Isovector And Pairing Properties Of The Gogny Force In The Context Of Neutron Stars

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For Mum, Dad and Andrew.
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Abstract

11 of the Gogny forces available in literature have been studied in order to evaluate their isovector properties and to comment on their viability for beyond $^1S_0$ pairing channels, and neutron star cooling calculations. I find that, even within a relatively narrow set of Gogny functionals, there is a large variation in isospin properties. In particular, I find that the density dependence of the symmetry energy provided by Gogny forces is too soft and lies outside of currently accepted values [1, 2]. This points to poor constraints in the isovector sector, that should be improved in future fitting protocols.

In addition to this, the pairing properties of the Gogny forces have been examined for the $^1S_0$, $^3S_1$, $^3P_2$ and $^1P_1$ channels. Although most forces are in keeping with literature for the $^1S_0$ gaps, the remaining channels are not particularly constrained and many of the Gogny forces produce unphysical results. The pairing gaps generated for the D1P parametrisation have been used to calculate neutron star cooling curves as a proof-of-concept for the Gogny force. Successful cooling curves incorporating superfluid effects have been produced using the NSCool software package.

Although most Gogny forces perform poorly in both the isovector and pairing sectors, it is shown that consistent neutron star models cooling models can be generated with the Gogny force. Suggestions are made as to the fitting considerations for a future force.
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Chapter 1

Introduction

1.1 Infinite Nuclear Matter

Theoretical nuclear physics is concerned with many currently developing fields of research. These range from experiments performed on the very small, such as measurements of neutron skin thicknesses [3, 4, 5], to more macroscopic phenomena such as properties of neutron stars and neutron star mergers [6, 7]. As more experimental and observational data become available they can be used to evaluate and improve current theories. One relevant area of interest is Infinite Nuclear Matter (INM) [8].

INM can be modelled as a degenerate Fermi liquid of an infinite number of homogeneously distributed nucleons. The matter behaves similarly to a standard liquid drop model [9], having a compressibility and other bulk properties, but without a finite radius. In contrast to the usual description of finite nuclei, the Coulomb force is not considered in INM. This force is weak in comparison to the strong nuclear force and its inclusion adds an infinite number of contributions to quantities that are not physically relevant. INM also neglects electrons and only considers the strong force contributions between nucleons. The infinite size of INM removes the need for a surface term and the homogeneous distribution of nucleons means that the system is considered at the thermodynamic limit and so is constant throughout. It is also possible for matter to
be infinite yet inhomogeneous, such as in a neutron star crust, however I am only considering homogeneous nuclear matter in this thesis. The Equation of State (EoS) describes how pressure changes with density and the assumptions made in INM allow the EoS to be expressed as a function solely of density and proton fraction. Temperature is not considered here either. The justification for this is that at the temperatures considered in this work, the thermal energy is very small compared to the single particle energies, and so can reasonably be approximated at $T = 0$ K. The EoS has large implications for the behaviour of compact objects and so one natural application of INM is to neutron stars [10]. The isovector properties of INM determine the ratio of protons to neutrons, known as the isospin asymmetry, as a function of density. In particular the isospin asymmetry in a neutron star can have large effects on its observable properties [11, 12] and so using the isovector properties to calculate the EoS at the appropriate asymmetry allows for a more realistic description of the neutron star interior.

1.1.1 Mean Field Approximation

INM properties can be calculated within different theoretical frameworks. Popular models in nuclear theory include Relativistic Mean Field (RMF) Theory [13] and Chiral Effective Field Theory [14]. In this text the focus will be on a microscopic non-relativistic Hartree-Fock (HF) mean field (MF) approximation.

Except for exchange effects, such as the Pauli exclusion principle, HF ignores particle correlations. The lack of correlations between particles in HF means that, rather than considering explicitly the interactions between each nucleon and each other nucleon, one can treat the particles as if they were immersed in a MF. The MF assumption is that the many-body interactions that a particle feels from its neighbours can be accurately approximated by an average potential well. For instance, if a proton were surrounded by several other nucleons, then rather than explicitly calculating the
strong force interaction between each particle, one could assume the proton to be in an approximated field, greatly simplifying the calculations required to develop accurate predictions for the behaviour of this system. As such, an effective two-body interaction, which approximates the many-particle correlations in the external field, can be used in the HF Hamiltonian.

The approximating interactions discussed in this work are phenomenological in nature, made up of several free parameters that are adjusted to reproduce properties of known nuclei. These interactions are comprised of a single-particle kinetic term and a two-body nucleon-nucleon (NN) interaction. Although their use is widespread, these interactions are not without their problems and even commonly used parametrisations can fail to reproduce the properties of neutron-rich nuclei and highly asymmetric nuclear matter [15]. Different forces are parametrized for different purposes and it is possible that some of these forces are, unjustifiably, used outside of the model space in which they are valid [16]. It is for this reason that this work focusses on evaluating several of these forces and comments on the validity of their use in the field of Neutron Star Matter (NSM).

1.1.2 Isovector And Isoscalar Properties

The isovector and isoscalar properties of INM play an important role in the behaviour of neutron stars and neutron-rich nuclei. It is useful here to express the isospin asymmetry parameter, $\beta$, where

$$\beta = \frac{\rho_n - \rho_p}{\rho_T},$$

(1.1)

$\rho_n$ and $\rho_p$ are the neutron and proton densities respectively, and $\rho_T$ is the total density of the system, $\rho_n + \rho_p$. For brevity, I will use $\rho = \rho_T$ as the nomenclature for the remainder of this thesis. Where $\beta = 0$ the system contains equal densities of protons
and neutrons and so the matter is referred to as symmetric. The extreme, $\beta = 1$, represents a purely neutronic system. The other extreme, $\beta = -1$, represents a purely protonic system.

An isovector property parametrizes the asymmetry dependence, whereas an isoscalar one relates to isospin-independent properties. This means that a property at a given constant density e.g. saturation density, $\rho = \rho_0$, will exhibit a non-zero derivative with respect to $\beta$. Isoscalar properties are properties that will evolve with density but do not depend on asymmetry. Altering the neutron to proton ratio will change isovector values. Calculations based on the energy difference between Symmetric Nuclear Matter (SNM) and Pure Neutron Matter (PNM), known as the symmetry energy, play a large role in calculating properties of highly asymmetric matter, as does the density dependence of this energy difference, known as the slope parameter.

Isovector and isoscalar properties will have a large impact on the observables of neutron stars. For example, the mass and radius are heavily linked to the stiffness of the EoS [17], which is an isovector property. These links mean that neutron star observations can be used to constrain the possible values for INM bulk properties. These constraints can help to evaluate currently existing theory as well as offer guidance for creating more accurate models.

1.2 Neutron Stars

Neutron stars are one of the possible end points of stellar evolution. They come from progenitor stars of between around 9 and 25 $M_\odot$ [18], which go supernova. After the supernova the neutron star has a mass in the region of 1-2 $M_\odot$ [19]. Despite their mass
they have radii of only around 10 km \[20\], making them incredibly dense\(^1\). This disparity between their radius now and the radius they had when a fully fledged star means that as they collapse they can spin up to very high speeds to conserve angular momentum. They can also be spun-up by accretion of material \[21\]. The fastest known neutron star spins at 716 Hz \[22\]. These rotating neutron stars can emit radiation from x-rays to microwaves from their poles \[21\]. This radiation led to their first observation in the form of “pulsars” in 1967 \[23\], gaining Hewish and Ryle a Nobel prize\(^2\).

Because neutron stars are left over after a supernova occurs they are initially hot objects, with internal temperatures of 20 - 50 MeV \[24\], corresponding to 232 - 580 billion Kelvin. The neutron stars cool rapidly but rather than cooling from photon radiation and convection at the surface, as would be expected for every day objects such as a cup of tea \[25\], they cool from the inside-out via neutrino radiation arising from several subatomic processes \[26\]. These processes are explored further in Chapter 5.

The cores of neutron stars can be approximated as being comprised of INM. Although the star is finite in size it is large compared to the range of the inter-nucleon interaction. It is for this reason that the surface effects can be considered to offer a negligible contribution to the properties of the bulk matter. Charge neutrality is maintained by the presence of negatively charged particles e.g. electrons and muons \[27, 28\] shielding the charge of the protons.

A cross section of a neutron star depicting the different phases of matter is displayed in figure 1.1. Although the neutron stars I consider throughout the thesis are treated as INM, it is useful for illustration to explore the more complex compositions

\(^1\)A single teaspoon of neutron star matter contains as much mass as much as the entire human population of the earth!

\(^2\)Despite being the one to make the actual observation and initial analysis Jocelyn Bell-Burnell was not included as a co-recipient, much to the disdain of many prominent astronomers
Figure 1.1: Artist’s impression of a Neutron Star cross section, depicting the multiple phases of matter present in most neutron star models. Image Credit: Dany Page

that can occur in neutron star structure. At densities above $10^4 \text{ g/cm}^3$, matter is fully ionized [29]. The outer edge of a neutron star is comprised of a thin atmosphere of finite nuclei. The exact composition of this atmosphere varies from star-to-star and depends on many factors e.g. post-formation accretion and thermonuclear reactions [30]. Determining the composition is challenging but fits can be made to observed spectra of the neutron star [30]. Beneath the atmosphere is the crust, which consists of finite nuclei of a broad range of masses. Near the bottom of the crust, these nuclei can form a frustrated crystal lattice [31]. The nuclei are at very high densities and the exotic isotopes produced may exceed those that have been identified experimentally. At some density the neutrons will begin to drip from the finite nuclei [32, 33] and a Fermi liquid of neutrons will coexist with the crystal lattice of nuclei. At the
transition boundary between the lattice of nuclei and the Fermi seas the NSM can develop through several “pasta phases” [31]. In the pasta phases nearest the surface free nucleons will start to form bubbles within the lattice of nuclei. As the density increases the bubbles will combine and elongate to form spaghetti- and lasagne-style structures. Delving deeper still, the Fermi sea now dominates the composition and the remaining nuclei form bubbles and ultimately disappear entirely [34].

Finally, the matter consists of a homogenous sea of neutrons, protons and negative charge carriers [35]. In the interest of brevity in the remainder of this discussion I consider the negative charge carriers to be entirely electrons. This part of the neutron star is the most justifiably linkable to INM due to the lack of finite nuclei. The charge neutrality is maintained by the protons and electrons and the relative homogeneity in the system. It is during this region that the matter passes saturation density, becoming ever-more compacted as one descends within the star. This is also the region thought to be responsible for some observable neutron star phenomena, such as cooling [36].

Descending further one encounters the inner core. There is a large disagreement in literature about the nature of the inner core and many open questions remain. Models can describe nuclear matter cores consisting of neutrons and protons [37], hyperonic cores where heavier baryons with strange content begin to freeze out [38], and even quark-gluon-plasma cores whereby the inner core is highly exotic material consisting of a sea of deconfined quarks [39, 40]. The aim of this work is to explore nuclear degrees of freedom only. As such, in this work I have only considered neutrons and protons, neglecting any possible contribution from hyperons or more exotic matter.

One field currently interested in INM and NSM is astrophysics, after the discovery of a $2M_\odot$ neutron star [41, 42]. This measurement indicated that many of the EoS available were not accurate, because they could not account for that high a mass. Stiffer equations of state, those with a steeper pressure-density relation, will allow
for larger masses of neutron stars \cite{43} and so are favoured by current observations. Masses of neutron stars can be measured using Shapiro delay, a method using general relativistic orbit effects with a binary partner \cite{44}. Although measuring the radius of a neutron star is more difficult than measuring the mass, it is not impossible; the thermal flux and the colour temperature can sometimes be observed and if measured in conjunction with the redshift, both the mass and radius can be inferred \cite{45}. Knowing both the mass and the radius would allow one to evaluate the accuracy of a given EoS \cite{46}.

It is also known that the proton fraction within neutron stars would need to be above a critical number if certain cooling processes are allowed to occur, namely the Direct Urca process \cite{47}. The EoS of INM is able to give information about the predicted proton fraction at a given density and so can make predictions as to whether this cooling process should exist in observed neutron stars. Techniques have also been proposed to estimate the neutron star EoS from gravitational wave observations \cite{48} which will hopefully be provided in the future at facilities such as the Laser Interferometer Gravitational-Wave Observatory (LIGO) \cite{49}. Neutron star mergers have been named as a site for r-process nucleosynthesis \cite{50, 51}. It has been shown that neutron stars are unlikely to be the dominant process \cite{52}, however improving our knowledge of neutron stars can help us to understand the abundances in the interstellar medium.

Despite the approximations and assumptions, INM is also highly relevant to finite nuclei. The behaviour of the EoS of INM will have large implications for the understanding of neutron-rich nuclei. As experimental facilities broaden our ability to explore new regions of the Segré chart INM also provides a good theoretical playground to describe the interior of extremely heavy nuclei and even to help us understand islands of stability towards the drip line \cite{53}. Thus, INM is able to bridge the gap between the micro- and macro-scopic and provides a channel for astrophysical observations to improve our predictions of the behaviour of finite nuclei. One other factor
that can be considered in the context of NSM is the impact of superfluidity.

1.2.1 Superfluidity And Its Role in Neutron Stars

In 1911, Kamerlingh Onnes discovered the first evidence of superconductivity when investigating mercury atoms as part of his experiments with helium [54]. The experiment produced previously unseen properties, where the resistivity of Mercury appeared to drop to zero. Later, in the early 1930s, this was recognised as being superfluidity. The term superconductivity is used to describe charged particles and relates to the current flow within the material. Superfluidity can be used to describe both charged and non-charged particles, and describes a frictionless fluid. When superconductivity takes place, one effect is the resistivity of a material dropping to zero, making it a “perfect conductor”. A perfect superconductor also contains no magnetic field. This effect was explained by Meissner [55] in 1933. It is due to a complete ejection from the material of the magnetic flux and the formation of a perfect diamagnet. The expulsion of this magnetic field cannot be explained by classical physics alone, and so superconductivity is necessarily a quantum mechanically driven phenomena.

In 1957, Bardeen, Cooper and Schrieffer (BCS) [56] constructed their theory of superconductivity. It was shown that a superconductive phase was favourable when the attractive weak interaction caused by phonon exchange between electrons [57] became stronger than the Coulomb repulsion. The electrons will then be drawn into Cooper pairs: pairs of opposite spin and momentum. Breaking this pair requires a certain amount of energy and this value is known as the pairing gap. Although constructed with superconductivity in mind, the BCS principles can be also applied to nucleons to calculate more generic superfluidity, where the particles are not necessarily charged. Since 1958 the BCS theory of superfluidity [56] has been applied to nuclear structure calculations [58]. The pairing gap between nucleons predicted by this theory has
significance in many areas of physics, including the structure of finite nuclei \([59, 60]\) and the properties of neutron stars \([61, 62, 63]\). The magnitude of the pairing gap is related to the critical temperature of the superfluid phase and has implications for the thermal transport properties in INM \([64]\).

The cooling profile of neutron stars can reveal much about the physics occuring within. In particular the rate of cooling can be linked to the direct and modified Urca processes \([65]\) and observations can suggest which, if either, are occurring at a given time. The neutron star Cassiopeia A has been measured to be cooling at a high rate \([66]\) and, although measurements disagree on the precise cooling rate, they nearly all declare far more rapid temperature drop than can be explained by either of the Urca processes alone \([67]\). One possible explanation is the appearance of a superfluid phase within the neutron star. Neutrino emission can be enhanced by the continuous breaking and formation of Cooper pairs, which can occur just below the critical temperature \([68]\). This formation and breaking produces neutrino-anti-neutrino pairs as well as photon pairs. The neutron star interior is opaque to photons, however shortly after birth, neutron stars are transparent to neutrinos \([69]\). As such, onset of neutron superfluidity may be responsible for the rapid cooling that has been measured in Cassiopeia A \([70, 36]\).

In a degenerate Fermi liquid an overall attractive interaction in any channel will lead to nucleon pairing \([71]\). Effective mean field interactions can be used to calculate the values of these pairing gaps and so make predictions on the behaviour of some neutron star observables. Comparing these predictions to observation can therefore help to compare and evaluate different nuclear functionals. Along with the isovector and isoscalar properties, superfluid pairing gaps can be used to probe the validity of different models in the context of neutron stars and neutron-rich nuclei.
The forces examined in the following are Gogny\textsuperscript{1} forces [72], which are finite-range interactions, and the energy-per-particle and derived isovector and isoscalar properties will be extracted via a Hartree-Fock approximation.

1.3 The Gogny Force

The Gogny force is a well-known and extensively used effective nuclear interaction [73]. Unlike the popular Skyrme density functional, which parametrizes the dependence on the relative distance by contact interactions and derivatives, the Gogny force has a built-in finite range [74] term. This brings the Gogny force closer in spirit to realistic interactions. Moreover, the non-zero range is essential to avoid spurious truncations in the pairing channel within Hartree-Fock-Bogolyubov (HFB) nuclear structure calculations, and it was the main motivation behind its inception in the 1980’s by the Bruyères group [72]. A sum of two Gaussians was first used to describe finite range interactions in 1967 [75], but this could not correctly reproduce the binding energies from a Hartree-Fock approximation [76]. Gogny then introduced a density dependence and a spin-orbit term which correctly reproduced the binding energies of spherical nuclei [77]. In the decade that followed, Gogny forces were particularly used in nuclear fission studies [78]. In this context, Gogny interactions are still a popular starting point for a variety of reasons [79]. Heavy nuclei, deformation and multipolar collective degrees of freedom have also been studied using Gogny HFB [80, 81]. Recently, even Gogny time-dependent calculations have become available [82].

INM has usually been taken as a reference in the fitting procedure of Gogny functionals [74]. This includes isoscalar properties, such as saturation energies and saturation density. The compressibility of nuclear matter, of crucial importance for a variety of nuclear structure observables, has also been extensively studied with the Gogny

\textsuperscript{1}Named after Daniel Marc Gogny (1928-2015)
functional [83]. In contrast, the isovector properties of the Gogny parametrizations have hardly ever been discussed. Typically, only the symmetry energy is considered, if anything [72]. The remaining isovector dependence is expected to be captured by the fit to finite nuclei. Studies have shown, however, that in addition to nuclei, neutron-rich infinite matter is also needed to constrain the isovector sector [84, 85, 86]. One could therefore put into question the predictive power of Gogny interactions in isovector-dominated properties, such as neutron skins, or neutron-rich systems.

1.4 Outline of the thesis

The isovector properties of Skyrme functionals have been extensively studied [87, 88], with a very wide variety of criteria to characterise their quality [89]. In the following, I aim at providing a generic description of isospin asymmetric nuclear matter with the Gogny interaction.

Where possible, I have compared the functionals with existing values of isoscalar and isovector properties [90, 91]. As a specific aspect of the isovector sector, I discuss neutron star properties as predicted by the present generation of Gogny forces [92]. On the one hand, this might go beyond the scope of applicability of part of the Gogny functionals. On the other, a new generation of observations is starting to put severe constraints on the equation of state of neutron-rich matter [93, 94, 95]. Ideally, these constraints should also be considered in fitting procedures of future functionals [85]. In line with the poor reproduction of bulk isovector properties, my calculations indicate that it is difficult to produce sufficiently massive neutron stars with the present generation of Gogny functionals [92].

In the hope of providing a more comprehensive review of the Gogny force in INM, I have investigated the $^1S_0$ pairing gap as predicted by the available Gogny forces.
Values for this pairing channel are readily available in literature [96, 97, 98, 99] and provide a good preliminary benchmark. Despite its significance, there is a dearth of literature regarding values for pairing gaps beyond the $^1S_0$ channel and so these extra gaps have also been explored.

Finally, I have examined the neutron star cooling curves generated by the Gogny functional known as D1P [100]. In particular I have focussed on the effect of cooling enhancement and suppression due to superfluids appearing in the neutron star core. I have used these results to evaluate the possibility of using the Gogny force to produce a comprehensive and realistic description of neutron star behaviour without the need to employ multiple models for calculating the star profile, EoS, composition and pairing.
Chapter 2

The Gogny Force

2.1 Mean Field Approximation

When examining subatomic interactions between nucleons, the complicated interaction of the strong force between multiple nucleons presents a many-body problem that cannot be analytically solved due to the non-perturbative nature of QCD and so it becomes necessary to employ an approximation to accurately estimate the energy-per-particle. There are multiple ways of handling the many-body problem presented by nuclear systems. Here I briefly describe two of these calculation frameworks, Density Functional Theory (DFT) and Hartree-Fock (HF). In terms of density matrices, rather than densities, DFT can be seen to be equivalent to the method used in this work \cite{?}, and so it provides a useful introduction to the topic.

2.1.1 Density Functional Theory

Density functional theory was originally devised by Hohenberg and Kohn in the context of atomic physics \cite{101}. I will briefly explore their formalism here. One begins by considering the Schrödinger equation,

$$E\psi = \hat{H}\psi,$$  \hspace{1cm} (2.1)
where $E$ is the total energy, $\psi$ is the many-body wavefunction of the system and $\hat{H}$ is the Hamiltonian. The many-body wavefunction, $\psi$, is a function of the position of each particle. Here, I am neglecting any time-dependence, and so one can say that the wavefunction of an $N$ body system is

$$\psi = \psi (r_1, r_2, ..., r_N),$$  \hspace{1cm} (2.2)

where $r_i$ is the position of particle $i$. The Hamiltonian takes the form

$$\hat{H} = \hat{T} + \hat{U} + \hat{V}$$  \hspace{1cm} (2.3)

with $\hat{T}$, $\hat{U}$ and $\hat{V}$ representing the kinetic energy operator, external field operator and particle-particle interaction operator, respectively. When considering the many body case the problematic term is the particle-particle interaction which for a two-body interaction, can be described as

$$\left\langle \psi_1, \psi_2, ..., \psi_N | \hat{V} | \psi_1, \psi_2, ..., \psi_N \right\rangle = \sum_{i < j}^N \left\langle \psi_1, \psi_2, ..., \psi_N | V (r_i, r_j) | \psi_1, \psi_2, ..., \psi_N \right\rangle.$$  \hspace{1cm} (2.4)

This interaction contains a large number of terms that require knowledge of the wavefunction of the system, including the position of each particle. Density Functional Theory, however, helps one to circumvent this problem. The number density of the system can be expressed in terms of the wavefunction;

$$\rho (r_1) = N \int d^3 r_2 ... \int d^3 r_N \psi^* (r_1, r_2, ..., r_N) \psi (r_1, r_2, ..., r_N).$$  \hspace{1cm} (2.5)

The Hohenberg-Kohn theorem [101] states that the ground-state energy, so long as it is not degenerate, can be determined from the number density, and so is a function of a function\textsuperscript{1} of $r_1, r_2, ..., r_N$. This necessarily means that the potential, $\hat{V}$, is a unique

\textsuperscript{1}One refers to the function of a function as a functional
functional of density. I briefly show a proof of this here.

Suppose that there are two potentials, $\hat{V}$ and $\hat{V}'$, leading to two Hamiltonians, $\hat{H}$ and $\hat{H}'$. Also, suppose that the Hamiltonians have ground-state energies of $E$ and $E'$, respectively, from the corresponding ground-state wavefunctions $\psi$ and $\psi'$, but with the same ground-state density. We would therefore expect $\langle \psi' | \hat{H} | \psi' \rangle$ to give a larger energy than the ground state, $E$, and $\langle \psi | \hat{H}' | \psi \rangle$ to give a larger energy than $E'$. Let us examine this claim, under the assumption of a common density function $\rho(r)$:

$$E < \langle \psi' | \hat{H} | \psi' \rangle = \langle \psi' | \hat{H}' | \psi' \rangle + \langle \psi' | \hat{H} - \hat{H}' | \psi' \rangle = E' + \int d^3r \rho(r) [V(r) - V'(r)], \quad (2.6)$$

$$E' < \langle \psi | \hat{H}' | \psi \rangle = \langle \psi | \hat{H} | \psi \rangle + \langle \psi | \hat{H}' - \hat{H} | \psi \rangle = E + \int d^3r \rho(r) [V'(r) - V(r)] = E - \int d^3r \rho(r) [V(r) - V'(r)]. \quad (2.7)$$

Summing equations 2.6 and 2.7, the integral terms including $\hat{V}$ and $\hat{V}'$ cancel and we are left with the illogical inequality

$$E + E' < E + E', \quad (2.8)$$

which clearly cannot be correct. As such the two ground state energies, and the two potentials $\hat{V}$ and $\hat{V}'$, must have unique densities. Thus, the contribution from $\hat{V}$ can be expressed as solely a function of density:

$$\hat{V} = V(\rho(r)). \quad (2.9)$$
One can also state that

\[ \langle \psi (\rho) | \hat{V} | \psi (\rho) \rangle = \int d^3r \ V (\rho (r)) \rho (r). \] (2.10)

The contributions from \( V \) can therefore be accounted for without explicitly considering the wavefunction of the system. One can define a functional of density,

\[ F (\rho (r)) = \langle \rho (r) | (T (\rho (r)) + V (\rho (r))) | \rho (r) \rangle = E_T + E_V, \] (2.11)

which contains information about the kinetic energy of the system, \( E_T \), as well as the two-body potential energy, \( E_V \), yet only relies on the density, \( \rho (r) \). With this, one can define the total energy, \( E \), as a function of density:

\[ E (\rho) = \int U (r) \rho (r) \, d^3r + F (\rho (r)). \] (2.12)

For a given \( U \) and if \( F \) is the correct complete density functional, one could now determine the \( \rho \) for which the energy is a minimum: the ground state energy. Moreover, if the correct universal functional, \( F \), and external potential, \( U \), were known then this method can now describe the properties of a system solely as a function of density rather than needing to understand all of the correlations included in the many-body wavefunction. The significance of this becomes clear when one considers that classically, as well as the spin and isospin, \( V (r) \) depends on the x, y and z positions of each particle in the system and so represents a 3N variable problem. Relating the properties to a 3-dimensional variable, \( \rho (r) \) dramatically reduces the computational power required in calculations by reducing the many-body problem involved in calculating \( V (r) \) to a single-body problem dependent on \( \rho (r) \), completely circumventing
the many-body wavefunction.

If one does not know the exact functional, $F$, one can still calculate an effective approximation. I now follow the formalism of Kohn and Sham \cite{102} to describe the self-consistent method for approximating $F$. Under the assumptions of Eqs. (2.10) and (2.12),

$$E_V = \int d^3r \rho(r) \epsilon_V(\rho(r)),$$

where $\epsilon_V$ is the single-particle interaction energy, and that

$$\int d^3r \delta \rho(r) = 0,$$

where $\delta \rho(r)$ is a small change in density, I can construct the equation

$$\int \delta \rho(r) \left[ \phi(r) + \frac{\delta T(\rho)}{\delta \rho(r)} + \mu_{ec}(\rho(r)) \right] dr = 0,$$

where $\phi(r)$ is the sum of external potential and interaction potentials,

$$\phi(r) = U(r) + V(\rho(r)),$$

and $\mu_{ec}$ is the exchange and correlation contribution to the chemical potential,

$$\mu_{ec} = \frac{d (\rho \epsilon_V(\rho))}{d\rho}.$$

As such, for a given $\phi$ and $\mu$ one constructs the Kohn-Sham equation:

$$\left( -\frac{\hbar^2}{2m} \nabla^2 + [\phi(r) + \mu_{ec}(\rho(r))] \right) |\psi_i(r)\rangle = \epsilon_i |\psi_i(r)\rangle.$$
Calculating $\rho$ as

$$\rho(r) = \sum_{i=1}^{N} \langle \psi_i(r) | \psi_i(r) \rangle, \quad (2.19)$$

one can now perform a self-consistent routine starting with a function for $\rho(r)$. This value can be fed into Eqs. (2.16) and (2.17) which in turn can use the Kohn Sham equation, Eq. (2.18), to provide a new $\rho(r)$. This method allows one to determine the many-body properties of a system without knowing the wavefunction itself.

### 2.1.2 Hartree-Fock

In this work I have used a HF method which, rather than circumventing the wavefunction entirely, ignores many-body correlations between nucleons and assumes the wavefunction of the system is expressed as a single Slater determinant [103]. For the discussion on Hartree-Fock I now follow the formalism of Blaizot and Ripka [104]. Under the assumption that the system can be described by a single Slater determinant, $|\phi\rangle$, one can construct an energy density matrix,

$$\rho_{ij} = \langle i | \rho | j \rangle = \langle \phi | a_j^\dagger a_i | \phi \rangle, \quad (2.20)$$

where $a_i^\dagger$ and $a_i$ are the creation and annihilation operators of state $|\lambda_i\rangle$. The density operator $\rho$ is

$$\rho = \sum_{\nu} n_\nu |\lambda_\nu\rangle \langle \lambda_\nu|, \quad (2.21)$$
where $n_{\nu}$ is the occupation number of state $\nu$. One can express the expectation value of the energy as a function of density:

$$E(\rho) = \langle \phi | \hat{H} | \phi \rangle = \sum_{\nu} n_{\nu} \langle \lambda_{\nu} | \hat{T} | \lambda_{\nu} \rangle + \frac{1}{2} \sum_{\mu,\nu} \langle \lambda_{\mu}, \lambda_{\nu} | \hat{V} | \lambda_{\mu}, \lambda_{\nu} \rangle A_{\mu \nu} n_{\mu} n_{\lambda}. \tag{2.22}$$

For Fermions $n_{\nu}$ can be zero or unity. The density dependence in the right-hand-side occurs because the number of states is a function of the density,

$$N = \sum_{\nu} n_{\nu} = O(\rho). \tag{2.23}$$

The current unknown is now which single particle states are occupied. One can now utilise the single-particle Hartree-Fock Hamiltonian,

$$\hat{h}_{ij} = \frac{\delta E(\rho)}{\delta \rho_{ij}} \tag{2.24}$$

and the Hartree-Fock equation,

$$\left[ \hat{h}, \rho \right] = 0, \tag{2.25}$$

(that is to say that the Hartree-Fock Hamiltonian and the density matrix commute) so that it becomes possible to determine the occupied states iteratively. One can supply a test wavefunction of $N$ occupied states and calculate the density matrix. This can then be used to determine the Hartree-Fock Hamiltonian. One can then solve the eigenvalue equation

$$\hat{h} |\lambda_{\nu}\rangle = e_{\nu} |\lambda_{\nu}\rangle \tag{2.26}$$

to diagonalise the Hartree-Fock Hamiltonian and determine a new set of states $|\lambda_{\nu}\rangle$. 
These new states can be used to create a new density matrix and the process repeats.
With each iteration the occupied states will, hopefully, stabilise to within a desired
accuracy.

2.1.3 The Hartree-Fock Hamiltonian

One important equation to know for these calculations is the form of the HF equa-
tion, so that one can solve Eq. (2.26). The HF equation is not dissimilar from the
Kohn-Sham (KS) equation of 2.18. However, the KS Schroedinger equation does not
explicitly include exchange correlations. The HF approximation does not separately
deal with the three- and higher-body interactions and considers only the two-body con-
tributions, although higher order effects can be captured in density dependent terms.
Dealing with the kinetic energy term and the two-body potential term we can express
the Hamiltonian in second quantized form:

\[
\hat{h} = \sum_{\alpha\beta} t_{\alpha\beta} \hat{a}_\alpha^\dagger \hat{a}_\beta + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta\gamma\delta} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\gamma.
\] (2.27)

The HF is a first approximation to the full many-body propagator [105]. It consists
of a “direct” term plus an “exchange” term. The Hartree, or direct, part of the
potential assumes that all bodies feel the effect of the many-body mean-field. A
weakness of this Hartree approximation is that it does not account correctly for the
anti-symmetry of the system [106]. To account for this discrepancy the “Fock” part of
the full many-body propagator is also included. This represents the exchange of two
particles within the mean field and is subtracted from the direct term in order to leave
a fully antisymmetrised set of matrix elements,

\[
\langle a, b | \hat{O} | a, b \rangle_A = \langle a, b | \hat{O} | a, b \rangle - \langle a, b | \hat{O} | b, a \rangle,
\] (2.28)

where the subscript A denotes that the ket is antisymmetrised and \( \hat{O} \) is a generic
two-body operator. The practical advantage to this is that one can account for an-
tisymmetrisation by including the antisymmetrised ket in direct formalism without
having to explicitly consider a direct plus exchange term.

Equation (2.28) clearly shows that if $a = b$, that is that the two Fermions are iden-
tical, then upon exchange of the two particles the Fock term is equal to the Hartree
term and the overall matrix element will be zero. This allows the full HF framework
to deal explicitly with the Pauli exclusion principle.

2.1.4 Effective Interactions

In DFT one can use an appropriate approximating interaction, $V$, to determine the
properties of a system without explicitly considering the wavefunction. Alternatively,
one can employ a HF approximation, using Eq. (2.27) as the HF Hamiltonian and
employing a given two-body interaction for $V$ that reduces to a one-body function of
the density matrix when the wavefunction is a diagonalised Slater determinant. With
a given interaction it would now be possible to use a self-consistent Hartree-Fock tech-
nique to determine the nuclear structure of a finite nucleus [107], by starting with a
trial wavefunction and iterating the HF equation until a self-consistent solution arises.
When applying HF to infinite matter, however, the self-consistent interactions can be
avoided. This is because the homogeneous distribution of nuclear matter means that
all momentum states between zero and the Fermi surface are occupied. As such, there
is no need for explicit calculation of the wavefunction of the system and a sum over
single-particle states can be replaced by an integration over momentum space up to
the Fermi surface. For the sake of brevity, from here-on in this thesis I will refrain
from using the nomenclature of second quantization and instead use standard Dirac
notation.
One popular model employed in the field of nuclear structure is the Skyrme force, which is a phenomenological interaction involving 10 free parameters, each of which controls the strength of the nucleon-nucleon (NN) interaction. The Skyrme force was originally devised by T. Skyrme in the 1950s [108]. The components of the interaction depend on the spin and isospin of the nucleons involved and are zero range in nature, meaning that a nucleon will only interact via a contact potential. The different parameters are determined by fitting to known properties and minimising in a $\chi^2$ procedure. These properties include reproducing known ground states of doubly closed-shell nuclei [109] and or fitting to current neutron star models [110]. There are approximately 240 Skyrme forces and each parametrisation has been fitted to a different set of known properties. The most general form of the Skyrme force is [111]:

$$V_{12} = t_0 (1 + x_0 P_\sigma) \delta (\vec{r}) + t_1 (1 + x_1 P_\sigma) \left( \vec{k}'^2 \delta (\vec{r}) + \vec{k}^2 \delta (\vec{r}) \right) + t_2 (1 + x_2 P_\sigma) \vec{k} \delta (\vec{r}) \vec{k} + t_3 (1 + x_3 P_\sigma) \rho^a \delta (\vec{r})$$

(2.29)

where $\vec{k}$ and $\vec{r}$ are the single-particle momentum and positions, respectively. The single-particle momentum $\vec{k}$ is defined as $\vec{k}_1 - \vec{k}_2$ acting on the right, $\vec{k}'$ is $\vec{k}_1 - \vec{k}_2$ acting on the left and the single-particle position $\vec{r}$ is $\vec{r}_1 - \vec{r}_2$. $P_\sigma$ is the spin exchange operator and its function is to exchange the spin of a two-particle wavefunction:

$$\langle \sigma_1, \sigma_2 | P_\sigma | \sigma_1, \sigma_2 \rangle = \langle \sigma_1, \sigma_2 | \sigma_2, \sigma_1 \rangle = \delta_{\sigma_1, \sigma_2}.$$  

(2.30)

It is worth noting at this point there is also an isospin exchange operator, $P_\tau$, [112] which is not usually present in the Skyrme force but will be used later. Skyrme forces are inherently zero-range as can be inferred from the $\delta (r)$ function and whilst this assumption offers significant computational advantages over incorporating a finite range component, there are forces which exist that contain finite range terms. This inclusion of nonzero range terms smooths the HF field and is even more crucial in
extending models beyond HF, particularly when dealing with high momenta [113].

