Protected ground states in short chains of coupled spins in circuit quantum electrodynamics

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The two degenerate ground states of the anisotropic Heisenberg (XY) spin model of a chain of qubits (pseudospins) can encode quantum information, but their degree of protection against local perturbations is known to be only partial. We examine the properties of the system in the presence of nonlocal spin-spin interactions, possibly emerging from the quantum electrodynamics of the device. We find a phase distinct from the XY phase admitting two ground states which are highly protected against all local field perturbations, persisting across a range of parameters. In the context of the XY chain we discuss how the coupling between two ground states can be used to observe signatures of topological edge states in a small controlled chain of superconducting transmon qubits.

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There is much interest in the use of a topological ground-state degeneracy in quantum systems to realize a quantum bit where quantum information can be encoded [1]. In principle, such states can overcome significant obstacles to robust quantum computing since it is theorized that the topological nature of the encoding, essentially very nonlocal, would be protecting against sources of locally acting noise [2]. This scenario arises theoretically under certain conditions in models, such as the Kitaev fermion chain [3] in which the encoded information is protected against fluctuations in local electric potentials and can be relatively easy to isolate against external sources of charge. The Kitaev chain provided inspiration for looking at nonlocal encoding of qubits in the mathematically equivalent spin chain with nearest-neighbor (NN) interactions known as the anisotropic XY spin-chain model where related topologically nontrivial phases can appear [4−7]. Using spins (or pseudospins) to realize such states is attractive since there are several physical systems which can simulate spins, e.g., superconducting qubits, trapped ions, quantum dots, and impurities in silicon. Unfortunately spin chains are coupled to a gapless bosonic environment, and the ground-state qubit is not protected even against local uncorrelated noise. Consequently such noise can couple the two ground states and hence scramble the encoded information.

In this paper we explore a spin chain combining local and nonlocal spin-spin interactions and find that it admits a phase with a fully protected twofold degenerate ground-state manifold and gapped from the higher excited states. We show that, in sharp contrast to the XY chain, the quantum information encoded in these states will remain uncoupled from local external noise. We attribute this protection to the strong suppression of all matrix elements between the ground states for all possible local spin couplings and all the two-body couplings that arise from fluctuations in the Hamiltonian parameters. We demonstrate that the protection persists in a range of parameters remarkably even in short chains of 8−14 qubits which are subject to strong finite-size effects. We explore the boundaries of the phase and discover that it is separated from the XY model by a phase transition, controlled by the anisotropy parameter. Finally, employing a ground-state averaged entanglement entropy as a measure, we find that it acquires a universal value within the new phase, distinct from the universal value of the XY model.

The phase we find is considered here, while other schemes for encoding with different complex spin systems have been of theoretical interest recently [8−10]. Owing to its protection, it avoids many of the shortcomings of the XY model in encoding quantum information in the ground-state manifold. Although a full discussion of its realization is beyond the scope of this investigation, it is inspired by the types of Hamiltonians which typically arise for superconducting qubit-cavity systems. Superconducting qubits may prove an attractive venue due to the precision with which the interqubit coupling can be controlled [11−15] and due to progress in techniques for preparing, simulating, and measuring correlated qubit states [16−20]. Nevertheless, the model is general and can potentially be realized in other physical systems. The protection of the ground-state manifold to all local spin perturbations suggests a certain robustness of the phase against additional couplings that may arise in an actual physical system. Such perturbations are suppressed by the gap to the first excited state as predicted by second-order perturbation theory.

I. A MODEL OF HYBRID LOCAL AND NONLOCAL INTERACTIONS

We consider the following coupled spin-chain model:

\[ H_S = -\frac{1}{2} \sum_{j=0}^{N-2} \left[ (t + \Delta) \sigma_j^x \sigma_{j+1}^x + (t - \Delta) \sigma_j^y \sigma_{j+1}^y \right] \]

\[ + \frac{\lambda_{FF}}{2} \sum_{i,j=0}^{N-1} \left[ \sigma_i^- \sigma_j^+ + \sigma_i^+ \sigma_j^- \right] - \frac{\mu}{2} \sum_{j=0}^{N-1} \sigma_j^z. \]  

(1)

