Protected ground states in short chains of coupled spins in circuit quantum electrodynamics

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The quasi-degenerate ground state manifold of the anisotropic Ising spin model can encode quantum information, but its degree of protection against local perturbations is known to be only partial. We explain how the coupling between the two ground states can be used to observe signatures of Majorana zero modes in a small controlled chain of qubits. We argue that the protection against certain local perturbations persists across a range of parameters even away from the ideal point. Remarkably, when additional non-local interactions are considered the system enters a phase where the ground states are fully protected against all local field perturbations.

There is much interest in the use of a topological ground-state degeneracy, arising under certain conditions in models such as the Kitaev fermion chain [1], as the basis for a non-locally encoded qubit. This could help overcome a significant obstacle to robust quantum computing [2] by protecting the qubit from some sources of locally-acting noise. In the Kitaev chain the qubit is encoded in the state of the Majorana zero-modes (MZMs) \[ \mathcal{M}_\sigma \], which arise at topological defects which are non-abelian anyons [3] and appears at the edges of the chain. The realisation and detection of MZMs in fermionic systems has been discussed extensively in recent years [4–11], with signatures compatible with MZMs observed in two experimental systems [12, 13]. Although robust against certain types of local noise sources, some processes are predicted to still cause decoherence of the encoded qubit [14–18].

In parallel an alternative approach is being discussed in the direction of realising topological protection with multiple qubits [15]. The realisation of the Kitaev chain model with superconducting qubits has been discussed recently [20] as it can be modelled as chains of coupled spins [21] [22], also referred to as the XY model. This is an attractive direction due to the precision with which the inter-qubit coupling can be controlled [22] and of techniques for preparing, simulating and measuring correlated qubit states [22–27]. However, the perturbations affecting the spin chains are different in nature than the perturbations affecting the fermionic Kitaev chain itself. The former originate from the coupling to a bosonic reservoir while the latter from the coupling to a fermion reservoir. In addition, the MZM quasi-particle operator becomes non-local when translated into the spin representation and no longer has a simple interpretation [22–27]. Understanding the effect of perturbations in the spin chain is important both for the question of decoherence as well as for detecting signatures of edge states in the spin system.

Here we start by comparing the coupling patterns of the degenerate ground states for small, finite size chains of interacting spins (qubits) analogous to those arising in the fermion Kitaev chain model. We show that away from an ideal point of parameter space the robustness against certain local spin perturbations persists and it shows a weak edge vs. bulk effect. It is compared to the strong edge effect of the fermion chain and we demonstrate that the latter could be observed in a small spin chain. When non-local spin-spin interactions are added, the coupling between the degenerate ground states is suppressed for all the local spin coupling channels. This state shows spectral characteristics of an essentially different topological state, namely a closing of the gap at the edges and a quantised entanglement entropy.

We consider the coupled spin-chain model

\[
H_S = -\frac{1}{2} \sum_{j=0}^{N-2} \left[ (t + \Delta) \sigma_j^x \sigma_{j+1}^x + (t - \Delta) \sigma_j^y \sigma_{j+1}^y \right] + \lambda_{FF} \sum_{i,j=1,N} \left[ \sigma_i^z \sigma_j^z + \sigma_i^+ \sigma_j^- \right] - \frac{\mu}{2} \sum_{j=0}^{N-1} \sigma_j^z. \tag{1}
\]

with nearest-neighbour interaction with hopping \( t > 0 \) that can be realised for example in a system of superconducting qubits coupled to each capacitively inductively [20, 21], and a route to realising a general anisotropic coupling was discussed recently [23]. Related spin-spin interactions have been successfully realised also in trapped ions [30]. Non-local interactions \( \lambda_{FF} > 0 \) can arise when the qubits are all strongly coupled to a common superconducting resonator and are well detuned from its resonance frequency [37] [38] and \( \mu \) is usually taken to be the detuning between the qubit transition frequency and the drive or measurement tones. We show below that the effects of the non-local interaction are most pronounced in the isotropic case of the nearest-neighbour couplings \( \Delta \ll t, \lambda_{FF} \) and close to \( \mu = 0 \), i.e. in the rotating frame. We first turn to discuss the case \( \lambda_{FF} = 0 \) in detail.