2.2 The Gogny Force - Beyond Zero Range

The first Gogny force was the “D1” interaction, constructed in 1980 [113] and is similar to the Skyrme force in that it is phenomenological in nature. The Gogny force is comprised of 16 free parameters (including two in a spin-orbit term) which act as coefficients for a density dependent term and a Gaussian term. It has a spin and isospin dependent structure and contains, along with the finite range term, a zero range potential. This zero range potential is the same many-body potential from the Skyrme force.

Each parameter is calculated by fitting the properties of the force to known properties. There are currently 11 Gogny interactions available in literature. D1, the first Gogny force, was introduced to provide a finite range component and allow the introduction of explicit pairing in Hartree-Fock-Bogolyubov calculations [113]. D1S was first introduced in 1991 to study large amplitude collective motions without a cutoff [114]. In 1999 Farine et al. devised D1P to improve the agreement of the predicted depth of the optical potential with experimental results. Interestingly for the context of this work, it also aimed to reproduce a more realistic EoS for pure neutron matter (PNM) at high densities [100]. D1M was introduced in 2009 to reproduce masses at an accuracy similar to the best mass formulae. It also sought to reproduce properties of INM and PNM in agreement with calculations from microscopic two- and three-body forces [115]. A set of forces designed to reproduce specific compressibilities were produced by Blaizot et al.; D250, D260, D280, D300 in theory will produce compressibilities of 250 MeV, 260 MeV, 280 MeV and 300 MeV respectively [116]. D1N was also parametrised to reproduce a realistic PNM equation of state, but also to reduce drift in binding energies for parts of the isotopic chains [15]. D1AS is an extension of D1, which has been used in the context of transport calculations [117]. A major motivation for this force was to provide a stiffer symmetry energy, but its nuclear structure
properties have not been explored to my knowledge. GT2, in contrast, was developed to provide realistic nuclear structure calculations including a tensor term, to account for changing shell structure in neutron-rich systems [118].

The Gogny functional has the form:

\[
V(\vec{r}) = \sum_{i=1,2} \left( W_i + B_i P_\sigma - H_i P_\tau - M_i P_\sigma P_\tau \right) e^{-r^2/\mu_i^2} \\
+ \sum_{i=1,2} t_0^i \left( 1 + x_0^i P_\sigma \right) \rho^{\sigma_i} \delta(\vec{r}) \\
+ iW_0^i (\vec{\sigma}_1 + \vec{\sigma}_2) [\vec{k}' \times \delta(\vec{r})\vec{k}],
\]

(2.31)

where each character with a subscript \(i\) represents a free parameter. Regarding the finite range component there are 5 different two dimensional parameters. The first two of these, \(\mu_1\) and \(\mu_2\), control the effective range of the force and usually have values which bracket 1 fm. Due to the fact that the force is Gaussian in structure, chosen because of computational advantages when calculating in a harmonic oscillator basis [119], \(\mu_i\) is the variance of that Gaussian.

The remaining 4 two-dimensional parameters \((W_i, B_i, H_i, M_i)\) comprising the finite range force each have a different effect on the matrix elements of the two-body interaction depending on the spin and isospin exchange operators involved.

The only other parameter not coupled with an exchange operator is \(W_i\), and so for the direct term it describes a spin and isospin independent coupling and will provide a contribution at a given density that is independent of asymmetry. Conversely, the exchange term will only include \(W_i\) interactions between particles with identical spin and isospin.

\(B_i\) is coupled with the spin exchange operator and so the direct term will only
include contributions of $B_i$ for particles with the same isospin, making the total $B_i$ contribution asymmetry dependent but unaffected by spin polarisation. The exchange term is independent of asymmetry, but is sensitive to polarisation effects.

$H_i$ has the opposite behaviour to $B_i$ due to the isospin exchange operator, causing the direct term to be sensitive to polarisation and the exchange term to be sensitive to asymmetry.

$M_i$ is coupled to both the spin and the isospin exchange operators and so acts in an opposite manner to $W_i$; the direct term will only involve interactions between particles of the same spin and isospin whereas the exchange term would be entirely density dependent and unaffected by changes in asymmetry. A full table of the Gogny parametrisations featured in this work can be found in Tables 2.1 and 2.2. The effective ranges, $\mu_i$ are similar for all forces. The parameter $\alpha_1$ is either $\frac{1}{2}$ or $\frac{2}{3}$ for all forces and $\alpha_2$ is unity for all except D1P. The zero-range parameter $t_{02}$ is set to zero for all parameters bar D1P and D1AS, however in D1AS the value is comparatively close to zero. Conversely the parameters $W_i$, $B_i$, $H_i$ and $M_i$ show large model dependence in both magnitude and sign.

The 3rd line of Eq. (2.31) is the spin-orbit component of the Gogny force. In this work we consider only INM, which has constant density within a local volume, and the spin-orbit term depends on the gradient of the density which is zero in the INM regime. It is clear that this term is therefore zero and is not considered. Tensor components have been previously considered [118] but are neglected here for the same reason as the spin-orbit term.

The finite ranged nature of the force causes it to be more computationally intensive when calculating properties of finite nuclei using self-consistent methods such as HF. This increase in processing time is circumvented when calculating properties of INM
<table>
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<th>$W_1$</th>
<th>$W_2$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$M_1$</th>
<th>$M_2$</th>
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Table 2.2: Available Gogny parametrisations used in this work (Part 2)

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</table>

as there is no need for self-consistent iteration. This advantage comes from the homogeneous nature of INM, where all single-particle momentum states up to the Fermi surface are occupied. The homogeneous matter also replaces a discrete sum, of self consistent single-particle momenta, with a continuous integral between zero and the Fermi surface.

For nuclear systems with different isospin contributions, like isospin-polarized matter, one can group the spin-isospin prefactors in the Gogny matrix elements into diff-
ferent terms. The zero-range contribution direct and exchange parts that, in practice, are computed together. For the finite-range terms, however, it is convenient to split the contribution into direct terms, which will be proportional to densities, and exchange terms, which involve more complicated functions of Fermi momenta. For the zero-range and direct terms, for instance, one can differentiate between isoscalar (0 subscript):

\[
A_i^0 = \frac{\pi^{3/2} \mu_i^3}{4} \left[ 4W_i + 2B_i - 2H_i - M_i \right], \tag{2.32}
\]
\[
C_i^0 = \frac{3}{4} t_0, \tag{2.33}
\]

and isovector (1 subscript):

\[
A_i^1 = \frac{\pi^{3/2} \mu_i^3}{4} \left[ -2H_i - M_i \right], \tag{2.34}
\]
\[
C_i^1 = -\frac{1}{4} t_0 \left( 1 + 2x_0 \right), \tag{2.35}
\]

contributions. For the finite-range exchange contribution, it is useful to consider terms associated to equal and unequal isospin pairs:

\[
B_{nn}^i = B_{pp}^i = -\frac{1}{\sqrt{\pi}} \left[ W_i + 2B_i - H_i - 2M_i \right], \tag{2.36}
\]
\[
B_{np}^i = \frac{1}{\sqrt{\pi}} \left[ H_i + 2M_i \right]. \tag{2.37}
\]

It is also useful to introduce the exchange isoscalar and isovector terms:

\[
B_0^i = B_{nn}^i + B_{np}^i
= -\frac{1}{\sqrt{\pi}} \left[ W_i + 2B_i - 2H_i - 4M_i \right], \tag{2.38}
\]
\[
B_i^i = B_{nn}^i - B_{np}^i = -\frac{1}{\sqrt{\pi}} \left[ W_i + 2B_i \right]. \tag{2.39}
\]

In infinite matter, and within the HF approximation employed in this work, the
bulk properties have contributions associated with the exchange term that are proportional to double integrals of Gaussians over the Fermi surfaces of neutrons and protons. These integrals can be computed analytically, and give rise to a series of polynomial and Gaussian functions. These are discussed later in Chapter 3.

![Figure 2.1: Matrix elements of Eqs. (2.32) to (2.37) for the 11 Gogny functionals under consideration. The first (second) coefficient of every couple corresponds to } i = 1 (i = 2). For the zero range terms, C_0 and C_1, we only present } i = 1. The units are MeVfm^3 for } A_i’s; MeV for } B_i’s and MeVfm^{3α_i+3} for } C_i’s.

The numerical parameters appearing in Eq. (2.31) are obtained by a fitting procedure of the Gogny functional. With these, one can compute the different matrix elements. We note that the parameters can be degenerate, in the sense that only linear combinations enter the fitting procedure. In addition, the separation of matrix elements in zero-range, direct finite-range and exchange terms is arbitrary. Therefore, the independent values of these matrix elements are not necessarily meaningful. Some specific parameters, however, do determine physical properties, and hence it can be interesting to find out their values. We provide a plot with the values of these matrix elements in Fig.2.1, which provides, at a glance, an explanation of the different isovec-
tor and isoscalar parameters for the functionals under consideration.

There are, for instance, common trends that can impact isoscalar and isovector properties of matter. Most forces prefer a large positive $A_1^1$ and a negative $A_1^2$, suggesting cancellations in the isovector, direct finite-range part of the functional. In contrast, the majority of forces prefer negative $B_{xy}^{1,2}$, which suggests that the exchange terms act as overall attractive contributions. The isospin singlet zero-range, $C_0$, term is repulsive, as expected from the usual density-dependent terms of the functionals. In contrast, all forces, except for D1AS, present attractive $C_1$ contributions. This suggests a dominance of attractive terms in the isovector channels which, as I shall show, hampers the development of stiff symmetry energies.

In addition, we find a relatively large spread for most parameters. This is a sign of large functional dependence or systematic uncertainty [120]. In particular, all the short-range parameters $A_1^1$ and $B_1^1$ present a much larger variability than their long-range counterparts, $A_1^2$ and $B_1^2$. In terms of functionals, D1M is an outlier as compared to most other parametrizations, with extreme values of finite-range exchange parameters, $B_{nn}^1$ and $B_{np}^1$. The specific optimisation procedure of this force should most likely account for these large differences [121]. Similarly, GT2 also shows a distinct behaviour for $A_1^{1,2}$ and $B_{nn}^{1,2}$, as already acknowledged in the original publication [122].

The isoscalar zero range matrix elements $C_0^1$ are, as expected, all repulsive, whereas their isovector counterparts, $C_1^1$, are attractive and of a similar order of magnitude\(^1\). We do not show the $C_2^x$ parameters, since they are zero for all forces except for D1P, where they are repulsive $C_2^x \approx 192$ MeVfm\(^4\).

With the Gogny force and its various parametrisations defined, we can now explore the isovector and isoscalar properties of the Gogny force.

---

\(^1\)All forces have $\alpha_i = \frac{1}{3}$ except for D250 and D300 which have $\alpha_i = \frac{2}{3}$.\(^{30}\)
Chapter 3

Isoscalar and Isovector Properties

The isoscalar and isovector properties of a force can play a large role in both the very small, such as neutron skins, and very large, such as neutron stars. As such, good constraint on these properties is needed if one wishes to construct a consistent model of a neutron star, detailing both the micro- and macro-physics with one single force. In this chapter I explore the isovector and isoscalar properties of the Gogny interaction and discuss what the results mean for the Gogny force in neutron star calculations.

3.1 Introduction to Bulk Properties and Single-Particle Potentials

3.1.1 Energy-Per-Particle

The energy-per-particle \( (E/A) \) at a given density and asymmetry is used to identify whether or not a system is bound. A system will be bound if \( E/A \) is negative and unbound if \( E/A \) is positive. The way the energy-per-particle changes with particle number has large implications for the structure of finite nuclei, as well as the behaviour of nuclear matter. As such, I now relate the Bethe-Weizsäcker mass formula to INM:

\[
B.E. = a_v A - a_s A^{2 \frac{2}{3}} - a_e Z (Z - 1) A^{-\frac{1}{3}} - a_{sym} \frac{(A - 2Z)^2}{A} + \delta, \tag{3.1}
\]
where $a_v$ is the volume term, $a_s$ is the surface term, $a_c$ is the Coulomb term, $a_{sym}$ is the symmetry term, $\delta$ is the pairing term and $A$ is the number of nucleons [123]. A simple division by $A$ yields the binding energy per particle:

$$
\frac{B.E.}{A} = a_v - \frac{a_s}{A^{\frac{2}{3}}} - \frac{a_c}{A^{\frac{2}{3}}} \frac{Z(Z-1)}{A^{\frac{2}{3}}} - a_{sym} \frac{(A-2Z)^2}{A^2} + \frac{\delta}{A}.
$$

(3.2)

In INM $A \to \infty$ and so the surface, Coulomb, and pairing terms all go to zero. Considering symmetric $N = Z$, matter the symmetry term also becomes zero because $A - 2Z = 0$. Thus Eq. (3.2) simplifies to:

$$
\frac{B.E_{INM}}{A} \approx a_v.
$$

(3.3)

In the context of INM one can determine from Eq. (3.3) that $E/A$ represents the volume term, $a_v$. The Bethe-Weizsäcker formula constituent terms have been intensively studied and so the volume term has been constrained in the region of -15.5 MeV [123]. Correspondingly the energy per particle for INM is well known and has been constrained both experimentally and in theory, often cited in the region of $16 \pm 1$ MeV [124].

$E/A$ of INM as a function of density forms a parabolic curve. The minimum of $E/A$ occurs at the saturation density, $\rho_0$. This, being the point of minimal energy, is the density to which INM will tend. As mentioned above, the value of $E/A$ at the saturation density is expected to be around -16 MeV, and the curvature is described by the incompressibility. The range of empirical constrains on the saturation density place it at around 0.16 fm$^{-3}$ (typical examples of which are $0.17 \pm 0.03$ fm$^{-3}$ in [125], and $0.156 \pm 0.015$ fm$^{-3}$ in [124]). The saturation density also has a role to play in finite nuclei calculations. Following the textbook by Krane [126], the central density is
almost the same for most nuclei, though nuclei can be more or less dense\(^1\). Other than
at the surface of nuclei, the number of nucleons per unit volume is relatively constant;
that is to say,

\[
\frac{A}{\frac{4}{3} \pi R^3} \approx \text{Constant},
\]

(3.4)

and

\[
R = R_0 A^{\frac{1}{3}}.
\]

(3.5)

Electron scattering measurements have given \(R_0\) to be approximately 1.2 fm [126].
Taking this measurement one would expect a single nucleon to occupy an area bounded
by radius \(R_0\), and as such the nucleon density would be \((\frac{4}{3} \pi 1.2^3)^{-1}\), which is equal to
0.138 fm\(^{-3}\). This is slightly lower than the value given above but this is due to the fact
that INM does not decrease in density towards the surface, whereas finite nuclei do,
which drives their average density down. The nuclear matter saturation density enters
into the derivations in section 3.1.4, so any model aiming to reproduce properties of
INM should predict a saturation density within this empirically defined region.

### 3.1.2 Single-particle potentials

I continue my discussion by looking at a series of single-particle properties of asym-
metric nuclear matter as obtained by different Gogny functionals at different densities.
All these properties characterise, in one way or another, the single-particle potential
of a neutron or a proton with momentum \(k\), denoted by \(U_{\tau}(k)\). The isospin index, \(\tau\),
corresponds to a neutron, \(\tau = +\), or a proton, \(\tau = -\). I work within the Hartree-Fock
approximation and, in asymmetric infinite matter, the single-particle potential is the

\(^1\)A list of nuclear central densities can be found at [www.astro.ulb.ac.be/bruslib/](http://www.astro.ulb.ac.be/bruslib/)
result of an integral and spin average over the neutron and proton Fermi surfaces:

\[ U_\tau(k) = \frac{1}{2} \sum_{\sigma, \tau'} \int \frac{d^3 k'}{(2\pi)^3} \langle k \sigma \tau; k' \sigma' \tau' | V | k \sigma \tau; k' \sigma' \tau' \rangle_A, \tag{3.6} \]

so that the integral runs only for \( k' < k_F' \). The subscript \( A \) denotes an antisymmetrization in the matrix element.

Using the expression for the Gogny functional in Eq. (2.31), one can find analytical expressions for \( U_\tau(k) \). The zero-range and the direct terms, for instance, are momentum independent and can be integrated straight away. This gives rise to two terms proportional to the density, or to \( \rho^{\alpha_i+1} \), in case of the zero-range contribution. The exchange term, arising from the antisymmetrization, is more cumbersome. It involves an integral over a Gaussian momentum factor, which includes at least one angular integration. This integral can be computed in a closed form, and I provide the explicit expression in Appendix A, Eq. (B.1). The isovector contribution is proportional to the isospin-asymmetry, \( \beta = \frac{\rho_n - \rho_p}{\rho} \). The result of the integral in the exchange finite-range contribution depends on the Fermi momenta of each species, \( k_F' = \left( \frac{3\pi^2}{2} \rho \left[ 1 \pm \beta \right] \right)^{1/3} \).

I do not provide the rearrangement term, Eq. (B.4), explicitly here, but include it in all figures that need it. The function \( u(q, q_F) \), given in Eq. (B.2), involves Gaussians and error functions. This encodes the momentum dependence of the single-particle potential, as all the other terms are constants with respect to \( k \).

I show the single-particle potentials, \( U_\tau(k) \), as a function of momentum in Fig. 3.1. The different panels correspond to the different Gogny parametrizations\(^1\). The results have all been computed at \( \rho = 0.16 \text{ fm}^{-3} \) at different isospin asymmetries (see Figure caption for details). I highlight with symbols the neutron (triangles) and proton (circles) potentials at the respective Fermi surfaces. These contributions are relevant for the understanding of the evolution of isospin in single-particle properties.

\(^1\)We ignore D1AS for the time being, as its momentum dependence is identical to D1.
Figure 3.1: Single-particle potential of neutrons (solid triangles) and protons (solid circles) as a function of momentum for 6 different isospin asymmetries: $\beta = 0$ (solid line), 0.2 (long dashed line), 0.4 (short dashed line), 0.6 (dotted line), 0.8 (dashed-dotted line) and 1.0 (double-dotted dashed line). The results have been obtained at $\rho = 0.16 \text{ fm}^{-3}$. The symbols denote the single-particle potential of a neutron (triangles) or a proton (circles) at the respective Fermi momentum, $k_F$.

The single-particle potential for symmetric matter (solid lines) is rather well con-
strained at low momenta. All forces predict values $U_\tau(0) \approx 70 - 80$ MeV at zero momentum and $\rho = 0.16$ fm$^{-3}$. At low momentum, below about 2 fm$^{-1}$, the symmetric matter single-particle potentials are similar. As a function of momentum, $U(k)$ generally increases with $k$ in this region. Above $k \approx 2$ fm$^{-1}$, for some functionals the potential saturates (D1N, D1P or GT2), decreases (D1S, D1M, D250) or increases (D260, D280, D300). For symmetric matter, the momentum dependence is dictated by the sum of two terms, governed by $B_0^i$ and the function $u(q, q_F)$ evaluated at the Fermi momentum, $q_F = \mu_i k_F$, of symmetric matter. I note that in the $k \gg 1$ limit, the momentum-dependent exchange term becomes negligible and the direct and zero-range terms dominate. Hence, the high-momentum value is entirely dominated by momentum-independent terms.

The interplay between asymmetry and momentum is also relevant, particularly in transport calculations [91, 127, 128]. The data presented in Fig. 3.1 shows that, at low momentum, the isospin asymmetry dependence of single-particle potentials is relatively well constrained, at least around saturation density. In all cases, I observe an increase of the neutron potential, $U_n(k)$, as a function of asymmetry, whereas the proton potential, $U_p(k)$, decreases. This corresponds to the physically intuitive idea that neutrons (protons) are less (more) bound in neutron-rich systems. In the low momentum region, below $\approx 2$ fm$^{-1}$, the dependence in asymmetry is rather monotonous. For an increase of 0.2 in asymmetry, I find a steady decrease of $U_p$ by around $\approx 10$ MeV. In contrast, the neutron potential increases by about $5 - 7$ MeV in a pattern that is less linear. This suggests that, in the limit of neutron-rich systems, the non-linear exchange terms dominate the isospin dependence of the single-particle momentum. For neutrons in neutron matter, the single-particle potential at the Fermi surface is $U_n \approx -32$ MeV, with a spread of around $5 - 10$ MeV. In the limit of extreme isospin imbalance ($\beta = 1$), I find that most forces predict a similar value for the zero-momentum proton potential, $U_p(0) \approx -115$ MeV, with a spread of around 10 MeV. A proton impurity is a particularly interesting system, as its momentum dependence is entirely
The asymmetry dependence in the high momentum region, \( k \gtrsim 2 \text{ fm}^{-1} \), is less constrained. One finds a large variety of results. For D1, for instance, the single-particle momentum of neutrons for \( k > 3 \text{ fm}^{-1} \) is lower than that of protons. This inversion occurs also for D260, D270, D300 and GT2 in a region ranging between 2 and 3 fm\(^{-1}\). The results obtained with D1S, D1P and D250 suggest that the asymmetry dependence of the neutron potential is very weak above 2.5 fm\(^{-1}\). In stark contrast, D1M suggests a strong increase (decrease) of \( U_n (U_p) \) with asymmetry in the large momentum region. As a matter of fact, for this force at large asymmetries, \( U_p \) decreases rather steeply as a function of momentum. I will see the consequences of these behaviours in the analysis of the effective mass that follows. All in all, Fig. 3.1 suggests that the isospin asymmetry dependence of the high-momentum single-particle properties is not constrained in the Gogny functional. One could foresee improvements in this direction by using fitting protocols that take into account the information available from realistic many-body calculations in isospin asymmetric nuclear matter [131, 132, 133].

Up to this point, I have only displayed results computed at a single density, \( \rho = 0.16 \text{ fm}^{-3} \). One would like to know whether similar issues are found at other densities. Rather than showing the whole asymmetry dependence of the single-particle potentials for different densities, I opt for displaying separately the isovector and isoscalar components of the single-particle potential. As described in Appendix B, one can introduce an isoscalar potential, \( U_0(k) \), which is essentially the average of the neutron and proton potentials. This basically corresponds to the single-particle potential of symmetric matter, which one would expect to be well constrained at sub-saturation densities by nuclear data. The isovector component, \( U_{1\text{sym}}(k) \), is the first derivative with respect to \( \beta \) of the single-particle potentials. It therefore encodes information on how the potentials evolve with asymmetry.
Figure 3.2: Isoscalar (top panels) and isovector (bottom panels) components of the single-particle potential as a function of momentum. Results for all Gogny functionals are displayed at 3 densities: $\rho = 0.08$ fm$^{-3}$ (left panels), $\rho = 0.16$ fm$^{-3}$ (central panels) and $\rho = 0.24$ fm$^{-3}$ (right panels) are displayed. The grey band in the bottom central panel is the allowed region of saturation isovector single-particle potentials obtained in Ref. [134]. The arrows mark the position of the Fermi momentum at each density.

We show the isoscalar potential for three characteristic densities in the top panels of Figure 3.2. At half saturation (top left panel), the low-momentum part of the single-particle potential is very well constrained, in the sense that all functionals predict very similar values. At $k = 0$, for instance, one finds $U_0(0) \approx -50$ MeV. The increase of $U_0(k)$ with momentum is rather mild and, at $k = 2$ fm$^{-1}$, most potentials are around $-30$ MeV. As momentum increases past this point, though, a relatively large spread of values develops.

At saturation (central panels), one finds qualitatively equivalent results. The low-momentum region of the potential is well constrained, although larger differences between functionals are observed. In the region $k \approx 1$ fm$^{-1}$, all isoscalar potentials...
are close to $U_0 \approx -60$ MeV. Above this momentum region, large divergences appear. Whereas some functionals increase indefinitely with momentum, others saturate or even decrease. For densities above saturation, in contrast, there is a relatively wide spread of potentials both at low and at high momenta. For instance, the zero-momentum single-particle potential is predicted to range between $-90$ and $-70$ MeV. As with other densities, the spread increases substantially above $k > 2$ fm$^{-1}$.

Regarding the momentum dependence of the different functionals, it is interesting to note that it is (up to density-dependent normalizations) essentially the same at different densities. D1S, for instance, predicts at all density an isoscalar potential that increases, then saturates and further on decreases with momentum. In contrast, the D1N isoscalar potential increases monotonously at low $k$ and saturates rather quickly at all densities. D280 is extreme, in that its momentum dependence is the steepest, with a large increase in $U$ as $k$ becomes larger.

We note that the functionals with largest $U_0$ at high momentum (D280, D260 and D300) are those that displayed large $A_0^i$ values in Fig.2.1. At large $k$, one expects the exchange term in $U_0$ to become negligible. The isoscalar single-particle potential hence tends to the momentum-independent value:

$$U_0(k \gg 1) \approx \sum_{i=1,2} \left[ A_0^i + C_0^i \rho^{\alpha_i} \right] \rho.$$  \hspace{1cm} (3.7)

Forces with large $A_0^i$ and $C_0^i$ matrix elements will eventually show large momentum-independent contributions in the high-momentum region of $U_0(k)$. Moreover, since most zero-range isoscalar couplings $C_0^i$ are positive, those functionals that have large and positive $A_0^i$ will develop strongly repulsive single-particle potentials at high momenta and large densities. Note that D1P was precisely fit to have $U_0(k)$ cross zero around $k \approx 3.2$ fm$^{-1}$ and tend to an asymptotic value of 30 MeV for $k \gg 1$ [135].
The bottom panels of Fig. 3.2 show a strikingly different pattern. Even at sub-saturation densities (bottom left panel), the low-momentum components of the isovector potentials coming from different functionals are rather different. At zero momentum, I find values ranging from $U_1(0) \approx 19$ to $35$ MeV, with significant divergences. At saturation (bottom central panel), most functionals predict $U_1(0) \approx 33$ MeV, except for GT2, D1M and D1N. Between $k = 1.4$ and $1.6$ fm$^{-1}$, there is an area of overall agreement between functionals, with values of $U_1 \approx 25$ MeV, but results diverge even more than in the isoscalar sector as momentum increases. The isovector potentials above saturation (bottom right panel) cover a wide range of values, which indicates that they are not constrained by the parameter fitting procedure.

A recent analysis of the isovector optical potential and its connection to bulk properties suggests that $U_1$ should decrease with momentum [127]. I show with a grey band the allowed region of the saturation density isovector single-particle potential as obtained from the optical potential fits of Ref. [134]. At low momentum, the allowed region is above all single-particle potentials. The steep decrease as a function of momentum suggested by this analysis is only reproduced by a minority (D260, D280 and GT2) of extreme functionals. Empirical values and theoretical predictions seem to have somewhat similar momentum dependences, but the absolute values of $U_0(k)$ are somewhat too low. This shift in absolute values could be due to the lack of non-locality corrections in $U_0(k)$ [136, 137, 134]. I note that the results for $U_0$ and $U_1$ agree with those presented in Ref. [91] for the corresponding parametrizations.

The high momentum asymmetry of $U_1$ is also determined entirely by the direct and zero-range matrix elements:

$$ U_1(k \gg 1) \approx \sum_{i=1,2} \left[ A_1^i + C_1^i \rho^i \right] \rho. \quad (3.8) $$

Hence, the extremely large and positive $A_1^i$ values of D1M and D1N dictate limiting
values of $U_1$ which are large, positive and increasing with density. In contrast, because $A_1^1 \ll 0$ for GT2, its isovector potential becomes very negative as momentum increases.

To some extent, the large differences among functionals are not surprising. The fitting procedure includes only a series of points of finite nuclei, which are typically sub-saturation systems, and bulk, zero-temperature saturation matter properties. All of these data are essentially determined by single-particle properties at (a) densities below saturations and (b) momenta below the Fermi momentum, which is typically of the order $k_F \approx 1 - 1.3 \text{ fm}^{-3}$. In addition, since most of these systems are almost isospin-symmetric, the single-particle isovector properties are rather poorly constrained. Consequently, the single-particle potential is only well constrained at low momentum, below saturation and near symmetric systems.

### 3.1.3 Effective masses

The effective mass provides a sensitive characterisation of the momentum dependence of the single-particle potential. I note again that the momentum dependence of the mean-field is exclusively due to the exchange term, as both the direct and zero-range contributions are constants as a function of $k$. In other words, the function $u(q, q_F)$, given in Eq. (B.2) is entirely responsible for the non-trivial effective mass. As a consequence, the effective mass, $m^*_\tau$, is only proportional to the matrix elements $B^{nn}$ and $B^{np}$:

$$\frac{m_N}{m^*_\tau} = 1 + \frac{m_N}{2\hbar^2} \frac{\partial U_\tau(k)}{\partial k} = 1 + \frac{m_N}{2\hbar^2} \sum_{i=1,2} \left\{ B^{ii}_{nn} m(\mu_i k, \mu_i k_F^+) + B^{ii}_{np} m(\mu_i k, \mu_i k_F^-) \right\},$$

(3.9)

where $M_N$ is the nucleon rest mass. The function $m(q, q_F)$, given in Eq. (B.6), is essentially a momentum derivative of the $u(q, q_F)$ appearing in the single-particle potential.

In analogy to Fig. 3.1, I present in the 10 panels of Figure 3.3 the results for the
Figure 3.3: Effective mass of neutrons (solid triangles) and protons (solid circles) as a function of momentum for 6 different isospin asymmetries. The results have been obtained at $\rho = 0.16$ fm$^{-3}$. See Fig. 3.1 for further explanations.

The effective mass as a function of momentum at $\rho = 0.16$ fm$^{-3}$ for different functionals. I explore different asymmetries and show the corresponding Fermi momentum effective masses of neutrons (protons) with solid triangles (solid circles). D1AS is a momentum independent extension of D1, so their effective masses are the same.
In this case, the results of Fig. 3.3 indicate that all effective masses at the Fermi surface are similar. I note, however, that the momentum dependence of $m^*_\tau(k)$ is relatively disparate for the different functionals. D1 shows a steady increase, saturating to $m^*\approx m$ at large momenta. Qualitatively similar results are found for D1N, D1P, D260, D300 and GT2. In contrast, D1S, D1M and D250 go through a maximum at $k\approx 3.5\text{ fm}^{-1}$ before settling down at $m^*\approx m$. D280 is unique, in that the effective mass at low momentum is a decreasing function of momentum.

The asymmetry dependence of the effective masses is also rather heterogeneous. For most forces, the neutron effective mass increases with asymmetry at low momentum. For D1P, for instance, an increase of 0.2 in asymmetry results in an increase in the effective mass of around 0.02. The decrease in proton effective mass is slightly smaller. If one excludes D1N, the effective mass of a proton impurity in neutron matter at $k=0$ falls within $m_p^*(k=0)\approx 0.48 - 0.62$. In contrast, the neutron effective mass in neutron matter changes from $m_n^*(k_F)\approx 0.68$ to 1.26. Again, I point out that the impurity system could be used to constrain the value of $B_{np}^{*}[129, 130]$.

D1N and D250 are exceptional, in the sense that their neutron effective masses at low momenta hardly depend on asymmetry. The momentum dependence of the neutron effective mass for D1M is striking, as it changes drastically from low to large asymmetries. In particular, the proton effective mass at large momenta increases substantially, to values above $m_p^*\approx 1.7m_N$ in neutron matter. GT2 (bottom right panel) shows a similarly large dependence in asymmetry, but in this case it is the neutron effective mass that grows substantially with asymmetry.

Note also that D1S, D1M and D250 show a crossing point, above which the neutron effective masses is smaller than the proton effective mass. Again, because this high-momentum, high-asymmetry region is relatively unconstrained, it is not surprising to
Effective mass splitting, \( \frac{m^*_n - m^*_p}{m_N} \)

Asymmetry, \( \beta \)

D1
D1S
D1N
D1M
D1P
D250
D260
D280
D300
GT2

Figure 3.4: Isovector mass splitting at the Fermi surface as a function of isospin asymmetry at \( \rho = 0.16 \text{ fm}^{-3} \). Most functionals show a linear dependence at small asymmetries.

find this variety of results. It would be interesting to explore whether these reordering with momentum has any impact on transport calculations [138] and observables of low energy nuclear collisions [128].

The effective mass is often characterised by its value at the Fermi surface, \( m^*_\tau(k = k_F^\tau) \), rather than by its full momentum dependence. This allows for a simple characterisation of the variation of momentum dependence with asymmetry, at least close to the respective Fermi surfaces. Since \( m \) is symmetric in its two arguments, the isovector effective mass splitting is proportional to \( B_{nn}^i \) only and to the difference of two symmetric \( m \) functions evaluated at the two Fermi surfaces:

\[
\frac{m_N}{m^*_n} - \frac{m_N}{m^*_p} = \sum_{i=1,2} B_{nn}^i \left[ m(\mu_i k_F^n, \mu_i k_F^n) - m(\mu_i k_F^p, \mu_i k_F^p) \right].
\]  

(3.10)

Because of its simplicity, this splitting can be analysed rather straightforwardly.
For symmetric arguments, $m(q_F, q_F)$ is a decreasing function of its argument up to $q_F \approx 2.38$. Consequently, as long as $\mu_i k_F^i \lesssim 2.378$ and for $k_F^n > k_F^p$, the difference of $m$ functions will be positive. In these conditions, the sign of the isovector splitting will be proportional to that of $B_{nn}$.

These findings are confirmed by the results displayed in Figure 3.4, where I show the isovector effective mass splitting as a function of isospin asymmetry at $\rho = 0.16$ fm$^{-3}$. 10 functionals have positive splittings, with a neutron effective mass larger than a proton effective mass, $m_n^* > m_p^*$. The exception is D1N, which has a very small but negative isovector splitting [139]. In contrast, since GT2 has a large and positive $B_{nn}^2$, its isovector mass splitting is almost twice as large as the results from other functionals. For the typical isospin asymmetry of heavy nuclei, $\beta = 0.2$, the isovector mass splitting predicted by most forces is of the order $m_n^* - m_p^* \approx 0.04 - 0.06$.

It is also interesting to note that most functionals show a linear dependence in $\beta$ for small asymmetries. This weak asymmetry dependence is entirely due to the shift in Fermi momenta within the $m$ functions in Eq.(3.10). The slope of the curve could potentially be used to constrain $B_{nn}^2$, if accurate data for the isovector effective mass splitting was available. Efforts in this direction using optical model potentials are already underway [136, 137, 134]. A stronger dependence in asymmetry appears as the neutron matter limit is approached. At that point, the functional form of $m$ becomes relevant and a non-linear shape is expected.

### 3.1.4 Symmetry Energy

One property of infinite nuclear matter to be explored is the Symmetry Energy, $S$. It is worth mentioning at this point that it is not uncommon in the literature for the symmetry energy to take the nomenclature of $S_2$ or $J$. This notation will not feature
in this work but may feature in the references within. $S$ is the energy it would take to turn all the protons within an initially isospin symmetric\footnote{same number of neutrons and protons} system into neutrons. Around saturation density the symmetry energy $S(\rho_0)$ is relatively well known but its value at other densities has not yet been constrained [140]. To calculate the symmetry energy one can take the absolute difference between $E/N$ for PNM and SNM. The symmetry energy calculated in this manner will be referred to as “Sym$_{Diff}$”:

$$Sym_{Diff} = E_{PNM} - E_{SNM}. \quad (3.11)$$

The symmetry energy plays a large role in both microscopic and macroscopic areas of physics, with its density dependence having significance in nuclear astrophysics [141] as well as in experimental nuclear physics such as that involving Isobaric Analogue States [142], nuclei which have the same number of nucleons and differ by 1 unit of isospin but share a resonant excitation [143]. The symmetry energy as a function of density governs the proton fraction predicted in NSM. The proton fraction of NSM is known to hint at the cooling history of the neutron star. If the proton fraction exceeds $11 - 15\%$ then the powerful Direct Urca process can occur, significantly affecting the cooling gradient of the neutron star via neutrino emission [144].