Here NNs’ interaction \( t > 0 \) can be realized, for example, in a system of superconducting qubits coupled to each other capacitively or inductively [6,7]. A route for realizing a general anisotropic coupling was discussed recently [21] (anisotropic interactions are not critical for realizing our proposed phase but play an essential role in simulating the XY model). Related
spin-spin interactions have successfully been realized also in trapped ions [22]. Nonlocal “flip-flop” interactions $\lambda_{FF} > 0$ arise naturally in the context of superconducting circuit QED setups when the qubits are all strongly coupled to a common superconducting resonator and are well detuned from its resonance frequency [23,24]. For a homogeneous case the interactions can be expressed with a total pseudospin operator $S_{i,j}^{x,y,z} = \sum_{i=1,N}^{\sigma_i^{x,y,z}} + \sum_{j=1,N}^{\sigma_j^{x,y,z}} = S_i^x S_j^x + S_i^y S_j^y = 2(S_0^x - S_0^y)$. The magnetic-field $\mu$ usually is taken to be the detuning between the qubit transition frequency and the drive or measurement tones. Spin chains with negative NNs and positive next-to-NN interactions were considered recently in quantum magnetism [25].

We perform an exact diagonalization study of the Hamiltonian $H_S$ and identify a new phase of the spin chain which is most pronounced in the isotropic case of the NN couplings $\Delta \ll t, \lambda_{FF}$ and close to $\mu = 0$, i.e., in the rotating frame. In the following we characterize the main properties of this phase and of its phase boundaries.

II. CHARACTERIZATION OF THE NEW PHASE

The first property of interest is the appearance of a quasidegenerate doublet of ground states which is separated by a gap from the excited states. This occurs for a finite chain with open boundary conditions when the nonlocal coupling strength $\lambda_{FF}$, which couples each spin to $N - 1$ other spins, is on the order of $t/N$ and within a finite range of the anisotropy $\Delta$ close to $\Delta = 0$ (see Fig. 2). For larger values of $\Delta$ it can be seen that the bulk gap closes and reopens at the boundaries of the protected region (see Figs. 2 and 3), suggesting that the nonlocal interactions have introduced a distinct correlated phase. As is apparent from the figure, the XY model also possesses a doublet of ground states. However, as we now turn to discuss, it is distinguished by its degree of protection to external perturbations.

The difference between the (noninteracting) $XY$ phase and the new phase manifests most strikingly in the matrix elements of the local spin operators between the degenerate ground states and their dependence on the system parameters. These coupling patterns for the $XY$ chain are presented in Fig. 1 (left) and are more or less constant and seem fairly unremarkable. The efficient coupling between the two ground states, seen in Figs. 1 and 5, can be understood in the case of $\lambda = 0$, $\Delta = t$ within the quantum Ising model with a transverse field $\mu$. The two ground states are symmetric and antisymmetric superpositions of $| \uparrow \cdots \uparrow \rangle$ and $| \uparrow \cdots \downarrow \rangle$ where $| \uparrow \rangle$ and $| \downarrow \rangle$ denote eigenstates of $\sigma^z$. Hence it is easy to see that a local operator $\sigma_i^x$ at site $i$ can change one ground state into the other. In contrast, when turning $\lambda_{FF}$ on all three spin coupling strengths are suppressed within the new correlated phase, so the ground-state manifold is protected completely against external perturbations of type $\sigma^x\sigma^y\sigma^z$. This manifests as the striking black regions in Fig. 1 (right-hand side) which are calculated for open boundary conditions [26]. Due to this complete cancellation in all spin directions, other choices of basis states must remain decoupled. Interestingly the protected phase survives in some finite region of parameters even in the presence of a moderate amount of 10% disorder in either site or NN coupling energies (see Appendix F). It is also worth noting that the two-body perturbations, most likely to be introduced by fluctuations in the model parameters [6], such as $t, \lambda$, here, do not couple the two degenerate ground states. This is in contrast to what is seen in related models, such as the Majumdar-Ghosh model [27] where interactions extend only to next-nearest neighbors.

We note that the splitting of near-degenerate ground states is much smaller than the other energy scales in the system. By inspecting chains of different lengths $N = 6, 8, 10, 12, 14, 16$ we see that the residual doublet splitting decreases with system size and most features sharpen. This suggests that the residual splitting is due to the finite size of the chain, and it trends towards an exact degeneracy for the longer chains (see Appendix D). This phase indeed requires a long-range interaction in order to appear which however can fall off gradually (see Appendix E). The transition into this phase can be explored both from the direction of reducing the anisotropy $\Delta$ or from the direction of reducing the spin-polarizing energy $\mu$. In the former ($\Delta$) first the degeneracy splits, and the upper state (Fig. 2, blue curve) switches place with another excited state which descends from above (Fig. 2, green curve). In the latter ($\mu$) a transition from a single ground state into a gapped doublet appears at a certain critical value (see Fig. 3), whereas $\Delta$ is zero, distinguishing it from the $XY$ phase. Increasing $\mu$, positively or negatively, eventually leads to the closing of the gap and transition into a more polarized and less correlated states with a single ground state (see circles).