By performing the Jordan-Wigner transformations described by \( a_j^+ = \prod_{i=0}^{j-1} (-1)^{a_i} \sigma_i^+ \) on Eq. (1) yields the

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Kitaev fermion chain Hamiltonian [1]

\[
H = -\mu \sum_{j=0}^{N-1} (a_j^\dagger a_j - \frac{1}{2}) + \lambda_{FF} \sum_{i,j=1}^{N} \left[ a_i^\dagger T_{ij} a_j + a_i T_{ij} a_j^\dagger \right] - \sum_{j=0}^{N-2} t \left( a_j a_{j+1} + a_{j+1}^\dagger a_j \right) + \sum_{j=0}^{N-2} \Delta \left( a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger \right),
\]

where \(a_j, a_j^\dagger\) are fermionic annihilation and creation operators respectively, \(\mu\) is a chemical potential, \(t\) is a hopping rate between adjacent sites and \(\Delta\) is a pairing amplitude for Cooper pairs entering and exiting the chain to and from the bulk superconductor, and \(T_{ij} = \prod_{n=\min(i,j)}^{\max(i,j)-1} (2a_n a_n^\dagger - 1)\) is the string operator. The system can be formally decomposed into MF operators \(\gamma_j, \eta_j\), where \(\gamma_j^\dagger = \gamma_j, \eta_j^\dagger = \eta_j\) by the substitution \(a_j = \frac{1}{2}(\gamma_j + i\eta_j)\). Taking the parameters of the ideal topological point, \(\mu = 0\) and \(\Delta = t\), reduces Eq. (2) to

\[
H_{\text{ideal}} = 2t \sum_{i=1}^{N-1} \left( d_i^\dagger d_i - \frac{1}{2} \right) \text{ where } d_i = \frac{1}{2}(\eta_j - i\gamma_{j+1}).
\]

The missing operator \(d_0 = \frac{1}{2}(\gamma_0 - i\eta_{N-1})\) accounts for the topological features of the system.

**Signatures of Majorana edge states.** When the system couples to a fermionic bath, a gauge invariant measure for the induced transitions between degenerate ground states is given by the following quantity

\[
f_j = | \langle 0 | a_j^\dagger | 1 \rangle |^2 + | \langle 0 | a_j | 1 \rangle |^2
\]

at each site \(j\), where \(|0\rangle, |1\rangle\) are the lowest two states of the system.

The quantity \(f_j\), as defined in Eq. (3) has been calculated numerically for an edge site \(j = 0\) and a bulk site \(j = 1\) for an 8-site spin chain. \(f_0\) and \(f_1\) are shown in Fig. 1.

Fig. 1 shows a clear edge effect in the context of the Kitaev chain: it appears much easier to couple the 2 ground-states over most of the parameter range at the edge site than in the middle of the chain. This is consistent with the Majorana fermion picture: the MFs constituting the MZM are individually localised on the edge sites, rendering them easier to affect there.

In fact, it can be seen that the edge site transition is strongest at the ideal points \((\mu = 0, \Delta = \pm 1)\), where the localisation is perfect; away from this point, the MFs decay into the bulk of the chain and \(f_0\) is reduced. Conversely, the bulk site transition \(f_1\) is 0 at the ideal point, where it can have no effect on the MFs, and gradually increases away from this point as the MFs begin to decay into the bulk. These phenomena are qualitatively consistent with the conductivity measured in an experiment like the one in [12]. Additionally, these phenomena might be recoverable in the spin-chain by recognising that \(a_j^\dagger, a_j\) translate into the fictitious spin perturbations \(\Pi_{j=0}^{j-1} - \sigma_j^x \) \(\sigma_j^+, \sigma_j^-\). These might be experimentally accessible for \(j = 0\) by perturbing with \((\sigma_j^+, \sigma_j^-)\), respectively, and \(j = 1\), by perturbing with \(-\sigma_0^x, \sigma_0^+\). Extracting the response of the spin-system could constitute a test for Kitaev chain behaviour in the spin system.