To explain how the symmetry energy fits into the overall INM equation of state it is useful for us to examine the Taylor expansion of the energy around saturation density and around isospin symmetric matter. To do this, I begin with the expansion of energy in both density-space and asymmetry-space up to the 3rd order. To allow us to construct density and asymmetry dependent equations I first define some physical values. As well as using the asymmetry parameter defined above, I also introduce a new parameter, $X$: 
\[ X = \frac{\rho - \rho_0}{3\rho_0} \]  

which is useful to describe the density deviation from \( \rho_0 \). The division by 3 simplifies the calculation for the Taylor expansion, and so one can write the Taylor expansion for the energy around \( \beta = 0 \) and \( \rho = \rho_0 \) (or \( X = 0 \)):

\[
E(\rho, \beta) = E(\rho = \rho_0, \beta = 0) \\
+ 3\rho_0 \frac{\delta E(\rho, \beta)}{\delta \rho} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \beta \frac{\delta E(\rho, \beta)}{\delta \beta} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \frac{9\rho_0^2}{2!} \frac{\delta^2 E(\rho, \beta)}{\delta \rho^2} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \frac{\beta^2}{2!} \frac{\delta^2 E(\rho, \beta)}{\delta \beta^2} \bigg|_{\rho=\rho_0, \beta=0} \\
+ 3\rho_0 \frac{\delta^2 E(\rho, \beta)}{\delta \rho \delta \beta} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \frac{9\rho_0^2}{2!} \frac{\delta^3 E(\rho, \beta)}{\delta \rho^3} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \frac{3\rho_0 \beta^2}{2!} \frac{\delta^3 E(\rho, \beta)}{\delta \rho^2 \delta \beta} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \frac{27\rho_0^3}{3!} \frac{\delta^3 E(\rho, \beta)}{\delta \rho^3} \bigg|_{\rho=\rho_0, \beta=0} \\
+ \frac{\beta^3}{3!} \frac{\delta^3 E(\rho, \beta)}{\delta \beta^3} \bigg|_{\rho=\rho_0, \beta=0}. 
\]  

(3.13)

It is possible to simplify Equation (3.13) by examining the expansion at limits of \( \rho \) and \( \beta \). The expansion is taken to be at saturation density \( (\rho = \rho_0) \) which is by definition at the minimum of the curve. This is where the gradient of the energy per particle with respect to density is 0. This limit immediately removes the first density derivative from the Taylor expansion. Further terms can be removed by working under
the assumption that the nuclear force is isospin symmetric [145]. Isospin symmetry means that the system remains the same when all protons are exchanged for neutrons and vice versa. Consequently $E(\rho, \beta) = E(\rho, -\beta)$ and therefore the energy should be an even function of asymmetry, which implies that there can be no odd powers of $\beta$ within the expansion. Removing these terms gives a modified Taylor expansion:

\[
E(\rho, \beta) = E(\rho = \rho_0, \beta = 0) + \frac{9\rho_0^2 X^2}{2!} \frac{\delta^2 E(\rho, \beta)}{\delta \rho^2} \bigg|_{\beta=0} + \frac{\beta^2}{2!} \frac{\delta^2 E(\rho, \beta)}{\delta \beta^2} \bigg|_{\rho=\rho_0} + \frac{3\rho_0 X \beta^2}{2!} \frac{\delta^3 E(\rho, \beta)}{\delta \rho \delta \beta^2} \bigg|_{\rho=\rho_0, \beta=0} + \frac{27\rho_0^3 X^3}{3!} \frac{\delta^3 E(\rho, \beta)}{\delta \rho^3} \bigg|_{\rho=\rho_0}.
\]

(3.14)

Now that I have this expansion for asymmetric matter around saturation density I can investigate the isovector properties of INM. As I am currently interested in the symmetry energy I will examine the difference in energy of SNM and PNM by setting $\rho$ to be $\rho_0$ and comparing the Taylor expansion up to 3rd order at $\beta = 0$ and $\beta = 1$.

$\beta = 0$:

\[
E(\rho_0, 0) = E(\rho = \rho_0, \beta = 0),
\]

(3.15)

$\beta = 1$:
\[ E(\rho_0, 0) = E(\rho = \rho_0, \beta = 0) \]
\[ + \frac{1}{2!} \left. \frac{\delta^2 E(\rho, \beta)}{\delta \beta^2} \right|_{\beta=0} \]

and inserting Eqs. (3.15) and (3.16) into Eq. (3.11) one can find an analytic approximation for the symmetry energy:

\[ S(\rho) = \frac{1}{2!} \left. \frac{\delta^2 E(\rho, \beta)}{\delta \beta^2} \right|_{\beta=0}. \]  

(3.17)

Because this equation represents the neutron matter case minus the symmetric matter case it can be approximated by \( Sym_{Diff} \). The two are not precisely equal because in the above case I have truncated the Taylor expansion to 3\(^{rd}\) order, neglecting higher order components of \( Sym_{Diff} \). In this thesis I consider the symmetry energy to be the one defined in Eq. (3.17), \( S(\rho) \). Having expressed the symmetry energy it is now possible to investigate the density dependence of \( S \), known as the “Slope Parameter”, \( L \).

### 3.1.5 Slope Parameter

To define the slope parameter, \( L \), consider the first order derivative of the symmetry energy with respect to density:

\[ L = 3\rho_0 \left. \frac{\delta S(\rho)}{\delta \rho} \right|_{\rho=\rho_0} = 3\rho_0 \frac{\delta^2 E(\rho, \beta)}{2 \delta \beta^2 \rho} \left|_{\rho=\rho_0, \beta=0} \right. . \]

(3.18)

Because \( L \) is the first derivative of \( S \) with respect to density it also represents the first order term in the Taylor expansion of \( S(\rho) \). The differentiation with respect to \( \rho \) in Eqn. 3.14 means that all of the density independent terms will disappear.

The slope parameter is a measure of the density dependence of the symmetry en-
ergy and is the following non-zero term in the Taylor expansion of the energy with respect to density and asymmetry. The slope parameter has a significant impact on the behaviour of heavily isospin asymmetric matter and effects many facets of nuclear physics. Its effect ranges from the very small, such as neutron skin thicknesses [5] where the slope parameter of the EoS is heavily correlated to the skin thickness produced, to the very large, such as astrophysical observations, where the softness or stiffness of an EoS governs the maximum radius of a neutron star through the relationship between $L$ and pressure [146]. The latter is a matter of particular interest at this time due to the recent observation of a 2 solar mass neutron star [41], ruling out an EoS that is particularly soft at all densities. In most realistic frameworks a soft EoS places a strict lower bound on $L$. It is worth noting that a soft EoS at saturation density does not necessarily prevent a stiffer EoS at other densities.

Nuclear reactions involving neutron rich nuclei and investigation of radionuclide structure provide an opportunity to constrain the symmetry energy [147]. The constraints on the slope parameter have large uncertainties at this time. Listed here are a selection of those constraints which will be used later to evaluate the reliability of the different Gogny forces when it comes to predicting they symmetry energy and slope of INM. The Pygmy Dipole Resonance (PDR) is a low lying electric dipole excitation. Along with the Giant Dipole Resonance (GDR) the PDR is a useful tool to examine symmetry energy due to the isovector dependence of these resonances. Investigating these properties in finite nuclei [145] determined $S = 32.3 \pm 1.3$ MeV and cites the slope envelope as $L = 64.8 \pm 15.7$ MeV. Constraints from neutron star observations have offered predictions for the symmetry energy of $31.6 \pm 1.9$ MeV [148] and neutron star matter EoS predict $34 \pm 4$ MeV [149]. Tsang et al in [150] compared several different constraints on the symmetry energy and gave a final estimation of $S = 30.85 \pm 0.75$ MeV. Measurements from neutron skin thicknesses predict a slope parameter of $L = 58 \pm 18$ MeV [151], whilst isospin diffusion measurements predict $L = 88 \pm 25$ MeV [152].
The slope parameter is heavily correlated to the thickness of the neutron skin [153]. Recent model dependent experiments have constrained the neutron skin thickness relatively well [154], however there have, as of yet, been no experiments that successfully constrain the neutron skin thickness in a model independent manner, without large error bars [155]. An experiment has been carried out at JLAB, using parity violating electron scattering in lead (PREX) to determine a model independent neutron skin measurement [156]. Unforseeable issues adversely affected the statistics gleaned from the data and provided very large error bars in the results and so a very precise measurement from the same method will not be available until the completion of PREX-2, which will hopefully provide a narrow constraint on the skin thickness and therefore also the slope parameter. Pion photoproduction has recently provided a measurement of skin thickness with an electromagnetic probe [157] and these results could also offer constrains on the slope parameter. A larger skin thickness will imply a lower liquid to solid transition density in neutron rich matter and so a precise measurement will have implications for the crust structure of neutron stars [158, 159]. It is also hoped that in the future, radioactive beam experiments far from stability will help to further constrain the envelope in which the slope parameter can lay [160].

3.1.6 Extra Bulk Properties

With the symmetry energy and the slope parameter defined it is possible to investigate further the isovector and isoscalar properties of INM by examining the higher order components of the expansion of energy in Eq. 3.14. These higher order terms will provide a greater insight into the bulk properties of INM and will help to calculate behaviour of this material at high asymmetries and both sub- and supra-saturation densities, such as those occurring within neutron star crusts and cores.

Utilising the parameter $X$, as defined in (3.12), it is possible to extend (3.13) to
include terms up to $\rho^3$ and $\beta^2$:

\[
E(\rho, \beta) = E(\rho = \rho_0, \beta = 0) + S\beta^2 + L\beta^2
+ \left[ K_0 + K_{sym}\beta^2 \right] X^2
+ \left[ Q_0 + Q_{sym}\beta^2 \right] X^3,
\]

where

\[
K_0 = \frac{9\rho_0^2}{2} \frac{\delta^2 E(\rho, \beta)}{\delta \rho^2} \bigg|_{\rho = \rho_0, \beta = 0}
\]

\[
K_{sym} = \frac{9\rho_0^2}{4} \frac{\delta^3 E(\rho, \beta)}{\delta \rho^2 \delta \beta^2} \bigg|_{\rho = \rho_0, \beta = 0}
\]

\[
Q_0 = \frac{27\rho_0^2}{6} \frac{\delta^3 E(\rho, \beta)}{\delta \rho^3} \bigg|_{\rho = \rho_0, \beta = 0}
\]

\[
Q_{sym} = \frac{27\rho_0^2}{12} \frac{\delta^5 E(\rho, \beta)}{\delta \rho^5 \delta \beta^2} \bigg|_{\rho = \rho_0, \beta = 0}.
\]

The parameters are in units of MeV. $K_0$ can be viewed as the compressibility (or
curvature) of INM which is defined as the 2nd derivative with respect to density and plays a role in describing the softness or stiffness in the equation of state. The forces D250, D260, D280 and D300 were fit to reproduce compressibilities of 250, 260, 280 and 300 respectively [116]. The compressibility of INM is a factor in determining the collective breathing modes of finite nuclei [161]. It is also linked to the IsoScalar Giant Monopole Resonance (ISGMR) [162]. In the context of neutron stars, $K_0$ also has a bearing on the critical proton density required for the Direct Urca process to occur [47]. This is because far from saturation density the higher-order terms of the energy Taylor expansion become significant, particularly in calculating $\beta$ equilibrium. The Urca process requires a proton fraction of at least 10%, and so the density at which this occurs is rather sensitive to the compressibility term [163]. Because the Gogny forces are fit to properties at and below saturation density one would expect differences in compressibilities to have larger effects as the density increases above saturation.

The parameter $Q_0$ is the density dependence of $K_0$ and therefore the skewness of the energy-per-particle. It is of interest in supernovae calculations as the gradient of the decrease in $K_0$ with density has a large effect on hydrodynamic models [164]. $Q_0$ also has an effect on the fluctuation of $K_0$ around saturation and contributes to the softening of the equation of state when higher asymmetries lower the saturation density [165].

In kind, the parameters $K_{Sym}$ and $Q_{Sym}$ can be viewed as the compressibility of the symmetry energy and slope parameter respectively, in the asymmetry plane. Neutron skin thickness is very heavily correlated to $L$ and it is also correlated with $K_{Sym}$ [5]. As part of the Taylor expansion of the $S$, both $K_{Sym}$ and $Q_{Sym}$ can be viewed as higher order corrections to the symmetry energy when calculated far from saturation density, such as in neutron star and supernova modelling.

The full equations for the bulk properties with the Gogny force can be found in
Appendix A.

3.2 Calculations and Results

3.2.1 Energy-Per-Particle

An expression for the energy-per-particle using the Gogny force can be found in Appendix C, and in [166]. Using the parameters $A_0^0$, $C_0^0$, $A_1^1$, $C_1^1$, $B_{nn}^i$ and $B_{np}^i$ from Section 2.2 one can express a complete analytic formula for the energy, dependent on the density and asymmetry:

$$E(\rho, \beta) = \frac{3}{5} \hbar^2 k_F^2 \left[ \frac{(1 + \beta)^{\frac{5}{3}} + (1 - \beta)^{\frac{5}{3}}}{2} \right] + \frac{1}{2} \sum_i \left[ C_0^0 \rho^{\alpha_i+1} + A_0^0 \rho \right] + \frac{\beta^2}{2} \sum_i \left[ C_1^1 \rho^{\alpha_i+1} + A_1^1 \rho \right] + \frac{1}{2} \sum_i \frac{B_{nn}^i}{\pi^2 \mu_i^3} \left[ \frac{1 + \beta}{2} g(q_+) + \frac{1 - \beta}{2} g(q_-) \right] + \frac{1}{2} \sum_i \frac{B_{np}^i}{\pi^2 \mu_i^3} h(q_+, q_-). \quad (3.25)$$

All the terms in Eq. (3.25) are both density and asymmetry dependent. Part of the density and asymmetry dependence of the $g$ and $h$ functions lies in that the definition of $q$ involves both $\rho$ and $\beta$. In fact, $h$ only appears in the asymmetric case because $h(q, q) = g(q)$. The Fermi momentum, $k_F$, gives rise to a density dependence.

In the limit of $\beta = 0$ (SNM) the asymmetry dependent multiplier of the kinetic term becomes unity. The isovector elements of the zero range and finite range direct term disappear, leaving only the isoscalar $A_0^0$ and $C_0^0$. The two terms, $g(q_+)$ and $g(q_-)$ become equal and the $h$ term also simplifies down to this same value.

Calculations were numerically performed at both sub and super-saturation densities. The energy-per-nucleon, $E/N$, was determined for isospin symmetric matter.
Figure 3.5: Energy Per Nucleon for different Gogny forces and a Free Fermi Gas (FFG) in isospin symmetric matter.

(\(\beta = 0, \nu_\sigma = 2\)), pure neutron matter (\(\beta = 1, \nu_\sigma = 2\)) and polarised neutron matter (\(\beta = 1, \nu_\sigma = 1\)) where \(\nu_\sigma\) is the number of spin states allowed. A value of \(\nu_\sigma = 1\) means all particles possess the same spin. The graph for \(E/N\) in symmetric matter obtained for each of the different Gogny forces can be found in Fig. 3.5. It can be clearly seen that the different forces are very similar for symmetric matter. They do not, however, produce precisely the same saturation density. The saturation density for each of the functionals, and the properties corresponding to this density, are tabulated in table (3.1).

The \(E/N\) of Free Fermi Gas (FFG) is positive at all densities and so the system is unbound. Therefore, all of the attractive force in the non FFG cases is from the Gogny interaction. At \(\rho = 0.32 \text{ fm}^{-3} \approx 2 \rho_0\) the different forces all have noticeably different \(E/N\), which illustrates the large variation in compressibility across the force range (as discussed in section 3.1.6). Symmetric matter is expected to deviate from the FFG case by a larger amount than neutron matter. One reason for this is that the Pauli ex-
Figure 3.6: Energy per particle as a function of density for pure neutron matter for all the Gogny functionals. The shaded region enclosed by a dotted (dashed) line corresponds to quantum Monte Carlo [167] (self-consistent Green’s functions [168, 169]) calculations based on chiral potentials. The points with error bars on the left panel correspond to three representative points used in Ref. [86].

Inclusion principle limits the phase space over which nucleons can interact. This means that SNM, having the smallest fraction of identical fermions, feels the largest potential.

Neutron matter provides a particularly sensitive test to the isovector behaviour of nuclear energy density functionals [94], which is interesting because it can be straightforwardly connected to neutron stars [85]. Present and future generation observations will eventually constrain the mass-radius relation for astrophysical compact objects, and this information can be fed back into the neutron matter EoS [93]. Even before that happens, essential tests regarding the consistency and isospin dependence of energy density functionals should be performed. I note that this task has been already carried out in extensive studies of Skyrme functionals [87, 89].

Gogny functionals are particularly poorly determined in the isovector sector. I
therefore start my discussion looking at the energy per particle of neutron matter,

\[
e_{\text{PNM}}(\rho) = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m_n} + \frac{1}{2} \sum_{i=1,2} \left\{ A_i^1 + C_i^1 \rho^a_i \right\} \rho + \frac{1}{2} \sum_{i=1,2} B_{nn}^i (\mu_i k_F^a) ,
\]

where the Fermi momentum is that of neutron matter, \( k_F^n = (3\pi^2 \rho)^{1/3} \). I show in Figure 3.6 the Gogny functional predictions for the energy per particle of neutron matter. The left panel concentrates on the low-density regime. In addition to the Gogny data, I show results for microscopic determinations of the energy per particle of neutron matter at low densities. In the left panel, I show a band obtained from a recent Auxiliary Field Diffusion Monte Carlo calculation using N2LO local chiral interactions [167]. The width of the band reflects the uncertainty in the original interaction, but I note that missing 3 nucleon forces would increase the average value at higher densities. The three points with error bands correspond to the representative data used in Ref. [86]. This is obtained as an educated guess, based on a variety of many-body calculations in this low-density regime.

At sub-saturation densities, I find relatively similar results for the different parametrizations. None of the Gogny functionals reproduces the two lowest density points of the Brown-Schwenk analysis. A few are able to fit within the relatively large error band of the third. Fitting Skyrme density functionals to nuclei and to these three sub-saturation points in neutron matter yields a good quality density functional in the isovector sector [86]. This could be a good starting point for future fitting protocols of Gogny functionals. D1AS results go below the many-body data, which indicates that the functional form of Gogny interactions can in principle be amenable to reproduce such low-density points.

The right panel of Figure 3.6 concentrates on a wider density regime, up to slightly above twice saturation density. As expected, the differences between functionals here are quite marked. While around saturation the variations among functionals are ac-
commodated within about 5 MeV, at 0.24 fm$^{-3}$ one finds quite larger discrepancies. As a matter of fact, above saturation one can distinguish between two classes of Gogny functionals. On the one hand, a majority of functionals have a rather soft density dependence and predict energies which are well below 30 MeV for $\rho \lesssim 0.32$ fm$^{-3}$. On the other, D280, D1AS and D1P increase far more quickly with density. This is important, as the density increase will reflect in a substantially larger pressure which, in turn, will allow for larger and heavier neutron stars.

Interestingly, the subgroup of Gogny functionals that have a stiff neutron matter energy per particle follow closely the band enclosed by dashed lines. This band is the result of a realistic many-body calculation, obtained within the self-consistent Green’s functions approach using an N3LO chiral two-body interaction supplemented with a three-body force at N2LO [168, 169]. Because of the non-perturbative nature of the resummation scheme, these calculations have been performed with unrenormalized interactions and up to twice saturation density. Again, the width of the band is a reflection of the uncertainties in the underlying low-energy constants of the chiral interactions. I find that the stiff Gogny functionals are indeed well within this band up to 0.32 fm$^{-3}$. Note, however, that these very same functionals do not reproduce the many-body results at low densities.

Figure 3.7 displays the energy-per-particle of polarised neutron matter. The polarised neutron matter is more constrained than the non-polarised neutron matter. The forces all behave in a similar manner to one another, contrary to non-polarised neutron matter where there is a significant divergence above saturation density. In both cases, GT2 is the only force which exhibits a negative gradient below 0.32 fm$^{-3}$. In the polarised neutron matter regime the nucleons are all of the same type, exhibiting identical spin and isospin. This severely constrains the phase space in which the interaction can act. In fact, in the Gogny force, the polarisation leads to the zero range direct and exchange terms being equal and as such cancelling and only the finite
Figure 3.7: Energy per particle as a function of density for polarised neutron matter for all the Gogny functionals.

range contributes. This is why D1 and D1AS provide the same result: they share identical finite range parameters. D280 [116] exhibits a noticeably larger repulsive $E/N$ than the remaining models. It is interesting that D280 exhibits this feature but D250, D260 and D300 do not as all of these forces were fit in the same procedure but with different compressibilities in mind [116]. D1M initially is less repulsive than the other forces but soon after saturation density the repulsive force is greater than its counterparts. This could be because the density dependence of the curve is strongly linked to the range of the Gaussian, which for D1M is narrower than other forces. It is also worth noting that GT2 reaches a maximum around $\rho = 0.24$ fm$^{-3}$ and begins to decline. The GT2 maximum is lower than the values reached by the other forces. The difference may lay in the fact that GT2 was designed with a tensor term in mind and so it is perhaps unsurprising that its behaviour does not mirror the other Gogny forces.
3.2.2 Isovector properties at saturation

Table 3.1: Properties of INM at Saturation Density, units in MeV unless otherwise stated

<table>
<thead>
<tr>
<th>Model</th>
<th>(\rho_0 \text{ [fm}^{-3})</th>
<th>(E/N)</th>
<th>(S)</th>
<th>(L)</th>
<th>(K_0)</th>
<th>(K_{sym})</th>
<th>(Q_0)</th>
<th>(Q_{sym})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D1)</td>
<td>0.1665</td>
<td>-16.32</td>
<td>31.90</td>
<td>21.11</td>
<td>229.5</td>
<td>-461.1</td>
<td>-274.7</td>
<td>617.0</td>
</tr>
<tr>
<td>(D1S)</td>
<td>0.1634</td>
<td>-16.02</td>
<td>31.94</td>
<td>22.23</td>
<td>203.0</td>
<td>-516.2</td>
<td>-241.6</td>
<td>644.5</td>
</tr>
<tr>
<td>(D1P)</td>
<td>0.1698</td>
<td>-15.27</td>
<td>33.99</td>
<td>53.10</td>
<td>254.3</td>
<td>-341.0</td>
<td>-159.4</td>
<td>408.4</td>
</tr>
<tr>
<td>(D1M)</td>
<td>0.1648</td>
<td>-16.04</td>
<td>29.72</td>
<td>24.64</td>
<td>225.1</td>
<td>-459.4</td>
<td>-133.2</td>
<td>735.9</td>
</tr>
<tr>
<td>(D250)</td>
<td>0.1581</td>
<td>-15.86</td>
<td>32.36</td>
<td>24.82</td>
<td>250.1</td>
<td>-362.3</td>
<td>-289.5</td>
<td>484.6</td>
</tr>
<tr>
<td>(D260)</td>
<td>0.1602</td>
<td>-16.27</td>
<td>31.85</td>
<td>24.28</td>
<td>259.6</td>
<td>-373.4</td>
<td>-298.9</td>
<td>539.7</td>
</tr>
<tr>
<td>(D280)</td>
<td>0.1526</td>
<td>-16.34</td>
<td>34.88</td>
<td>53.23</td>
<td>285.4</td>
<td>-298.8</td>
<td>-212.0</td>
<td>326.3</td>
</tr>
<tr>
<td>(D300)</td>
<td>0.1562</td>
<td>-16.23</td>
<td>32.43</td>
<td>29.76</td>
<td>299.3</td>
<td>-237.8</td>
<td>-315.2</td>
<td>359.7</td>
</tr>
<tr>
<td>(D1N)</td>
<td>0.1613</td>
<td>-15.98</td>
<td>30.13</td>
<td>31.92</td>
<td>225.8</td>
<td>-438.6</td>
<td>-168.6</td>
<td>440.2</td>
</tr>
</tbody>
</table>

The characterisation of isovector bulk properties for Gogny functionals is one of the main aims of this thesis. I provide the values for some macroscopic isovector properties, computed at saturation, in columns 2-9 of Table 3.1. The symmetry energy, \(S\), has been fitted in some of the functionals (\(D1\) [72], \(D1AS\) [170], \(D1M\) [121]). In others, it is indirectly constrained by fits to the neutron matter equation of state (\(D1P\) [135], \(D1N\) [171]). In contrast, fitting procedures for functionals like the D250-D300 family [83] have not explicitly considered isovector bulk properties. In spite of these differences, I find that the values of the saturation symmetry energy are generally within the range \(S \approx 30 - 34\) MeV. I note that \(D1M\) has a particularly low value for the symmetry energy, imposed during the fitting procedure [121].

The slope parameter of the symmetry energy is particularly relevant for neutron star physics, since it determines the pressure of neutron matter close to saturation [172]. The variation in values of \(L\) is extremely wide, and I find that most Gogny functionals are outside the empirically constrained ranges. More details are provided below, but let us highlight the extremely low values of \(L \approx 5\) MeV for GT2 and \(L \approx 18\) MeV for both \(D1\) and \(D260\). Inevitably, this will have a negative impact on the neutron star properties associated to these functionals.
In the compressibility, $K_0$, column it can be seen that D250, D260 and D300 have compressibilities of approximately 250, 260 and 300 respectively. This is unsurprising as that was the main parameter that these forces were fitted to. However, it is notable that D280 produces a compressibility of 5 MeV larger than the 282 MeV reported in [116] though this may be due to the slightly higher saturation density determined in this work.

Columns 7 and 9 of Table 3.1 show the values of $K_{\text{sym}}$ and $Q_{\text{sym}}$ at saturation for all the functionals. These properties are less constrained by empirical data. In general, I find negative large values of $K_{\text{sym}}$, in the range $-90$ to $-446$ MeV. These are far more negative than the values predicted by microscopic calculations [133]. Similarly, Gogny functionals predict positive and relatively large values of $Q_{\text{sym}}$, in contrast to the negative and small values obtained by Brueckner–Hartree–Fock calculations [133]. I find the expected general trends obtained in other mean-field approaches [174, 175]. I show in Fig. 3.8 the values of $K_{\text{sym}}$ (left panel) and $Q_{\text{sym}}$ (right panel) as a function
of the slope parameter. Gogny functionals with lower values of $L$, like GT2, D300, D260 or D1, tend to have more negative values of $K_{\text{sym}}$ and more positive values of $Q_{\text{sym}}$. These follow the generic correlations of Skyrme and relativistic mean-field calculations, which I show in small squares and triangles in the Figure [133]. All in all, this suggests that the correlations arise from basic underlying isovector physics. For the Gogny functionals, the correlations must arise from a few of the underlying parameters, as shown in the expressions of Eqs. (A.13)-(A.15). Note, in particular, that both $K_{\text{sym}}$ and $Q_{\text{sym}}$ are independent of the isovector direct finite-range matrix elements, $A_{i1}$.

A combination of parameters depending on $K_{\text{sym}}$, $K_{\tau} \equiv K_{\text{sym}} - 6L - (Q_0/K_0)L$, has been extracted from Giant Monopole Resonance experiments [172, 176]. The value $K_{\tau} = -550 \pm 100$ MeV is nowadays generally accepted [177], but there is still much debate surrounding this value [178]. Most Gogny forces, as found in column 11 of Table 3.1, sit somewhat above the less negative end, with values between $K_{\tau} \approx -450$ and $-300$ MeV. As a matter of fact, taking this experimental value seriously, only GT2 and D300 would be valid, in spite of their very low slope parameters. I note that Gogny functionals indicate a preference for less negative values of $K_{\tau}$, in agreement with the theoretical analysis presented in Ref. [175]. Further research on finite nuclei properties, particularly resonances, with Gogny interactions will provide a further consistency check.

In contrast to the relatively poorly constrained $K_{\tau}$, there is an abundance of empirical evidence that helps restrict the values of the saturation symmetry energy and its slope. These two properties are generally tightly correlated, so their independent determination is difficult [120]. I show in Fig. 3.9 the values of $S$ and $L$ for all Gogny functionals under consideration. In addition, following Refs. [179, 173], I show some of the empirical constraints obtained from a variety of methods. The 68% confidence ellipse labeled “Masses” is obtained by propagating the fit errors in a density functional calculation using the UNEDF0 functional [180], with the choice $\sigma = 1$ MeV [173]. The
Sn neutron skin thickness results come from the analysis of Chen and co-authors [181]. From this analysis, one can also fit a relation between the skin thickness in lead and the corresponding symmetry energy and slope parameters of a variety of functionals. In addition, there is a tight linear correlation between the dipole polarizability and the skin thickness [182]. If one uses the fit parameters of Ref. [179] for the former and of Ref. [182] for the latter, one finds the “polarizability” band in the figure. Note that the position and the width of the band can change if different fits are used for either correlation.

Isospin diffusion studies in heavy-ion collisions, labelled HIC, provide additional constrains in the region of small symmetry energies [1]. The band labelled “neutron stars” is obtained considering the 68% confidence values for $L$ obtained from
a Bayesian analysis of simultaneous mass and radius measurements of neutron stars [183]. Finally, a narrow and small diagonal region above $S > 30$ MeV is obtained from simultaneous constraints of Skyrme–Hartree–Fock calculations of isobaric analog states and the $^{208}$Pb neutron skin thickness [184].

Strikingly, six of the eleven Gogny parametrizations fall outside of all empirical determinations of $S$ and $L$. In all cases, the value of $L$ is below the expected results. D1N and D1M sit within the polarizability and the masses constraints, but their slopes are still small compared to the neutron stars constraints. Only D1P, D1AS and D280 have large enough slopes to fit within the polarizability, neutron skin and astrophysical constraints. It is therefore particularly interesting that it is D280 that produced the lowest saturation density, and D1P which produced the highest, rather than having similar saturation points. Conversely, D1P produces the lowest value for the energy per particle, whereas D280 produces the highest. As for $S$, these functionals predict values within 1 MeV of the average value of all observations, $S \approx 31$ MeV. I note that no parametrization falls within the IAS+$\Delta r_{np}$ region or within the joint constraint region.

Figure 3.9 demonstrates graphically one of the main conclusions of this thesis. Few, if any, Gogny functionals show a good simultaneous reproduction of the symmetry energy, $S$, and its slope, $L$, at saturation as of the present empirical constraints. An analysis of the density dependence of these quantities in the following subsection will demonstrate that this is largely a consequence of the poor constraints on the isovector finite-range part of the functional. Future fits of Gogny functionals risk lacking quality in the isovector sector unless these properties are fitted consistently. I point out, however, that the comparison with empirical constraints should be performed with some caution. Most of these constraints have been obtained using zero-range functionals of the Skyrme type [180, 181, 182, 184]. A consistent determination of the correlations depicted in the figure using finite-range functionals is still missing.
3.2.3 Density dependence of isovector properties

Figure 3.10: Symmetry energy as a function of density for all Gogny functionals. The shaded region corresponds to the constraints arising from Isobaric Analog States of Ref. [184].

In addition to the values at saturation, there have been a wide range of attempts to determine the density dependence of the symmetry energy and, generally speaking, of all isovector properties. With Gogny functionals, an analytic expression of the symmetry energy, $S(\rho)$, is provided in Eq. (A.12). Note that, in addition to the standard isovector zero- and direct finite-range terms, the exchange finite range contribution provides a non-trivial density dependence. The density dependence of the symmetry energy is explored in Fig. 3.10. I show results for all Gogny functionals and highlight the value of $S$ at saturation. These results are compared to the constraints from the Isobaric Analog States analysis with Skyrme functionals [184].

The sub-saturation region has a relatively small systematic error. Most functionals fall within, or very close to, the empirically constrained region. GT2, however, has a relatively large symmetry energy below $\rho_0$ and overshoots the constraints in this region.
In contrast, D1AS predicts relatively low symmetry energies for $\rho < 0.14$ fm$^{-3}$, but it has a much stiffer density dependence above $\rho_0$. The fact that most Gogny functionals reproduce the low-density symmetry energy indicates that it is well-constrained by data from finite nuclei.

The largest discrepancies between functionals are observed above saturation density, as expected. I stress the fact that most functionals show either a maximum or a plateau in the symmetry energy as a function of density and that this occurs, in most cases, right above saturation. As a consequence, the values of the slope, $L$, are relatively small for most functionals. The exceptions are D1AS and D1P, which have a rather stiff character. Above saturation, several functionals display a sharp decrease in density, which will eventually lead to negative values of $S$ at densities beyond the range displayed in Fig. 3.10. For GT2, the change in sign of the symmetry energy happens already around $\rho \approx 0.38$ fm$^{-3}$. The presence of this isospin instability would have consequences in both neutron stars [185, 186, 187] and heavy ion collisions [188]. Note, however, that microscopic calculations do not predict any signs of such a transition [133].

A complementary perspective of these results is obtained by analysing the density dependence of the slope parameter, $L$. I provide an expression for this quantity in Eq. (A.13). Fig. 3.11 shows the density dependence of the slope parameter for all Gogny functionals. I also display the “neutron star” band, $L \approx 44–66$ MeV, discussed already in the context of Fig. 3.9. Most Gogny functionals show a peak and a subsequent decrease of $L$ around half saturation density. Consequently, the value of $L$ at saturation becomes relatively small, falling below the empirical estimates. One therefore expects Gogny forces to predict rather small neutron skin thicknesses in $^{208}$Pb. This is the case for both D1S and D1N, as observed in Ref. [189]. More importantly, the pressures for isospin asymmetric matter as predicted by these functionals will be smaller than empirical estimates suggest. The structure of neutron-rich isotopes will be affected by
Figure 3.11: Slope parameter as a function of density for all Gogny functionals. The shaded region corresponds to the combined constraints obtained by Lattimer and Steiner in Ref. [173].

the unrealistic prediction of slopes. Note also that about half the functionals predict negative slopes within 0.05 fm$^{-3}$ of around saturation, in accordance with the very soft symmetry energies observed around saturation in Fig. 3.10. It is worth mentioning that D1M shows a rather unique density dependence for the slope, with a nuanced decrease in density above saturation. For this functional, $L$ does not become negative below $\rho > 0.42$ fm$^{-3}$.

Three functionals fall within the empirical range predicted by neutron star physics. For D280 and D1P, which have large enough values of $L$ at saturation, the slope is close to a maximum at saturation. The decrease of $L$ with density eventually leads to decreasing symmetry energies at relatively large densities (0.26 fm$^{-3}$ for D280 and 0.42 fm$^{-3}$ for D1S). In contrast, D1AS is the only functional predicting a monotonically increasing slope parameter - and a consequently stiff symmetry energy. I note that
this was achieved by construction, adding an additional density-dependent zero-range term to the functional [170]. Because of their large pressures for neutron-rich matter around saturation, one would expect relatively large neutron stars associated to these functionals. I will confirm this tendency in the following section.