Finally, we explore the bipartite entropy of the ground state which is obtained by splitting the chain to two equal left and right parts and is defined as the von Neumann entropy of the reduced density matrix of one side. In the Kitaev phase, for the mixed state of the two ground states $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ it is quantized at the value of $\ln(2)$ due to the topological nature of the states and their distinct parity symmetries. Here it is modified by the interactions, see Fig. 4, but still approaches the value of $\ln(2)$ asymptotically as $\Delta$ is increased. Interestingly, as $\Delta$ is decreased and the system enters the protected phase

FIG. 1. The effect of introducing cavity-mediated (nonlocal) interactions on the coupling of the ground states. The total coupling $K_j = |\langle 0|\sigma_i^j|1\rangle|^2 + |\langle 0|\sigma_j^0|1\rangle|^2 + |\langle 0|\sigma_j^1|1\rangle|^2$ at the end of the chain (upper) vs at a bulk (bottom) site without (left) vs with (right) additional nonlocal interactions $U_{\sigma_i^\mu,\sigma_j^\alpha}$ for all $i, j$ and $\lambda_{FF} = 0.15$ on an $N = 8$ chain (similar features appear for $N = 6$ and $N = 10$). When the nonlocal interactions are added a region of only very weak total coupling ($\sim 10^{-3} - 10^{-4}$) opens at the central region of the parameters space. We set $t = 1$ for all cases.
FIG. 2. Low-lying energy eigenvalues of the spin chain as a function of parameters show transition into the protected doublet phase shown as a function of decreasing anisotropy $\Delta$ (upper) or increasing nonlocal interactions (bottom) as the bulk gap closes and reopens at the transition point (black circle) leaving a quasidegenerate ground state for a finite range. The plots are of differences between the $n$th lowest-energy level $E_n$ (for $n = 1–5$) and the lowest-energy level $E_0$ for an $N = 8$ spin chain with $\mu = 0$, $t = 1$ as a function of $\Delta$ (top, $\lambda_{FF} = 0.15$) and of $\lambda_{FF}$ (bottom, $\Delta = 0$). The solid arrows show the splitting between nearly degenerate ground states, and the dotted arrows show the bulk gap. Plots are differentiated by colors/shades for clarity, and $t = 1$ for all cases. The cases of $N = 10, 12$ are shown in Appendix D.

FIG. 3. Low-lying energy eigenvalues of the spin chain as a function of the spin-polarizing energy $\mu$ are shown to have different behaviors inside and outside the protected phase. The $n$th lowest-energy level $E_n$ (for $n = 0–5$) for an $N = 8$ spin chain with $\Delta = 0$, $\lambda_{FF} = 0.15$ as a function of $\mu$. A transition from a single ground state into a gapped doublet appears at a certain critical value (black circle) passing through a quasidegenerate ground state for a finite range. The plots are differentiated by colors/shades for clarity, and $t = 1$ for all cases.

FIG. 4. The correlations within the spin chain increase sharply as the system enters the protected phase. This is demonstrated with the entanglement entropy (EE) for partitioning the system into two equal chains, presented as function of the parameter $\Delta/t$. Using the same parameters as in Fig. 2, the EE presents a $2 \ln 2$ plateau within the new phase, which rapidly drops at the phase transition. We set $t = 1$ for all cases.

The same measure of EE jumps to a higher value of $\ln(4)$. It is plausible that the enhanced degree of correlations within the ground states, generated by long-range interactions, make it less sensitive to local perturbations.

Due to the small splitting between the ground states, the system will be found in a thermal mixture state. Initial pure state preparation is conceivable by preferentially driving the system to an excited state from which it decays to the ground states (optical pumping) or by adiabatically turning on the interactions. We observe that the expectation values of $\langle 0 | \sigma_z^j | 0 \rangle$ and $\langle 1 | \sigma_z^j | 1 \rangle$ are generally different in most of the parameter space. Therefore preparation of a ground state followed by a local measurement of $\sigma_z^i$ can reveal the relaxation time. By crossing the protected phase boundary in the $\Delta$ direction, a significant difference should be observed as the states become coupled by external perturbations.

