On the Stability and Basins of Attraction of Forced Nonlinear Oscillators

by

James A. Wright

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University of Surrey
Department of Mathematics

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Declaration

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Abstract

We consider second order ordinary differential equations describing periodically forced dynamical systems with one and a half degrees of freedom. First, we study Hill’s equation where we investigate boundedness of solutions. We also construct a class of equations, including a class of Hill’s equation, for which we can obtain closed form solutions. We continue to analyse the boundedness of solutions to nonlinear systems with sufficiently high and low frequency forcing, utilising averaging techniques and KAM theory. We then focus on the study of dissipative systems with parameters chosen in a region of parameter space for which the equilibrium points of the linearised systems are Lyapunov stable. In dissipative systems, many of the solutions which exist in the absence of dissipation are destroyed, leaving a finite set of attractive solutions. We investigate the basins of attraction of the attractive equilibrium points and periodic orbits. In particular we study how the basins of attraction change when the coefficient of dissipation is allowed to initially vary as a function of time. Although it is the final value of the dissipative coefficient which determines which attractors eventually exist, the sizes of their corresponding basins of attraction are found to depend strongly on the full evolution of the coefficient. We study the dynamics of systems with the dissipative parameter modeled by both linearly increasing and decreasing functions of time with various gradients. In this instance we outline four cases pertaining to the sets of attractors at both the initial and final values of the coefficient of dissipation. For each scenario we state our expectations which are illustrated by means of numerical simulation for the systems of the pendulum with vertically oscillating support and the pendulum with periodically varying length. Further to this, a method which allows the fast numerical computation of basins of attraction for a system with an initially varying coefficient of dissipation is identified. This method is also applied in explaining a phenomenon in which the basins of attraction can drastically change when the coefficient of dissipation is a function of time.
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Chapter 1

Introduction

Throughout history the study of physical systems and the equations which describe their motion has been the focus of a wide field of mathematics. A large number of systems can be modelled by ordinary differential equations which depend on time. However, more often than not, the systems in which we are interested have equations of motion for which, even the solutions to the linearised equations cannot be found in a closed form.

Consider the ordinary differential equations

\[ \ddot{x} + G(x, t) + \gamma \dot{x} = 0, \quad \ddot{\theta} + F(\theta, t) + \gamma \dot{\theta} = 0, \]  

(1.1)

where \((x, \dot{x}) \in \mathbb{R}^2\) and \((\theta, \dot{\theta}) \in T \times \mathbb{R}\), with \(T = \mathbb{R}/2\pi\mathbb{Z}\). The functions \(F\) and \(G\) are smooth and 2\(\pi\)-periodic in time \(t\) (\(F\) is also 2\(\pi\)-periodic in \(\theta\)); the dots denote derivative with respect to time \(t\). Equations to describe the motion of systems with one-and-a-half degrees of freedom are often of this form, in which case the functions \(G(x, t)\) and \(F(\theta, t)\) can be considered as an external driving force and the parameter \(\gamma\) represents the damping coefficient (also commonly referred to as the coefficient of dissipation), which for physical reasons is assumed to be positive.

Setting \(\gamma = 0\), the systems (1.1) are non-dissipative and have an infinite set of solutions, uniquely defined by their initial conditions. The space of possible position and velocity values (and hence all possible initial conditions) for a system is known as
the phase space. For $\gamma > 0$, it is expected that only finitely many (asymptotically) stable solutions exist and between them they attract a full measure set of phase space; this was conjectured by Jacob Palis [133], although no proof exists. Solutions which attract a nonzero measure set of phase space are known as attractors. The attractors are either fixed points, periodic curves, quasi-periodic curves or more complicated solutions such as chaotic strange attractors. Here and throughout the rest of this thesis we refer to equilibrium solutions of the equations (1.1) (with $\dot{x} = 0$, $\dot{\theta} = 0$) as fixed points of the systems, as they appear fixed in $(x, \dot{x})$, and $(\theta, \dot{\theta})$ co-ordinates, respectively. The set of initial conditions in phase space whose trajectories tend toward a given attractor as $t \to +\infty$, is called the attractors associated basin of attraction.

In many physical systems one or more of the coefficients vary with time in a non-periodic manner. Examples of this include the angular velocity of a rotating system with an automatic balancing device, the forcing amplitude of a mechanical oscillator [37] and tidal dissipation in spin-orbit systems of heavenly bodies. In this thesis we focus on systems with varying coefficient of dissipation.

Generally in the literature, such systems are studied in the scenario of constant damping coefficient, and in some cases a time periodic coefficient has been considered. However, as mentioned above, in many physical systems the damping coefficient varies with time in a non-periodic manner. This can be due to several factors, such as the heating/cooling of a mechanical system, due to wear or rust on mechanical parts, build up of deposits in joints or the polishing of surfaces which rub against one another. Despite this, usually, models and numerical simulations of such systems only take the final value of dissipation into account when calculating basins of attraction. The variation in dissipation can be nonlinear, and need not be monotonic; however for simplicity we shall consider monotonic, linear variations of the coefficient of dissipation.

There are a few different aspects included in the work contained in the chapters
which follow. We begin by studying forced non-dissipative systems, where the focus is on the boundedness of solutions. We first study linear systems using Hill theory and later extend some of the analysis to nonlinear systems using KAM theory. This work is later put in context of the systems which appear in the following chapters, as well as the appendices.

We then study systems with a positive coefficient of dissipation. Primarily the aim is to understand how the basins of attraction are altered when the coefficient of dissipation is initially a function of time. As previously mentioned, the motivation for this work has its roots in physical applications. One famous example is the spin-orbit system in celestial mechanics. That is, the study of the ratio between the number times a satellite spins on its own axis, to the number of orbits it completes around some central body. One of the main focuses when studying the spin-orbit model is to understand why satellites (for example the Moon around the Earth or Mercury around the Sun) fall into a particular $p : q$ resonance. In particular why many satellites have 1:1 resonance. In this instance the main dissipation under consideration is due to tidal friction caused by the liquid core of a heavenly body. Initially, the core is extremely hot and molten, therefore the resulting dissipation is relatively small. As the heavenly body ages the core cools down and becomes more viscous; this results in an increase in tidal dissipation. Finally the core will become solid and tidal dissipation no longer exists. Here it is clear that the dissipation is a nonconstant, nonperiodic function of time. In this instance, the system evolves over a very long time-span, for which we will see it is important to take the full evolution of dissipation into consideration if one wishes to obtain an accurate description of the dynamics.

In particular our focus is on the variation in the sizes of the basins of attraction relative to some region of phase space. That is, the probability of landing on a particular attractive solution. In the above example it is often the probability of achieving a
Figure 1.1: Basins of attraction for the pendulum with oscillating support described by \( \ddot{\theta} + (\alpha - \beta \cos(t)) \sin \theta + \gamma \dot{\theta} = 0 \), with \( \alpha = 0.5 \). In (a) the parameters are fixed as \( \beta = 1.18 \), \( \gamma = 0.05 \) for which two rotating attractors exist and have 1:1 resonance with the forcing. Namely rotations with \( \dot{\theta}(t) > 0 \) whose basins of attraction is shown in red and rotations with \( \dot{\theta}(t) < 0 \) shown in yellow. In (b) \( \beta = 1.28 \), \( \gamma = 0.28 \) for which four attractors exist; rotations with 1:1 resonance whose basins are coloured as before and two asymmetric oscillating attractors which have 1:2 resonance with basins shown in green and black.

particular solution which is of interest, i.e. why the probability of falling into 1:1 resonance in the spin-orbit system is so high. Furthermore, there are many systems for which the boundaries between basins of attraction are fractal; even in classical systems such as the pendulum with oscillating support, see Figure 1.1. This result is that the basins of attraction appear broken and there is sensitive dependence on initial conditions in phase space. In such cases it is impossible (in the real world) to give the system initial conditions with enough accuracy so as to determine the final solution. In such instances, knowing the probability of moving toward a particular attractor as \( t \to \infty \) is of more use than knowing the exact topology of the basins of attraction.

We build on and utilise information obtained for systems where the coefficient of dissipation is set at a constant value. Of course, we do not hope nor intend to offer a full understanding of systems with varying dissipation, indeed even the study of systems with constant dissipation is an open problem. Instead we aim to make clear the importance of considering the full time evolution of the damping coefficient and lay down a few basic
rules, which may serve as a foundation for further research on the topic.

The variations in the sizes of the basins of attraction are illustrated with the use of two forced pendula. In particular we study the pendulum with vertically oscillating support and the pendulum with variable length. This brings us on to the final motivation of the thesis. Although both systems are commonly studied perturbations of the simple pendulum, little work has been conducted to contrast their dynamics. In some instances, even as recent as 2014, the two systems have been mistaken for one-another. However, although the two systems are both perturbations of the simple pendulum, they can exhibit different dynamics. To avoid further confusion, we aim to make clear some of the differences between the two systems. In Section 6.5 we draw comparisons between the two systems, looking both at the similarities and differences, in light of the work completed in Chapters 5 and 6.

1.1 Publications


1.2 Drafts

- J.A. Wright, M. Bartuccelli, G. Gentile, *Comparisons between the pendulum with variable length and the pendulum with oscillating support*. Soon to be submitted for publication. A summary comparing the two pendulum systems, analogous to Section 6.5.

- M. Bartuccelli, G. Gentile, J.A. Wright, *Stable dynamics in forced systems with sufficiently high/low driving frequency*. Submitted for publication. Contains the work in Chapter 3.

1.3 Index of original results

- Theorem 2.5 in Section 2.1 is original.

- The work in Section 2.2 is original unless otherwise stated.

- Examples 2.3, 2.4 and 2.5 in Chapter 2 are original.

- Theorems 3.2 and 3.3 in Chapter 3 are original.

- The application of analytic continuation to a system with trigonometric nonlinearity is original, see Chapter 4.

- The four statements in the introduction of Chapter 5 concerning how the basins of attraction vary when the coefficient of dissipation is initially a function of time are original.

- To the best of my knowledge, the bifurcations of the pendulum with variable length, as studied in Section 6.4.1, are not currently available in the literature.

- The comparison between the two pendulum systems in Section 6.5 is original.
• All the ideas, applications and definitions in Chapter 7 are original.

1.4 Chapter outline

In Chapter 2 we consider Hill’s equation, focusing our analysis on (linear) stability and a few cases in which closed form solutions may be obtained. In the course of Chapter 2 we show that Hill’s equation is (linearly) stable provided the frequency of the forcing term is sufficiently high. However we are unable to obtain any information about nonlinear systems using the linearised Hill form. This work is continued in Chapter 3 where KAM theory is applied to nonlinear systems with either high or low frequency forcing. Chapter 4 details the numerical schemes utilised in the chapters which follow. Also a brief statistical justification is provided for the methods used to choose initial conditions when calculating basins of attraction. In Chapters 5, 6 and 7 we study how basins of attraction are affected when the coefficient of dissipation is initially a nonconstant function of time. We focus our study on two perturbations of the simple pendulum, although we expect our results are applicable to many more systems. At the end of Chapter 6 we compare the two pendulum systems, pointing out both similarities and differences in the dynamics. Finally, conclusions, interesting open problems and future work are given in Chapter 8. Appendix A is devoted to introducing background knowledge. In the process we also provide an introduction to some of the work contained in the earlier chapters.
Chapter 2

Hill’s equation

Hill’s equation (named after George William Hill due to his work on the equation, see [90] ) is a homogeneous linear second order ordinary differential equation of the form

\[ \ddot{y}(t) + Q(t)y(t) = 0, \quad (2.1) \]

where \( Q(t) \) is a real periodic function of \( t \). Though not strictly necessary the period of \( Q(t) \) is often set to \( \pi \), and many of the theorems stated throughout this chapter make this assumption. Hill’s equation has attracted much attention in applied mathematics, physics and engineering, as the equations of motion of many physical systems are perturbations of this form. Furthermore any second order ordinary differential equation of the form

\[ \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0, \quad (2.2) \]

with \( a(t) \) at least once differentiable can be transformed into an equation in the form of Hill’s equation using the substitution, see [116]

\[ y(t) = x(t)e^{\frac{1}{2}\int a(t)\,dt}, \quad (2.3) \]

which results in

\[ \ddot{y}(t) + Q(t)y(t) = 0, \]

\[ Q(t) = -\frac{1}{2}\dot{a}(t) - \frac{1}{4}a^2(t) + b(t). \]
This is particularly useful, as finding out properties for one of the equations and its solutions results in corresponding properties for the other. This is exploited to some extent in Section 2.2. Another common form of Hill’s equation is obtained by writing \( Q(t) = \lambda + \tilde{Q}(t) \), so that (2.1) becomes

\[
\ddot{y}(t) + \left( \lambda + \tilde{Q}(t) \right) y(t) = 0,
\]

(2.4)

where \( \tilde{Q}(t) \) is a real periodic function of \( t \), and again we set the period equal to \( \pi \).

The function \( \tilde{Q}(t) \) has zero mean value and the mean value of \( Q(t) \) in equation (2.1) is represented by the constant \( \lambda \). The two equations are thus equivalent, however explicitly stating the mean value \( \lambda \) is often preferable. As such, unless otherwise stated we shall continue to use equation (2.4) as the standard form for Hill’s equation, and drop the tilde over \( \tilde{Q}(t) \). As \( Q(t) \) is \( \pi \) periodic it may be represented by a Fourier series as

\[
\ddot{y}(t) + \left( \lambda + \sum_{n=1}^{\infty} c_n \cos(2nt) + \sum_{n=1}^{\infty} s_n \sin(2nt) \right) y(t) = 0,
\]

where the coefficients \( c_n \) and \( s_n \) are constants. A special case of Hill’s equation, known as Mathieu’s equation, can be obtained by taking only the term with coefficient \( c_1 \) in the above expansion to yield

\[
\ddot{y}(t) + \left( \lambda + c_1 \cos(2t) \right) y(t) = 0.
\]

(2.5)

Mathieu’s equation often appears in dynamics, as the linearised equations of motion of many mechanical systems can be written in this form. Further details on Mathieu’s equation are given in Appendix A.9. However, in general neither Hill’s equation nor Mathieu’s equation can be explicitly solved. In Section 2.1 we look at what information can be discovered about the stability of solutions to Hill’s equation, even when the solutions themselves cannot be found in a closed form. Section 2.2 focuses on a few
particular cases where explicit solutions can be found and in Section 2.3 we consider
perturbations of Hill’s equation.

2.1 Linear stability

One of the fundamental interests when studying solutions to any equation is whether or
not the solutions are “stable”. In this section we outline some necessary and sufficient
criteria for the solutions of Hill’s equation to be (linearly) stable. For systems of the
form (2.4) if all the solutions of (2.4) are bounded then we say that they are (linearly)
stable. If at least one solution is unbounded then the solutions are unstable, [116].
Further discussion of stability is given in Appendix A.8, see also Appendix A.9. Before
discussing theorems regarding (linear) stability we first introduce some notation and
definitions, which may also be found in [116].

The equation (2.4) has two continuously differentiable solutions $y_1(t, \lambda), y_2(t, \lambda)$ which
are determined uniquely by the initial conditions

$$y_1(0, \lambda) = 1, \quad \dot{y}_1(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad \dot{y}_2(0, \lambda) = 1.$$  

These are known as the normalised solutions to Hill’s equation, [116].

**Definition 2.1** The characteristic equation of (2.4) is

$$\mu^2 - \left(y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda)\right) \mu + 1 = 0,$$

and the characteristic exponent, $\rho$, is a number which satisfies

$$e^{i\rho \pi} = \mu_1, \quad e^{-i\rho \pi} = \mu_2,$$

where $\mu_1, \mu_2$ are the roots of the characteristic equation, known as characteristic numbers.

From Definition 2.1 we see that $\mu_1 \mu_2 = 1, \ 2 \cos(\rho \pi) = \mu_1 + \mu_2 = y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda)$ and
$\rho$ is defined up to an integer multiple of 2.
CHAPTER 2. HILL’S EQUATION

Theorem 2.1 (Floquet’s Theorem) If the roots $\mu_1$, $\mu_2$ of the characteristic equation are different from one another, then Hill’s equation has two linearly independent solutions

$$f_1(t, \lambda) = e^{i\rho t}p_1(t, \lambda), \quad f_2(t, \lambda) = e^{-i\rho t}p_2(t, \lambda),$$

where $p_1(t, \lambda)$ and $p_2(t, \lambda)$ are periodic functions with period $\pi$.

If the characteristic numbers are equal, then Hill’s equation has a nontrivial periodic solution with period $\pi$ if $\mu_1 = \mu_2 = 1$ or period $2\pi$ if $\mu_1 = \mu_2 = -1$. Let $p(t, \lambda)$ denote such a periodic solution and let $y(t, \lambda)$ be another solution linearly independent of $p(t, \lambda)$.

Then

$$y(t + \pi, \lambda) = \mu_1 y(t, \lambda) + Cp(t, \lambda),$$

for some constant $C$. The case $C = 0$ is equivalent to

$$y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda) = \pm 2, \quad y_2(\pi, \lambda) = 0, \quad \dot{y}_1(\pi, \lambda) = 0. \quad (2.6)$$

From Floquet’s theorem alone we are already able to deduce some properties of the solutions to Hill’s equation. If $\mu_1 \neq \mu_2$ and $\rho$ is a real number (i.e. $\mu_1$, $\mu_2$ lie within the unit circle) then the absolute value of every solution to (2.4) is bounded. That is

$$|y(t, \lambda)| < M,$$

for some constant $M$. Moreover the value of $M$ depends only on the initial conditions of the solution $y(t, \lambda)$ and not on $t$. If $\rho$ is not real then there exists a nontrivial unbounded solution of Hill’s equation. Finally if $\mu_1 = \mu_2$, then for all the solutions of Hill’s equation to be bounded and hence stable it is both necessary and sufficient that $y_1$, $y_2$ satisfy equation (2.6), i.e. $C = 0$ in Floquet’s theorem. Floquet’s theorem is later restated in a more applied manner in Appendix A.9, where it is used in finding the transition curves for Mathieu’s equation. Here we present some formal theorems which we refer back to later in the thesis, see also [116].
The solutions to \((2.4)\) are stable if and only if 
\[ |y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda)| < 2, \]
or
\[ y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda) = \pm 2, \quad y_2(\pi, \lambda) = 0, \quad \ddot{y}_1(\pi, \lambda) = 0. \]

Theorem 2.3 (Oscillation Theorem) For every differential equation of the form \((2.4)\) there are two monotonically increasing infinite sequences of real numbers
\[ \lambda_0, \quad \lambda_1, \quad \lambda_2, \quad \ldots, \]
\[ \lambda'_1, \quad \lambda'_2, \quad \lambda'_3, \quad \ldots, \]
which satisfy the inequalities
\[ \lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda'_3 \leq \lambda'_4 < \lambda_3 \leq \lambda_4 < \ldots, \]
and the relations
\[ \lim_{n \to \infty} \lambda_n^{-1} = 0, \quad \lim_{n \to \infty} (\lambda'_n)^{-1} = 0, \]
such that \((2.4)\) has a solution of period \(\pi\) if and only if \(\lambda = \lambda_n\) for \(n = 0, 1, 2, \ldots\) and a solution of period \(2\pi\) if and only if \(\lambda = \lambda'_n\) for \(n = 1, 2, \ldots\). The solutions are stable in the intervals
\[ (\lambda_0, \lambda'_1), \quad (\lambda'_2, \lambda_1), \quad (\lambda'_3, \lambda'_2), \ldots \] (2.7)
At the end points of the above intervals the solutions are in general unstable. The solutions for \(\lambda\) at a given end point are stable if and only if \(\lambda = \lambda_{2n+1} = \lambda_{2n+2}\) or \(\lambda = \lambda'_{2n+1} = \lambda'_{2n+2}\). Solutions for the value \(\lambda = \lambda_0\) are always unstable. Also if \(\lambda\) is complex then the solutions of \((2.4)\) are always unstable. The \(\lambda_n\) are the roots of the equation
\[ y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda) = 2, \]
and $\lambda'_n$ are the roots of

$$y_1(\pi, \lambda) + \dot{y}_2(\pi, \lambda) = -2.$$  

We shall refer to the intervals (2.7) as intervals of stability and the intervals $(\lambda_{2n+1}, \lambda_{2n+2})$, $(\lambda'_{2n+1}, \lambda'_{2n+2})$ as the intervals of instability. The interval $(-\infty, \lambda_0)$ is known as the zeroth interval of instability and $(\lambda'_1, \lambda'_2)$ the first interval of instability. It can be seen from Theorem 2.3 that the intervals of stability and instability can never shrink to a point and the intervals of stability can never vanish, but two of them can merge together if $\lambda_{2n+1} = \lambda_{2n+2}$ or $\lambda'_{2n+1} = \lambda'_{2n+2}$. However if $Q(t)$ is taken to be constant the intervals of instability, with the exception of the zeroth interval of instability, cease to exist. Furthermore this can only happen when $Q(t)$ is constant, see [43]. It can be shown, using Lemma 2.5 and Corollary 2.1 from [116] in conjunction with Theorem 2.2 that the solutions of (2.4) are stable for $\lambda = \lambda_{2n+1}$ (or $\lambda = \lambda'_{2n+1}$) if and only if $\lambda_{2n+1}$ (or $\lambda'_{2n+1}$) is a double root of the equation

$$\left( y_1(\pi, \lambda) - \dot{y}_2(\pi, \lambda) \right)^2 + 4\dot{y}_1(\pi, \lambda)y_2(\pi, \lambda) = 0. \quad (2.8)$$

This condition is equivalent to $\lambda_{2n+1} = \lambda_{2n+2}$ (or $\lambda'_{2n+1} = \lambda'_{2n+2}$), as per Theorem 2.3. The following theorem by Beurling (which was not published but may be found in [116]) allows one to check the stability of a class of equations in an incredibly simple way.

**Theorem 2.4** If $c$ and $d$ are real numbers and

$$c^2 \leq \lambda + Q(t) \leq d^2,$$

then solutions of (2.4) will be stable for all possible $\lambda + Q(t)$ satisfying the above condition if and only if the interval $(c^2, d^2)$ does not contain the square of an integer.

Note, Theorem 2.4 states that, if the interval $(c^2, d^2)$ is such that it contains the square of an integer, then there exists at least one function $\lambda + Q(t)$ for which equation (2.4)
does not have stable solutions. However, it is still possible to find particular choices of $\lambda + Q(t)$ for which the solutions of (2.4) are stable. The result is that the intervals of stability (and also the stable regions for the Mathieu’s equation, see Appendix A.9) will always contain the regions in which Theorem 2.4 is satisfied.

Using Theorem 2.4 we are able to show that high frequency forcing can stabilise the solutions to Hill’s equation. Consider Hill’s equation given by

$$\ddot{y}(t) + \left(\lambda + Q(\omega t)\right)y(t) = 0,$$

where $\lambda$ is greater than the minimum of $Q(\omega t)$, so that $\lambda + Q(\omega t) \geq 0$ and the frequency of $Q$ is $\omega \gg 1$ such that the period is $T = \pi/\omega$. If $T \neq \pi/\omega$, we may write $T = \Omega \pi/\omega$ for some $\Omega$. Then define $\dot{\omega} = \omega/\Omega$ and require $\dot{\omega} \gg 1$, and drop the hat. Transforming the system to time $\tau = \omega t$ yields

$$y''(\tau) + \frac{1}{\omega^2}\left(\lambda + Q(\tau)\right)y(\tau) = 0,$$

where $Q(\tau)$ is now periodic with period $\pi$ and the dashes denote differentiation with respect to time $\tau$. Provided $\omega$ is large enough that

$$0 \leq \frac{1}{\omega^2}\left(\lambda + Q(\tau)\right) \leq 1,$$

Theorem 2.4 is satisfied and all the solutions are bounded and hence stable. This leads us to the following result

**Theorem 2.5** Consider Hill’s equation given by (2.9) where $Q(\omega t)$ has zero mean value, is periodic with period $\pi/\omega$ and $\lambda + Q(\omega t) \geq 0$. Provided $\omega$ satisfies

$$\omega^2 \geq \lambda + Q(t),$$

for all $t$, then all the solutions to (2.9) are bounded and hence stable.
If we consider Mathieu’s equation (see Appendix A.9) we can plot the transition curves which separate the regions of stable and unstable dynamics. This allows us to view Theorem 2.5 pictorially and illustrate the phenomenon clearly. Figure 2.1 shows the transitions curves for Mathieu’s equation \( \ddot{y} + (1 + \delta \cos(\omega t))y = 0 \), in \( \omega, \delta \) space. The first region of instability is shown by the shaded region furthest to the right. Subsequent regions of instability occur to the left of the first region. Here it is clear that, for fixed \( \delta \), by taking \( \omega \) large the parameters occupy a region of parameter space which is to the right of the first region of instability. Further details are given in Appendix A.9.

![Figure 2.1: The first seven stability tongues for Mathieu’s equation in \((\omega, \delta)\) parameter space. For parameters in the shaded region the fixed point \((y, \dot{y}) = (0, 0)\) is unstable and trajectories diverge. For parameters in the unshaded region, the fixed point is Lyapunov stable. The first stability tongue is situated on the right, emanating from \(\omega = 2\). See Appendix A.9 for further details.]

**Example 2.1**

Consider the system

\[
\ddot{y}(t) + (\lambda + \beta \sin(2\omega t))y(t) = 0, \tag{2.10}
\]

with \(\lambda = 3\) and \(\beta = 2\). Letting \(\tau = \omega t\) the function \(Q(\tau)\) is periodic in \(\tau\) with period \(\pi\).
Figure 2.2: On the left a solution in time $t$ and on the right in time $\tau = \omega t$. The top two images show an example trajectory in phase space and the bottom two show the trajectory plotted against time. The parameter values are $\alpha = 3, \beta = 2, \omega = 1.9 < \sqrt{5}$.

Figure 2.3: On the left a solution in time $t$ and on the right in time $\tau = \omega t$. The top two images show an example trajectory in phase space and the bottom two show the trajectory plotted against time. The parameter values are chosen as $\alpha = 3, \beta = 2, \omega = 2.24 > \sqrt{5}$.
Then according to Theorem 2.4 the solutions are stable provided

\[ 0 \leq \frac{1}{\omega^2} \left( 3 + 2 \sin(2\tau) \right) \leq 1. \]

Hence, if \( \omega \geq \sqrt{5} \) all the solutions are bounded and stable. In Figures 2.2 and 2.3 we show two numerical examples for the system (2.10), one where the solutions are unstable and one where they are made stable as a result of larger \( \omega \). Of course, one obvious question to ask is whether this may be extended to include nonlinear systems, such as the cubic equation or Duffing equation, where the linearisation is in the form of Hill’s equation. However the result does not follow through. This is demonstrated by the following example.

**Example 2.2 Cubic oscillator**

Consider the equation for a perturbed cubic oscillator given by

\[ \ddot{x}(t) + \left( 1 + \varepsilon f(\omega t) \right)x^3(t) = 0, \quad f(\omega t) = \sin(2\omega t), \quad (2.11) \]

where \( \varepsilon \ll 1 \) and \( \omega \gg 1 \). By transformation of the time co-ordinate to \( \tau = \omega t \) the equation becomes

\[ x''(\tau) + \frac{1}{\omega^2} \left( 1 + \varepsilon f(\tau) \right)x^3(\tau) = 0, \quad f(\tau) = \sin(2\tau). \]

Writing \( x(\tau) \) as a power series in \( \varepsilon \) as

\[ x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots, \]

we have \( x_0(\tau) = \alpha \text{cn} \left( \frac{\alpha}{\omega^2} \tau, \frac{1}{\sqrt{2}} \right) \) as a solution to the unperturbed system, i.e. \( x_0(t) = \alpha \text{cn} \left( \alpha t, \frac{1}{\sqrt{2}} \right) \) where \( \text{cn}(\cdot, k) \) is the elliptic cosine-amplitude with elliptic modulus \( k \), see Appendix A.3. From here on, as the elliptic modulus has been set equal to \( 1/\sqrt{2} \), the
elliptic integrals and functions can be assumed to be evaluated at this value, and we shall write \(\text{cn}(u), K\) and \(E(u)\) for short, where \(K\) and \(E(u)\) are the elliptic integrals of the first and second kinds respectively, see Appendix A.3. At every level of \(\varepsilon\) above the zeroth level we have

\[
x_j''(\tau) + \frac{3x_0^2(\tau)}{\omega^2}x_j(\tau) = [U]_{j-1},
\]

where \([U]_{j-1}\) consists of all the terms which appear at lower levels. The homogenous equation of (2.12) is of the form of Hills equation with

\[
\lambda + Q(\tau) = \frac{3x_0^2(\tau)}{\omega^2} = \frac{3\alpha^2 \text{cn}^2 \left( \frac{\alpha}{\omega} \tau \right)}{\omega^2},
\]

(2.13)

where

\[
\lambda = \frac{\alpha}{2\omega K} \int_0^{2\omega K/\alpha} \frac{3\alpha^2}{\omega} \text{cn}^2 \left( \frac{\alpha}{\omega} \tau \right) d\tau
\]

\[
= \frac{3\alpha^2}{\omega^2 K} \left[ E \left( \text{sn}(2K) \right) - K \right]
\]

\[
\approx 1.37 \frac{\alpha^2}{\omega^2},
\]

and

\[
Q(\tau) \in \left[ -\lambda, \frac{3\alpha^2}{\omega^2} - \lambda \right].
\]

The homogenous solutions at each level can be written as

\[
\hat{x}_j(\tau) = \alpha e^{i\rho\tau} p_1(\tau) + \beta e^{-i\rho\tau} p_2(\tau),
\]

(2.14)

where \(p_1(\tau), p_2(\tau)\) have the same period as \(x_0^2 = \alpha^2 \text{cn}^2 \left( \frac{\alpha}{\omega} \tau \right)\), i.e \(2K\omega/\alpha\). The parameter \(\rho\) is given by

\[
\rho = \frac{1}{T} \arccos \left( \frac{\hat{x}_* (T) + \hat{x}'_* (T)}{2} \right),
\]

(2.15)

with \(T = 2K\omega/\alpha\) and where \(\hat{x}_* (\tau)\) and \(\hat{x}'_* (\tau)\) are the normalised solutions of homogenous equation resulting from (2.12), see [116]. It may be thought that through the correct
choices of $\alpha$ and $\omega$, it is possible to satisfy Theorem 2.4 at each level of $\varepsilon$. Then one must show that the series for $x(\tau)$ converges. However due to the construction of the system, one of the two homogenous solutions may be written as $\hat{x}_{j_1}(\tau) = x'_0(\tau)$. The other linearly independent solution may be found using

$$\hat{x}_{j_2}(\tau) = \hat{x}_{j_1}(\tau) \int_0^\tau \frac{1}{(\hat{x}_{j_1}(s))^2} \, ds.$$ 

It may be seen that the second solution is always linear in $\tau$. This happens as the parameter $\rho$ is equal to zero. This property is not expected to be unique to the cubic system, but apply to similar systems of the form

$$\ddot{x}(t) + \left( \lambda + Q(t) \right) x^n(t) = 0,$$

with $n = 2m + 1$ for $m \in \mathbb{N}$. Hence using the approach of linearising the system so that it may be written in the form of Hill’s equation, does not allow one to use Theorem 2.4 to analyse the stability of the solutions to the full nonlinear system. We return to the problem of stability in nonlinear systems in Chapter 3. There are many other theorems relating to the stability of the solutions to Hill’s equation, for example Lyapunov’s theorem and extensions of Lyapunov’s theorem [116, 104, 14], however, as they are not used in later sections they have not been included here.

### 2.2 Some explicit solutions

As previously mentioned, in general we are unable to find explicit solutions to Hill’s equation. However there are some particular cases where solutions can be found. Using the transformation (2.3), it is possible to investigate a few such cases. The following examples, which were published in [29], illustrate a couple of instances where the transformation (2.3) can be utilised to find explicit solutions to interesting special cases.
Example 2.3 (Pendulum with varying length)

Consider the linearised system for a pendulum with periodically varying length $\ell = \ell(t)$ described by

$$\ddot{\theta}(t) + 2\frac{\dot{\ell}(t)}{\ell(t)}\dot{\theta}(t) + \frac{g}{\ell(t)}\theta(t) = 0,$$

(2.16)

where $g$ is the acceleration due to gravity, $\ell = \ell_0 + \ell_1 \varphi(\omega t)$ with $\ell_0 > \ell_1 > 0$ and $\varphi$ is a $2\pi$ periodic function of time $\tau = \omega t$ with mean value zero and $\|\varphi\|_\infty = 1$. Hence $\omega$ is the frequency at which the pendulum length varies. Transforming into the corresponding “Hamiltonian” form we have

$$\ddot{y}(t) + Q(t) y(t) = 0,$$

where

$$Q(t) = \frac{g - \ddot{\ell}(t)}{\ell(t)}.$$

If we take the particular case where $Q(t) = \omega^2 = \text{constant}$, then we can obtain a solution to the above system as

$$y(t) = \gamma \cos(\omega t) + \delta \sin(\omega t),$$

provided

$$\ell(t) = \alpha \cos(\omega t) + \beta \sin(\omega t) + \frac{g}{\omega^2}, \quad \ell_1 = \sqrt{\alpha^2 + \beta^2} < \frac{g}{\omega^2} = \ell_0.$$

Then transforming back to the system in $\theta(t)$ we have a solution to (2.16) as

$$\theta(t) = \frac{\gamma \cos(\omega t) + \delta \sin(\omega t)}{\alpha \cos(\omega t) + \beta \sin(\omega t) + \frac{g}{\omega^2}}.$$

Remark 2.1 The choice $Q = \omega^2 = \text{constant}$ effectively sets the mean length of the pendulum $l_0 = g/\omega^2$. In fact the choice of $Q$ equal to any positive constant results in a relation between the class of pendulum systems with periodically varying length about
$\ell_0$ with frequency $\omega = \sqrt{g/\ell_0}$, and the classical pendulum with length $\ell_0$. Furthermore, choosing $Q = \text{constant}$ results in the instability intervals vanishing (with the exception of the zeroth interval of instability) and consequently all solutions of this form are stable provided $g > g_0$ where $g_0$ satisfies $g_0 - \ddot{\ell}(t) = 0$. If we consider the inverted pendulum by instead linearising about $\theta = \pi$, it can be seen that $Q < 0$ and the solutions are hyperbolic, hence always unstable. Further details regarding the stability of similar systems can be found in [43].

