INERTIAL MANIFOLDS FOR 1D CONVECTIVE REACTION-DIFFUSION SYSTEMS

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ABSTRACT. An inertial manifold for the system of 1D reaction-diffusion-advection equations endowed by the Dirichlet boundary conditions is constructed. Although this problem does not initially possess the spectral gap property, it is shown that this property is satisfied after the proper non-local change of the dependent variable.

CONTENTS

1. INTRODUCTION

It is believed that the long-time behavior of many dissipative PDEs in bounded domains is essentially finite-dimensional. Thus, despite of the infinite-dimensionality of the initial phase space, the reduced dynamics on the so-called global attractor can be effectively described by finitely many parameters. This conjecture is partially supported by the fact that this global attractor usually has finite Hausdorff and box-counting dimension and, by the Mané projection theorem, can be embedded by a Hölder continuous homeomorphism into the finite-dimensional plane of the phase space. In turn, this allows us to describe the reduced dynamics on the attractor in terms of a finite system of ODEs - the so-called inertial form of the PDE considered, see [1, 3, 6, 13, 14, 19, 23] and references therein.

Unfortunately, the reduction based on the Mané projection theorem guarantees only the Hölder continuity of the vector field in the above mentioned inertial form and the regularity of this vector field seems to be crucial here. Indeed, as recent counterexamples show (see [4, 23]), this vector field cannot be made Lipschitz or log-Lipschitz continuous in general and this lack of regularity may lead to actual infinite-dimensionality of the reduced dynamics on the attractor despite the fact that the attractor has finite box-counting dimension. By this reason,

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understanding under what extra assumptions the considered PDE possesses an inertial form with more regular vector field becomes a central problem of the theory.

An ideal situation arises when the considered PDE possesses the so-called *inertial manifold* (IM) which is finite-dimensional invariant manifold of the phase space with exponential tracking (asymptotic phase) property which contains the global attractor. In this case, the desired inertial form is constructed by restricting the initial PDE to the manifold and has smoothness of the IM (usually $C^{1+\varepsilon}$), see [5, 12, 15, 23] and references therein. However, the standard theory of IMs requires the so-called *spectral gap* assumption which looks very restrictive and is not satisfied in many physically relevant examples, see also [7, 8, 10, 11, 23] for the examples where the spectral gap assumption may be relaxed using the so-called spatial averaging principle. By this reason, a lot of efforts has been made in order to improve the regularity of the inertial form without referring to the IMs, see [14, 23] and references therein. The most interesting from our point of view is the Romanov theory which gives necessary and sufficient conditions for the *Lipschitz* continuous embeddings of the attractor to finite-dimensional spaces and allows us to construct Lipschitz continuous inertial forms, see [16, 17]. The key application of this theory is related with the 1D convective reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + f(u, u_x) \tag{1.1}$$

which is considered on the interval $x \in (0, L)$ and is endowed by the proper boundary conditions. On the one hand, the spectral gap condition is not satisfied for this equation and the existence of the IM has not been proved before. On the other hand, the Romanov theory allows us to build up the Lipschitz continuous inertial form for equation (1.1) under more or less general assumptions on the non-linearity f, see also [9]. This example might be treated as an indication that a reasonable theory may be developed beyond the inertial manifolds. We mention here also an interesting attempt to extend the theory to the case of log-Lipschitz Mané projections and log-Lipschitz inertial forms, see [14], as well as the counterexamples to the existence of the IM in slightly more general than (1.1) classes of PDEs, see [18].

The main aim of the present paper is to show that 1D reaction-diffusion-advection problems actually possess IMs. Thus, similarly to all reasonable examples known to the authors, good properties of the inertial form here are also related with the existence of an IM. This somehow confirms the conjecture stated in [23] that the existence of an IM gives a *sharp* borderline between the finite and infinite-dimensional dynamics arising in dissipative PDEs.

Namely, we consider the following reaction-diffusion-advection problem:

$$\partial_t u + f(u)\partial_x u - \partial_x^2 u + g(u) = 0,\ x \in (0, L),\quad u\big|_{x=0} = u\big|_{x=L} = 0, \tag{1.2}$$

where $u = (u^1(t, x), \cdots, u^n(t, x))$ is an unknown vector-valued function and f and g are given smooth functions with finite support. Thus, we have assumed from the very beginning that the non-linearities are already cut-off outside of the global attractor $\mathcal{A}$ (which is a subset of C^1) and do not specify more or less general assumptions on f and g which guarantees global solvability, dissipativity and the existence of such global attractor, see e.g. [1, 2, 6, 19] and references therein for more details on this topic.

We also mention that the eigenvalues of the Laplacian in this case are $\lambda_d := \left(\frac{\pi}{L}\right)^2 d^2$ and the spectral gap condition for this equation reads

$$\frac{\lambda_{d+1} - \lambda_d}{\lambda_d^{1/2} + \lambda_{d+1}^{1/2}} = \frac{\pi}{L} \geq L_f, \tag{1.3}$$

where L_f is a constant related with the Lipschitz constant of the function f (the Lipschitz constant of non-linearity g is not essential here, so we prefer to state the spectral gap assumption for the particular case $g = 0$). Therefore, the standard methods give the existence of IM only under the assumption that the nonlinearity f is small enough.

The main idea of our method is to transform equation (1.2) (using the appropriate non-local in space change of variables u) in such way that the obtained new equation will have *small* nonlinearity f. To this end, we set

$$u(t,x) = N(t,x)v(t,x), \tag{1.4}$$

where $N(t,x) \in GL(n)$ is a matrix depending on the solution u. Then, equation (1.2) reads

$$\begin{aligned}\partial_t v - \partial_x^2 v = N^{-1}[-f(Nv)N + 2\partial_x N]\partial_x v + \\ + [N^{-1}\partial_x^2 N - N^{-1}\partial_t N - N^{-1}f(Nv)\partial_x N]v - N^{-1}g(Nv).\end{aligned} \tag{1.5}$$

The naive way to remove the term $\partial_x v$ would be to fix the matrix $N = N(v)$ as a solution of the following ODE:

$$\partial_x N = \frac{1}{2}f(Nv)N, \quad N\big|_{x=0} = \operatorname{Id}. \tag{1.6}$$

However, in this case

$$\partial_x^2 N = \frac{1}{4}f(Nv)^2 N + \frac{1}{2}f'(Nv)(\partial_x N v + N\partial_x v)N$$

and we see that the remaining terms in the RHS of (1.5) will lose smoothness due to the presence of the term $\partial_x v$ (the analogous thing happens also with $\partial_t N$) and we end up with similar to (1.3) spectral gap condition for equation (1.5) which is again not satisfied in general. It worth mentioning that this "naive" method may work if the nonlinearity $f(u)$ is non-local and smoothing. Then, the above mentioned terms remain of order zero and the convective terms can be completely removed, see [20, 21] for the application of this method to the so-called Smoluchowski equation. Similar idea based on the Cole-Hopf transform has been applied in [22] to the case of Burgers equation with low-wavenumber instability.