![Figure 3.12: Direct (left panels), exchange (central panels) and zero-range contributions (right panels) of the symmetry energy (top panels) and the slope parameter (bottom panels). Results for 11 functionals are shown. The symbols correspond to the respective saturation points. Note that the left and central panel share the same y-axis.](image)

What is the underlying cause for the rapid decrease of the symmetry energy with density in most Gogny functionals? One can attempt to answer this question by separating different contributions to both the symmetry energy and the slope parameter. In Fig. 3.12, I provide the contributions to both quantities arising from the direct finite-range, the exchange finite-range and the zero-range term (which includes both...
direct and exchange at once). The direct contribution to both quantities is:

\[
S_{\text{direct}}(\rho) = \frac{1}{2} \sum_{i=1,2} A_i^1 \rho. \tag{3.27}
\]

Note, in particular, that the term is linear in density and that only the sum of all isovector matrix elements plays a role. Because the values of these matrix elements are relatively widely spread I expect a relatively large systematic uncertainty. This is precisely what I find in the left panels of Fig. 3.12. D1M has extremely large and positive contributions. In contrast, GT2 provides a negative contribution as density increases. Because the combination \( \sum_i A_i^1 \) is very similar for some functionals, I find that D1P (and D250 to a lesser extent) and D1 have very similar values of \( S_{\text{direct}} \) and \( L_{\text{direct}} \). The direct contribution to the slope parameter (bottom left panel) is three times that of the symmetry energy (top left panel), \( L_{\text{direct}}(\rho) = 3 \times S_{\text{direct}}(\rho) \). Note also that, as expected, D1 and D1AS have the same direct finite-range contributions.

The exchange contribution to the isovector properties (central panels of Fig. 3.12) shows a density dependence that is, to a certain extent, the opposite of the direct term. GT2, for instance, is now large and positive, whereas D1M is large and negative. In general, these terms increase substantially with density, even though the dependence is not linear anymore. In fact, the density dependence is dictated by non-trivial functions of the Fermi momenta, see Eqs. (A.16) and (A.17) in Appendix A. With these equations, it is easy to show that in the low density limit the exchange contributions to \( S \) and \( L \) are (a) linear in density, (b) proportional to \( B_i^1 \) only and (c) related by \( L_{\text{exc}}(\rho) = 3 \times S_{\text{exc}}(\rho) \). The exchange contributions are of the same order as the direct term, which indicates that large cancellations are necessary to get isovector properties of a natural size.

The right panels of Fig. 3.12 show the density dependence of the zero-range contributions to the isovector properties. First, I note that the dependence on the functional
is largely reduced. The fitting procedure of Gogny functionals is such that part of the zero-range, density-dependent term is often fixed from the start\(^1\). Consequently, the parameter space for this term is substantially reduced and \(S_{ZR}(\rho)\),

\[
S_{ZR}(\rho) = \frac{1}{2} \sum_{i=1,2} C_i^{\alpha} \rho^{\alpha i + 1},
\]

(3.28)
is very similar for all functionals. Also, because in most parametrizations \(C_i^2 = 0\), the relation,

\[
L_{ZR}(\rho) = 3 (\alpha_i + 1) S_{ZR}(\rho),
\]

(3.29)
holds.

Second, and most important, the zero-range density-dependent term is negative at all densities. In the low density limit, the linear density dependence of both the direct and the exchange dominates (although absolute values are determined by large cancellations between terms). In the limit of large densities, in contrast, Eqs. (A.16) and (A.17) suggest that the exchange term is proportional to \(k_F \approx \rho^{1/3}\). In addition, for most functionals the zero-range term has a power \(\alpha_1 = \frac{1}{3}\) (and \(\alpha_2 = 0\)) which therefore involves \(S_{ZR} \rightarrow \sum_i C_i^1 \rho^{4/3}\). Together with the linear dependence on the direct term, it appears that the zero-range term inevitably dominates the density dependence of isovector properties at high densities. As a consequence, the symmetry energy of most parametrizations becomes soft at relatively low densities. I note that D1P, the functional with a second zero-range term (that is, with \(t_0^2 \neq 0\)), performs well in the isovector sector. Future parametrizations could use this freedom to improve the density dependence of isovector properties. Specifically, the fact that \(S_{exc}\) is not proportional to \(L_{exc}\) for \(t_0^2 \neq 0\) can help break the tension between these two isovector parameters.

\(^1\)For all functionals, \(\alpha_1\) is chosen to be \(\frac{1}{3}\) or \(\frac{2}{3}\). \(x_0\) is also often fixed, which leaves very little room for \(t_0^1\) to change substantially.
The decomposition in terms of direct, zero-range and exchange terms of the isovector properties is arbitrary to a large extent. Other decompositions can be used to pinpoint similar issues. The symmetry energy can be decomposed into two terms:

\[ S_1(\rho) = \frac{1}{3} \frac{\hbar^2 k_F^2}{2m_0^*(\rho)} , \]  
\[ S_2(\rho) = \frac{1}{2} U_1^{\text{sym}}(k_F) . \]  

The first term is a non-locality corrected kinetic contribution. The non-local correction, evidenced by the effective mass, is of the order of \( \approx 30\% \) at saturation and usually increases with density. The competition between the Fermi momentum and the effective mass can give rise to \( S_1 \) terms that decrease with density. The second term, \( S_2 \), is proportional to the value of the isovector single-particle potential at the Fermi surface. As observed in Fig. 3.2, the value of \( U_1 \) at the Fermi surface can vary substantially. For quite a few functionals, this has a tendency to become negative slightly above saturation and drives the decrease of \( S \) with density [91].

### 3.2.4 Equation of State

We have shown that GT2 exhibits a maximum of the energy per particle around saturation, and subsequently decreases as density increases. The decrease of energy as a function of density is a telltale sign of a mechanical instability, which is reflected in a negative pressure. I confirm this fact in Figure 3.13, where I show the pressure in neutron matter for all the Gogny functionals. GT2 indeed shows a negative pressure slightly above saturation. I also find that a few other functionals (D1, D1S, D250) predict negative pressures, although at much larger densities, \( \rho \gtrsim 0.64 \text{ fm}^{-3} \) and above. At this point, of course, one should discuss the predictive power of these functionals at such high densities. I do not expect Gogny functionals to describe the physics of the very high density regime, but they might be able to extrapolate some of the
Figure 3.13: EoS (pressure as a function of density) for pure neutron matter. The shaded region enclosed by a dotted (dashed) line corresponds to the stiff (soft) symmetry energy obtained with the experimental flow data of Ref. [190]. The shaded region enclosed by a full line is obtained from quiescent low-mass X-ray binary mass and radius observations [191].

The pressures of Fig. 3.13 are validated against three sets of constraints. The area enclosed by a solid line is obtained from the data in Fig. 9 of Ref. [191]. This delineates the EoS probability distribution for neutron stars assuming a baseline EoS and column densities that have atmospheres containing both hydrogen and helium, as preferred by some observations [191]. At densities above twice saturation, I compare the Gogny pressures to the flow data obtained assuming a stiff (grey dotted region) or soft (blue dashed region) symmetry energy [190]. As expected, the functionals falling close to these constraints are those with the highest values of $L$: D1AS, D1P and D280. The differences between models in the high-density regime are extreme.
Other than the already discussed mechanically unstable models, I find that D260 and D300 have equations of state that are almost an order of magnitude below the empirical constraints. These anomalous behaviours will impact the structure of the corresponding neutron stars, as I shall show next.

### 3.3 Neutron Star Masses and Radii

Neutron stars are highly degenerate stellar objects. The stars are very dense, up to several time saturation density. They are prevented from gravitationally collapsing by the nucleon degeneracy pressure. In this work I have calculated the internal structure of neutron stars under the assumption of hydrostatic equilibrium. I have not taken into account rotation of the star or crustal structures. Despite these approximations I am still able to calculate the maximum mass of a neutron star to a reasonable degree.

Masses are calculated using the Tolman-Oppenheimer-Volkov (TOV) equations (see [192] and references therein). The equations calculate the pressure required to maintain hydrostatic equilibrium in a star. They are derived from the Einstein equations and the main equation, taking full account of general relativistic effects, is

\[
\frac{dP}{dr} = -\frac{G\rho(r)M(r)}{c^2r^2} \left[ 1 + \frac{P(r)}{\epsilon(r)} \right] \times \left[ 1 + \frac{4\pi r^3 P(r)}{M(r)c^2} \right] \left[ 1 - \frac{2GM(r)}{c^2r} \right]^{-1},
\]

(3.32)

where \(G\) is the gravitational constant, \(M(r)\) is the mass enclosed at a radius, \(r\), and the mass density, \(\rho\), has been converted to an energy density, \(\epsilon\).

To determine the mass and radius of a neutron star, one begins by considering the very center of a neutron star with a central pressure, \(P_C\). Using the EoS of a given force, one is then able to deduce the density, \(\rho_C\). At the center of the star the mass enclosed is taken to be zero. One can now calculate a new mass enclosed by moving...
out to some radius, \( r \). The new mass enclosed can then be calculated as a function of the density,

\[
M_{\text{new}} = M_{\text{old}} + \frac{dM}{dr} (r_{\text{new}} - r_{\text{old}}),
\]

\( (3.33) \)

\[
\frac{dM}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2},
\]

where

\[
\frac{\epsilon(r)}{c^2} = \rho(r).
\]

Increasing the mass enclosed alters the gravitational attraction and so the pressure must adapt accordingly to fully oppose the inward pull. Therefore, solving the TOV equation provides a new pressure, which can be used to extract a new density. This process is repeated for an increasing radius until I arrive at a situation where the density, \( \rho(r) \), becomes zero. I define this as the edge of the star, \( r = R \). At the end of the calculation the entire star is in hydrostatic equilibrium. For a given central density it is now possible to determine the mass and radius, \( M(R, \rho_C) \) and \( R(\rho_C) \). It is expected that as \( \rho_C \) increases so too does the mass, until some critical density where the trend reverses and further increase of \( \rho_C \) yields lower mass neutron stars. The apex of this trend is therefore the maximum mass neutron star that a given EoS can support.

I have solved the TOV equations to study the structure of purely neutronic stars [193]. For simplicity, I have initially considered stars formed only of neutrons\(^1\). I stress once again that the results obtained with Gogny functionals at very high densities should be handled with care. I do not take them as realistic estimates, but rather as indications of the strengths and weaknesses of the functionals at the extremes of isospin asymmetry. Also, note that I use the Gogny functional throughout the star, without any low-density matching to a crust EoS. In consequence, the less massive

\(^1\)I have also performed calculations in \( \beta \)-equilibrium, which yield very similar results.
Figure 3.14: Mass-radius relation for pure neutron stars obtained with 11 Gogny functionals. The shaded region enclosed by a full line is obtained from quiescent low-mass X-ray binary mass and radius observations using atmosphere models that include both hydrogen and helium [191]. The shaded region enclosed by a dashed line corresponds to the 90% confidence region for the mass-radius relation of NGC 6397, as obtained in Ref. [95].
neutron stars have rather small radii.

I show the resulting mass-radius relations in Figure 3.14. I compare once again with the region (red continuous region) of most probable mass-radius of Ref. [191]. The blue area enclosed by a dashed line corresponds to the 90% confidence region probability contours for the mass and radius of the object NGC 6397 qLMXB, using a helium atmosphere (see right panel of Fig. 9 in Ref. [95]). In addition, I show in Fig. 3.14 the excluded region by causality [94] and the two measurements of neutron-star masses around $M \gtrsim 2M_\odot$ [41, 194].

Very few functionals predict maximum solar masses close to this limit, and only four achieve a maximum mass above $1.4M_\odot$. Those that do are essentially the parametrizations that fell within the constraints in the energy and the pressure of pure neutron matter. D1AS has the largest maximum mass, just about $M_{\text{max}}^{\text{D1AS}} \approx 2M_\odot$ at $R_{\text{max}}^{\text{D1AS}} \approx 10$ km. The maximum mass for D1P raises up to $M_{\text{max}}^{\text{D1P}} \approx 1.98M_\odot$ at $R_{\text{max}}^{\text{D1P}} \approx 9$ km. The mass-radius relation for these two functionals is relatively close to the Lattimer et al. [191] probability distribution and falls right through the constraints of Heinke and coauthors [95]. D280, in contrast, falls at the edge of the Lattimer constraints, as its EoS runs just below D1P and D1AS. Above $0.5M_\odot$, these three forces predict essentially constant radii, between 11 and 12 km. At the high mass end, however, D280 falls short and yields a maximum mass of $M_{\text{max}}^{\text{D280}} \approx 1.74M_\odot$.

In addition to these three functionals, D1M also produces a sizeable maximum mass, $M_{\text{max}}^{\text{D1M}} \approx 1.83M_\odot$. Since D1M has a substantially softer EoS, though, it reaches this mass by exploring far smaller radii, below $R \approx 10$ km. The only other parametrization that is able to support a neutron star above $1M_\odot$ is D1N. The maximum mass thus obtained is $M_{\text{max}}^{\text{D1M}} \approx 1.29M_\odot$. Note, however, that the mass-radius relation is biased towards small radii and falls well below the Lattimer predictions.
I show two more mass-radius relations in Fig. 3.14. I have already discussed the fact that D300 and D260 have EoS which fall well below the constraints obtained by a variety of empirical analysis. As expected, these two functionals cannot support heavy neutron stars. They also predict rather small objects, with minimum radii of the order of $6 - 8$ km. Finally, I note that none of the remaining 4 Gogny functionals (D1, D1S, D250, GT2) are able to sustain sizeable neutron stars. One can in principle trust these functionals up to the mechanical instability region, but their maximum masses are all below $0.16M_\odot$ and their radii lie above $17$ km. These unrealistic neutron star properties are a further indication of poor isovector properties in the Gogny functionals.

The isovector properties of the Gogny force have been calculated. One other property of interest in the field of neutron star structure is superfluidity and pairing. In the following chapter I discuss the BCS pairing properties of the Gogny forces and evaluate the physicality of the results.
Chapter 4

Pairing and Superfluidity

Superfluidity is a phenomena that has been experimentally investigated since the turn of the 20th century [54, 55]. The concept, when applied to Fermions, involves two Fermions forming a “pair” whereby the attractive forces between the two particles, e.g. phonon attraction between electrons [57], outweigh any repulsive components. Superfluidity was initially considered in the context of atomic physics. Shortly after its inception in the 1950s, the BCS Theory of Superconductivity [56] was applied to nuclear physics [58].

In this chapter I have calculated pairing properties of the Gogny forces in INM in a BCS framework. The Gogny force was itself constructed with pairing properties of finite nuclei in mind [72] and so it is interesting to study the pairing in INM. The Gogny force is an attractive choice for pairing calculations due to the behaviour of the finite range component at high densities. This is discussed in more detail later in this chapter. In Chapter 3 I have constructed neutron star profiles using the Gogny force. Whilst solving the TOV equations offers predictions for some neutron star observables (i.e. mass, radius) it does not give any information regarding observables that evolve with time, such as cooling. The cooling profile of neutron stars is sensitive to the superfluids, or lack thereof, present in the core [195]. This chapter therefore forms the basis for the calculation of cooling curves in the next chapter.
As well as neutron star properties, I will comment in this chapter on the information BCS pairing calculations can provide on the isovector properties of INM. Pairing behaviour is sensitive to the asymmetry of the system and so depends on the underlying isovector properties, which have been shown to be poorly constrained in the Gogny forces [92]. In particular I discuss the implications of single proton impurities, and the limiting case of a deuteron in INM framework. The results gained allow for discussion on the systematic uncertainty in pairing with the Gogny interaction and help one to evaluate whether the Gogny force can justifiably be used to calculate the superfluid properties of neutron stars.

### 4.1 Partial Wave Basis

In Eq. (2.31) the matrix elements are expressed in configuration space. For calculations of pairing gaps it is favourable to express the matrix elements in momentum space, with the quantum numbers $L$, $S$ and $T$ and relative momentum $q$, where $q$ is defined as

$$
\vec{q} = \frac{\vec{k}_1 - \vec{k}_2}{2},
$$

and $k_i$ is the single-particle momentum of particle $i$. The requirement for the particles to be in Cooper pairs means that $\vec{k}_1 = -\vec{k}_2$ and hence $q = k_1$. I henceforth drop the index for brevity and refer only to the single particle momentum, $k$. Appendix D shows the derivation I have performed to convert the Gogny matrix elements into
The partial wave matrix elements are

\[
V^{LST}(q, q') = 
\left[ 1 - (-)^{L+S+T} \right] \sum_i^2 \left[ Z_i^{LS} + F_i^{ST} e^{-\mu_2^2(q^2+q'^2)/4} I \left( \frac{\mu_2^2 qq'}{2} \right) \right],
\]

where

\[
Z_i^{LS} = \frac{\delta_{L,0}}{4\pi^2} \left[ t_i^0 \rho_i^0 \left( 1 - x_0^i (-)^S \right) \right],
\]

and

\[
F_i^{ST} = \frac{\mu_i^3}{2\pi^{3/2}} \left[ W_i + B_i (-)^S - H_i (-)^T - M_i (-)^{S+T} \right].
\]

In the Gogny interaction there is a zero range density dependent term which, amongst other things, accounts for the three- and higher-body interactions. This term does not tend to zero at high momentum differences. The INM pairing calculations used in this thesis include integrations between zero and infinity in momentum space. The finite range part of the Gogny force tends to zero at large momenta, but the zero range part does not exhibit the same behaviour. Thus, when including the zero range term the integral diverges [73]. The zero range component only contributes to the \( S \)-wave channels. As I explain here, in most Gogny forces the non-converging integral issue is avoided in the \( ^1S_0 \) channel by the choice of the free parameter \( x_0 \).

The prefactor obtained through antisymmetrizing the interaction matrix element is \( 1 - (-)^{L+S+T} \) and so clearly for a non-zero value, \( L + S + T \) must be odd. In the \( ^1S_0 \) partial wave channel case\(^1\) \( S = 0 \) and so I am left with a \((1 - x_0^i)\) factor in Eq. (D.24). In most Gogny forces the chosen value for \( x_0 \) is unity and the zero range term disappears. This holds for all forces except D1P and D1AS. All forces contain a zero

\(^1\)Where the notation is \( ^{2S+1}L_{L+S} \), with \( S \) the total spin and \( L \) the total isospin.
range component for the $^3S_1$ wave, which is non-zero for $T = 0$ n-p interactions. In this chapter I am interested only in the phenomena of pairing arising from finite range correlations. As such, any zero range components will give spurious contributions to the pairing, obfuscating the finite range effects. In order to avoid these unwanted parameters I have deactivated the zero range components in all forces for all partial waves, for which there is a precedent with the Gogny force [196].

The matrix elements for each partial wave with the D1S parametrisation, shown in Fig.4.1, can be grouped into distinct families. $^1S_0$, $^1D_2$ and $^1G_4$ behave qualitatively similarly. They all show a zero contribution at large separations of momentum, with a negative well forming as one moves higher in momentum along the $q_1 = q_2$ line. Further increasing in momentum leads to the dropoff of the well, although it is worth noting that in the momentum range considered it does not peter out to zero. This similar behaviour is not surprising, as these waves have identical prefactors and differ only in the order of the Bessel function. That is to say that in Eq. (D.23) the channels differ only by the value of L that factors into the modified Bessel function. This explains why, although the channels behave very similarly, the momentum at which the well is at its deepest increases with $L$, whilst the magnitude of the dip decreases. $^1S_0$, $^1D_2$ and $^1G_4$ are all $T = 1$, $S = 0$ waves, sharing a common $F_{ST}^i$ and as such also display similar relational behaviour. This same sort of relationship holds for $^3S_1$ and $^3D_3$ (the $S = 1$, $T = 1$ group), $^1P_1$ and $^1F_3$ (the $S = 0$, $T = 1$ group), and $^3P_2$ and $^3F_4$ (the $S = 1$, $T = 0$ group). A list of the Gogny forces and associated $F_{ST}^i$ prefactors can be found in Table 4.1.

Higher order modified Bessel functions tend to infinity with a shallower gradient than their lower order counterparts and so the apex of the force moves to higher $k$. The magnitude of the Bessel functions for a given argument decreases with angular momentum, $L$. The $^3S_1$ and $^3D_3$ waves have no positive components for D1S. All $T = 0$ waves are, compared to their $T = 1$ counterparts, large in magnitude and broad
Figure 4.1: Matrix elements of the D1S Gogny functional for different partial waves, given in MeV. Zero range terms have been switched off for the purpose of illustration.
in momentum space. In symmetric matter these will produce large and broad pairing gaps.

Table 4.1: $F_i^{ST}$ factors for the Gogny forces in MeV

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An attractive interaction over all $k$ in any channel will yield a finite pairing gap [71]. Table 4.1 gives the prefactors $F_i^{ST}$ for each combination of $T$ and $S$. It can therefore allow us to predict which forces will, or will not, yield a non-zero pairing gap for a given channel. The two components ($i = 1,2$) are not evenly weighted. As such it is not possible upon inspection to reliably declare that a channel will have a finite gap unless both components are attractive. For example, both components of D1P are attractive (negative) for the $T = 1, S = 0$ channels. This means that D1P must produce a non-zero pairing gap for channels such as $^3S_1$, $^3D_3$, $^3G_5$ and so forth. There are no forces in any channel that produce purely repulsive $F_i^{ST}$ for both values of $i$.
and so it is not possible at the outset to state conclusively that any pairing channels are explicitly forbidden.

It is difficult to compare channels overall due to the large variety of values amongst the Gogny forces. It can, however, be seen that the $T = S = 0$ channel appears to have the largest values for $F_{i}^{ST}$. Some of these components are several thousands of MeV. Factors of that magnitude are rare in the other channels and in fact do not occur at all for $T = 1, S = 0$. This may indicate a much stronger pairing in the $T = S = 0$ case than in any of the other channels. One can also see that the $T = 0$ channels appear to have larger variability than the $T = 1$ channels. This variance points towards a lack of constraint in the $T = 0$ channel. This is interesting, particularly in the context of the discussion in Chapter 3, because the $T = 0$ channels represent n-p pairing and, as such, one would expect them to be more sensitive to the isovector properties of the interaction and asymmetry of the system.

For the $T = 0, S = 1$ channels, most of the terms are attractive and so many forces must necessarily produce a finite gap in those channels. It is interesting that in all other channels each force has both a positive and a negative component of $F_{i}^{ST}$ (apart from the D1P exception mentioned above). There do not appear to be any rules to describe the relation between channels for a particular Gogny force. That is, knowing for example that $F_{1}^{00}$ is positive, one cannot deduce whether other channels will be positive or negative. This inconsistent behaviour once again highlights the systematic uncertainty that exists between the different Gogny forces.

### 4.2 The BCS Equations

After the work of Kamerlingh Onnes [54] it was clear that some materials exhibited extraordinary properties when cooled. Liquid mercury became superconducting when cooled to below what is known as the “critical temperature”. The explanation for
this was a gap in the single particle energies of the electrons [57]. The presence of a band gap suggested a condensing of the electrons which should not be possible because Fermions obey the Pauli exclusion principle. The mechanism for this was suggested by Cooper in 1956, who showed that two electrons could pair together to form a bosonic pair. The electrons are allowed to pair once the attractive force due to phonon lattice vibrations is greater than the repulsive Coulomb force [57].

Several early attempts were made to quantify superconductivity, but they were unsuccessful [56]. It was not until 1957 that a theory which correctly predicted a superconducting phase transition was devised by Bardeen, Cooper and Schreiffer in their seminal paper [56]. This theory is the famous BCS theory, on which this chapter is based.

The following description of the BCS formalism follows the textbook of Ring and Schuck [73]. The BCS theory begins with the assumption that the ground state wavefunction is comprised entirely of Cooper pairs. The pair creation operator in second quantized form is $a_k^*a_{-k}^*$, which creates a pair of Fermions with equal and opposite momentum. The BCS wavefunction is postulated as

$$|BCS\rangle = \Pi_k \left( u_k + v_k a_k^*a_{-k}^* \right) |0\rangle,$$  \hspace{1cm} (4.5)

where $u_k$ is the probability distribution of a hole at momentum $k$, $v_k$ is the probability of a pair with momenta $k$ and $-k$, and $|0\rangle$ is the vacuum. The parameters $u_k$ and $v_k$ are such that

$$|u_k|^2 + |v_k|^2 = 1.$$  \hspace{1cm} (4.6)
I define the Hamiltonian as

\[ H = \sum_{k_1,k_2<0} T_{k_1,k_2} a_{k_1}^{\dagger} a_{k_2} + \frac{1}{4} \sum_{k_1,k_2,k_3,k_4<0} V_{k_1,k_2,k_3,k_4} a_{k_1}^{\dagger} a_{k_2}^{\dagger} a_{k_4} a_{k_3}, \]  

(4.7)

where \( T \) and \( V \) are the kinetic and potential operators, respectively. One can then define a new Hamiltonian \( H' \):

\[ H' = H - \mu \hat{N}, \]  

(4.8)

where \( \mu \) is the chemical potential and \( \hat{N} \) is the particle number operator which satisfies

\[ \langle BCS | \hat{N} | BCS \rangle = 2 \sum_{k>0} v_k^2 = N, \]  

(4.9)

with \( N \) as the particle number of the system. With the wavefunction and Hamiltonian defined one can now determine the BCS expectation value of \( H' \):

\[ \langle BCS | H' | BCS \rangle = \sum_{k<0} \left[ (T_{k,k} - \mu) v_k^2 + \frac{1}{2} \sum_{k'<>0} V_{k,k',k,k'} v_k^2 v_{k'}^2 \right] + \sum_{k,k'>0} V_{k,-k,k'-k'} u_k v_k u_{k'} v_{k'}. \]  

(4.10)

The expectation value of \( H' \) is at a minimum and so

\[ \left( \frac{\delta}{\delta v_k} + \frac{\delta u_k}{\delta v_k} \frac{\delta}{\delta u_k} \right) \langle BCS | H' | BCS \rangle = 0. \]  

(4.11)

Performing this differentiation one finds that, for \( k > 0 \),

\[ 2\epsilon_k u_k v_k + \Delta^r (k,k) (v_k^2 - u_k^2) = 0, \]  

(4.12)
with
\[
\epsilon_k = \frac{1}{2} \left( T_{k,k} + T_{-k,-k} + \sum_{k' \leq 0} (V_{k,k,k,k'} + V_{-k,-k',-k,-k'}) v_{k'}^2 \right) - \mu. \tag{4.13}
\]

For real matrix elements one can also state that
\[
\Delta^\tau (k,k) = - \sum_{k' > 0} V_{k,-k,k',-k'} u_k v_{k'}. \tag{4.14}
\]

If \( \epsilon_k \) and \( \Delta^\tau (k,k) \) are fixed then using Eqs. (4.6) and (4.12) one can produce quadratic equations for \( u_k^2 \) and \( v_k^2 \) with the solutions:

\[
u_k^2 = \frac{1}{2} \left( 1 \pm \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^\tau (k,k)^2}} \right), \tag{4.15}\]

\[
v_k^2 = \frac{1}{2} \left( 1 \pm \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^\tau (k,k)^2}} \right). \tag{4.16}\]

Under the condition that with zero interaction there will be zero gap, and that \( v_k^2 = 1 \) for occupied states (\( \epsilon < 0 \)), the solutions are

\[
u_k^2 = \frac{1}{2} \left( 1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^\tau (k,k)^2}} \right), \tag{4.17}\]

\[
v_k^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^\tau (k,k)^2}} \right). \tag{4.18}\]

Inserting Eq. (4.18) into Eq. (4.14) leads to a self-consistent equation for calculating the pairing gap, \( \Delta^\tau (k,k_{F}) \). The final form of the BCS equation is

\[
\Delta^\tau_L(k,k_{F}) = - \frac{1}{\pi} \sum_{\tau'} \int_0^\infty \! dk' k'^2 \frac{\Delta^\tau_L(k',k_{F})(k;\tau,\tau'|V_L|k';\tau,\tau')_A}{\sqrt{\left( \epsilon^\tau' (k') - \epsilon^\tau (k_{F}) \right)^2 + \Delta^\tau'^2 (k',k_{F})}}, \tag{4.19}
\]

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where the total gap function is:

\[
\Delta^{\tau'}(k, k_F) = \sum_{L} \Delta^{\tau'}_{L}(k, k_F),
\]  

(4.20)

and \(\langle k; \tau, \tau' | V_L | k'; \tau, \tau' \rangle_A\) refers to the antisymmetrized matrix elements of an interaction between a particle with isospin \(\tau\), and a particle with \(\tau'\), with a relative momentum of \(k\). The subscript \(L\) denotes the fact that the interaction, \(V\), is decomposed into partial waves of angular momentum \(L\). The Gogny force has no active tensor component in INM. As such, it is diagonal in \(L\), meaning that there is no coupling between states of different angular momentum. It is worth noting that this is not the case for realistic interactions and so should not be considered generally true.

The gap term in the denominator is a sum over all \(L\), \(\sum_{L} \Delta^{\tau'}_{L}(k, k_F)\). Generally, the sum is ignored because, to a first approximation, the momentum range at which the gap for a given \(L\) is non-zero does not overlap with the momentum range for differing \(L\). That is, where

\[
\Delta^{\tau}_{L}(k, k_F) \neq 0, \quad \Delta^{\tau'}_{L \neq L}(k, k_F) \approx 0.
\]  

(4.21)

This approximation makes it possible to neglect the sum over \(L\) and replace \(\Delta^{\tau}(k, k_F)\) with \(\Delta^{\tau}_{L}(k, k_F)\) in the denominator. For the single particle energies one would usually use

\[
\epsilon^{\tau}(k) = \frac{k^2}{2m_{\tau}} + U_{\tau}(k) - \mu.
\]

(4.22)

It is, however, also possible to simplify matters by using a non-interacting potential such that

\[
\epsilon^{\tau}(k) = \frac{k^2}{2m_{\tau}}.
\]

(4.23)
I show now that, to a reasonable approximation, the difference between the interacting and non-interacting single-particle potential methods can be expressed as a function of the effective mass, $m^* (k)$ only. The non-zero region of the integrand within the BCS equation is focused heavily around the Fermi surface [197]. Examining the denominator further one can expand the term involving $\epsilon$:

$$\epsilon^\tau (k^\tau) - \epsilon^\tau (k^\tau_F) = \frac{k^\tau^2}{2m^\tau} - \frac{k^\tau_F^2}{2m^\tau} + U^\tau (k^\tau) - U^\tau (k^\tau_F).$$

(4.24)

For a slowly varying function, one can approximate $k$ to first order as $k_F + \delta_k$, where $\delta_k$ is a small deviation in momentum. Restating equation (4.24) with this approximation in mind and taking $\delta_k$ to be small, one finds

$$\epsilon^\tau (k^\tau) - \epsilon^\tau (k^\tau_F) = \frac{2\delta_k k^\tau_F}{2m^\tau} + \frac{\delta U^\tau (k^\tau)}{\delta k} \bigg|_{k^\tau_F} \delta_k$$

$$= \left( \frac{k^\tau_F}{m^\tau} + \frac{\delta U^\tau (k^\tau)}{\delta k} \bigg|_{k^\tau_F} \right) \delta_k$$

$$= \frac{k^\tau_F}{2m^*_\tau (k)} (k^\tau - k^\tau_F),$$

where $m^*$ is the effective mass. This means that to a first approximation, the difference between the free single particle gaps and the interacting single particle gaps relies only on $\frac{m^*_\tau}{m^\tau}$:

$$\Delta^\tau_{int} (k^\tau_F, k^\tau_F) \approx \frac{m^*_\tau (k^\tau_F)}{m^\tau} \times \Delta^\tau_{free} (k^\tau_F, k^\tau_F).$$

(4.25)

(4.26)

In this approximation, hereafter named the effective mass approximation, the gap computed with full single-particle energies can be approximated by a free-spectrum gap times the corresponding effective mass at $k = k_F$. For a discussion on the effective mass with the Gogny interaction, see Chapter 3. The forces exhibit varied effective masses at lower momenta but generally all have neutron effective masses less that 1. As such one would generally expect to see a suppression of the gap when the inter-
acting single-particle potential is considered. It is, however, worth noting that most of the neutron effective masses for the Gogny force tend to unity at high momentum. As such, one would not expect the interacting single-particle potential to alter the high-momentum results in a particularly significant manner.

To solve the BCS equation I have written a code in Fortran 95. I start with an initial arbitrary guess for the value of the pairing gap as a function of momentum. This guess is then used in the integrand to calculate the new pairing gap at a given momentum. The integral in Eq. (4.19) is calculated numerically using a modified Euler integration. Although the integral runs from zero to infinity preliminary tests have shown that the Gogny force matrix elements are effectively zero at momenta beyond 30 fm\(^{-1}\). As such, the code used integrates between 0 and 50 fm\(^{-1}\). It is worth noting at this point that this approximation of \(\int_0^\infty \approx \int_0^{50}\) is purely for the sake of numerics and not related to the necessary cutoffs employed with interactions such as Skyrme.

The mesh used for the integration is not linear in momentum space. Due to the focus of the integrand around the Fermi surface, the mesh widths change depending on the Fermi momentum, \(k_F\), under consideration. There is a Gaussian mesh of approximately 101 points generated between 0 and \(k_F\), a second Gaussian mesh of the same number of points between \(k_F\) and \(2k_F\), and finally a tangential mesh of the same number of points between \(2k_F\) and 50 fm\(^{-1}\). In the cases of small \(k_F\) a linear mesh is used at all times, because a Gaussian mesh is not necessarily useful in this instance.

Once a new gap has been calculated for each momentum, the process begins again. Each iteration, in theory, converges \(\Delta^L_L(k, k_F)\) closer to the correct solution. Thus, the BCS equation is solved self-consistently until the gap function generated has converged to within an acceptable accuracy.
4.3 Gaps in Literature

The $^1S_0$ gap in symmetric matter and neutron matter has been explored quite thoroughly in literature. This gap is usually the easiest to calculate and plays a large role in neutron star physics, and so it is perhaps unsurprising that this is the most commonly explored gap. Values for this pairing gap are relatively prolific and mostly agree with one another. Sedrakian et al. compared one of the Gogny forces, D1S, with a low-k cutoff potential [198]. Their work yielded a $^1S_0$ gap with a maximum value of just above 3 MeV, for a non-interacting single particle spectrum, with a finite gap presenting itself between 0.1 and 1.8 fm$^{-1}$. The Gogny results differed from the $V_{low-k}$ results by approximately 0.5 MeV but the momentum dependence was qualitatively similar. Introducing an interacting single particle spectrum caused the gap to shrink in magnitude and compress slightly in momentum space. This outcome is unsurprising considering the role of the effective mass as discussed above. Kennedy et al. [199] considered the BCS Gap using a Yamaguchi potential and their results agree closely with those of Sedrakian. The Gogny D1 interaction was examined by Garrido et al. [200] who calculated an interacting single particle D1 $^1S_0$ gap in symmetric matter that displayed a maximum of around 2.8 MeV. This result is slightly higher than that of D1S, but the momentum behaviour is once again qualitatively similar.

Although most models within BCS theory provide very similar results, larger model dependencies appear when using frameworks beyond BCS. Fig. 4.2 is taken from [201] and shows the variability of the $^1S_0$ neutron matter gap. This work considers only the BCS framework and so I do not explore these other models in further detail, but it is useful to illustrate the variance of models in literature. Although it may be interesting to explore the Gogny force in beyond-BCS settings, this work is designed as a first generic study of the variability within the Gogny force. As such, I have constrained myself to BCS pairing only and leave exploration of other methods for future work.
The \( ^1S_0 \) pairing gap in Neutron Matter for different calculation frameworks. Data taken from Wambach et al. [202] (PPM), Chen et al. [203] (CBF), Schulze et al. [204] (Realistic NN interaction), Schwenk et al. [205] (Realistic NN interaction), Cao et al. [206] (Brueckner), Gezerlis and Carlson [207] (QMC), and Margueron et al. [208] (Hartree-Fock).