The transformation (2.3) may also be used to transform from one second order differential equation to another as follows:

Consider the two equations

\[ \ddot{x}_1(t) + a_1(t)\dot{x}_1(t) + b_1(t)x_1(t) = 0, \]
\[ \ddot{x}_2(t) + a_2(t)\dot{x}_2(t) + b_2(t)x_2(t) = 0, \]

where $a_1(t)$ and $b_1(t)$ are such that the solution $x_1$ can be found explicitly. If $a_2(t)$, $b_2(t)$ satisfy the equation

\[ -\frac{1}{2}\dot{a}_1(t) - \frac{1}{4}a_1^2(t) + b_1(t) = -\frac{1}{2}\dot{a}_2(t) - \frac{1}{4}a_2^2(t) + b_2(t), \]

that is $Q_1(t) \equiv Q_2(t)$ when the equations are transformed into the form of Hill’s equation, the corresponding solution $x_2(t)$ is

\[ x_2(t) = x_1(t)e^{\frac{1}{2}\int (a_1(t) - a_2(t))dt}. \]

**Example 2.4**

Take $a_1(t) = \alpha t^{-1}$ and $b_1(t) = \beta t^{-2}$, hence picking the Cauchy-Euler equation. Then $a_2$ and $b_2$ must satisfy

\[ \frac{2\alpha - \alpha^2 + 4\beta}{t^2} = -2\dot{a}_2(t) - a_2^2(t) + 4b_2(t). \]
Let us choose $a_2(t) = \delta t^m$, hence we require

$$b_2(t) = \frac{\delta^2 t^{2m+2} + 2m\delta t^{m+1} - \alpha^2 - 2\alpha + 4\beta}{4t^2},$$

giving the equation for $x_2$ as

$$\ddot{x}_2(t) + \delta t^m \dot{x}_2(t) + \frac{\delta^2 t^{2m+2} + 2m\delta t^{m+1} - \alpha^2 - 2\alpha + 4\beta}{4t^2} x_2(t) = 0,$$

with solution

$$x_2(t) = t^\gamma e^{\int \alpha t^{-1} + \delta t^m \, dt}, \quad \text{with} \quad \gamma = \frac{1}{2} \left( \alpha - 1 \pm \sqrt{(\alpha - 1)^2 - 4\beta} \right).$$

In fact, it is possible to choose $a_1, b_1$ as constants and obtain solutions for systems with coefficients $a_2(t), b_2(t)$ which are functions of time.

In [25] a class of time-dependent equations were investigated for which solutions may be found. The following theorem and further details may be found in [25].

**Theorem 2.6** Given the second order linear time-dependent differential equation

$$\frac{d}{dt} \left( \frac{\dot{x}(t)}{\omega(t)} \right) + \omega(t)x(t) = 0, \quad (2.17)$$

where $\omega(t)$ is a differentiable positive function of time, then its general solution has the form

$$x(t) = \alpha \cos \left( \int \omega(t) \, dt \right).$$

The differential equation (2.17) may be written as

$$\ddot{x}(t) - \frac{\dot{\omega}(t)}{\omega(t)}\dot{x}(t) + \omega^2(t)x(t) = 0, \quad (2.18)$$
which is in the form of (2.2) so that the transformation (2.3) may be used. This results in a class of solvable Hill’s equations given by equation (2.4) with

\[ \lambda + Q(t) = \frac{1}{2} \frac{\ddot{\omega}(t)}{\omega(t)} - \frac{3}{4} \left( \frac{\dot{\omega}(t)}{\omega(t)} \right)^2 + \omega^2(t), \]

\[ y(t) = \frac{x(t)}{\sqrt{\omega(t)}}. \]

More generally in [25] the authors considered systems of the form

\[ \frac{d}{dt} \left( \frac{\dot{x}(t)}{\omega(t)} \right) + \omega(t) F(x(t)) = 0, \tag{2.19} \]

which have first integral of motion given by

\[ H(x(t), \dot{x}(t), t) := \frac{1}{2} \left( \frac{\dot{x}(t)}{\omega(t)} \right)^2 + U(x(t)) = E = \text{constant}, \tag{2.20} \]

where \( U(x) \) is a primitive of \( F(x) \). It can be seen that setting \( F(x) = x \) we get back the system (2.17). If we instead keep \( F(x) \) general and multiply (2.20) by \( \omega^2(t) \), we can obtain an equation for the energy given by

\[ \int \frac{dx}{\sqrt{E - U(x)}} = \pm \sqrt{2} \int \omega(t) \, dt. \tag{2.21} \]

The following example demonstrates one instance where Theorem 2.6 can be used to find solutions to an interesting problem with time dependent coefficients.

**Example 2.5 (Pendulum with varying length and vertically oscillating support)**

Consider the pendulum with vertically oscillating support given by \( \varphi(t) \) and varying length \( \ell(t) = \ell_0 + \ell_1(t) \) where \( \varphi(t) \) and \( \ell(t) \) are both periodic functions of time \( t \). The motion of the system can be described by

\[ \frac{d}{dt} \left( \ell^2(t) \dot{\theta}(t) \right) + \ell(t) \left( g - \ddot{\varphi}(t) \right) \sin(\theta(t)) = 0, \tag{2.22} \]
where \( \theta(t) \) is the angle the pendulum makes with the downward vertical and \( g \) represents the gravitational constant. The system obtained by linearising about the downward vertical position \( \theta = 0 \) is

\[
\frac{d}{dt}
\left(
\ell^2(t)\dot{\theta}(t)
\right) + \ell(t)
\left(g - \ddot{\varphi}(t)\right)\theta(t) = 0.
\] (2.23)

If we constrain ourselves to the particular case where

\[
\ell(t) = \left(\frac{1}{g - \ddot{\varphi}(t)}\right)^\frac{1}{3},
\] (2.24)

then equation (2.23) is in the form of (2.17) and the solution is given by

\[
\theta(t) = \alpha \cos \left(\int \ell(t)\left(g - \ddot{\varphi}(t)\right)dt + \beta\right),
\] (2.25)

for arbitrary constants \( \alpha, \beta \). As one example we can consider the physically interesting case \( \varphi(t) = a \cos(\omega t) \) which gives \( g - \ddot{\varphi}(t) = g + a\omega^2 \cos(\omega t) \) with a constant such that \( g > |a|\omega^2 \); the corresponding solution is given by

\[
\theta(t) = \alpha \cos \left(\int \left(g + a\omega^2 \cos(\omega t)\right)^\frac{2}{3} dt + \beta\right).
\] (2.26)

Note that \( \theta(t) \) is in general a quasi-periodic function of time. It is also possible to study the nonlinear system (2.22) using the more general form of the system (2.19). In this instance the system can be rearranged to obtain an equation of the form (2.21) so that the solutions depend on the elliptic integrals and elliptic functions.

**Remark 2.2** The relationship between \( \varphi(t) \) and \( \ell(t) \) given by (2.24) constrains the length and support of the pendulum to oscillate with the same frequency and the mean length of the pendulum to be governed by the gravitational constant. This result can be viewed as follows: although in general the pendulum with periodically, vertically
oscillating support and varying length exhibits complicated dynamics, if we relate the
mean length, \( \ell_0 \), of the pendulum to the gravitational constant and match the frequencies
at which the support oscillates and length varies then we can find an explicit solution to
the equation of motion.

Now let us reconsider Hill’s equation in the form of (2.4) and the transformed equation
(2.2). If Hill’s equation can be transformed into an equation of the form (2.18) then we
may proceed to find solutions using Theorem 2.6. When this is possible it is evident that
\( b(t) \) may be expressed in terms of \( a(t) \), and \( \lambda + Q(t) \) may be written as a function of \( a(t) \)
only, namely

\[
\lambda + Q(t) = -\frac{1}{2} \dot{a}(t) - \frac{1}{4} a^2(t) + e^{-2 \int a(t) dt}.
\]

In this form it is apparent that for the solution \( y(t) \) of (2.4) to be periodic, \( a(t) \) must
be a periodic function of \( t \) with zero mean value. Rewriting equation (2.2) in terms of
\( a(t) \) and recalling the solutions to (2.2) where \( a(t) = -\dot{\omega}(t)/\omega(t) \) are finite and periodic,
we see that all solutions found for the corresponding Hill’s equation are in turn bounded
and hence stable.

\[
\ddot{z}(t) + a(t) \dot{z}(t) + \left( e^{-2 \int a(t) dt} \right) z(t) = 0,
\]

\[
z(t) = \cos \left( \int \left( e^{-2 \int a(t) dt} \right) dt \right),
\]

\[
y(t) = \left( e^{\frac{1}{2} \int a(t) dt} \right) \cos \left( \int \left( e^{-2 \int a(t) dt} \right) dt \right).
\]

Using the definition of \( \omega(t) \) we can rewrite the solution \( y(t) \) as

\[
y(t) = \frac{1}{\sqrt{\bar{\omega}(t)}} \cos(\bar{\omega} t + \Omega(t)), \tag{2.27}
\]

where \( \bar{\omega} \) is the average of the function \( \omega(t) \) and \( \Omega(t) \) is a primitive of the zero-average
function \( \omega(t) - \bar{\omega} \). Note that in general \( y(t) \) is a quasi-periodic function, with two
frequencies \( \bar{\omega} \) and \( \omega_0 \), where \( \omega_0 \) is the frequency of \( \omega(t) \). The solution reduces to a periodic function if and only if \( \bar{\omega} \) is commensurate with \( \omega_0 \). More generally, we can consider the case in which \( \omega(t) \) is a quasi-periodic function with frequencies \( \omega_0, \ldots, \omega_n \). In this instance the solution \( y(t) \) can still be found and is of the form (2.27), but with \( \Omega(t) \) now a quasi-periodic function with the same frequencies as its derivative \( \omega(t) \). This provides an example where we can construct a quasi-periodic Hill’s equation which we can solve, though in general the study of the quasi-periodic Hill’s equation would require very delicate analysis.

As mentioned, all of the solutions found using Theorem 2.6 have been stable solutions. We now consider a slightly modified form of equation (2.17) given by

\[
\frac{d}{dt} \left( \frac{\dot{\theta}(t)}{C \omega(t)} \right) + C \omega(t) \theta(t) = 0, \tag{2.28}
\]

where \( C \) is constant. Using the transformation (2.3) we find the corresponding Hill’s equation for (2.28) given by

\[
\ddot{y} + \left[ \frac{1}{2} \frac{d}{dt} \left( \frac{\dot{\omega}(t)}{\omega(t)} \right) - \frac{1}{4} \left( \frac{\dot{\omega}(t)}{\omega(t)} \right)^2 + C^2 \omega^2(t) \right] y = 0, \tag{2.29}
\]

with corresponding solution

\[
y = \frac{1}{\sqrt{\omega(t)}} \cos \left( C \int \omega(t) \, dt \right).
\]

It can be seen from the solution that if \( C \) is real the solutions to (2.29) are bounded and stable. If \( C \) is purely imaginary, \( (C^2 < 0 \text{ and real}) \) then the solutions are hyperbolic and therefore unstable. Finally if \( C \) has nonzero real and imaginary parts (i.e \( C^2 \notin \mathbb{R} \)), then the solutions are always unstable, see [116]. Comparing equation (2.29) with Hill’s equation, (2.4) we find the relation between \( C \) and \( \lambda \) as

\[
\lambda = \frac{1}{2T} \int_0^T \frac{d}{dt} \left( \frac{\dot{\omega}(t)}{\omega(t)} \right) \, dt - \frac{1}{4T} \int_0^T \left( \frac{\dot{\omega}(t)}{\omega(t)} \right)^2 \, dt + \frac{C^2}{T} \int_0^T \omega^2(t) \, dt, \tag{2.30}
\]
where \( T \) is the period of \( \omega(t) \). Note that the first integral vanishes as the integrand has zero average. Using equation (2.30) we can see how the stability and instability of the solutions found correspond to the stable and unstable intervals of Hill’s equation mentioned in Theorem 2.3. We find that the choice of \( C^2 < 0 \) corresponds to the zeroth interval of instability, \((-\infty, \lambda_0)\) where \( \lambda_0 \) is given by

\[
\lambda_0 = -\frac{1}{4T} \int_0^T \left( \frac{\dot{\omega}(t)}{\omega(t)} \right)^2 dt.
\]

Choosing \( C^2 > 0 \) and purely real, all the solutions are stable. It is possible to generate stable and unstable intervals of (2.29) in \( C \) by modeling \( C \) as a function of another constant with respect to time which goes between real and imaginary values as the constant is varied, for example \( C = \sqrt{\sin \alpha} \), removing the values \( C = 0 \). However this is equivalent to moving in and out of the zeroth region of stability for \( \lambda \). Of course, provided the coefficient of \( y \) in (2.29) is nonconstant, further intervals of instability exist for real \( \lambda \), however they are not found by this class of solutions.

Note, multiplying equation (2.28) by \( C \), the corresponding equation in Hill form is the same. Hence the same properties hold, however now the value \( C = 0 \) need not be omitted.

### 2.3 Instability regions for perturbations of Hill’s equation

We now consider Hill’s equation with the inclusion of damping for which we wish to find the regions of (asymptotic) stability/instability. In this section we recap some of the material in [147, 149] which is later applied to the pendulum with variable length in Chapter 6 and Mathieu’s equation in Appendix A.9. We study systems of the form

\[
\ddot{y}(t) + \left( \lambda + \varepsilon Q(t) \right) \dot{y}(t) + \gamma \dot{y}(t) = 0,
\]

\[\text{(2.31)}\]
where $Q(t)$ is a piecewise continuous periodic function of $t$ and $\varepsilon$ is a parameter representing a smallness factor so the system may be considered as a perturbation of an integrable one. The parameter $\gamma > 0$ is the coefficient of dissipation, which again we shall assume to be small, typically of the order $\varepsilon$ with $\gamma = \varepsilon C$ for some constant $C$. We set the period of $Q(t)$ to $T = 2\pi$. This is done so as to match with the systems studied later in Chapters 5 and 6, though of course we could renormalise such that $T = \pi$ to correspond with the previous results. Taking both $\varepsilon$ and $\gamma$ equal to zero, the system reduces to that of the Harmonic Oscillator, which may be solved trivially. In this instance it is clear that the solution has natural frequency $\omega_0 = \sqrt{\lambda}$. The full system (2.31) can be written as a pair of coupled first order equations, represented in matrix form as

$$\dot{y} = A(t)y, \quad y = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -\lambda - \varepsilon Q(t) & -\gamma \end{pmatrix}. \quad (2.32)$$

**Definition 2.2** Consider the system $\dot{y} = A(t)y$. The Matrix $Y(t)$ which satisfies

$$\dot{Y}(t) = A(t)Y(t),$$

with initial condition

$$Y(0) = I,$$

is called the principal fundamental matrix or matricant of the system.

An equivalent statement to Definition 2.2 is; if we let $y_1, \ldots, y_n$ be the $n$ linearly independent solutions of an $n$ dimensional matrix system $\dot{Y}(t) = A(t)Y(t)$ then the fundamental matrix is $Y = [y_1, \ldots, y_n]$. Note, the matricant $\tilde{Y}$ of the system with $\tilde{A} = -A^T$ (minus the transpose of $A$) satisfies $\tilde{Y}^T Y = I$. 

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Definition 2.3 The matricant \( Y(t) \) taken at the period \( t = T \) gives the matrix

\[
F = Y(T),
\]

which is called the Floquet matrix or monodromy matrix.

It is shown in [149] that

\[
\frac{\partial F}{\partial s} = F_0 \int_0^T Y^{-1} \frac{\partial A}{\partial s} Y \, dt,
\]

where \( s \) is one of the parameters of the system (2.32) and \( F_0 \) is the Floquet matrix of the unperturbed system. Also note that

\[
det F = det Y(T) = det Y(0) = 1,
\]

so the product of the eigenvalues \( \mu_1, \mu_2 \) is equal to 1 and hence

\[
\mu_1 = \frac{1}{\mu_2}.
\]

As mentioned, the system (2.32) without damping and periodic forcing (\( \gamma = \varepsilon = 0 \)) has matrix \( A \) independent of \( t \) and the system can easily be solved. The unperturbed system has matricants

\[
Y_0 = \begin{pmatrix}
\cos (\omega_0 t) & \frac{1}{\omega_0} \sin (\omega_0 t) \\
-\omega_0 \sin (\omega_0 t) & \cos (\omega_0 t)
\end{pmatrix}, \quad \tilde{Y}_0 = \begin{pmatrix}
\cos (\omega_0 t) & \omega_0 \sin (\omega_0 t) \\
-\frac{1}{\omega_0} \sin (\omega_0 t) & \cos (\omega_0 t)
\end{pmatrix},
\]

and Floquet matrix

\[
F_0 = \begin{pmatrix}
\cos (2\pi \omega_0) & \frac{1}{\omega_0} \sin (2\pi \omega_0) \\
-\omega_0 \sin (2\pi \omega_0) & \cos (2\pi \omega_0)
\end{pmatrix}.
\]

Finding the eigenvalues of the Floquet matrix (2.34) yields

\[
\mu_{1,2} = \cos (2\pi \omega_0) \pm i \sin (2\pi \omega_0).
\]
CHAPTER 2. HILL’S EQUATION

If \( \omega_0 \neq k/2 \) for any integer \( k = 1, 2, \ldots \) the eigenvalues are simple, complex conjugate and lie on the unit circle in the complex plane. If \( \omega_0 = k/2 \) then the eigenvalues are double and satisfy

\[
\mu_1 = \mu_2 = (-1)^k.
\]

In this case the Floquet matrix is \( F_0 = (-1)^k I \) and the eigenvalues have two linearly independent eigenvectors. Values of \( \omega_0 = k/2 \) are called the resonance or critical values of the natural frequency. Using equation (2.33) we find the derivatives of \( F \) with respect to the parameters as

\[
\frac{\partial F}{\partial \omega_0} = \frac{1}{2\omega_0} F_0 \begin{pmatrix}
1 - \cos(4\pi\omega_0) & \frac{4\pi\omega_0 - \sin(4\pi\omega_0)}{\omega_0} \\
-\omega_0 (4\pi\omega_0 + \sin(4\pi\omega_0)) & \cos(4\pi\omega_0) - 1
\end{pmatrix},
\]

\[
\frac{\partial F}{\partial \varepsilon} = \frac{1}{2\omega_0} F_0 \int_0^{2\pi} \begin{pmatrix}
\sin(2\omega_0 t) & \frac{1 - \cos(2\omega_0 t)}{\omega_0} \\
-\omega_0 (1 + \cos(2\omega_0 t)) & -\sin(2\omega_0 t)
\end{pmatrix} Q(t) \, dt, \tag{2.35}
\]

\[
\frac{\partial F}{\partial \gamma} = \frac{1}{4\omega_0} F_0 \begin{pmatrix}
\frac{\sin(4\pi\omega_0) - 4\pi\omega_0}{\omega_0} & \frac{1 - \cos(4\pi\omega_0)}{\omega_0} \\
\omega_0 (1 - \cos(4\pi\omega_0)) & -\sin(4\pi\omega_0) - 4\pi\omega_0
\end{pmatrix},
\]

from which an approximation to the Floquet matrix of the system (2.32) with \( \gamma \neq 0, \varepsilon \neq 0 \) can be constructed by Taylor expansion in a neighbourhood of the unperturbed system. Doing so yields

\[
F = F_0 + \frac{\partial F}{\partial \omega_0} \left( \sqrt{\lambda} - \omega_0 \right) + \frac{\partial F}{\partial \varepsilon} \varepsilon + \frac{\partial F}{\partial \gamma} \gamma + h.o.t. \tag{2.36}
\]

The system is then (asymptotically) stable if and only if all the eigenvalues of the Floquet matrix \( F \), lie within the unit circle. If at least one eigenvalue lies outside of the unit circle the system is unstable. For non-critical values of the frequency, i.e \( \omega_0 \neq k/2 \), the
simple eigenvalues when $\gamma = 0$ satisfy $\mu_1 = 1/\mu_2$, thus cannot leave the unit circle when $\varepsilon$ is small. When dissipation is added the Floquet matrix satisfies
\[
\mu_1 \mu_2 = \det \mathbf{F} = 1 - 2\pi \gamma + h.o.t.
\]
Hence a positive coefficient of dissipation causes the eigenvalues to move further into the unit circle and the system becomes asymptotically stable (while $\gamma < 0$ causes them to move out and the system loses stability). For critical values of the natural frequency the eigenvalues satisfy
\[
\mathbf{F} = \begin{pmatrix}
1 + \pi k^{-1} b_k \varepsilon - \pi \gamma & 4\pi k^{-1}(\Delta \omega + 2k^{-1} \varepsilon (c_0 - a_k)) \\
-\pi k(\Delta \omega + 2k^{-1} \varepsilon (c_0 + a_k)) & 1 - \pi k^{-1} b_k \varepsilon - \pi \gamma
\end{pmatrix},
\]

\[
\mu_{1,2} = (-1)^k (1 - \pi \gamma) \pm \pi \sqrt{D},
\]

\[
D = \frac{r_k^2 \varepsilon^2}{k^2} - 4 \left( \sqrt{\lambda - \frac{k}{2}} + \frac{c_0}{k} \varepsilon \right)^2,
\]

where $r_k = \sqrt{a_k^2 + b_k^2}$ and $c_0$, $a_k$ and $b_k$ are defined by
\[
c_0 = \frac{1}{2\pi} \int_0^{2\pi} Q(t) \, dt,
\]
\[
a_k = \frac{1}{\pi} \int_0^{2\pi} Q(t) \cos(kt) \, dt,
\]
\[
b_k = \frac{1}{\pi} \int_0^{2\pi} Q(t) \sin(kt) \, dt. \tag{2.37}
\]

If $D < 0$ then the eigenvalues of $\mathbf{F}$ are complex conjugate and lie on the unit circle when $\gamma = 0$, outside the unit circle if $\gamma < 0$ and inside for $\gamma > 0$. When $D > 0$ the system is unstable if $\sqrt{D} - \gamma > 0$ and hence $D > \gamma^2$. Then the regions of instability in parameter space are where the following inequality is satisfied
\[
4 \left( \sqrt{\lambda - \frac{k}{2}} + \frac{c_0}{k} \varepsilon \right)^2 + \gamma^2 < \frac{r_k^2}{k^2} \varepsilon^2.
\]
Chapter 3

High and low frequency forcing

We return to the phenomena noticed in Section 2.1, where it was found that a class of Hill’s equation has (linearly) stable solutions provided the forcing frequency is high enough, see Theorem 2.5. Here we focus on nonlinear systems and show that an analogous result holds when the frequency of the forcing is either sufficiently high or sufficiently low. We consider integrable systems, which we perturb by adding a forcing term. When dealing with forced systems, it is well known that, in the extreme situations in which the oscillation of the forcing is either very slow or very fast, the dynamics can be strikingly different with respect to the case in which the forcing period is comparable with the motions of the unperturbed, integrable system [42, 135, 68, 11, 51, 32, 61]; a classical example is provided by the pendulum with oscillating support [160, 41, 98, 42, 26] which is a perturbation of the simple pendulum. The standard technique used to attack the problem is the averaging method; however, a rigorous implementation is quite nontrivial, as it involves KAM-like arguments. In particular we study the case where the forcing amplitude is far beyond the perturbation regime and show how to reduce the problem to results already available in the literature. In doing so we prove that most of the phase space is filled by KAM invariant tori when the period of the forcing is sufficiently large or sufficiently small. A similar scenario has been studied in [112, 113, 114], in which the authors consider KAM tori which are far from the origin, that is, $|x(t)| + |\dot{x}(t)|$ sufficiently
large and forcing with frequency $0 < \omega < 1$. In doing so they show that motions are bounded, as they are trapped inside the KAM tori and cannot move to infinity. However, to prove boundedness one has to prove the existence of confining KAM tori far away from the origin. Hence only the asymptotic behaviour of the potential really counts, and in general one needs a condition on the growth of the potential at infinity – besides smoothness conditions, see [112, 113, 114]. On the contrary, here we consider a region with distance of order $O(1)$ from the origin and show that, not only are the orbits of the perturbed system bounded, but that they remain close to those of the unperturbed system. To study the existence of KAM tori in a fixed region of phase space of a forced system, one needs information about the potential in that region: this explains why the assumptions we shall require on the potential are stronger, as they are not just asymptotic properties. As a concrete example, we study the model of a forced cubic oscillator, which has been extensively investigated in the literature [122, 18, 23, 68], however the results are formulated for general Hamiltonians of forced systems which satisfy a few conditions which we outline in Section 3.1.

Consider a cubic oscillator subject to a periodic driving force. The Hamiltonian describing the system is

$$
\mathcal{H}(y, x, t) = \frac{y^2}{2} + (1 + \mu f(\omega t)) \frac{x^4}{4},
$$

(3.1)

where $(y, x) \in \mathbb{R}^2$, $\mu \in \mathbb{R}$, $\omega \in \mathbb{R}$ is the frequency of the driving force and $f$ is a $2\pi$-periodic analytic function of $t$ with zero average and $\|f\|_{\infty} = 1$. The phase space for the system is $\mathbb{R}^2 \times T_\omega$, with $T_\omega = \mathbb{R}/(2\pi/\omega)\mathbb{Z}$ and the corresponding equations of motion are

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -(1 + \mu f(\omega t)) x^3, \\
\dot{t} &= 1,
\end{aligned}
$$

(3.2)

where the dots denote derivative with respect to time $t$. 

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For $\mu = 0$ the system is integrable, so that the full phase space is filled by invariant tori, with $t$ increasing linearly in time with frequency 1 and $(x, y)$ moving on a closed orbit in the plane with frequency $\Omega$ depending on the initial data.

Fix arbitrarily $\Omega_2 > \Omega_1 > 0$ and consider, for the unperturbed cubic oscillator, the closed orbits $C_1$ and $C_2$ with frequencies $\Omega_1$ and $\Omega_2$, respectively. Call $D_0 \subset \mathbb{R}^2$ the bounded region enclosed by $C_1$ and $C_2$ and set $D = D_0 \times T_\omega$. For $\mu$ small enough, we can apply KAM theorem [11], so as to obtain the existence of most of the invariant tori inside $D$, independently of the value of the frequency $\omega$, see Appendix A.6. This well known result, see for example [72, 137], is captured by the following theorem.

**Theorem 3.1** For $\mu$ small enough the set of existing invariant tori in $D$ for the system with Hamiltonian (3.1) leaves out a set with relative measure $O(\sqrt{\mu})$.

In this Chapter we want to show that the same kind of results extend by removing the condition of smallness on $\mu$, provided the frequency $\omega$ is either large enough or small enough. In particular, by increasing $\omega$ from 0 to infinity, a double transition regularity-chaos-regularity occurs. This is supported in Section 3.3 by numerical results, which also give evidence that the regularity regime extends to wider ranges of the parameters for which the analytical results do not apply. In Section 3.4 a brief discussion is provided on how to extend the analysis to more general systems.

### 3.1 High frequency regime

We consider first the case of $\omega$ large and we set $\omega = 1/\varepsilon$, with $\varepsilon$ small. We can formulate our result for such a case in a more general context. In action-angle variables, the Hamiltonian (3.1) becomes, see Appendix A.5

$$H(I, \varphi, t) = H_0(I) + \mu H_1(I, \varphi, \omega t),$$

(3.3)
where
\[
\mathcal{H}_0(I) = \frac{1}{4} \left( \frac{3I}{T} \right)^{4/3}, \quad \mathcal{H}_1(I, \varphi, t) = \frac{1}{4} \left( \frac{3I}{T} \right)^{4/3} \text{cn}^4(T \varphi) f(t),
\]
(3.4)
with \(\text{cn} \varphi := \text{cn}(\varphi, 1/\sqrt{2})\) and \(T := 2K(1/\sqrt{2})/\pi\), where \(\text{cn}(\varphi, k)\) and \(K(k)\) denote respectively the cosine-amplitude function and the complete elliptic integral of the first kind with elliptic modulus \(k\), see Appendix A.3.

The corresponding Hamilton equations are
\[
\begin{aligned}
\dot{\varphi} &= \Omega_0(I) + \mu \partial_I \mathcal{H}_1(I, \varphi, \omega t), \\
\dot{I} &= -\mu \partial_{\varphi} \mathcal{H}_1(I, \varphi, \omega t),
\end{aligned}
\]
(3.5)
where \(\Omega_0(I) := \partial_I \mathcal{H}_0(I) = (3I/T^4)^{1/3}\). In terms of the variables \((I, \varphi, t)\) the domain \(D\) defined above becomes \(D = \mathcal{I}_0 \times \mathbb{T} \times \mathbb{T}_\omega\), where \(\mathbb{T} = \mathbb{T}_1\) and \(\mathcal{I}_0 = \{I \in \mathbb{R}_+ : I_1 \leq |I| \leq I_2\}\), with \(I_1 = \Omega_3^3 T^4 / 3\) and \(I_2 = \Omega_3^3 T^4 / 3\).

More generally we shall consider Hamiltonians of the form (3.3) and make the following assumptions on \(\mathcal{H}_0\) and \(\mathcal{H}_1\) (trivially satisfied in the case (3.4)).

**Assumption 1** Assume the Hamiltonian function (3.3) to be real-analytic in a domain \(D := \mathcal{I}_0 \times \mathbb{T} \times \mathbb{T}_\omega\), where \(\mathcal{I}_0\) is an open subset of \(\mathbb{R}\).

**Assumption 2** Assume \(I \mapsto \Omega_0(I) := \partial_I \mathcal{H}_0(I)\) to be a local diffeomorphism on \(\mathcal{I}_0\).

**Assumption 3** Assume \(\langle \mathcal{H}_1(I, \varphi, \cdot) \rangle := \int_\mathbb{T} \frac{dt}{2\pi} \mathcal{H}_1(I, \varphi, t) = 0\), \(\forall (I, \varphi) \in \mathcal{I}_0 \times \mathbb{T}\).

In (3.3) rescale time \(t \to \tau = \omega t\). Then the equations of motion (3.5) become
\[
\begin{aligned}
\varphi' &= \varepsilon \Omega_0(I) + \varepsilon \mu \partial_I \mathcal{H}_1(I, \varphi, \tau), \\
I' &= -\varepsilon \mu \partial_{\varphi} \mathcal{H}_1(I, \varphi, \tau),
\end{aligned}
\]
(3.6)
where the dashes denote derivative with respect to time \(\tau\).
We can apply Neishtadt’s averaging theorem \cite{129, 130} to cast the system into the form

\[
\begin{align*}
\varphi' &= \varepsilon (\Omega_0(I) + \mu \partial_I V_\varepsilon(I, \varphi) + \mu \partial_I R_\varepsilon(I, \varphi, \tau)), \\
I' &= -\varepsilon (\mu \partial_\varphi V_\varepsilon(I, \varphi) + \mu \partial_\varphi R_\varepsilon(I, \varphi, \tau)),
\end{align*}
\tag{3.7}
\]

where $V_\varepsilon$ and $R_\varepsilon$ are suitable analytic functions, with $V_\varepsilon(I, \varphi) = \langle H_1(I, \varphi, \cdot) \rangle + O(\varepsilon)$ and $R_\varepsilon$ an exponentially small remainder, that is $|R_\varepsilon| \leq C_\exp(-c/\varepsilon)$ for some positive constants $c$ and $C$. The change of coordinates is canonical and $\varepsilon$-close to the identity — in order not to overwhelm the notations, we denote the new variables with the same letters as the old ones.

By Assumption 3 the average of $H_1$ vanishes, hence $V_\varepsilon$ is a correction of order $\varepsilon$ to $H_0$. So one can perform a further close-to-identity change of coordinates which leads to the equations

\[
\begin{align*}
\varphi' &= \varepsilon \overline{\Omega}_\varepsilon(I) + \varepsilon \mu \partial_I \overline{R}_\varepsilon(I, \varphi, \tau), \\
I' &= -\varepsilon \mu \partial_\varphi \overline{R}_\varepsilon(I, \varphi, \tau),
\end{align*}
\tag{3.8}
\]

where $\overline{\Omega}_\varepsilon(I) = \Omega_0(I) + O(\varepsilon)$ and $\overline{R}$ is still exponentially small (and once more we still denote by $(I, \varphi)$ the transformed coordinates). The corresponding Hamiltonian is

\[
\overline{H}(I, \varphi, \varepsilon) = \varepsilon \left( \overline{H}_\varepsilon(I) + \mu \overline{R}_\varepsilon(I, \varphi, \tau) \right).
\tag{3.9}
\]

The overall change of coordinates leading to (3.9) is close to the identity within $O(\varepsilon)$ and hence, up to a region with measure $O(\varepsilon)$, the domain $\mathcal{D}$ is transformed into a region enclosed between two KAM invariant tori. By studying the Hamiltonian $\overline{H}(I, \varphi, \varepsilon)/\varepsilon$ we see that we can apply once more KAM theorem and conclude that most of the unperturbed tori for the Hamiltonian $\overline{H}_\varepsilon(I)$ still exist when the perturbation $\overline{R}_\varepsilon(I, \varphi, \tau)$ is switched on. Since now the perturbation is exponentially small, the relative measure of the tori which are destroyed is exponentially small in $\varepsilon$. To go back to the original coordinates, we have to scale back time. So we obtain the following result.
Theorem 3.2 Consider a system with Hamiltonian (3.3) which satisfies Assumptions 1 to 3. For any value of \( \mu \), for \( \omega \) large enough the domain \( \mathcal{D} \) is filled by KAM invariant tori, up to a region whose relative measure is \( O(1/\omega) \). Apart from a thin region close to the boundary, the invariant tori leave out a region with measure exponentially small in \( 1/\omega \).

Note that Assumption 3 is needed here, contrary to Theorem 3.1, to ensure that the averaged system is integrable (such a condition is automatically satisfied for \( \mu \) small, without any assumption on \( \mathcal{H}_1 \)).

3.2 Low frequency regime

Now consider (3.1) with \( \omega = \varepsilon \). We can reason as in [106] (see also [70, 71, 64]). Fix \( |\mu| < 1 \), so that \( 1 + \mu f(\omega t) > 0 \). We rewrite the equations of motion (3.2) as

\[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -a(\varepsilon t) x^3, \\
&\quad a(t) := 1 + \mu f(t).
\end{aligned}
\]  

(3.10)

Then the argument proceeds through the following steps.

First, through a time-dependent canonical change of coordinates \((x, y) \mapsto (I, \phi)\), with

\[
\begin{aligned}
x &= \left(\frac{3I}{T}\right)^{1/3} (a(\varepsilon t))^{-1/6} \text{cn}(T\phi), \\
y &= -\left(\frac{3I}{T}\right)^{2/3} (a(\varepsilon t))^{1/6} \text{sn}(T\phi) \text{dn}(T\phi),
\end{aligned}
\]  

(3.11)

where \( \text{sn} \phi \) and \( \text{dn} \phi \) are the sine-amplitude and delta-amplitude functions with modulus \( k = 1/\sqrt{2} \), respectively, see Appendices A.3 and A.5, one writes (3.10) as the Hamilton equations corresponding to the Hamiltonian

\[
\begin{aligned}
\mathcal{H}(I, \phi, t) &= \mathcal{H}_0(I, \phi, t) + \frac{\varepsilon}{6} \frac{3I}{T} \text{cn}(T\phi) \text{sn}(T\phi) \text{dn}(T\phi) \frac{b(\varepsilon t)}{a(\varepsilon t)}, \\
\mathcal{H}_0(I, \phi, t) &= \frac{1}{4} \left(\frac{3I}{T}\right)^{4/3} (a(\varepsilon t))^{1/3},
\end{aligned}
\]  

(3.12)
where \( b(t) = \dot{a}(t) \) and \( T = 2K/\pi \). Next, in order to eliminate the dependence on time in the leading term one makes the change of coordinates \( (I, \varphi, t) \mapsto (p, q, s) \), with

\[
p = \mathcal{H}(I, \varphi, t), \quad q = t, \quad s = \varphi,
\]

which leads to the Hamiltonian, see Appendix A.5

\[
\mathcal{A}(p, q, s) = \mathcal{A}_0(p, \varepsilon q) + \varepsilon \mathcal{A}_1(p, \varepsilon q, s),
\]

\[
\mathcal{A}_0(p, q) := \frac{T (4p)^{3/4}}{3 (a(q))^{1/4}},
\]

for some function \( \mathcal{A}_1 \) of order 1 in \( \varepsilon \). Then one may perform a further change of variables \((p, q, s) \mapsto (J, \phi, s)\) into action-angle variables for \( \mathcal{A}_0 \), so yielding the Hamiltonian

\[
\mathcal{B}(J, \phi, s) = \mathcal{B}_0(J) + \varepsilon \mathcal{B}_1(J, \varepsilon \phi, s),
\]

\[
\mathcal{B}_0(J) = (\kappa J)^{3/4}, \quad \frac{1}{\kappa} := \frac{1}{4} \left( \frac{3}{T} \right)^{4/3} \int_0^{2\pi} (a(q))^{1/3} \frac{dq}{2\pi},
\]

for a suitable function \( \mathcal{B}_1 \) of order 1 in \( \varepsilon \).