Crucial observation which allows us to handle the general case of equation (1.2) is that we *need not* to remove the gradient term $\partial_x v$ in the RHS of (1.5) completely. Instead, as mentioned before, it is enough to make it *small* and this is possible to do in such way that $N = N(v)$ will be *smoothing* operator and the problem with the other terms in the RHS of (1.5) will not arise. Namely, as will be shown, we may modify equation (1.6) as follows:

$$\partial_x N = \frac{1}{2}f(P_K(Nv))N, \quad N\big|_{x=0} = \operatorname{Id}, \tag{1.7}$$

where P_K is an orthoprojector on the first K eigenvalues of the Laplacian $-\partial_x^2$ on $(0, L)$ with Dirichlet boundary conditions and $K \gg 1$ (other smoothing operators are also possible). Then, the spectral gap assumption will be satisfied for the modified equation (1.5) and, since this equation is equivalent to the initial problem (1.2), we end up with the following theorem which can be treated as the main result of the paper.

Theorem 1.1. *Let the non-linearities f and g be smooth and have finite supports. Then equation* (1.2) *possesses an IM in the phase space $H_0^1(0, L)$ and this manifold is $C^{1+\varepsilon}$-smooth where $\varepsilon > 0$ is small enough.*

The paper is organized as follows. The properties of the diffeomorphism generated by equation (1.7) are studied in section 2. The transformation of equation (1.3) to the analogous equation with respect to the new dependent variable v is made in section 3. The properties of the nonlinearities involved in this equation is also studied there. The main theorem on the existence of an inertial manifold is proved in section 4. Finally, the cases of different boundary conditions (including equations (1.1)), more general equations, etc., are discussed in section 5.

2. The Auxiliary Diffeomorphism

The aim of this section is to study the change of variables generated by equation (1.7) and formula $u = N(v)v$. We start with the basic properties of solutions of equation (1.7).

Lemma 2.1. *Let the above assumptions hold. Then, for any $v \in H^1(0,L)$ and any $K \in \mathbb{N}$, there exists at least one solution $N : [0,L] \to GL(n,\mathbb{R})$ and the following estimate hold:*

$$\|N\|_{W^{1,\infty}(0,L)} + \|N^{-1}\|_{W^{1,\infty}(0,L)} \le C, \tag{2.1}$$

where the constant C is independent of K and v. Moreover, for sufficiently large $K \ge K_0(\|v\|_{H^1})$, the solution is unique.

Proof. Let $A(x) := \frac{1}{2} f(P_K(Nv))$. Then, $N(x)$ is a solution matrix of the ODE

$$y'(x) = A(x)y(x), \quad y \in \mathbb{R}^n, \tag{2.2}$$

i.e., every solution $y(x)$ of this equation has the form $y(x) = N(x)y(0)$. By this reason, $N(x)$ is invertible and, since the matrix f is globally bounded, by the standard estimates for the ODE (2.2), we see that $N(x)$ is also uniformly bounded with respect to $x \in [0,L]$. Moreover, since the inverse matrix $M = [N^{-1}(x)]^t$ solves the equation

$$\frac{d}{dx}M = -\frac{1}{2} f(P_K(Nv))^t M, \quad M\big|_{x=0} = \mathrm{Id}, \tag{2.3}$$

the analogous estimate holds also for $N^{-1}(x)$. Finally, from (1.7) and (2.3), we establish the estimate for the derivative of N. Thus, estimate (2.1) is proved and we only need to check the solvability.

The existence of a solution is an immediate corollary of the fact that the operator

$$N \mapsto \mathrm{Id} + \frac{1}{2}\int_0^x f(P_K(Nv))N(y)\,dy$$

is compact and continuous, say, as an operator in $L^2(0,L)$ and we have uniform a priori bounds for the solution (e.g., the Leray-Schauder degree theory can be used to verify the existence of a solution).

Let us prove the uniqueness. Let N_1 and N_2 be two solutions of equation (1.7) and $\bar N := N_1 - N_2$. Then, this function solves

$$\frac{d}{dx}\bar N = \frac{1}{2} f(P_K(N_1 v))\bar N + \frac{1}{2}[f(P_K(N_1 v)) - f(P_K(N_2 v))]N_2. \tag{2.4}$$

Integrating this equation and using that f' is bounded, together with estimate (2.1), we get

$$\|\bar N(x)\| \le C\int_0^x \|\bar N(y)\| + \|P_K(\bar N v)(y)\|\,dy.$$

Denoting $Q_K = \mathrm{Id} - P_K$, estimating

$$\|\bar N(y)v(y)\| \le \|\bar N(y)\|\|v(y)\| \le C\|v\|_{H^1}\|\bar N(y)\|,$$

and using the Gronwall inequality, we have

$$\|\bar N\|^2_{L^\infty(0,L)} \le C\|Q_K(\bar N v)\|^2_{L^2(0,L)}, \tag{2.5}$$

where C depends on $\|v\|_{H^1}$, but is independent of K. Moreover, inserting this estimate to (2.4), we have

$$\|\bar N\|^2_{H^1(0,L)} \le C\|Q_K(\bar N v)\|^2_{L^2(0,L)}. \tag{2.6}$$

Finally, using that

$$\|Q_K z\|^2_{L^2(0,L)} \le \lambda_K^{-1/4}\|Q_K z\|^2_{H^{1/4}(0,L)} \le CK^{-1/2}\|z\|_{H^{1/4}(0,L)},$$

we get

$$\|Q_K(\bar{N}v)\|^2_{L^2(0,L)} \le CK^{-1/2}\|\bar{N}v\|^2_{H^{1/4}(0,L)} \le CK^{-1/2}\|v\|^2_{H^1}\|\bar{N}\|^2_{H^1(0,L)} \tag{2.7}$$

and (2.6) guarantees that $\bar{N} = 0$ if $K = K(\|v\|_{H^1})$ is large enough. Lemma 2.1 is proved. □

Thus, we have proved that, for every $R > 0$, equation (1.7) defines a map

$$v \mapsto N(v), \quad N : B(R, 0, H^1) \to W^{1,\infty}(0, L; GL(n, \mathbb{R}))$$

if $K \ge K_0(R)$ is large enough (here and below, we denote by $B(R, x, V)$ the ball of radius R in the space V centered at x). The next lemma gives the Lipschitz continuity of this map.