The quantity \(S_j^x\), as defined in Eq. (4) has been calculated numerically calculated for parameters in the range \(-2 \leq \mu < 2\) and \(-1 \leq \Delta < 1\) for an edge site \(j = 0\) and a bulk site \(j = 1\) for an 8-site spin chain. \(S_0^x\) and \(S_1^x\) are shown in Fig. 1.

**Intrinsic edge signatures of the spin chain.** In [20], three kinds of perturbations \(\sigma_j^x, \sigma_j^y\) and \(\sigma_j^x \sigma_{j+1}^x\) which affect a superconducting qubits array with nearest-neighbour coupling were discussed, qualitatively. It was argued that \(\sigma_j^x\) can only drive the system out of the ground-space into higher energy states, such that for a sufficiently large bulk gap this perturbation cannot cause decoherence within the ground state sector. It is further argued that the multi-qubit perturbation \(\sigma_j^x \sigma_{j+1}^x\) is usually much smaller than \(\sigma_j^x\), meaning that \(\sigma_j^x\) is the dominant perturbation. The action of this operator on the spin chain is to flip the value of the spin on the site \(j\). The propensity of this perturbation to induce transitions between the ground-states can be characterised by the coupling value

\[
S_j^x = | \langle 0 | \sigma_j^x | 1 \rangle |^2
\]

at each site \(j\). Fig. 1(bottom) shows an edge effect when
the local spin operator $\sigma_j^z$ is calculated. Both plots show a region of strong transition (for positive $\Delta$), but this region is more pronounced for the bulk site than for the edge site. Conversely, both plots show a region of weak transition (for negative $\Delta$), but this region is more pronounced for the edge site than for the bulk site. This increased coupling in the bulk of the chain is an opposite edge effect than that of the Kitaev chain, where the coupling reduces in the bulk of the chain. Nevertheless, the difference in the response between the edge and bulk for spins can be shown to be directly related to the edge-localised Majorana wave function in the analogue fermionic chain. If for simplicity we focus on the immediate neighbor of the edge site, the two probes $\langle 0 | \sigma_0^+ \sigma_1^- | 1 \rangle$ and $\langle 0 | \sigma_1^+ \sigma_1^- | 1 \rangle$ can be represented by the Jordan-Wigner transformation as $s^0_1 = \langle 0 | (a_0^+ a_1^+ + a_1^+ a_0^+) | 1 \rangle$ and $s^1_1 = \langle 0 | (2a_0 a_0^+ - 1)(a_1 + a_1^+) | 1 \rangle$, respectively. The former can be represented, using the Bogoliubov-de-Gennes (BdG) formalism, as simply the combined strength of the wave function amplitudes of the zero modes $u_0^{(-)} + v_0^{(-)}$ and $v_0^{(-)}$ at the edge site, where $u_i^{(-)}$, $v_i^{(-)}$ are the wave function amplitudes for the $n$th single-particle excitation state on site $i$ (see supplementary). The presence of the string operator of the JW transformation in the latter means that $\sigma^z_i$ does not simply probe the strength of the wave function of the MZM as is. Using BdG we can represent this coupling as a weighted sum of the amplitudes $u_i^{(-)}$, $v_i^{(-)}$ of the zero mode on all the sites of the chain including the edge

$$s^1_1 = \sum_{i=0}^{1} \left[ w_{a,i}^{(-)} u_i^{(-)} + w_{v,i}^{(-)} v_i^{(-)} \right]$$  \hspace{1cm} (5)

where the weight factors can be expressed via the BCS correlation functions $C_{i,j} = \langle 0 | a_i^+ a_j^+ | 0 \rangle$, $F_{i,j} = \langle 0 | a_i^+ a_j^+ | 0 \rangle$, $w_{u,0}^{(-)} = 2(C_{0,1} + F_{0,1})$, $w_{v,0}^{(-)} = 2(F_{1,0} - C_{1,0})$, $w_{u,1}^{(-)} = 1 - 2C_{0,0}$ \hspace{1cm} (6)

and hence the edge-bulk difference is expected to be much less pronounced.