Finally we integrate the Hamiltonian equations corresponding to (3.15) between \( s = 0 \) and \( s = 2\pi \). Denote by \((J(s), \phi(s))\) the solution; then, defining \( \psi(s) = \varepsilon \phi(s) \) and setting \((\hat{J}, \hat{\psi}) = (J(2\pi), \psi(2\pi))\) and \((J, \psi) = (J(0), \psi(0))\), we obtain the twist map

\[
\begin{cases}
\hat{J} = J + \varepsilon^2 F(J, \psi), \\
\hat{\psi} = \psi + \varepsilon \Omega_0(J) + \varepsilon^2 G(J, \psi), \quad \Omega_0(J) = \frac{3\pi}{2} \kappa^{3/4} J^{-1/4},
\end{cases}
\]

for suitable analytic functions \( F \) and \( G \).

Therefore we can apply Moser’s twist theorem, see Appendix A.6, and conclude that any invariant curve with Diophantine rotation number still exists for \( \varepsilon \) small enough. For fixed \( \varepsilon \), the relative measure of the invariant curves which still exist in a given region of the cylinder is \( O(\sqrt{\varepsilon}) \) [108, 160].

Coming back to the original coordinates we obtain the existence of a large measure set of invariant tori for the continuous flow which has the twist map (3.16) as Poincaré
section at times which are multiples of $2\pi$. We can summarise the discussion by the following statement.

**Theorem 3.3** Consider the system with Hamiltonian (3.1) and fix $\mu \in (-1, 1)$. For $\omega$ small enough the domain $D$ is filled by KAM invariant tori, up to a region whose relative measure is $O(\sqrt{\omega})$.

### 3.3 Numerical results

Numerically we may illustrate the scenarios considered in the previous sections as well as provide some insight into cases which are not covered by the analysis. We study the system (3.1) with $f(\omega t) = \cos \omega t$. In particular we consider three situations, $|\mu| < 1$, $|\mu| > 1$ and $|\mu| = 1$, with both high and low frequency forcing.

To check the dynamics we take 10 000 pseudo-random initial conditions within a square $[-2, 2] \times [-2, 2]$ from the phase plane $(x, y)$. The chosen numerical integration method is a Störmer-Verlet scheme with variable step size. The Störmer-Verlet method is a second order symplectic scheme, details of which may be found in [110].

After an initial transient period, the trajectories are checked to ascertain how their asymptotic behaviour has changed with respect to the trajectories of the unperturbed system with the same initial conditions. If most of the orbits have remained close to the corresponding orbits of the unperturbed system, then we say that the system is “stable”\(^1\). This is expected to occur when the system is well within the KAM regime: the majority of the unperturbed tori still exist, albeit slightly deformed, so that every orbit either lies on a torus or is trapped between two surviving tori. However it is possible that the trajectories of the perturbed system do not remain close, but are still bounded. This

\(^1\)Note that here the use of the term “stable” is different from linear stability used in Chapter 2 and asymptotic stability which is used in later chapters. Here the use of the term stability is more close to that of structural stability, although it is still not the same; as the system is non-dissipative the equilibrium point is not hyperbolic.
can happen as we are moving out of the KAM regime; most of the tori are destroyed, with a few of them still existing and undergoing much larger deformations. We refer to such a case by saying that the system is “bounded”. Numerically it is difficult to classify trajectories as unbounded, as a trajectory which appears unbounded may be bounded within a very large region. Therefore, we class the trajectories as unbounded once their amplitude exceeds 30 in either the $x$ or $y$ direction, and class the system as “unbounded” if any trajectory is found to be so. When this happens, nearly all (if not all) KAM tori are expected to be destroyed, at least in the region investigated, otherwise any of them would confine the orbits inside. We note, however, that, even though KAM theory no longer applies in this case, one can still have invariant curves of a different kind, such as cantori (see for instance [144]), so that it is quite possible for some trajectories to be unbounded whilst others remain bounded (and possibly even close to those of the unperturbed system).

First we consider the scenario where $|\mu| < 1$, in particular we choose $\mu = 0.8, 0.9$ and 0.95; some numerical results are shown in Table 3.1. As proved in the previous sections, for $\omega$ sufficiently large or small, the dynamics are stable, whilst between these two extremes, the orbits move away from those of the unperturbed system and there exists unbounded trajectories. In Figure 3.1 we show some example orbits corresponding to the initial conditions $(x, y) = (1, 1)$ with $\mu = 0$ in Figure 3.1(a) and $\mu = 0.8$ in (b), (c) and (d). It may be seen that the perturbed orbits remain close to the unperturbed system for suitable $\omega$. With $\mu$ getting close to 1, one has to take $\omega$ increasingly small or increasingly large for the system to be stable.

The results in Table 3.1 show that for $\mu = 0.8$ the orbits become unstable when $\omega > 0.2$. We see in Figure 3.2(b) and (c) that as $\omega$ is increased from 0.2 to 0.4, many of the tori are broken and cantori separate the invariant KAM tori, creating larger and larger gaps. However, as many of the tori still exist in the region investigated, the orbits
Table 3.1: Stability and boundedness results for choices of $|\mu| < 1$ and various values of $\omega$. Orbits are classed as stable if they remain close to the corresponding orbit of the unperturbed system with the same initial conditions, unbounded if their amplitude is greater than 30 in either direction and bounded otherwise, see text for details.

<table>
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<tr>
<th>$\omega$</th>
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<th>$\mu = 0.9$</th>
<th>$\mu = 0.95$</th>
</tr>
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<td>Bounded</td>
<td>Bounded</td>
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<tr>
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<tr>
<td>14.0</td>
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</tr>
</tbody>
</table>

Figure 3.1: Orbits for the system (3.2) with initial conditions $(x, y) = (1, 1)$. In Figure (a) $\mu = 0$. In Figures (b), (c) and (d) $\mu = 0.8$ and $\omega = 0.0001$, 0.2 and 14 respectively.
CHAPTER 3. HIGH AND LOW FREQUENCY FORCING

Figure 3.2: Poincaré maps for the system (3.1) with $\mu = 0.8$ and various initial conditions, showing the break up of the KAM tori. The parameter $\omega = 0.2$ and $0.4$ in (a) and (b) respectively.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\mu = 1$</th>
<th>$\mu = 1.2$</th>
<th>$\mu = 2$</th>
<th>$\mu = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
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</tr>
<tr>
<td>0.0002</td>
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<td>Stable</td>
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</tr>
<tr>
<td>0.0003</td>
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<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
</tr>
<tr>
<td>0.0004</td>
<td>Bounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
</tr>
<tr>
<td>0.0005</td>
<td>Bounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
</tr>
<tr>
<td>0.0010</td>
<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
</tr>
<tr>
<td>13</td>
<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
<td>Unbounded</td>
</tr>
<tr>
<td>14</td>
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<td>Stable</td>
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<td>Unbounded</td>
</tr>
<tr>
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<td>Unbounded</td>
</tr>
<tr>
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<td>Unbounded</td>
</tr>
<tr>
<td>20</td>
<td>Stable</td>
<td>Stable</td>
<td>Stable</td>
<td>Stable</td>
</tr>
</tbody>
</table>

Table 3.2: Stability and boundedness results for choices of $|\mu| \geq 1$ and various values of $\omega$. Orbits are classed as stable if they remain close to the corresponding orbit of the unperturbed system with the same initial conditions, unbounded if their amplitude is greater than 30 in either direction and bounded otherwise, see text for details.

remain bounded. Increasing $\omega$ further causes more of the KAM tori to break and for $\omega = 0.8$ the orbits are no longer confined in the region $[-30, 30] \times [-30, 30]$.

For $|\mu| \geq 1$ the analysis in the previous sections can only be applied when the system undergoes high frequency forcing. Numerically we find that the system is also stable with low frequency forcing, however $\omega$ must be taken considerably smaller than the cases where $|\mu| < 1$. This is not true for high frequency forcing, where similar orders of $\omega$ (compared with the cases where $|\mu| < 1$) are sufficient for the dynamics to become
stable. Some numerical results are presented in Table 3.2. In Figure 3.3 we show some example orbits with $\mu \geq 1$. Similarly to the case where $|\mu| < 1$ it is evident that the perturbed orbits remain close to that of the unperturbed system provided $\omega$ is suitably chosen. In Figure 3.3(f) we see that, although the system is classed as unbounded for $\omega = 14$ and $\mu = 5$, it is still possible to find initial conditions for which the orbit remains close to the unperturbed system.

![Figure 3.3: Orbits for the system (3.2) with initial conditions $(x, y) = (1, 1)$. In Figures (a), (b) and (c) $\omega = 0.0001$ and in Figures (d), (e) and (f) $\omega = 14$. The parameter $\mu = 1, 2$ and 5 in Figures (a, d), (b, e) and (c, f) respectively.](image)

### 3.4 Further remarks

More generally one can write the Hamiltonian of a forced cubic oscillator as

$$ H(y, x, t) = \frac{1}{2} my^2 + (a + \mu f(\omega t)) \frac{x^4}{4}. \quad (3.17) $$

However, the form (3.1) is not restrictive, since we can reduce (3.17) to that form by rescaling both variables $x$ and $y$ and redefining the parameter $\mu$. 
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Other generalisations could be easily envisaged. For instance any potential $V(x)$ yielding closed orbits in a region encircling the origin could be considered. For example one could take a potential $x^{2n}/2n$, $n \geq 1$ instead of $x^4/4$: the unperturbed system would still be integrable, so that the analysis of Section 3.1 would apply immediately. Also the discussion in Section 3.2 could be easily adapted to cover such a case; see [71, 106] for details. Also, less regularity can be required for the driving force. Finally, one could consider quasi-periodically forced systems, as in [114], in the case in which all components of the frequency vectors are small or large.

Coming back to our system (3.1), the condition that $f$ has zero average could be relaxed. In fact, for Theorem 3.2 to hold, what we really need is that $1 + \mu \langle f(\cdot) \rangle > 0$, so that the averaged system is integrable. On the other hand, Theorem 3.3 requires $1 + \mu f(\cdot) > 0$. In particular, if $\langle f(\cdot) \rangle = 0$ and $\|f\|_\infty = 1$, this excludes the cases $\mu \geq 1$.

However, if the average of $1 + \mu f(\omega t)$ is positive, one can argue that the potential remains positive for most of time, so one can conjecture that the condition $1 + \mu \langle f(\cdot) \rangle > 0$ might also be sufficient in the low frequency regime. Furthermore, we have shown numerically that the orbits remain close to those of the unperturbed system if the forcing frequency is sufficiently low, even when $|\mu| \geq 1$. It would be interesting to investigate the issue in more detail by means of analysis, see the comments in Chapter 8. Further comments are also given in Appendix A.9, where the results of this chapter are related to the stability regions of Mathieu’s equation.
Chapter 4

Numerical analysis

This chapter is devoted to the details of the numerical simulations which were used to obtain the data presented in Chapters 5, 6 and 7. In particular, we explain some of the techniques used and the justification of choices made regarding numerical simulations. The chapter is organised as follows. In Section 4.1 the method used to classify the attractors is outlined. In Section 4.2 we provide details about the selection of initial conditions, as well as a justification of the number of initial conditions used. In Section 4.3 we talk about the numerical integration schemes utilised, in particular Section 4.3.1 gives an in depth explanation of analytic continuation, which is the primary integration method used in the later chapters. Finally in Section 4.4 we discuss the technique of cell mapping.

4.1 Classification of initial conditions

All of the simulations used in Chapters 5, 6 and 7 utilised a solution library to classify the limiting behaviour of trajectories. Originally, solutions were classified using the area of phase space they occupied in conjunction with their period. This method posed a number of disadvantages, and was discarded early on in favour of the solution library. One of the main disadvantages of the original approach is that the user is required to know all of the attractive solutions which exist for the system under the given parameter
values, prior to running the simulation. This of course results in extra work for the user and can lead to solutions being missed if the user is not aware of them. Furthermore, as the parameter values are varied, the attractive solutions also vary. Slight variations in the amplitude and location in phase space of attractors or bifurcations which the user is unaware of may cause problems in the classification of trajectories limiting behaviour. A library of solutions, which trajectories can be checked against and is built up as new solutions are found during the course of the simulation, overcomes all of the above mentioned problems.

4.2 Selection of initial conditions

In Chapters 5 and 6 initial conditions were chosen using a stream of pseudo-random numbers, which were scaled so as to uniformly cover the desired region of phase space. A pseudo-random sequence is one that is generated by a deterministic process, but shows no discernable pattern. As such the same sequence of numbers may be used for each simulation. In Chapter 7, for ease of coding, we also took initial conditions from a uniform grid covering the region of phase space investigated. In the literature, both approaches have been used, for example in [18, 155] the authors use pseudo-random initial conditions, and in [22, 109] the authors used a grid with uniform spacing. There appears to be no justification either way, and in much of the literature the authors do not specify how initial conditions were selected. One of the advantages of a pseudo-random stream over a uniform mesh of initial conditions is that it is easier to add extra data from more initial conditions if greater accuracy is required. Further to this, if the simulation is unexpectedly interrupted, the data obtained is still distributed across all of the desired region of phase space. This is not true for a mesh of initial conditions, where unexpected interruptions result in only partial coverage of the phase space, for which it is much more difficult to top up. However, as mentioned previously, in Chapter 7 a mesh
of initial conditions is used. Therefore in Section 4.2.1 we compare the results obtained using the two methods and show that statistically the methods are equivalent.

Since basins of attraction are computed by taking a large but finite discrete set of initial conditions (either on a mesh or uniformly and randomly distributed), an error is always produced by using the fraction of trajectories converging towards a given attractor to estimate the corresponding relative area. The following statistical result gives the confidence interval for results obtained by Monte Carlo simulation; see [127]. If \( p \) and \( \hat{p} \) are the actual and estimated probability, respectively, of landing in a given basin of attraction, then we have

\[
\hat{p} - z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{N}} < p < \hat{p} + z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{N}},
\]

where \( N \) is the sample size (the number of initial conditions used for the simulation);

<table>
<thead>
<tr>
<th>N</th>
<th>10000</th>
<th>50000</th>
<th>100000</th>
<th>200000</th>
<th>300000</th>
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<th>500000</th>
<th>600000</th>
<th>1000000</th>
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<td>0.1910</td>
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<td>0.0780</td>
<td>0.0675</td>
<td>0.0604</td>
<td>0.0551</td>
<td>0.0427</td>
</tr>
<tr>
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<td>0.1859</td>
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<td>0.1074</td>
<td>0.0930</td>
<td>0.0832</td>
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<td>0.0588</td>
</tr>
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<td>0.1107</td>
<td>0.0990</td>
<td>0.0904</td>
<td>0.0700</td>
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<td>0.1431</td>
<td>0.1240</td>
<td>0.1109</td>
<td>0.1012</td>
<td>0.0784</td>
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<td>0.1640</td>
<td>0.1420</td>
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<td>0.1160</td>
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<td>0.1550</td>
<td>0.1386</td>
<td>0.1265</td>
<td>0.0980</td>
</tr>
</tbody>
</table>

Table 4.1: Largest expected error for various basin sizes and number of initial conditions \( N \) using equation (4.1) with a 95% confidence level.

the variable \( z_{\alpha/2} \) is the so-called \( z \) value, which ensures a \((1 - \alpha) \times 100\%\) confidence level [127]. A confidence level of 95% results in the error values shown in Table 4.1. For example if we take \( N = 10\,000 \) we can see that if an estimated basin of attraction covers 99.5% of the phase space, then we can state with 95% confidence that the actual basin
of attraction will be \((99.5 \pm 0.1386)\%\). Similarly, if the estimated basin covers 0.5% of
the phase space, the actual basin of attraction will also be \((0.5 \pm 0.1386)\%\); although
the error is the same, the error relative to the basin size is much greater for small basins
of attraction. Thus when the basins of attraction are small, it is necessary to use more
initial conditions. From equation (4.1) it is evident that the error is proportional to one
over the square root of the sample size. Therefore it is necessary to take many more
initial conditions to improve the estimates with any significance.

### 4.2.1 Grid versus pseudo-random

As previously mentioned, selecting initial conditions from a stream of pseudo-random
numbers is often advantageous when compared to taking a mesh of initial conditions.
However, for the numerics carried out in Chapter 7, using a uniform mesh of initial
conditions is preferable as it removes many complexities in numerical implementation.
In the following we compare the use of uniformly distributed pseudo-random initial
conditions against that of a uniform mesh of initial conditions. The aim is to verify
whether the two methods of selecting initial conditions are statistically equivalent. To
check that the two methods of initial condition selection are equivalent, we use data
obtained for the sizes of basins of attraction for the system of the pendulum with
oscillating support. This system is chosen so as to fit with the numerical studies later
on, in particular Chapter 7. Throughout this section the chosen numerical integration
scheme is MATLAB's ODE113. Although it is found in later sections that ODE113 is not
the best numerical integrator for the problem, it is easy to use and is sufficient for the
purpose of comparing the two methods of selecting initial conditions.

To make comparisons, we use two sets of random initial conditions, taken from
a stream of pseudo-random numbers. The results using pseudo-random numbers are
compared with those created by using different meshes of initial conditions and a \(\chi^2\) test
is implemented to check if there is evidence that the results from the two approaches are significantly different.

First data using both methods of initial condition selection are obtained. This we refer to as the “observed” results and denote the observed results in the $i$-th row and $j$-th column by $O_{i,j}$.

<table>
<thead>
<tr>
<th>Number of points in BOA</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>Row sum</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Random1</strong></td>
<td>$O_{1,1}$</td>
<td>$O_{1,2}$</td>
<td>$O_{1,3}$</td>
<td>$RS_1$</td>
</tr>
<tr>
<td><strong>Mesh1</strong></td>
<td>$O_{2,1}$</td>
<td>$O_{2,2}$</td>
<td>$O_{2,3}$</td>
<td>$RS_2$</td>
</tr>
<tr>
<td><strong>Column sum</strong></td>
<td>$CS_1$</td>
<td>$CS_2$</td>
<td>$CS_3$</td>
<td>$N$</td>
</tr>
</tbody>
</table>

Table 4.2: Example table of observed results for random initial conditions and those chosen from a mesh, with three attractors present, $A_1$, $A_2$ and $A_3$.

The “expected” results $E_{i,j}$ are then calculated as

$$E_{i,j} = \frac{RS_i \times CS_j}{N},$$

where $RS_i$ is the sum of the $i$-th row, $CS_j$ is the sum of the $j$-th column and $N$ is the total, i.e

$$N = \sum_i RS_i = \sum_j CS_j.$$

<table>
<thead>
<tr>
<th>Number of points in BOA</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Random</strong></td>
<td>$E_{1,1}$</td>
<td>$E_{1,2}$</td>
<td>$E_{1,3}$</td>
</tr>
<tr>
<td><strong>Mesh</strong></td>
<td>$E_{2,1}$</td>
<td>$E_{2,2}$</td>
<td>$E_{2,3}$</td>
</tr>
</tbody>
</table>

Table 4.3: Example table of expected results for random initial conditions and those chosen from a mesh, with three attractors present, $A_1$, $A_2$ and $A_3$.

Then the $\chi^2$ test statistic is defined as, [84]

$$\chi^2 = \sum_i \sum_j \frac{(O_{i,j} - E_{i,j})^2}{E_{i,j}}.$$
To understand the meaning of the value obtained for test statistic $\chi^2$, we must first calculate the number of degrees of freedom. The number of degrees of freedom is equal to $(\text{rows} - 1) \times (\text{columns} - 1)$, in our case 2. Under the null hypothesis, that the results do not depend on the approach used to generate the initial conditions, the test statistic is a random observation from the $\chi^2$ distribution with 2 degrees of freedom. Then the probability that under the null hypothesis the deviation of the observed results from the expected results is due to chance alone can be determined. It is standard to have a critical region corresponding to a probability of 5%, [84]. This means that the test statistic is compared against the 95\textsuperscript{th} percentile of $\chi^2(2)$, i.e. 5.991. The hypothesis that the results from a uniform mesh of initial conditions are equivalent to those generated using uniformly distributed random initial conditions is rejected if the test statistic is greater than 5.991.

Tables 4.4 to 4.7 show the observed results for the system of the pendulum with oscillating support. For the chosen parameter values there exist three (asymptotically) stable attractive solutions which we refer to as UFP, DO2 and DO4, further details on these attractors may be found in Section 5.5. The values of $\chi^2$ are calculated separately for each mesh, and each set of pseudo-random initial conditions, so that the observed data may be written as in Table 4.2. Tables 4.4 to 4.7 show that, provided a large enough number of initial conditions is chosen, the hypothesis that the results produced with the two methods are statistically equivalent is not rejected.

<table>
<thead>
<tr>
<th>Number of points in BOA</th>
<th>UFP</th>
<th>DO2</th>
<th>DO4</th>
<th>$\chi^2_{R1}$</th>
<th>$\chi^2_{R2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random1</td>
<td>1721</td>
<td>7945</td>
<td>335</td>
<td>N/A</td>
<td>5.515</td>
</tr>
<tr>
<td>Random2</td>
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<td>7940</td>
<td>282</td>
<td>5.515</td>
<td>N/A</td>
</tr>
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<td>1.790</td>
<td>11.653</td>
</tr>
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<td>1.084</td>
</tr>
<tr>
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<td>8104</td>
<td>368</td>
<td>0.999</td>
<td>9.970</td>
</tr>
</tbody>
</table>

Table 4.4: The observed results obtained using approximately 10,000 initial conditions. “Random 1” and “Random 2” correspond to 2 different sets of uniformly distributed pseudo-random initial conditions. Mesh 1, 2 and 3 correspond to meshes of initial conditions with spacing $(\Delta\theta, \Delta\theta') = (0.0628, 0.08), (0.0785, 0.0640)$ and $(0.1256, 0.04)$ respectively.
4.3 Numerical integrators

In Chapters 5, 6, and 7, three numerical methods are used to integrate the trajectories which evolve from the chosen initial conditions. Of these, the most easy to implement is MATLAB’s ode113. ode113 uses a variable order Adams-Bashforth-Moulton method which is a standard approach for solving ODEs of the form (5.1) and (6.1). However,
whilst MATLAB offers ease of implementation, the simulations often run much more slowly than programs written in C. This is the case as MATLAB compiles each line of code as it runs, whereas C compiles the whole code at the start. Also used is a Runge-Kutta integration scheme which was implemented in C. Runge-Kutta is also a standard method applied to solve ODEs of the form (5.1) and (6.1). The primary numerical integration method used in Chapters 5 and 6 is analytic continuation, which is a numerical implementation of the Frobenius method [48, 55, 78]. Analytic continuation is the preferred scheme as it allows the user greater control over the error induced in a given time-step and proved to be the fastest of the three methods. The details of analytic continuation, as well as application to the pendulum with vertically oscillating support are explained in Section 4.3.1.

4.3.1 Analytic continuation

Analytic continuation is a method which can be used to extend the domain over which a power series representation of an analytic function is valid. Given an analytic function $g(z)$, defined on a region of the complex plane, the extension of its domain is achieved by constructing a locally convergent power series about a point, $z_0 \in \mathbb{C}$, for which the function and sufficient derivatives of the function are already known. This may be written in the form

$$g(z) = \sum_{i=0}^{\infty} g_i \hat{z}^i, \quad \hat{z} = z - z_0, \quad (4.2)$$

where the coefficients $g_i$ are constants to be found. The expansion (4.2) is convergent within a disk whose radius is governed by the nearest singular point of $g(z)$ to $z_0$. Picking another point in the complex plane, $z_1$, within the radius of convergence about $z_0$, the process may be iterated to extend the domain of the power series representation of $g(z)$, see [48, 100, 158]. An illustration of this is shown in Figure 4.1.
By taking $g(z)$ as a solution of an ordinary differential equation, analytic continuation can be utilised to solve initial value problems. In this instance the coefficients $g_i$ are determined by the derivatives of $g(z)$, evaluated at the point about which the function is expanded, i.e. $z_0$. For this reason the method is often referred to as the Taylor Series Method or Frobenius method. The use of Taylor series expansions to solve initial value problems in ordinary differential equations has been well known as both a classical and a numerical method for many years. It has been rediscovered and applied in many areas of mathematics, for example in Celestial Mechanics it is known as the recurrent power series method [16, 146]. Previous examples may be found in the literature [16, 17, 146, 59, 120].

An advantage that the Taylor series method can often have, is that it enables one to take large steps compared with integrators such as Runge-Kutta, which typically uses a step size of the order $10^{-2}$ or less. This may result in faster numerical integrators, despite the necessary computational overhead. It is not surprising then that much work already exists in the literature to determine where singular points of functions are in the complex plane and the possible step sizes that the Taylor series method can attain. These include both classical and numerical approaches, see for example [60, 86].

Calculating the locations of the singular points can require a lot of computation. Furthermore it is not enough to compute the singular points of the system one time only. The singular points move depending on the initial conditions, and hence with each step...
the integrator takes. This may be illustrated even with a system as simple as

\[ \dot{x} - x^2 = 0, \]  

(4.3)

with initial condition \( x_0 \). Separating the variables and integrating, the solution is found as

\[ x = \frac{x_0}{1 - tx_0}, \]  

(4.4)

from which it is explicitly apparent that the singular point varies with \( x_0 \). In the process of numerical integration, each step takes the result from the previous step as an initial condition. Hence the values of \( x_0 \) and \( t \) at the start of each step vary non-predictably, and so also do the locations of the singular points. For this reason, in practice it is not always the best way to decide the length of each time step, and instead the error inherent in a time-step is used.

The ability to directly check the error inherent in a given time-step is another substantial advantage that analytic continuation has over other numerical integrators. Often, numerical integrators with an adaptive step size use comparisons between integrations with different sized steps to decide on an appropriate step size to choose. This is achieved by integrating forward one step, then completing the same integration again with more, smaller steps. Once the difference between the two integrations is less than a given tolerance the error is assumed to be acceptable. A consequence of using the Taylor series method is that, from a power series for the solution the user can deduce a series expansion for the left-hand side of the ODE. Time-steps may be substituted into this series and the error can be directly calculated using the value of the right-hand side. By taking all forcing and constant terms to the left-hand side of the system, the right-hand side is zero and the value of the series expansion, \( \mathcal{E}(\hat{z}) \), is a measure of the error in the solution.

However for any finite series approximation to the solution of an ODE system, \( \mathcal{E}(\hat{z}) \)
will be an $n$-th degree polynomial, which has $n$ roots, some of which may be real. Substituting values of $\hat{z}$ close to one of the roots into the approximation of $E(\hat{z})$ may give the appearance of small error when it is not the case. It is possible to construct a series for $dE(\hat{z})/d\hat{z}$ and obtain the gradient of the error. We will later see that the series expansion for the derivative of a function can be quickly computed using the series expansion of the function itself. In cases where $E(\hat{z})$ gives false appearance of small error, the gradient, $E'(\hat{z})$, will be large. It is only possible for both $E(\hat{z})$ and $E'(\hat{z})$ to falsely look small if the Taylor series expansion has no linear term. Hence by checking the series expansions for $E(\hat{z})$ and $E'(\hat{z})$ against tolerances, the user has greater control over the error induced by a step $\hat{z}$ than is possible with many other integration schemes.

**Application to the pendulum with oscillating support**

In previous literature the Taylor series method has been utilised to solve systems with finite polynomial nonlinearities [55, 78, 88, 143]. To the best of our knowledge it has not yet been applied to systems with trigonometric nonlinearities. In the following we apply the Taylor series method to a pendulum system. In the process we show how addition formulae may be used in the application of analytic continuation to systems with trigonometric nonlinearities. In fact, we shall see that it is possible to apply analytic continuation to a system which has polynomial nonlinearities of infinite degree, provided there exists addition formula for the nonlinearities which allow the constant terms to be separated from the rest of the series expansion. Consider the system

$$\ddot{\theta}(t) + f(\theta, t) + g(\dot{\theta}, t) = 0, \quad (4.5)$$

where $\theta(t)$ is a generalised co-ordinate depending on time $t$ and the dots denote derivatives with respect to $t$. The functions $f(\theta, t)$ and $g(\dot{\theta}, t)$ are analytic in their arguments and we shall assume that the nonlinearities in both functions are either finite.
polynomials or trigonometric functions. Given initial conditions $\theta(t_0)$ and $\dot{\theta}(t_0)$, analytic continuation may be applied by first expanding the functions $f(\theta, t)$, $g(\dot{\theta}, t)$ as well as $\theta(t)$ and its derivatives as infinite power series in time, around $t = t_0$. Letting $\tau = (t - t_0)$ the power series may be written as

$$
\theta(t) = \sum_{i=0}^{\infty} \theta_i \tau^i,
$$

(4.6)

$$
\dot{\theta}(t) = \sum_{i=0}^{\infty} (i+1)\theta_{i+1} \tau^i,
$$

(4.7)

$$
\ddot{\theta}(t) = \sum_{i=0}^{\infty} (i+1)(i+2)\theta_{i+2} \tau^i,
$$

(4.8)

$$
f(\theta, t) = \sum_{i=0}^{\infty} f_i \tau^i,
$$

(4.9)

$$
g(t) = \sum_{i=0}^{\infty} g_i \tau^i.
$$

(4.10)

For numerical integration these series may be truncated to a finite order which achieves the desired level of accuracy. Setting $t = t_0$ it becomes explicitly obvious that the initial conditions $\theta(t_0)$ and $\dot{\theta}(t_0)$ give us the coefficients $\theta_0$ and $\theta_1$, respectively. The coefficients $f_i$ and $g_i$ will in general be dependent on the coefficients in the series for $\theta(t)$ and its derivatives. Substituting equations (4.6) to (4.10) into (4.5) it is clear that the coefficients for $\theta(t)$ may be calculated iteratively using the relation

$$
\theta_{i+2} = -\frac{g_i + f_i}{(i+1)(i+2)},
$$

(4.11)

provided that $f_i$ and $g_i$ only depend on $\theta_k$ for $k = 0, \ldots, i + 1$. If either of $f_i$ or $g_i$ depend on $\theta_k$ for $k \geq i + 2$ it is not possible to iteratively construct a series for $\theta(t)$ in this way. In systems where the nonlinearities in $f(\theta, t)$ and $g(\dot{\theta}, t)$ are finite polynomials
it is clear that this requirement is satisfied. However, if at least one of the functions contains trigonometric nonlinearities, by simply expanding we find that the entire series for \( \theta(t) \) is required to calculate the function at each order of \( \tau = (t - t_0) \). This becomes explicitly apparent when expanding, for example \( \sin(\theta_0 + \theta_1 \tau + \ldots) \) in the Horner form. Letting \( \theta(t) = \theta_0 + \dot{\theta}(t) \) and using the addition formulae for trigonometric functions we may write

\[
\sin \theta = \sin (\theta_0) \cos (\dot{\theta}) + \cos (\theta_0) \sin (\dot{\theta}), \quad (4.12)
\]

\[
\cos \theta = \cos (\theta_0) \cos (\dot{\theta}) - \sin (\theta_0) \sin (\dot{\theta}), \quad (4.13)
\]

where \( \dot{\theta} \) is defined by

\[
\dot{\theta}(t) = \theta(t) - \theta_0 = \sum_{i=1}^{\infty} \theta_i \tau^i. \quad (4.14)
\]

Now the coefficients at the \( i \)-th order in \( \tau \) of \( \cos(\dot{\theta}) \) and \( \sin(\dot{\theta}) \) (and hence \( \cos \theta \) and \( \sin \theta \)) are dependent only on coefficients of lower order in the series for \( \theta(t) \). Thus (4.11) may be used iteratively.

The system of the damped pendulum with oscillating support may be described by equation (4.5) with \( f(\theta, t) = (\alpha - \beta \cos t) \sin \theta \) and \( g(\dot{\theta}, t) = \gamma(t) \dot{\theta} \). For simplicity in this example we take \( \gamma(t) \) to be a constant and write \( \gamma(t) = \gamma \), though it may be easily extended to include time-dependency. In this instance equation (4.11) becomes

\[
\theta_{i+2} = -\frac{\gamma(i+1)\theta_{i+1} + f_i}{(i+1)(i+2)}. \quad (4.15)
\]

Expanding \( \cos t \) and \( \sin \theta \) using the addition formula the expansion of \( f(\theta, t) \) can be...
written as

\[ \sum_{i=0}^{\infty} f_i \tau^i = \left[ \alpha - \beta \left( \cos (t_0) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \tau^{2n} - \sin (t_0) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \tau^{2n+1} \right) \right] \times \left[ \sin (\theta_0) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \left( \sum_{k=1}^{\infty} \theta_k \tau^k \right)^{2n} + \cos (\theta_0) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \sum_{k=1}^{\infty} \theta_k \tau^k \right)^{2n+1} \right]. \]

It may be checked that the coefficients \( f_i \) depend only on coefficients \( \theta_k \) for \( k \leq i \) and hence equation (4.11) may be used. As mentioned previously the expansion of \( \theta(t) \) is valid on a disc in the complex plane, centred about the point \( t_0 \in \mathbb{C} \). The radius of convergence is determined by the nearest singular point of the system to \( t = t_0 \). Although the expansion is valid for values of time in the complex plane, it is often preferable to view the solutions \( \theta(t) \), in real time. For this reason we often specify that \( t \in \mathbb{R} \) although in general it is possible to take values of \( t \in \mathbb{C} \). From the physical properties, we know that the pendulum system does not begin to move infinitely fast at any point in real time. Hence there are no singular points on the real axis and the expansion is always valid on the real time axis for some radius greater than zero.

The Taylor series method is used in later chapters to integrate both the system of the pendulum with oscillating support and that of the pendulum with variable length, with both constant and linear coefficients of dissipation. In both cases, the time taken to complete numerical integrations was faster than that of a Runge-Kutta integrator, once appropriate possible step sizes were chosen. This was also observed in [22] for the system of a forced cubic oscillator. In particular, for the pendulum with variable length the Taylor series was able to take time steps with length larger than 6, which is 600 times greater than the largest time step used by Runge-Kutta.
4.4 Cell-to-cell mapping

Numerically estimating basins of attraction can be computationally heavy due to the
large number of initial conditions in phase space required for accurate estimates, see
Table 4.1. This has motivated research into faster numerical integrators and methods to
speed up computation time. One such method, originally proposed by C.S. Hsu [92, 93]
is cell-to-cell mapping; often referred to as CCMT (cell-to-cell mapping technique) or
simply CM (cell mapping). In this section we give an overview of the method and also
some reasons why it was not utilised in Chapters 5 and 6.