III. COMPARISON TO THE KITAEV MODEL

It is instructive to compare the situation to the fermionic Kitaev chain model where the qubit is encoded in the state of two Majorana zero modes (MZMs) [28–30] and the realization of MZMs in fermionic systems has been discussed extensively [31–35] with signatures compatible with MZMs observed recently [36,37]. Although robust against certain types of local noise sources, some processes are predicted to still cause decoherence of the encoded qubit. The system is mostly sensitive to a fermion bath coupling at low energies. Such baths could couple mostly to the edge of the chain [38–42]. As noted above the $XY$ model was considered for potential realization for the Kitaev chain as it can be straightforwardly simulated in highly controllable quantum systems, such as superconducting qubits [6]. Although formally equivalent to the Kitaev chain, the two models were understood to be physically different due to the nonlocality of the mathematical mapping between them (see Appendix A). Consequently, typical external perturbations affecting the $XY$ spin-chain model would act essentially different as compared to the perturbations affecting the fermionic Kitaev chain. In the former perturbations originate from the coupling to a bosonic reservoir, whereas in the latter from the coupling to a fermion
ground states for the edge ($f_j$) perturbation in the analog spin model (bottom). Transition strengths in injection perturbation (upper) vs coupling induced by a spin-flipping ADAM CALLISON, EYTAN GROSFELD, AND ERAN GINOSSAR PHYSICAL REVIEW B

At each site $j$, where $\ket{0,1}$ are the lowest two states of the system and $a_j, a_j^\dagger$ are fermionic annihilation and creation operators of the spinless chain at site $j$, respectively. The quantity $f_j$, as defined in Eq. (2), has been calculated numerically for an edge site ($j = 0$) and a bulk site ($j = 1$) for an eight-site Kitaev chain. $f_0$ and $f_1$ are shown in Fig. 5. This figure also shows a clear edge effect in the context of the Kitaev chain: It appears much easier to couple the two ground states over most of the parameter range at the edge site than in the middle of the chain. This is consistent with the Majorana fermion (MF) picture: The MFs constituting the MZM are localized individually on the edge sites, rendering them easier to affect there. In fact, it can be seen that the edge site transition is strongest at the ideal points ($\mu = 0, \Delta = \pm 1$) where localization is perfect; away from this point, the MFs decay into the bulk of the chain, and $f_0$ is reduced. Conversely, the bulk site transition $f_1$ is 0 at the ideal point where it can have no effect on the MFs and gradually increases away from this point as the MFs begin to decay into the bulk. This is consistent with the conductivity measured in an experiment, such as the one in Ref. [37]. Importantly, this response could be observed in the spin chain by recognizing that $(a_j^\dagger, a_j)$ translate into the fictitious spin perturbations $[\prod_{j=1}^{j+1} -\sigma_j^x(\sigma_j^+, \sigma_j^-)]. $ These are experimentally accessible for $j = 0$ by perturbing with $(\sigma_j^+, \sigma_j^-)$, respectively, and $j = 1,$ by perturbing with $-\sigma_j^z(\sigma_j^+, \sigma_j^-).$ Extracting the response of the spin system is a test for the Kitaev chain behavior in the spin system.

As noted above, local spin perturbations can strongly couple these ground states in the $XY$ model. This can be seen and studied through the quantity $S_j^z$, defined in Eq. (3),

$$S_j^z = \braket{0,1}_j^z, \quad (3)$$

which has been calculated numerically for parameters in the range of $-2 \leq \mu < 2$ and $-1 \leq \Delta < 1$ for an edge site ($j = 0$) and a bulk site ($j = 1$) for an eight-site spin chain. $S_0^z$ and $S_1^z$ are shown in Fig. 5 (bottom). Although for negative $\Delta$ the states appear uncoupled for $\sigma_j^x$, they are coupled for the $\sigma_j^z$ in this region, namely, the graph is symmetrically inverted. For $\sigma_j^x$ the states are uncoupled, however, overall the states are unprotected for the spin-$XY$ model.

Figure 5 also shows some edge effect for the local spin operator $\sigma_j^x$. The difference in the response between the edge and the bulk for spins can be shown to be directly related to the edge-localized Majorana wave function in the analog fermionic chain using the Bogoliubov–de Gennes (BdG) formalism (see Appendices A–C).

In conclusion, the results indicate the existence of a phase with a highly correlated spin ground-state doublet in the presence of nonlocal interactions. We studied the properties of this phase and discovered a remarkable resilience to perturbations that extends beyond the protection offered by the $XY$ model. These properties are suggestive of a topological phase which is distinct from the phase of the $XY$ model. This phase may offer resource-efficient quantum state encoding with enhanced protection against uncorrelated local perturbations. The nonlocal interactions appear naturally in circuit quantum electrodynamics but may be realized in other systems with effective long-range exchange interactions.