Overall, these results illustrate, despite the formal equivalence, just how differently each of the two systems respond to their natural perturbations. In the fermionic system, there are strong coupling regions at the chain ends and weak coupling regions in the bulk, all of which are independent of the sign of $\Delta$. This $\Delta$ symmetry exists because the bulk gap in which the MZM lives depends only on $|\Delta|$, and not on the sign. In the spin-system, however, it can be seen from equation (1) that the sign of $\Delta$ determines the orientation of the interqubit coupling axis in the $x, y$ plane. This determines to which perturbation operators the system is resilient. The most striking difference is the presence of the large, negative-$\Delta$, weak-coupling region at all sites including the chain edge. This is fundamentally different from the fermionic chain, in which edge-site perturbations can easily couple the ground-states. This follows from the realisation [33] that the chain ends are relatively unimportant in the spin-system. The asymmetric spin response can be understood in terms of the Kitaev chain by recognising that $\sigma^z_i$ translates into the fictitious perturbation $\prod_{i=0}^{N-1} a_i^+ a_i - 1$ at the ideal points. For $\Delta > 0$, this can be shown to be equivalent to $\prod_{i=0}^{N-1} (-i \gamma n_i \gamma_n) = \gamma_0 \prod_{i=n=1}^{N-1} (2d_i d_i - 1)$. Thus, $S^z_j = \langle 0 | \gamma_0 \prod_{i=0}^{N-1} 2d_i d_i - 1 | 1 \rangle |^2$. At the ideal point, it is clear that $d_j d_j = 0$ for all $0 < i < N$. Thus, $S^z_j = \langle 0 | \gamma_0 d_i d_i - 1 | 1 \rangle |^2 = \langle 0 | \gamma_0 d_i d_i - 1 | 1 \rangle |^2$ at all $j$. This is equivalent to $S^z_j = \langle 0 | \gamma_0 d_i d_i - 1 | 1 \rangle |^2 = \frac{1}{2} \gamma_0 d_i d_i - 1 | 0 \rangle - i \langle 0 | \gamma_0 d_i d_i - 1 | 0 \rangle |^2$ and thus $S^z_j = \frac{1}{2} \gamma_0 d_i d_i - 1 | 0 \rangle - i \langle 0 | \gamma_0 d_i d_i - 1 | 0 \rangle |^2$. If, however, the opposite point is taken, where $\Delta = -1$, it is found that $\sigma^z_j = -i \prod_{i=0}^{N-1} 2a_i a_i - 1 \eta_i$. By performing similar MF operator algebra, it can be shown that $S^z_j = \langle 0 | \gamma_0 d_i d_i - 1 | 0 \rangle |^2$. These two MF operators are completely uncorrelated at the ideal points, and thus $S^z_j = -1$. This explains the asymmetry along $\Delta$ in terms of Majorana formalism. Away from these ideal points, the values move away from 1 and 0 as correlations begin to break or build up for positive or negative $\Delta$ respectively. However, the boundary between these regions is sharply defined around the line of $\Delta = 0$, where the bulk gap closes in the Kitaev model [33]. This is further support for the features being related to the topological order of the Kitaev chain.

Protected ground states with cavity-mediated interactions. The inclusion of non-local spin-spin interactions has interesting consequences for the quantities that we study and is experimentally feasible for superconducting circuit QED setups where a non-local ‘flip-flop’ interaction can be enabled by the strongly coupled superconducting resonator. Related phases of spin chains with negative nearest neighbour with positive next to nearest neighbour interactions were considered recently in the context of quantum magnetism [40]. For the ideal case of totally homogeneous interactions the interactions of strength $\lambda_{FF}$ can be expressed with a total pseudo-spin operator $J(x,y,z) = \sum_{i=1}^{N} \sum_{x,y,z} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y$ as $\sum_{i,j=1}^{N} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y = S^x S^x + S^y S^y = 2(S^2 - S^z)$ . For a finite chain and when the spin-pinning coupling strength is approximately equivalent to the single particle energy $t/N \sim \lambda_{FF}$, close to zero pairing $\Delta = 0$, an interesting quasi-degenerate ground states appears which is gapped from the rest of the excited states. The quasi-degenerate ground state space persists for a small region of $\mu, \Delta \neq 0$ for a relatively small interaction e.g. $\lambda_{FF} = 0.15$ in a finite region of the $\Delta,t$ parameter space. Interestingly, in this region (see Fig. 3) all three spin
closely coupled strengths are suppressed so that the manifold is completely protected against external perturbations of the type $\sigma_{x,y,z}$ as opposed to the case of the XY-model. It is easy to show that, due to this complete cancellation, also other choices of basis states are also uncoupled.