The \( ^3P_2 \) values are far sparser than their \( ^1S_0 \) counterparts. Part of this is, as discussed above, realistic forces will couple channels of differing angular momentum. The Gogny force does not do this and so will only generate \( ^3P_2 \) gaps, rather than the \( ^3P-F_2 \) gaps more usually discussed in literature. Takatsuka and Tamagaki started with the BCS equation and compared the \( ^3P_2 \) gap of different potentials [209]. The largest of these gaps exhibits a peak of just below 2 MeV, at approximately \( \rho = 12 \times 10^{14} \text{ g/cm}^3 \) which corresponds to a \( k_F \) of 2.7 fm\(^{-1}\). This is a higher Fermi momentum region, and therefore density region, than that associated with the \( ^1S_0 \). Most of the potentials explored in their work yield \( ^3P_2 \) gaps of the order 0.5 MeV; smaller than \( ^1S_0 \) but not by many orders of magnitude. Vonderfecht et al. [210] utilised the Reid soft-core potential and determined a maximum \( ^3P-F_2 \) gap of the order 0.5 MeV, spanning a momentum range between \( k_F = 1.4 \) and 2.6 fm\(^{-1}\). This is interesting for neutron star cooling calculations; the presence of a small but finite gap in density regions where \( ^1S_0 \)
is smaller or non existent could lead to enhancement of cooling, followed by a suppression of cooling, at earlier times than would occur with just $^1S_0$. If the gap were much smaller than the $^1S_0$, then it is likely that the critical temperature for superfluidity would be far lower than the temperatures generally experienced by a neutron star. If the gap were much larger, then nucleons would preferentially pair in the $^3P_2$ channel if possible, and the comparatively high density range of the $^3P_2$ gap would have profound effects on the cooling profile of the neutron star.

Vonderfecht et al. also explored the coupled $^3S_D1$ channel in symmetric matter and found it to be surprisingly large, peaking at 15 MeV. The momentum range is from 0 up to 2.6 fm$^{-1}$, broader than the $^1S_0$ and the $^3P_2$. It is worth pointing out that this channel is non-zero at $k_F \to 0$, but in fact exhibits a value of around 1 MeV. This is a $T = 0$ channel and so corresponds to n-p pairing. One would expect, therefore, that at the zero density limit the neutron pairing gap would equal half of the deuteron binding energy and that is, indeed, what is produced in this instance [210]. Baldo et al. [211] used the Paris potential to examine the $^3S_D1$ channel and found a maximum of 12 MeV at $k_F = 1.3$ fm$^{-1}$ for the non-interacting single particle spectrum. Including the interacting single particle spectrum the gap was shifted to a maximum of 10 MeV at $k_F = 0.8$ fm$^{-1}$.

The $^1S_0$ channel appears to be relatively well constrained and has been the subject of many calculations in the literature. It therefore provides a good preliminary benchmark for evaluating the pairing properties of the Gogny force in INM. The $^3P_2$ gap is less widely cited and exhibits a greater model dependence than the $^1S_0$. The density dependence is also unclear, but most models seem to agree with a high-momentum onset, with the $^3P_2$ gap appearing in the same density region that the $^1S_0$ disappears. Although evaluation of the magnitude of the $^3P_2$ gap may be difficult, it is useful to investigate the non-zero momentum region calculated with the Gogny interaction. Beyond the $T = 1$ channels it is clear that the exact magnitude and momentum behaviour
of the \(^3S_1\) gap is poorly constrained at this time.

### 4.4 Pairing Gaps with the Gogny Interaction

The Gogny force is quite appealing for BCS calculations. One advantage is that, unlike many other phenomenological forces, they do not require an explicit cutoff in the momentum integration. For example, the matrix elements of the Skyrme force do not tend to zero at large momentum differences. This can cause a problem for gap calculations because the BCS equation contains an integral up to infinity in momentum space and so an explicit cutoff is required to prevent the integral from diverging [212]. The form of the Gogny force, when one has deactivated the zero range terms, allows one to perform BCS pairing gap calculations without the need to include an explicit cutoff. This is due to the Gaussian finite-range term in real space ensuring a finite extent in momentum space. As such, the finite-range force rapidly tends to zero at large momentum differences. The Gogny force’s finite-range component also brings it closer in spirit to the strong force than pure zero-range interactions.

#### 4.4.1 \(^1S_0\)

The \(^1S_0\) gap in neutron matter, calculated with a free single-particle spectrum for all Gogny forces, is displayed in Fig.4.3. The gaps produced are very similar. D1AS is necessarily the same as D1 and so has been ommitted for this, and all other gap figures in this thesis\(^1\). All forces peak around similar values of \(k_F\), approximately 1 fm\(^{-1}\). D1P and D1M peak slightly higher, at approximately 1.2 fm\(^{-1}\). The forces exhibit peaks ranging from D1P at approximately 2.8 MeV, to D1, which is in the region of 3.8 MeV. Most forces produce gaps that cover a similar Fermi momentum range, disappearing at approximately 2 fm\(^{-1}\). D1P, however, is finite at several fm\(^{-1}\) higher than the other

\(^1\)D1P and D1AS would normally have a zero range contribution but these have been set to zero as discussed in section 4.1. This causes D1AS to perfectly match D1, as the differences between the two forces are purely in the zero range channel.
forces. As discussed above, D1P is the only force with two negative $F_i^{01}$ factors. This doubly-attractive component could be the reason D1P is allowed to persist to higher densities, as the attractive component is not having to compete with a repulsive counterpart.

It is useful to calculate the gap with both a free and interacting single particle spectrum so that their comparison can be used to evaluate the sensitivity of the gap to dressing the single-particle spectra in the denominator of the gap equation. Fig. 4.4 shows the effect of including the interacting single particle spectrum. The inclusion lowers the peak of the gap by around 0.5 MeV but has little effect on the structure of the gap itself. The model dependence of the gap appears to be suppressed with the inclusion of the interacting spectrum. D1P becomes noticeably more compressed.

Figure 4.3: The $^1S_0$ pairing gap in Neutron Matter for each Gogny force, using a free single particle spectrum. Pairing gaps are given as $\Delta (k_F, k_F)$.
in momentum space, bringing it more in-line with the remaining forces. The peak of all forces is shifted slightly towards lower Fermi momentum. The Fermi momentum at which the forces experience gap closures is also lowered, particular prominently in D1M. This is not surprising when considering the effective mass approximation discussed in section 4.2. At this point it is worth showing a figure of the effective proton and neutron masses in pure neutron matter. This is seen in Fig. 4.5.

In particular, Fig. 4.5 goes some way to explain why some models are affected more than others: the effective masses in neutron matter are somewhat model dependent. Some of the forces have effective masses that are consistently close to unity. Considering the effective mass approximation described above it is understandable that these forces will not alter much with the inclusion of an interacting spectrum. Some models, however, show larger deviations from unity and so the interacting spectrum will act to
Figure 4.5: The proton and neutron effective masses with the Gogny interactions in pure neutron matter. The solid (dashed) line represents the neutron (proton) effective mass in neutron matter. The effective mass is given at a density of $\rho = 0.16 \text{ fm}^{-3}$.

suppress their pairing gaps. In particular, the neutron effective mass of D1M behaves qualitatively differently to the other forces. The proton effective masses (in this case a proton polaron) is particularly unconstrained. A comparison of the gaps using interacting and non-interacting single particle energies, utilising the D1S and D1P forces, is shown in Fig.4.6.

Fig.4.7 shows the $^1S_0$ gap as a function of $k$ and $k_F$ for an interacting single-particle spectrum. It is useful to examine the pairing gap in this two-dimensional manner to give one further insight as to the behaviour of the pairing gap between particles with different momentum. It is clear from this that moving away from the $k = k_F$ line suppresses the gap. This is due to the finite-range nature of the Gogny force and the highly peaked function of $k'$ in (4.19). The maximum of each plot is, interestingly, not along the diagonal $k = k_F$ line. The structure of the BCS equation is such that one would expect the integrand to be focussed around the Fermi surface. The pairing gap
Figure 4.6: The $^1S_0$ pairing gap with the D1S force for both the free and interacting single particle energies.

does not reflect such a focus and instead appears strongest when the nucleons do not have the same momentum. As such, the maxima of the 2D plots are larger than those in the 1D $\Delta^\tau (k_F, k_F)$ plots. It is also interesting that although all forces give a similar peak gap in the $k = k_F$ case, mostly within 0.5 MeV of one another, the 2D plots have model dependent maxima that vary by several MeV. The gap is more diffuse in the $y$-axis than in the $x$-axis. Most of the forces exhibit gap closures at $k_F \approx 1.5$ fm$^{-1}$. However, for a given $k_F < 1.5$ the gap $\Delta^\tau (k, k_F)$ can be finite for $k > 1.5$. It is clear also that some forces have finite gaps over a larger momentum space than others. For instance D1P covers a larger area than D1N.

The $^1S_0$ neutron gap, as a function of Fermi momentum and utilising the free single particle spectrum in symmetric nuclear matter, is the same as that in neutron matter. This is because the matrix elements are the same for both nn pairs and pp pairs. As
Figure 4.7: The $^1S^0$ pairing gap in Neutron Matter for each Gogny force, using an interacting single particle spectrum. Pairing gaps are given as $\Delta (k, k_F)$ in MeV.

such, the integrand over all momenta is unchanged by altering the asymmetry. The relationship between $k_F$ and density, however, depends on asymmetry, $\beta$, and so the symmetric ($\beta = 0$) and neutron ($\beta = 1$) free spectrum gaps will not be the same when expressed as a function of density.

The interacting single particle energies produce different results due to the effect of
Figure 4.8: The $^1S_0$ pairing gap in Symmetric Nuclear Matter for each Gogny force, using an interacting single particle spectrum. For a key, see figure 4.3.
asymmetry on the single particle energies. Fig.4.8 shows the $^1S_0$ gap for the interacting symmetric matter case. The introduction of protons has suppressed the magnitude of the gap, reducing the maxima by around 0.5 MeV, but does not appear to have majorly changed the $k_F$ dependence of the gap. That is to say, the momentum at which the force experiences a gap closure is not particularly altered by the inclusion of protons. The peak momentum for the gap also remains relatively unchanged. The $^1S_0$ gap is known to exist in a density region relevant to finite nuclei [196]. Gaps in symmetric matter are of interest in finite nuclei. Terrestrially accessible matter is, compared to neutron star matter, close to the $N = Z$ line. Therefore, in finite nuclei the $^1S_0$ pairing correlations may reflect those in INM [213]. Conversely, neutron matter is highly relevant to neutron star physics because the proton fraction within neutron stars is predicted to be small, if finite at all. As I have shown, when using a free single particle spectrum, with $T = 1$ gaps, the neutron matter gap is identical to the symmetric matter gap. For this reason I will show the free neutron matter case, the interacting neutron matter case, and the interacting symmetric matter case for each of the pairing gaps considered, omitting the free symmetric matter calculations.

One Gogny force, GT2, has been omitted from the above figures. This is because GT2 gives a far higher value for the $^1S_0$ gap (several tens of MeV). The GT2 parametrization contains an adjustment to include a tensor term [118]. Because tensor terms have been ignored for these BCS calculations the GT2 result will be missing the tensor term for which it was designed. As such, the results produced by GT2 may not be informative here.

4.4.2 $^3P_2$

As well as the $^1S_0$ pairing gaps I have calculated the $^3P_2$ pairing gaps for each of the Gogny forces. In other theoretical models the $^3P_2$ is coupled with the $^3F_2$. This
connection is not considered here because the coupling between the two channels is achieved using the tensor interaction which, as discussed, is absent from the Gogny interaction in INM. These gaps can be found in Fig.4.9 for the free single particle spectrum. The model dependence is striking, particularly in view of the logarithmic scale. The differences are much more acute than in the $^{1}S_0$ case.

![Figure 4.9: The $^{3}P_2$ pairing gap in Neutron Matter for each Gogny force, using a free single particle spectrum. Pairing gaps are given as $\Delta(k_F,k_F)$. Not all forces produce finite gaps.](image)

The Gogny forces D1, D1S, D250, D260 and D1AS do not generate finite $^{3}P_2$ pairing gaps. D1M, however, generates particularly large gaps, of the order 1000 MeV! The D1M gap is also finite across a wide range of momenta, opening near $k_F = 0.1$ fm$^{-1}$ and remaining finite past $k_F = 10$ fm$^{-1}$. Both of these scenarios are inconsistent with current, if broadly varying, $^{3}P_2$ gaps found in literature [214, 209, 215, 216]. The D1P force and the D300 force are the only two of the Gogny parametrizations which
produce $^3P_2$ gaps of the same magnitude as the $^1S_0$ gaps.

![Graph showing $^3P_2$ pairing gap in Neutron Matter for each Gogny force.](image)

Figure 4.10: The $^3P_2$ pairing gap in Neutron Matter for each Gogny force, using an interacting single particle spectrum. Pairing gaps are given as $\Delta (k_F, k_F)$. Not all forces produce finite gaps.

When considering the interacting single-particle spectrum, in Fig. 4.10, I find a significant suppression of the $^3P_2$ gap. D1P drops by almost an order of magnitude. As with the $^1S_0$ case, the gap is compressed in Fermi momentum dependence with both D1P and D300 exhibiting a lower gap closure than with a free single-particle spectrum.

### 4.4.3 $^3S_1$

Going beyond the $T = 1$ $^1S_0$ and $^3P_2$ gaps, $T = 0$ gaps can emerge when protons begin to appear and therefore $T = 0$ channels can be useful for astrophysics. The n-p
interactions in neutron star matter are not expected to produce large gaps, due to the large asymmetry and the large difference in proton and neutron chemical potentials. This large difference suppresses significantly the strength of the $T = 0$ pairing. The $L = 0$ channel of these is the $^{3}\! \! S_1$ pairing gap. Before displaying the pairing results it is worth discussing the physical implications of pairing between protons and neutrons at high asymmetries. Because $T = 0$ states require n-p pairing, the large difference in proton and neutron Fermi momenta in highly asymmetric matter, or indeed neutron matter, should heavily suppress these pairing channels. In neutron matter there will be a proton density of zero, but not necessarily a vanishing gap. At first this appears counter intuitive considering that the $^{3}\! \! S_1$ wave is n-p and in the case of $\beta = 1$ the proton density is necessarily zero. The presence of this finite gap is in fact a single proton impurity, or polaron [217], existing in the infinite neutron sea.

It is worth briefly mentioning the implications of vanishing density. In the case of symmetric matter at zero Fermi momentum a single neutron and proton would exist as two particles in infinite volume and so can be considered to have a density of zero. The BCS equation reduces to the bound-state equation in the low $k_F$ limit. As such, the equivalent of the pairing gap in the two-body system is the binding energy. Therefore, the symmetric matter pairing gap at zero Fermi momentum can be seen as equivalent to half of the binding energy of the deuteron.

As discussed in section 4.1, the zero range term of each force has been deactivated. The presence of this term would require an explicit momentum cutoff [218] and could drastically alter the gaps produced for the $^{3}\! \! S_1$ channel and so the values presented here should be viewed with caution. I begin with the case of neutron matter, i.e. a proton impurity. The values for this gap can be seen in Fig. 4.11.

The $^{3}\! \! S_1$ gaps produced by the Gogny forces are all particularly large, of the order of hundreds of MeV. The extreme magnitudes here could be to do with the large $F_i^{ST}$
values discussed in Table 4.1. It is also worth noting that this is the channel where even with the choice of \( x_0 = 1 \) there is a non-negligible zero-range contribution. The zero range part of the Gogny force is always positive and therefore repulsive. Thus it is feasible that inclusion of zero range terms would reduce the gap at all densities, bringing the results into a more physical realm. In any case, the lack of zero range contributions means that the extreme \( ^3S_1 \) results are not necessarily meaningful in this instance.

The \( ^3S_1 \) gaps generated are unusual in that the gap is finite at \( k_F = 0 \) (and therefore zero density). The physical meaning of this is, as mentioned above, the equivalence with a deuteron. In this zero density limit one would therefore expect the gap to reproduce half the deuteron binding energy (approximately 1.1 MeV). This is not the case with any of the Gogny forces. This may be because the zero-range components
of the forces have been de-activated, leaving the Gogny force abnormally attractive in the $^3S_1$ channel.

Mathematically, the existence of a finite gap at zero density would be prohibited in higher order channels due to the modified Bessel function taking the value of zero when the argument is zero. In the $L = 0$ modified Bessel function, an argument of zero yields a value of unity and so finite matrix elements are not prohibited. In Chapter 3 it was discussed that polaron quasiparticle energies could be useful for future Gogny forces, improving the isovector sector. It would be interesting to see if such a consideration had a significant impact on the $^3S_1$ gaps produced. This is likely to have an impact, because the matrix elements of the finite-range part of the Gogny force between a proton and neutron of opposite spin\(^1\) are

$$
\left\langle k_1 \uparrow \frac{1}{2}, k_2 \downarrow -\frac{1}{2} \left| V_{\text{Finite}}^{\text{Gogny}} \right| k_1 \uparrow \frac{1}{2}, k_2 \downarrow -\frac{1}{2} \right\rangle_A = \sum_{i=1}^{2} [W_i - B_i - H_i + M_i] e^{-\frac{\mu q^2}{4}},
$$

(4.27)

where the prefactor $[W_i - B_i - H_i + M_i]$ is the same as $F_{10}^{10}$. This is the prefactor for the $^3S_1$ channel and so one would expect polaron considerations to influence the $^3S_1$ gaps for the Gogny force.

A large pairing gap in the $^3S_1$ channel would have large consequences for finite nuclei. Pairing of this strength between neutrons and protons is not currently supported by experimental evidence [219]. Despite the unphysical results, I present them here for completeness so that they may be a part of the first step to exploring pairing properties with the Gogny force. The identification of these unusual results also helps suggest inputs for future Gogny functional fits, particularly in functionals designed for use in pairing.

\(^1\)One requirement for Cooper pairs is that they have opposite spin.
Fig. 4.12 shows the effect of an interacting single particle spectrum. The effect of this inclusion is very model dependent. D1, D1S, D1P, D250 and D1AS are barely altered by the inclusion whereas D1M exhibits a far steeper momentum dropoff and by a $k_F$ of 5 fm$^{-1}$ is half of its free spectrum counterpart. Looking again to Fig. 4.5 this behaviour is understandable. The forces that are least altered by the interacting single particle spectrum (D1, D1S, D1P, D250, D1AS) are those with effective masses that quickly tend to unity. As such, their effect is limited to the low momentum end of the BCS integrand and the effect is minimal. Conversely, D1M has a neutron effective mass that actually drops as the momentum increases, reaching a minimum of 0.7 MeV near 3 fm$^{-1}$. This unusual behaviour provides a significant suppression of the pairing gap.

Moving from neutron matter to symmetric matter the situation is markedly differ-
Figure 4.13: The $^3S_1$ pairing gap in symmetric matter for each Gogny force, using an interacting single particle spectrum.

ent, as displayed in fig. 4.13. The dropoff shown by D1M in neutron matter is far less severe than in symmetric matter. All other forces show a decreasing gap with increasing momentum, with D280 in particular exhibiting a pronounced decay. The effective masses as a function of asymmetry, shown in Fig. 3.3, can go some way to explaining this behaviour. Particularly in D1M, it is clear that for symmetric matter the effective mass behaves qualitatively similarly to those of the other forces. However, altering the asymmetry has a particularly pronounced influence and the neutron effective mass becomes very large. Using the reasoning from section 4.2 this will cause the gap to become small, which is the behaviour seen in the $^3S_1$ channel.

When discussing the symmetric $^3S_1$ channel in literature there is normally a coupling to the $^3D_1$. This is not considered here for the same reason that I do not consider the $^3P$-$F_2$ coupling. Although there is no current consensus in literature, the values I
have generated for the $^3S_1$ gap are an order of magnitude higher than $^3$S-D$_1$ calculations usually suggest [219, 210, 209, 220].

For $T = 1$ gaps the Fermi surface of each nucleon involved is the same, $\tau = \tau'$, and therefore $k_{F}^\tau = k_{F}^{\tau'}$. In symmetric matter, the Fermi surfaces are aligned for $T = 0$ pairs, but as the asymmetry increases the proton and neutron Fermi surfaces drift apart and the off-diagonal values of the interaction, $k_{F}^p \neq k_{F}^n$, become important. In the previous sections I have discussed only the extreme cases of neutron matter ($\beta = 1$) or symmetric matter ($\beta = 0$). To be useful in the context of neutron stars, one must be able to discuss the gaps for asymmetric matter ($0 < \beta < 1$) whereby the nuclear matter is predominantly, but not exclusively, neutrons. I treat the Gogny force as charge independent and employ no Coulomb term in the gap calculations. That is to say that

$$\Delta (k_{F}^N, k_{F}^p, \beta) = \Delta (k_{F}^p, k_{F}^N, -\beta).$$

A proton dominated system is not expected to be realised physically, but I include the comment here for completeness.

A plot of the $^3S_1$ gap for several different asymmetries can be found in Fig. 4.14. What is particularly striking is that as asymmetry increases, and therefore the difference in proton and neutron Fermi surfaces increases, the maximum value for the gap also increases. One would normally expect a divergence in Fermi surfaces to suppress the gap. The asymmetry dependence of the gap closely follows the asymmetry dependence of the effective mass: D1P has a particularly stable effective mass with respect to asymmetry, and as such exhibits only a small shift in gap as the proton fraction increases. Conversely, D1M has an effective mass with a very large isovector dependence which is reflected in the diverse values for the gap as a function of asymmetry.
Figure 4.14: The proton $^3S_1$ pairing gap in asymmetric nuclear matter for each Gogny force, using an interacting single particle spectrum.
Neutron star matter, whilst predominantly neutrons, contains a non-zero number of protons at certain densities. For this reason the gap as a function of asymmetry is of importance when considering neutron star physics. Usually, the $^3S_1$, and in fact all gap, contributions is considered to be small due to the large difference between the proton and neutron Fermi surfaces [220, 221, 222, 209]. This is because the neutron gap decreases with asymmetry and so is expected to be small at the proton fractions present in neutron star matter. Impurity protons are still able to provide a finite gap, but because of their extremely small abundances they are not expected to give a noticeable contribution to the star’s properties. These assumptions may not be accurate with the Gogny force; the large proton impurity gaps will not have much bearing due to their small abundance. However, the abnormally large neutron gaps mean that even at large asymmetries the $^3S_1$ neutron gap is relevant.

Usually one would expect nucleons to pair in the channel with the largest gap. Therefore, if a $^3S_1$ gap of hundreds of MeV were present at the momenta relevant to the $^1S_0$ gaps then one would expect all protons to pair in the p-n $^3S_1$ channel. This would mean that no p-p pairing would appear in this region. For each proton paired in this way a neutron would also be paired and so the population of neutrons found in the $^1S_0$ state would be suppressed. In the limiting case of $\beta = 0$ all neutrons and protons would be paired in the $^3S_1$ state. As is discussed in the next chapter, this would have a profound impact on the cooling behaviour of neutron stars.

**4.4.4 $^1P_1$**

Finally, for completeness, I have examined the $^1P_1$ channel, which is the lowest order channel exhibiting $T = S = 0$. A graph of the free spectrum neutron matter case is displayed in Fig. 4.15. There are several striking features here. There is a large model dependence in the results. All forces begin at a zero gap at zero $k_F$, which is
Figure 4.15: The $^1P_1$ pairing gap in neutron matter for each Gogny force, using a free single particle spectrum.

to be expected due to the behaviour of the $L = 1$ Bessel function. Most of the forces give abnormally high gap results, of the order of thousands of MeV. D1M yields a far larger gap than any of the other models. Most of the gaps produced are finite for a broad range of momentum. To explain this behaviour I direct the reader to Table 4.1, where I have already discussed that the $T = S = 0$ channel, of which $^1P_1$ is a member, appears to be the least constrained. It is also the channel with the largest $F_{ST}^i$ elements. D1M has extremely large values of $F_{ST}^i$ in comparison to other forces, which could be the cause for its magnitude. It is possible that the broad range of $F_{ST}^i$ values in the $T = S = 0$ channel are responsible for the large variance in $^1P_1$ gaps generated.

Fig. 4.16 displays the interacting single particle potential for neutron matter. As with the $^1S_0$ case, the qualitative behaviour of the gaps is not dramatically changed.
Figure 4.16: The $^1P_1$ pairing gap in neutron matter for each Gogny force, using an interacting single particle spectrum.

The magnitude of most gaps has been noticeably reduced. Conversely, D1S, D1M and D250 appear almost unaffected by the inclusion of the single particle potentials. At this point it is worth reminding the reader that the $^1P_1$ channel is a $T = 0$ gap and so exists between protons and neutrons. As such, the neutron matter values represent the interaction with a single proton impurity. The $^1P_1$ gap near saturation density is relevant for finite nuclei [223] and so it is interesting to see the symmetric matter case.

Fig. 4.17 shows the interacting symmetric matter $^1P_1$ gaps. Interestingly there is almost no change from the neutron matter case with any of the forces. The $^1S_0$, $^3S_1$ and $^3P_2$ channels have shown considerable suppression by the shift in asymmetry, but $^1P_1$ is markedly unaffected. Whilst spanning a wide range of $k_F$, the D1 force (and by extension the D1AS force) gives a far smaller gap than the other Gogny counterparts. Although the maximum is in the region of 30 MeV at around $k_F = 4$ fm$^{-1}$, the force
is between 0 and 1 MeV in the momentum region near saturation density (near $k_F \approx 1.5 \text{ fm}^{-1}$). Despite the feasibility of a gap of this magnitude around saturation, the gap increases rapidly with increasing $k_F$ and is of the order of 10 MeV when still within the momentum region relevant for the study of neutron stars.

### 4.5 Consequences for the Gogny force

The pairing gaps produced for the $^1S_0$ case are in keeping with those found in the literature. However, beyond the $^1S_0$ channel the values of the gaps are striking. Many of the forces produce gaps that are unphysical, particularly in the $^3S_1$ channel. These results are extreme, particularly considering that many of the gaps produced are above 100 MeV. This energy scale is already beyond the limits of non-relativistic equations used here. Moreover it may be beyond the scope of the BCS theory itself. This initial
broad study of the Gogny forces suggests that the current functionals are generally inadequate for $L > 0$ pairing calculations in INM.

The effect of an interacting single particle potential on these gaps varies hugely between forces. This variance can be, at least in part, explained by way of comparing effective masses. Beyond this, the $^3S_1$ gaps show very large model dependence of their behaviour with asymmetry; some are very resilient to changes in the proton fraction whereas others are very sensitive. When compared to Fig. 4.1 it becomes clear that those with larger values of $B^{np}$ exhibit larger asymmetry dependence. The dependence on $B^{np}$ is interesting, because the mass splitting depends only on the $B^{nn}$ parameters defined in Chapter 2. As such, the asymmetry dependence of the pairing gaps could help to provide a mechanism for constraining the finite-range exchange values of the Gogny interaction and $B^{np}$ may provide a mechanism for tuning the isovector behaviour of the pairing gaps whilst mass splitting calculations help tune $B^{nn}$.

The Gogny force was devised with pairing in mind, though in the context of finite nuclei. It is also worth noting that of the pairing channels considered in the D1 fit [119] only $^1S_0$ is presented in this work. Although pairing was taken into account in the original parametrisation the beyond $^1S_0$ pairing is poorly constrained for INM. This may in part be due to the BCS equations exploring matrix elements for momenta far above saturation. It is also worth noting that in the pairing fitting procedure the magnitude of pairing gaps was not considered. In fact, the fitting requirement was simply for the matrix elements of the pairing interactions to be negative. Tightening this constraint may provide a mechanism for improvement in future fits. The forces produce sensible gap values for the $^1S_0$, which is also the only gap considered that is localised to momenta near saturation density. The effective masses also experience their greatest model dependence above saturation, which can help to explain the erratic behaviour of the asymmetry dependence of the gaps leaking into the high momentum regions.
In the $^3P_2$ case, of the forces that produce finite gaps, most models give non-zero gaps at lower $k_F$ than are suggested by literature. Several of these small $k_F$ $^3P_2$ gaps are finite in the same momentum region as the $^1S_0$ gaps and in the case of D1M is larger than the $^1S_0$ channel at all momenta. As such, one other possible cause of the unusual gaps is the assumption made in Eq. (4.21) that only one gap is present at a given momentum. As can be seen from the figures above, this assumption is invalid with the Gogny force. If this is invalid, then so too is the approximation of the chemical potential being equal to the Fermi energy. Inclusion of all finite gaps in the denominator of the BCS equation could yield very different gap behaviour. The consideration of the gaps would lead to an increase in the denominator of the BCS equation and so will cause an overall suppression of the gap. The difficulty in including these other channels is because their incorporation would add another layer of self-consistency to the BCS equation. Each channel would need to be calculated separately, and then re-calculated with the other channels incorporated. These new values would again be fed back into the equations. I have not attempted to solve this problem in this work but mention it as a reasonable future extension of this investigation.

Obtaining the pairing gap as a function of $k_F$ is a powerful tool for astrophysics. Critical temperatures can be extracted [224] and compared to the cooling behaviour of neutron stars, such as Cassopeia A [225]. Superfluid behaviour at a given $k_F$ can also be connected to a superfluid at a given density. Using a Gogny force one could now determine a neutron star mass-radius relationship and calculate the cooling of that neutron star with the same Gogny force.

The D1P force is the one that gives the most reasonable, if not necessarily always physical, isovector properties and beyond-$^1S_0$ pairing gaps. As such, the following chapter utilises the D1P parametrization to investigate neutron star cooling with $^1S_0$ and $^3P_2$ superfluidity.
Chapter 5

Neutron Star Cooling

5.1 Neutron Star Cooling

At birth, neutron stars have a lot of residual heat left over from their progenitor supernova. Within the first few milliseconds after the supernova bounce, the neutron star will be in hydrostatic equilibrium [226]. From hereon in, the neutron star does not exhibit uniform cooling, but instead passes through several cooling regimes whereby different cooling processes are enhanced or suppressed, depending on the state of the matter within. The behaviour of the neutron star cooling curves can therefore tell us about the internal composition and structure of the neutron star.

For the first few seconds after birth the neutron star is opaque to both photons and neutrinos [227] and so heat transport occurs through diffusion, whereby the neutrinos will preferentially heat the core [228]. After the first few seconds the star remains opaque to photons, but becomes transparent to neutrinos. At this point there is a strong enhancement of the cooling due to the uninhibited neutrinos carrying heat out of the star [229].

For several years after the birth of the neutron star the crust and the core are thermally decoupled, exhibiting a large temperature gradient across the crust-core
boundary. The core of the star cools whilst the crust remains hot and so it can be described as a cooling wave propogating from the center of the star outwards \[230\]. For this reason, in the first few years after birth the temperature of the surface offers little insight into the temperature of the core. After around 100 years the crust and core become thermally adjusted and most of the star can be considered isothermal \[229\].

Up until around 100,000 years the cooling of the star is predominantly via neutrino emission. After this point, the neutrino luminosity can drop below the photon luminosity and x-ray emission at the surface becomes the most important mechanism for cooling \[231\].

5.1.1 Cooling mechanisms

There are several mechanisms available for generating these neutrinos. One of these methods is known as the direct Urca process and was first postulated in 1940 by Gamow and Schoenberg \[232, 233\] \(^1\). The Urca process occurs when neutrons (protons) spontaneously decay into protons (neutrons), releasing an electron (positron) and an anti (normal) neutrino. These neutrinos can then escape the star, carrying heat away with them.

\[
\begin{align*}
 n & \rightarrow p + e^- + \bar{\nu}_e \\
 p & \rightarrow n + e^+ + \nu_e.
\end{align*}
\]

The neutrino capture cross-section is small \[234\] and so effects of neutrino absorption are small in comparison to the above processes and can be ignored. The star is assumed to be in \(\beta\) equilibrium and so the rate of \(p \rightarrow n\) is assumed to be equal to

\(^1\)The Urca process is not an abbreviation but in fact takes the name of the Casino de Urca in Rio De Janiero, where Gamow and Schoenberg first mused as to the existence of the process
the rate of $n \rightarrow p$. Therefore the direct Urca process does not have an impact on the composition of the star. The direct Urca process requires a strong proton fraction in order to occur. This is due to the need for conservation of momentum:

\[ k_F^n < k_F^p + k_F^e, \]  

(5.3)

where $k_F^n$, $k_F^p$ and $k_F^e$ are the Fermi momenta of neutrons, protons and electrons respectively. The neutrino momenta are not considered here because $k_\nu \approx k_B T / c$, where $k_B$ is the Boltzmann constant, $T$ is the temperature and $c$ is the speed of light, and $kT/c << k_F^p [24]$. The calculations of Lattimer et al. suggest that a proton fraction of above 11% is necessary, and that corrections due to the presence of neutrons raises this further [24]. The existence of this large a proton fraction is not predicted by all models.

The emissivity arising from the direct Urca process, if the proton threshold satisfies the conservation of momentum equation, Eq. (5.3), is (Eq. (7) in [24]):

\[ \epsilon_{Urca} = 4 \times 10^{27} \left( \frac{(1 - \beta) \rho}{2 \rho_0} \right)^{\frac{1}{3}} T_9^6 \text{ erg cm}^{-3} \text{ s}^{-1}, \]  

(5.4)

where $\rho$ is the number density, $\rho_0$ is the saturation density, 0.16 fm$^{-3}$, $T_9$ is the temperature in 10$^9$ K. Although muons are not considered in this work it is worth noting for completeness that if the muon Urca process were to occur the emissivity would increase by a factor of 2 [24].

One cooling channel that does not require such a large proton fraction is the modified Urca process. This is similar to the direct Urca process, but requires a spectator nucleon, N, in order to aid with conservation of momentum:

\[ N + n \rightarrow N + p + e^- + \bar{\nu}_e \]  

(5.5)

\[ N + p + e \rightarrow N + n + \nu_e. \]  

(5.6)
Note that the isospin of the spectator nucleon remains unchanged in the process. Although the modified Urca can occur in a larger asymmetry space than the direct Urca the cooling provided is not as strong. From Eq. (8) in [24] the emissivity of the modified Urca process is

$$\epsilon_{\text{ModUrca}} = 10^{22} \left( \frac{1 - \beta}{2 \rho_0} \right)^{\frac{1}{3}} T^8_g \text{ erg cm}^{-3} \text{ s}^{-1}. \tag{5.7}$$

From the two equations one can see that the ratio of emissivity between the two channels is

$$\frac{\epsilon_{\text{Urca}}}{\epsilon_{\text{ModUrca}}} = \frac{4 \times 10^{27} \left( \frac{1 - \beta}{2 \rho_0} \right)^{\frac{1}{3}} T^6_g}{10^{22} \left( \frac{1 - \beta}{2 \rho_0} \right)^{\frac{1}{3}} T^8_g} \text{ erg cm}^{-3} \text{ s}^{-1}$$

$$= \frac{4 \times 10^5}{T^2_g} \text{ erg cm}^{-3} \text{ s}^{-1}. \tag{5.8}$$

At the temperatures considered here $T^2_g << 4 \times 10^5$ and so cooling with the direct Urca is much stronger than the modified Urca process. As such, after the point of neutrino transparency, the cooling with direct Urca processes present will be more pronounced than without. Monitoring the cooling of an isolated neutron star in the centuries after birth could therefore be used to classify it as inhabiting a slow or rapid cooling regime. This in turn could be used to place limits on the proton fraction present within the star. Although other cooling mechanisms can exist, the Urca processes are the only non-superfluid cooling channels that I will consider in the following discussion.