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APPENDIX A: BOGOLIUBOV–DE GENNES METHOD

1. Quasiparticle operators in the local operator basis

For completeness we review the BdG method, which is used to obtain Fig. 5. The Kitaev Hamiltonian, defined with
fermion creation and annihilation operators $a_j^\dagger, a_j$ for site $j$ is

$$H = -\mu \sum_{j=0}^{N-1} (a_j^\dagger a_j - \frac{1}{2}) \quad + \sum_{j=0}^{N-2} \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j \quad - \sum_{j=0}^{N-2} t(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j),$$

and can be written as

$$H = \mathcal{C}^\dagger \mathcal{H} \mathcal{C},$$

where

$$\mathcal{C} = \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \\ a_0^\dagger \\ \vdots \\ a_{N-1}^\dagger \end{pmatrix}.$$

and $\mathcal{H}$ is a $2N \times 2N$ (where $N$ is the number of chain sites) Hermitian matrix which depends on $\mu$, $t$, and $\Delta$. Let the $2N$ eigenvalues of $\mathcal{H}$ be written as $E_{a,b}$ for $0 \leq n \leq N-1$ and be indexed such that $E_a \leq E_b$ if $a < b$. These eigenvalues represent the single-particle spectrum of the Kitaev model, symmetric about eigenvalues $\pm \Delta$. The corresponding eigenvectors $d_{a,n}^\dagger$ where $d_{1,a}^\dagger = d_{a,0}^\dagger$ are the creation and annihilation operators for the elementary excitation quasiparticles, expressed in the $a_n, a_n^\dagger$ basis. This picture is illustrated in Fig. 6.

The unitary transformation $U$, which diagonalizes $\mathcal{H}$ with $U^\dagger \mathcal{H} U$ can be written as

$$U = \begin{pmatrix} u_0^{(-N+1)} & \ldots & u_0^{(-0)} & u_0^{(+0)} & \ldots & u_0^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(-N+1)} & \ldots & u_1^{(-0)} & u_1^{(+0)} & \ldots & u_1^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_0^{(-N+1)} & \ldots & v_0^{(-0)} & v_0^{(+0)} & \ldots & v_0^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_1^{(-N+1)} & \ldots & v_1^{(-0)} & v_1^{(+0)} & \ldots & v_1^{(N-1)} \end{pmatrix},$$

and so the fermionic quasiparticle operators can be found by the matrix-vector multiplication,

$$\begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \\ a_0^\dagger \\ \vdots \\ a_{N-1}^\dagger \end{pmatrix} = U \begin{pmatrix} d_{N-1} \\ \vdots \\ d_0 \\ d_{N-1}^\dagger \\ \vdots \\ d_0^\dagger \end{pmatrix}.$$

It is worth noting that, by comparing $d_{a,n}^\dagger = \sum_{i=0}^{N-1} u_i^{(a,n)} a_i + v_i^{(a,n)} a_i^\dagger$ with $(d_{a,n})^\dagger = d_{a,n}^\dagger = \sum_{i=0}^{N-1} u_i^{(a,n)} a_i^\dagger + v_i^{(a,n)} a_i$, it can be seen that $u_i^{(a,n)} v_i^{(a,n)} = v_i^{(a,n)} u_i^{(a,n)}$.

2. Ground-state coupling for fermions

To contrast the simplicity of applying the BdG method to the fermion system with the complexity of doing the same to the spin system, we first turn our attention to the quantity $f_j = |\langle 0|a_j^\dagger |1 \rangle|^2 + |\langle 0|a_j|1 \rangle|^2$, which we write as

$$f_j = |f_j^+|^2 + |f_j^-|^2,$$

where $(f_j^+, f_j^-) = \langle 0|a_j^\dagger, a_j|1 \rangle$. The operators of interest are

$$(a_j^\dagger, a_j) = \sum_{n=0}^{N-1} (u, v_j)^{(a,n)} d_n + (u, v_j)^{(a,n)} d_n^\dagger,$$

and so $(f_j^+, f_j^-) = \langle 0| \sum_{n=0}^{N-1} (u_j^{(-n)}, v_j^{(-n)}) d_n d_n^\dagger + (u_j^{(n)}, v_j^{(n)}) d_n^\dagger d_n |0 \rangle$. Since the $d_n$’s (for all $0 \leq n < N - 1$) are eigenoperators of $\mathcal{H}$ and $|0\rangle, |1\rangle$ are eigenstates of $\mathcal{H}$, the only nonvanishing term in the sum is the one containing $d_0 d_0^\dagger$. Thus, $f_j^+, f_j^- = u_j^{(-0)}, v_j^{(-0)}$.