Lemma 2.2. *Under the assumptions of Lemma 2.1, the map N satisfies*

$$\|N(v_1) - N(v_2)\|_{W^{1,\infty}(0,L)} + \|N^{-1}(v_1) - N^{-1}(v_2)\|_{W^{1,\infty}(0,L)} \le C\|v_1 - v_2\|_{H^1(0,L)}, \tag{2.8}$$

for all two functions $v_1, v_2 \in B(R, 0, H^1)$. Moreover, the constant C depends on R, but is independent of $K \ge K_0(R)$.

Proof. Let $N_1 = N_1(v_1)$ and $N_2 = N_2(v_2)$ be two solutions of (1.7) and $\bar{N} = N_1 - N_2$. Then, this matrix solves

$$\frac{d}{dx}\bar{N} = \frac{1}{2}f(P_K(N_1v_1))\bar{N} + \frac{1}{2}[f(P_K(N_1v_1)) - f(P_K(N_2v_2))]N_2. \tag{2.9}$$

Arguing as in the proof of Lemma 2.1, we derive the following analogue of inequality (2.6):

$$\|\bar{N}\|_{H^1(0,L)} \le C\|Q_K(\bar{N}v_1)\|_{L^2(0,L)} + C\|N_2(v_1 - v_2)\|_{L^2(0,L)} \tag{2.10}$$

which, together with (2.7) gives

$$\|\bar{N}\|_{H^1(0,L)} \le C\|v_1 - v_2\|_{L^2(0,L)} \tag{2.11}$$

and the constant C is independent of $K \ge K_0$. Using now the equation (2.9) together with the fact that $\|P_Kw\|_{L^\infty} \le C\|w\|_{H^1}$ where the constant C is independent of K, we prove that

$$\|N(v_1) - N(v_2)\|_{W^{1,\infty}(0,L)} \le C\|v_1 - v_2\|_{H^1(0,L)}. \tag{2.12}$$

The estimate for the inverse matrix N^{-1} can be obtained analogously using the fact that the matrix $M(x) := [N^{-1}(x)]^t$ solves equation (2.3). Thus, Lemma 2.2 is proved. □

Let us consider now the map $v \mapsto u$ given by

$$u = U(v) := N(v)v. \tag{2.13}$$

According to Lemmas 2.1 and 2.2, this map is Lipschitz continuous as the map from $B(R, 0, H^1)$ to H^1 if $K \ge K_0(R)$ and

$$\|U(v_1) - U(v_2)\|_{H^1(0,L)} \le C\|v_1 - v_2\|_{H^1(0,L)}, \tag{2.14}$$

where the constant C depends on R, but is independent of $K \ge K_0$.

We now describe the inverse map $v = V(u)$ which defined via $V(u) = N^{-1}(u)u$ and $N = N(u)$ solves the linear ODE

$$\frac{d}{dx}N = \frac{1}{2}f(P_Ku)N, \quad N\big|_{x=0} = \mathrm{Id}\,. \tag{2.15}$$

Arguing analogously to Lemma 2.1 and 2.2 (but a bit simpler since equation (2.15) is *linear*), we see that the analogues of estimates (2.1) and (2.8) hold for $N(u)$ as well (also for all K independently of R). In addition, clearly, since the function f is smooth, $N(u)$ is a C^∞-map in $H^1(0, L)$. Therefore, we have proved that the inverse map $V = V(u)$ belongs to $C^\infty(H^1, H^1)$ and

$$\|V(u_1) - V(u_2)\|_{H^1(0,L)} \le C\|u_1 - u_2\|_{H^1(0,L)}, \quad u_i \in B(R, 0, H^1), \tag{2.16}$$

where the constant C depends only on R, but is independent of K. Thus, we have proved the following result.

Lemma 2.3. *The above defined map $V : H^1(0,L) \to H^1(0,L)$ is C^∞-diffeomorphism between $B(R,0,H^1)$ and $V(B(R,0,H^1)) \subset H^1$ if $K \geq K_0(R)$. Moreover, the norms of V and $U = V^{-1}$ as well as their derivatives are independent of $K \geq K_0(R)$.*

Indeed, all assertions except of differentiability of the inverse $U = V^{-1}$ are checked above and the differentiability of U can be easily derived via, say, inverse function theorem.

Remark 2.4. Note that, according to (2.13), the maps U and V act not only from $H^1(0,L)$ to $H^1(0,L)$, but also from $H^1_0(0,L)$ to $H^1_0(0,L)$. In other words, the above constructed diffeomorphism preserves the Dirichlet boundary conditions. Moreover, as not difficult to show, this diffeomorphism also preserves the regularity. In particular, it maps $H^2(0,L)$ to $H^2(0,L)$.

Moreover, if we assume in addition that

$$f(0) = 0, \tag{2.17}$$

then

$$f(P_K(Nv))\big|_{x=0} = f(P_K(Nv))\big|_{x=L} = f(0) = 0 \tag{2.18}$$

and, together with equation (1.7), this implies the boundary conditions

$$\partial_x N\big|_{x=0} = \partial_x N\big|_{x=L} = 0 \tag{2.19}$$

for the matrix N. In turn, this guarantees that the above constructed diffeomorphism preserves also the Neumann or Robin boundary conditions.

3. Transforming the Equation

The aim of this section is to rewrite equation (1.2) in terms of the new dependent variable $v = V(u)$. To do this, we first remind the standard properties of solutions of this problem.

Proposition 3.1. *Let f and g be smooth functions with finite support. Then, for every $u_0 \in H^1_0(0,L)$, problem* (1.3) *possess a unique solution*

$$u \in C([0,T], H^1_0(0,L)) \cap L^2(0,T; H^2(0,L)), \quad T > 0, \tag{3.1}$$

satisfying $u\big|_{t=0} = u_0$. Moreover, the following dissipative estimate holds:

$$\|u(t)\|_{H^1} \leq C\|u_0\|_{H^1} e^{-\alpha t} + C_*, \tag{3.2}$$

where the positive constants α, C and C_* are independent of t and u_0.