Looking at the lower part of the many-body energy spectrum, it can be seen that the bulk gap closes and reopens at the boundaries of the protected region (see Figs. 3, 4), suggesting that the non-local interactions has introduced a new correlated phase. It can also be seen that the energy of the splitting of the near-degenerate ground states is much smaller than the other energy scales.

By inspecting chains of different lengths $N = 6, 8, 10$ we see that the doublet splitting decreases and this suggests that it is due to the finite size of the chain. The transition into this phase can be explored both from the direction of reducing the pairing, or anisotropy $\Delta$ or from the direction of reducing the spin polarizing energy, or chemical potential $\mu$. In the former ($\Delta$) first the degeneracy splits and the upper state (Fig. 3 blue curve) switches place with another excited state which descends from above (Fig. 3 green curve). In the latter ($\mu$) a transition from a single ground state into a gapped doublet appears at a certain critical value (see Fig. 4) while the pairing energy $\Delta$ is zero, distinguishing it from the Kitaev phase. Increasing, positively or negatively, the polarising energy $\mu$ eventually leads to the closing of the gap and a transition into a more polarised and less correlated states with a single ground state (see circles). Finally, it is interesting to discuss the entanglement entropy (EE) of the ground state. In the Kitaev phase, for the mixed state of the two ground states $\rho = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|)$ it is quantised at the value of $\log(2)$ due to the topological nature of the states and their distinct parity symmetries. Here it is modified by the interactions, see Fig. 5 but still approaches the value of $\log(2)$ asymptotically as $\Delta$ is increased. Interestingly, as $\Delta$ is decreased and the system enters the protected phase the same measure of EE jumps to a higher value of $\log(4)$.

Preparation and measurement. Due to the small splitting between the ground states the system is likely to be found in a thermal mixture state. Initial pure state preparation is conceivable by preferentially driving the system to an excited state from which it decays the ground states (optical pumping) or by adiabatically turning on the interactions. We observe that the expectation values of $\langle 0 | \sigma^x_j | 0 \rangle$ and $\langle 1 | \sigma^x_j | 1 \rangle$ are generally different in most regions in parameter space. Therefore a preparation of a ground state followed by a local measurement of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The effect of introducing cavity-mediated (non-local) interactions on the coupling of the ground states. The total coupling $K_j = |\langle 0 | \sigma^x_j | 1 \rangle|^2 + |\langle 0 | \sigma^y_j | 1 \rangle|^2 + |\langle 0 | \sigma^z_j | 1 \rangle|^2$ at the end of the chain (upper) vs. at a bulk (bottom) site without (left) vs. with (right) additional non-local interactions $U \sigma_i \sigma_j$ for all $i,j$ and $\lambda_{FF} = 0.15$ on an $N = 8$ chain (similar features appear for $N = 6$ and $N = 10$). When the non-local interactions are added a region of very weak total coupling ($\sim 10^{-3} - 10^{-4}$) opens at the central region of the parameters space.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The transition into the protected doublet phase shown as a function of decreasing anisotropy $\Delta$ (upper) or increasing non-local interactions (bottom). The difference between the $n$th lowest energy level, $E_n$ (for $n = 1, 2, 3, 4, 5$) and the lowest energy level $E_0$ for an $N = 8$ spin-chain with $\mu = 0$, $t = 1$ as a function of $\Delta$ (top, $\lambda_{FF} = 0.15$) and of $\lambda_{FF}$ (bottom, $\Delta = 0$). The points where the bulk gap closes are circled. The solid arrows shows the splitting between nearly degenerate ground-states and the dotted arrows shows the bulk gap.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The transition into the protected doublet phase shown as a function of the spin polarising energy $\mu$. The $n$th lowest energy level, $E_n$ (for $n = 0, 1, 2, 3, 4, 5$) for an $N = 8$ spin-chain with $\Delta = 0, t = 1, \lambda_{FF} = 0.15$ as a function of $\mu$. A transition from a single ground state into a gapped doublet appears at a certain critical value even though the pairing energy $\Delta$ is zero and there are no Kitaev chain zero modes.}
\end{figure}
ference should be observed as the states become coupled. 