An alternative way to achieve the same expression, which involves the BCS correlators, is to instead use the substitution

FIG. 6. A schematic of the quasiparticle spectrum in the Kitaev chain model. The quasiparticles are created ($d_j^\dagger$) with a minimal energy of $E_1 = \Delta$. 

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\[ d_0^j = \sum_{n=0}^{N-1} n u_n^{(+0)} a_n + v_n^{(+0)} a_0^j. \]

Thus, \[ f_j^+ + f_j^- = (0 \sum_{n=0}^{N-1} n u_n^{(+0)} (a_n a_0^j + v_n^{(+0)} (a_n a_0^j) |0 \rangle = \sum_{n=0}^{N-1} u_n^{(+0)}(F_{i,j} C_{j,i}) + v_j^{(+0)}(C_{i,j} F_{i,j}), \]

where \[ (C_{i,j}, F_{i,j}) = (0 | a_i^j a_i^j a_i|^0 \rangle = \sum_{n=0}^{N-1} v_j^{(-n)}(u_j^{(-n)} a_i a_i^j)^0), \]

and

\[ F_{i,j}' C_{i,j} = (F_{i,j} \delta_{ij} - C_{j,i}) = (0 | a_i a_i a_i|^0 \rangle = \sum_{n=0}^{N-1} a_i^{(-n)}(u_j^{(-n)} a_i a_i^j)^0) \]

Thus, \[ f_j^+ + f_j^- = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} n u_n^{(+0)} (u_j^{(-m)} a_j^{(m)} a_j^{(m)} v_j^{(-m)} a_j^j \]

\[ = \sum_{m=0}^{N-1} (u, v)^{(m)} a_j^{(m)} a_j^j \]

\[ = \sum_{m=0}^{N-1} (u, v)^{(m)} (d_m^+, d_m^0) \]

Additionally, this equivalence proves that

\[ (u_i^{(-0)}, v_i^{(-0)}) = \sum_{j=0}^{N-1} u_j^{(+0)} (F_{i,j} C_{j,i}) + v_j^{(+0)} (C_{i,j} F_{i,j}). \]

Thus, expressions have been found for the couplings \( f_j^+ + f_j^- \), and thus for \( j \).

3. Ground-state coupling for spins

The BdG formalism can also be useful for spins by recognizing that \( \sigma_j^z \) can be related to \( a_i^j \) and \( a_j \) by the Jordan-Wigner transformation (JWT). That is,

\[ S_j^z = |\langle 0 | \sigma_j^z |1 \rangle|^2 = \frac{1}{2} \prod_{n=0}^{j-1} (2 a_n a_n^0 - 1)(a_n^0 + a_n) |1 \rangle|^2. \]

For the first site of the chain \( j = 0 \), the translation is straightforward

\[ S_0^z = |\langle 0 | a_0^0 + a_0^0 |1 \rangle|^2 = |A_0^+ + A_0^-|^2 \]

where \( A_0^- = \langle 0 | a_0^0 |1 \rangle \) and \( A_0^+ = \langle 0 | a_0^0 |1 \rangle \). At the second site \( j = 1 \), the translation is

\[ S_1^z = |\langle 0 | 2 a_0 a_0^0 - 1)(a_1^0 + a_1) |1 \rangle|^2 \]

\[ = |2 (0 | a_0 a_0^0)(a_1^0 + a_1) |1 \rangle - (A_1^+ + A_1^-)|^2, \]

which is more complicated but still simple enough to analyze. An analysis of this reveals that

\[ S_j^z = \sum_{m=0}^{N-1} u_m^{(+0)} \left[ F_{0m} (F_{01} + C_{01}) - C_{0m} (F_{01} + C_{01}) \right] \]

\[ + (C_{lm} + F_{lm}) (C_{00} - \frac{1}{2}) \]

\[ + \sum_{m=0}^{N-1} v_m^{(+0)} \left[ C_{0m} (F_{01} + C_{01}) - F_{0m} (F_{01} + C_{01}) \right] \]

\[ + (C_{lm} + F_{lm}) (C_{00} - \frac{1}{2}) \] ². \( \text{(A3)} \)