Proof. We give below only the derivation of the dissipative estimate (3.2). The rest statements are straightforward and are left to the reader, see also [1, 13, 19] for more details.

We first obtain the L^2-analogue of estimate (3.2). To this end, we multiply equation (1.3) by u and integrate over $x \in (0,L)$. Then, after standard transformations, we end up with

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\partial_x u(t)\|^2_{L^2} + (f(u)\partial_x u, u) + (g(u), u) = 0. \tag{3.3}$$

Using now that f and g have finite support, we have

$$|(g(u), u)| \leq C, \quad |(f(u)\partial_x u, u)| \leq \frac{1}{2}\|\partial_x u\|^2_{L^2} + \frac{1}{2}\|[f(u)]^* u\|^2_{L^2} \leq C + \frac{1}{2}\|\partial_x u\|^2_{L^2}. \tag{3.4}$$

Thus,

$$\frac{d}{dt}\|u(t)\|^2_{L^2} + \|\partial_x u(t)\|^2_{L^2} \leq C_*. \tag{3.5}$$

Using now the Poincare inequality together with the Gronwall inequality, we derive that

$$\|u(t)\|^2_{L^2}+\int_t^{t+1}\|\partial_x u(s)\|^2_{L^2}\,ds\le C\|u_0\|^2_{L^2}e^{-\alpha t}+C_* \tag{3.6}$$

and the L^2-analogue of the desired dissipative estimate is obtained.

At the next step, we multiply equation (1.3) by $\partial_x^2 u$ and integrate over $x\in(0,L)$ to obtain

$$\frac12\frac{d}{dt}\|\partial_x u(t)\|^2_{L^2}+\|\partial_x^2 u(t)\|^2_{L^2}=(f(u)\partial_x u,\partial_x^2 u)+(g(u),\partial_x^2 u). \tag{3.7}$$

Using again that f and g are globally bounded, we arrive at

$$(f(u)\partial_x u,\partial_x^2 u)+(g(u),\partial_x^2 u)\le\frac12\|\partial_x^2 u(t)\|^2_{L^2}+C(\|\partial_x u(t)\|^2_{L^2}+1) \tag{3.8}$$

and, consequently,

$$\frac{d}{dt}\|\partial_x u(t)\|^2_{L^2}+\|\partial_x^2 u(t)\|^2_{L^2}\le C(1+\|\partial_x u(t)\|^2_{L^2}). \tag{3.9}$$

Using again the Poincare inequality and the Gronwall inequality together with estimate (3.6), we finally have

$$\|\partial_x u(t)\|^2_{L^2}+\int_t^{t+1}\|\partial_x^2 u(s)\|^2_{L^2}\,ds\le C\|u_0\|^2_{H^1}e^{-\alpha t}+C_*. \tag{3.10}$$

Thus, the desired dissipative estimate is verified and the proposition is proved. $\square$

According to the proved proposition, equation (1.2) generates a solution semigroup $S(t)$ in the phase space $\Phi:=H^1_0(0,L)$ via

$$S(t):\Phi\to\Phi,\quad S(t)u_0:=u(t). \tag{3.11}$$

Moreover, according to (3.2), this semigroup is dissipative and possesses an absorbing ball

$$\mathbb{B}:=\{u_0\in\Phi,\quad \|u_0\|_\Phi\le R/2\} \tag{3.12}$$

if R is large enough. Our next task is to establish the existence of a global attractor for this semigroup.

Definition 3.2. Recall that a set $\mathcal{A}$ is a global attractor of the semigroup $S(t)$ in Φ if the following conditions are satisfied:

1. The set $\mathcal{A}$ is compact in Φ.
2. The set $\mathcal{A}$ is strictly invariant, i.e., $S(t)\mathcal{A}=\mathcal{A}$ for all $t\ge0$.
3. It attracts the images of all bounded sets as time tends to infinity, i.e., for every bounded set $B\subset\Phi$ and every neighbourhood $\mathcal{O}(\mathcal{A})$ of the attractor $\mathcal{A}$, there exists time $T=T(\mathcal{O},B)$ such that

$$S(t)B\subset\mathcal{O}(\mathcal{A})$$

for all $t\ge T$.

Proposition 3.3. *Let the above assumptions hold. Then, the solutions semigroup $S(t)$ generated by equation* (1.2) *possesses a global attractor $\mathcal{A}$ in the phase space Φ. Moreover, this attractor is bounded in $H^2(0,L)$ and is generated by all complete bounded solutions of equation* (1.2)*:*

$$\mathcal{A}=\mathcal{K}\big|_{t=0}, \tag{3.13}$$

where $\mathcal{K}\subset C_b(\mathbb{R},\Phi)$ consists of all solutions $u(t)$ of problem (1.2) *which are defined for all $t\in\mathbb{R}$ and bounded.*

Proof. Indeed, according to the abstract theorem on the attractors existence (see e.g., [1]), we need to check that the maps $S(t) : \Phi \to \Phi$ are continuous for every fixed time t and that the semigroup possesses a compact absorbing set. The continuity is obvious in our case. Moreover, the ball (3.12) is the absorbing set for the solution semigroup if R is large enough. However, this ball is not compact in Φ. In order to overcome this difficulty, it is enough to note that the set $S(1)\mathbb{B}$ is also absorbing and that, due to the parabolic smoothing property, $S(1)\mathbb{B}$ is bounded in $H^2(0, L)$ and, by this reason, is compact in Φ. Thus, all of the conditions of the above mentioned abstract theorem are verified and the existence of a global attractor is also verified. Formula (3.13) is a standard corollary of this theorem and the fact that $\mathcal{A}$ is bounded in H^2 follows from the fact that the attractor is always a subset of the absorbing set. Thus, the proposition is proved. □

Remark 3.4. Recall that the assumptions that f and g have finite support are not physically relevant. More realistic would be to assume, for instance, that f and g are polynomials in u (e.g., $f(u) = u$ in the case of Burgers equation). However, verification of the key dissipative estimate (3.2) is much more difficult in this case and requires extra assumptions especially in the case of systems, see the discussion in [2] for the case of coupled Burgers equations. On the other hand, since any trajectory enters the absorbing ball in finite time and never leaves it, the solutions outside of this ball are not important to study the long-time behavior, so the non-linearities may be cut off outside of the absorbing ball and this reduces the general case to the case of non-linearities with finite support. Since this trick is standard for the theory of inertial manifolds, we do not discuss here general assumptions on f and g which guarantee the validity of the dissipative estimate (3.2), assuming instead that the cut off procedure is already done, and start from the very beginning with the non-linearities with finite support.