### 5.1.2 Superfluidity in cooling

Superfluid neutrons and protons have been predicted in the interior of neutron stars since Migdal in 1959 [235]. Superfluid pairing can effect the cooling of neutron stars through several mechanisms and can be responsible for both suppression and enhancement of cooling.
One of the most prominent effects of superfluidity can occur as the temperature of the star cools to near the critical temperature of the superfluid gap. As the temperature decreases and the matter begins its transition from fluid to superfluid there can be a spontaneous pair formation and breaking \cite{224} process. This process can occur at the phase transition between a normal and superconducting phase. In this process, nucleons will form Cooper pairs and, just below the critical temperature, they can be excited into quasiparticle excitations (broken Cooper pairs), which can then recombine. For the remainder of this work I will refer to the Pair Breaking and Formation process as PBF. The formation and breaking of these pairs gives rise to neutrino and photonic bremsstrahlung,

\[
NN \rightarrow N + N \rightarrow NN + \gamma \gamma \quad (5.9) \\
NN \rightarrow N + N \rightarrow NN + \nu \bar{\nu}. \quad (5.10)
\]

The photonic contribution is neglected when considering cooling because, as discussed above, the star is opaque to photons. The neutrino component, however, can escape the star and remove heat in much the same manner as in the Urca processes. This leads to a pronounced enhancement of the cooling in the vicinity of the critical temperature. The presence of this enhancement can therefore be used as evidence for the existence of a superfluid within the neutron star.

Once the star has cooled below the critical temperature the enhancement from PBF ceases and is replaced instead by a strong suppression of the cooling. The pairing of nucleons limits the phase space in which nucleon-dependent cooling processes can occur. This is because paired nucleons do not contribute to Cooling processes without being first broken. Paired nucleons are therefore removed from the pool of nucleons available to cooling mechanisms. Thus, processes such as the direct Urca process will shut off when the nucleons are all superfluid \cite{236}.
The presence of multiple pairing gaps can further contribute to the shape and quantitative structure of the cooling curve. In stars containing matter across a wide range of densities it is entirely feasible that, above a certain density, one gap will become smaller than the gap of another channel and so the second gap will become dominant (e.g. a $^1S_0$ gap giving way to a $^3P_2$ gap [237]). This can lead to different regions of the star providing different critical temperatures, and so a PBF cooling enhancement could occur, followed by a suppression, followed by a second round of enhancement and suppression. The question of pairing is further complicated when multiple species of baryon are present. Isospin asymmetry will separate the Fermi surfaces of the neutrons and protons and so it is possible for each species to be paired in different gaps within the same density region. For example, protons could exist in a low-density $^1S_0$ state in the same region in which neutrons were paired in a comparatively small high-density $^3P_2$ state.

In addition to its direct effect on the cooling, superfluidity within the neutron star will have implications for the thermal conductivity across the crust-core boundary and thermal transport properties of the star [238]. The existence of strongly paired nucleons at this boundary will act to reduce the specific heat capacity when below the critical temperature and as a consequence decrease the time taken for the cooling wave to propagate through the crust [239].

5.1.3 Other Effects

Neutron stars are not necessarily isolated and can be found in binary partners [240]. Estimating the precise fraction of neutron stars in binary systems is difficult. When in a binary system it is possible that a compact neutron star will accrete mass from a binary partner [241]. The cooling profile of a neutron star in such a situation is more
complicated than in the isolated case, as the accretion of matter can contribute significantly to crustal heating of the neutron star [242], obscuring the effects of internal processes.

Although the star rapidly becomes cold by astrophysical standards, nuclear fusion can still occur under the right conditions [243]. These thermonuclear reactions can heat the surface of the star, further disconnecting the temperature behaviour of the crust from that of the core. The accretion and fusion of material on the surface of neutron stars is the current favoured scenario for several astrophysical phenomena, such as production of X-ray bursts [244].

In the crust, one of the important mechanisms for thermal transport is via the Fermi-gas of electrons [245]. Neutron stars can produce very strong magnetic fields [246], with some magnetars producing fields of greater than $10^{14}$ Gauss [247]. The strength of the magnetic field can restrain the motion of free electrons perpendicular to the magnetic field [248]. Therefore, these magnetic fields can lead to non-spherically symmetric temperature distribution [249]. Surplus to this distortion, magnetic field decay can itself contribute to heating of a neutron star [250]. The anisotropy of this heating calls into question the validity of using spherically symmetric models and so I have neglected magnetic field effects from the calculations and discussion presented below. This omission means that magnetars are not considered or affected by this work.

5.1.4 Cooling Observables

As discussed above, the main cooling mechanisms within a neutron star occur within the core rather than the crust. Terrestrial detection of photons is still far easier than detection of neutrinos and so the surface temperature of the neutron star is the most accessible observable.
During the cooling of a neutron star from around ten years after birth the surface will continuously emit x-ray photons. The surface will remain decoupled from the core for several decades, until the cooling wave from the core reaches the surface and sharply drops the effective surface temperature [251]. The gravitational forces of the neutron star are so great that general relativistic effects will make the temperature appear different to a distant observer than to an observer at the star surface. As such, the “effective temperature” refers to the temperature as measured by a distant observer. If the star remains warm enough (that is to say that there is no direct Urca process), then the star will remain x-ray visible for millions of years. The inferred surface temperature depends on the composition of the envelope and atmosphere of the neutron star [231]. Light elements near the surface of the star are still able to undergo pyconuclear fusion, significantly raising the effective temperature. The more prolific the light elements, the more fusion can occur and the higher the effective temperature will be [252]. The metallicity of the star surface has been shown to have a large effect on the effective temperature [253] and so must be determined to a good degree of accuracy in order to comment on the internal thermal behaviour of the star.

Aside from physically altering the effective surface temperature the composition is important for understanding the x-ray spectrum that we can observe terrestrially. Neutron stars do not radiate as perfect black bodies and the composition of the surface will affect the spectrum emitted [254]. One proposed method for determining this composition is from multi-wavelength analysis of isolated neutron stars observations [255]. Ho and Heinke compared existing data from the Chandra X-ray telescope with various atmospheric models and determined that the young neutron star Cassiopeia A has an atmosphere of mainly carbon, with a low magnetic field [30].

If present, the direct Urca process will allow for rapid cooling of the neutron star in the first century or so of life. As mentioned above, the proton fraction must be
sufficiently high for direct Urca to be permissible within the star. Identifying cooling due to the direct Urca, or indeed the lack thereof, can therefore be useful to place limits on the asymmetry of the neutron star matter. Checking for the suppression of cooling processes in a neutron star between one hundred and a few thousand years old can provide evidence for the onset of superfluidity within a neutron star [236]. There are already several neutron stars within this age range that have provided interesting results, such as the 961 year old Crab pulsar (surface temperature $2.5 \times 10^6$ K) [256], the 835 year old PSR J0205+6449 (1×10$^6$ K) [257] and the 335 year old Cassiopeia A (2×10$^6$ K) [225]. Cassiopeia A in particular is interesting due the rapid cooling revealed by repeated measurements over the last 15 years. This cooling and its implication is discussed in further detail later in this thesis.

At this point, whilst discussing neutron star observables, it is worth briefly discussing one of the important relationships when examining pulsars: the $P$-$\dot{P}$ relation. Although this is not explicitly used in this work it is useful to give the reader context of the state of neutron star observations at the time of writing. $P$ is the spin period of the pulsar and so $\dot{P}$ is the derivative of the spin period with time. Pulsars will slowly lose rotational energy over time and so $\dot{P}$ should be positive. Several properties can be gleaned from the $P$-$\dot{P}$ relationship [258]. In many ways the $P$-$\dot{P}$ diagram for neutron stars is synonymous with the Hertzsprung-Russel diagram of normal stars. An example of a $P$-$\dot{P}$ diagram is shown in Fig. 5.1, taken from [259].

Using the $P$-$\dot{P}$ relation one can make links to the characteristic age of the neutron star, as well as the surface magnetic field. This is because neutron stars will slow their rotation over time, and the rate of deceleration also slows. As such, neutron stars with long spin periods and slow spin decay are likely to be older than their rapidly spinning, rapidly decaying counterparts. Regarding the magnetic field, stars with a high spin frequency are likely to generate a larger magnetic field than those with a slow spin frequency. Larger magnetic fields are also associated with larger losses in
Figure 5.1: Spin period against spin period derivative for a selection of pulsars. Using the diagram one can estimate the magnetic field strength, age, and spin-down power of the pulsar. Image Credit: “Handbook of Pulsar Astronomy”, Lorimer and Kramer

rotational energy and so generally exhibit a higher $\dot{P}$. One can see from Fig. 5.1 that pulsars fall into several distinct categories. For instance, the pulsars associated with supernova remnants are all relatively young. Possible explanations for this include the supernova giving the neutron star a sufficient kinetic “kick” that it departs the remnant over time, or simply the remnant cooling to such a scale that it is invisible.

In particular it is relevant that the $P$-$\dot{P}$ relationship can give insight into the age of the neutron star. For cooling calculations this is useful as one could feasibly observe the surface temperature of the star, the period, and the period derivative and thus the star can be compared to theoretical cooling curves. Whilst there are plenty of other phenomena discernable from a $P$-$\dot{P}$ diagram this thesis is concerned only with
the cooling and so I do not discuss the $P - \dot{P}$ relation further in this work.

## 5.2 Current Observations

![Sample of currently existing neutron star cooling data](image)

There have, so far, been a handful of observations of the effective temperatures of isolated neutron stars. Fig. 5.2 is taken from Yakovlev et al. [229] and shows the effective temperature for a distant observer as a function of age for 11 isolated neutron stars. Also depicted is a sample theoretical cooling curve for the purpose of illustration. The large error bars and small number of data points make it difficult to draw quantitative conclusions from the graph. The data do seem to support the idea of neutron stars undergoing a period of rapid cooling somewhere in the first few thousand years of life and then remaining comparatively isothermal thereafter. The long timescales involved with cooling in the later stages of a neutron star life and
the expected dominance of x-ray photon cooling means that little information on the internal structure is likely to be extracted from cooling rate measurements of elderly neutron stars. Equally, very young neutron stars still have interiors that are thermally decoupled from the outer crust and so the internal structure is still thermally obscured from view. In lieu of efficient neutrino detectors it therefore seems that the prime age for neutron star cooling to reveal the properties of their inner structure is somewhere between 100 and 10,000 years.

The Cassiopeia A neutron star (hereon referred to as Cas. A) is inside the young supernova remnant in the constellation of Cassiopeia. Observations of the expansion of the remnant have dated the star to be $334 \pm 19$ years [260]. The young age of this neutron star makes it interesting to study the cooling rate over time. Heinke and Ho have analysed 10 years worth of Chandra observations and have determined Cas. A to exhibit a steady cooling of the surface temperature of approximately 4% [261]. This measurement was then confirmed by Shternin et al. [36], but after consideration of calibration issues was revised to 2% by Elshamouty et al. [67]. The rate of cooling is too large to be easily explained with either of the Urca processes and so Page et al. proposed that the rapid cooling was due to the PBF process and as such Cas. A provides evidence of superfluidity at supra-nuclear densities [225]. Further examination of young neutron stars can therefore help to constrain the magnitude of the pairing gaps present, and thus refine our understanding of pairing in INM.

5.3 Neutron Stars Generated with D1P

In this section I will investigate neutron star properties constructed from the D1P Gogny force. This force is interesting to study because, as seen in Chapter 3 it is one of the only Gogny forces to provide a slope parameter large enough to satisfy many experimental constraints. It is also one of the only forces that yields a neutron star maximum mass large enough to incorporate the 2 solar mass measurements of Demor-
D1P is one of only two forces which provide a finite $^3P_2$ gap that is of the order of 0.5 MeV\(^1\). As such, D1P is the only Gogny force currently available that holds a large enough slope parameter and maximum mass as well as a small enough yet finite $^3P_2$ gap that it is not ruled out by literature.

Figure 5.3: Neutron star mass in solar masses as a function of central pressure for the D1P Gogny force

Fig. 5.3 shows the relationship between central pressure and neutron star mass, as calculated with the D1P interaction. A 1.6 solar mass neutron star comprised of beta equilibrated matter corresponds to a central pressure of approximately $2 \times 10^{35}$ dyne/cm\(^3\) and the maximum mass of 2 solar masses is provided at a central pressure of $1.4 \times 10^{36}$ dyne/cm\(^3\). The mass changes very little with increasing central pressure beyond $1 \times 10^{36}$ dyne/cm\(^3\). To further investigate neutron stars constructed with D1P I have focussed on a 1.6 solar mass neutron star but have included Fig. 5.3 for completeness. The figure also helps to illustrate that a 1.6 $M_\odot$ neutron star is not at the extreme of the central pressures accessible by the Gogny force.

\(^1\)The other force being D300
Figure 5.4: Neutron star mass enclosed in solar masses as a function of radial coordinate for the D1P Gogny force. The curve displayed is for a 1.6 $M_\odot$ neutron star.

Fig. 5.4 shows the mass enclosed as a function of radius for a 1.6 solar mass neutron star generated with D1P. As the radius increases linearly the mass enclosed exhibits more complex behaviour. One can imagine building the star up in shells of a given density with a small arbitrary thickness. Near the core of the star the volume enclosed by each consecutive shell of constant thickness is quite small due the small radial coordinate. Therefore, despite the large densities in this region, the mass does not increase much with radial coordinate. As one moves further from the center the radial coordinate increases and so the mass enclosed within a shell increases. For a large portion of the star, the mass enclosed increases almost linearly with radius. It is worth noting that if the star were to have a constant density then the mass enclosed would increase cubically with radius. The lack of this cubic behaviour implies that as the radial coordinate is getting larger the density is diminishing. Towards the outer edge of the star the density is very small and so the final 1000 meters of the star contributes little to the overall mass. It is worth noting that in these structure calculations I have not included a crust. The entire star is therefore currently modelled as INM and the inclusion of a crust may alter the picture near the surface of the star.
Understanding the evolution of density with radial coordinate is important for cooling calculations as it helps to identify the regions in the star where cooling variations will show the largest effect.

### 5.3.1 Density Dependent Properties

Fig. 5.5 depicts several neutron star properties as a function of density. The top panel displays the radial coordinate as a function of density. At zero density the radius is at a maximum: the edge of the star. An increase in density corresponds to moving deeper into the star. The radial coordinate initially decreases very little as the density increases, corresponding to a large density gradient as one moves further into the star. As the density increases further the radius begins to shrink at more rapid a rate, until ultimately towards the core of the star the radius dropoff becomes comparatively sharp and the central 1000 meters is at an almost constant density. This makes sense when compared with Fig. 5.4, which shows that as the radial coordinate increases from the center outwards the mass enclosed rises at a very shallow rate. The shallow rise in mass means that the gravitational attraction felt in the first few thousand meters of the star does not increase by much. The near constant gravity means a near constant pressure is required to support the star and as such the corresponding density changes very little. This is quite interesting for cooling calculations because regions of similar density will have similar sizes of pairing gap and so one would expect most of the central kilometer of the inner core to transition to a superfluid phase at the same temperature.

The central panel shows the proton fraction. At the densities present in the 1.6 $M_\odot$ neutron star all finite density regions contain protons. This is particularly significant for cooling properties. It is worth noting that there is no region of the star in which the proton fraction is large enough to satisfy the 11% threshold that Lattimer et al. [24] have suggested is necessary for the direct Urca process to occur. The modified Urca process is allowed at all densities with a finite proton fraction and so can occur
Figure 5.5: Radial coordinate, proton fraction and pairing gap as a function of density for a 1.6 solar mass neutron star generated with the D1P interaction. All quantities are calculated using an interacting particle spectrum throughout.
throughout the star described above. This process can, however, be inhibited where the temperature of the star is lower than the critical temperature for superfluidity.

The lower panel displays three of the pairing gaps as a function of density, namely the $^1S_0$ neutron gap, $^1S_0$ proton gap and $^3P_2$ neutron gap. All pairing gaps discussed from hereon in are calculated with an interacting single particle spectrum, using the asymmetry shown in the middle panel, unless stated otherwise. It is notable that although the neutron $^1S_0$ gap peaks at relatively low densities (and therefore towards the edge of the star), it is pervasive throughout all the densities present here. Interestingly, the proton $^1S_0$ has a maximum that, whilst having a peak value of approximately $1/3$ of that of the neutron, covers a far broader density range. The $^3P_2$ gap emerges at approximately $3.5 \times 10^{14}$ g/cm$^3$ and increases steadily in magnitude, becoming dominant over the $^1S_0$ at around $8 \times 10^{14}$ g/cm$^3$. This crossover point corresponds to a radial coordinate of approximately 6 km. Therefore, within a central 6 km sphere (a significant portion of the star), the matter will be paired in the $^3P_2$ channel. The overlap in the $^3P_2$ and $^1S_0$ gaps means that other than at the surface, where the gap peaks at around 3 MeV, neutrons throughout the star will always experience a pairing gap of no lower than 1 MeV. It is also useful to note at this stage that, for the majority of the star, protons experience a $^1S_0$ pairing gap near 0.5 MeV. Higher order pairing gaps are not considered at this stage. Based on the assumption that the proton Fermi energy will never be large enough for beyond $^1S_0$ gaps to emerge. The validity of this assumption is discussed later in this chapter.

Having established the presence of the proton and neutron pairing gaps it is useful to relate these to their corresponding critical temperatures in order to comment on the possible effects on cooling. The critical temperature, $T_c$ can be calculated from the
pairing gap at zero temperature from the relation [224]:

\[ \Delta = 1.76k_B T_c, \]  

(5.11)

where \( k_B \) is the Boltzmann constant. This approximation is valid for calculations at zero temperature and, as my calculations are carried out for \( T = 0 \) K, can be applied to the \( ^1S_0 \) and \( ^3P_2 \) gaps [214]. This allows for a comparison of critical temperatures as a function of density. Fig. 5.6 displays the critical temperature for each pairing gap as a function of density.

Figure 5.6: Critical temperature of different pairing channels as a function of density calculated using the D1P interaction.

From the discussion above one would expect that if the star were to cool isothermally then, as it cooled and reached the neutron critical temperature at approximately 12 billion Kelvin, there would be a significant cooling enhancement from the PBF mechanism. As the cooling continues one would expect the majority of the star to fall below the neutron \(^1S_0\) critical temperature. At this point a significant cooling suppres-
sion would be anticipated, caused by the strong neutron pairing heavily decreasing the phase space available for proton dependent cooling processes, e.g. the modified Urca process. At some further time one would expect the star to reach the proton $^1S_0$ and, at the high-density region where $^3P_2$ dominates, $^3P_2$ critical temperatures. This would once again enhancing the cooling through PBF. Below this temperature the PBF would cease. One would expect very little extra cooling suppression at this point, because processes requiring multiple species of baryon would already have ceased due to proton superfluidity. It is worth mentioning that the star is unlikely to cool isothermally and so it may be that superfluidity occurs in a different order to that stated above. For example the first superfluid to emerge may be a mid-density proton gap, rather than the larger low-density neutron gap.

5.4 Cooling Curves Using NSCool

In this section I have used the data generated over the previous sections to explore neutron star cooling for the D1P interaction. To create these cooling curves I have made use of the software “NSCool”, created by Dany Page from the Instituto de Astronomia in Mexico. The code and its relevant documentation is hosted at http://www.astroscu.unam.mx/neutrones/NSCool/. It is written in Fortran 77 and it was necessary to edit the code in order to accommodate the Gogny interaction.

I now give a brief explanation of how NSCool operates. Cooling curves are determined in spherical symmetry by calculating cooling/heating rates at a given radius and then solving the thermal transport equations throughout the star. The code, working from the initial conditions, solves the fully general relativistic equations of conservation of energy and energy transport. The internal temperature and internal
luminosity are taken in their red-shifted form:

\[ T_{RS} \equiv e^{\phi T}, \quad (5.12) \]

\[ L_{RS} \equiv e^{2\phi L}, \quad (5.13) \]

where \( \phi \) is the gravitation redshift. The temperature and luminosity are functions of time, \( t \) and radial coordinate, \( r \), and so the energy balance equation can be written as

\[ \frac{\delta T_{RS} (t, r)}{\delta t} = F \left( Q_h, Q_v, C_v, \frac{\delta L_{RS} (t, r)}{\delta r} \right), \quad (5.14) \]

where \( Q_h \) represents any energy sources, \( Q_v \) represents any energy sinks, and \( C_v \) is the specific heat capacity. The thermal transport equation can be given as

\[ L_{RS} (t, r) = G \left( \lambda, \frac{\delta T_{RS} (t, r)}{\delta r} \right), \quad (5.15) \]

where \( \lambda \) is the thermal conductivity. The functions \( F \) and \( G \) depend on the active heating and cooling processes within the star, such as accretion, Urca processes and superfluidity. Boundary conditions can be set on luminosity such that it is zero at the centre of the star, \( r = 0 \), (due to no mass being enclosed) and the surface of the star, \( r = R \), complies with the Stefan-Boltzmann equation:

\[ L_{RS} (t, r = 0) \equiv 0, \quad (5.16) \]

\[ L_{RS} (t, r = R) \equiv 4\pi R^2 \sigma_{SB} T_{e\infty}^4, \quad (5.17) \]

using the Stefan-Boltzmann constant \( \sigma_{SB} \), and the effective surface temperature at infinity, \( T_{e\infty} \). NSCool advances the temperature through time using the gradient of \( T_{RS} \) shown above multiplied by the time-step, \( dt \), and the old temperature. The heat sources, sinks, specific heat capacity and thermal conductivity all rely on \( T_{RS} \) and so
one can express the advancement through time as

$$T_{RS}^{New} = T_{RS}^{Old} + dt \times F\left( T_{RS}^{New}, \frac{\delta L_{RS}^{New}}{\delta r} \right).$$  (5.18)

The new luminosity can then be found from Eq. 5.15. The appearance of the new temperature on both the right- and left-hand side of Eq. 5.18, and the appearance of $L_{RS}^{New}$ on the right-hand side, present a self-consistent equation. NSCool solves for the new time iteratively, choosing a starting guess for the temperature, $T_{RS}^{New(0)}$. This guess is fed into Eq. 5.18 to find a new guess, which is fed back into the equation to generate yet another guess:

$$T_{RS}^{New(i+1)} = T_{RS}^{Old} + dt \times F\left( T_{RS}^{New(i)}, \frac{\delta L_{RS}^{New(i)}}{\delta r} \right).$$  (5.19)

Each value for the temperature guess differs from the previous by some amount, $\delta T_{RS}^{New(i+1)}$, where

$$\delta T_{RS}^{New(i+1)} = T_{RS}^{New(i+1)} - T_{RS}^{New(i)}. \quad (5.20)$$

The iterations continue until the difference between each iteration is smaller than some desired accuracy for the temperature, $A_T$, and a desired accuracy for the luminosity, $A_L$:

$$\frac{\delta T_{RS}^{New(i+1)}}{T_{RS}^{New(i+1)}} < A_T, \quad (5.21)$$

$$\frac{\delta L_{RS}^{New(i+1)}}{L_{RS}^{New(i+1)}} < A_L. \quad (5.22)$$

NSCool uses this procedure to iterate through the star from the centre outwards in steps of some radius, $dr$. Once an acceptable solution is found, NSCool moves to the next time step and the process begins again. This process has the condition that the total radius of the star is known at the outset and so the package also includes a
separate TOV solving code. I have not made use of this aspect of the software. Instead I have written my own TOV solver for use with the Gogny force, which was used for the neutron star mass-radius calculations discussed in Chapter 3.

5.4.1 Necessary Inputs

Before performing calculations with NSCool several parameters must be calculated and stored as inputs. I now briefly explain the motivation behind each input parameters and the values chosen. Although altering these parameters can have significant effects on the cooling curves, this work is not concerned with the individual parameter effects, but rather the viability of the Gogny force for consistent neutron star structure and cooling calculations. As such, only one parameter set is considered in this work. It is worth noting that the reason for choosing D1P was its performance in the isovector sector and the magnitude of the $^3P_2$ pairing gap produced, both of which show large model-dependence. Therefore whilst some of the input parameters will be insensitive to the chosen parameter set, an examination over the full parameter space would be a reasonable future project.

5.4.1.1 Crust Core Boundary

An important parameter for cooling is the density at which the neutron star transitions from the crust to the core. The thermal transport across the crust-core boundary is sensitive to the pairing gaps present and so it is important to know the density of this transition. To determine the crust core boundary density I have used Eq.(18) from [158]:

$$\rho_{\text{CrustCore}} = 7.46 \times 10^{-4} \times (S - 0.558 \times L) + 0.0754,$$

where $S$ and $L$ are the symmetry energy and slope parameter discussed in Chapter 3. $S$ and $L$ are in MeV and $\rho_{\text{CrustCore}}$ is in g/cm$^3$. For D1P this transition density corresponds to $1.39 \times 10^{14}$ g/cm$^3$. NSCool uses the crust core boundary to determine
where to switch between a core of homogeneous nuclear matter and a crust including finite nuclei.

5.4.1.2 Star Profile

To calculate the cooling for an individual star, NSCool requires an input file detailing several parameters as functions of radius. The necessary input parameters are the baryon number density, mass density, pressure, enclosed mass, gravitational redshift and enclosed baryonic mass. To provide this input file I have written a separate TOV solving code to give the mass density, baryon number density, pressure and enclosed mass as a function of radius.

The gravitational redshift is calculated by numerically solving the differential equation

$$ \frac{d\phi}{dr} = - \frac{dP}{d\rho} \left( \rho + P \right), \quad (5.24) $$

where $P$ is the pressure and $\rho$ is the density. The baryonic mass is also a differential equation. Based on Eqs. (3) and (4) from [262], one can express this differential equation as

$$ \frac{dM_B (r)}{dr} = m_N \times \frac{4\pi r^2 n_{\text{bar}} (r)}{\sqrt{1 - \left( \frac{2GM(r)}{cr^2} \right)}}, \quad (5.25) $$

where $n_{\text{bar}}$ is the number density, $G$ is the gravitational constant, $m_N$ is the nucleon mass and $M$ is the gravitational mass enclosed. $M_B$ is different from $M$ because whilst $M$ describes the full gravitational mass present $M_B$ provides only the sum of the mass of the baryons. The file generated is specific to an input central pressure and so corresponds to a star of a particular maximum mass. As well as this profile, NSCool requires an equation of state data file, which depends only on the force used and is independent of central pressure.
5.4.1.3 Equation Of State

The EoS file generated supplies NSCool not only with the pressure as a function of density, but also model-specific information on the composition of the INM used. The parameters required as a function of density are: pressure, baryon number density, electrons per nucleon, neutron fraction, proton fraction, neutron effective mass and proton effective mass. Although NSCool has functionality to consider hyperons and muons I have ignored these from my calculations and so their properties have been set to zero in all instances, though I mention them here for completeness. The pressure and baryon number density are calculated as described in Chapter 3. The lack of other charged particles, and the requirement for the INM to be charge neutral, means that the electron fraction is equal to the proton fraction. The proton and neutron fraction can be defined in terms of the asymmetry parameter, $\beta$, where

\[ n_p = \frac{(1 - \beta)}{2}, \quad (5.26) \]
\[ n_n = \frac{(1 + \beta)}{2}. \quad (5.27) \]

The value for $\beta$ is calculated by meeting the condition that the neutron chemical potential is equal to the proton plus electron chemical potential,

\[ \mu_n (\rho, \beta) = \mu_p (\rho, \beta) + \mu_e (\rho, \beta), \quad (5.28) \]

where the chemical potential, $\mu_n$, is the same as that discussed in Chapter 4.

This criteria must be met to ensure that the rate of $\beta^-$ decay is equal to the rate of $\beta^+$ decay to keep the proton fraction stable. If the neutron chemical potential were larger than the electron-plus-proton chemical potential then it would be more energetically favourable for the neutrons to decay to protons than for protons to decay to neutrons. As such, the decay rates would be uneven, altering the asymmetry of the system.
5.4.1.4 Surface Composition and External Heating

As mentioned above, the outer crust and atmospheric composition of a neutron star can have a big influence on the effective temperature measured by a distant observer. As such, NSCool allows for several different surface compositions. In my calculations I have elected for a crust comprised mainly of iron. One way that the surface temperature can be decoupled from the core is through nuclear fusion reactions occurring at the surface. This is particularly likely for lighter elements that will fuse their way up to iron [263]. Modelling using an iron surface therefore allows one to assume that there is no accretion and no pyconuclear fusion occurring in the star [30]. This work has been carried out to evaluate the feasibility of using a Gogny type force for neutron star property calculations. The calculations performed are therefore illustrative of neutron star modelling rather than exhaustive. As such, although it may be interesting to explore the effect of different crust models this is beyond the scope of the current work but may be of interest for future consideration.

NSCool’s modular layout allows for the consideration of effects such as accretion, and highly magnetised neutron stars. These are not considered here and so I have assumed zero accretion and no extra cooling effects from the presence of a magnetic field.

5.4.2 Surface Temperature

Fig. 5.7 shows the red-shifted surface temperature for a 1.6 $M_\odot$ neutron star as a function of time in years. Several curves are shown, corresponding to the presence of different combinations of pairing gaps. The curve marked as “none” (red full line) is the D1P cooling curve with no pairing gaps. It is worth drawing attention to the log scale of the x-axis. The rapid initial cooling onset represents the star cooling to thermal equilibrium and happens over the timescale of days. All cooling curves agree quite closely during this phase of the cooling.
Figure 5.7: Effective surface temperature as a function of time for a 1.6 solar mass neutron star calculated with the D1P interaction.

In this first phase, the proton-only curve (orange dashed line) matches the no pairing case exactly. This agreement continues throughout the first hundred years. Setting the initial temperature of the star such that it is above the critical temperature for proton pairing, one would expect that at some latter time the cooling would be enhanced by the PBF process after which there would be a cooling suppression. In Fig. 5.7 there is no discernable cooling enhancement, but there does come a point at which the presence of superfluid protons appears to noticeably suppress the cooling. The low proton fraction could mean that the predominance of neutron cooling makes it difficult to identify the small proton cooling component. The core is not fully thermally
coupled to the crust until after the first century and this may also help to mask any cooling enhancement.

Also initially in alignment with the no pairing case is the $^3\!P_2$ only case (green dotted line). This curve sticks to the zero gap case for several hundred years longer than the proton only curve, after which there is a large cooling enhancement. This is understandable from Fig. 5.6 which shows that the $^3\!P_2$ critical temperature is, compared to $^1\!S_0$, quite small. It therefore is predictable that the onset of superfluidity would happen at a lower temperature, and therefore a later time, than in the proton superfluidity case. Once the temperature drops sufficiently low the PBF process can begin. Because the $^3\!P_2$ gap is pervasive through a large portion of the star there are many neutrons participating in the PBF enhancement. The magnitude of the gap changes with radius and so not all of the neutrons become superfluid at the same temperature; instead the PBF process is allowed to occur in at least some part of the star for many years. This could explain why no cooling suppression is noticeable within the timeframe considered in this work.

The neutron $^1\!S_0$ only channel (black broadly dashed line) exhibits a slightly faster cooling rate than the no gap case, from the first few days. This enhancement is due to the PBF from the comparably large critical temperature. The proximity of the gap peak to the star surface could also go some way to link the behaviour of the crust and core: the PBF can occur in the neutron fluid present in the inner crust. After the first year, the cooling has levelled off and is now slower than the zero gap, but shortly thereafter the cooling reactivates, taking the temperature once again below the no pairing case. This coincides with the time at which the internal cooling wave would be expected to propagatae to the surface and so this could be the ghost of the original PBF enhancement near the core finally reaching the edge of the star.

Allowing both $^1\!S_0$ and $^3\!P_2$ neutrons to exist (light blue dashed-dotted line) in
the star yields a cooling curve initially very similar to the $^1S_0$ only case. This is not surprising: at the initial temperatures considered only the $^1S_0$ has a relevant critical temperature. As the star cools further, the nucleons star to become superfluid in the $^3P_2$ channel. There is a very small but finite enhancement of the cooling around 300 years when compared to the $^1S_0$ case, followed by a noticeable suppression of the cooling. This is where the $^3P_2$ gap has given a burst of PBF enhancement at a slightly higher temperature than $^1S_0$ would, and a cooling suppression at an earlier time than $^1S_0$ would.

The final curve (dark blue dashed-dotted) is one with all of the available pairing gaps switched on. Several of the behaviours of individual pairing gaps can be seen, but the total cooling enhancement occurs at an earlier time than any of the other pairing scenarios. The curve is initially in line with the neutron $^1S_0$, which is not to say that the protons are not pairing, but their effect could be very small for the reasoning described above. After ten years there is a sudden enhancement of pairing which cools the star rapidly over a short period of time. At this point there may well be a large portion of the star pairing in the $^1S_0$ channel of both protons and neutrons, leading to a very large PBF enhancement. After this there is a strong suppression of the cooling, with the surface temperature of the star almost constant for many thousands of years. Although it appears on the graph as though there is another sudden temperature dropoff I would draw the reader’s attention to the log scale of $x$: the final cooling leg is of the order of millions of years. The full pairing regime reaches this final stage faster than the other scenarios. This is explained by the fact that with all pairing channels active, and the existence of a finite pairing gap at all densities for both protons and neutrons, the star becomes completely superfluid and cooling processes can be suppressed at an earlier time than with any other pairing regime. Justification for this can be found by examining the internal temperature profile of the neutron star.
5.4.3 Temperature Profile

To explore the effect of the pairing gaps in a more depth it is necessary to examine the temperature profile of the star rather than simply its surface temperature. NSCool has functionality to output the temperature at any given density (and therefore radius) at each time step. Using this functionality one can produce a graph at multiple time ticks of NSCool. These graphs can be found in Fig. 5.8. The “time zero” curve is the uppermost one and with each tick of NSCool the temperature drops. As such, each curve occurs at a later time than the one above it.

The results presented here confirm the conclusions drawn from Fig. 5.7. It is clear that the neutron $^1S_0$ gap has the most pronounced effects, showing clear PBF enhancement as the temperature reaches the critical temperature. Fig. 5.8 also makes it clear that the proton $^1S_0$ gap is pervasive throughout the entire region of the star.

In Fig. 5.8 one can easily identify the discontinuity between crust and core. There is a strong decoupling at early times, whereby the temperature drops far more rapidly in the core than in the crust. It is also interesting to note that the top left panel, which displays matter with no superfluidity, is almost isothermal throughout the core. There do not appear to be parts of the core that are cooling preferentially to others.

The inclusion of the neutron pairing gaps has a definite effect on the cooling profile. The top right panel shows the case of including only neutron $^1S_0$ pairing. There is a new discontinuity showing: where superfluidity appears. As a particular region of the star enters superfluidity there is the characteristic temperature drop that one would associate with a PBF cooling enhancement and the region becomes cooler than the matter still above the critical temperature. Shortly after the region has entered the superfluid phase the cooling is suppressed. The leads to the superfluid components of the core cooling slower than the non-superfluid region.
Figure 5.8: Temperature profile at several times for a 1.6 solar mass neutron star calculated with the D1P interaction. The curves are at the following times in years: $10^{-10}$, $10^{-9}$, $10^{-8}$, $10^{-7}$, $10^{-6}$, $10^{-5}$, $10^{-4}$, $3 \times 10^{-4}$, $10^{-3}$, $3 \times 10^{-3}$, $10^{-2}$, $3 \times 10^{-2}$, $10^{-1}$, $3 \times 10^{-1}$, 1, 3, 10, 30, $10^2$, $3 \times 10^2$, $10^3$, $3 \times 10^3$, $10^4$, $3 \times 10^4$, $10^5$, $3 \times 10^5$, $10^6$, $3 \times 10^6$.