Consider an ordinary differential equation given by

\[ \dot{x} = f(x, \alpha), \quad (4.16) \]

where \( x \) is the co-ordinate vector, \( \alpha \) is a vector of the parameters (which we assume
to be fixed) and the dot denotes derivative with respect to time \( t \). We assume without
loss of generality that the system is autonomous, since a non-autonomous system can be
converted into an autonomous one by letting \( t \) be an additional co-ordinate. Then, for
any trajectory starting from an initial condition \( x(t) = x_0 \) in phase space, the arrival
point, \( x_1 \), at time \( t + \Delta t \) is uniquely determined by [31]

\[ x_1(x_0, \alpha) = \int_t^{t+\Delta t} f(x(s), \alpha) ds \quad (4.17) \]

The idea of cell-to-cell mapping is to partition the region of phase space we wish to
study into cells, say \( V_j \) for \( j = 1, 2, \ldots, J \). Note that this region of phase space should
contain all the attractive solutions for the system under the parameters \( \alpha \). The dynamics
of the system may then be thought of as a mapping between these cells, the time interval
\( \Delta t \) is then the mapping step. There are two techniques to study this mapping, simple
cell mapping and generalised cell mapping.
Simple cell mapping (SCM)

For simple cell mapping, one departure point in phase space is chosen in each cell, this is usually the centre points of the cells. If we consider a particular cell, $V_h$, we wish to find its image under the mapping. This is done by integrating the departure point in the cell and determining the arrival point using equation (4.17). Whichever cell, $V_i$, the arrival point falls into is considered the image cell of $V_h$, see Figure 4.2. Since there is only one departure point chosen in $V_h$, the probability that an initial condition chosen in $V_h$ maps to $V_i$ is $p(h, i) = 1$. If the image of a cell is the cell itself, the cell is called a sink cell. Sink cells correspond to fixed points of the system. A cell is called a periodic cell of period $K$ if it maps back into itself in $K$ mappings; therefore a sink cell is a period cell with $K = 1$. If a cell is not a periodic cell then it is called a transient cell. In this way we are able to quickly build up a picture of the basins of attraction.

![Figure 4.2: Illustration of the cell-to-cell mapping technique. The green cell $V_h$, is mapped to the blue cell $V_i$.](image)

However, the method of single cell mapping requires a large number of cells, as trajectories are essentially averaged to the centre point of cells after each mapping. If the cells are not small enough this averaging at each step can cause a number of problems in estimating the basins of attraction of the original, continuous system. For example,
in Figure 4.3(a) the yellow cell is found to be the image of the red cell after three applications of the map. However, if the cells are not small enough, the averaging which occurs after each application of the map may result in inconsistencies compared to the continuous system. Furthermore it is possible for cell mapping to find “fake solutions” [57]. A fake solution is a solution found by cell mapping which does not occur in the continuous system. For example a cell may be found to be a sink due to slow dynamics or a periodic solution may be found as a result of the averaging process, see Figure 4.3(b). Thus the accuracy and reliability of simple cell mapping is limited [62]. One technique to try and avoid such inaccuracies is generalised cell mapping.

![Figure 4.3: In figure (a) we illustrate the multiple applications of averaging that occur which can lead to errors when compared with the continuous system. In (b) we show how a periodic solution is constructed; here we show a period 4 solution. Due to the averaging of the trajectories after each mapping, the continuous system may not have such a periodic solution.](image)

**Generalised cell mapping (GCM)**

In generalised cell mapping, multiple departure points are allowed from each cell, and therefore each cell may have several image cells. The procedure proposed by Hsu [93] is as follows. Choose multiple departure points in the cells, typically both interior and boundary points. Then find the fraction of points starting in a particular cell $V_h$ which
map to a particular cell $V_i$ during the interval $[t, t + \Delta t]$. The probability $p(h, i)$ that an initial condition starting in $V_h$ is mapped to $V_i$ is then defined as this fraction.

In generalised cell mapping a cell $V_h$ “leads” to a cell $V_i$ if the probability that $x(t + k\Delta t) \in V_i$ given that $x(t) \in V_h$ is positive for some $k$ i.e. after $k$ mappings. A cell $V_h$ is said to “communicate” with a cell $V_i$ if and only if $V_h$ leads to $V_i$ and $V_i$ leads to $V_h$. A cell $V_h$ is called a “persistent cell” if a point starting in $V_h$ eventually returns to $V_h$. The set of persistent cells which communicate with each other form a “persistent group”. Persistent groups then correspond to fixed points or periodic solutions of the continuous system.

The method of generalised cell mapping results in more algorithmic complexity compared with simple cell mapping. In practice generalised and simple cell mapping are employed together [62]. However, even with generalised cell mapping the important structures in phase space (such as persistent groups and boundaries of basins of attraction) are often too crude and need to be improved via refinement techniques [62]. In particular, for systems which are sensitive to slight variations in the initial conditions, such as those considered in Chapters 5, 6 and 7, a very large number of cells must be used to attain accuracy when determining the basins of attraction. Furthermore, as the trajectories are averaged after each application of the map, if too few cells are used the errors induced by averaging are repeatedly compounded. For these reasons cell mapping is not utilised to calculate basins of attraction in Chapters 5 and 6.
Chapter 5

The Pendulum with vertically oscillating support

We now extend the system of the simple pendulum (see Appendix A.4) to that of the pendulum with vertically oscillating support and additional dissipative forces. The pendulum with vertically oscillating support has attracted much attention over the years and is arguably one of the most studied perturbations of the simple pendulum, see for example [26, 27, 28, 40, 65, 66, 154]. First, we recap some analysis of the linearised system which can be found in the literature [27, 96], after which we focus on the full nonlinear system. In particular we extend the action-angle variables for the simple pendulum to those for the forced, damped system. These “perturbed” action-angle variables are then used to calculate the thresholds of dissipation (commonly referred to as the thresholds of friction [18, 22]), that is, the largest coefficient of dissipation under which a given periodic solution exists. However, it shall be seen that the method used to calculate the thresholds of dissipation, due to its nature and roots in perturbation theory, can only be applied in very limited cases.

We then continue to investigate the pendulum outside of this regime by means of numerical simulation. We focus our attention on the scenario where dissipation linearly increases or decreases to some constant value over an initial period of time. The
motivation is to investigate the effects on the sizes of the basins of attraction when
the coefficient of dissipation is initially a function of time, see Chapter 1 for further
details. Similar work has been conducted in [22], *Attractiveness of periodic orbits in*
*parametrically forced systems with time-increasing friction*, in which the results suggest
that, although the final value of dissipation determines which solutions exist and are
attractive, the evolution of dissipation to this final value affects the relative sizes of the
basins of attraction. However in [22] the only functions considered for the dissipative
coefficient are ones which increase linearly from zero to a given constant value. Contrary
to this, we consider both increasing and decreasing functions which need not start from
zero. As such, many of the observations which follow in this chapter are not included in
[22].

Let us illustrate in more detail the phenomenology. Suppose that for two values \( \gamma_0 \)
and \( \gamma_1 \) of the damping coefficient, with \( \gamma_0 \neq \gamma_1 \), the same attractors exist, that is, the sets
of attractors for the two values \( \gamma_0, \gamma_1 \) are the same. Provided the difference between the
two values is sufficiently large, the relative sizes of the basins of attraction under the two
coefficients will in general be appreciably different. If we allow the damping coefficient \( \gamma \)
to depend on time, \( \gamma = \gamma(t) \), and vary from \( \gamma_0 \) to \( \gamma_1 \) over an initial period of time \([0, T_0]\),
after which it remains constant at the value \( \gamma_1 \), then the sizes of the basins of attraction
will be different from those where the system has constant coefficient \( \gamma_1 \) throughout.
Moreover if \( T_0 \) is taken larger, the sizes of the basins of attraction tend towards those
for the system under constant \( \gamma = \gamma_0 \); this reflects the fact that the damping coefficient
remains close to \( \gamma_0 \) for longer periods of time.

Now consider two values \( \gamma_0 \) and \( \gamma_1 \) of the damping coefficient for which the
corresponding sets of attractors \( A_0 \) and \( A_1 \) are not the same. As a system evolving
under dissipation is expected to have only finitely many attractors, there can only be a
finite number of attractors which exist for one of the two values and not for the other
one. What happens is that, by varying $\gamma(t)$ from $\gamma_0$ to $\gamma_1$, an attractor can either appear or disappear, and in the latter case it can disappear either by being replaced with a new attractor by bifurcation or leaving behind trajectories which are attracted by the remaining attractors. Suppose, for instance, that the only difference between $A_0$ and $A_1$ is that the attractor $a_0 \in A_0$ simply disappears, that is $A_0 \setminus A_1 = \{a_0\}$; then, if the time $T_0$ over which $\gamma(t)$ varies is large, each remaining attractor tends to have a basin of attraction not smaller than that it has for $\gamma$ fixed at $\gamma_0$. The reason being, again, that the damping coefficient remains close to $\gamma_0$ for a long time and, moreover, the trajectories which would be attracted by $a_0$ at $\gamma = \gamma_0$ will move towards some other attractor when $a_0$ disappears. If, instead, the only difference between the sets of attractors $A_0$ and $A_1$ is that the attractor $a_0 \in A_0$ is replaced by an attractor $a_1$, say by period doubling bifurcation, then, letting $\gamma(t)$ vary from $\gamma_0$ to $\gamma_1$ over a sufficiently large time $T_0$ causes the size of the basin of attraction of $a_1$ to tend towards that of the basin of attraction that $a_0$ has for $\gamma = \gamma_0$.

We summarise our results by the following statements:

1. If $A_0$, the set of attractors at $\gamma = \gamma_0$, is a subset of $A_1$, the set of attractors which exist at $\gamma = \gamma_1$, that is $A_0 \subseteq A_1$, then, as the time $T_0$ over which $\gamma(t)$ is varied from $\gamma_0$ to $\gamma_1$ is taken larger, the basins of attraction tend towards those when $\gamma$ is kept constant at $\gamma = \gamma_0$. In particular, if an attractor belongs to $A_1 \setminus A_0$, then the larger $T_0$ the more negligible is the corresponding basin of attraction.

2. If the set of attractors at fixed $\gamma = \gamma_1$ is a proper subset of those which exist at $\gamma = \gamma_0$, that is $A_1 \subset A_0$, then, as $T_0$ is taken larger, the basins of attraction for the attractors which exist at both $\gamma_0$ and $\gamma_1$ change so that for $\gamma(t)$ varying from $\gamma_0$ to $\gamma_1$ they tend to become greater than or equal to those for constant $\gamma = \gamma_0$.

3. If an attractor $a_0$ exists for $\gamma = \gamma_0$ but is destroyed as $\gamma(t)$ tends towards $\gamma_1$, and a
new attractor \( a_1 \) is created from it by bifurcation (we will explicitly investigate the case of saddle-node and period doubling bifurcations), then the size of the basin of attraction of \( a_1 \), as \( T_0 \) is taken larger, tends towards that of \( a_0 \) at constant \( \gamma = \gamma_0 \).

4. If \( A_{01} \) is the set of attractors which exist at both \( \gamma = \gamma_0 \) and \( \gamma = \gamma_1 \), that is \( A_{01} = A_0 \cap A_1 \), and none of the elements in \( A_0 \setminus A_{01} \) are linked by bifurcation to elements in \( A_1 \setminus A_{01} \), then, as \( T_0 \) is taken larger, the phase space covered by the basins of attraction of the attractors which belong to \( A_{01} \) tends towards 100%.

Moreover, all such attractors have a basin of attraction larger than or equal to that they have when the coefficient of dissipation is fixed at \( \gamma = \gamma_0 \).

The system of the pendulum differs from that of the cubic oscillator in many ways. For example, the existence of a separatrix allows for rotating solutions and greatly complicates the dynamics. Another important difference with respect to the results in [22] is the following. In [22], if an attractor exists for some value of \( \gamma \), it is also found to exist for smaller values of \( \gamma \). This is not always true for the pendulum considered in the following sections, where we will see that, at least for some values of the parameters, both increasing and decreasing \( \gamma \) can destroy attractors as well as create new ones. This is related to the choice of parameters and the bifurcation structure of the system, we refer to sections 5.4.1 and 5.5.1 for further details. However, for the parameter values which we choose to investigate, this only occurs away from the perturbation regime, where the system can no longer be considered as a perturbation of an integrable one; the appearance and disappearance of attractors is expected to occur also in the case of the cubic oscillator for larger values of the forcing coefficient. Therefore, in such cases more possibilities must be taken into account when considering how the relative sizes of the basins of attraction change when \( \gamma \) is allowed to initially vary with time. In addition to the case of increasing dissipation studied in [22], here we also include the case where
the damping coefficient decreases to a constant value, which is appropriate for physical systems where joints are initially tight and require time to loosen. In this case similar phenomena are expected. For instance, as the value of variation time $T_0$ is taken larger, the amount of phase space covered by each of the basins of attraction should tend towards that corresponding to original value $\gamma_0$ of $\gamma$, provided the set of attractors remains the same.

5.1 Linearised systems

The system of the non-linear pendulum with vertically oscillating support can be described by

$$\ddot{\theta}(t) + f(t) \sin \theta(t) + \gamma \dot{\theta}(t) = 0, \quad f(t) = \left( \frac{g}{l} - \frac{b\omega^2}{l} \cos(\omega t) \right), \quad (5.1)$$

where the parameters $l, b, \omega$ and $g$ represent the length of the pendulum, the amplitude and frequency of the forcing oscillations and the gravitational coefficient respectively, all of which remain constant. The parameter $\gamma$ represents the coefficient of dissipation, which remains constant with respect to time until Section 5.4.2. Note that the high frequency result of Chapter 3 do not apply to this system, as the coefficient of forcing contains the term $\omega^2$. By the same reasoning, the low frequency result applies trivially. Equation (5.1) may be considered as a pair of coupled first order non-autonomous differential equations by letting $x = \theta$ and $y = \dot{x}$. The system can then be written as

$$\begin{cases} 
\dot{x}(t) = y(t), \\
\dot{y}(t) = -\left( \frac{g}{l} - \frac{b\omega^2}{l} \cos(\omega t) \right) \sin x(t) - \gamma y(t).
\end{cases}$$

The pendulum system can be non-dimensionalised by defining dimensionless parameters

$$\alpha = \frac{g}{l\omega^2}, \quad \beta = \frac{b}{l}, \quad \tau = \omega t,$$
and rewriting equation (5.1) as

\[ \theta''(\tau) + f(\tau) \sin \theta(\tau) + \gamma \theta'(\tau) = 0, \quad f(\tau) = (\alpha - \beta \cos \tau), \quad (5.2) \]

or equivalently, as a system of first order differential equations

\[ q' = p, \quad p' = -f(\tau) \sin q - \gamma p, \quad (5.3) \]

where the dashes represent differentiation with respect to the new time \( \tau \) and \( \gamma \) has been normalised so as not to contain the frequency. The system in this form is preferable for numerical analysis as it has a reduced number of parameters. It is evident, due to the properties of \( \cos \tau \) that a change in the sign of \( \beta \) is equivalent to a shift of \( \pi \) in time \( \tau \), hence the invariant curves do not depend on the sign of \( \beta \). It can also be seen from the way in which the parameters are defined, that changing the sign of \( \alpha \) has the effect of negating the length. While physically this makes little sense, for the purpose of analysis and computer simulations, the result is that the same equation can be used for both the downward and inverted configurations, simply by changing the sign of \( \alpha \). The linearisation of the system about either one of its fixed points results in a system in the form of the Mathieu’s equation, a special case of Hill’s equation discussed in Chapter 2. Linearising about the downward and upward fixed points (\( \theta = 0, \theta = \pi \)) respectively results in the two equations

\[ \psi''_1(\tau) + (\alpha - \beta \cos \tau)\psi_1(\tau) + \gamma \psi'_1(\tau) = 0, \quad (5.4) \]
\[ \psi''_2(\tau) - (\alpha - \beta \cos \tau)\psi_2(\tau) + \gamma \psi'_2(\tau) = 0, \quad (5.5) \]

where \( \psi_1, \psi_2 \) represent the angle between the pendulum and the downward or upward verticals respectively. However as stated, for numerical computation it is possible to consider only the first of the two equations by changing the sign of \( \alpha \). Setting \( \gamma = 0 \) in the above linearisations, one can use the methods described in [96] to obtain the
transition curves (or stability curves) of the system, see Appendix A.9 for details. Figure 5.1 shows both the $2\pi$ and $4\pi$ periodic solution curves. Regions of instability have been shaded and the regions which are stable (also known as the stability tongues) are white.

Choosing values of the parameters $\alpha$ and $\beta$ inside the stable regions, the fixed point $\psi = 0$ is (linearly) stable. Choosing parameters inside the unstable regions, the fixed point $\psi = 0$ is (linearly) unstable and the solutions grow exponentially; this corresponds to Lyapunov instability of the fixed point $\theta = 0$ in the nonlinear system. In the nonlinear system resonance causes the amplitude of the motion to increase, as a result the relation between the period and amplitude causes the resonance to detune. Hence we do not see exponential growth in the nonlinear system (as one might expect from a physical point of view). Choosing parameter values in the unstable regions of Figure 5.1 we observe oscillatory and rotating solutions in the nonlinear system [79, 118, 141]. Further details on stability and Mathieu’s equation are given in Appendices A.8 and A.9, respectively.

Figure 5.1: Stability curves for the linearised system of the pendulum with oscillating support. The broken line shows the curves of the $2\pi$ periodic solutions and the solid line those of the $4\pi$ periodic solutions. The regions shaded in grey are those in which the fixed point $\psi = 0$ is unstable and exponential growth occurs. In the nonlinear system, solutions in these regions are typically oscillatory, rotating or chaotic.
5.2 Condition for the origin to attract a full measure set of phase space

To compute the conditions for which the origin attracts the entire phase space, up to a zero measure-set, we use the approach outlined in [24]; see also [23]. We define $f(\tau)$ as in (5.2) and require $f(\tau) > 0$; the consequences of this restriction are that the method can only be applied to the pendulum centred about the downward hanging position, with $\alpha > \beta$. This restriction is necessary to apply the Liouville transformation to the time parameter. In terms of the physical parameters, this restricts our consideration to cases where the coefficient of forcing in (5.1), that is $b\omega^2$, is not too large i.e. the system maybe be considered as a perturbation of an integrable one. Then we apply the Liouville transformation

$$\tilde{\tau} = \int_0^\tau \sqrt{f(s)}ds,$$

and write our equation (5.2) in terms of the new time $\tilde{\tau}$ as

$$\theta_{\tilde{\tau}\tilde{\tau}} + \left(\frac{\tilde{f}(\tilde{\tau})_{\tilde{\tau}}}{2\tilde{f}(\tilde{\tau})} + \frac{\gamma}{\sqrt{\tilde{f}(\tilde{\tau})}}\right)\theta_{\tilde{\tau}} + \sin \theta = 0,$$

where the subscript $\tilde{\tau}$ represents derivative with respect to the new time $\tilde{\tau}$ and $\tilde{f}(\tilde{\tau}) := f(\tau)$. This can be represented as the two-dimensional system on $\mathbb{T} \times \mathbb{R}$, by setting $x(\tilde{\tau}) = \theta(\tilde{\tau})$ and writing

$$x_{\tilde{\tau}} = y, \quad y_{\tilde{\tau}} = -\frac{y}{\sqrt{\tilde{f}}} \left(\frac{\tilde{f}_{\tilde{\tau}}}{2\sqrt{\tilde{f}}} + \gamma\right) - \sin x,$$

for which we have the energy $E(x,y) = 1 - \cos x + \frac{y^2}{2}$. By setting $\mathcal{H}(\tilde{\tau}) = E(x(\tilde{\tau}), y(\tilde{\tau}))$, one finds

$$\mathcal{H}_{\tilde{\tau}} = -\frac{y^2}{\sqrt{\tilde{f}}} \left(\frac{\tilde{f}_{\tilde{\tau}}}{2\sqrt{\tilde{f}}} + \gamma\right), \quad (5.6)$$
thus $\mathcal{H} \leq 0$, i.e $x$, $y$ are bounded provided $\gamma$ satisfies

$$\gamma > - \min_{\tau \geq 0} \frac{\dot{f}_\tau}{2\sqrt{f}} = - \min_{\tau \geq 0} \frac{f'}{2f},$$

(5.7)

Moreover we have that for all $\tilde{\tau} > 0$

$$\mathcal{H}(\tilde{\tau}) + \int_0^{\tilde{\tau}} \frac{y^2}{\sqrt{f}} \left(\frac{\dot{f}_\tau}{2\sqrt{f}} + \gamma\right) \, ds = \mathcal{H}(0),$$

so that, as $\tilde{\tau} \to \infty$, using the properties above we can arrive at

$$\min_{s \geq 0} \left[\frac{1}{\sqrt{f}} \left(\frac{\dot{f}_s}{2\sqrt{f}} + \gamma\right)\right] \int_0^{\infty} y^2(s) \, ds < \infty.$$

Hence $y \to 0$ as time tends to infinity. There are two regions of phase space to consider.

Any level curve of $\mathcal{H}$ strictly inside the separatrix\(^1\) of the simple pendulum is the boundary of a positively invariant set $D$ containing the origin, since $S = \{(x(\tilde{\tau}), y(\tilde{\tau})) : \mathcal{H}(\tilde{\tau}) = 0\} \cup D$ consists purely of the origin, we can apply the local Barbashin-Krasovskii-La Salle theorem [103], to conclude that every trajectory that begins strictly inside the separatrix will converge to the origin as $\tilde{\tau} \to +\infty$.

Outside of the separatrix we may use equation (5.6), which shows the energy to be strictly decreasing while $y \neq 0$, provided $\gamma$ is chosen large enough, coupled with $y \to 0$ as time tends to infinity. The result is that all trajectories tend to the invariant points on the $x$-axis as time tends to infinity. One of two cases must occur: either the trajectory moves inside the separatrix or it does not. In the first instance we have already shown that the limiting solution is the origin. In the latter there is only one possibility. As all points on the $x$-axis are contained within the separatrix other than the unstable fixed

\(^1\)A separatrix is a heteroclinic orbit which separates two modes of behaviour. In the simple pendulum the orbit connects the saddle points at $\theta = \pm \pi$ and separates the rotating and oscillating dynamics.
point, the trajectory must move onto such a fixed point and hence belongs to its stable manifold, which is a zero-measure set. Therefore we conclude that a full measure set of initial conditions are attracted by the origin. Reverting back to the original system with time $\tau$, we conclude that also for that system the basin of attraction of the origin has full measure, provided $\beta < \alpha$ and $\gamma$ satisfies (5.7).

### 5.3 Threshold values of $\gamma$

The following method to calculate the threshold values of $\gamma$ below which given attractors exist follows that described in [18, 22], where it was applied to the damped quartic oscillator and the spin-orbit model. We consider the system (5.3), with $\beta = \varepsilon$ and $\gamma = C_1 \varepsilon$, where $\varepsilon, C_1 > 0$. This approach is well suited to compute the leading order of the threshold values. In general, it would be preferable to write $\gamma$ as a function of $\varepsilon$ of the form $\gamma = C_1 \varepsilon + C_2 \varepsilon^2 + \ldots$ (bifurcation curve), and fix the constants $C_k$ by imposing formal solubility of the equations to any perturbation order, see [80]; however this only produces higher order corrections to the leading order value.

For $\varepsilon = 0$ the system reduces to the simple pendulum $\theta'' + \alpha \sin \theta = 0$, which admits periodic solutions inside the separatrix (librations or oscillations) and outside the separatrix (rotations). In terms of the variables $(q, p)$ the equations (5.3) become $q' = p, p' = -\sin q$: the librations are described by

$$
\begin{align*}
q_{\text{osc}}(\tau) &= 2 \arcsin[k_1 \text{sn}(\sqrt{\alpha}(\tau - \tau_0), k_1)], \\
p_{\text{osc}}(\tau) &= 2 k_1 \sqrt{\alpha} \text{cn}(\sqrt{\alpha}(\tau - \tau_0), k_1),
\end{align*}
$$

where $\text{cn}(\cdot, k)$, $\text{sn}(\cdot, k)$ and $\text{dn}(\cdot, k)$ are the Jacobi elliptic functions with elliptic modulus

$$
\begin{align*}
\left\{ \begin{array}{ll}
q_{\text{rot}}(\tau) &= 2 \arcsin[\text{sn}(\sqrt{\alpha}(\tau - \tau_0)/k_2, k_2)], \\
p_{\text{rot}}(\tau) &= 2 k_2^{-1} \sqrt{\alpha} \text{dn}(\sqrt{\alpha}(\tau - \tau_0)/k_2, k_2),
\end{array} \right. \\
&\quad \quad k_2 < 1,
\end{align*}
$$

where $\text{cn}(\cdot, k)$, $\text{sn}(\cdot, k)$ and $\text{dn}(\cdot, k)$ are the Jacobi elliptic functions with elliptic modulus

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$k$ [44, 54, 107, 165], and $k_1$ and $k_2$ are such that $k_1^2 = (E + \alpha)/2\alpha$ and $k_2^2 = 1/k_1^2$, with $E$ being the energy of the pendulum, see Appendices A.3 and A.4. From (5.8) and (5.9) it can be seen that the solutions are functions of $(\tau - \tau_0)$, so that the phase of a solution depends on the initial conditions. We can fix the phase of the solution to zero without loss of generality by instead writing $f(\tau)$ in equation (5.3) as $f(\tau - \tau_0)$. This moves the freedom of choice in the initial condition to the phase of the forcing.

The dynamics of the simple pendulum can be conveniently written in terms of action-angle variables $(I, \varphi)$, for which we obtain two sets of variables: for the librations inside the separatrix one expresses the action as

$$I = \frac{8}{\pi} \sqrt{\alpha} \left[ (k_1^2 - 1)K(k_1) + E(k_1) \right],$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds respectively, and writes

$$q = 2 \arcsin \left[ k_1 \text{sn} \left( \frac{2K(k_1)}{\pi} \varphi, k_1 \right) \right], \quad p = 2k_1 \sqrt{\alpha} \text{cn} \left( \frac{2K(k_1)}{\pi} \varphi, k_1 \right),$$

with $k_1$ obtained by inverting (5.10), while for the rotations outside the separatrix one expresses the actions as

$$I = \frac{4}{k_2 \pi} \sqrt{\alpha} E(k_2),$$

and writes

$$q = 2 \arcsin \left[ \text{sn} \left( \frac{K(k_2)}{\pi} \varphi, k_2 \right) \right], \quad p = \frac{2}{k_2} \sqrt{\alpha} \text{dn} \left( \frac{K(k_2)}{\pi} \varphi, k_2 \right),$$

with $k_2$ obtained by inverting (5.12); further details can be found in Appendix A.4.

For $\varepsilon$ small, in order to compute the thresholds values, we first write the equations of motion for the perturbed system in terms of the action-angle coordinates $(I, \varphi)$ of the
simple pendulum, then we look for solutions of the form

\[ I(\tau) = \sum_{n=0}^{\infty} \varepsilon^n I^{(n)}(\tau), \quad \varphi(\tau) = \sum_{n=0}^{\infty} \varepsilon^n \varphi^{(n)}(\tau), \tag{5.14} \]

where \( I^{(0)}(\tau) \) and \( \varphi^{(0)}(\tau) \) are the solutions to the unperturbed system, that is, see Appendix A.4,

\[ I^{(0)}(\tau) = I_{osc} = \frac{8}{\pi} \sqrt{\alpha} \left[ (k_1^2 - 1)K(k_1) + E(k_1) \right], \quad \varphi^{(0)}(\tau) = \varphi_{osc}(\tau) = \frac{\pi}{2K(k_1)} \sqrt{\alpha \tau}, \]

\[ I^{(0)}(\tau) = I_{rot} = \frac{4}{k_2^2 \pi} \sqrt{\alpha} E(k_2), \quad \varphi^{(0)}(\tau) = \varphi_{rot}(\tau) = \frac{\pi}{K(k_2)} \sqrt{\alpha \tau}, \]

in the case of oscillations and rotations respectively, with given \( k_1 = k_1^{(0)} \) and \( k_2 = k_2^{(0)} \).

As the solution (5.14) is found using perturbation theory, its validity is restricted to the system where \( \varepsilon \) is comparatively small. In particular this limitation has the consequence that the calculations of the threshold values are not valid for the inverted pendulum, where large \( \varepsilon \) is required to stabilise the system. On the other hand the regime of small \( \varepsilon \) has the advantage that we can characterise analytically the attractors and hence allows a better understanding of the dynamics with respect to the case of large \( \varepsilon \), where only numerical results are available.

### 5.3.1 Librations

We first wish to write the equations of motion (5.3) in action-angle variables. By equation (A.4.13) one has

\[
\begin{pmatrix}
\partial \varphi / \partial q & \partial \varphi / \partial p \\
\partial I / \partial q & \partial I / \partial p
\end{pmatrix}
= \begin{pmatrix}
\partial p / \partial I & -\partial q / \partial I \\
-\partial p / \partial \varphi & \partial q / \partial \varphi
\end{pmatrix}.
\]
Taking into account the forcing term \(-\beta \cos \tau \sin \theta\) in (5.2), and shortening \((\cdot) = \left(\frac{2K(k_1)\varphi}{\pi}, k_1\right)\) as in Appendix A.4, one finds

\[
P' = \frac{\partial I}{\partial q} p' + \frac{\partial I}{\partial p} q' = -\frac{\partial p}{\partial \varphi} q' + \frac{\partial q}{\partial \varphi} p' = -\frac{8\beta k_1^2 K(k_1)}{\pi} \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot),
\]

\[
\varphi' = \frac{\partial \varphi}{\partial q} q' + \frac{\partial \varphi}{\partial p} p' = \frac{\partial p}{\partial I} q' - \frac{\partial q}{\partial I} p' = \frac{\pi \sqrt{\alpha}}{2K(k_1)} - \frac{\pi \beta}{2K(k_1) \sqrt{\alpha}} \left[ \text{sn}^2(\cdot) + \frac{2E(k_1) \varphi \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi k_1^2} \right. \\
\left. + \frac{k_1^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_1^2} - \frac{E(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_1^2} \right] \cos(\tau - \tau_0),
\]

where we have used the properties of the Jacobi elliptic functions in Appendix A.3 as well as the relation \(\sin(2 \arcsin(kx)) = 2kx \sqrt{1 - k^2 x^2}\). The shorthand \(I\) and \(\varphi\) have been used to represent \(I_{\text{osc}}\) and \(\varphi_{\text{osc}}\) respectively. Adding the dissipative term \(\gamma \theta'\) the system becomes

\[
p'(\tau) + (\alpha - \beta \cos(\tau)) \sin\left(q(\tau)\right) + \gamma q'(\tau) = 0.
\]

We then have

\[
q'(\tau) = \frac{\partial q}{\partial I} I' + \frac{\partial q}{\partial \varphi} \varphi' = 2k_1 \sqrt{\alpha} \text{cn}(\cdot).
\]

Note that, all the terms containing \(\varepsilon\), that is all the perturbed terms, have canceled leaving \(q'\) exactly as it is for the unperturbed system, namely \(q' = p\). Hence the action-
angle variables satisfy

\[ I' = \frac{8\beta k_1^2 K(k_1)}{\pi} \cos(\tau - \tau_0) \sin(\cdot) \cos(\cdot) - \frac{8\gamma k_1^2 \sqrt{\alpha} K(k_1)}{\pi} \cos(\cdot) - \frac{8\gamma k_1^2 \sqrt{\alpha} K(k_1)}{\pi} \cos(\cdot), \]

\[ \varphi' = \frac{\pi \sqrt{\alpha}}{2K(k_1)} - \frac{\pi \beta}{2K(k_1) \sqrt{\alpha}} \left[ \frac{\sin^2(\cdot)}{\sqrt{1 - k_1^2}} + \frac{2E(k_1) \sin(\cdot) \cos(\cdot) \sinh(\cdot)}{\pi (1 - k_1^2)} \right] \cos(\tau - \tau_0) \]

\[ + \frac{\gamma \pi \cos(\cdot)}{2K(k_1)} \left[ \frac{\sin(\cdot)}{\cosh(\cdot)} + \frac{2E(k_1) \sin(\cdot) \cosh(\cdot)}{\pi (1 - k_1^2)} + \frac{k_1^2 \sin(\cdot) \cosh(\cdot)}{(1 - k_1^2) \sin(\cdot)} - \frac{E(\cdot) \sin(\cdot)}{1 - k_1^2} \right]. \]

Using the property that \( E(u, k) = E(k) u / K(k) + Z(u, k) \) and setting \( \beta = \varepsilon, \gamma = C_1 \varepsilon \) we arrive at the equations

\[ \varphi' = \frac{\pi \sqrt{\alpha}}{2K(k_1)} - \frac{\varepsilon \pi}{2K(k_1) \sqrt{\alpha}} \left[ \frac{\sin^2(\cdot)}{\sqrt{1 - k_1^2}} + \frac{k_1^2 \sin^2(\cdot) \cosh^2(\cdot)}{1 - k_1^2} - \frac{Z(\cdot) \sin(\cdot) \cosh(\cdot) \sinh(\cdot)}{1 - k_1^2} \right] \cos(\tau - \tau_0) \]

\[ + \frac{C_1 \varepsilon \pi \cos(\cdot)}{2K(k_1)} \left[ \frac{\sin(\cdot)}{\cosh(\cdot)} + \frac{k_1^2 \sin(\cdot) \cosh(\cdot)}{(1 - k_1^2) \sin(\cdot)} - \frac{Z(\cdot) \cos(\cdot)}{1 - k_1^2} \right], \]

\[ I' = \frac{8\varepsilon k_1^2 K(k_1)}{\pi} \cos(\tau - \tau_0) \sin(\cdot) \cosh(\cdot) \sinh(\cdot) - \frac{8C_1 \varepsilon k_1^2 \sqrt{\alpha} K(k_1)}{\pi} \sin^2(\cdot), \]

(5.15)

where \( Z(\cdot) \) is the Jacobi zeta function, see [107]. Note that in (5.15), the dependence on \( I \) of the vector field is through the variable \( k_1 \), according to (5.10). It is thus also possible to carry out the following calculations using the variable \( k_1 \) instead of the action \( I \). The coordinates for the unperturbed system \( (\varepsilon = 0) \) satisfy

\[ \varphi' = \frac{dE}{dI} := \Omega(I) = \frac{\pi \sqrt{\alpha}}{2K(k_1)}, \quad I' = 0. \]

Linearising around \( (\varphi^{(0)}(\tau), I^{(0)}(\tau)) = (\Omega(I^{(0)}) \tau, I^{(0)}) \), we have

\[ \delta \varphi' = \frac{\partial \Omega}{\partial I} (I^{(0)}) \delta I, \quad \delta I' = 0, \]

(5.16)
where, see Appendix A.4,

\[ \zeta := \frac{\partial \Omega}{\partial I}(I) = -\frac{\pi^2}{16k_1^2K_3(k_1)} \left[ \frac{E(k_1)}{1 - k_1^2} - K(k_1) \right]. \quad (5.17) \]