Our next task is to transform equation (1.2) to the analogous equation with respect to the new dependent variable $v = V(u)$. To this end, we fix the radius R in such a way that the set $\mathbb{B}$ is an absorbing ball for the semigroup $S(t)$ and also fix $K_0 = K_0(R)$ as in Lemma 2.3. Then, the map V is a C^∞-diffeomorphism in the neighborhood $B(R, 0, \Phi)$ of the attractor $\mathcal{A}$ and, therefore, the change of variables $v = V(u)$ is well-defined and one-to-one in this neighborhood. We now study equation (1.5) which due to (1.7) has the form

$$\partial_t v - \partial_x^2 v = F_1(v)\partial_x v + F_2(v), \tag{3.14}$$

where

$$F_1(v) = N^{-1}(v)[f(P_K(N(v)v)) - f(N(v)v)]N(v) \tag{3.15}$$

and

$$F_2(v) := [N^{-1}\partial_x^2 N - N^{-1}\partial_t N - N^{-1}f(Nv)\partial_x N]v - N^{-1}g(Nv). \tag{3.16}$$

Recall also that the map N as well as F_i depend on a parameter $K \geq K_0$. We start with the most complicated operator F_2.

Lemma 3.5. *For sufficiently large $K \geq K_0$ the map F_2 belongs to $C^\infty(B(r, 0, H^1))$, where r is chosen in such way that*

$$V(B(R, 0, H^1)) \subset B(r, 0, H^1)$$

and, in particular,

$$\|F_2(v_1) - F_2(v_2)\|_{H^1(0,L)} \leq C_K\|v_1 - v_2\|_{H^1_0(0,L)}, \quad v_i \in B(r, 0, H^1_0), \tag{3.17}$$

where the constant C_K depends on K.

Proof. Actually, definition (3.16) of the operator F_2 involves two non-trivial terms $\partial_x^2 N(v)$ and $\partial_t N(v)$, the other terms can be estimated in a straightforward way using Lemmas 2.1 and 2.2. Let us first treat $\partial_x^2 N$. Differentiating equation (1.7) in x, we get

$$\frac{d^2}{dx^2}N = \frac{1}{2}f(P_K(Nv))\partial_x N + \frac{1}{2}f'(P_K(Nv))(\partial_x P_K(Nv))N. \tag{3.18}$$

The estimate for the L^∞-norms of the terms $N(v_1) - N(v_2)$ as well as for the terms $\partial_x N(v_1) - \partial_x N(v_2)$ are obtained in Lemma 2.2. Furthermore, since

$$P_K(Nv) = \sum_{k=1}^{K}(Nv, e_k)e_k \tag{3.19}$$

and all e_k are smooth, we also have

$$\|P_K(N(v_1)v_1 - N(v_2)v_2)\|_{L^\infty(0,L)} \le C_K(\|v_1 - v_2\|_{L^2(0,L)} + \|N(v_1) - N(v_2)\|_{L^2(0,L)})$$

and, therefore,

$$\|\partial_x^2 N(v_1) - \partial_x^2 N(v_2)\|_{L^\infty(0,L)} \le C_K\|v_1 - v_2\|_{H^1(0,L)}.$$

Moreover, differentiating equation (3.18) once more in x and arguing analogously, we derive that

$$\|\partial_x^3 N(v_1) - \partial_x^3 N(v_2)\|_{L^\infty(0,L)} \le C_K\|v_1 - v_2\|_{H^1(0,L)} \tag{3.20}$$

which is sufficient for estimating the H^1 norm of $F_2(v_1) - F_2(v_2)$.

Let us now treat the term $\partial_t N(v)$. Being pedantic, this term is even not defined yet since we need to use equation (1.2) or (1.5) in order to evaluate the time derivative. To define it, we differentiate equation (1.7) in time and write

$$\frac{d}{dx}\partial_t N = \frac{1}{2}f(P_K u)\partial_t N + \frac{1}{2}f'(P_K u)(P_K\partial_t u)N, \quad \partial_t N\big|_{x=0} = 0, \tag{3.21}$$

where $u = U(v)$. Then, using equation (1.2), we write

$$P_K\partial_t u = \sum_{k=1}^{k}[(u, \partial_x^2 e_k) - (f(u)\partial_x u, e_k) - (g(u), e_k)]e_k. \tag{3.22}$$

Thus, we *define* operator $\partial_t N(v)$ as a solution of the ODE (3.21) where $u = U(v)$ and $P_K\partial_t u$ is defined by (3.22). Moreover, using (2.14), we see that

$$\|P_K\partial_t u(v_1) - P_K\partial_t u(v_2)\|_{L^\infty(0,L)} \le C_K\|v_1 - v_2\|_{H^1(0,L)}. \tag{3.23}$$

Then, analogously to the proof of Lemma 2.2, we deduce from equation (3.21) that

$$\|\partial_t N(v_1) - \partial_t N(v_2)\|_{W^{1,\infty}(0,L)} \le C_K\|v_1 - v_2\|_{H^1(0,L)}. \tag{3.24}$$

Thus, the second non-trivial term $\partial_t N$ is also treated and (3.17) follows in a straightforward way from Lemma 2.2 and estimates (3.20) and (3.24). Lemma 3.5 is proved. □

We now return to the operator $F_1(v)$.

Lemma 3.6. *Under the above assumptions the operator $F_1(v)$ satisfy the following estimates:*

$$\|F_1(v)\|_{L^\infty(0,L)} \le CK^{-1/2} \tag{3.25}$$

and

$$\|F_1(v_1) - F_1(v_2)\|_{L^\infty(0,L)} \le CK^{-1/2}\|v_1 - v_2\|_{H_0^1(0,L)}, \tag{3.26}$$

where $v, v_1, v_2 \in B(r, 0, H_0^1)$ and the constant C depends on r, but is independent of K.