The middle left panel gives the case where only the protons are superfluid. The effects from this superfluidity are barely noticeable. This is perhaps unsurprising considering the low proton fraction within the star. Cooling enhancements from PBF will only be occurring in a small amount of matter and so the overall cooling profile is relatively unaffected.

In all of the panels with the neutron $^1S_0$ channel active the decoupling between
the crust and the core appears to break down. Although there is still a noticeable difference between the crust and core temperatures, the transition between the two is far less abrupt: the onset of superfluidity has enhanced thermal conductivity across the crust-core boundary.

The neutron \(^3\!\!P_2\) critical temperatures present in the star are low compared to the neutron and proton \(^1\!\!S_0\) cases. The middle right panel shows the case of only \(^3\!\!P_2\) pairing. There is a small effect on the cooling profile in the inner core, but this is present at far lower temperatures than effects from neutron \(^1\!\!S_0\) superfluidity. The \(^3\!\!P_2\) gap does not open within the crust and so the decoupling experienced in the no pairing case is also present here. As such, the onset of \(^3\!\!P_2\) superfluidity has little effect on the temperature of the surface.

The bottom right panel shows the case with all available pairing channels active. The cooling profiles are very similar to the case of neutron \(^1\!\!S_0\) only. There are, however, noticeable effects from the \(^3\!\!P_2\) onset in the inner core, where \(^3\!\!P_2\) becomes dominant over \(^1\!\!S_0\). The proton pairing also appears to influence the thermal coupling between crust and core. This increase in thermal coupling helps to clarify the cooling scenario in Fig. 5.7, where the full cooling regime cools faster than any individual process. Interestingly, the enhancement of cooling in the crust is larger here than in the case where only proton pairing is active. It is worth noting that as the temperature drops past the proton critical temperature all of the neutrons are already paired. The small cooling burst from PBF could, despite its size, be the strongest cooling process available in parts of the inner crust.

The different pairing scenarios provide several different pictures of the neutron star cooling profile. Overall, the largest influence is the onset of neutron \(^1\!\!S_0\) superfluidity. This particular gap dwarfs the effect of the other channels explored, however that may be subjective to the densities explored in this 1.6 \(M_\odot\) star. It would be an interesting
extension of this project to examine how the predictions change for neutron stars with higher central densities.

5.5 Discussion and Comments

Earlier in this chapter it was touched upon that NSCool does not include proton $^3P_2$ pairing. The proton $^1S_0$ gap increases as one moves deeper into the star until some critical density where the gap begins to drop. The maximum magnitude of the proton gap should be comparable to that of the neutron gap, and so the large disparity in the maximum gaps present indicates that the proton density saturates and then decreases before the maximum gap is achieved. Under the assumption that the proton gaps behave similarly to the neutron gaps, that is to say that the $^3P_2$ does not become dominant until well after the peak $^1S_0$ density has been passed, then the proton $^3P_2$ gap is unlikely to be relevant at the densities considered. It is therefore justifiable to restrict the proton pairing to $^1S_0$. However, this situation is specific to D1P and does not apply to all of the Gogny forces.

D1M exhibits a $^1S_0$ that closely agrees with D1P and in fact all other Gogny forces except GT2. Beyond $^1S_0$, D1M produces an extremely high and extremely broad $^3P_2$ gap and it can be seen from the figures in Chapter 4 that there is in fact no density at which the proton $^1S_0$ is larger than the $^3P_2$. This would lead to exceptionally strong $^3P_2$ pairing throughout the star which would likely give rise to a cooling suppression throughout the star’s life. In fact, with neutron $^3P_2$ pairing also considered, all nucleons would be strongly paired, the critical temperature would be very high throughout the star and it is unlikely there would ever be a PBF enhancement phase. Because of this dramatic $^3P_2$ gap, for complete consideration of D1M and other similarly behaving forces incorporation of beyond $^1S_0$ proton gaps would be necessary.

It is also worth noting that even at large asymmetries the $^3S_1$ n-p pairing gap does
not disappear for any force until at least $k_{F_N} = 5$ fm$^{-1}$. The gaps are also extremely large, several hundreds of MeV. As such, for all Gogny forces including D1P, the effect of $^3S_1$ pairing should not be ignored. Large n-p pairing may suppress the cooling from the start, if the initial temperature is already below the critical temperature, as would be expected from the extremely high values of the gap.

An interesting feature of the results is that the different superfluid regimes are discernable from one another within the first few hundred years. This is relevant because the young neutron star Cas. A is around such an age [260]. Monitoring of the cooling curve of Cas. A over the past decade has provided us with an idea of the current cooling behaviour [261]. One could use these, and future, observations to establish whether Cas. A currently contains superfluid components and thus evaluate the predictions made by the Gogny forces.

One of the main aims of this thesis is to investigate the possibility of using the Gogny force as a single consistent force for neutron star property calculations. In this chapter I have shown cooling curves that have been generated for a single 1.6 $M_\odot$ neutron star. The neutron star EoS, composition, effective masses and superfluid pairing calculations have all been performed with one single Gogny force: D1P. The curves generated are designed to be illustrative rather than exhaustive; there is scope for future work to investigate the cooling in different mass stars and with Gogny forces other than D1P. In particular it may be interesting to see the effect of using forces with large broad $^3P_2$ gaps on the cooling profile. The poor isovector constraints on the Gogny force mean that there is high systematic uncertainty in the neutron star matter properties they produce, however this chapter has shown that the Gogny force can, in principle, be used for a consistent description of the structure and evolution of a neutron star.
Chapter 6

Conclusions and Outlook

6.1 Isovector Properties

As discussed in Chapter 3 the high momentum, high asymmetry region of the Gogny forces is relatively unconstrained. It has been shown that the effective masses of certain forces, namely D1S, D1M and D250, exhibit a crossing point whereby neutron effective mass becomes lower than the proton effective mass. It would therefore be useful to investigate the effect on transport calculations \cite{264} and observables of low energy nuclear collisions \cite{265}.

Although the Gogny forces give symmetry energy values that are within the currently accepted constraints, the slope parameter has high variability. Most of the Gogny forces are unable to produce a large enough slope to fit with the accepted envelope. The few forces that are able to meet the slope criteria are still at the low end of the constraining window.

Most of the Gogny forces are unable to predict neutron star maximum masses of a large enough magnitude to account for current observations. Some forces do not predict stable neutron stars at all. This discrepancy can be linked to the slope parameters generated by the forces; most are far below the experimentally constrained envelope. Only a handful of forces provide isovector properties within this envelope,
and they are at the lower end of the accepted range.

6.2 Pairing Properties

6.2.1 The $^1S_0$ Pairing Gap

The $^1S_0$ gap generated with the Gogny interaction is relatively model-independent. The values generated agree with those in literature for BCS $^1S_0$ pairing [198, 199, 200]. Most Gogny forces are comprised of $x_0$ values of unity, which eliminate the zero range term for $^1S_0$ pairing. The D1P interaction, however, does not provide the same cancellation and so it is necessary to manually disable the zero-range component in order to extract sensible $^1S_0$ gaps. Despite this truncation, the D1P force still generates a $^1S_0$ gap in keeping with the other forces. It is good to confirm, but not surprising, that the Gogny forces perform well in this channel. The force was originally constructed with S-wave pairing in mind. The pairing gap also appears near to, and below, saturation density. As such, these calculations are within the comfort zone of the Gogny force.

6.2.2 The $^3P_2$ Pairing Gap

Beyond $^1S_0$ the pairing channels become far more model-dependent. The $^3P_2$ pairing gaps range from zero to several hundred MeV. There is also a clear relationship between maximum magnitude of the gap and the momentum range it covers. The largest gap, D1M, is finite immediately after zero and remains finite up until many times saturation density. Such a large gap is not supported by any currently accepted literature. A zero $^3P_2$ gap is also unsupported, yet D1, D1AS, D1S, D250 and D260 all fail to produce gaps in the $^3P_2$ channel. Only D1P and D300 give values that are near to the wide array of $^3P_2$ gaps suggested in the literature [209, 210]. D1P was constructed with some INM properties in mind [100] and it is perhaps this consideration that has constrained the $^3P_2$ value to a sensible level.
It is interesting that D300 should provide such a reasonable \(^3\)P\(_2\) channel when, in theory, the only difference between D250, D260, D280 and D300 is the fit to compressibilities [116]. This suggests that it could be interesting future work to examine more closely the role of compressibility in pairing with the Gogny force.

With most models there is an overlap in Fermi momentum space between the \(^1\)S\(_0\) and the \(^3\)P\(_2\) gaps. In fact, in many models there is a finite neutron gap in at least one of the channels throughout the entirety of the star. As such, there would be a temperature at which the entirety of the star is predicted to have all neutrons paired. This comprehensive pairing would ultimately suppress heavily the cooling of the star after a certain temperature. Such a dramatic suppression could indicate that the entirety of the neutron star core is in one superfluid state or another. With the Gogny force there is a clear relationship between the \(^3\)P\(_2\) peak maximum and the \(^3\)P\(_2\) gap width, and the \(^1\)S\(_0\) gap is well constrained. As such, with the Gogny forces, the minimum neutron gap present in the core depends entirely on the point at which the \(^3\)P\(_2\) gap intersects with the \(^1\)S\(_0\) gap, which in turn depends on the width of the \(^3\)P\(_2\) gap and therefore the maximum gap. If neutron stars were shown to be superfluid throughout then determining the minimum neutron gap in a neutron star should effectively constrain the \(^3\)P\(_2\) gap in the Gogny force. Furthermore if some regions of the core are shown to lack a superfluid phase, and therefore proving that the \(^3\)P\(_2\) and \(^1\)S\(_0\) neutron pairing channels do not overlap, then D1P pairing results can be shown to be unphysical.

### 6.2.3 The \(^3\)S\(_1\) Pairing Gap

The Gogny forces all produce finite \(^3\)S\(_1\) gaps. These gaps are finite at zero momentum, a property only possible with this particular channel and supported by literature [210]. They are, however, all far higher than those reported in the available literature [211]
and span an extremely large momentum space. There is very large model dependence displayed and once again it is interesting that the D250, D260, D280 and D300 family show such a large variation.

Although D1P produced literature-compliant $^3P_2$ values, the $^3S_1$ gap is still abnormally large. This discrepancy is disappointing for the D1P force considering its relative success in the $^3P_2$ instance. The $^3S_1$ is, however, a proton-neutron channel and (as has been shown above the Gogny force) is poorly constrained in the isovector sector. This could be reflected in the proton-neutron pairing channels where a wide range of momentum differences is explored in the BCS integrand.

Also reflecting the systematic uncertainty in the Gogny interaction is the asymmetry dependence of the $^3S_1$ channel. The forces behave in very different ways; some forces experience very little change with asymmetry (D1S, D250, D1N), some find a higher asymmetry to significantly enhance the gap at most momenta (D1, D1P, D260, D280, D300) and another finds higher asymmetries significantly suppress the gap (D1M). Furthermore, within the momentum range examined some forces experience an inversion point whereby large asymmetry begins to impede instead of enhance the gap (D1S, D1P, D250, D1N).

Of particular interest is the relation of the $^3S_1$ matrix elements and those entering into calculations of the proton polaron single particle energies. This link means that experimental information on the polaron could be used to constrain the Gogny force in a channel that directly effects $T = 0$ pairing. In pure neutron matter the effective mass splitting relies only on the parameter $B_{np}$ and so this provides further mechanisms for information on the polaron to constrain the isovector behaviour of the Gogny force.
6.2.4 The $^1P_1$ Pairing Gap

The $^1P_1$ pairing gap with the Gogny force is extremely unconstrained. Many of the forces open the $^1P_1$ gap almost immediately above zero density and are finite, and large, well above saturation. The D1 gap, the earliest of the Gogny forces, is the only force which gives close to a sensible pairing gap near saturation. The extreme gaps generated at high density are not supported by literature.

The model dependence of the $^1P_1$ gap is explained using the very wide range of values for $F_{i0}^{00}$ exhibited by the Gogny force. This channel is clearly the least constrained. Interestingly, the $F_{i1}^{11}$ channel is the most constrained, but the channels only differ by $2B_i - 2H_i$. It would therefore imply that these are the parameters most in need of constraint. This is interesting because $B_i$ is important for the isovector exchange term, as can be seen in Chapter 3 and as such further isovector constraints may improve the $^1P_1$ pairing channel.

6.3 Viability for Cooling

6.3.1 Critical Temperatures

The pairing gaps of the Gogny force exhibit large model dependence and therefore so too does the critical temperature. Several of the Gogny forces (D1, D1AS, D1S, D250, D260) do not generate finite $^3P_2$ gaps, which is contrary to what is currently suggested by the literature [225]. Conversely, some of the forces (D1M, D280, D1N) provide $^3P_2$ gaps that are far larger than would be suggested in the literature. These large gap producers also have very broad $^3P_2$ channels; the $^3P_2$ gap is active and dominant in the same region as the $^1S_0$. Proton $^3P_2$ pairing is not expected to occur in neutron stars. D1M, D280 and D1N, all predict proton P-wave pairing at relatively low momenta and so these forces exhibit poorly constrained critical temperatures. D1P and D300 give critical temperatures that are much more reasonable in terms of their maximum
mass and momentum range.

It is perhaps also worth noting that many pairing gaps and their critical temperatures are in coupled channels. For example the $^3P_2$ channel is often coupled with the $^3F_2$. This is not considered in this work because I have neglected tensor components, hence the omission of GT2 from the current analysis. It may, however, be interesting for future work to examine the effect that inclusion of tensor components and coupling of different angular momenta has on the critical temperatures generated by the Gogny force.

### 6.3.2 Cooling Curves

The D1P force has been used to calculate neutron star cooling curves, including effects from pairing. The curves produced are similar in structure to those found in the literature. The curves generated include features that can be attributed to particular pairing gaps and the differences between different pairing configurations are discernable at timescales relevant to young neutron stars such as Cas. A [260]. This means that accurate observations of neutron star cooling rates and masses could provide a viable method for constraining the pairing properties of the Gogny interaction. In particular, if the onset of PBF cooling could be identified and attributed to a particular channel, and the mass of the neutron star were known, then this information could be used to constrain future parametrisations. The neutron star mass-radius relations presented in Chapter 3 are not degenerate as a function of central pressure, that is to say that each mass of astrophysical relevance corresponds to a unique central pressure, and so for a neutron star of a known mass the corresponding central pressure within a given parametrisation can be determined. This can be used to generate a cooling profile prediction for that neutron star. Comparing this prediction to the observed sections of a cooling profile would allow one to evaluate the predictive power of the force. Moreover, a new parametrisation could include the onset of PBF for a given
mass star in the fitting procedure.

The cooling study has been performed with a single Gogny force. The D1P force was selected because of its comparable sensible pairing gaps and slope parameter. It is therefore likely to be the force that provides the most sensible results in the cooling sector. There is, however, scope for future work to investigate the remaining Gogny forces and, in particular, test the sensitivity of the cooling to the magnitude of the $^{3}\!P_2$ channel.

The neutron star discussed in Chapter 5 is at 1.6 $M_\odot$. With D1P, the central density of such a neutron star is still sufficiently low that protons are present. It would be interesting as an extension of this work to investigate the effect of using more massive neutron stars, perhaps where protons are no longer present in the inner core. Also, it may be interesting to investigate the cooling behaviour of a lighter neutron star, where the core density may be insufficient for $^{3}\!P_2$ cooling to occur at any point throughout the star.

It is worth mentioning that none of the Gogny forces provide a proton fraction that meets the 11% suggested by Lattimer et al. [24] for the direct Urca cooling process. As such, if future observations were to suggest the presence of this rapid cooling regime, and therefore a strong proton fraction, then future Gogny forces will need to be able to generate larger proton fractions to match.

### 6.4 Outlook for the Gogny Force

Most of the Gogny forces have failed to perform well in reproducing isovector properties or pairing properties at saturation density. As the density increases, so too does the model-dependence of the Gogny force. Although there are only a handful of different Gogny parametrisations at this time this study has shown that they exhibit
a large amount of variability, particularly in their pairing properties. As such, the Gogny forces exhibit large systematic uncertainty. Despite the fact that several of the Gogny parameters have been fitted with pure neutron matter in mind, there are still very few that provide sensible high-density high-asymmetry properties. As such, their use in neutron star matter is difficult to justify at this stage.

Current fitting procedures are typically sub-saturation and are not particularly asymmetric. It is perhaps unsurprising, therefore, that the forces exhibit poor isovector constraints and large divergences at high density. To parametrise a new Gogny interaction that could be sensibly used for neutron star calculations extra considerations should be added to the $\chi^2$ fitting procedure. The slope parameter produced, in particular, would be a useful consideration. Improving the stiffness of the Gogny force could help to produce larger maximum masses, which may be necessary to explain future observational evidence. It is worth noting that the D1AS case was adjusted to produce stiffer equations of state. However, it shares all of its finite range properties with the D1 interaction and so shares its weaknesses in the pairing channels.

The pairing channel is dependent on two parameter sets, $i = 1, 2$. In particular the width of the Gaussian, $\mu_i$, usually takes one short-range and one-longer range value, usually in the region of 0.7 and 1.2 fm. It is possible that the parameter space of the Gogny force should be expanded to allow for greater control over the pairing terms. Alterations to the form of the Gogny force are already being attempted [266]. Although this thesis is concerned solely with the classical structure of the Gogny force it may be an interesting extension to examine the effect of expanding the parameter space.

D1P has been shown to perform well in the isovector sector. The consideration of PNM at higher densities in its fitting procedure [100] may also go some way to explaining why the high-density behaviour of D1P does not produce the same extreme
pairing values in the $^3P_2$ channel as some of its counterparts, in part due to the high-momentum behaviour of the Gogny interaction in the integrand of the BCS equation.

When examining the $T=0$ n-p pairing channels the D1P force gives extremely large anomalous gaps that are not currently justified in the literature. For asymmetric matter the neutron and proton Fermi surfaces are separate and so the isovector properties of the Gogny interaction play a stronger role than in the isospin triplet state. As such, despite D1P being the most appropriate of the Gogny forces in the context of NSM, it is far from ideal and further isovector constraint would be necessary for a robust NSM-ready Gogny force.

The D1M force provides particularly extreme results in the pairing channels explored, and could act as a cautionary tale for future parametrisations. D1M gives particularly large finite range exchange parameters, $B_{nn}^1$ and $B_{np}^1$. Because the pairing properties and effective masses are very sensitive to these parameters their unbridled nature could be the cause of the anomalous nature of D1M. In particular the large positive $B_{np}^1$ provides for extreme asymmetry dependence of the effective mass. Because many neutron star properties depend on the proton fraction, any force being utilised should include stringent asymmetry considerations, which do not appear to be present in D1M.

The D280 force also displays some properties that, in light of the shared fitting procedure with D250, D260 and D280, could be considered abnormal. There is a stronger repulsive energy-per-particle for polarised neutron matter, when compared with the sibling forces. It is also the only one of the four that does not exhibit an inversion in the ordering of the neutron and proton single particle energies in the $k_F = 2 - 3$ fm$^{-1}$ range. These could be linked to its comparatively stiff equation of state and may go some way to explaining why D280 gives a much larger maximum mass than any of the other three forces in this cluster, despite the lack of isovector bulk property consid-
erations in the fitting procedure [83]. Examining the parameters that make up these 4 forces shows that their makeup is very similar. The only notable difference is that D280 has a larger negative $M_1$ value, -2401 MeV, compared to the D250, D260 and D300 values of 1009, -1973 and -1037 MeV respectively. This is particularly relevant when considering that, as stated in Chapter 3, the effective mass splitting is entirely governed by $B_{np}^i$ which depends heavily on $M_i$.

Although this work is focussed on INM, the isovector weaknesses of the current Gogny parametrisations could well extend to finite nuclei, impacting on isovector properties such as neutron skins and neutron-rich nuclei. New facilities are expanding the asymmetries that can be terrestrially probed. Upcoming experiments are also increasing our knowledge of other isovector behaviours. For example, PREX-2 [267] will provide model independent values for the neutron skin thickness in $^{208}$Pb. It would therefore be an interesting and relevant extension of this work to perform a systematic investigation of the isovector and properties of the Gogny force in finite nuclei.

The D1P force produces a neutron star maximum mass in keeping with current observations, but does not leave much flexibility for future measurements. The force produces sensible $^1S_0$ gaps as well as a $^3P_2$ gap comparable to those suggested in the literature. The ability of the D1P force to produce cooling curves that are at least qualitatively comparable to literature shows that it is within the ability of the Gogny force to produce sensible cooling profiles. As such, although the D1P force does not produce results perfectly in keeping with the literature it shows that in principle a Gogny force could be tuned to give, simultaneously, accurate isovector properties, large enough maximum masses, and realistic enough cooling profiles for use in neutron star calculations.

For future fits of the Gogny force, if one is interested in studying neutron star matter, there are several considerations that should be taken into account during the fitting
procedure. The single-particle properties of the current Gogny forces are relatively unconstrained at high densities. One method of constraining the high-momentum single particle properties is through fitting to realistic many-body calculations in isospin asymmetric nuclear matter, such as in [131, 132, 133]. It is also worth repeating that the most successful force, D1P, utilises the $i = 2$ zero-range channel, which could provide a mechanism for improving the stiffness of the EoS. Furthermore, because this does not directly interfere with the finite-range components one could feasibly tune the slope parameter without disrupting pairing considerations in which the zero-range terms are neglected. Whilst utilising a different Gogny force for the EoS and pairing calculations is possible, a new Gogny parametrisation would allow for a wholly consistent description of a neutron star and so is the preferably solution. In terms of pairing the D1 Gogny force was originally fitted without considering the magnitude of the pairing involved, only the sign of the matrix elements. Future fits may want to consider the explicit magnitudes expected in order to prevent unphysically large gaps from appearing. Future Gogny fits should also aim to constrain the $^3P_2$ gap to momenta in keeping with the literature. If they are available then observations of neutron star masses and PBF onset times should be taken into account in the fitting procedure. Finally, proton polaron effects could be used to constrain the effective mass splitting and $T = 0$ pairing.

The effective masses are entirely dependent on the finite range exchange terms, $B_{\tau\tau'}^{t}$, of the Gogny interaction and so effective mass related behaviour may provide a good parameter for future fittings of Gogny interactions, particularly in the isovector sector. The mass splitting itself depends only on $B_{np}^{t}$ and can therefore provide an even more specific limitation on the Gogny parametrisation.

Although I have proposed several experimental and observational constraints for future fits, it is worth noting that not all of these data are easily obtainable. For instance, proton polarons in nuclear matter would be hard to observe, and there is no
real analog in nuclei. The polaron problem does, however, feature in solid-state atomic physics [268] and so information from polarons in a crystalline lattice could be relevant to nuclear physics. Our ability to observe neutron stars is increasing rapidly. Accurate radius measurements could help to narrow the allowed bands on the mass-radius diagram, this allowing for more confident evaluations of a force’s EoS. Continued observations of surface temperature throughout several decades are already taking place [261]. As the data grows so too does our ability to estimate cooling histories and superfluid influences. Future terrestrial experiments will help to narrow constraints on the slope parameter. In particular the PREX-2 experiment should provide an accurate, model independent probe of the neutron skin thickness in lead [267]. Measurements of this kind would directly contribute to our knowledge of the slope parameter.

In conclusion, although most of the Gogny forces are not currently robust enough in both the pairing and isovector sectors for reliable neutron star property calculations, I have shown that it is at least possible to produce neutron stars of sufficient maximum masses, with sensible \( ^1S_0 \) and \( ^3P_2 \) pairing gaps in a BCS framework with the Gogny force. As such, there is scope for future parametrisations to generate a Gogny force that has acceptable isovector properties and beyond \( ^1S_0 \) pairing gaps, for use in comprehensive and consistent neutron star structure calculations.
Appendices
Appendix A

Analytical expressions for bulk properties

At zero temperature, the thermodynamical properties of asymmetric nuclear matter with the Gogny interaction can be expressed analytically. These involve double integrations of the matrix elements over Fermi spheres. Let us start with the energy per particle of asymmetric nuclear matter. The full expression for the Gogny force reads [139]:

\[
\begin{align*}
\epsilon (\rho, \beta) &= \frac{3}{5} \frac{\hbar^2 k_F^2}{2m_N} \frac{1}{2} \left\{ \frac{1}{2} \beta \frac{5}{3} \rho^2 \right\} + \frac{1}{2} \sum_{i=1,2} \left\{ A_i^0 C_i^0 \rho^2 + A_i^1 C_i^1 \rho^2 \right\} + \frac{1}{2} \sum_{i=1,2} \left\{ B_{nn}^i \left[ \frac{1}{2} + \frac{1}{2} \frac{1}{2} g (\mu_i k_F^n) + g (\mu_i k_F^p) \right] + B_{np}^i h (\mu_i k_F^n, \mu_i k_F^p) \right\} .
\end{align*}
\]

(A.1)

The function \( g(q) \),

\[
g (q) = \frac{2}{q^3} - \frac{3}{q} - \left( \frac{2}{q^3} - \frac{1}{q} \right) e^{-q^2} + \sqrt{\pi} \text{erf}(q) ,
\]

(A.2)

is the result of a double integration of the exchange matrix elements over the same
Fermi surface. It is the sum of a Gaussian and an error function, where the error function is:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}. \]  

This function appears as well in the completely degenerate cases and has been quoted in the literature [72].

The second function,

\[ h(q_1, q_2) = 2 \frac{q_1^2 - q_1 q_2 + q_2^2}{q_1^2 + q_2^2} - 2 \frac{q_1^2}{q_1^2 + q_2^2} e^{-\frac{(q_1 + q_2)^2}{4}} - 2 \frac{q_2^2}{q_1^2 + q_2^2} e^{-\frac{(q_1 - q_2)^2}{4}} \]

\[ - \sqrt{\pi} \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2} \text{erf} \left( \frac{q_1 - q_2}{2} \right) + \sqrt{\pi} \text{erf} \left( \frac{q_1 + q_2}{2} \right), \]

is the result of integrating over two different Fermi surfaces and is hence unique to the polarized case. Note that, as expected, \( h \) is a symmetric function of its arguments. In symmetric conditions, \( h \) reduces to \( g \), \( h(q, q) = h(0, 2q) = h(2q, 0) = g(q) \).

Using the latter property, one can easily find the energy per particle of the fully (un)polarised cases. In the case of symmetric nuclear matter, one obtains:

\[ e_{\text{SNM}}(\rho) = \frac{3}{5} \frac{\hbar^2 k_F^2}{2 m_N} + \frac{1}{2} \sum_{i=1,2} \left( A_i^0 + C_i^0 \rho \alpha_i \right) \rho + \frac{1}{2} \sum_{i=1,2} B_i^0 g(\mu_i k_F). \]

Taking derivatives with respect to the density, one can easily find expressions for the
pressure, the compressibility and the skewness:

\[
P_{\text{SNM}}(\rho) = \rho^2 \frac{\partial e_{\text{SNM}}}{\partial \rho} = \frac{2}{5} \frac{\hbar^2 k_F^2}{2 m_N} \rho + \frac{1}{2} \sum_{i=1,2} \left\{ A_i' + (\alpha_i + 1) C_0^i \rho^\alpha \right\} \rho^2
\]

\[
+ \sum_{i=1,2} B_i^0 p(\mu_i k_F) \rho, \tag{A.6}
\]

\[
K_0(\rho) = 9 \rho^2 \frac{\partial^2 e_{\text{SNM}}}{\partial \rho^2} = -\frac{6}{5} \frac{\hbar^2 k_F^2}{2 m_N} + \frac{9}{2} \sum_{i=1,2} (\alpha_i + 1) \alpha_i C_0^i \rho^\alpha + 1
\]

\[
- 3 \sum_{i=1,2} B_i^0 k(\mu_i k_F), \tag{A.7}
\]

\[
Q_0(\rho) = 27 \rho^3 \frac{\partial^3 e_{\text{SNM}}}{\partial \rho^3} = \frac{24}{5} \frac{\hbar^2 k_F^2}{2 m_N} + \frac{27}{2} \sum_{i=1,2} (\alpha_i + 1) \alpha_i (\alpha_i - 1) C_0^i \rho^\alpha + 1
\]

\[
+ 3 \sum_{i=1,2} B_i^0 q(\mu_i k_F). \tag{A.8}
\]

I have introduced the dimensionless functions:

\[
p(q) = -\frac{1}{q^3} + \frac{1}{2q} + \left( \frac{1}{q^3} + \frac{1}{2q} \right) e^{-q^2}, \tag{A.9}
\]

\[
k(q) = -\frac{6}{q^3} + \frac{2}{q} + \left( \frac{6}{q^3} + \frac{4}{q} + q \right) e^{-q^2}, \tag{A.10}
\]

\[
q(q) = \frac{54}{q^3} + 14q + \left( \frac{54}{q^3} + \frac{40}{q} + 13q + 2q^2 \right) e^{-q^2}, \tag{A.11}
\]

which are a combination of derivatives of the original g function over its arguments. I note here that neither \(K_0\) nor \(Q_0\) receive any contribution from the direct term of the finite range part.

In the case of isospin-imbalanced matter, Eq. (C.38) provides the full expression for the energy. Successive derivatives of this function over density and isospin asymmetry yield the isovector bulk properties of the system. These can, again, be written...
analytically. The expressions read:

$$S(\rho) = \frac{1}{3} \frac{e^2 k_F^2}{2m_N} + \frac{1}{2} \sum_{i=1,2} \left\{ A_i^i + C_i^i \rho^i \right\} \rho$$

$$+ \frac{1}{6} \sum_{i=1,2} \left\{ B_{nn}^i s_1 (\mu_i k_F) + B_{np}^i s_2 (\mu_i k_F) \right\}, \quad (A.12)$$

$$L(\rho) = \frac{2}{3} \frac{e^2 k_F^2}{2m_N} + \frac{3}{2} \sum_{i=1,2} \left\{ A_i^i + (\alpha_i + 1) C_i^i \rho^i \right\} \rho$$

$$+ \frac{1}{6} \sum_{i=1,2} \left\{ B_{nn}^i l_1 (\mu_i k_F) + B_{np}^i l_2 (\mu_i k_F) \right\}, \quad (A.13)$$

$$K_{\text{sym}}(\rho) = -\frac{2}{3} \frac{e^2 k_F^2}{2m_N} + \frac{9}{2} \sum_{i=1,2} (\alpha_i + 1) \alpha_i C_i^i \rho^{i+1}$$

$$- \frac{2}{3} \sum_{i=1,2} \left\{ B_{nn}^i k_1 (\mu_i k_F) + B_{np}^i k_2 (\mu_i k_F) \right\}, \quad (A.14)$$

$$Q_{\text{sym}}(\rho) = \frac{8}{3} \frac{e^2 k_F^2}{2m_N} + \frac{27}{2} \sum_{i=1,2} (\alpha_i + 1) \alpha_i (\alpha_i - 1) C_i^i \rho^{i+1}$$

$$+ \frac{1}{3} \sum_{i=1,2} \left\{ B_{nn}^i q_1 (\mu_i k_F) + B_{np}^i q_2 (\mu_i k_F) \right\}, \quad (A.15)$$

with the functions:

$$s_1(q) = \frac{1}{q} - \left( \frac{1}{q} + q \right) e^{-q^2},$$

$$s_2(q) = \frac{1}{q} - q - \frac{1}{q} e^{-q^2}, \quad (A.16)$$

$$l_1(q) = -\frac{1}{q} + \left( \frac{1}{q} + q + 2q^3 \right) e^{-q^2},$$

$$l_2(q) = -\frac{1}{q} - q + \left( \frac{1}{q} + 2q \right) e^{-q^2}, \quad (A.17)$$

$$k_1(q) = -\frac{1}{q} + \left( \frac{1}{q} + q + \frac{q^3}{2} + q^5 \right) e^{-q^2},$$

$$k_2(q) = -\frac{1}{q} - \frac{q}{2} + \left( \frac{1}{q} + \frac{3}{2}q + q^3 \right) e^{-q^2}, \quad (A.18)$$

$$q_1(q) = -\frac{14}{q} + \left( \frac{14}{q} + 14q + 7q^3 + 4q^5 + 4q^7 \right) e^{-q^2},$$

$$q_2(q) = -\frac{14}{q} - 5q + \left( \frac{14}{q} + 19q + 12q^3 + 4q^5 \right) e^{-q^2}. \quad (A.19)$$

I note that $s_i$ and $l_i$ ($k_i$ and $q_i$) functions are proportional to $q^3$ ($q^5$) in the limit
$q \rightarrow 0$. Moreover, the limits are such that the exchange terms are determined entirely by $B_1^i = B_{nn}^i - B_{np}^i$. In the opposite extreme, $q \gg 1$, the linear term in $s_2$, $l_2$, $k_2$ and $q_2$ dominates. In particular, this implies that in the high density limit: (a) the exchange contributions grow indefinitely as $\rho^{1/3}$ and (b) the matrix elements $B_{np}^i$ determine the high-density behaviour of the exchange term of isovector properties.
Appendix B

Analytical expressions for single-particle properties

Single-particle potentials with the Gogny force are obtained by integrating the effective interaction over one Fermi sea in the Hartree–Fock approximation. I express the results in terms of the total density, \( \rho \), and the isospin asymmetry parameter, \( \beta = \frac{\rho_n - \rho_p}{\rho} \). The Fermi momenta are defined as usual, with: \( k_F^n = k_F (1 + \beta)^{1/3} \) and \( k_F^p = k_F (1 - \beta)^{1/3} \), with \( k_F = \left( \frac{3\pi^2 \rho}{2} \right)^{1/3} \), the Fermi momentum of the corresponding isospin symmetric system. For both the zero-range and the direct, finite-range terms the single Fermi sphere integration leads to trivial density factors. The integral of the exchange term can be performed analytically and expressed in terms of Gaussian and error functions.

One finds the following expression for the single-particle potential \([96, 181]\):

\[
U_\tau(k) = \sum_{i=1,2} \left\{ [A^i_0 + C^i_0 \rho^{\alpha_i}] \rho + \tau [A^i_1 + C^i_1 \rho^{\alpha_i+1}] \rho \beta \right.
\]
\[
+ B^i_{nn} u(\mu_i k, \mu_i k_F^\tau) + B^i_{np} u(\mu_i k, \mu_i k_F^\tau) \left. \right\}, \tag{B.1}
\]
where I have introduced the dimensionless function:

\[
\begin{align*}
u(q, q_F) &= \frac{1}{q} \left[ e^{-\frac{(q+q_F)^2}{4}} - e^{-\frac{(q-q_F)^2}{4}} \right] + \frac{\sqrt{\pi}}{2} \left[ \text{erf} \left( \frac{q + q_F}{2} \right) - \text{erf} \left( \frac{q - q_F}{2} \right) \right].
\end{align*}
\] (B.2)

The error function is defined as in Eq. (A.3). The zero-momentum limit of the \( u \) function is finite,

\[
u(0, q_F) = -q_F e^{-\frac{q_F^2}{4}} + \sqrt{\pi} \text{erf} \left( \frac{q_F}{2} \right),
\] (B.3)

and provides information on the single-particle potential at low momentum. Note that the derivative of this expression with respect to \( q_F \) is positive, which suggests that the exchange term grows indeﬁnitely with density. The limit \( q_F \gg 1 \) leads to a constant value, indicating a saturation of the exchange term with density.

In some cases of interest, one needs to consider the rearrangement term in the single-particle potential. This is a momentum-independent contribution which arises from the density dependence of the interaction. It is the same for both neutrons and protons and in asymmetric matter it reads:

\[
U_R = \frac{1}{2} \sum_{i=1,2} \left[ C_0^i + C_1^i \beta^2 \right] \rho^{\alpha_i+1}.
\] (B.4)

Note that the isovector contribution, \( C_1^i \), enters with a \( \beta^2 \) dependence in this term.