Now, analytical expressions for sites \( j = 0, 1 \) with straightforward interpretations have been found. Using Eq. (A2) multiple times, Eq. (A3) can be simplified to

\[ S_j^z = 2 (F_{01} + C_{01}) u_{0}^{(-0)} - 2 (F_{01} + C_{01}) v_{0}^{(-0)} \]

\[ + 2 \left( C_{00} - \frac{1}{2} \right) \left( u_{1}^{(-0)} + v_{1}^{(-0)} \right)^2 \]

\[ = \sum_{i=0}^{1} u_{i}^{(-0)} u_{i}^{(-0)} + w_{i}^{(-0)} v_{i}^{(-0)}, \]

where

\[ u_{0,0}^{(-0)} = 2 (F_{01} + C_{01}), \]

\[ u_{1,0}^{(-0)} = -2 (F_{01} + C_{01}), \]

\[ w_{0,1}^{(-0)} = w_{1,0}^{(-0)} = 2 C_{00} - 1, \]

or, equivalently,

\[ u_{0,0}^{(-0)} = 2 (F_{01} + C_{01}), \]

\[ u_{1,0}^{(-0)} = -2 (F_{01} + C_{01}), \]

\[ w_{0,1}^{(-0)} = w_{1,0}^{(-0)} = 1 - 2 C_{00}. \]

APPENDIX B: INTRINSIC EDGE SIGNATURES OF THE SPIN CHAIN

If for simplicity we focus on the immediate neighbor of the edge site, the two probes \( (0 | \sigma_0^0 |1 \rangle \) and \( (0 | \sigma_1^0 |1 \rangle \) can be represented by the JWT as \( S_0^z = (0 | a_0^0 + a_1^0 |1 \rangle \) and \( S_1^z = (0 | 2 a_0 a_0^0 - 1)(a_1^0 + a_1^0 |1 \rangle \), respectively. The former can be represented using the BdG formalism as simply the combined strength of the amplitudes of the zero mode \( u_0^{(-0)} + v_0^{(-0)} \) at the edge site, where \( u_i^{(-0)}, v_i^{(-0)} \) are the wave-function amplitudes for the nth single-particle excitation state on site i. The presence of the string operator of the JWT in the latter means that \( \sigma_i^z \) does not simply probe the strength of the wave function

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of the MZM as is. Using the BdG method we can represent this coupling as a weighted sum of the amplitudes $u_{i,j}^{(0)}$, $v_{i,j}^{(0)}$ of the zero mode on all the sites of the chain including the edge,

$$S_1^i = \sum_{j=0}^{1} \left[ u_{i,j}^{(0)} u_{i}^{(0)} + u_{i,j}^{(0)} v_{i}^{(0)} \right],$$

(B1)

where the weight factors can be expressed via the BCS correlation functions $C_{i,j} = \langle 0|a_i^\dagger a_j^\dagger|0\rangle$, $F_{i,j} = \langle 0|a_i^\dagger a_j^\dagger|0\rangle$, $F_{i,j}^{\dagger} = \langle 0|a_i a_j^\dagger|0\rangle$,

$$w_{i,0}^{(0)} = 2(C_{0,1} + F_{0,1}),$$

$$w_{i,0}^{(0)} = -2(F_{1,0}^\dagger - C_{1,0}),$$

$$w_{i,1}^{(0)} = w_{i,0}^{(0)} = 1 - 2C_{0,0},$$

and hence the edge-bulk difference is expected to be much less pronounced.

**APPENDIX C: THE ASYMMETRIC SPIN RESPONSE TO LOCAL SPIN PERTURBATIONS IN THE XY CHAIN**

The asymmetric spin response can be understood in terms of the Kitaev chain by recognizing that $\sigma_j^s$ translates into the fictitious perturbation $\prod_{n=0}^{N-1} 2d_i^\dagger d_i - 1|a_i^\dagger + \frac{\Delta}{\eta} a_i^\dagger a_i \rangle$ at the ideal points. For $\Delta > 0$, this can be shown to be equivalent to $\prod_{n=0}^{N-1} (-i\eta \eta_n \eta_{n+1}) = \eta_0 \prod_{n=1}^{N-1} (-i\eta \eta_n \eta_{n+1}) = \eta_0 \prod_{n=0}^{N-1} (2d_i^\dagger d_i - 1)$. Thus, $S_j^y = |\langle 0|\eta_0 \prod_{n=0}^{N-1} 2d_i^\dagger d_i - 1|1\rangle|^2$. At the ideal point, it is clear that $d_i^\dagger d_i|1\rangle = 0$ for all $0 < i < N$. Thus, $S_j^y = |\langle 0|(-1)^j \eta_0 |1\rangle|^2 = |\langle 0|\eta_0 |1\rangle|^2$ at all $j$. This is equivalent to $S_j^y = |\langle 0|\eta_0 d_i^\dagger d_i |0\rangle|^2 = \frac{1}{2}|\langle 0|\eta_0 d_i |0\rangle|^2$ and thus $S_j^y = \frac{1}{2}|\langle 0|(-2d_i^\dagger d_i - 1)|0\rangle - i\langle 0|\eta_0 |0\rangle|^2 = 1$. If, however, the opposite point is taken, where $\Delta = -1$, it is found that $\sigma_j^y = -i\prod_{n=0}^{N-1} 2d_i^\dagger a_i - 1|\eta_j$. By performing similar MF operator algebra, it can be shown that