Proof. Indeed, due to Lemma 2.2, it is sufficient to verify the above estimates for the operator

$$F_0(v) = f(Nv) - f(P_K(Nv)) = \int_0^1 f'(Nv - sP_K(Nv))\,ds\,Q_K(Nv). \tag{3.27}$$

To this end, we use that

$$\|Q_K(Nv)\|_{L^\infty(0,L)} \le \\ \le C\|Q_K(Nv)\|^{1/2}_{L^2(0,L)}\|Q_K(Nv)\|^{1/2}_{H^1_0(0,L)} \le CK^{-1/2}\|Nv\|_{H^1_0(0,L)} \le CK^{-1/2} \tag{3.28}$$

and, analogously,

$$\|Q_K(N(v_1)v_1 - N(v_2)v_2)\|_{L^\infty(0,L)} \le CK^{-1/2}\|v_1 - v_2\|_{H^1_0(0,L)}. \tag{3.29}$$

Estimates (3.28) and (3.29) together with Lemma 2.2 allow us to deduce (3.25) and (3.26) as an elementary calculation. Lemma 3.6 is proved. □

Remark 3.7. Note that in general F_2 does not preserve the Dirichlet boundary conditions since

$$F_2(v)\big|_{x=0,L} = N^{-1}(v)\big|_{x=0,L}g(0) \neq 0.$$

However, it will map H^1_0 to H^1_0 if we assume in addition that

$$g(0) = 0. \tag{3.30}$$

4. The Inertial Manifold

We are now ready to construct the desired inertial manifold for equation (1.2). As shown in the previous section, this equation is equivalent to (3.14) at least in the neighbourhood of the absorbing set $\mathbb{B}$. By this reason, we may construct the inertial manifold for equation (3.14) instead. However, the non-linearities F_1 and F_2 in this equation are still not globally defined on Φ. To overcome this problem, we need, as usual, to cut-off the nonlinearities outside of a large ball making them globally Lipschitz continuous.

Namely, we introduce a smooth cut-off function $\varphi(z)$ such that $\varphi(z) \equiv 1$ for $z \le r_1^2$ and $\varphi(z) = 0$, $z \ge r^2$, where r_1 is such that $V(\mathbb{B}) \subset B(r_1, 0, H^1_0)$ and $r > r_1$ and $K_0 = K_0(r)$ are chosen in such way that the inverse map $U = U(v)$ is a diffeomorphism on $B(r, 0, H^1_0)$ and the assertions of Lemma 2.3 hold for every $K \ge K_0$.

Finally, we modify equation (3.14) as follows:

$$\partial_t v - \partial_x^2 v = \varphi(\|v\|^2_{H^1})F_1(v)\partial_x v + \varphi(\|v\|^2_{H^1})F_2(v) := \mathcal{F}_1(v) + \mathcal{F}_2(v). \tag{4.1}$$

Then, according to Lemma 3.5 and 3.6, we have

$$\|\mathcal{F}_1(v_1) - \mathcal{F}_1(v_2)\|_{L^2(0,L)} \le CK^{-1/2}\|v_1 - v_2\|_{H^1_0(0,L)} \tag{4.2}$$

and

$$\|\mathcal{F}_2(v_1) - \mathcal{F}_2(v_2)\|_{H^1(0,L)} \le C_K\|v_1 - v_2\|_{H^1_0(0,L)}, \tag{4.3}$$

where $v_i \in H^1_0(0, L)$ are arbitrary and the constant C is independent of K. Moreover, under the additional technical assumption (3.30), the map $\mathcal{F}_2$ will act from H^1_0 to H^1_0.

Thus, instead of constructing the inertial manifold for the initial equation (1.3), we will construct it for the transformed problem (4.1). For the convenience of the reader, we recall the definition of the inertial manifold for this equation.

Definition 4.1. A finite-dimensional submanifold $\mathcal{M}$ of the phase space Φ is an inertial manifold for problem (4.1) if the following conditions are satisfied:

1. The manifold $\mathcal{M}$ is strictly invariant with respect to the solution semigroup $\tilde{S}(t)$ of equation (4.1), i.e., $\tilde{S}(t)\mathcal{M} = \mathcal{M}$ for all $t \ge 0$.

2. The manifold $\mathcal{M}$ is a graph of a Lipschitz continuous function $M : P_d\Phi \to Q_d\Phi$ for some $d \in \mathbb{N}$. Here and below we denote by P_d the orthoprojector in Φ to the first d-eigenvalues of the Laplacian and $Q_d = \mathrm{Id} - P_d$.

3. The manifold $\mathcal{M}$ possesses the so-called exponential tracking property, i.e., for every trajectory $v(t)$, $t \ge 0$, of problem (4.1) there is a trajectory $\tilde{v}(t)$ belonging to $\mathcal{M}$ such that

$$\|v(t) - \tilde{v}(t)\|_\Phi \le C\|v(0) - \tilde{v}(0)\|_\Phi e^{-\theta t} \tag{4.4}$$

for some positive C and θ.

We are now ready to state and prove the main result of the paper.

Theorem 4.2. *Under the above assumptions on f and g, equation* (4.1) *possesses an inertial manifold $\mathcal{M} \subset \Phi$ of smoothness $C^{1+\varepsilon}$, for some $\varepsilon > 0$.*

Proof. We first explain the main idea of constructing the inertial manifold restricting ourselves to the special case when $g(0) = 0$, see next section on the explanations on how to remove this technical assumption. As known, in order to do so, we need to verify the so-called spectral gap conditions, see [5, 23] and references therein. Indeed, the nonlinearity $\mathcal{F}_1$ decreases the smoothness by one, so the spectral gap condition for it reads

$$\frac{\lambda_{d+1} - \lambda_d}{\lambda_{d+1}^{1/2} + \lambda_d^{1/2}} > L_1, \tag{4.5}$$

where L_1 is the Lipschitz constant of $\mathcal{F}_1$ as the map from H^1_0 to L^2. On the other hand, the nonlinearity $\mathcal{F}_2$ is globally bounded in H^1_0, so the spectral gap condition for it reads

$$\lambda_{d+1} - \lambda_d > 2L_2, \tag{4.6}$$

where L_2 is the Lipschitz constant of $\mathcal{F}_2$ as the map from H^1_0 to H^1_0.