Since \( I = I(k_1) \), that is the action is a function of \( k_1 \), setting \( I = I^{(0)} \) fixes \( k_1 = k_1^{(0)} \), yielding \( \zeta = \zeta^{(0)} \), with \( \zeta^{(0)} \) given by (5.17) with \( k_1 = k_1^{(0)} \). The linearised system (5.16) can be written in compact form as

\[ \begin{pmatrix} \delta \varphi' \\ \delta I' \end{pmatrix} = \begin{pmatrix} 0 & \zeta^{(0)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \varphi \\ \delta I \end{pmatrix}. \]

The Wronskian matrix \( W(\tau) \) is a solution of the unperturbed linear system

\[ W'(\tau) = \begin{pmatrix} 0 & \zeta^{(0)} \\ 0 & 0 \end{pmatrix} W(\tau), \quad W(0) = \mathbb{I}, \]

where \( \mathbb{I} \) is the \( 2 \times 2 \) identity matrix. Hence

\[ W(\tau) = \begin{pmatrix} 1 & \zeta^{(0)} \tau \\ 0 & 1 \end{pmatrix}. \quad (5.18) \]

We now look for periodic solutions \((\varphi(\tau), I(\tau))\) with period \( T = 2\pi q = 4K(k_1)p/\sqrt{\alpha} \), with \( p/q \in \mathbb{Q} \), of the form (5.14); see also [80, 81] for a more general discussion. A solution of this kind will be referred to as a \( p:q \) resonance. The functions \((\varphi^{(n)}(\tau), I^{(n)}(\tau))\) are formally defined by, see [128]

\[ \begin{pmatrix} \varphi^{(n)}(\tau) \\ I^{(n)}(\tau) \end{pmatrix} = W(\tau) \begin{pmatrix} \varphi^{(n)}(0) \\ I^{(n)}(0) \end{pmatrix} + W(\tau) \int_0^\tau d\sigma W^{-1}(\sigma) \begin{pmatrix} F^{(n)}_1(\sigma) \\ F^{(n)}_2(\sigma) \end{pmatrix}, \quad (5.19) \]
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where \( \varphi^{(n)} \) and \( I^{(n)} \) are the \( n \)-th order in the \( \varepsilon \)-expansion of the initial conditions for \( \varphi \) and \( I \), respectively, while \( F_1^{(n)}(\tau) \) and \( F_2^{(n)}(\tau) \) are given by

\[
F_1^{(n)}(\tau) = \left[ \frac{\pi \sqrt{\alpha}}{2K(k_1)} - \zeta^{(0)} I \right]^{(n)} + \left[ \frac{\pi}{2K(k_1) \sqrt{\alpha}} \left\{ \text{sn}^2(\cdot) + \frac{k_1^2}{1 - k_1^2} \text{sn}^2(\cdot) \text{cn}^2(\cdot) \right. \right. \\
\left. \left. - \frac{Z(\cdot)}{1 - k_1^2} \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) \right\} \cos(\tau - \tau_0) \\
+ C_1 \frac{\pi}{2K(k_1)} \left[ \frac{\text{sn}(\cdot)}{\text{dn}(\cdot)} + \frac{k_1^2 \text{sn}(\cdot) \text{cn}(\cdot)}{(1 - k_1^2) \text{dn}(\cdot)} - \frac{Z(\cdot) \text{cn}(\cdot)}{1 - k_1^2} \right] \right]^{(n-1)},
\]

\[
F_2^{(n)}(\tau) = \left[ \frac{8k_1^2K(k_1)}{\pi} \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) - \frac{8C_1 k_1^2 \sqrt{\alpha K(k_1)} \text{cn}(\cdot)}{\pi} \right]^{(n-1)}.
\]

The notation \([\ldots]^{(n)}\) means that one has to take all terms of order \( n \) in \( \varepsilon \) of the function inside \([\ldots]\). In the last term of (5.19) we have

\[
W(\tau) \int_0^\tau d\sigma W^{-1}(\sigma) \left( \frac{F_1^{(n)}(\sigma)}{F_2^{(n)}(\sigma)} \right) = \int_0^\tau d\sigma W(\tau - \sigma) \left( \frac{F_1^{(n)}(\sigma)}{F_2^{(n)}(\sigma)} \right).
\]

This yields

\[
\varphi^{(n)}(\tau) = \varphi^{(n)} + \zeta^{(0)} \tau I^{(n)} + \int_0^\tau d\sigma F_1^{(n)}(\sigma) + \zeta^{(0)} \int_0^\tau d\sigma \int_0^\sigma d\sigma' F_2^{(n)}(\sigma'),
\]

\[
I^{(n)}(\tau) = I^{(n)}(\tau) + \int_0^\tau d\sigma F_2^{(n)}(\sigma).
\]

For a periodic function \( g \), let us denote the average of \( g \) with \( \langle g \rangle \) and the zero-average function \( g - \langle g \rangle \) with \( \bar{g} \). Suppose that

\[
\langle F_2^{(n)} \rangle := \frac{1}{T} \int_0^T d\tau F_2^{(n)}(\tau) = 0,
\]

where \( T = 4K(k_1) \pi \); we will check later on the validity of (5.21). Then we may write

\[
\mathcal{F}_1^{(n)}(\tau) = \int_0^\tau d\sigma F_1^{(n)}(\sigma) = \langle F_1^{(n)} \rangle \tau + \int_0^\tau d\sigma \bar{F}_1^{(n)}(\sigma),
\]

\[
\mathcal{F}_2^{(n)}(\tau) = \int_0^\tau F_2^{(n)}(\sigma) d\sigma = \int_0^\tau d\sigma \bar{F}_2^{(n)}(\sigma),
\]

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and subsequently rewrite (5.20) as
\[
\varphi^{(n)}(\tau) = \varphi^{(n)} + \zeta^{(0)} \tau \bar{I}^{(n)} + \int_0^\tau d\sigma \bar{F}^{(n)}_1(\sigma) + \zeta(0) \langle \mathcal{F}^{(n)}_2 \rangle \tau + \zeta(0) \int_0^\tau d\sigma \mathcal{F}^{(n)}_2(\sigma),
\]
\[
\bar{I}^{(n)}(\tau) = \bar{I}^{(n)} + \int_0^\tau d\sigma \bar{F}^{(n)}_2(\sigma),
\]
in which all the terms which are not linear in \(\tau\) are periodic. If we choose our initial conditions \(\bar{I}^{(n)}\) such that they satisfy
\[
\bar{I}^{(n)} = -\frac{1}{\zeta(0)} \langle F^{(n)}_1 \rangle - \langle \mathcal{F}^{(n)}_2 \rangle,
\]
the above reduces to
\[
\varphi^{(n)}(\tau) = \varphi^{(n)} + \int_0^\tau d\sigma \bar{F}^{(n)}_1(\sigma) + \zeta(0) \int_0^\tau d\sigma \mathcal{F}^{(n)}_2(\sigma),
\]
\[
\bar{I}^{(n)}(\tau) = \bar{I}^{(n)} + \int_0^\tau d\sigma \bar{F}^{(n)}_2(\bar{\tau}),
\]
so that both \(\varphi^{(n)}(\tau)\) and \(\bar{I}^{(n)}(\tau)\) are periodic functions with period \(T\), provided (5.21) holds.

**Lemma 5.1** Consider the series (5.14). If \(p/q = 1/2m, m \in \mathbb{N}\) and \(C_1\) is small enough, then it is possible to fix the initial conditions \((\varphi^{(n)}, \bar{I}^{(n)})\) in such a way that (5.21) holds for all \(n \geq 1\). If \(p/q \neq 1/2m\) for all \(m \in \mathbb{N}\), then (5.21) can be satisfied only for \(C_1 = 0\).

**Proof** For \(n = 1\) we have
\[
F^{(1)}_2(\tau) = \frac{8k_2^2 K(k_1)}{\pi} \cos(\tau - \tau_0) \text{sn}(\sqrt{\alpha} \tau, k_1) \text{cn}(\sqrt{\alpha} \tau, k_1) \text{dn}(\sqrt{\alpha} \tau, k_1)
\]
\[
- \frac{8C_1 k_2^2 \sqrt{\alpha} K(k_1)}{\pi} \text{cn}^2(\sqrt{\alpha} \tau, k_1),
\]
with \(k_1 = k_1^{(0)}\) here and henceforth. Moreover set, see Appendix A.3
\[
\Delta := \frac{\sqrt{\alpha}}{4K(k_1)} \int_0^{4K(k_1)/\sqrt{\alpha}} d\tau \text{cn}^2(\sqrt{\alpha} \tau, k_1)
\]
\[
= \frac{1}{2K(k_1)} \int_0^{2K(k_1)} d\tau \text{cn}^2(\tau, k_1) = \frac{1}{k_1^2} \left[ \frac{1}{2K(k_1)} E(2K(k_1), k_1) - (1 - k_1^2) \right],
\]

(5.22)
where \( E(u, k) \) is the incomplete elliptic integral of the second kind, and

\[
\Gamma_1(\tau_0; p, q) = \frac{1}{T} \int_0^T \text{sn}(\sqrt{\alpha \tau}) \text{cn}(\sqrt{\alpha \tau}) \text{dn}(\sqrt{\alpha \tau}) \cos(\tau - \tau_0) \, d\tau \\
= \frac{\cos(\tau_0)}{T} \int_0^T \text{sn}(\sqrt{\alpha \tau}) \text{cn}(\sqrt{\alpha \tau}) \text{dn}(\sqrt{\alpha \tau}) \cos(\tau) \, d\tau \\
+ \frac{\sin(\tau_0)}{T} \int_0^T \text{sn}(\sqrt{\alpha \tau}) \text{cn}(\sqrt{\alpha \tau}) \text{dn}(\sqrt{\alpha \tau}) \sin(\tau) \, d\tau \\
= \frac{\sin(\tau_0)}{T} \int_0^T \text{sn}(\sqrt{\alpha \tau}) \text{cn}(\sqrt{\alpha \tau}) \text{dn}(\sqrt{\alpha \tau}) \sin(\tau) \, d\tau,
\]

where \( T = 2\pi q = 4K(k_1)p \) and the integral multiplying \( \cos(\tau_0) \) vanishes due to parity.

Thus \( \Gamma_1(\tau_0; p, q) := \sin(\tau_0) G_1(p, q) \), with

\[
G_1(p, q) = \frac{1}{T} \int_0^T \text{sn}(\sqrt{\alpha \tau}, k_1) \text{cn}(\sqrt{\alpha \tau}, k_1) \text{dn}(\sqrt{\alpha \tau}, k_1) \sin(\tau)
\]

\[= \frac{1}{4K(k_1)p} \int_0^{4K(k_1)p} d\tau \text{sn}(\tau, k_1) \text{cn}(\tau, k_1) \text{dn}(\tau, k_1) \sin(\tau/\sqrt{\alpha}). \tag{5.23}\]

Under the resonance condition \( \pi \sqrt{\alpha}/2K(k_1) = p/q \), one has

\[
\sin(\tau/\sqrt{\alpha}) = \sin \left( \frac{\pi \tau}{2K(k_1)} \frac{q}{p} \right),
\]

where \( p \) and \( q \) are relatively prime. By expanding the Jacobi elliptic functions in Fourier series, see Appendix A.3, we find that \( p, q \) must also satisfy the condition

\[
p \left( \pm(2m_1 - 1) \pm (2m_2 - 1) \pm 2m_3 \right) \pm q = 0,
\]

for \( G_1(p, q) \) to be non-zero. Thus \( q = 2mp, m \in \mathbb{N} \), that is \( p = 1 \) and \( q \in 2\mathbb{N} \), and \( \langle F_2^{(1)} \rangle = 0 \) provided \( C_1 \) and \( \tau_0 \) satisfy

\[
C_1 = \frac{\sin(\tau_0)}{\sqrt{\alpha \Delta}} G_1(p, q). \tag{5.24}
\]
Note that the existence of a value of $\tau_0$ satisfying (5.24) is possible only if

$$|C_1| \leq C_1(p/q) := \frac{1}{\sqrt{\alpha\Delta}} G_1(p,q).$$

<table>
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<tr>
<th>$q$</th>
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Table 5.1: Constants $C_1$ for the oscillating attractors with $\alpha = 0.5$.

For all $n \geq 2$ we can write $F_2^{(n)}(\tau)$ as

$$F_2^{(n)}(\tau) = \frac{8k_1^2 K(k_1)}{\pi} \frac{\partial}{\partial \varphi} \left( \cos(\tau - \tau_0) \sin(\cdot) \cos(\cdot) - \sqrt{\alpha} C_1 \cos^2(\cdot) \right) \bigg|_{\varphi=\varphi^{(n-1)}} + R^{(n)}(\tau),$$

where $R^{(n)}(\tau)$ is a suitable function which does not depend on $\varphi^{(n-1)}$. It can be seen that $\langle F_2^{(n)} \rangle = 0$ if and only if

$$\langle R^{(n)} \rangle = -\frac{16k_1^2 K^2(k_1)}{\sqrt{\alpha} \pi^2} \cos(\tau_0) G_1(p,q) \bar{\varphi}^{(n-1)}.$$

This can be rewritten as

$$\langle R^{(n)} \rangle = -\frac{16k_1^2 K^2(k_1)}{\sqrt{\alpha} \pi^2} \cos(\tau_0) G_1(p,q) \bar{\varphi}^{(n-1)},$$

using the evaluation of the integrals

$$\frac{1}{T} \int_0^T \frac{2K(k_1)}{\sqrt{\alpha} \pi} \frac{\partial}{\partial \tau} \left( \cos^2(\sqrt{\alpha}\tau) \right) d\tau = 0,$$
and
\[
\frac{1}{T} \int_0^T 2K(k_1) \frac{\partial}{\partial \tau} \left( \text{sn}(\sqrt{\alpha} \tau) \text{cn}(\sqrt{\alpha} \tau) \text{dn}(\sqrt{\alpha} \tau) \right) \cos(\tau - \tau_0) \, d\tau
\]
\[
= \frac{1}{T} \int_0^T 2K(k_1) \frac{\partial}{\partial \tau} \text{sn}(\sqrt{\alpha} \tau) \text{cn}(\sqrt{\alpha} \tau) \text{dn}(\sqrt{\alpha} \tau) \sin(\tau - \tau_0) \, d\tau
\]
\[
= \frac{\cos(\tau_0)}{T} \int_0^T 2K(k_1) \frac{\partial}{\partial \tau} \text{sn}(\sqrt{\alpha} \tau) \text{cn}(\sqrt{\alpha} \tau) \text{dn}(\sqrt{\alpha} \tau) \sin(\tau) \, d\tau
\]
\[
- \frac{\sin(\tau_0)}{T} \int_0^T 2K(k_1) \frac{\partial}{\partial \tau} \text{sn}(\sqrt{\alpha} \tau) \text{cn}(\sqrt{\alpha} \tau) \text{dn}(\sqrt{\alpha} \tau) \cos(\tau) \, d\tau,
\]
where \( T = 2\pi q = 4K(k_1)p \). The integral multiplying \( \sin(\tau_0) \) vanishes due to parity and hence
\[
2K(k_1) \frac{\partial}{\partial \tau} \left( \text{sn}(\sqrt{\alpha} \tau) \text{cn}(\sqrt{\alpha} \tau) \text{dn}(\sqrt{\alpha} \tau) \right) \cos(\tau - \tau_0) \, d\tau = \frac{2K(k_1)}{\sqrt{\alpha} \pi} \cos(\tau_0) \, G_1(p, q).
\]

The coefficient of \( \tilde{\varphi}^{(n-1)} \) is non-vanishing for \( \tau_0 \) chosen such that (5.24) is satisfied. Therefore it is possible to fix the initial conditions \( \tilde{\varphi}^{(n-1)} \) in such a way that one has \( \langle F_2^{(n)} \rangle = 0 \) at all orders, thus completing the proof of the lemma.

Lemma 5.1 implies that the threshold values of the \( p:q \) resonances are \( \gamma(p/q) = C_1(p/q) \varepsilon \) for \( p = 1 \) and \( q \) even, with the constants \( C_1(p/q) \) in Table 5.1, while the threshold values of the other resonances are at least \( O(\varepsilon^2) \).

5.3.2 Rotations

Similarly, for the rotating scenario the presence of the forcing term leads to the equations
\[
I' = -\frac{\partial p}{\partial \varphi} q' + \frac{\partial q}{\partial \varphi} p' = \frac{4\beta K(k_2)}{\pi} \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot),
\]
\[
\varphi' = \frac{\partial p}{\partial I} q' - \frac{\partial q}{\partial I} p' = \frac{\pi \sqrt{\alpha}}{k_2 K(k_2)} + \frac{\pi k_2 \beta}{\sqrt{\alpha} K(k_2)} \left[ \frac{E(k_2) \varphi \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi(1-k_2^2)} \right. \\
+ \left. \frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1-k_2^2} - \frac{E(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1-k_2^2} \right] \cos(\tau - \tau_0),
\]
where here and throughout this subsection we set 
\[
(\cdot) = \left(\frac{K(k_2)}{\pi} \varphi, k_2\right).
\]
Again, adding a dissipative term, we arrive at
\[
I' = \frac{4\beta K(k_2)}{\pi} \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) - \frac{4\gamma \sqrt{\alpha} K(k_2)}{\pi k_2} \text{dn}^2(\cdot),
\]
\[
\varphi' = \frac{\pi \sqrt{\alpha}}{k_2 K(k_2)} + \frac{\pi k_2 \beta}{\sqrt{\alpha} K(k_2)} \left[ \frac{E(k_2) \varphi \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi(1 - k_2^2)} \right. \\
+ \left. \frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_2^2} - \frac{E(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_2^2} \right] \cos(\tau - \tau_0) \\
- \frac{\gamma \pi}{K(k_2)} \left[ \frac{E(k_2) \varphi \text{dn}^2(\cdot)}{\pi(1 - k_2^2)} + \frac{k_2^2 \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_2^2} - \frac{E(\cdot) \text{dn}^2(\cdot)}{1 - k_2^2} \right].
\]

As for the libration case we use that
\[
E(u,k) = E(k) u / K(k) + Z(u,k)
\]
and set
\[
\gamma = C_1 \varepsilon
\]
so that the above can be written as
\[
\varphi' = \frac{\pi \sqrt{\alpha}}{k_2 K(k_2)} + \frac{\varepsilon \pi k_2}{\sqrt{\alpha} K(k_2)} \left[ \frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_2^2} - \frac{Z(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_2^2} \right] \cos(\tau - \tau_0) \\
- \frac{C_1 \varepsilon \pi}{K(k_2)} \left[ \frac{k_2^2 \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_2^2} - \frac{Z(\cdot) \text{dn}^2(\cdot)}{1 - k_2^2} \right],
\]
\[
I' = \frac{4\varepsilon K(k_2)}{\pi} \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) - \frac{4C_1 \varepsilon \sqrt{\alpha} K(k_2)}{\pi k_2} \text{dn}^2(\cdot).
\]

In this scenario, the Wronskian matrix \(W(\tau)\) can be written as in (5.18), with \(\zeta^{(0)}\) given by, see Appendix A.4,
\[
\zeta = -\frac{\pi^2 \sqrt{\alpha}}{4K^3(k_2)} \left[ \frac{E(k_2)}{1 - k_2^2} - K(k_2) \right],
\]
for \(k_2 = k_2^{(0)}\). We again look for solutions \((\varphi(\tau), I(\tau))\), this time with period \(T = 2\pi q = 4k_2 K(k_1)p / \sqrt{\alpha}\) corresponding to a resonance \(p : q\), of the form (5.14). The functions
\( \varphi^{(n)}(\tau) \) and \( I^{(n)}(\tau) \) are defined as in (5.19), with \( F_1^{(n)}(\tau) \) and \( F_2^{(n)}(\tau) \) defined as

\[
F_1^{(n)}(\tau) = \left[ \frac{\pi \sqrt{\alpha}}{k_2 K(k_2)} - \zeta^{(0)} \right]^{(n)} + \left[ \frac{\pi k_2}{\sqrt{\alpha} K(k_2)} \left( \frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_2^2} \right. \right.
\]
\[
- \frac{Z(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_2^2} \cos(\tau - \tau_0)
\]
\[
- \left. \frac{C_1 \pi}{K(k_2)} \left[ \frac{k_2^2 \text{sn}(\cdot) \text{cn}(\cdot)}{1 - k_2^2} - \frac{Z(\cdot) \text{dn}(\cdot)}{1 - k_2^2} \right] \right]^{(n-1)},
\]
\[
F_2^{(n)}(\tau) = \left[ \frac{4K(k_2)}{\pi} \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) - \frac{4C_1 \sqrt{\alpha} K(k_2)}{\pi k_2} \text{dn}^2(\cdot) \right]^{(n-1)}.
\]

The theory goes through exactly as previously shown for the case of libration and we must show that \( \langle F_2^{(n)} \rangle = 0 \).

**Lemma 5.2** Consider the series (5.14). If \( p/q = 1/2m, m \in \mathbb{N} \), and \( C_1 \) is small enough, then it is possible to fix the initial conditions \( (\varphi^{(n)}, I^{(n)}) \) in such a way that \( \langle F_2^{(n)} \rangle = 0 \) for all \( n \geq 1 \). If \( p/q \neq 1/2m \) for all \( m \in \mathbb{N} \) then \( \langle F_2^{(n)} \rangle = 0 \) only when \( C_1 = 0 \).

**Proof** One has

\[
F_2^{(1)}(\tau) = \frac{4K(k_2)}{\pi} \cos(\tau - \tau_0) \text{sn} \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right) \text{cn} \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right) \text{dn} \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right)
\]
\[
- \frac{4C_1 \sqrt{\alpha} K(k_2)}{\pi k_2} \text{dn}^2 \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right),
\]

with \( k_2 = k_2^{(0)} \) here and henceforth. Define, see Appendix A.3

\[
\Delta := \frac{\sqrt{\alpha}}{2k_2 K(k_2)} \int_{0}^{2k_2 K(k_2)/\sqrt{\pi}} d\tau \ \text{dn}^2 \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right) = \frac{1}{2K(k_2)} \text{E}(2K(k_2), k_2),
\]

and \( \Gamma_1(\tau_0; p, q) := \sin(\tau_0) G_1(p, q) \), where

\[
G_1(p, q) = \frac{1}{T} \int_{0}^{T} d\tau \ \text{sn} \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right) \text{cn} \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right) \text{dn} \left( \frac{\sqrt{\alpha}}{k_2}, k_2 \right) \sin(\tau)
\]
\[
= \frac{1}{4K(k_2)p} \int_{0}^{4K(k_2)p} d\tau \ \text{sn}(\tau, k_2) \text{cn}(\tau, k_2) \text{dn}(\tau, k_2) \sin(k_2 \tau / \sqrt{\alpha}),
\]
then use the resonance condition to set

\[
\sin \left( \frac{k_2 \tau}{\sqrt{\alpha}} \right) = \sin \left( \frac{\pi \tau}{2K(k_2)} p \right).
\]

On inspection of the above we see that \( \Gamma_1(\tau_0; p, q) \) can be calculated similarly to the case inside the separatrix. It follows that the same applies and \( p/q = 1/2m \) for \( m \in \mathbb{N} \). Then

\[
\langle F_2^{(1)} \rangle = 0 \quad \text{if} \quad C_1 = \frac{k_2 \sin(\tau_0)}{\sqrt{\alpha \Delta}} G_1(p, q), \quad (5.27)
\]

which requires

\[
|C_1| \leq C_1(p/q) := \frac{k_2}{\sqrt{\alpha \Delta}} G_1(p, q).
\]

<table>
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<th>( q )</th>
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Table 5.2: Constants \( C_1 \) for the rotating attractors with \( \alpha = 0.5 \).

For \( n \geq 2 \) one has

\[
F_2^{(n)}(\tau) = \frac{4K(k_2)}{\pi} \frac{\partial}{\partial \varphi} \left( \cos(\tau - \tau_0) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) - \sqrt{\alpha} C_1 \text{dn}^2(\cdot) \right) \bigg|_{\varphi = \varphi^{(n-1)}} + R^{(n)}(\tau),
\]

where again \( R^{(n)}(\tau) \) will be a suitable function which does not depend on \( \varphi^{(n-1)} \).

Similarly to the case of libration, \( \langle F_2^{(n)} \rangle = 0 \) if and only if

\[
\langle R^{(n)} \rangle = -\frac{4K(k_2)}{\pi} \left( \frac{1}{T} \int_0^T \frac{k_2 K(k_1)}{\sqrt{\alpha \pi}} \frac{\partial}{\partial \tau} \left( \text{sn}(\sqrt{\alpha \tau}) \text{cn}(\sqrt{\alpha \tau}) \text{dn}(\sqrt{\alpha \tau}) \right) \cos(\tau - \tau_0) \, d\tau \right.
\]

\[
- \frac{\sqrt{\alpha} C_1}{T} \int_0^T \frac{k_2 K(k_1)}{\sqrt{\alpha \pi}} \frac{\partial}{\partial \tau} \left( \text{dn}^2(\sqrt{\alpha \tau}) \right) d\tau \bigg) \varphi^{(n-1)} = -\frac{4k_2 K^2(k_2)}{\sqrt{\alpha \pi^2}} \cos(\tau_0) G_1(p, q) \varphi^{(n-1)},
\]

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so that the coefficient of $\bar{\varphi}^{(n-1)}$ turns out to be non-vanishing for $\tau_0$ chosen such that equation (5.27) is satisfied. Therefore it is possible to fix the initial conditions $\bar{\varphi}^{(n-1)}$ in such a way that one has $\langle F_2^{(n)}(\tau) \rangle = 0$ at all orders, thus completing the proof. □

Lemma 5.2 implies that the threshold values of the $p:q$ resonances are $\gamma(p/q) = C_1(p/q)\varepsilon$ for $p = 1$ and $q$ even, with the constants $C_1(p/q)$ in Table 5.2, while the threshold values of the other resonances are at least $O(\varepsilon^2)$.

Note, in Tables 5.1 and 5.2 it is apparent that, for $\alpha = 0.5$, increasing $q$ causes the value of $C_1(1/q)$ to converge to 0.487577 in both cases. However this does not mean that for $\gamma < 0.487577\varepsilon$ there are infinitely many attracting solutions with increasing period; this would be a counter-example to Palis’ conjecture! The explanation for this seeming paradox is as follows: as $q$ increases the solutions move closer and closer to the separatrix (this can be seen by the corresponding values of $k_1$ and $k_2$), where the power series expansions (5.14) for the solutions $I(\tau)$ and $\varphi(\tau)$ which were constructed with perturbation theory converge only for very small values of $\varepsilon$; the larger $q$, the smaller $\varepsilon$ must be. In particular, for any fixed $\varepsilon$ there is only a finite number of periodic solutions which can be studied by perturbation theory. In particular, for the chosen parameters the only periodic solution corresponds to the resonance 1:2 inside the separatrix. We also note that the above analysis applies only to periodic attractors with $p = 1$ and $q$ even. However we shall see that the numerical simulations provide also rotating attractors with period $2\pi$, that is the same period as the forcing; we expect that continuing the analysis to second order and writing $\gamma = C_2\varepsilon^2$, see [18], would give the threshold values for these periodic attractors.

5.4 Numerics for the pendulum centred about $$(\theta, \theta') = (0, 0)$$

We investigate the system (5.2) in the same region of phase space as studied in [27, 50, 91], namely $\theta \in [-\pi, \pi], \theta' \in [-4, 4]$ and calculate the relative areas of the basins of attraction,
that is the percentage of phase space they cover, relative to this region. Throughout this section we fix the parameters $\alpha = 0.5$ and $\beta = \varepsilon = 0.1$. These parameter values, also investigated in [27], are inside a region of the stability tongues for the corresponding linearised system (linearised about $\theta = 0$), for which the fixed point $\theta = 0$ is Lyapunov stable, see Figure 5.1, Section 5.1 and [96]. Therefore the downward configuration is (Lyapunov) stable even in the absence of dissipation. Adding dissipation causes the unstable regions to move up away from the $\beta = 0$ axis, hence the parameter values never end up inside the unstable tongues, see Appendix A.9 for further details. However, as the full system is nonlinear we also observe periodic orbits which coexist with the fixed point attractor. The chosen values for the damping coefficient span values between $\gamma = 0.002$ and 0.06, of which only $\gamma = 0.03$ was previously investigated in [27]. For some values of $\gamma$, the system exhibits three non-fixed-point attractors, examples of which are shown in Figure 5.2, as well as the downward fixed point attractor $(\theta, \theta') = (0, 0)$, which we denote by DFP. Here and henceforth, for brevity, we shall say that a solution has period $n$ if it comes back to its initial value after $n$ periods of the forcing. Of course the upward fixed point (UFP) also exists as a solution to the system, however it is unstable for the chosen parameter values and thus does not attract any non-zero measure subset of phase space. In Section 5.5 we will consider parameter values for which the upward vertical position (UFP) is stable.

It can also be seen from Figure 5.2 that, for $\alpha = 0.5$ and $\beta = 0.1$ the attractive solutions are near the separatrix of the unperturbed, simple pendulum system; this is evident as the curves described by the two rotating attractors are close to that of the oscillating attractor and the separatrix lies between them. This observation confirms the reasoning as to why the computation of the threshold values in Section 5.3 can only produce valid results for the period 2 oscillating attractor with the chosen parameter values (see the concluding remarks in Section 5.3).
For $\gamma < \bar{\gamma}_0$, for a suitable $\bar{\gamma}_0 \in (0.4, 0.5)$, the system has three periodic attractors, in addition to the downward fixed point: one oscillating and two rotating attractors. For $\gamma \geq \bar{\gamma}_0$ the two rotating attractors no longer exist, leaving just the oscillatory attractor and the fixed point. The basins of attraction for $\gamma = 0.02, 0.03, 0.04$ and $0.05$ are shown in Figure 5.3, from which we can see that the entire phase space is covered; this suggests that no other attractors exist, at least for the values of the parameters considered. It is also apparent that some regions of the basins of attraction appear sparse and sensitive to slight perturbations in initial conditions. This happens when the boundaries of the basins of attraction are fractal, see Appendices A.10 and A.11 for further details. The corresponding relative areas, as estimated by the numerical simulations, are given in Table 5.3 and plotted in Figure 5.4. The relative areas of the basins of attraction for positive and negative rotations have been listed in the same column; numerical simulations found a difference in size no greater than $10^{-2}\%$ and, due to the symmetries of the system, it is expected that this difference is numerically.
induced by the selection of initial conditions. The basins of attraction were estimated using numerical simulations with 600,000 pseudo-random initial conditions in phase space. This results in an error in the first or second decimal place. More notes on the numerics used can be found in Chapter 4. It can be seen that the results in Table 5.3 are in agreement with the calculations for the threshold value for the period 2 oscillatory attractor. The calculations in Section 5.2 predict that, for the chosen values $\alpha = 0.5$ and $\beta = 0.1$, taking $\gamma > \bar{\gamma}_1 \approx 0.1021$ ensures that the origin captures a full measure set of initial conditions.

![Basins of attraction for the system (5.2) with $\alpha = 0.5$, $\beta = 0.1$ and (a) $\gamma = 0.02$, (b) $\gamma = 0.03$, (c) $\gamma = 0.04$ and (d) $\gamma = 0.05$. The downward fixed point (DFP) is shown in blue, the positive and negative rotating solutions (PR and NR) are shown in red and yellow, respectively, and the oscillating solution (DO2) in green.](image)

Figure 5.3: Basins of attraction for the system (5.2) with $\alpha = 0.5$, $\beta = 0.1$ and (a) $\gamma = 0.02$, (b) $\gamma = 0.03$, (c) $\gamma = 0.04$ and (d) $\gamma = 0.05$. The downward fixed point (DFP) is shown in blue, the positive and negative rotating solutions (PR and NR) are shown in red and yellow, respectively, and the oscillating solution (DO2) in green.
From Table 5.3 a stronger result emerges numerically; the fixed point attracts the full phase space, up to a zero-measure set, for $\gamma \geq \bar{\gamma}_2 \approx 0.06$. Upon comparing results in Table 5.3, we see that, essentially, the basin of attraction for the fixed point becomes smaller with an increase in $\gamma$, up to approximately 0.035, after which it grows again. Similarly, by increasing the value of $\gamma$, the basins of attraction of the oscillating and rotating solutions attractors increase initially, up to some value (about $\gamma = 0.025$ and 0.035, respectively), after which they become smaller. Furthermore, the variations of the relative areas of the basins of attraction are never monotonic, as one observes slight oscillations for small variations of $\gamma$. These features seem contrary to systems such as the cubic oscillator, where decreasing dissipation seems to cause the relative area of the basin of attraction of the fixed point to decrease monotonically, while the basins of attraction of the periodic attractors reach a maximum value, after which their relative areas
Table 5.4: Relative areas of the basins of attraction of the main attractors for the system $\ddot{x} + (1 + \varepsilon \cos t)x^3 + \gamma \dot{x} = 0$, with $\varepsilon = 0.1$ and $\gamma$ around 0.0005. The basins of attraction were estimated using numerical simulations with 1000000 random initial conditions in the square $[-1, 1] \times [-1, 1]$ in phase space.

<table>
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<th>1/2</th>
<th>1/4</th>
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<th>1b</th>
<th>1/6</th>
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<td>1.24</td>
<td>1.66</td>
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<td>1.28</td>
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<td>1.30</td>
<td>1.75</td>
<td>0.34</td>
<td>0.34</td>
</tr>
</tbody>
</table>

slightly decrease, see for instance Table III in [22]. We note, however, that a more detailed investigation shows that oscillations occur also in the case of the cubic oscillator. This was already observed for some values of the parameters (see Table IX in [22]), but the phenomenon can also be observed for the parameter values of Table III, simply by considering smaller changes of the value of $\gamma$ with respect to the values in [22]. For instance, by varying slightly $\gamma$ around 0.0005 (see Table III in [22] for notations), one finds for the main attractors the relative areas in Table 5.4.

In conclusion, for the pendulum, apart from small oscillations, by decreasing the value of $\gamma$ from 0.06 to 0.002, the basins of attraction of the periodic attractors, after reaching a maximum value, becomes smaller. A similar phenomenon occurs also in the cubic oscillator (albeit less pronounced). However, a new feature of the pendulum, with respect to the cubic oscillator, is that the basin of attraction of the origin after reaching a minimum value increases again. In the case of the cubic oscillator the slight decrease in the sizes of the basins of attraction of the periodic attractors was due essentially to the appearance of new attractors and their corresponding basins of attraction. This is not so for the pendulum system, where the decrease is in fact related to the bifurcation structure, and the region of parameter space in which the parameters are chosen; a more detailed explanation is given at the end of Section 6.5.
5.4.1 Bifurcation Structure

The bifurcation structure of the pendulum system (5.2) has previously been studied in [49, 79, 36, 66, 67, 56]. The authors studied the bifurcation structure in relation to the amplitude and frequency of the forcing, keeping the coefficient of dissipation fixed at a constant value. First we consider the bifurcation structure of the pendulum system (5.2) in $\alpha$-$\beta$ space, with constant $\gamma$. This frames the results of [49, 79, 66, 67, 56] in the context of our parameters. We note that, due to the use of different parameters, the bifurcation structure appears reversed when compared with the results in [49, 36, 35, 66, 67]. We then fix $\alpha = 0.5$, $\beta = 0.1$ and study the bifurcations leading to the disappearance of the attractors PR, NR and DO2 as $\gamma$ increases, see Table 5.3 and Figure 5.4.