$$S_j^y = |\langle 0|\eta_j a_i^\dagger a_i |0\rangle|^2.$$ These two MF operators are completely uncorrelated at the ideal points, and thus $S_j^y = 0$. This explains the asymmetry along $\Delta$ in terms of Majorana formalism. Away from these ideal points, the values move away from 1 and 0 as correlations begin to break or build up for positive or negative $\Delta$, respectively. However, the boundary between these regions is defined sharply around the line of $\Delta = 0$ where the bulk gap closes in the Kitaev model. This is further support for the features being related to the topological order of the Kitaev chain.

**FIG. 7.** The transition towards the protected doublet phase and increased degeneracy is shown as a function of the strength of nonlocal interactions $\lambda_{FF}$ for different chain lengths. The plots show energy differences $(E - E_0)$ between the $n$th level denoted $E$ (plotted for levels $E = E_0$, $n = 1–5$) and the lowest-energy level $E_0$ with parameters $\mu = 0, t = 1, \Delta = 0$ as a function of the interaction strength $\lambda_{FF}$. The range of the parameter $\lambda_{FF}$ where strong degeneracy and the protected phase exist is seen to increase with the length of chain $N$.

**FIG. 8.** The transition towards the protected doublet phase shown as a function of the anisotropy parameter $\Delta$. The difference between the $n$th lowest-energy level $E$ (for $n = 1–5$) and the lowest-energy level $E_0$ for an $N = 10$ spin chain with $\mu = 0, t = 1, \lambda_{FF} = 0.15$.
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FIG. 10. Examples of the influence of disorder in the parameters $\mu$ and $t$ on the region of protection in parameter space. In the upper panel a case with specific realization of disorder (10% variation) in $\mu$ is shown, and in the lower panel the disorder is in the coupling parameter $t$.

APPENDIX D: RESULTS FOR LONGER CHAINS ($N > 8$)

One of the important features that was discussed is the appearance of a gapped phase with a double quasidegeneracy. This happens when long-range interactions of the flip-flop type are added with sufficient strength and the correct positive sign. One of the immediate questions that is raised is whether the degeneracy is not absolute because of a finite-size effect. The trend can be seen clearly, Fig. 7, by looking at the spectra for increasingly large chains, e.g., $N = 8–14$ scanning on the interaction parameter, $\lambda$ indicating that the phase is robust for larger chains.

In Fig. 8 the spectrum is shown as a function of the anisotropy parameter $\Delta$ and when compared to Fig. 2 that even the incremental increase from $N = 8$ to $N = 10$ has a strong effect of strengthening the degeneracy.

APPENDIX E: REQUIRED EXTENT OF THE INTERACTION

Realistic interaction potentials often have an effective range and a gradual decay, and it is instructive to check the influence of the interaction range on the extent of the protected states phase. For example, the results of choosing an exponentially decaying potential are shown in Fig. 9. We observe that a significant extent of the interaction, measured in sites is required in order to observe the protected state but it does not need to cover the whole chain. We have checked other models of algebraic decay, and we find similar results.

APPENDIX F: RESILIENCE AGAINST DISORDER

In realistic physical systems we expect a certain variation in the local Hamiltonian parameters $\mu$, $t$, and $\lambda$. Since inside the protected phase there is a gap between the degenerate ground states and the first excited state we expect that small variations will not destroy the protection against external fields at least within some finite region.

In Figs. 10 and 11 we show an example of this. In Fig. 10 a specific random realization of the Hamiltonian parameters $\mu$, $t$, and $\lambda$ is shown where either $\mu$ or $t$ is varied to an extent of $\pm 10\%$ and the protected region is plotted. We see that even in the presence of such significant disorder there are regions of protection. In Fig. 11 the resilience of the degeneracy is tested and since perturbations will lift it when these are on the order of the spectral gap a variation of $\pm 5\%$ of the gap of $N = 8$ is chosen. It can be seen that the degeneracy and gap remain resilient at this finite level of disorder.


[26] Central site perturbations do couple the ground states, however this coupling diminishes with the chain length and vanishes altogether for the case of periodic boundary conditions.