In our case, we have an extra parameter $K \in \mathbb{N}$ involved and

$$L_1 = CK^{-1/2}, \quad L_2 = C_K. \tag{4.7}$$

Moreover, the eigenvalues of the Laplacian

$$\lambda_d = \frac{\pi^2}{L^2}d^2 \text{ and } \frac{\lambda_{d+1} - \lambda_d}{\lambda_{d+1}^{1/2} + \lambda_d^{1/2}} = \frac{\pi}{L}. \tag{4.8}$$

Thus, fixing K being large enough, we may make the Lipschitz constant $L_1 = CK^{-1/2}$ of the nonlinearity $\mathcal{F}_1$ small enough to satisfy the spectral gap condition (4.5). Then, since

$$\lambda_{d+1} - \lambda_d = \frac{\pi^2}{L^2}(2d+1), \tag{4.9}$$

we may find $d = d(K)$ large enough so that the Lipschitz constant $L_2 = C_K$ of the second nonlinearity $\mathcal{F}_2(v)$ satisfies the spectral gap condition (4.6). Thus, the spectral gap conditions are satisfied and the IM for equation (4.1) exists. However, the standard theory works with only one nonlinearity ($\mathcal{F}_1$ or $\mathcal{F}_2$) and its validity for the case where both nonlinearities are simultaneously present in the equation should be verified/explained. By this reason, we briefly recall below the proof of the inertial manifold existence and show that the slight modification of assumptions (4.5) and (4.6) works indeed for the case where both nonlinearities are involved simultaneously.

Following the Perron method, the desired manifold is found by solving the backward in time boundary value problem

$$\partial_t v - \partial_x^2 v = \mathcal{F}_1(v) + \mathcal{F}_2(v), \quad P_d v\big|_{t=0} = v_0, \quad t \le 0 \tag{4.10}$$

in the weighted space $L^2_{e^{-\theta t}}(\mathbb{R}_-,\Phi)$ with $\theta:=\frac{\lambda_{d+1}+\lambda_d}{2}$. The solution of this equation is usually constructed by Banach contraction theorem and the desired map $M:P_d\Phi\to Q_d\Phi$ is then defined via

$$M(v_0):=Q_dv(0), \tag{4.11}$$

see [23] for the details. To apply the Banach contraction theorem, we introduce the function $w=w(v_0)$ as a solution of the linear problem

$$\partial_t w-\partial_x^2 w=0,\quad P_dw\big|_{t=0}=v_0. \tag{4.12}$$

Then, as not difficult to see that this problem is uniquely solvable in $L^2_{e^{-\theta t}}(\mathbb{R}_-,\Phi)$, so the associated linear operator

$$w:\ P_d\Phi\to L^2_{e^{-\theta t}}(\mathbb{R}_-,\Phi)$$

is well-defined. Introducing the function $z=v-w$, we transform (4.10) to

$$\partial_t z-\partial_x^2 z=\mathcal{F}_1(z+w)+\mathcal{F}_2(z+w),\quad P_dz\big|_{t=0}=0,\quad t\le 0. \tag{4.13}$$

Furthermore, as also not difficult to show, the linear non-homogeneous problem

$$\partial_t z-\partial_x^2 z=h(t),\quad t\in\mathbb{R}, \tag{4.14}$$

is uniquely solvable in the space $L^2_{e^{-\theta t}}(\mathbb{R},\Phi)$ for any $h\in L^2_{e^{-\theta t}}(\mathbb{R},\Phi)$, so the linear operator

$$\mathcal{R}:\ L^2_{e^{-\theta t}}(\mathbb{R},\Phi)\to L^2_{e^{-\theta t}}(\mathbb{R},\Phi)$$

is well defined. Using this operator, problem (4.13) can be rewritten in the equivalent form as follows:

$$z=\mathcal{R}(\chi_-\mathcal{F}_1(z+w)+\chi_-\mathcal{F}_2(z+w)), \tag{4.15}$$

where $\chi_-(t)=0$ for $t\ge 0$ and $\chi_-(t)=1$ for $t<0$, see [23] for the details.

To estimate the Lipschitz constant of the right-hand side of (4.15), we need the following lemma.

Lemma 4.3. *Under the above assumptions the following estimates for the norms of $\mathcal{R}$ hold:*

$$\|\mathcal{R}\|_{\mathcal{L}(L^2_{e^{-\theta t}}(\mathbb{R},\Phi),L^2_{e^{-\theta t}}(\mathbb{R},\Phi))}\le\frac{2}{\lambda_{d+1}-\lambda_d} \tag{4.16}$$

and

$$\|\mathcal{R}\|_{\mathcal{L}(L^2_{e^{-\theta t}}(\mathbb{R},L^2),L^2_{e^{-\theta t}}(\mathbb{R},\Phi))}\le\frac{2\lambda_{d+1}^{1/2}}{\lambda_{d+1}-\lambda_d}. \tag{4.17}$$

Indeed, these estimates are the straightforward corollaries for the key estimate

$$\|y\|^2_{L^2_{e^{-\theta t}}(\mathbb{R})}\le\frac{1}{(\lambda_k-\theta)^2}\|h\|^2_{L^2_{e^{-\theta t}}(\mathbb{R})}$$

for the solution of the 1st order ODE

$$\frac{d}{dt}y+\lambda_k y=h(t),$$

see [23] for the details.

Thus, the Lipschitz constant of the right-hand side of (4.15) can be estimated by

$$\operatorname{Lip}\le\frac{2\lambda_{d+1}^{1/2}L_1}{\lambda_{d+1}-\lambda_d}+\frac{2L_2}{\lambda_{d+1}-\lambda_d} \tag{4.18}$$

and this constant is indeed less than one (which allows us to apply the Banach contraction theorem) if

$$\frac{\lambda_{d+1}-\lambda_d}{\lambda_{d+1}^{1/2}} > 4L_1, \quad \lambda_{d+1}-\lambda_d > 4L_2. \tag{4.19}$$

Therefore, since $L_1 = CK^{-1/2}$, we may fix K to be large enough to satisfy the first condition of (4.19) for all $d \in \mathbb{N}$. Then, since $L_2 = C_K$ and $\lambda_{d+1} - \lambda_d \sim Cd$, we always may find d in such way that the second condition of (4.19) will be also satisfied. Thus the desired inertial manifold could be indeed constructed by the Banach contraction theorem. As shown in [23] both the exponential tracking and $C^{1+\varepsilon}$-regularity of this manifold are also the straightforward corollaries of this contraction and the theorem is proved. □

5. Generalizations and Concluding Remarks

In this concluding section, we briefly discuss possible extensions of the obtained result to more general equations as well as indicate some open problems. We start with discussing different boundary conditions.