The bifurcation structure may be studied numerically by starting on a particular attractor and allowing a parameter of the system to slowly vary until the attractor is either destroyed or bifurcates into another attractor. This method is simplistic, and will only allow the user to view stable solutions, since trajectories will not converge toward unstable solutions. More rigorous numerical techniques exist which allow the user to view both the stable and unstable solutions which are created at a bifurcation and thus classify a bifurcation with more accuracy. However as there is already a wealth of literature on the bifurcations of the pendulum with oscillating support, we already know the bifurcations which occur and need only to locate them in parameter space. Therefore we are able to use this simplistic approach coupled with knowledge from the literature.

Note that the bifurcation structure is symmetric through the line $\alpha = 0$. Hence, the results in this section may also be applied when the pendulum is centred about the upward vertical position $\theta = \pi$, see Section 5.5. The bifurcation structure of the rotating solutions PR and NR with $\gamma = 0.05$ and 0.2 is shown in Figure 5.5. The choice $\gamma = 0.2$
Figure 5.5: Bifurcation structure of the period 1 rotating attractors for the system (5.2). At line A the rotations appear at a saddle node bifurcation. Line B corresponds to a period doubling bifurcation, where a period 2 rotation is born. Line C corresponds to the start of a period doubling cascade which leads to chaos. The start of chaos is marked by the line slightly above C. In (a) $\gamma = 0.05$ and in (b) $\gamma = 0.2$.

is much larger than the values studied throughout the rest of this section. The reason for also plotting the curves for large $\gamma$ is to illustrate the shifting of the curves for different values of $\gamma$ and so that the results fit with those in Section 5.5.

For fixed $\alpha$, by increasing $\beta$ from zero the 1:1 rotations appear by a saddle-node bifurcation at the curve A. Following the notation in [67] we can classify a solution by how many rotations $r$, it completes in $n$ cycles of the forcing. In [67], the authors note that any $(r,n)$ orbit can only bifurcate from another rotating solution which has the same $r/n$ ratio; hence rotating solutions cannot bifurcate from fixed point solutions or oscillating solutions, which have $r = 0$. For $\alpha$ large enough, by increasing $\beta$ further the rotations undergo a series of period doubling (also called flip) bifurcations leading to chaos. The first and second period doubling bifurcations are marked by lines B and C, respectively. The line above the line C marks the start of chaotic dynamics, which exists in a very small interval of parameter values and disappears at a catastrophic bifurcation (also called crisis bifurcation, see [89]). It is evident in Figure 5.5(b) that for $\alpha$ slightly less than 0.25, by increasing $\beta$ only the first period doubling bifurcation is observed. By
increasing $\beta$ further the period 2 rotating solutions undergo a period halving bifurcation as the line B is crossed a second time. For $\alpha$ less than $\alpha \approx 0.15$ no period doubling bifurcations were observed in the range of the parameter values investigated. The same is also observed in Figure 5.5(a), albeit less pronounced. Note that in Figure 5.5(a), the values $\alpha = 0.5, \beta = 0.1$ lie below the curve A, this is in agreement with Figure 5.3(d) where we can see that the rotating attractors do not exist for $\gamma = 0.05$.

The bifurcation structure of the oscillating solution (DO2) is shown in Figure 5.6. In Figure 5.6(a) $\gamma = 0.05$ and in Figure 5.6(b) $\gamma = 0.2$. Here the focus is on the region of parameter space around the primary instability tongue in Mathieu’s equation (the curve marked as F in the figure). The appearance of the period 2 oscillating attractor of the pendulum with vertically oscillating support has been previously studied in [36, 49, 66, 79]. At the curve F the origin loses stability by a period doubling bifurcation, which is supercritical to the left and subcritical to the right of $c$ [36]. Indeed, if we fix $\alpha$ and let $\beta$ increase, to the left of $c$ the stable fixed point bifurcates into a stable symmetric period 2 oscillation at the curve $F$, while to the right of $c$, along the curve $D$, a stable

![Figure 5.6](image)

Figure 5.6: The line D corresponds to a saddle-node bifurcation, where a stable period 2 oscillation appears, together with an unstable period 2 oscillating solution. The line F is where the fixed point loses stability. At the line E, the oscillating solution undergoes a symmetry breaking bifurcation. The asymmetric oscillations period double at the line G, leading to a period doubling cascade and chaotic dynamics. In (a) $\gamma = 0.05$ and in (b) $\gamma = 0.2$.  

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period 2 oscillation appears by a saddle-node bifurcation [66], together with an unstable period 2 oscillation. The unstable oscillation coalesces with the stable fixed point at the curve F. Note that for fixed $\alpha$ the curve D can be obtained analytically from the perturbation theory calculations in Section 5.3, which express the threshold value for $\gamma$ in terms of $\beta$. In practice, for any fixed value of $\gamma$, we can invert the expression to deduce the value of $\beta$ at which the bifurcation occurs in terms of $\alpha$. In that way, we construct the curve D in $\alpha$-$\beta$ space. It is evident that for $p = 1$, $q = 2$ and $\alpha < 1/4$ the resonance condition $\pi \sqrt{\alpha} q = 2K(k_1)p$ can no longer be satisfied for $k_1 \in (0, 1)$, since $K(k_1) > \pi/2$ for $k_1 > 0$ [107]. This agrees with the fact that the curve D terminates at $\alpha = 1/4$, that is in correspondence of the point c. The line E is where a symmetry breaking bifurcation takes place. Here the symmetric period 2 oscillation bifurcates into two stable asymmetric period 2 oscillations, which are reflections of each other.

Figure 5.7: Regions of existence of four attractors for $\gamma = 0.2$. Lines D to G are as in Figure 5.6(b). The line H (blue) corresponds to a pitchfork bifurcation, where the point $\theta = \pi$ becomes stable. The line K (lower red line) is where the fixed point $\theta = \pi$ becomes unstable at a period doubling bifurcation. The lines L (top two red lines) correspond to a symmetry breaking bifurcation and the start of a period doubling cascade. The lines are coloured for clarity purposes.
through the origin [79]. This is observed later for the inverted pendulum, see the comments in Section 5.5.1. At the line G the asymmetric oscillations undergo a period doubling bifurcation. This is closely followed by a period doubling cascade which leads to chaos, again further comments are given in Section 5.5.1.

Finally in Figure 5.7 the regions of existence of four attractors when $\gamma = 0.2$ are shown, namely DFP, DO2, the upward pointing fixed point (UFP) $\theta = \pi$ and oscillations about the upward pointing fixed point. Lines D to G are labeled as in Figure 5.6. Between the lines H and K the fixed point $(\theta, \dot{\theta}) = (\pi, 0)$ is asymptotically stable. The stability of the inverted position has been previously studied in great detail in [99]. Crossing line H as $\beta$ increases (or $\alpha$ decreases), the fixed point $\theta = \pi$ becomes stable at a reverse pitchfork bifurcation, see [99]. The line K is a continuation of the line F. Here the fixed point $\theta = \pi$ loses stability and bifurcates into a stable symmetric period 2 solution which oscillates about the upward fixed point, see [99] for further details. The two lines L mark a symmetry breaking bifurcation and the first period doubling bifurcation of the asymmetric oscillations. Shortly after the first period doubling bifurcation the solutions transition to chaos, which exists only in a small range of parameter values before being destroyed by a catastrophic bifurcation.

We now fix $\alpha = 0.5$, $\beta = 0.1$ and consider the bifurcations in relation to the parameter $\gamma$. As for the bifurcations in $\alpha$-$\beta$ space, numerical simulations can be used to find estimates for the values of $\gamma$ at which solutions appear/disappear. This is done by starting with initial conditions on a solution, then allowing the parameter $\gamma$ to slowly vary. In this way we are able to see at which value of $\gamma$ the solution vanishes. By starting on a rotating solution and gradually increasing $\gamma$, we can follow the solution and determine where the bifurcation takes place. As $\gamma$ increases past $\gamma \approx 0.043$ the rotating attractors PR/NR disappear. If we consider $\gamma$ decreasing then the rotating attractors appear as $\gamma$ moves below $\gamma \approx 0.043$, although we would be unable to see them using this
Figure 5.8: Saddle-node bifurcation leading to the creation of the positive rotation PR as $\gamma$ decreases. By increasing $\gamma$ we see that the rotations vanish, in this instance the trajectories are then attracted to the equilibrium DFP. (a) The amplitude of Poincaré points on the $\theta$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

approach with $\gamma$ decreasing, since the fixed point (DFP) remains stable. Decreasing $\gamma$ is analogous to increasing the forcing term, although the link is not direct, see Figure 5.5 and the comments later in Section 5.5.1. In particular, as the parameters $\alpha$, $\beta$ are chosen in a region where the fixed point is stable even for $\gamma = 0$, as $\gamma$ decreases the system will never cross the line F in Figure 5.6. Never the less, by decreasing $\gamma$ we cross the line A in Figure 5.5 and the rotating attractors appear by saddle-node bifurcation at $\gamma \approx 0.043$. Figure 5.8 shows the points of a Poincaré section plotted against $\gamma$.

As mentioned earlier, and predicted by the threshold calculations, increasing $\gamma$ through $\gamma \approx 0.05979$ the oscillating attractor disappears. Again if we consider $\gamma$ decreasing the oscillating attractor is created at $\gamma \approx 0.05979$. By considering the location of the parameter values $\alpha$, $\beta$ in Figure 5.6, we see that we are to the right of c, hence as $\gamma$ is decreased the parameters move above the line D. Therefore the period 2 oscillations appear at a saddle node bifurcation, see earlier comments and [66] for a more detailed description. It is expected that, if $\gamma$ is taken small enough, a symmetry breaking bifurcation will occur, see Figure 5.6 and [79, 66, 49], and for yet smaller values of $\gamma$ we would observe a period doubling cascade leading chaotic oscillations, which exist
Figure 5.9: Saddle-node bifurcation leading to the creation of the oscillating attractor DO2 as $\gamma$ decreases. By increasing $\gamma$ we see that the oscillations vanish, in this instance the trajectories are then attracted to the equilibrium DFP. (a) The amplitude of Poincaré points on the $\theta$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

only in a small window of values for $\gamma$. Such behaviour is observed in Section 5.5, where the amplitude of the forcing is taken much larger.

### 5.4.2 Increasing dissipation

In this section we investigate the case where dissipation increases with time, up to a time $T_0$, after which it remains constant. In [37, 38] the authors studied the scenario in which the forcing amplitude initially increases. As mentioned in Section 5.4.1 increasing (decreasing) the coefficient of dissipation is analogous to decreasing (increasing) the forcing amplitude, although the link is not direct. In [37, 38] the equation studied was a damped Duffing equation, with additive forcing. The focus was on escape from a potential well and the authors studied the “safe basin of attraction” which ensured that trajectories remained inside the well. Rather than varying the initial time over which the variation happens, the authors varied the final amplitude of the forcing, keeping the initial amplitude fixed. Although the parameter varied is the forcing amplitude and the forcing is additive rather than parametric, we still expect the four statements outlined at the start of this chapter to apply. Indeed this is evident from the results in [37, 38].
where the increase in amplitude is small. In this case the gradient of the ramped forcing is shallow which is analogous to considering large values of \( T_0 \). We expect that also that for a large increase in forcing amplitude, if the time \( T_0 \) were to be increased and thus the gradient made more shallow, the authors would have seen similar behaviour to that of a small increase in forcing amplitude.

Here we will consider a linear increase in dissipation from an initial value \( \gamma_i \) at time \( t = 0 \) up to a final value \( \gamma_f \) at time \( T_0 \), that is (see Figure 5.10)

\[
\gamma = \gamma(\tau) = \begin{cases} 
\gamma_i + (\gamma_f - \gamma_i) \frac{\tau}{T_0}, & 0 \leq \tau < T_0, \\
\gamma_f, & T_0 \leq \tau.
\end{cases}
\]

(5.28)

Figure 5.10: Plot of equation (5.28) with \( \gamma_i = 0.02, \gamma_f = 0.03 \) and varying \( T_0 \).

Although this is a greatly simplified model of what might take place in reality, it serves the purpose of demonstrating the significant effects of initially non-constant dissipation on the final basins of attraction. Below we will consider explicitly the cases \( \gamma_i = 0.02 \) and \( \gamma_f = 0.03, 0.04 \) and 0.05.

As previously mentioned, we expect that increasing the value of \( T_0 \) results in the relative areas of the basins of attraction moving along the curves plotted for constant \( \gamma \) in Figure 5.4. The movement along these curves starts at \( \gamma_f \) and goes towards \( \gamma_i \). In
particular, any values of the relative area of a basin of attraction for constant values of the damping coefficient between $\gamma_f$ and $\gamma_i$ are traced as the value of $T_0$ varies for time-dependent dissipation. This movement along the curve is not linear with the value of $T_0$ but asymptotic, with the relative area of the basin of attraction tending towards the value at $\gamma = \gamma_i$ as $T_0 \to \infty$, provided the attractors existing at $\gamma = \gamma_i$ also exist at $\gamma_f$. When the attractors which exist at $\gamma = \gamma_f$ are a proper subset of those which exist at $\gamma_i$, we expect the attractors which exist at $\gamma = \gamma_i$ to absorb the remaining phase space left by the attractors which have disappeared; thus their basins of attraction should be greater than or equal to those at $\gamma = \gamma_i$. For the values of the parameters in the chosen range, only these two cases may occur as the solutions which exist for $\gamma = \gamma_f$ also exist at $\gamma = \gamma_i$, see Table 5.3.

The results in Tables 5.5, 5.6 and 5.7 are in agreement with the expectations above. It can be seen from Tables 5.5 and 5.6 that the relative areas of the basins of attraction trace those of constant $\gamma$. In particular, the relative area of the basins of attraction of the rotating attractors, tends towards that at $\gamma = \gamma_i$ from above, despite having a smaller basin of attraction for the chosen values of $\gamma_f$. More precisely, the larger $T_0$, the closer

<table>
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<th>DO2</th>
</tr>
</thead>
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<td>21.23</td>
</tr>
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</tr>
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<td>69.40</td>
<td>4.64</td>
<td>21.33</td>
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<td>4.85</td>
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<tr>
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<td>71.17</td>
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Table 5.5: Relative areas of the basins of attraction with $\gamma_i = 0.02$, $\gamma_f = 0.03$ and $T_0$ varying.

Figure 5.11: Plot of the relative areas of the basins of attraction as per Table 5.5.
the relative area of the basin of attraction is to the value it has at \( \gamma = \gamma_i \). However, the convergence to the asymptotic value is rather slow; for instance in Table 5.5, even \( T_0 = 2000 \) is not enough to reach the values corresponding to \( \gamma = 0.02 \). The simulations for time varying dissipation have in general taken 300 000 or 400 000 initial conditions in phase space. In some cases more points were used for additional accuracy. In Table 5.7, we see that for \( \gamma_i = 0.02 \) and \( \gamma_f = 0.05 \) only two attractors are present; indeed the rotating attractors no longer exist for \( \gamma = 0.05 \). Hence, when \( \gamma(\tau) \) increases and crosses the value at which those attractors disappear, all the trajectories which up to this time
were converging towards them, will fall into the basins of attraction of the attractors which still exist, that is the fixed point and the oscillating solution. In particular the corresponding basins of attraction will acquire relative areas larger than those they have at constant $\gamma = \gamma_i$, because of the absorption of all these trajectories. It is difficult to predict how such trajectories are distributed among the remaining attractors. In the case of Table 5.7 it is expected that they are attracted to the fixed point. This phenomenon is studied further in Chapter 7.

5.4.3 Decreasing dissipation

In this section we conversely look at the damping coefficient decreasing from some value $\gamma_i > \gamma_f$, with different rates of decrease, see Figure 5.14. We will consider the cases $\gamma_i = 0.04$ and $\gamma_f = 0.02$, $\gamma_i = 0.04$ and $\gamma_f = 0.03$, $\gamma_i = 0.05$ and $\gamma_f = 0.02$.

In this situation it is possible that more attractors exist at $\gamma_f$ than at $\gamma_i$, see Table 5.3. We again expect that increasing $T_0$ causes the relative areas of the basins of attraction to tend towards those at $\gamma_i$. As a consequence, solutions which do not exist at $\gamma_i$ will attract less and less of the phase space as $T_0$ increases, and for $T_0$ large enough their basins of attraction will tend to zero.

Figure 5.14: Plot of equation (5.28) with $\gamma_i = 0.04$, $\gamma_f = 0.03$ and varying $T_0$.  

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Tables 5.8 and 5.9 illustrate cases in which the system admits the same set of attractors for both values $\gamma_i$ and $\gamma_f$ of the damping coefficient. An example of what happens when an attractor exists at $\gamma_f$ but not at $\gamma_i$ can be seen in the results of Table 5.10, where $\gamma(\tau)$ varies from 0.05 to 0.02. As the damping coefficient starts off at a larger value, then decreases to some smaller value, we also expect the change in the basins of attraction to happen over shorter values of $T_0$, compared with the case where $\gamma(t)$ increases from a smaller value to a larger value. The reasoning for this is simply that larger values of dissipation cause trajectories to move onto attractors in less time.

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<tr>
<td>2000</td>
<td>73.11</td>
<td>1.63</td>
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</table>

Table 5.8: Relative areas of the basins of attraction with $\gamma_i = 0.04$, $\gamma_f = 0.02$ and $T_0$ varying.

Figure 5.15: Plot of the relative areas of the basins of attraction as per Table 5.8.

<table>
<thead>
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<th>PR/NR</th>
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<tr>
<td>1000</td>
<td>73.11</td>
<td>1.63</td>
<td>23.64</td>
</tr>
</tbody>
</table>

Table 5.9: Relative areas of the basins of attraction with $\gamma_i = 0.04$, $\gamma_f = 0.03$ and $T_0$ varying.

Figure 5.16: Plot of the relative areas of the basins of attraction as per Table 5.9.
### 5.5 Numerics for the inverted pendulum, centred about \((\theta, \theta') = (\pi, 0)\)

The upward fixed point (UFP) of the pendulum can be made stable for values of \(\beta\) which are large relative to \(\alpha\), see Figure 5.7 and [99], this also corresponds to high frequency \(\omega\), see equation (5.1). In this section we numerically investigate the system (5.2) for parameter values for which this happens. Previous investigation of this regime has been carried out in [27, 65, 126]. For simplicity, as mentioned in the introduction, we refer to this case as the inverted pendulum. It can be more convenient to set \(\xi = \theta + \pi\), so as to centre the origin at the upward position of the pendulum. Then the equations of motion become

\[
\xi'' + f(\tau) \sin \xi + \gamma \xi' = 0, \quad f(\tau) = -(\alpha + \beta \cos \tau),
\]

\[
\alpha = \frac{g}{\ell \omega^2}, \quad \beta = \frac{b}{\ell}, \quad \tau = \omega t.
\]

The difference between equations (5.2) and (5.29) is that here the parameter \(\alpha\) has changed sign.
CHAPTER 5. THE PENDULUM WITH VERTICALLY OSCILLATING SUPPORT

The stability of the upward fixed point creates interesting dynamics to study numerically, however it means that the system is no longer a perturbation of the simple pendulum system. This in turn has the result that the analysis in Section 5.3 to compute the threshold values of $\gamma$ cannot be applied. However, we shall see that the very idea that attractors have a threshold value below which they always exist does not apply to the inverted pendulum; both increasing and decreasing the damping coefficient can create and destroy attractors. This happens by crossing bifurcation curves as $\gamma$ is decreased, we refer to sections 5.4.1 and 5.5.1 for details.

For numerical simulations of the inverted pendulum we shall take parameters $\alpha = 0.1$ and $\beta = 0.545$ throughout, which again are within the stable regime for the upward position $\xi = 0$ even in the absence of dissipation. For these parameter values the function $f(\tau)$ changes sign. As such, the analysis in Section 5.2 cannot be applied. Again these particular parameter values were also investigated in [27], but with a small value for the damping coefficient, that is $\gamma = 0.08$, where only three attractors appeared in the system: the upward fixed point and the positive and negative rotating solutions. We have opted to focus on larger dissipation because the range of values considered for $\gamma$ allows us to incorporate already a wide variety of dynamics, in which remarkable phenomena occur, and, at the same time, larger values of $\gamma$ are better suited to numerical simulation because of the shorter integration times. We note that, for the chosen values of the parameters, no strange attractors arise; numerically, besides the fixed points, only periodic attractors are found.

For constant dissipation we provide results for $\gamma \in [0.05, 0.6]$. These values of $\gamma$ are considered to correspond to large dissipation, however non-fixed-point attractors still exist due to the large coefficient of the forcing term, $\beta$. Some examples of the non-fixed-point attractors can be seen in Figure 5.18; of course, the exact form of the curves depends on the particular choices of $\gamma$. For $\gamma$ varying in the range considered
the following attractors arise (we follow the same convention as in previous section when saying that a solution has period $n$): the upward fixed point (UFP), the downward fixed point (DFP), a positively rotating period 1 solution (PR), a negatively rotating period 1 solution (NR), a positively rotating period 2 solution (PR2), a negatively rotating period 2 solution (NR2), an oscillating period 2 solution (DO2) and an oscillating period 4 solution (DO4). However, as we will see, the solution DO2 deserves a separate, more detailed discussion.

The basins of attraction corresponding to the values $\gamma = 0.2, 0.23$ and 0.2725 are shown in Figure 5.19. The relative areas of the basins of attraction for $\gamma \in [0.05, 0.6]$. 

![Graphs of Attracting Solutions](image)

Figure 5.18: Attracting solutions for the system (5.29) with $\alpha = 0.1$ and $\beta = 0.545$. Figure (a) shows an example of the positive and negative rotating attractors with period 2 (PR2 and NR2), taken for $\gamma = 0.05$. Figure (b) shows the positive and negative attractors with period 1 (PR and NR) when $\gamma = 0.23$. Figures (c) and (d) show the oscillatory attractors with periods 2 (DO2) and 4 (DO4) respectively when $\gamma$ is taken equal to 0.2725. These solutions oscillate about the downward fixed point $\xi = \pi$ and the axis has been shifted to show a connected curve in phase space. The period of each solution can be deduced from the circles corresponding to the Poincaré map.
Figure 5.19: Basins of attraction for constant dissipation with $\gamma = 0.2$, 0.23 and 0.2725 from left to right respectively. The upward fixed point (UFP) is shown in blue, the positively rotating solution (PR) in red, the negatively rotating solution (NR) in yellow, the downward oscillation with period 2 (DO2) in green and finally the downward oscillation with period 4 (DO4) in orange.

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Table 5.11: Relative areas of the basins of attraction with constant damping coefficient $\gamma$. The solutions are named as per Figures 5.18 and 5.19 with the addition of the downward fixed point (DFP) and the rotating period 2 solutions (PR2/NR2). For details on the solution DO2 we refer to the text.
are listed in Table 5.11 and plotted in Figure 5.20. Again the positive and negative rotations have been listed together as any difference in the size of their basins of attraction is expected to be due to numerical inaccuracies.

In Figure 5.20, there is a large jump in the relative area of the basin of attraction for the upward fixed point ($\xi = 0$) UFP between the values of $\gamma = 0.22$ and 0.225, approximately; this is due to the appearance of the oscillatory solution which oscillates about the downward pointing fixed point ($\xi = \pi$). For values of $\gamma$ slightly larger than 0.22 large amounts of phase space move close to the solution DO2, where they remain for long periods of time; however they do not land on the solution and are eventually attracted to UFP. The percentage of phase space which does this is marked in Figure 5.20 by a dotted line, which becomes solid when the trajectories remain on the solution for all time (however, see comments below). When the parameter values are chosen such that they are near to parameter values for which the system undergoes a bifurcation, the speed of attraction to solutions decreases; this is known as critical slowing down [162]. This slow rate of attraction is the reason behind the phenomenon where trajectories take a long time to land on the attractor UFP. The closer the parameters are to the values at which the bifurcation occurs, the longer it takes for trajectories to move onto the attractors. The parameter values used in Table 5.11 remained far enough away from the bifurcation that a transient period of 10 000 was long enough to deduce the asymptotic behaviour of trajectories. The parameter values in Table 5.11 were also used to plot Figure 5.20, i.e. the curves are plotted by joining together a discrete set of points, not by continually varying the parameters. Hence the parameters do not pass through any values for which a bifurcation occurs.

The solution DO4 listed in Table 5.11 is found to exist only in the interval $[0.272, 0.27422]$, where it only attracts a small amount of the phase space. As such it has not been included in Figure 5.20.
5.5.1 Bifurcation Structure

As mentioned in Section 5.4.1 the bifurcation structure is symmetric in $\alpha$, $\beta$ parameter space through the line $\alpha = 0$. Hence the bifurcation structure for the system (5.29) is as shown in Figures 5.5 and 5.7. For the upward position $\xi = 0$ ($\theta = \pi$) to become stable we take $\alpha$ small and $\beta$ relatively large, see [99].

In Table 5.11, we see that for $\gamma$ small enough we observe period 2 rotating solutions.
Looking at Figure 5.5(b) one may not expect to observe period a doubling bifurcation when \( \alpha = 0.1 \). However as mentioned in Section 5.4.1, although decreasing (increasing) the coefficient of dissipation is analogous to increasing (decreasing) the forcing amplitude, the link is not direct. In fact, as \( \gamma \) decreases (increases) the bifurcation curves for the rotating solutions move down and to the left (up and to the right). This may be seen by comparing Figure 5.5(a) with Figure 5.5(b). The bifurcation curves in \( \alpha - \beta \) space when \( \gamma = 0.09 \) are shown in Figure 5.21. Here it is clear that the values \( \alpha = 0.1, \beta = 0.545 \) lie in the region where period 2 rotations exist. For \( \gamma = 0.1 \) the chosen parameter values are also in this region, however the period doubling is far less prominent. Furthermore, by increasing \( \beta \) we find that the solution does not undergo a second period doubling bifurcation, instead the rotations cross back over the line B and bifurcate back to period 1 rotations. The amplitude of the Poincaré points as \( \beta \) varies is shown in Figure 5.22, with \( \gamma = 0.09 \) in Figure 5.22(a) and \( \gamma = 0.1 \) in Figure 5.22(b).

We now consider fixed \( \alpha \) and \( \beta \) and study the bifurcations as \( \gamma \) varies. In Figure 5.23 the transition from the period 2 negatively rotating solution to the period 1 negatively rotating solution is shown; by moving towards smaller values of \( \gamma \), this corresponds to
a period doubling bifurcation [85, 63] (period halving, if we think of $\gamma$ as increasing). Period doubling bifurcations occur when one of the Floquet multipliers decreases through negative 1. Figure 5.24 shows the amplitude of the Poincaré points plotted against the value of $\gamma$. It is expected that by decreasing $\gamma$ further a period doubling cascade would be observed, leading to rotating chaos which exists only in a small window of parameter values, see also [67]. Similarly, starting on one of the period 1 rotating attractors and increasing further $\gamma$, the solution disappears at $\gamma \approx 0.2694$. Again if we consider $\gamma$ decreasing (and hence analogous to increasing the forcing amplitude) we find that the period 1 rotating attractors are created at $\gamma \approx 0.2694$. The rotations are created by

![Figure 5.23: The transition from the period 2 rotating solutions to the period 1 rotating solutions. $\gamma = 0.09, 0.094, 0.096$ and 0.098 from (a) to (d) respectively.](image)
saddle-node bifurcation as the line A in Figures 5.5 and 5.21 is crossed. The saddle-node bifurcation is illustrated in Figure 5.25 by use of points in the Poincaré section, plotted against $\gamma$. See [67] for further discussion on the phenomenon. We note that the oscillating attractors DO2 and DO4 oscillate about the downward fixed point $\xi = \pi$. 

Figure 5.24: Period doubling bifurcation of the positive rotating solution as $\gamma$ decreases. (a) The amplitude of Poincaré points on the $\xi$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

Figure 5.25: Saddle-node bifurcation leading to the creation of the positive rotation PR as $\gamma$ decreases. By increasing $\gamma$ we see that the rotations vanish, in this instance the trajectories are then attracted to one of the asymmetric period 2 oscillations DO2. (a) The amplitude of Poincaré points on the $\xi$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.
(\theta = 0). Oscillations about upward the fixed point \(\xi = 0\) are possible and have been previously studied in [3, 65]. However this does not occur for the parameter values investigated throughout Section 5.5. Looking at Figure 5.7 we see that the chosen values of \(\alpha = 0.1, \beta = 0.545\) are below the region where such solutions exist, when \(\gamma = 0.2\). Furthermore, by decreasing \(\gamma\) from 0.2 this region does not move downward significantly, see Figure A.7 in Appendix A.9. Therefore even for the smallest value of \(\gamma\) which we investigate (\(\gamma = 0.05\)) we remain inside the region where \(\xi = 0\) is (asymptotically) stable and no upward oscillations are observed.

![Figure 5.26](image)

**Figure 5.26:** The solution DO2 for different values of time independent \(\gamma\). As the damping coefficient is increased, the solution becomes more clearly defined: this is due to a period halving bifurcation which stops when the period becomes 2. The damping coefficient is \(\gamma = 0.23, 0.235, 0.239\) and 0.24 from (a) to (d), respectively. The position of the trajectory at every \(2\pi\), i.e the period of the forcing, is shown by circles.
We find that the downward oscillatory solution labeled DO2 exists for $\gamma$ in the interval $[0.222, 0.46]$, approximately. However such a solution is really a period 2 solution only for $\gamma$ greater than $\gamma \approx 0.24$. In the interval $[0.222, 0.24]$ the trajectory is “thick”, see Figure 5.26. Furthermore in the interval $[0.24, 0.27]$ there are two asymmetric period 2 solutions which are reflections of one another through the fixed point $\xi = \pi (\theta = 0)$. The curves of the two solutions move closer together as $\gamma$ increases to 0.27. For $\gamma \approx 0.27$ the period 2 oscillations become one single solution as shown in Figure 5.18(c). The splitting of the solution into two asymmetric solutions is known as symmetry breaking.

Figure 5.27: Plots showing the symmetry breaking bifurcation of the period 2 solution DO2 and the transition to chaos via period doubling cascade. (a) and (b) show Poincaré points on the $\xi$ axis plotted against $\gamma$ whilst (c) and (d) show Poincaré points on the $\theta'$ axis. Figures on the right are zoomed in versions of those on the left.
see [79, 66, 65, 49]. The symmetry breaking may be seen by plotting the amplitude of
the Poincaré points against \( \gamma \), see Figure 5.27. The point at which the symmetry is
broken is marked on Figure 5.27(b) by the line \( S_b \). Only due to similarity to the solution
DO2 and to prevent Table 5.11 having yet more columns, the basin of attraction of these
solutions in the range \([0.225, 0.46]\) have also been listed under that of DO2. Nevertheless,
by decreasing \( \gamma \) starting from 0.24 we have a sequence of period doubling bifurcations,
corresponding to values of \( \gamma \) closer and closer to each other. Similar work was conducted
in [79] for the pendulum centred about the downward position (\( \theta = 0 \)), in which the
authors studied asymmetric periodic oscillations as the forcing amplitude in the system
was increased. They noted that a series of period doubling bifurcations follow after a
symmetry breaking bifurcation, see also [35, 65, 66, 49] and Figures 5.6 and 5.7. A
period doubling cascade is expected to lead to a chaotic attractor, which, however, may
survive only for a tiny window of values of \( \gamma \) (at \( \gamma = 0.221 \) it has already definitely
disappeared at a catastrophic bifurcation) and has a very small basin of attraction (for
\( \gamma \) getting closer to the value 0.221 its relative area goes to zero). The period doubling
cascade may be seen in Figure 5.27 as \( \gamma \) decreases from 0.24. A similar figure is also
available in [79], however as the authors looked at varying the forcing amplitude rather
than the dissipative parameter, the figure in [79] appears reversed when compared with
Figure 5.27. The appearance of chaotic attractors for small sets of parameters and with
small basins of attraction has been observed in similar contexts of multistable dissipative
systems close to the conservative limit [75]. For the value \( \gamma = 0.223 \) numerical simulations
find that trajectories remain in the region of phase space occupied by DO2 for a long
time, before eventually moving onto the fixed point. It is expected that this is related
to the chaotic attractor [40].

As \( \gamma \) increases further towards \( \gamma \approx 0.46 \), the amplitude of the period 2 oscillatory
solution gradually decreases and taking \( \gamma \) larger causes a slow spiral into the downward
fixed point, which now becomes stable. Looking at the bifurcation as $\gamma$ decreases, the fixed point $\xi = \pi$ ($\theta = 0$) becomes unstable and a period 2 oscillation is born at a supercritical period doubling bifurcation, see Section 5.4.1 and reference [66] for further

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure528}
\caption{Bifurcation diagram showing the amplitude of Poincaré points as $\gamma$ is varied. (a) The amplitude on the $\xi$ axis of the solution DO2 as it bifurcates into the fixed point $\xi = \pi$. (b) The amplitude on the $\theta'$ axis.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure529}
\caption{Schematic bifurcation diagram for the oscillations DO2. Solid and dashed lines represent stable and unstable solutions, respectively. As $\gamma$ decreases the downward fixed point at $\xi = \pi$ becomes unstable at $\gamma = \gamma_1$ and a period 2 oscillation is created, see Section 5.4.1. At $\gamma = \gamma_s$ the period 2 oscillation undergoes symmetry breaking, creating two mirror image asymmetric stable oscillations, still with period 2. Between $\gamma = \gamma_p$ and $\gamma_c$ the asymmetric solutions undergo a cascade of period doubling bifurcations which lead to chaos before a catastrophic bifurcation at $\gamma = \gamma_c$.}
\end{figure}
details. The amplitude of the Poincaré points is plotted in Figure 5.28. A schematic bifurcation diagram illustrating the full bifurcation structure of the oscillations DO2 is shown in Figure 5.29.

Finally we consider the period 4 oscillations DO4. The attractor DO4 exists only in a small window of values, see Table 5.11, and attracts only a small region of phase space. Such solutions have previously been studied in [66]. Applying the same numerical analysis as was done for the period 1 rotations and period 2 oscillations, we find that as $\gamma$ is decreased the attractor bifurcates into a chaotic oscillation. It is expected that this is a result of a period doubling cascade, however it happens over such a small window of $\gamma$ that it is difficult to see on first inspection. For $\gamma \in [0.2705, 0.2710]$ this chaotic solution vanishes at a catastrophic bifurcation. In the instances studied, after the catastrophic bifurcation the trajectories – which were tending towards the chaotic solution prior to the bifurcation – are attracted to the nearby solution DO2, see Figure 5.30. One may also expect that DO4 undergoes a symmetry breaking bifurcation, see the comments for DO2. However, numerically it is difficult to observe if this is the case as the attractor only exists in a small window of values for $\gamma$. For $\gamma > 0.2725$ the period 4 oscillation is destroyed by a saddle-node bifurcation, see Figure 5.31.

