t)b^q) \geq tp\log(a) + (1-t)q\log b = \log(ab).
\] (7.2)
Taking exponents of both sides establishes the result.

It is also possible to vary the constants in Young’s inequality, so for a positive real number $\epsilon$,
\[
ab = \epsilon a \frac{b}{\epsilon} \leq \epsilon^\frac{a^p}{p} + \frac{b^q}{q}.
\] (7.3)

Hölder’s inequality

Hölder’s inequality is a generalisation of the Cauchy-Schwarz inequality (which occurs when $p = q = 2$).

Lemma 7.2 For two functions $u \in L^p$ and $v \in L^q$ it is true that $uv \in L^1$ and the following estimate holds,
\[
\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^q}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.
\] (7.4)

Proof
Using Young’s inequality,
\[
\frac{1}{\|u\|_{L^p} \|v\|_{L^q}} \int uv \, dx \leq \int \frac{u^p}{p\|u\|_{L^p}^p} + \frac{v^q}{q\|v\|_{L^q}^q} \, dx = \frac{1}{p} + \frac{1}{q} = 1.
\] (7.5)

Poincaré’s inequality (also known as Friedrich’s inequality)
Lemma 7.3 For a function \( u \in H^1_0(\Omega) \), which means \( u|_{\partial \Omega} = 0 \), on a bounded domain \( \Omega \subset \mathbb{R}^n \),

\[
\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}. \tag{7.6}
\]

**Proof**

Let \( u(x^1, x') \in C^\infty_0(\Omega) \) where \( \Omega \subset \mathbb{R}^n \) and \( x' = x^2, x^3 \ldots x^n \). Moreover let \( u(a, x') = u(b, x') = 0 \) for some finite interval \([a, b]\) with \( b > a > 0 \). Then

\[
u(x, x') - u(a, x') = \int_a^x \partial_1^1 u(x, x') \, dx \leq \int_a^b |\partial_1^1 u(x, x')| \, dx \leq (b - a)^2 \int_a^b |\partial_2^2 u(x, x')|^2 \, dx. \tag{7.7}
\]

Squaring both sides and integrating with respect to \( x \) and \( x' \) gives the result,

\[
\|u\|^2_{L^2} \leq (b - a)^2 \|\nabla u\|^2_{L^2}. \tag{7.8}
\]

Finally the result for \( H^1_0 \) can be established by approximation by smooth functions. \( \Box \)

**Interpolation inequality**

Recall in the \( W \) notation the first index is the number of derivatives and the second is the power, \( H^2 = W^{2,2} \) and \( L^6 = W^{0,6} \) etc.

Lemma 7.4 For \( u \in W^{l,p} \) on a smooth domain \( \Omega \subset \mathbb{R}^n \) it is true that,

\[
\|u\|_{W^{l,p}} \leq \|u\|_{W^{l_1,p_1}}^{\theta} \|u\|_{W^{l_2,p_2}}^{1-\theta}, \tag{7.9}
\]

where

\[
l = \theta l_1 + (1 - \theta) l_2, \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}, \tag{7.10}
\]

and \( \theta \in [0, 1] \)

**Proof**

See [75] page 182. \( \Box \)

**Sobolev inequality**

Lemma 7.5 On a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) the space \( W^{l_1,p_1}(\Omega) \) is embedded in \( W^{l_2,p_2}(\Omega) \) in dimension \( n \) if

\[
l_1 \frac{1}{n} - \frac{1}{p_1} \geq l_2 \frac{1}{n} - \frac{1}{p_2}. \tag{7.11}
\]
and if so the following inequality is valid,

\[ \|u\|_{W^{l_2,p_2}} \leq C \|u\|_{W^{l_1,p_1}}. \] (7.12)

Moreover if the inequality is strictly greater than the embedding is compact.

**Proof**

See [75] page 203. \[ \square \]

**Grönwall's inequality**

This inequality is a generalisation of this simple ODE to inequalities,

\[ y' + \alpha y = 0, \text{ therefore } y(t) = e^{-\alpha t}y(0). \] (7.13)

**Lemma 7.6** Let \( u \) and \( v \) be absolutely continuous real valued functions and \( \alpha \) and \( g \) be constants and let \( u \) be differentiable on the interval \([0,t]\), if

\[ u' + v \leq \alpha u + g, \text{ then } u(t + 1) + \int_t^{t+1} v \, dt \leq e^{\alpha(t+1)}u(0) - \frac{1}{\alpha}g. \] (7.14)

**Proof**

This proof is a modification of the proof from [37], also see [44].

First take the following differential and use the inequality in (7.14),

\[ \frac{d}{dt}(u(t)e^{-\alpha t}) = e^{-\alpha t}(u'(t) - \alpha u(t)) \leq e^{-\alpha t}g, \] (7.15)

now integrate this expression to obtain,

\[ u(t)e^{-\alpha t} \leq u(0) + \int_0^t e^{-\alpha s}g \, ds = u(0) - \frac{1}{\alpha}e^{-\alpha t}g, \] (7.16)

\[ u(t) \leq e^{\alpha t}u(0) - \frac{1}{\alpha}g. \]

Now integrate (7.14) itself to obtain,

\[ u(t + 1) + \int_t^{t+1} v \, dt \leq u(t) + \int_t^{t+1} \alpha u(s) \, ds + \int_t^{t+1} g \, ds \leq \]

\[ e^{\alpha t}u(0) - \frac{1}{\alpha}g + \int_t^{t+1} \alpha e^{\alpha s}u(0) - g \, ds + \int_t^{t+1} g \, ds \leq \]

\[ e^{\alpha(t+1)}u(0) - \frac{1}{\alpha}g. \] (7.17)

\[ \square \]
Lemma 7.7 If \( u \in L^6 \) and \( \nabla u \in L^2 \) then the surprising inequality,

\[
\| u \|_{L^6} \leq C \| \nabla_x u \|_{L^2},
\]

holds in the whole space \( \mathbb{R}^3 \).

Proof

In a ball of radius 1 the Sobolev embedding theorem and Poincaré inequalities give, as a limit case,

\[
\| u \|_{L^6(B_1)} \leq C \| \nabla u \|_{L^2(B_1)}.
\]

Introducing the change of variables \( y = Rx \), giving \( dy = R^3 \, dx \), it is true that,

\[
\nabla_x u(Rx) = R \nabla_y(y)
\]

and therefore

\[
\left( \frac{1}{R^3} \int_{\mathbb{R}^3} |u(y)|^6 \, dy \right)^{\frac{1}{6}} \leq C \left( \frac{R^2}{R^3} \int_{\mathbb{R}^3} |\nabla_y u(y)|^2 \, dy \right)^{\frac{1}{2}},
\]

where the constant \( R \) cancels on both sides. For any function \( u \in C_0^\infty(\mathbb{R}^3) \) this can be extended to the whole space,

\[
\| u \|_{L^6(\mathbb{R}^3)} \leq C \| \nabla u \|_{L^2(\mathbb{R}^3)},
\]

and therefore, using approximation by these smooth functions with compact support, it is true for functions in the closure of \( C_0^\infty(\mathbb{R}^3) \) in the norm \( \| \nabla u \|_{L^2(\mathbb{R}^3)}^2 \).

\[\Box\]
References


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