\[ \sigma_{\Delta} \]

\[ \text{Figure 5. The entanglement entropy (EE) for partitioning} \]

\[ \text{the system into two equal chains, presented as function of the} \]

\[ \text{parameter } \Delta. \text{ Using the same parameters as in Fig. 3 the EE} \]

\[ \text{presents a } 2 \log 2 \text{ plateau within the new phase, which rapidly} \]

\[ \text{drops at the phase transition.} \]

\[ \sigma_{\Delta} \text{can reveal the relaxation time. By crossing the protected} \]

\[ \text{phase boundary in the } \Delta \text{ direction, a significant difference should be observed as the states become coupled by external perturbations.} \]

In conclusion, as is well appreciated, in building a quantum simulator for the Kitaev chain with locally coupled spins the protection of the degenerate ground state is compromised by a local perturbation. We analyzed this exactly and showed how the response of the spin chain is related to, and indeed can measure, aspects of the Majorana zero mode wavefunction in the analogue fermion state. We also showed that this spin chain shows its own edge effect with respect to the natural spin-flipping coupling which is related to the Majorana zero modes. Finally, we showed that when non-local interactions are present a new highly correlated spin ground state doublet which has maximal protection against local spin perturbations and is suggestive of a topological phase.

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Quasiparticle operators in the local operator basis

The BdG method begins by recognising that the Kitaev Hamiltonian

\[ H = -\mu \sum_{j=0}^{N-1} \left( a_j^\dagger a_j - \frac{1}{2} \right) + \sum_{j=0}^{N-2} \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger - \sum_{j=0}^{N-2} t \left( a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right) \]  

(1)

can be written as

\[ H = C^\dagger H C \]

where

\[ C = \begin{pmatrix} a_0 & \cdots & a_{N-1} \\ a_{N-1}^\dagger & \cdots & a_0^\dagger \end{pmatrix} \]

and \( H \) is a 2N-by-2N (where \( N \) is the number of chain sites) hermitian matrix involving \( \mu, t \) and \( \Delta \). Let the 2N eigenvalues of \( H \) be written \( E_{\pm n} \) for \( 0 \leq n < N - 1 \) and be indexed such that \( E_a \leq E_b \) if \( a < b \). These eigenvalues represent the single-particle spectrum of the Kitaev model, symmetric about \( E = 0 \), where \( E_{-n} = -E_n \). The corresponding eigenvectors, \( d_{\pm n}^\dagger \) where \( d_{-n}^\dagger = d_n \), are the creation and annihilation operators for the elementary excitation quasiparticles, expressed in the \( a_n, a_n^\dagger \) basis. This picture is illustrated in fig. 1.

![Figure 1. A schematic of the quasi-particle spectrum in the Kitaev chain model](image)

If the eigenvector elements are defined as \( u_i^{(\pm n)} \) and \( v_i^{(\pm n)} \) such that \( d_{\pm n}^\dagger = \sum_{i=0}^{N-1} u_i^{(\pm n)} a_n + v_i^{(\pm n)} a_n^\dagger \), where it is
clear that the unitary transformation, $U$, which diagonalises $\mathcal{H}$ with $U^\dagger \mathcal{H} U$ can be written as

$$U = \begin{pmatrix} u_0^{(-N+1)} & \ldots & u_0^{(-1)} & u_0^{(+0)} & \ldots & u_0^{(N-1)} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ u_{N-1}^{(-N+1)} & \ldots & u_{N-1}^{(-1)} & u_{N-1}^{(+0)} & \ldots & u_{N-1}^{(N-1)} \\ v_0^{(-N+1)} & \ldots & v_0^{(-1)} & v_0^{(+0)} & \ldots & v_0^{(N-1)} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ v_{N-1}^{(-N+1)} & \ldots & v_{N-1}^{(-1)} & v_{N-1}^{(+0)} & \ldots & v_{N-1}^{(N-1)} \end{pmatrix}$$

and so the fermionic quasiparticle operators can be found by the matrix-vector multiplication

$$\begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \\ a_0^\dagger \\ \vdots \\ a_{N-1}^\dagger \end{pmatrix} = U \begin{pmatrix} d_0 \\ \vdots \\ d_{N-1} \end{pmatrix}.$$  