![Figure 5.30: Bifurcation plot showing the amplitude of Poincaré points as $\gamma$ is varied. (a) The amplitude on the $\xi$ axis of the solution DO4 which becomes chaotic as $\gamma$ decreases, leading to a catastrophic bifurcation. (b) The amplitude on the $\theta'$ axis.](image-url)
5.5.2 Increasing dissipation

As mentioned in Section 5.4.2, as $\gamma(\tau)$ increases from $\gamma_i$ to $\gamma_f$ it is expected that taking $T_0$ larger causes the sizes of the basins of attraction to tend towards the sizes of the corresponding basins when $\gamma = \gamma_i$, when the set of attractors remains the same for all values in between. If an attractor is replaced by a new attractor (by bifurcation), then the new attractor inherits the basin of attraction of the old one.

We shall begin by fixing $\gamma_f = 0.2$ and $\gamma_i \in [0.05, 0.2]$, as for $\gamma = \gamma_f$ the basins of attraction are not so sensitive to initial conditions, see Figure 5.19(a), and for $\gamma$ in that range the set of attractors consists only of the the upward fixed point and two rotating solutions; moreover the profiles of the corresponding relative areas plotted in Figure 5.20 are rather smooth and do not present any sharp jumps.

Tables 5.12, 5.13 and 5.14 show the relative area of each basin of attraction as $\gamma$ increases from 0.05, 0.1 and 0.17, respectively, to 0.2 with varying $T_0$. It can be seen from the results in Tables 5.13 to 5.14 that the numerical simulations are in agreement with the above expectation. With the exception of Table 5.12 the relative areas of the
basins of attraction tend towards those when $\gamma$ is kept constant at $\gamma_i$. The exception of the case of Table 5.12 is due to the fact that the set of attractors has changed as $\gamma$ passes from 0.05 to 0.2; the period 2 rotating solutions have been destroyed and replaced by the period 1 rotating solutions. However, when the transition occurs, the new attractors are located in phase space very close to the previous ones and we find that the initial conditions which were heading towards or had indeed landed on the period 2 rotating solutions move onto the now present period 1 rotating solutions. On the other hand, when the damping coefficient crosses the value $\gamma \approx 0.1$, the attractor undergoes topological changes, but, apart from that, the transition is rather smooth; the location in phase space and the basin of attraction change continuously. In conclusion, we find that the relative areas of the basins of attraction for the two period 1 rotating attractors (PR/NR) tend towards those that the (now destroyed) period 2 rotating attractors (PR2/NR2) had at $\gamma = \gamma_i$. As in Section 5.4 we expect that the sizes of the basin of attraction at $\gamma = \gamma_i$ are recovered asymptotically as $T_0 \to \infty$. Nevertheless, once more, the larger $T_0$ the smaller is the variation in the relative area; for instance in Table 5.12 for $T_0 = 100$ the relative area of the basin of attraction of the fixed point has become nearly 1/4 of the value for

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Table 5.12: Relative areas of the basins of attraction with $\gamma_i = 0.05$, $\gamma_f = 0.2$ and $T_0$ varying.

Figure 5.32: Plot of the relative areas of the basins of attraction as per Table 5.12.
We now consider the case where either $\gamma_i = 0.2$ and $\gamma_f = 0.23$ or $\gamma_i = 0.2725$ and $\gamma_f = 0.2725$. Such values for $\gamma$ offer more complexities as not only are there more attractors to consider, but one may have attractors (PR and NR) that are destroyed without leaving any trace. When this happens it is not obvious which of the remaining attractors will inherit their basins of attraction. The result of this could cause the final basins of attraction to be drastically different from those for constant $\gamma$ and even not
monotonically increasing or decreasing as the value of $T_0$ is increased.

In Table 5.15 we see that initially, for values of $T_0$ not too large, the basin of attraction of UFP slightly reduces in size, while those of the rotating solutions PR/NR increase substantially. Instead, for larger values of $T_0$, the basin of attraction of UFP increases appreciably, while those of PR/NR increase very slowly. Apparently, the rotating solutions react more quickly as $\gamma$ is varied, attracting phase space faster, so that the relative areas of their basins of attraction tend towards the values at $\gamma = \gamma_i$ for shorter initial times $T_0$. It would be interesting to study further this phenomenon.

When $\gamma(\tau)$ varies from $\gamma = 0.2$ to $\gamma = 0.2725$ and from $\gamma = 0.23$ to $\gamma = 0.2725$, the rotating solutions PR/NR disappear, so that their basins of attractions are absorbed by the remaining attractors. In Table 5.16 one sees a very slow movement towards global attraction of the upward fixed point, which is the only attractor which exists at both $\gamma_i$ and $\gamma_f$. However even taking $T_0 = 5000$ is not enough for the asymptotic behaviour.

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</table>

Table 5.15: Relative areas of the basins of attraction with $\gamma_i = 0.2$, $\gamma_f = 0.23$ and $T_0$ varying.
to be approached. The results in Table 5.17 show that, by taking $T_0$ larger and larger, the relative areas of the basins of attraction of UFP and DO4 both tend to the values corresponding to $\gamma_i = 0.23$ (in particular the basin of attraction of DO4 becomes negligible). For nearly all values of $T_0$, the trajectories which were converging towards the rotating solutions before the latter disappeared are attracted by the period two oscillations. We will see in Chapter 7 that this is a result of the size and position in phase space of the basin of attraction associated with DO2.

However, the more striking feature of Figures 5.36 and 5.37 are the jumps corresponding $T_0 = 1000$ in the prior and $T_0 = 100$ and $T_0 = 500$ in the latter. Moreover such jumps are very localised; for instance in Figure 5.37 for $T_0 = 99$ and $T_0 = 100$ the

<table>
<thead>
<tr>
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</table>

Table 5.16: Relative areas of the basins of attraction with $\gamma_i = 0.2$, $\gamma_f = 0.2725$ and $T_0$ varying.

<table>
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<th>$T_0$</th>
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<th>DO4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3.35</td>
</tr>
<tr>
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</tr>
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<td>16.04</td>
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<td>0.60</td>
</tr>
<tr>
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<td>18.39</td>
<td>80.27</td>
<td>1.34</td>
</tr>
<tr>
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<td>0.03</td>
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<tr>
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<td>55.83</td>
<td>0.03</td>
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<tr>
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<td>0.03</td>
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<td>20.54</td>
<td>79.42</td>
<td>0.03</td>
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<td>20.40</td>
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</tr>
<tr>
<td>1000</td>
<td>24.81</td>
<td>75.19</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 5.17: Relative areas of the basins of attraction with $\gamma_i = 0.23$, $\gamma_f = 0.2725$ and $T_0$ varying.
basins of attraction of UFP and DO2 are found to be about 44% and 56%, respectively, whereas by slightly increasing or decreasing $T_0$ they settle around 20% and 80%. The quantity of phase space exchanged in these instances is roughly equal to that attracted to the rotating solutions for $\gamma_i$. For particular values of $T_0$ when the rotating attractors disappear their trajectories move to the upward fixed point rather than the period 2 oscillations. The reason behind this will be studied in Chapter 7, though it shall be seen that it is difficult to predict for which values of $T_0$ such jumps occur.

### 5.5.3 Decreasing dissipation

Tables 5.18, 5.19, 5.20 and 5.21 and the corresponding Figures 5.38, 5.39, 5.40 and 5.41 illustrate the cases when dissipation decreases over an initial period of time $T_0$. We have considered the cases with $\gamma_i = 0.23, 0.2725$ and $\gamma_f = 0.2$, with $\gamma_i = 0.2725$ and $\gamma_f = 0.23$ and with $\gamma_i = 0.3$ and $\gamma_f = 0.2725$.

In particular they show that if the set of attractors at $\gamma = \gamma_f$ is a proper subset of the set of attractors which exist at $\gamma = \gamma_0$, then, as $T_0 \to \infty$, the basin of attraction of each attractor which exists at $\gamma_f$ turns out to have a relative area which tend to be greater than or equal to that found for $\gamma = \gamma_i$. In Table 5.18 we consider the situation in which the attractor DO2, which has a large basin of attraction for $\gamma_i = 0.23$, is no longer
## Chapter 5. The Pendulum with Vertically Oscillating Support

### Table 5.18: Relative areas of the basins of attraction with $\gamma_i = 0.23$, $\gamma_f = 0.2$ and $T_0$ varying.

<table>
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</table>

### Figure 5.38: Plot of the relative areas of the basins of attraction as per Table 5.18.

### Table 5.19: Relative areas of the basins of attraction with $\gamma_i = 0.2725$, $\gamma_f = 0.2$ and $T_0$ varying.

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<td>500</td>
<td>17.03</td>
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</tbody>
</table>

### Figure 5.39: Plot of the relative areas of the basins of attraction as per Table 5.19.

### Table 5.20: Relative areas of the basins of attraction with $\gamma_i = 0.2725$, $\gamma_f = 0.23$ and $T_0$ varying.

<table>
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<td>17.03</td>
<td>0.76</td>
<td>81.45</td>
</tr>
</tbody>
</table>

### Figure 5.40: Plot of the relative areas of the basins of attraction as per Table 5.20.
present when \( \gamma(t) \) has reached the final value \( \gamma_f = 0.2 \), as a consequence the trajectories which would be attracted by DO2 at \( \gamma = \gamma_i \) end up moving onto the other attractors; in fact most of them are attracted by the fixed point.

We also notice the interesting features in Table 5.19, as UFP is the only attractor which exists for both \( \gamma_i \) and \( \gamma_f \), we find that as \( T_0 \) increases its basin of attraction tends towards 100%, which corresponds to attraction of the entire phase space, up to a zero-measure set. This happens despite the fact that \( \gamma(\tau) \) does not pass through any value for which global attraction to UFP is satisfied. It also suggests that it is possible to provide conditions on the intersection of the two sets \( A_i \) and \( A_f \) of the attractors corresponding to \( \gamma_i \) and \( \gamma_f \), respectively, in order to obtain that all trajectories move towards the same attractor when the time \( T_0 \) over which \( \gamma(\tau) \) is varied is sufficiently large. In particular, it is remarkable that it is possible to create an attractor for almost all trajectories by suitably tuning the damping coefficient as a function of time.

In Table 5.20, the relative areas of the rotating solutions, which are absent at \( \gamma = \gamma_i \), tend to become negligible when \( T_0 \) is large. Similarly, in Table 5.21, the basin of attraction of the period 4 oscillating attractor, which exists only for the final value \( \gamma_f \) of the damping coefficient, tends to disappear when \( T_0 \) is taken large enough. This confirms the general expectation: the basin of attraction of the disappearing attractor is absorbed by the

<table>
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Table 5.21: Relative areas of the basins of attraction with \( \gamma_i = 0.3, \gamma_f = 0.2725 \) and \( T_0 \) varying.
closer attractor, that is the solution DO2 in this case.
Chapter 6

The pendulum with variable length

The pendulum with periodically varying length is less studied than the pendulum with oscillating support, however it is still a well known and much studied perturbation of the simple pendulum [6, 21, 33, 147, 148, 136]. Rather than the support of the pendulum oscillating vertically, the length of the pendulum oscillates about a given mean length. The two systems are sometimes said to be similar, and in some instances even mistaken for one another [77, 53, 131, 132]. However, this is not the case in all aspects of the dynamics, for example it is already well known that the pendulum with variable length cannot be stabilised about the upward vertical fixed point \((\theta, \dot{\theta}) = (\pi, 0)\) [13, 34]. In this chapter we study the system of a variable length pendulum in the same way that the pendulum with oscillating support was studied in Chapter 5. In Section 6.5 comparisons are drawn between the two systems, highlighting some aspects of the systems which are indeed similar and some where they are different.

6.1 Linearised system

Although the system of the pendulum with variable length has attracted less attention than the system with oscillating support, in recent years much work has been carried out
in an attempt to better understand the dynamics. We have already seen in Section 2.2 that if the parameters of the system satisfy particular conditions, a closed form solution can be obtained. However in general the system exhibits very complicated dynamics and we are unable to obtain an explicit solution. For now we focus our attention on summarising some of the properties of the system which are already known and can be found in the literature [33, 147, 148, 149]. We then extend our analysis to the perturbed system in a similar way as was done in Chapter 5.

The motion of the pendulum with variable length can be described by the equation

\[
\ddot{\theta}(t) + \frac{2\dot{\ell}(t)}{\ell(t)} \dot{\theta}(t) + \frac{g}{\ell(t)} \sin \theta(t) = 0,
\]  

(6.1)

where \( \ell(t) = \ell_0 + \ell_1 \phi(\omega t) \), \( \ell_1 < \ell_0 \), \( \phi \) is a periodic function of time \( t \) with zero mean value and \( \omega \) is the frequency at which the length varies. One immediately obvious difference between this system and that of the pendulum with oscillating support is that, here the equation of motion has a dissipative term resulting naturally from the Lagrangian. Furthermore, the coefficient of dissipation is already time dependent. This dissipation occurs as the length of pendulum varies. When the pendulum extends, the mass moves further away from the centre of rotation. Due to the conservation of momentum the angular velocity is reduced. Similarly, when the pendulum retracts, the mass moves closer to centre of rotation and the angular velocity increases.

The system may be non-dimensionalised by setting \( \tau = \omega t \) and defining dimensionless variables \( \alpha = g/\ell_0 \omega^2 \) and \( \varepsilon = \ell_1/\ell_0 < 1 \), which results in the equation

\[
\theta''(\tau) + \left( \frac{2\varepsilon \phi'(\tau)}{1 + \varepsilon \phi(\tau)} \right) \theta'(\tau) + \frac{\alpha}{1 + \varepsilon \phi(\tau)} \sin \theta(\tau) = 0.
\]  

(6.2)

Using the transformation (2.3), the system may be put into a form which does not include
a term containing $\theta'$. This results in the corresponding system

$$\hat{\theta}(\tau) = e^{\frac{1}{2} \int^{2\ell'/\ell} \theta(\tau) = \ell(\tau)\theta(\tau),}$$

$$\hat{\theta}''(\tau) - \frac{\varepsilon \phi''(\tau)}{1 + \varepsilon \phi} \hat{\theta}(\tau) + \alpha \ell_0 \sin \left( \frac{\hat{\theta}(\tau)}{\ell_0(1 + \varepsilon \phi)} \right) = 0. \quad (6.3)$$

Linearising about the downward fixed point we get an equation in the form of Hill’s equation given by

$$\psi''(\tau) + \frac{\alpha - \varepsilon \phi''(\tau)}{1 + \varepsilon \phi} \psi(\tau) = 0, \quad (6.4)$$

where $\psi(\tau)$ represents the angle between the pendulum and the downward vertical. It may be seen by setting the perturbation equal to zero, that is $\ell_1 = \varepsilon = 0$, that $\sqrt{\alpha} = \sqrt{\alpha_0} = \sqrt{g/(\ell_0 \omega^2)}$ is the natural frequency of the system in time $\tau$. Hence the stability tongues should emanate from the same values of $\alpha$ as in the system of the pendulum with oscillating support, see Appendix A.9. From equation (6.4) it may appear that we are able to calculate the transition curves in the same way as was done for the pendulum with vertically oscillating support in Section 5.1, see Appendix A.9 for the methodology. However, attempting to do so leads to coefficients in the infinite determinant being of order $O(1)$, thus convergence can not be guaranteed. Instead we first expand as

$$\frac{1}{1 + \varepsilon \phi} = \sum_{k=0}^{\infty} (-\varepsilon \phi)^k,$$

under the assumption $\varepsilon \ll 1$. Keeping only the terms at first order in $\varepsilon$ yields

$$\psi''(\tau) + \left( \alpha - \varepsilon (\phi''(\tau) + \alpha \phi(\tau)) \right) \psi(\tau) = 0. \quad (6.5)$$

The above equation is of the form of Hill’s equation and no longer has a division by $\varepsilon$, as such we can now calculate the transition curves in the same way as was done in Section 5.1. Choosing $\phi = \cos(\tau)$ (and hence (6.5) is in the form of Mathieu’s equation) we find
the regions of stability and instability shown in Figure 6.1(a). There are four important points to note about the transition curves shown in Figure 6.1(a):

1. They do not appear as the classical transition curves for Mathieu’s equation, see Appendix A.9. This is a consequence of the forcing coefficient in (6.5) also depending on $\alpha$. More specifically we have that $\zeta_n$ in equation (A.9.3) and $\eta_n$ in (A.9.4) become

\[
\zeta_n = -\frac{\varepsilon(\alpha - 1)}{2(\alpha - n^2)}, \quad \eta_n = -\frac{\varepsilon(\alpha - 1)}{2(\alpha - \frac{1}{4}n^2)}. \tag{6.6}
\]

Therefore when $\alpha = 1$ the forcing term vanishes and hence the stability tongue which originates at $\alpha = 1$ disappears.

2. The curves are plotted as far as $|\varepsilon| = 1$, however they only give us meaningful insight for $|\varepsilon| \ll 1$. Indeed the equation (6.3) is not defined at $\varepsilon = 1$, and the expansion of $1/1 + \varepsilon \cos \tau$ becomes inaccurate for $\varepsilon$ too large.

3. More care must be taken when considering the stability of the upward position $\theta = \pi$.

The transformation $\hat{\theta} = \ell \theta$ does not preserve both fixed points of the system (6.2): under that transformation one has $\theta = 0 \rightarrow \hat{\theta} = 0$, while $\theta = \pi \rightarrow \hat{\theta} = \pi \ell_0(1 + \varepsilon \cos t)$, thus the hyperbolic fixed point $\theta = \pi$ corresponds to a hyperbolic orbit of order $\varepsilon$ in the $\hat{\theta}$ coordinates. However, if we consider the original system (6.2) and transform to the coordinates $\Theta = \theta - \pi$ we obtain the system

\[
\ddot{\Theta}(t) - \frac{2\varepsilon \sin t}{1 + \varepsilon \cos t} \dot{\Theta}(t) - \frac{\alpha}{1 + \varepsilon \cos t} \sin \Theta(t) = 0. \tag{6.7}
\]

It is clear that equation (6.7) is the same as equation (6.2) (with $\phi = \cos \tau$), except for a change in sign of $\alpha$. Then transforming to coordinates $\hat{\Theta}(t) = \ell(t)\Theta(t)$ and linearising around $\hat{\Theta} = 0$ we obtain a system of the form (6.4), where only the sign of $\alpha$ is changed. Therefore, by changing the sign of $\alpha$ in equation (6.4) we are still able to obtain the stability of the inverted position $\theta = \pi$. However, the stable regions shown for $\alpha < 0$ in
Figure 6.1(a) do not correspond to the stability of the fixed point $\theta = \pi$ in the system (6.2), see the next comment.

4. The first order approximation to (6.4) results in a system of the form of Mathieu’s equation. Therefore it is not surprising that there are stable regions for $\alpha < 0$, however this does not correspond to the stability of the upward vertical position of the pendulum with varying length. In Section 6.3 we shall see that first order approximations are not sufficient when considering the pendulum with variable length. Approximating the linearised system (6.4) as far as $O(\varepsilon^8)$, we find the stability curves shown in Figure 6.1(b). It may be seen that for $\alpha > 0$ and $\varepsilon \ll 1$ the the curves show good agreement with those in Figure 6.1(a), whilst for $\alpha < 0$ the stable region has vanished. It is already well known that the fixed point $\theta = \pi$ in the system (6.2) is unstable for all values of the parameters, see [13, 34].

The curve emanating from $\alpha = 0$ in Figure 6.1(b) is still not vertical, as one expects it to be given that the pendulum with variable length cannot be stabilised about the fixed point $\theta = \pi$.

Figure 6.1: Stability regions for the linearised system of the pendulum with periodically varying length (6.4) with $\phi(\tau) = \cos \tau$. In (a) the system is approximated at first order in $\varepsilon$ given by equation (6.5). In (b) the system (6.4) is approximated at $O(\varepsilon^8)$. The broken line shows the curves of the $2\pi$ periodic solutions and the solid line those of the $4\pi$ periodic solutions. Shaded regions are those in which the solutions are unbounded.
upward vertical position. However, we are able to show that the curve tends towards a vertical line as higher orders of $\varepsilon$ are considered.

The transition curves for Hill’s equations, such as (6.4), are obtained by fixing $\alpha = \alpha(\varepsilon)$ in such a way that the corresponding solutions $\psi(t) = \psi(t, \varepsilon)$ are periodic with periods either $2\pi$ or $4\pi$ [116, 12, 96]. For $\varepsilon$ small enough we look for solutions which depend analytically on $\varepsilon$. Therefore we write [96]

$$\alpha(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \alpha_k, \quad \psi(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \psi_k(t), \quad (6.8)$$

and fix recursively the constants $\alpha_k$ in such a way that the functions $\psi_k(t)$ all have the same period (which is either $2\pi$ or $4\pi$). By inserting (6.8) into (6.4) (with $\phi = \cos \tau$) we obtain a sequence of equations,

$$\ddot{\psi}_0(t) + \alpha_0 \psi_0(t) = 0,$$

$$\ddot{\psi}_k(t) + \sum_{k_1, k_2, k_3 \geq 0 \atop k_1 + k_2 + k_3 = k} \alpha_{k_1} (-\cos t)^{k_2} \psi_{k_3}(t) + \cos t \sum_{k_1, k_2 \geq 0 \atop k_1 + k_2 = k-1} (-\cos t)^{k_1} \psi_{k_2}(t) = 0, \quad k \geq 1. \quad (6.9)$$

For $k = 0$ one requires $\alpha_0 = n^2/4$, with $n = 0, 1, 2, \ldots$, since the curves must pass through these values at $\varepsilon = 0$ for the solution to be periodic with period $2\pi$ or $4\pi$, see Appendix A.9. We are interested in the transition curve emanating from $\alpha = \varepsilon = 0$ which gives, by equation (6.9), $\psi_0 = 1$. In fact $\psi_0$ can be any constant value, however we can set $\psi_0 = 1$ without loss of generality. For $k = 1$, (6.9) gives

$$\dot{\psi}_1(t) + \alpha_0 \psi_1(t) - \alpha_0 \cos t \psi_0(t) + \alpha_1 \psi_0(t) + \cos t \psi_0(t)$$

$$= \ddot{\psi}_1(t) + \alpha_1 + \cos t = 0,$$

which yields $\alpha_1 = 0$ and $\psi_1(t) = \cos t$. Then, one can easily prove by induction that $\alpha_k = 0$ and $\psi_k(t) = 0$ for all $k \geq 2$. Indeed, for $k = 2$ (6.9) becomes

$$\ddot{\psi}_2(t) + \alpha_2 - \cos^2 t + \cos^2 t = \ddot{\psi}_2(t) + \alpha_2 = 0,$$
while for \( k > 2 \), by using the inductive hypothesis, one finds

\[
\ddot{\psi}_k(t) + \alpha_k + \cos t \left( (-\cos t)^{k-1} + (-\cos t)^{k-2} \psi_1(t) \right) = \ddot{\psi}_k(t) + \alpha_k = 0.
\]

This shows that \( \alpha(\varepsilon) = 0 \) and \( \psi(t, \varepsilon) = 1 + \varepsilon \cos t \). The solution has been found formally by assuming \( \varepsilon \) to be small, however, as the series reduces to a polynomial, it is in fact defined and solves the equation for all values \( |\varepsilon| < 1 \). Therefore the first transition curve is defined by the vertical line \( \alpha(\varepsilon) = 0 \) (and as a by-product of the construction, the corresponding periodic solution is \( \psi(t, \varepsilon) = 1 + \varepsilon \cos t \)).

Adding a dissipative term to the equation (6.5) results in the equation

\[
\psi''(\tau) + \gamma \psi'(\tau) + \left( \alpha - \varepsilon (\phi''(\tau) + \alpha \phi(\tau)) \right) \psi(\tau) = 0. \tag{6.10}
\]

Taking \( \dot{\phi} = -(\phi''(\tau) + \alpha \phi(\tau)) \) we can apply the stability analysis outlined in Section 2.1 which incorporates dissipation into the stability, see also [33, 148]. In [33, 148], the system was studied and written as a pair of coupled first order differential equations in such a way which allows for the stability analysis to be applied with the integrals \( a_k, b_k \), see Section 2.3, depending only on \( \phi \) and not its derivatives. However as we chose \( \phi = \cos(\tau) \) and thus \( \phi'' = -\phi \) this is not necessary. Following from Section 2.1 we know that instability occurs near frequencies \( \sqrt{\alpha_0} = k/2 \) for \( k \in \mathbb{N} \). For \( \phi = \cos(\tau) \) we find the instability domains are in the regions of parameter space which satisfy the inequality

\[
4 \left( \sqrt{\alpha} - \frac{k}{2} \right)^2 + \gamma^2 < \frac{r_k^2}{4} \left( 1 - \frac{k^2}{4} \right)^2 \varepsilon^2,
\]

where \( r_k \) is defined as \( \sqrt{a_k^2 + b_k^2} \), see equation (2.37). Taking \( k = 1 \), we find the first instability domain is where

\[
\gamma^2 < \frac{9}{16} \varepsilon^2 - (2\sqrt{\alpha} - 1)^2. \tag{6.11}
\]
Note that, as $\gamma$ was added after the transformation to time $\tau$ we have $\gamma = \sqrt{\alpha_0}\beta = k\beta/2$
for some constant $\beta$ which corresponds to the coefficient of dissipation for the system in time $t = \tau/\omega$.

It is also interesting to note that, if the dissipative coefficient is added to the system
(6.2), that is, before applying the transformation (2.3), the Taylor expansion resulting
from the linearised system remains the same as equation (6.10). Hence the analysis
of the instability domains is unchanged by adding the dissipative coefficient before or
after applying the transformation (2.3). In Figure 6.2 we show the first instability
domain corresponding to equation (6.11), with numerical calculations of largest Lyapunov
exponents, see [156, 163, 164]. Figure 6.2 is a recreation of a figure which may be found
in [33]. Throughout the remainder of the chapter we use the full nonlinear system (6.2)
with the inclusion of added dissipation, given by

$$
\theta''(\tau) + \left( \frac{2\varepsilon \phi'(\tau)}{1 + \varepsilon \phi(\tau)} + \gamma \right) \theta'(\tau) + \frac{\alpha}{1 + \varepsilon \phi(\tau)} \sin \theta(\tau) = 0. \quad (6.12)
$$

Figure 6.2: Parameter space of Lyapunov exponents for the system (6.2) with added
dissipative term $\gamma \theta'(\tau)$. The value of $\gamma$ is fixed at 0.0335, as in [33]. The white line
marks the region of the first instability domain predicted by equation (6.11). Black
regions of phase space have Lyapunov exponents $\leq 0$ while positive lyapunov exponents
are coloured by magnitude, increasing from red through to yellow.
6.2 Condition for the origin to attract a full measure set of phase space

The conditions for the origin to attract a full measure set of initial conditions are calculated in a similar way to those for the pendulum with oscillating support in Section 5.2. Therefore we only give an overview here and refer the reader to Section 5.2 for details of the method, see also [24, 23]. We define \( f(\tau) \) and \( g(\tau) \) as

\[
f(\tau) = \frac{\alpha}{1 + \varepsilon \phi(\tau)}, \quad g(\tau) = \frac{2\varepsilon \phi'(\tau)}{1 + \varepsilon \phi(\tau)},
\]

and make the assumption \( \alpha > 0 \) and \( \varepsilon < 1 \) so that \( f(\tau) > 0 \). Applying the Liouville transformation

\[
\tilde{\tau} = \int_0^\tau \sqrt{f(s)} \, ds,
\]

the system may be written in terms of the new time \( \tilde{\tau} \) as

\[
\theta_{\tilde{\tau}} + \left( \frac{2\tilde{g}(\tilde{\tau}) + \tilde{f}(\tilde{\tau})_{\tilde{\tau}}}{2\tilde{f}(\tilde{\tau})} + \frac{\gamma}{\sqrt{\tilde{f}(\tilde{\tau})}} \right) \theta_{\tilde{\tau}} + \sin \theta = 0,
\]

where the subscript \( \tilde{\tau} \) represents derivative with respect to the new time \( \tilde{\tau} \) and \( \tilde{f}(\tilde{\tau}) := f(\tau) \), \( \tilde{g}(\tilde{\tau}) := g(\tau) \). Representing the system in two dimensions on \( \mathbb{T} \times \mathbb{R} \), with \( x(\tilde{\tau}) = \theta(\tilde{\tau}) \) yields

\[
x_{\tilde{\tau}} = y, \quad y_{\tilde{\tau}} = -\frac{y}{\sqrt{\tilde{f}}} \left( \frac{2\tilde{g} + \tilde{f}_{\tilde{\tau}}}{2\sqrt{\tilde{f}}} + \gamma \right) - \sin x,
\]

for which we have the energy \( E(x, y) = 1 - \cos x + y^2/2 \). Setting \( \mathcal{H}(\tilde{\tau}) = E(x(\tilde{\tau}), y(\tilde{\tau})) \) the time derivative of \( \mathcal{H} \) is found as

\[
\mathcal{H}_{\tilde{\tau}} = -\frac{y^2}{\sqrt{\tilde{f}}} \left( \frac{2\tilde{g} + \tilde{f}_{\tilde{\tau}}}{2\sqrt{\tilde{f}}} + \gamma \right),
\]
thus $\mathcal{H}_{\tilde{x}} \leq 0$, i.e, $x, y$ are bounded provided $\gamma$ satisfies

$$\gamma > -\min_{\tilde{\tau} \geq 0} \frac{2\tilde{g} + \tilde{f}_x}{2\sqrt{\tilde{f}}} = -\min_{\tilde{\tau} \geq 0} \frac{2g\sqrt{f} + f'}{2f}. \quad (6.13)$$

Furthermore for all $\tilde{\tau} > 0$ we have that

$$\mathcal{H}(\tilde{\tau}) + \int_{0}^{\tilde{\tau}} \frac{y^2}{\sqrt{f}} \left( \frac{2\tilde{g} + \tilde{f}_x}{2\sqrt{f}} + \gamma \right) ds = \mathcal{H}(0),$$

so that, as $\tilde{\tau} \to \infty$, using the properties above we can arrive at

$$\min_{s \geq 0} \left[ \frac{1}{\sqrt{f}} \left( \frac{2\tilde{g} + \tilde{f}_x}{2\sqrt{f}} + \gamma \right) \right] \int_{0}^{\infty} y^2(s) ds < \infty.$$

Hence $y \to 0$ as time tends to infinity. The same argument used in Section 5.2 may be applied to show that the origin attracts a full measure set of initial conditions provided $f(\tau) > 0$ and $\gamma$ satisfies (6.13).

### 6.3 Threshold values of $\gamma$

The calculations for the threshold values of $\gamma$ for the pendulum with variable length relate very closely to those of the pendulum with oscillating support in Section 5.3. As such the calculations here will be brief and the reader should refer to Section 5.3 for details of the approach used, see also [18, 22]. We begin by writing the system (6.12) in the standard Hamiltonian notation with co-ordinates $p, q$ and set the coefficient of dissipation to be of the same order as the perturbation, yielding

$$p' + \left( \frac{2\varepsilon\phi'}{1 + \varepsilon\phi} + \gamma \right) p + \frac{\alpha}{1 + \varepsilon\phi} \sin q = 0, \quad \gamma = \varepsilon C. \quad (6.14)$$

As in Section 5.3, the calculations for solutions inside and outside of the separatrix are computed separately, we begin with the motion inside of the separatrix.
6.3.1 Librations

It is first necessary to find the action-angle variables for the perturbed system in the absence of dissipation. The time derivative of the action is

\[
I' = -\frac{\partial p}{\partial \varphi} q' + \frac{\partial q}{\partial \varphi} p' = 4\sqrt{\alpha k_1 K(k_1)} \frac{\text{sn}(\cdot) \text{dn}(\cdot)}{\pi} - \frac{4k_1 K(k_1)}{\pi} \text{cn}(\cdot) \left[ \frac{2\varepsilon \phi'}{1 + \varepsilon \phi} p + \frac{\alpha}{1 + \varepsilon \phi} \sin q \right],
\]

where here and throughout this subsection we use the shorthand \((\cdot) = \left( \frac{2K(k_1)}{\pi} \varphi, k_1 \right)\).