5.1. **Neumann boundary conditions.** Let us consider system of equations (1.2) with the Neumann boundary conditions

$$\partial_n u\big|_{x=0,L} = 0. \tag{5.1}$$

Then, differentiating the equation in x and denoting $w = \partial_x u$, we end up with the following system of equations:

$$\begin{cases} \partial_t u + f(u)w + g(u) = \partial_x^2 u, \quad \partial_n u\big|_{x=0,L} = 0, \\ \partial_t w + f'(u)[w,w] + g'(u)w + f(u)\partial_x w = \partial_x^2 w, \quad w\big|_{x=0,L} = 0. \end{cases} \tag{5.2}$$

Since this system does not contain the terms $\partial_x u$, we may transform only the second component w of this system via $w = N(u)v$, where $N(u)$ solves

$$\frac{d}{dx}N = \frac{1}{2}f(P_K u)N, \quad N\big|_{x=0} = \mathrm{Id}$$

and, in contrast to the previous theory, P_K is an orthoprojector to first K eigenvectors of the Laplacian with *Neumann* boundary conditions. Then, repeating word by word the above arguments, we obtain the existence of an inertial manifold for problem (5.2).

5.2. **Removing assumption $g(0) = 0$.** Recall that we have proved the main theorem on inertial manifold existence for equations (1.2) under the additional assumption that $g(0) = 0$. This assumption is posed only in order to have $F_2(v) \in H_0^1(0,L)$ if $v \in H_0^1(0,L)$ and can be easily removed. Indeed, in the general case, the boundary conditions are not preserved, but F_2 still maps H_0^1 to H^1. Using that $H_0^s = H^s$ for $s < 1/2$, we may treat the nonlinearity F_2 as the Lipschitz map from the phase space $\Phi = H_0^1$ to, say, $H^{1/4} = D((-\partial_x^2)^{3/8}\Phi)$. The spectral gap condition for such nonlinearities reads

$$Cd^{1/4} \sim \frac{\lambda_{d+1}-\lambda_d}{\lambda_d^{3/8}+\lambda_{d+1}^{3/8}} > L_2$$

and still can be satisfied by choosing $d = d(K)$ large enough.

5.3. **More general equations.** Consider the system of equations

$$\partial_t u + f(u, \partial_x u) = \partial_x^2 u, \quad u\big|_{x=0,L} = 0, \tag{5.3}$$

where the nonlinearity satisfies $f(0, \partial_x u) \equiv 0$. Then, differentiating this equation with respect to x and denoting $w = \partial_x u$, we end up with

$$\partial_t w + f'_u\,(u, w)\, w + f'_{\partial_x u}\,(u, w)\, \partial_x w = \partial_x^2 w, \quad \partial_n w\big|_{x=0,L} = 0. \tag{5.4}$$

The assumption $f(0, z) \equiv 0$ is essential here since it guarantees that $\partial_n w\big|_{x=0,L} = 0$. Actually, we do not know how to remove it.

Furthermore, differentiating (5.4) by x once more and denoting $\theta = \partial_x w$, we finally arrive at

$$\begin{cases} \partial_t u + f(u, \partial_x u) = \partial_x^2 u, \quad u\big|_{x=0,L} = 0, \\ \partial_t w + f'_u(u, w)w + f'_{\partial_x u}(u, w)\theta = \partial_x^2 w, \quad \partial_n w\big|_{x=0,L} = 0, \\ \partial_t \theta + f''_{uu}(u, w)[w, w] + 2f''_{u,\partial_x u}(u, w)[w, \theta]+ \\ +f''_{\partial_x u,\partial_x u}(u, w)[\theta, \theta] + f'_u(u, w)\theta + f'_{\partial_x u}(u, w)\partial_x \theta = \partial_x^2 \theta, \quad \theta\big|_{x=0,L} = 0. \end{cases} \tag{5.5}$$

These equations have the form of (5.2) and only the third equation contains a convective term, so we need to change only the θ-variable where the boundary conditions are the Dirichlet ones. Therefore, repeating word by word the above arguments, we may construct the inertial manifold for this equation.

5.4. **Periodic boundary conditions.** In contrast to the previous cases, the case of periodic boundary conditions looks much more complicated. Indeed, in order to preserve the periodic boundary conditions, we need to find *periodic* solution $N(x)$ of problem (1.7) and this is possible not for all non-linearities f. Indeed, in the scalar case $n = 1$, it is possible if and only if the mean value

$$\langle f(P_K(Nv))\rangle := \frac{1}{L}\int_0^L f(P_K(Nv))\,dx = 0,$$

so in this case problem (1.7) should be replaced by

$$\frac{d}{dx}N = \frac{1}{2}(f(P_K(Nv)) - \langle f(P_K(Nv))\rangle)N. \tag{5.6}$$

This leads to the presence of the extra term $\langle f(P_K(Nv))\rangle\,\partial_x v$ in equations (3.14) which is not dangerous from the point of view of the inertial manifold construction. Thus, for the scalar case $n = 1$, we still able to cover the case of periodic boundary conditions. However, we do not know the analogue of the solvability condition for equation (1.7) in the vector case $n > 1$, so constructing the inertial manifolds for this case remains an open problem.

5.5. **IM as a graph over the lower Fourier modes.** Recall that the standard definition of an inertial manifold, see Definition 4.1, usually assumes that the IM can be presented as a graph over the linear subspace generated by the lower Fourier modes. In our case, it is so for the transformed equation (4.1). However, if we return back to equation (1.7), the associated invariant manifold $U(\mathcal{M})$ a priori does not have this structure since it is not necessarily can be nicely projected to the linear subspace generated by the lower Fourier modes of this equation. On the other hand, as follows from the Romanov theory, see [16,17] for the details, the attractor $\mathcal{A}$ of equation (1.7) can be projected in a bi-Lipschitz way to the plane $P_d\Phi$ if d is large enough. Since the manifold $U(\mathcal{M})$ is smooth and contains the attractor, we may expect that a sufficiently small neighbourhood of the attractor in $U(\mathcal{M})$ can be nicely projected to $P_K\Phi$. By this reason, we may a posteriori expect that the constructed inertial manifold is still a graph over the lower Fourier modes. We will return to this problem somewhere else.

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