It is worth noting that, by comparing $d_{\pm n} = \sum_{i=0}^{N-1} (u_i^{(\pm n)} a_n + v_i^{(\pm n)} a_n^\dagger)$ with $(d_{\pm n}^\dagger)^\dagger = d_{\mp n} = \sum_{i=0}^{N-1} (u_i^{(\mp n)^*} a_n^\dagger + v_i^{(\mp n)^*} a_n)$, it can be seen that $u_i^{(\pm n)}, v_i^{(\mp n)} = u_i^{(\mp n)^*}, v_i^{(\mp n)^*}$.

**Ground-state coupling for fermions**

To contrast the simplicity of applying BdG to the fermion system with the complexity of doing the same to the spin system, we first turn our attention to the quantity $f_j = | \langle 0 \mid a_j^\dagger \mid 1 \rangle |^2 + | \langle 0 \mid a_j \mid 1 \rangle |^2$, which we write as

$$f_j = | f_j^+ |^2 + | f_j^- |^2$$

where $(f_j^+, f_j^-) = (0 \mid \begin{pmatrix} a_j^\dagger \\ a_j \end{pmatrix} \mid 1)$.

The operators of interest are

$$(a_i, a_i) = \sum_{n=0}^{N-1} (u_i^{(-n)} d_n + (u_i^{(n)} a_n + v_i^{(n)} a_n^\dagger)$$

and so $(f_j^+, f_j^-) = \langle 0 \mid \sum_{n=0}^{N-1} (u_j^{(-n)} d_n, v_j^{(-n)} d_n^\dagger + (u_j^{(n)} v_j^{(n)} d_n^\dagger d_n^\dagger) | 0 \rangle$. Since the $d_{\pm n}$ (for all $0 \leq n < N - 1$) are eigenoperators of $\mathcal{H}$ and $| 0 \rangle, | 1 \rangle$ are eigenstates of $\mathcal{H}$, the only non-vanishing term in the sum is the one in containing $d_0 d_0^\dagger$. Thus,

$$f_j^+, f_j^- = u_j^{(-0)}, v_j^{(-0)}$$

An alternative way to achieve the same expression, which involves the BCS correlators, is to instead use the substitution $d_0^\dagger = \sum_{n=0}^{N-1} u_n^{(+0)} a_n + v_n^{(+0)} a_n^\dagger$. Now,

$$(f_j^+, f_j^-) = \langle 0 \mid \sum_{j=0}^{N-1} u_j^{(+0)} | 0 \rangle | 0 \rangle (a_j^\dagger a_j | 0 \rangle + v_j^{(+0)} (a_j^\dagger a_j | 0 \rangle + v_j^{(+0)} (a_j^\dagger a_j | 0 \rangle$$

$= \sum_{j=0}^{N-1} u_j^{(+0)} (C_{i,j} F_{i,j}) + v_j^{(+0)} (C_{i,j}^* F_{i,j})$$

where

$$(C_{i,j}, F_{i,j}) = \langle 0 \mid a_i^\dagger (a_j, a_j^\dagger) \mid 0 \rangle$$

$= \sum_{n=0}^{N-1} u_n^{(-n)} (u_j^{(n)} v_j^{(n)})$$
and
\[
F'_{i,j}, C'_{i,j} = (F_{j,i}^*, \delta_{ij} - C_{j,i})
\]
\[
= \langle 0 \mid a_i \left( a_j, a_j^\dagger \right) \mid 0 \rangle
\]
\[
= \sum_{n=0}^{N-1} u_i^{(-n)}(u_j^{(+m)}, v_j^{(+m)}).
\]

Thus,
\[
f_i^+, f_i^- = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} u_n^{(+0)} \left( u_i^{(-m)} u_j^{(+m)}, v_i^{(-m)} u_j^{(+m)} \right)
\]
\[
+ v_n^{(+0)} \left( u_i^{(-m)} v_j^{(+m)}, v_i^{(-m)} v_j^{(+m)} \right)
\]
\[
= \sum_{m=0}^{N-1} \langle u, v \rangle_i^{(-m)} \sum_{m=0}^{N-1} u_n^{(+0)} u_j^{(+m)} + v_n^{(+0)} v_j^{(+m)}
\]
\[
= \sum_{m=0}^{N-1} \langle u, v \rangle_i^{(-m)} \{ d_m, d_0 \}
\]
\[
= u_i^{(-0)} v_i^{(-0)}.
\]