With these analytical expressions at hand, one can compute further single-particle properties. The effective mass, for instance, characterises the momentum dependence of the single-particle potential. In asymmetric matter and at arbitrary momentum, it
is given by:

\[
\frac{m_N(k)}{m^*_\tau} = 1 + \frac{m_N}{2\hbar^2} \sum_{i=1,2} \mu_i^2 \left\{ B^i_{nn} m(\mu_i k, \mu_i k^F) + B^i_{np} m(\mu_i k, \mu_i k^F) \right\} ,
\]

(B.5)

with

\[
m(q, q_F) = \frac{1}{q^3} \left[ (2 - qq_F) e^{-\frac{(q-q_F)^2}{4}} - (2 + qq_F) e^{-\frac{(q+q_F)^2}{4}} \right] .
\]

(B.6)

This function is essentially obtained as a momentum derivative of Eq. (B.2). The effective mass is often computed at the respective Fermi surface of a given particle species. The term with two equal Fermi momenta reduces to:

\[
m(q_F, q_F) = \frac{1}{q^2_F} \left[ 2 \left( 1 - e^{-q^2_F} \right) - q^2_F \left( 1 + e^{-q^2_F} \right) \right] .
\]

(B.7)

This function also determines the effective mass at the Fermi surface for symmetric and pure neutron matter. Let us note that, at zero Fermi momentum, both functions, \( u(q, 0) = m(q, 0) = 0 \), vanish. This is relevant for both the low-density and the impurity regimes.

Using the Hugenholtz-van Hove theorem, one can relate the bulk properties of asymmetric nuclear matter to a combination of parameters associated to derivatives of the single-particle potential at the Fermi surface of the symmetric system [91]. All these parameters can be obtained analytically in the case of the Gogny interaction. In particular, it is useful to separate the isoscalar and isovector contributions of the single-particle potential. The isoscalar potential is obtained in the symmetric matter limit:

\[
U_0(k) = \sum_{i=1,2} \left\{ \left[ A^i_0 + C^i_0 \rho^\alpha \right] \rho + B^i_0 u(\mu_i k, \mu_i k^F) \right\} .
\]

(B.8)

I note that, because both the momentum dependence and the rearrangement terms
are non-linear in asymmetry, this isoscalar potential is only approximately equal (but very close) to the combination $\frac{U_n(k) + U_p(k)}{2}$ in arbitrarily isospin asymmetric matter.

The isovector potential in nuclear matter can in principle be obtained analogously, from the difference of single-particle contributions. Alternatively, the asymmetry dependence of the single-particle potential can be Taylor-expanded. The coefficients of the expansion encode the isospin dependence of the single-particle potentials:

$$U_i^{\text{sym}}(k) = \frac{1}{i!} \frac{\partial^i U_n(k)}{\partial \beta^i} \bigg|_{\beta=0} = \frac{(-1)^i}{i!} \frac{\partial^i U_p(k)}{\partial \beta^i} \bigg|_{\beta=0}. \quad (B.9)$$

The first coefficient carries most of the information and it is analogous to the Lane potential [91]. The expression reads:

$$U_1^{\text{sym}}(k) = \sum_{i=1,2} \left[ A_i + C_i \rho^\alpha \right] \rho + \frac{1}{3} B_i u_1(\mu_i k, \mu_i k_F) \right] \right], \quad (B.10)$$

with

$$u_1(q, q_F) = -\frac{q_F^2}{2q} \left[ e^{-\frac{(q+q_F)^2}{4}} - e^{-\frac{(q-q_F)^2}{4}} \right]. \quad (B.11)$$

I note that the rearrangement contribution cancels exactly in this case.

The connection with macroscopic observables is obtained by evaluating these coefficients and some of its momentum derivatives at $k = k_F$. I have computed analytically all the coefficients needed to find the slope parameter $L$. These read:

$$\left. \frac{\partial U_0}{\partial k} \right|_{k_F} = \sum_{i=1,2} B_0^i u_0' (\mu_i k_F) \quad (B.12)$$

$$\left. \frac{\partial U_1}{\partial k} \right|_{k_F} = \frac{1}{6} \sum_{i=1,2} B_1^i u_1' (\mu_i k_F) \quad (B.13)$$

$$\left. \frac{\partial^2 U_0}{\partial k^2} \right|_{k_F} = \frac{1}{2} \sum_{i=1,2} B_0^i u_2' (\mu_i k_F). \quad (B.14)$$
with

\[ u'_0(q) = \frac{1}{2q} \left[ 2 - q^2 - (2 + q^2) e^{-q^2} \right], \]  \hspace{1cm} (B.15) \\

\[ u'_1(q) = q \left[ -1 + (1 + q^2) e^{-q^2} \right], \]  \hspace{1cm} (B.16) \\

\[ u'_2(q) = \frac{1}{q} \left[ -4 + q^2 + (4 + 3q^2 + q^4) e^{-q^2} \right]. \]  \hspace{1cm} (B.17)
Appendix C

Calculating the Energy Per Particle with the Gogny Force

To determine the energy-per-particle given by the Gogny force I express the matrix elements of the force, where each of the particles is described using three quantum numbers: momentum ($k$), isospin ($\tau$) and spin ($\sigma$). I will then sum the matrix elements over momentum space from 0 to the $k_F$, the Fermi momentum. In INM when summing the energy levels up to the Fermi momentum, the sum becomes an integration.

\[
E = \sum_{\sigma_1 \tau_1} \int_0^{k_F} \frac{d^3\vec{k}_1}{(2\pi)^3} \langle \vec{k}_1 \tau_1 \sigma_1 | \hat{T} | \vec{k}_1 \tau_1 \sigma_1 \rangle \\
+ \frac{1}{2} \sum_{\sigma_1 \sigma_2 \tau_1 \tau_2} \int_0^{k_F} \frac{d^3\vec{k}_1}{(2\pi)^3} \int_0^{k_F} \frac{d^3\vec{k}_2}{(2\pi)^3} \langle \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 | \hat{V}_{NN} | \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 \rangle_A.
\]

(C.1)

Note that this equation provides the total energy and not the energy-per-particle. This will be explicitly dealt with at the end of the derivation. I begin by examining only the zero range component of the force, dealing initially with the direct term. The Gogny force is comprised of two parameter channels, $i = 1, 2$, which are linearly independent. As such I ignore the sum for brevity and include it again at the end of
The exchange component is calculated in much the same way as the direct term but instead starting with a ket with the two particles exchanged:

\[
\langle \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 | t_0 (1 + x_0 P_\sigma) \rho^\alpha (\vec{r}) \delta (\vec{r}) | \vec{k}_2 \tau_2 \sigma_2, \vec{k}_1 \tau_1 \sigma_1 \rangle
\]

and this can be divided into three factors:

\[
t_0 \rho^{\alpha i} \langle \vec{k}_1, \vec{k}_2 | \delta (\vec{r}) | \vec{k}_2, \vec{k}_1 \rangle \tag{C.4}
\]

\[
\langle \sigma_1, \sigma_2 | (1 + x_0 P_\sigma) | \sigma_2, \sigma_1 \rangle \tag{C.5}
\]

and

\[
\langle \tau_1, \tau_2 | \tau_2, \tau_1 \rangle \tag{C.6}
\]

Beginning with Eq. (C.4) and using \( \vec{r} = \vec{x}_1 - \vec{x}_2 \):

\[
t_0 \rho^{\alpha i} \langle \vec{k}_1, \vec{k}_2 | \delta (\vec{r}) | \vec{k}_2, \vec{k}_1 \rangle = t_0 \rho^{\alpha i} \int d^3 \vec{x}_1 \int d^3 \vec{x}_2 e^{i \vec{k}_1 \cdot \vec{x}_1} e^{i \vec{k}_2 \cdot \vec{x}_2} e^{-i \vec{k}_2 \cdot \vec{x}_1} e^{-i \vec{k}_1 \cdot \vec{x}_2} \delta (\vec{x}_1 - \vec{x}_2)
\]

\[
= t_0 \rho^{\alpha i} \int d^3 \vec{x}_1 e^{i (\vec{k}_1 + \vec{k}_2) \cdot \vec{x}_1} e^{-i (\vec{k}_2 + \vec{k}_1) \cdot \vec{x}_1} = t_0 \rho^{\alpha i} \Omega. \tag{C.7}
\]
Continuing to Eq. (C.5),

\[
\langle \sigma_1, \sigma_2 | (1 + x_0 P_\sigma) | \sigma_2, \sigma_1 \rangle \\
= \langle \sigma_1, \sigma_2 | 1 | \sigma_2, \sigma_1 \rangle + x_0 \langle \sigma_1, \sigma_2 | P_\sigma | \sigma_2, \sigma_1 \rangle \\
= \delta_{\sigma_1, \sigma_2} + x_0 \langle \sigma_1, \sigma_2 | \sigma_1, \sigma_2 \rangle \\
= \delta_{\sigma_1, \sigma_2} + x_0 . \tag{C.8}
\]

The final piece is Eq. (C.6):

\[
\langle \tau_1, \tau_2 | \tau_2, \tau_1 \rangle = \delta_{\tau_1, \tau_2} \tag{C.9}
\]

meaning that the final equation for the exchange term is:

\[
\left\langle \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 | t_0 (1 + x_0 P_\sigma) \rho^\alpha (r) \delta (r) | \vec{k}_2 \tau_2 \sigma_2, \vec{k}_1 \tau_1 \sigma_1 \right\rangle \\
= t_0 \Omega \rho^\alpha \left( \delta_{\sigma_1, \sigma_2} \delta_{\tau_1, \tau_2} + x_0 \delta_{\tau_1, \tau_2} \right) . \tag{C.10}
\]

To express a fully antisymmetrised equation for the zero range component of the Gogny interaction I subtract the exchange term from the direct term:

\[
\left\langle \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 | t_0 (1 + x_0 P_\sigma) \rho^\alpha (r) \delta (r) | \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 \right\rangle_A \\
= t_0 \Omega \rho^\alpha \left( 1 + x_0 \delta_{\sigma_1, \sigma_2} \right) - t_0 \Omega \rho^\alpha \left( \delta_{\sigma_1, \sigma_2} \delta_{\tau_1, \tau_2} + x_0 \delta_{\tau_1, \tau_2} \right) \\
= t_0 \Omega \rho^\alpha \left[ (1 - \delta_{\sigma_1, \sigma_2} \delta_{\tau_1, \tau_2}) + x_0 (\delta_{\sigma_1, \sigma_2} - \delta_{\tau_1, \tau_2}) \right] . \tag{C.11}
\]

I now move on to the more complex finite range element of the Gogny force. For brevity, exchanges made by the \( P_\sigma \) and \( P_\tau \) operators have not been explicitly shown and bra-ket pairs are evaluated as either unity or a corresponding Kronecker \( \delta \) function.
\[
\left\langle \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 \left| \left[ W_i + B_i P_\sigma - H_i P_\tau - M_i P_\sigma \tau \right] e^{\frac{\vec{r}^2}{\mu_i^2}} \right| \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 \right\rangle = \langle \tau_1 \sigma_1, \tau_2 \sigma_2 \left| \left[ W_i + B_i P_\sigma - H_i P_\tau - M_i P_\sigma \tau \right] \right| \tau_1 \sigma_1, \tau_2 \sigma_2 \rangle
\times \left\langle \vec{k}_1, \vec{k}_2 \left| e^{\frac{\vec{r}^2}{\mu_i^2}} \right| \vec{k}_1, \vec{k}_2 \right\rangle,
\]
(C.12)

where the spin and isospin exchange terms yield:

\[
= W_i + B_i \delta_{\sigma_1 \sigma_2} - H_i \delta_{\tau_1 \tau_2} - M_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2}
\]
(C.13)

and the space-momentum part becomes:

\[
= \Omega \int d^3 r e^{\frac{-\vec{r}^2}{\mu_i^2}}.
\]
(C.14)

Substituting \( \vec{r} \) for \( \vec{x} \), \( \vec{y} \) and \( \vec{z} \):

\[
\Omega \int d\vec{x} e^{\frac{-\vec{x}^2}{\mu_i^2}} \int d\vec{y} e^{\frac{-\vec{y}^2}{\mu_i^2}} \int d\vec{z} e^{\frac{-\vec{z}^2}{\mu_i^2}}
\]
(C.15)

\[
= \Omega \pi \frac{3}{2} \mu_i^3
\]
(C.16)

and so the complete formula for the finite range direct component is

\[
\left[ W_i + B_i \delta_{\sigma_1 \sigma_2} - H_i \delta_{\tau_1 \tau_2} - M_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} \right] \Omega \pi \frac{3}{2} \mu_i^3.
\]
(C.17)

The final component to determine is the finite range exchange component needed to fully antisymmetrise the system. This is the least trivial of the energy-per-nucleon components due to the exponential factor. As before, exchange operations and bra-ket evaluations are not shown explicitly.
\[
\left\langle \vec{k}_1 \tau_1 \sigma_1, \vec{k}_2 \tau_2 \sigma_2 \left| [W_i + B_i P_\sigma - H_1 P_\tau - M_i P_{\sigma \tau}] e^{\frac{-\vec{r}_i^2}{\mu^2}} \right| \vec{k}_2 \tau_2 \sigma_2, \vec{k}_1 \tau_1 \sigma_1 \right\rangle
\]

\[
= \left\langle \tau_1 \sigma_1, \tau_2 \sigma_2 \left| [W_i + B_i P_\sigma - H_1 P_\tau - M_i P_{\sigma \tau}] \right| \tau_2 \sigma_2, \tau_1 \sigma_1 \right\rangle
\]

\[
\times \left\langle \vec{k}_1, \vec{k}_2 \left| e^{\frac{-\vec{r}_i^2}{\mu^2}} \right| \vec{k}_2, \vec{k}_1 \right\rangle
\]

(C.18)

where (C.18)

\[
= [W_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} + B_i \delta_{\tau_1 \tau_2} - H_1 \delta_{\sigma_1 \sigma_2} - M_i],
\]

(C.20)

and (C.19)

\[
= \int d^3 \vec{x}_1 \int d^3 \vec{x}_2 e^{i(\vec{k}_1 - \vec{k}_2)(\vec{x}_1 - \vec{x}_2)} e^{\frac{-\vec{r}_i^2}{\mu^2}}
\]

(C.21)

To evaluate (C.21) it is necessary to define a parameter:

\[
\vec{q} = \vec{k}_1 - \vec{k}_2.
\]

(C.22)

Substituting these into (C.21) one can fully evaluate (C.19):

\[
\int d^3 \vec{x}_1 \int d^3 \vec{x}_2 e^{i(\vec{k}_1 - \vec{k}_2)(\vec{x}_1 - \vec{x}_2)} e^{\frac{-\vec{r}_i^2}{\mu^2}}
\]

\[
= \int d^3 \vec{x}_1 \int d^3 \vec{x}_2 e^{i\vec{q} \cdot \vec{r}_i} e^{\frac{-\vec{r}_i^2}{\mu^2}}
\]

\[
= e^{-\frac{\vec{q}^2 \mu^2}{4}} \int d^3 \vec{x}_1 \int d^3 \vec{x}_2 e^{-\left(\frac{\vec{x}_1 \cdot \vec{q}}{\mu_i} + \frac{\vec{x}_2 \cdot \vec{q}}{\mu_i}\right)^2}.
\]

(C.23)

Employing the same techniques as in (C.15) one can show this to be

\[
\Omega \pi^2 \mu^3 e^{-\frac{\vec{q}^2 \mu^2}{4}}
\]

(C.24)
and so the full finite range exchange term becomes:

$$\left[ W_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} + B_i \delta_{\tau_1 \tau_2} - H_i \delta_{\sigma_1 \sigma_2} - M_i \right] \Omega \pi^2 \mu_i^3 e^{-\frac{q^2}{4}}. \quad (C.25)$$

Combining the direct and exchange terms we can give a full antisymmetrised description of the finite range component of the Gogny force:

$$\langle k_1^\tau_1 \sigma_1, k_2^\tau_2 \sigma_2 \mid [W_i + B_i P_{\sigma} - H_i P_{\tau} - M_i P_{\sigma \tau}] e^{-\frac{q^2}{4}} \mid k_1^\tau_1 \sigma_1, k_2^\tau_2 \sigma_2 \rangle_A = [W_i + B_i \delta_{\sigma_1 \sigma_2} - H_i \delta_{\tau_1 \tau_2} - M_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2}] \Omega \pi^2 \mu_i^3 e^{-\frac{q^2}{4}}$$

$$= [W_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} + B_i \delta_{\tau_1 \tau_2} - H_i \delta_{\sigma_1 \sigma_2} - M_i] \Omega \pi^2 \mu_i^3 e^{-\frac{q^2}{4}}$$

$$= \Omega \pi^2 \mu_i^3 \left[ W_i \left(1 - \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} e^{-\frac{q^2}{4}}\right) + B_i \left(\delta_{\tau_1 \tau_2} - \delta_{\sigma_1 \sigma_2} e^{-\frac{q^2}{4}}\right) \right]$$

$$- \Omega \pi^2 \mu_i^3 \left[ H_i \left(\delta_{\sigma_1 \sigma_2} - \delta_{\tau_1 \tau_2} e^{-\frac{q^2}{4}}\right) + M_i \left(\delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} - e^{-\frac{q^2}{4}}\right) \right]. \quad (C.26)$$

With the antisymmetrized matrix elements defined it is now possible to calculate the energy-per-particle resulting from the Gogny interaction. This can be found by taking integrals over the momenta of the particles involved, as shown in Eq. (C.1). The zero-range direct and exchange terms, and the finite-range direct term are all independent of momentum. As such I group them under the arbitrary constant, $A$, where

$$A = t_0 \Omega \rho^N \left[(1 - \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2}) + x_0 (\delta_{\sigma_1 \sigma_2} - \delta_{\tau_1 \tau_2})\right]$$

$$+ \Omega \pi^2 \mu_i^3 \left[W_i + B_i \delta_{\tau_1 \tau_2} - H_i \delta_{\sigma_1 \sigma_2} - M_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2}\right]. \quad (C.27)$$

The Gaussian dependent finite-range exchange term prefactors can be grouped in a similar manner:

$$B = \Omega \pi^2 \mu_i^3 \left[-W_i \delta_{\sigma_1 \sigma_2} \delta_{\tau_1 \tau_2} - B_i \delta_{\sigma_1 \sigma_2} + H_i \delta_{\tau_1 \tau_2} + M_i\right]. \quad (C.28)$$
The Gogny contribution to the total energy-per-particle then becomes

\[
E_{\text{Gogny}} = \frac{1}{2} \sum_{\sigma_1, \sigma_2, \tau_1, \tau_2} \left[ A \int_0^{k_{\tau_1}^2} \frac{d^3 k_1}{(2\pi)^3} \int_0^{k_{\tau_2}^2} \frac{d^3 k_2}{(2\pi)^3} + B \int_0^{k_{\tau_1}^2} \frac{d^3 k_1}{(2\pi)^3} \int_0^{k_{\tau_2}^2} \frac{d^3 k_2}{(2\pi)^3} e^{-\vec{q}^2/4} \right].
\]

(C.29)

The momentum independent integral over \(A\) simply provides a density contribution:

\[
A \int_0^{k_{\tau_1}^2} \frac{d^3 k_1}{(2\pi)^3} \int_0^{k_{\tau_2}^2} \frac{d^3 k_2}{(2\pi)^3} = A \rho_{\tau_1} \rho_{\tau_2}.
\]

(C.30)

Using the notation from Chapter 3 one can perform the sum over \(\sigma_1, \sigma_2, \tau_1, \tau_2\) and express \(A\) in terms of its different isospin combinations:

\[
\sum_{\sigma_1, \sigma_2, \tau_1, \tau_2} A \rho_{\tau_1} \rho_{\tau_2} = \Omega \left[ A_0^0 + C_0^0 \rho_{\alpha_1} + A_1^1 \beta_2 + C_1^1 \rho_{\alpha_1} \beta_2 \right] \rho^2.
\]

(C.31)

The momentum dependent integral over the Gaussian has a more complex solution, the formula for which is the function \(h\), where

\[
h(q_1, q_2) = 2q_1^2 - q_1 q_2 + q_2^2 - 2e^{-\frac{(q_1 + q_2)^2}{4}} - 2q_1^2 + q_1 q_2 + q_2^2 - 2e^{-\frac{(q_1 - q_2)^2}{4}}
\]

\[
- \sqrt{\pi} \frac{q_1^3 - q_2^3}{q_1^2 + q_2^2} \text{erf} \left( \frac{q_1 - q_2}{2} \right) + \sqrt{\pi} \text{erf} \left( \frac{q_1 + q_2}{2} \right),
\]

and so

\[
B \int_0^{k_{\tau_1}^2} \frac{d^3 k_1}{(2\pi)^3} \int_0^{k_{\tau_2}^2} \frac{d^3 k_2}{(2\pi)^3} e^{-\vec{q}^2/4} = B h(q_1, q_2).
\]

(C.32)

When the Fermi surfaces are equal the function \(h\) reduces to the function \(g\),

\[
g(q) = \frac{2}{q^3} - 3 - \left( \frac{2}{q^3} - \frac{1}{q} \right) e^{-q^2} + \sqrt{\pi} \text{erf}(q),
\]

and so one can break \(B\) into its contribution from different isospin combinations. One
can now write

$$
\sum_{\sigma_1, \sigma_2, r_1, r_2} B_h (q_1, q_2) = \Omega \rho B_{nn}^i \left[ \frac{1 + \beta}{2} g (\mu_i k_F^{n}) + \frac{1 - \beta}{2} g (\mu_i k_F^{p}) \right] + \Omega \rho B_{np}^i h (\mu_i k_F^{n}, \mu_i k_F^{p}) .
$$

(C.35)

The energy contribution from the Gogny force, $E_{Gogny}$, then becomes

$$
E_{Gogny} = \frac{1}{2} \left[ \Omega \left[ A_i^0 + C_i^0 \rho^{\alpha_i} + A_i^1 \beta^2 + C_i^1 \rho^{\alpha_i} \beta^2 \right] \rho^2 
+ \Omega \rho B_{nn}^i \left[ \frac{1 + \beta}{2} g (\mu_i k_F^{n}) + \frac{1 - \beta}{2} g (\mu_i k_F^{p}) \right] + \Omega \rho B_{np}^i h (\mu_i k_F^{n}, \mu_i k_F^{p}) \right].
$$

(C.36)

Up until here I have calculated the total energy of the system. A divide by the volume element $\Omega$ provides the energy-per-volume element. Converting to the energy-per-particle is relatively straightforward:

$$
\rho = \frac{N}{\Omega}, \\
\frac{E}{N} = \frac{E}{\Omega} \frac{\Omega}{N} = \frac{E}{\Omega} / \rho.
$$

(C.37)

Performing this conversion, re-inserting the sum over $i$, and adding the kinetic energy-per-particle contribution defined in reference [139] one arrives at the analytic formula for the energy-per-particle with the Gogny force:

$$
e(\rho, \beta) = \frac{3 k_F^2 k_F^2}{5 2 m_N} 2 \left\{ (1 + \beta)^{5/3} + (1 - \beta)^{5/3} \right\} 
+ \frac{1}{2} \left\{ [A_i^0 + C_i^0 \rho^{\alpha_i}] \rho + [A_i^1 + C_i^1 \rho^{\alpha_i}] \rho \beta^2 \right\} 
+ \frac{1}{2} \left\{ B_{nn}^i \left[ \frac{1 + \beta}{2} g (\mu_i k_F^{n}) + \frac{1 - \beta}{2} g (\mu_i k_F^{p}) \right] + B_{np}^i h (\mu_i k_F^{n}, \mu_i k_F^{p}) \right\}. 
$$

(C.38)
Appendix D

Partial Wave Conversion

To examine the partial wave behaviour of the Gogny force it is necessary to convert the matrix elements from a Single Particle (SP) basis, dependent on momentum, \( \vec{k} \), spin, \( \sigma \), and isospin, \( \tau \):

\[
\left\langle \vec{k}_\sigma \tau \vec{k} \sigma \tau | V | \vec{k}_\sigma \tau \vec{k} \sigma \tau \right\rangle_A , \quad (D.1)
\]

into Partial Wave (PW) basis dependent on relative momentum, \( \vec{q} \), angular momentum, \( L \), total spin, \( S \), and total isospin, \( T \):

\[
\langle \vec{q} LS | V_{LL'}^{ST} | \vec{q} S'T' \rangle_A . \quad (D.2)
\]

The Gogny force is initially stated in position space. For the purposes of this decomposition I require the force to be expressed in momentum space and so I perform a Fourier Transform (FT) on the matrix elements. I first consider the zero range, \( t_0 (1 + x_0 P_\sigma ) \rho^a \):
\[ \int d^3x_1 \int d^3x_2 \, e^{i \vec{k} \cdot x_1} e^{-i \vec{k} \cdot x_2} t_0 (1 + x_0 P_\sigma) \rho^\alpha \delta (x_1 - x_2) \]
\[ = \int d^3x_1 \, e^{i \vec{k} \cdot x_1} e^{-i \vec{k} \cdot x_1} t_0 (1 + x_0 P_\sigma) \rho^\alpha \]
\[ = \Omega t_0 (1 + x_0 P_\sigma) \rho^\alpha, \quad (D.3) \]

and the Finite Range, \([W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau}] e^{-\vec{r}^2/\mu^2} : \]
\[ \int d^3x_1 \int d^3x_2 \, e^{i \vec{k} \cdot x_1} e^{-\vec{r}^2/\mu^2} e^{-i \vec{k} \cdot x_2} [W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau}] \]
\[ = \int d^3x_1 \int d^3x_2 \, e^{i \vec{k} \cdot \vec{r}} e^{-\vec{r}^2/\mu^2} [W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau}], \quad (D.4) \]

where \(\Omega\) is a volume element and \(\vec{r} = x_1 - x_2\). To progress I use a slight rearrangement,
\[ i \vec{k} \cdot \vec{r} - \frac{\vec{r}^2}{\mu^2} = - \left( \frac{\vec{r}}{\mu} - \frac{i \vec{k} \mu}{2} \right)^2 - \frac{\vec{k}^2 \mu^2}{4}, \quad (D.5) \]

to give
\[ \int d^3x_1 \int d^3x_2 \, e^{i \vec{k} \cdot \vec{r}} e^{-\vec{r}^2/\mu^2} [W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau}] \]
\[ = e^{-\frac{\vec{r}^2}{4 \mu^2}} [W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau}] \int d^3x_1 \int d^3x_2 \, e^{-\left( \frac{\vec{r}_1^2 - \vec{r}_2^2}{\mu^2} \right) - \frac{\vec{k}^2 \mu^2}{4}} \]
\[ = \Omega \mu^2 \pi^2 \left[ W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau} \right] e^{-\frac{\vec{r}_2^2}{4 \mu^2}}. \quad (D.6) \]

I now consider the spin and isospin exchange dependent terms:

\[ \langle SM_s TM_T | V | S'M'_s T'M'_T \rangle = \]
\[ \langle SM_s TM_T | \Omega t_0 (1 + x_0 P_\sigma) \rho^\alpha | S'M'_s T'M'_T \rangle \]
\[ + \langle SM_s TM_T | \Omega \mu^2 \pi^2 \left[ W_i + B_i P_\sigma - H_i P_\tau - M_i P_{\sigma\tau} \right] e^{-\frac{\vec{r}_2^2}{4 \mu^2}} | S'M'_s T'M'_T \rangle. \quad (D.7) \]
The spin and isospin exchange operators lead to -1 with exponents of S and T,
\[
\langle S | P_x | S \rangle = (-1)^S, \quad (D.8)
\]
and so
\[
\langle SM_TM_T | V | SM_TM_T \rangle = 
\Omega t_0 \left(1 - x_0 (-1)^S\right) \rho^x
+ \Omega \mu^2 \pi^{3/2} \left[ W_i - B_i (-1)^S + H_i (-1)^T - M_i (-1)^S (-1)^T \right] e^{-\frac{\vec{r}^2}{4}} \quad (D.9)
\]

Now that I have dealt with the S and T terms, dealing with the aspect of the interaction dependent on the angular momentum, L, is slightly more complex. I will simplify by replacing the momentum independent terms with constants:
\[
A^S \equiv \Omega t_0 \left(1 - x_0 (-1)^S\right) \quad \text{and} \quad \quad B^{ST} \equiv \Omega \mu^2 \pi^{3/2} \left[ W_i - B_i (-1)^S + H_i (-1)^T - M_i (-1)^S (-1)^T \right] \quad (D.10)
\]

Initially I examine the zero range component as this is arguably the more straight forward of the two. The projection into partial waves is acheived by means of the following integral:
\[
V_L(k, k') = 2\pi \int_{-1}^{1} d(\cos \theta) \; V(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \; P_L(\cos \theta), \quad (D.11)
\]
where \(P_L\) is the Legendre polynomial [269] which relates to the spherical harmonics via
\[
P_L(\cos \theta_{kk'}) = \frac{4\pi}{2L + 1} \sum_{M_L = -L}^{L} Y_{LM_L}(\hat{k}) Y_{LM_L}^*(\hat{k'}), \quad (D.12)
\]
and make use of the fact that $A^S$ is independent of $k$ and $k'$:

\[
V_L (k; k') = 2\pi \int_{-1}^{1} d\cos\theta \, V \left( \vec{k},\vec{k}' \right) P_L (\cos\theta) \\
= 2\pi A^S \int_{-1}^{1} d\cos\theta \, P_L (\cos\theta) \\
= 4\pi A^S \delta_{L0}.
\]

(D.13)

This means that only the $L = 0$ channels can contain a zero range component. For the finite range term I use a slightly different method;

\[
\left\langle \vec{k}LM_L \right| V \left| \vec{k}'L'M'_L \right\rangle = \\
\int d\Omega_k \left\langle LML_k | \hat{k} \right\rangle \int d\Omega_{k'} \left\langle L'M'_{L_k} | \hat{k}' \right\rangle \left\langle \hat{k} | V (r) | \hat{k}' \right\rangle.
\]

(D.14)

The resulting integrals can be expressed in terms of spherical harmonics from the relation

\[
\left\langle LM_L | \hat{k} \right\rangle = Y^*_{LM_L} \left( \hat{k} \right).
\]

(D.15)

The Gaussian term can be split into partial waves from a known identity:

\[
\left\langle \vec{k} | V_{\text{Gogny}} | \vec{k}' \right\rangle = B_{ST} \frac{\pi^3}{2} \mu^3 e^{-\frac{\mu^2 \left( \vec{k} - \vec{k}' \right)^2}{4}} \\
= B_{ST} \frac{\pi^3}{2} \mu^3 e^{-\frac{\mu^2 (|k|^2 + |k'|^2)}{4}} \sum_{L=0}^{\infty} i^{-L} (2L + 1) j_L \left( i \frac{\mu^2}{4} kk' \right) P_L (\cos\theta_{kk'}). 
\]

(D.16)

The function $j_L (x)$ is the standard Bessel function [269]. The Bessel functions are valid for complex arguments and so the occurrence of $i$ in the argument is not problematic. Note that the Gogny force will actually have two components, $i = 1, 2$, which are summed together. Here I have dropped the sum over $i$ for brevity of the
derivation. This makes the finite range expression

\[
\left\langle \vec{k}L M_L | V | \vec{k}' L' M'_L \right\rangle = 4\pi B^{ST} \frac{\pi^2}{\mu^3} e^{\frac{\nu^2(|k|^2+|k'|^2)}{4}} \frac{L''}{M''_L} \sum_{L''} \sum_{M''_L} i^{-L''} j_{L''} \left( i\frac{\mu^2}{2} k k' \right) 
\times \int d\Omega_k \int d\Omega_{k'} Y_{L''M''_L}^{*} \left( \hat{k} \right) Y_{L'M_L} \left( \hat{k}' \right) Y_{L''M''_L}^{*} \left( \hat{k}' \right) Y_{L'M_L} \left( \hat{k}' \right). \tag{D.17}
\]

Performing the angular integrations gives us Kronecker delta functions over angular momentum which allow me to express the matrix elements as

\[
\left\langle \vec{k}L | V | \vec{k}' L' \right\rangle = 4\pi i^{L - L'} B^{ST} e^{\frac{\nu^2(|k|^2+|k'|^2)}{4}} j_{L} \left( i\frac{\mu^2}{2} k k' \right). \tag{D.18}
\]

One can utilise the modified Bessel function \[269\],

\[
i_L (x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma (m + L + 1)} \left( \frac{x}{2} \right)^{2m+L} , \tag{D.19}
\]

where \( \Gamma (x) \) is the gamma function. The modified Bessel can be related to a spherical Bessel function, \( j_L (x) \), by

\[
i_L (x) = i^{-L} j_L (ix). \tag{D.20}
\]

I am now left with

\[
\left\langle \vec{k}LST | V | \vec{k}' L'S'T' \right\rangle = 4\pi A^{S} \delta_{L0} + B^{ST} e^{\frac{\nu^2(|k|^2+|k'|^2)}{4}} i_L \left( \frac{\mu^2}{2} k k' \right). \tag{D.21}
\]

Finally I antisymmetrize the wavefunction. In partial wave basis the wavefunction is antisymmetrized via

\[
\left\langle \vec{k}LST | V | \vec{k}' L'S'T' \right\rangle_A = \left[ 1 - (-1)^{L' + S' + T'} \right] \left\langle \vec{k}LST | V | \vec{k}' L'S'T' \right\rangle , \tag{D.22}
\]

allowing only odd values for \( L' + S' + T' \) to have non-zero values. For the Gogny force
there is no coupling between different angular momentum states and the interaction
does not change the angular momentum. As such $L$ must equal $L'$, $S$ equal $S'$ and $T$
equal $T'$. From hereon I drop the primes for brevity. Expressing $V$ as $V^{LST}$ and using
the relative momentum, $\vec{q}$, I can describe the Gogny force in partial wave basis:

$$V^{LST}(q,q') = \left[ 1 - (-)^{L+S+T} \right] \sum_{i}^{2} \left[ Z_{i}^{LS} + F_{i}^{ST} e^{-\frac{\mu_{i}^{2}(|q|^{2}+|q'|^{2})}{4}} i \left( \frac{\mu_{i}^{2}qq'}{2} \right) \right].$$

(D.23)

$Z_{i}^{LS}$ and $F_{i}^{ST}$ depend on total spin, $S$, and total isospin, $T$, and represent $A^{S}$ and $B^{ST}$ combined with extra coefficients gained through the partial wave expansion:

$$Z_{i}^{LS} = \frac{\delta_{L0}^{i}}{4\pi^{3}} \left[ t_{0}^{i} \delta_{\alpha i}^{0} \left( 1 - x_{0}^{i} (-)^{S} \right) \right],$$

(D.24)

$$F_{i}^{ST} = \frac{\mu_{i}^{3}}{2\pi^{2}} \left[ W_{i} + B_{i} (-)^{S} - H_{i} (-)^{T} - M_{i} (-)^{S+T} \right].$$

(D.25)

It should be noted that the zero range term, $Z_{i}^{LS}$, does not exhibit an isospin
dependence. In fact, because of the Kronecker $\delta$ function between $L$ and zero, the zero
range term only contributes to the S waves. Because $S$ and $T$ can each take values
of 0 or 1 there are four different values that $F_{i}^{ST}$ can take and two values that $Z_{i}^{LS}$
can take. Because the Bessel functions shown in Eqn. D.19 tend to infinity when
their argument is large there is a competition between the exponentially increasing
Bessel function and the exponentially decaying Gaussian term. At large momentum
separations the Gaussian term is able to suppress the Bessel at all orders and so the
matrix elements will tend to zero.
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