Additionally, this equivalence proves that
\[
\left( \langle u_i^{(-0)}, v_i^{(-0)} \rangle = \sum_{j=0}^{N-1} u_j^{(+0)} \langle F'_{i,j}, C_{i,j} \rangle + v_j^{(+0)} \langle C'_{i,j}, F_{i,j} \rangle \right)
\]
\[
(2)
\]

Thus, expressions have been found for the couplings \( f_i^+, f_i^- \), and thus for \( f_j \).

### Ground-state coupling for spins

The BdG formalism can also be useful for spins by recognising that \( \sigma_j^x \) can be related to \( a_j^\dagger \) and \( a_j \) by the JWTs. That is,
\[
S_j^x = | \langle 0 \mid \sigma_j^x \mid 1 \rangle |^2 = | \langle 0 \mid \prod_{n=0}^{j-1} (2a_n a_n^\dagger - 1) \left( a_j^\dagger + a_j \right) \mid 1 \rangle |^2.
\]

For the first site of the chain, \( j = 0 \), the translation is straightforward
\[
S_0^x = | \langle 0 \mid a_0^\dagger + a_0 \mid 1 \rangle |^2
\]
\[
= | A_0^+ + A_0^- |^2
\]
\[
= | u_0^{(+0)} + v_0^{(-0)} |^2
\]

At the second site, \( j = 1 \), the translation is
\[
s_1^x = | \langle 0 \mid \left( 2a_0 a_0^\dagger - 1 \right) \left( a_1^\dagger + a_1 \right) \mid 1 \rangle |^2
\]
\[
= | 2 \langle 0 \mid a_0 a_0^\dagger \left( a_1^\dagger + a_1 \right) \mid 1 \rangle |^2 - \langle C_1^+ + C_1^- \rangle^2
\]

which is more complicated but still simple enough to analyse.

An analysis of this reveals that
\[
S_j^x = \left| 2 \sum_{m=0}^{N-1} u_m^{(+0)} \left[ F'_{0m} (F_{01} + C_{01}) - C_{0m} (F'_{01} + C_{01}) + (C_{1m} + F'_{1m}) \left( C_{00} - \frac{1}{2} \right) \right] \right|^2
\]
\[
+ 2 \sum_{m=0}^{N-1} v_m^{(+0)} \left[ C'_{0m} (F_{01} + C_{01}) - F_{0m} (F'_{01} + C_{01}) + (C'_{1m} + F_{1m}) \left( C_{00} - \frac{1}{2} \right) \right]^2.
\]
Now, analytical expressions for sites $j = 0,1$ with straightforward interpretations have been found. Using eq. 2 multiple times, eq. 3 can be simplified to

$$S_j^r = \left| 2 (F_{01} + C_{01}) u_0^{(-0)} - 2 (F_{01} + C_{01}) v_0^{(-0)} + 2 \left( C_{00} - \frac{1}{2} \right) \left( u_1^{(-0)} + v_1^{(-0)} \right) \right|^2$$

$$= \sum_{i=0}^{1} w_{u,i}^{(-0)} u_i^{(-0)} + w_{v,i}^{(-0)} v_i^{(-0)}$$

where

$$w_{u,0}^{(-0)} = 2 \left( F_{01} + C_{01} \right)$$
$$w_{v,0}^{(-0)} = -2 \left( F_{01} + C_{01} \right)$$
$$w_{u,1}^{(-0)} = w_{v,1}^{(-0)} = 2C_{00} - 1$$

or, equivalently

$$w_{u,0}^{(-0)} = 2 \left( F_{01} + C_{01} \right)$$
$$w_{v,0}^{(-0)} = -2 \left( F_{10} - C_{10} \right)$$
$$w_{u,1}^{(-0)} = w_{v,1}^{(-0)} = 1 - 2C_{00}$$