Using \(\sin(2 \arcsin(kx)) = 2kx \sqrt{1 - k^2 x^2}\), we may rewrite the above as

\[
I' = \frac{8k_1^2 \alpha K(k_1)}{\pi} \frac{\text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 + \varepsilon \phi} - \frac{16k_1^2 \sqrt{\alpha K(k_1)} \varepsilon \phi' \text{cn}(\cdot)^2}{\pi(1 + \varepsilon \phi)}
\]

Furthermore using

\[
\frac{1}{1 + \varepsilon \phi} = \sum_{n=0}^{\infty} (-\varepsilon \phi)^n,
\]

the derivative of the action may be written in the more applicable form

\[
I' = -\frac{8k_1^2 \alpha K(k_1)}{\pi} \frac{\text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 + \varepsilon \phi} \sum_{n=1}^{\infty} (-\varepsilon \phi)^n - \frac{16k_1^2 \sqrt{\alpha K(k_1)} \varepsilon \phi' \text{cn}(\cdot)^2}{\pi(1 + \varepsilon \phi)} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n.
\]

Similarly the angle variable satisfies

\[
\varphi' = \frac{\partial p}{\partial I} q' - \frac{\partial q}{\partial I} p' = \frac{\pi \sqrt{\alpha}}{2K(k_1)} \left[ 1 + \left( \frac{\text{sn}(\cdot)^2}{\pi(1 - k_1^2)} + \frac{k_1^2 \text{sn}(\cdot)^2 \text{cn}(\cdot)^2}{1 - k_1^2} \right) \right.
\]

\[
- \frac{\text{E}(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_1^2} \sum_{n=1}^{\infty} (-\varepsilon \phi)^n \right] + \frac{\pi \varepsilon \phi' \text{cn}(\cdot)}{K(k_1)} \left[ \frac{\text{sn}(\cdot)}{\text{dn}(\cdot)} + \frac{2\text{E}(k_1) \varphi \text{cn}(\cdot)}{\pi(1 - k_1^2)} \right]
\]

\[
+ \frac{k_1^2 \text{sn}(\cdot) \text{cn}(\cdot)^2}{(1 - k_1^2) \text{dn}(\cdot)} - \frac{\text{E}(\cdot) \text{cn}(\cdot)}{1 - k_1^2} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n.
\]
where again we use the shorthand $I, \varphi$ to represent $I_{osc}, \varphi_{osc}$ respectively. Taking into account the dissipative term we have

$$q' = \frac{\partial q}{\partial I}I' + \frac{\partial q}{\partial \varphi} \varphi' = 2k_1 \sqrt{\alpha} \text{cn}(\cdot), \quad (6.16)$$

so that the action-angle variables satisfy

$$I' = -\frac{8k_1^2 \alpha K(k_1) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi} \sum_{n=1}^{\infty} (-\varepsilon \phi)^n$$

$$- \frac{16k_1^2 \sqrt{\alpha} K(k_1) \varepsilon \varphi' \text{cn}^2(\cdot)}{\pi} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n - \frac{8\gamma k_1^2 \sqrt{\alpha} K(k_1) \text{cn}^2(\cdot)}{\pi},$$

$$\varphi' = \frac{\pi \sqrt{\alpha}}{2K(k_1)} \left\{ 1 + \left[ \text{sn}^2(\cdot) + \frac{2E(k_1) \varphi \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi (1 - k_1^2)} + \frac{k_1^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_1^2} \right] \sum_{n=1}^{\infty} (-\varepsilon \phi)^n \right\} + \frac{\pi \varepsilon \varphi' \text{cn}(\cdot)}{K(k_1)} \left[ \text{sn}(\cdot) \right] \sum_{n=0}^{\infty} (-\varepsilon \phi)^n$$

$$+ \frac{2E(k_1) \varphi \text{cn}(\cdot)}{\pi (1 - k_1^2)} + \frac{k_1^2 \text{sn}(\cdot) \text{cn}^2(\cdot)}{(1 - k_1^2) \text{dn}(\cdot)} - \frac{E(\cdot) \text{cn}(\cdot)}{1 - k_1^2} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n$$

$$+ \frac{\gamma \sqrt{\alpha} \pi \text{cn}(\cdot)}{2K(k_1)} \left[ \text{cn}(\cdot) - \frac{2E(k_1) \varphi \text{sn}(\cdot) \text{dn}(\cdot)}{\pi (1 - k_1^2)} \right]$$

$$- \frac{k_1^2 \text{sn}^2(\cdot) \text{cn}(\cdot)}{1 - k_1^2} + \frac{E(\cdot) \text{sn}(\cdot) \text{dn}(\cdot)}{1 - k_1^2} \right].$$

As in Section 5.3 the angle variable may be simplified using the property $E(u, k) = E(ku/K(k)) + Z(u, k)$, where $Z(u, k)$ is the Jacobi zeta function, see [107, 45]. Note that the equation (6.16) is the same as in Section 5.3.1, a consequence of the two systems having the same unperturbed system. As noticed for the pendulum with oscillating support, it is possible to use the variable $k_1$ instead of the action $I$; as $I = I(k_1)$ depends
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solely on $k_1$. Here we choose to use the variable $k_1$ and use the transformation

$$
k'_1 = \left( \frac{dI}{dk_1} \right)^{-1} = \frac{\pi}{8k_1 \sqrt{\alpha K(k_1)}} I',
$$

which results in the equation

$$
k'_1 = -k_1 \sqrt{\alpha} \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) \sum_{n=1}^{\infty} (-\varepsilon \phi)^n - 2k_1 \varepsilon \phi' \text{cn}^2(\cdot) \sum_{n=0}^{\infty} (-\varepsilon \phi)^n - \varepsilon C k_1 \text{cn}^2(\cdot).
$$

The co-ordinates for the unperturbed system satisfy

$$
\varphi' = \frac{dE}{dI} = \Omega(k_1) = \frac{\pi \sqrt{\alpha}}{2K(k_1)}, \quad k'_1 = 0.
$$

Linearising around $(\varphi^{(0)}, k_1^{(0)}) = (\Omega(k_1^{(0)}), k_1^{(0)})$ we have

$$
\delta \varphi' = \frac{\partial \Omega}{\partial k_1}(k_1^{(0)}) \delta k_1, \quad \delta k'_1 = 0,
$$

where

$$
\zeta = \frac{\partial \Omega}{\partial k_1}(k_1) = -\frac{\pi \sqrt{\alpha}}{2K(k_1)k_1} \left( \frac{E(k_1)}{1 - k_1^2} - K(k_1) \right), \quad (6.17)
$$

The Wronskian matrix is given as per equation (5.18) with $\zeta^{(0)}$ given by setting $k_1 = k_1^{(0)}$ in equation (6.17). We then look for solutions $(\varphi(\tau), k_1(\tau))$ with period $T = 2\pi q = 4K(k_1)p/\sqrt{\alpha}$ with $p/q \in \mathbb{Q}$, of the form

$$
k_1(\tau) = \sum_{n=0}^{\infty} \varepsilon^n k_1^{(n)}(\tau), \quad \varphi(\tau) = \sum_{n=0}^{\infty} \varepsilon^n \varphi^{(n)}(\tau). \quad (6.18)
$$

The functions $(k_1^{(n)}(\tau), \varphi^{(n)}(\tau))$ are given by, see [128]

$$
\begin{pmatrix}
\varphi^{(n)}(\tau) \\
k_1^{(n)}(\tau)
\end{pmatrix} = W(\tau) \begin{pmatrix}
\varphi^{(n)}_1(n) \\
\varphi^{(n)}_2(n)
\end{pmatrix} + W(\tau) \int_{0}^{\tau} d\sigma W^{-1}(\sigma) \begin{pmatrix}
F_1^{(n)}(\sigma) \\
F_2^{(n)}(\sigma)
\end{pmatrix}, \quad (6.19)
$$

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where \( \varphi^{(n)} \) and \( \tilde{k}_1^{(n)} \) are the \( n \)th order in the \( \varepsilon \)-expansion of the initial conditions for \( \varphi \) and \( k_1 \), respectively, while \( F_1^{(n)}(\tau) \) and \( F_2^{(n)}(\tau) \) are given by

\[
F_1^{(n)}(\tau) = \left[ \frac{\pi \sqrt{\alpha}}{2K(k_1)} - \chi^{(0)} k_1 \right]^{(n)} + \sum_{j=1}^{n} \left[ \frac{\pi \sqrt{\alpha}}{2K(k_1)} \left( \sin^2(\cdot) + \frac{k_1^2 \sin^2(\cdot) \cos^2(\cdot)}{1 - k_1^2} \right)
- \frac{Z(\cdot) \sin(\cdot) \cos(\cdot) \cosh(\cdot)}{1 - k_1^2} \right]^{(n-j)} \left( -\phi^j \right)

+ \sum_{s=0}^{n-1} \left[ k_1 \sqrt{\alpha} \sin(\cdot) \cos(\cdot) \cosh(\cdot) (-\phi)^s \right]^{(n-s-1)} \left( -\phi^s \right)

+ \left[ \frac{C \sqrt{\alpha} \pi \sin(\cdot)}{2K(k_1)} \left( \cos(\cdot) - \frac{k_1^2 \sin^2(\cdot) \cos(\cdot)}{1 - k_1^2} + \frac{Z(\cdot) \sin(\cdot) \cosh(\cdot)}{1 - k_1^2} \right) \right]^{(n-1)},
\]

\[
F_2^{(n)}(\tau) = -\sum_{j=1}^{n} \left[ k_1 \sqrt{\alpha} \sin(\cdot) \cos(\cdot) \cosh(\cdot) (-\phi)^j \right]^{(n-j)} \left( -\phi^j \right)

- \sum_{s=0}^{n-1} \left[ 2k_1 \phi' \cos^2(\cdot) (-\phi)^s \right]^{(n-s-1)} - \left[ Ck_1 \cos^2(\cdot) \right]^{(n-1)}.
\]

The theory is the same as in Section 5.3 and we proceed to prove Lemma 5.1 for this instance.

**Proof** As in Section 5.3 we must find \( C \) such that \( \langle F_2^{(n)} \rangle = 0 \). Setting \( n = 1 \) we have

\[
F_2^{(1)}(\tau) = \left[ k_1 \sqrt{\alpha} \sin(\cdot) \cos(\cdot) \cosh(\cdot) \phi - 2k_1 \phi' \cos^2(\cdot) - Ck_1 \cos^2(\cdot) \right]^{(0)}.
\]

Taking \( \phi = \cos(\tau - \tau_0) \) and writing \( k_1 \) as shorthand for \( k_1^{(0)} \) we proceed by defining

\[
\Delta := \frac{\sqrt{\alpha}}{4K(k_1)} \int_0^{4K(k_1)/\sqrt{\alpha}} d\tau \ \cos^2(\sqrt{\alpha}\tau, k_1)

= \frac{1}{2K(k_1)} \int_0^{2K(k_1)} d\tau \ \cos^2(\tau, k_1) = \frac{1}{k_1^2} \left[ \frac{1}{2K(k_1)} \text{E}(2K(k_1), k_1) - (1 - k_1^2) \right],
\]

(6.20)
\[ \Gamma_1(\tau_0; p, q) := \sin(\tau_0) G_1(p, q), \text{ with} \]
\[ G_1(p, q) = \frac{1}{4K(k_1)p} \int_0^{4K(k_1)p} d\tau \, \text{sn}(\tau, k_1) \, \text{cn}(\tau, k_1) \, \text{dn}(\tau, k_1) \sin(\tau/\sqrt{\alpha}), \quad (6.21) \]
and \[ \Lambda_1(\tau_0; p, q) := \sin(\tau_0) H_1(p, q) \]
\[ H_1(p, q) = \frac{1}{4K(k_1)p} \int_0^{4K(k_1)p} d\tau \, \text{cn}^2(\tau, k_1) \cos(\tau/\sqrt{\alpha}). \]

Under the resonance condition \( \pi \sqrt{\alpha}/2K(k_1) = p/q \), one has
\[ \sin(\tau/\sqrt{\alpha}) = \sin \left( \frac{\pi \tau}{2K(k_1) p} \right), \]
with \( p \) and \( q \) relatively prime. Expanding the Jacobi elliptic functions in Fourier series, see Appendix A.3, we find that for \( G_1(p, q) \) to be non-zero, \( p, q \) must satisfy
\[ p \left( \pm (2m_1 - 1) \pm (2m_2 - 1) \pm 2m_3 \right) \pm q = 0, \]
hence \( p/q = 1/2m \) for \( m \in \mathbb{N} \). Then choosing \( C_1 \) and \( \tau_0 \) such that
\[ C_1 = \frac{\sqrt{\alpha} G_1(p, q) - 2H_1(p, q)}{\Delta} \sin \tau_0, \quad (6.22) \]
we have \( \langle \bar{F}_2^{(n)} \rangle = 0 \). Of course the existence of such a \( \tau_0 \) is only possible if
\[ |C_1| \leq C_1(p/q) := \frac{\sqrt{\alpha} G_1 - 2H_1}{\Delta}. \]

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<td>0.243436</td>
<td>0.407121</td>
<td>0.896916</td>
</tr>
<tr>
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<td>0.077675</td>
<td>0.109849</td>
<td>0.224342</td>
<td>0.734474</td>
</tr>
<tr>
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<td>0.051734</td>
<td>0.073163</td>
<td>0.150043</td>
<td>0.731424</td>
</tr>
<tr>
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<td>0.038800</td>
<td>0.054873</td>
<td>0.112539</td>
<td>0.731354</td>
</tr>
<tr>
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<td>0.043902</td>
<td>0.090055</td>
<td>0.731279</td>
</tr>
<tr>
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<td>0.025861</td>
<td>0.036584</td>
<td>0.075020</td>
<td>0.731747</td>
</tr>
</tbody>
</table>

Table 6.1: Values of \( C_1 \) corresponding to oscillations with periods which are even multiples of the forcing period for \( \sqrt{\alpha} = 0.5 \).
For all \( n \geq 2 \) we can write \( F_2^{(n)}(\tau) \) as

\[
F_2^{(n)}(\tau) = -k_1 \sum_{j=1}^{n} \left( -\cos(\tau - \tau_0) \right)^j \frac{\partial}{\partial \varphi} \left. \left( \text{sn}(\cdot) \, \text{cn}(\cdot) \, \text{dn}(\cdot) \right) \right|_{\varphi = \varphi^{(0)}} \left( \varphi^{(n-j)} \right) \\
+ 2k_1 \sum_{s=0}^{n-1} \left( \sin(\tau - \tau_0) \left( -\cos(\tau - \tau_0) \right)^s \frac{\partial}{\partial \varphi} \left( \text{cn}^2(\cdot) \right) \right|_{\varphi = \varphi^{(0)}} \left( \varphi^{(n-s-1)} \right) \\
- C_1 k_1 \frac{\partial}{\partial \varphi} \left( \text{cn}^2(\cdot) \right) \left|_{\varphi = \varphi^{(0)}} \right. \left. + R^{(n)}(\tau), \right.
\]

where \( R^{(n)}(\tau) \) is a suitable function which does not depend on \( \varphi^{(n-1)} \). In order that \( \langle F_2^{(n)} \rangle = 0 \), \( R^{(n)}(\tau) \) must satisfy

\[
\langle R^{(n)} \rangle = \sum_{j=1}^{n} \left[ \frac{1}{T} \int_0^T \frac{2k_1 K(k_1)}{\sqrt{\alpha \pi}} \frac{\partial}{\partial \varphi} \left( \text{sn}(\sqrt{\alpha \tau}) \, \text{cn}(\sqrt{\alpha \tau}) \, \text{dn}(\sqrt{\alpha \tau}) \right) \left( -\cos(\tau - \tau_0) \right)^j \, d\tau \right] \varphi^{(n-j)} \\
- \sum_{s=0}^{n-1} \left[ \frac{1}{T} \int_0^T \frac{4k_1 K(k_1)}{\sqrt{\alpha \pi}} \frac{\partial}{\partial \varphi} \left( \text{cn}^2(\sqrt{\alpha \tau}) \right) \sin(\tau - \tau_0) \left( -\cos(\tau - \tau_0) \right)^s \, d\tau \right] \varphi^{(n-s-1)} \\
+ \left[ \frac{C_1}{T} \int_0^T \frac{2k_1 K(k_1)}{\sqrt{\alpha \pi}} \frac{\partial}{\partial \varphi} \left( \text{cn}^2(\sqrt{\alpha \tau}) \right) \, d\tau \right] \varphi^{(n-1)},
\]

where \( T = 2\pi q = 4K(k_1)p \). This may be simplified by noticing that the last integral vanishes; furthermore by expanding the term \( \sin(\tau - \tau_0) \) in the second integral one obtains

\[
\langle R^{(n)} \rangle = \sum_{j=1}^{n} \left[ \frac{1}{T} \int_0^T \frac{2k_1 K(k_1)}{\sqrt{\alpha \pi}} \frac{\partial}{\partial \varphi} \left( \text{sn}(\sqrt{\alpha \tau}) \, \text{cn}(\sqrt{\alpha \tau}) \, \text{dn}(\sqrt{\alpha \tau}) \right) \left( -\cos(\tau - \tau_0) \right)^j \, d\tau \right] \varphi^{(n-j)} \\
+ \sum_{s=0}^{n-1} \left[ \frac{\cos(\tau_0)}{T} \int_0^T \frac{4k_1 K(k_1)}{\pi} \text{sn}(\sqrt{\alpha \tau}) \, \text{cn}(\sqrt{\alpha \tau}) \, \text{dn}(\sqrt{\alpha \tau}) \, \sin(\tau) \left( -\cos(\tau - \tau_0) \right)^s \, d\tau \right] \varphi^{(n-s-1)}.
\]

We again conclude that the coefficient of \( \varphi^{(n-1)} \) is non-vanishing for \( \tau_0 \) chosen such that equation (6.22) is satisfied. Hence it is possible to fix the initial conditions \( \varphi^{(n-1)} \) in such a way that \( \langle F_2^{(n)} \rangle = 0 \) at all orders. \( \square \)
6.3.2 Rotations

The action-angle variables for the rotations of the perturbed system without dissipation satisfy

\[
I' = -\frac{\partial p}{\partial \varphi} q' + \frac{\partial q}{\partial \varphi} p' \\
= -\frac{4\alpha K(k_2) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi} \sum_{n=1}^{\infty} (-\varepsilon \phi)^n - \frac{8K(k_2)\varepsilon \phi' \sqrt{\alpha} \text{dn}^2(\cdot)}{\pi k_2} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n,
\]

\[
\varphi' = \frac{\partial p}{\partial I} q' - \frac{\partial q}{\partial I} p' = \frac{\pi k_2 \sqrt{\alpha}}{K(k_2)} \left[ \frac{1}{k_2^2} - \left( \frac{\varphi \text{E}(k) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi(1 - k_2^2)} \right) \sum_{n=1}^{\infty} (-\varepsilon \phi)^n \right] \\
+ \frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_2^2} - \frac{\text{E}(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{(1 - k_2^2)} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n,
\]

where here and throughout this subsection we set \( (\cdot) = \left( \frac{K(k_2)}{\pi} \varphi, k_2 \right) \) for convenience.

Note that in finding the expression for \( \varphi' \) we use the following equality, which may be derived using the identities in Appendix A.3

\[
\frac{\text{dn}^2}{k^2} - \frac{k^2 \text{sn}^2 \text{cn}^2}{1 - k^2} + \frac{\text{sn}^2 \text{dn}^2}{1 - k^2} = \frac{\text{dn}^2}{k^2} - \frac{k^2 \text{sn}^2(1 - \text{sn}^2)}{1 - k^2} + \frac{\text{sn}^2 \text{dn}^2}{1 - k^2} \\
= \frac{\text{dn}^2}{k^2} + \frac{\text{sn}^2}{1 - k^2} \left[ \text{dn}^2 - k^2 + k^2 \text{sn}^2 \right] \\
= \frac{\text{dn}^2}{k^2} + \frac{\text{sn}^2}{1 - k^2} \left[ 1 - k^2 \right] \\
= \frac{1}{k^2} \left[ \text{dn}^2 + k^2 \text{sn}^2 \right] = \frac{1}{k^2}.
\]

Again we find the same expression for \( q' \) as in Section 5.3, given by

\[
q' = \frac{\partial q}{\partial I} I' + \frac{\partial q}{\partial \varphi} \varphi' = \frac{2}{k_2} \sqrt{\alpha} \text{dn}(\cdot).
\]
Thus the action-angle variables for the pendulum with variable length with the inclusion of added dissipation satisfy

\[
I' = -\frac{\partial p}{\partial \varphi} q' + \frac{\partial q}{\partial \varphi} p' = -\frac{4\alpha K(k_2) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi} \sum_{n=1}^{\infty} (-\varepsilon \phi)^n \\
- \frac{8K(k_2)\varepsilon' \sqrt{\alpha}}{\pi k_2} \sum_{n=0}^{\infty} (-\varepsilon \phi)^n - \frac{4\gamma \sqrt{\alpha} K(k_2) \text{dn}^2(\cdot)}{k_2 \pi},
\]

\[
\varphi' = \frac{\partial p}{\partial I} q' - \frac{\partial q}{\partial I} p' = \frac{\pi k_2 \sqrt{\alpha}}{K(k_2)} \left[ \frac{1}{k_2^2} - \left( \frac{\varphi E(k) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{\pi (1 - k_2^2)} \right) \sum_{n=1}^{\infty} (-\varepsilon \phi)^n \right] \\
+ \frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_2^2} - \left( \frac{E(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{1 - k_2^2} \right) \sum_{n=1}^{\infty} (-\varepsilon \phi)^n \\
- \frac{2\varepsilon \pi \phi' \text{dn}(\cdot)}{K(k_2)} \left[ \frac{\varphi E(k_2) \text{dn}(\cdot)}{\pi k_2 (1 - k_2^2)} + \frac{k_2 \text{sn}(\cdot) \text{cn}(\cdot)}{1 - k_2^2} - \frac{E(\cdot) \text{dn}(\cdot)}{k_2 (1 - k_2^2)} \right] \sum_{n=0}^{\infty} (-\varepsilon \phi)^n \\
+ \frac{\gamma \sqrt{\alpha} \pi k_2 \text{dn}(\cdot)}{K(k_2)} \left[ \frac{\text{dn}(\cdot)}{k_2^2} + \frac{E(k_2) \varphi \text{sn}(\cdot) \text{cn}(\cdot)}{\pi (1 - k_2^2)} + \frac{\text{sn}^2(\cdot) \text{dn}(\cdot)}{1 - k_2^2} - \frac{E(\cdot) \text{sn}(\cdot) \text{cn}(\cdot)}{1 - k_2^2} \right].
\]

As in Section 5.3 the angle may be simplified using the property \( E(u, k) = E(k) u / K(k) + Z(u, k) \), see [107]. Similar to the case of libration, we find that it is easier to use \( k'_2 \) as a co-ordinate rather than \( I' \). Transforming to \( k'_2 \) we find

\[
k'_2 = \left( \frac{dI}{dk_2} \right)^{-1} I' = \frac{-\pi k_2^2}{4\sqrt{\alpha} K(k_2)} I',
\]

hence

\[
k'_2 = \sqrt{\alpha} k_2^2 \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) \sum_{n=1}^{\infty} (-\varepsilon \phi)^n + 2\varepsilon \phi' k_2 \text{dn}^2(\cdot) \sum_{n=0}^{\infty} (-\varepsilon \phi)^n + \gamma k_2 \text{dn}^2(\cdot).
\]

The coordinates for the unperturbed system (\( \varepsilon = 0 \)) satisfy

\[
\varphi' = \Omega(k_2) = \frac{\pi \sqrt{\alpha}}{k_2^2 K(k_2)}, \quad k'_2 = 0.
\]
linearising around \((\varphi(0)(\tau), k_2(0)(\tau)) = (\Omega(k_2(0))\tau, k_2(0))\), we have

\[
\delta \varphi' = \frac{\partial \Omega}{\partial k_2}(k_2(0)) \delta k_2, \quad \delta k_2' = 0,
\]

where

\[
\zeta = \frac{\partial \Omega}{\partial k_2}(k_2) = -\frac{\pi \sqrt{\alpha}E(k_2)}{k_2^2(1 - k_2^2)K(k_2)}.
\] (6.23)

Fixing \(k_2 = k_2(0)\) in equation (6.23) yields \(\zeta(0)\). The Wronskian matrix \(W(\tau)\) can again be written as equation (5.18) in Section 5.3 with \(\zeta(0)\) given by equation (6.23). As before, we look for solutions of \((\varphi(\tau), k_2(\tau))\), this time with period

\[
T = \frac{2\pi q}{4k_2^2K(k_1)p/\sqrt{\alpha}}
\]

corresponding to a resonance \(p : q\), of the form (6.18), replacing \(k_1\) with \(k_2\). The functions \(\varphi^{(n)}(\tau)\) and \(k_2^{(n)}(\tau)\) are defined as in equation (6.19), with \(F_1^{(n)}(\tau)\) and \(F_2^{(n)}(\tau)\) defined as

\[
F_1^{(n)}(\tau) = \left[\frac{\pi \sqrt{\alpha}}{k_2K(k_2)} - \zeta(0)k_2\right]^{(n)} - \sum_{j=1}^{n} \left[\frac{\pi k_2 \sqrt{\alpha}E(k_2)}{K(k_2)} \right]^{(n-j)} \left[\frac{k_2^2 \text{sn}^2(\cdot) \text{cn}^2(\cdot)}{1 - k_2^2} - \frac{\text{Z}(\cdot) \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)}{(1 - k_2^2)}\right]^{(n-j)} (-\phi)^j \right.
\]

\[
-\sum_{s=0}^{n-1} \left[\frac{2\pi \phi' \text{dn}(\cdot)}{K(k_2)} \left(\frac{k_2 \text{sn}(\cdot) \text{cn}(\cdot)}{1 - k_2^2} - \frac{\text{Z}(\cdot) \text{dn}(\cdot)}{k_2(1 - k_2^2)}\right)(-\phi)^s\right]^{(n-s-1)}
\]

\[
+ \left[\frac{C \sqrt{\alpha} k_2 \text{dn}(\cdot)}{k_2^2(1 - k_2^2)} \left(\frac{\text{dn}(\cdot)}{k_2^2} + \frac{\text{sn}^2(\cdot) \text{dn}(\cdot)}{1 - k_2^2} - \frac{\text{Z}(\cdot) \text{sn}(\cdot) \text{cn}(\cdot)}{1 - k_2^2}\right)\right]^{(n-1)}
\]

\[
F_2^{(n)}(\tau) = \sum_{j=1}^{n} \left[\sqrt{\alpha} k_2^2 \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot)(-\phi)^j\right]^{(n-j)} + \sum_{s=0}^{n-1} \left[2\phi' k_2 \text{dn}^2(\cdot)(-\phi)^s\right]^{(n-s-1)} + \left[C k_2 \text{dn}^2(\cdot)\right]^{(n-1)}.
\]
The theory follows through exactly the same as for all previous cases and we wish to prove Lemma 5.2 for the system of the pendulum with variable length.

**Proof** As in Section 5.3 we require $\langle F_{2}^{(n)} \rangle = 0$ for all $n$. For $n = 1$ we have

$$F_{2}^{(1)} = -\sqrt{\alpha} \left( k_{2}^{(0)} \right)^{2} \text{sn}(\cdot) \text{cn}(\cdot) \text{dn}(\cdot) \phi + 2\phi' k_{2}^{(0)} \text{dn}^{2}(\cdot) + C k_{2}^{(0)} \text{dn}^{2}(\cdot).$$

We now set $\phi(\tau) = \cos(\tau - \tau_{0})$ and to save on clutter, we shall continue by writing $k_{2}$ instead of $k_{2}^{(0)}$. Define

$$\Delta = \frac{\sqrt{\alpha}}{2k_{2}K(k_{2})} \int_{0}^{2k_{2}K(k_{2})} d\tau \text{dn}^{2} \left( \frac{\sqrt{\alpha}}{k_{2}} \tau, k_{2} \right) = \frac{1}{2K(k_{2})} E \left( 2K(k_{2}), k_{2} \right),$$

$$\Gamma_{1}(\tau_{0}; p, q) := \sin(\tau_{0}) G_{1}(p, q),$$

where

$$G_{1}(p, q) = \frac{1}{4K(k_{2})p} \int_{0}^{4K(k_{2})p} d\tau \text{sn}(\tau, k_{2}) \text{cn}(\tau, k_{2}) \text{dn}(\tau, k_{2}) \sin(k_{2}\tau/\sqrt{\alpha}),$$

and $\Lambda_{1}(\tau_{0}; p, q) := H_{1}(p, q)$, where

$$H_{1}(p, q) = \frac{1}{4K(k_{2})p} \int_{0}^{4K(k_{2})p} \text{dn}^{2}(\tau, k_{2}) \cos(k_{2}\tau/\sqrt{\alpha}).$$

We then use the resonance condition to set

$$\sin \left( \frac{k_{2}\tau}{\sqrt{\alpha}} \right) = \sin \left( \frac{\pi \tau}{2K(k_{2})p} \right).$$

As in the previous calculations, when expanding in Fourier Series we find that $p/q$ must satisfy $p/q = 1/2m$ for $m \in \mathbb{N}$. Then $\langle F_{2}^{(1)} \rangle = 0$ if

$$C_{1} = \frac{\sqrt{\alpha}k_{2}G_{1} - 2H_{1}}{\Delta} \sin(\tau_{0}),$$

hence

$$|C_{1}| \leq C_{1}(p/q) = \frac{\sqrt{\alpha}k_{2}G_{1} - 2H_{1}}{\Delta}.$$
For \( n \geq 2 \) we find

\[
P_2^{(n)}(\tau) = \sqrt{\alpha} k_2^2 \sum_{j=1}^{n} \left[ \left( \cos(\tau - \tau_0) \right)^j \frac{\partial}{\partial \varphi} \left( \sin(\varphi) \cos(\varphi) \right) \right] \bigg|_{\varphi = \bar{\varphi}(0)} \tilde{\varphi}^{(n-j)} - 2k_2 \sin(\tau - \tau_0) \sum_{s=0}^{n-1} \left[ \left( \cos(\tau - \tau_0) \right)^s \frac{\partial}{\partial \varphi} \left( \cos^2(\varphi) \right) \right] \bigg|_{\varphi = \bar{\varphi}(0)} \tilde{\varphi}^{(n-s-1)} + C_1 k_2 \frac{\partial}{\partial \varphi} \left( \cos^2(\varphi) \right) \bigg|_{\varphi = \bar{\varphi}(0)} + R^{(n)}(\tau),
\]

where once again \( R^{(n)}(\tau) \) is a suitable function which does not depend on \( \tilde{\varphi}^{(n-1)} \). It can be seen that \( \langle P_2^{(n)} \rangle = 0 \) if and only if

\[
\langle R^{(n)} \rangle = -\sum_{j=1}^{n} \left[ \frac{1}{T} \int_{0}^{T} k_2^2 \frac{\partial}{\partial \tau} \left( \sin(\sqrt{\alpha} \tau) \cos(\sqrt{\alpha} \tau) \right) \left( \cos(\tau - \tau_0) \right)^j d\tau \right] \tilde{\varphi}^{(n-j)} - \sum_{s=0}^{n-1} \left[ \frac{1}{T} \int_{0}^{T} 2k_2^2 \frac{\partial}{\partial \tau} \left( \sin^2(\sqrt{\alpha} \tau) \right) \left( \cos(\tau - \tau_0) \right)^s d\tau \right] \tilde{\varphi}^{(n-s-1)} + \left[ \frac{C_1}{T} \int_{0}^{T} k_2^2 \frac{\partial}{\partial \tau} \left( \sin^2(\sqrt{\alpha} \tau) \right) d\tau \right] \tilde{\varphi}^{(n-1)},
\]

where \( T = 2\pi q = 4K(k_1) \). The last integral of \( \langle R^{(n)} \rangle \) is vanishing and the second integral may be simplified to obtain

\[
\langle R^{(n)} \rangle = -\sum_{j=1}^{n} \left[ \frac{1}{T} \int_{0}^{T} k_2^2 \frac{\partial}{\partial \tau} \left( \sin(\sqrt{\alpha} \tau) \cos(\sqrt{\alpha} \tau) \right) \left( \cos(\tau - \tau_0) \right)^j d\tau \right] \tilde{\varphi}^{(n-j)} - \sum_{s=0}^{n-1} \left[ \frac{\cos(\tau_0)}{T} \int_{0}^{T} 2k_2^2 \frac{\partial}{\partial \tau} \sin(\sqrt{\alpha} \tau) \cos(\sqrt{\alpha} \tau) \sin(\tau) \left( \cos(\tau - \tau_0) \right)^s d\tau \right] \tilde{\varphi}^{(n-s-1)}.
\]

Table 6.2: Values of \( C_1 \) corresponding to rotations with periods which are even multiples of the forcing period for \( \sqrt{\alpha} = 0.5 \)

<table>
<thead>
<tr>
<th>( q )</th>
<th>( k_2 )</th>
<th>( G_1(1/q) )</th>
<th>( H_1(1/q) )</th>
<th>( \Delta )</th>
<th>( C_1(1/q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.924397052341</td>
<td>0.156774</td>
<td>0.204949</td>
<td>0.474414</td>
<td>0.648007</td>
</tr>
<tr>
<td>4</td>
<td>0.998899257272</td>
<td>0.077612</td>
<td>0.109639</td>
<td>0.225808</td>
<td>0.728313</td>
</tr>
<tr>
<td>6</td>
<td>0.99998693601</td>
<td>0.051734</td>
<td>0.073162</td>
<td>0.150063</td>
<td>0.731309</td>
</tr>
<tr>
<td>8</td>
<td>0.99999846887</td>
<td>0.038800</td>
<td>0.054871</td>
<td>0.112540</td>
<td>0.731365</td>
</tr>
<tr>
<td>10</td>
<td>0.99999998199</td>
<td>0.031040</td>
<td>0.043897</td>
<td>0.090032</td>
<td>0.731363</td>
</tr>
<tr>
<td>12</td>
<td>0.999999999978</td>
<td>0.025893</td>
<td>0.036567</td>
<td>0.075026</td>
<td>0.729694</td>
</tr>
</tbody>
</table>
Then the coefficient of $\bar{\varphi}^{(n-1)}$ is non-vanishing provided $\tau_0$ is chosen such that equation (6.24) is satisfied. Therefore it is possible to fix the initial conditions $\bar{\varphi}^{(n-1)}$ so as $\langle F_2^{(n)} \rangle = 0$ at all orders, thus completing the proof. □

Similarly to the calculations in Section 5.3, we find that for $\alpha = 0.5$, increasing $q$ causes the value of $C_1(1/q)$ to converge to a constant, in this instance approximately 0.73. The explanation for this is the same as for the pendulum with oscillating support, see Section 5.3: As $q$ increases the corresponding solutions move closer to the separatrix, where the power series expansions of the action-angle variables for the perturbed system converge only for very small $\varepsilon$. As for the pendulum with oscillating support, the values of $k_1$ and $k_2$ are related to the choice of $\alpha$, in particular $K(k_1) = \pi \sqrt{\alpha q}/2p$ and $k_2 K(k_2) = \pi \sqrt{\alpha q}/2p$ where $p = 1$. Hence by choosing smaller values of $\alpha > 1/4$ it is possible to obtain threshold results for solutions corresponding to higher values of $q$.

6.4 Numerics for the pendulum centred about $(\theta, \theta') = (0, 0)$

We continue to investigate the system of the pendulum with variable length by means of numerical simulation. The study is analogous to that in Section 5.4, in the sense that we follow a similar format and investigate the same region of phase space, namely $\theta \in [-\pi, \pi]$, $\theta' \in [-4, 4]$. We also fix the parameters $\alpha = 0.5$, $\varepsilon = 0.1$ throughout this section. Again the chosen parameter values lie in a region of parameter space for which the system linearised about the fixed point $(\theta, \theta') = (0, 0)$ is (Lyapunov) stable, even for $\gamma = 0$. Further to this, for $\gamma \in [0.005, 0.0903]$, numerically we only find the fixed point attractor and periodic orbits.

We use the same convention as in Section 5.4 when referring to a solution of period $n$, that is the period is $n$ times the period of the forcing. Tables 6.4 to 6.6 show numerical estimates to the relative sizes of the basins of attraction when $\gamma$ is constant throughout
the evolution of the system. For the range of \( \gamma \) investigated, \( \gamma \in [0.003, 0.0903] \), we find a wider variety of attractors than were observed in Section 5.4 for the pendulum with oscillating support. Namely: the downward fixed point (DFP), a positively rotating period 1 solution (PR), a negatively rotating period 1 solution (NR), a second positively rotating period 1 solution (PR1), a second negatively rotating period 1 solution (NR1), a positively rotating period 3 solution (PR3), a negatively rotating period 3 solution (NR3), a positively rotating period 8 solution (PR8), a negatively rotating period 8 solution (NR8), an oscillating period 2 solution (DO2), an oscillating period 4 solution whose curve occupies the left of phase space (OSC L), and an oscillating period 4 solution whose curve occupies the right of phase space (OSC R), examples of which are shown in Figure 6.3. Of these, only solutions similar to DFP, PR, NR and DO2 were observed in the investigation carried out in Section 5.4. The sizes of the basins of attraction for the positively and negatively rotating solutions with period \( n \), as well as OSC L and OSC R, have been listed in the same column as their counter part. This was done as the numerical results showed very small differences between the estimated sizes of their basins of attraction. Furthermore, given the symmetry of the system we expect these small differences to be numerical artifacts only.

It is clear from Figure 6.3(c) that the period 2 oscillating attractors (DO2) cross the rotating attractors (PR/NR) when projected into the phase plane. This happens as the length of the pendulum varies. Through conservation of momentum, when the pendulum extends (or retracts) the angular velocity \( \dot{\theta} \) decreases (or increases). Thus an oscillating solution may temporarily move faster than a rotating solution, without escaping the potential well. This is quite different from the pendulum with oscillating support investigated in Chapter 5, where the curves of the rotating and oscillating attractors did not cross when mapped in to phase plane.

The period one rotating attractors PR1 and NR1, shown in Figure 6.4(a), are found
to exist for particularly small values of $\gamma$, in particular they were observed for $\gamma = 0.003$ and $\gamma = 0.005$. These attractors are interesting as they complete two rotations in each period of the forcing. In [67] the authors studied rotating solutions of the pendulum with

![Pendulum diagrams](image)

Figure 6.3: Examples of the attracting solutions for the system (6.12) with $\alpha = 0.5, \beta = 0.1$. Figures (a) and (b) show the period 1 rotations (PR/NR1) and the period 8 rotations (PR/NR8) respectively for $\gamma = 0.003$. Figure (c) shows the positive and negative rotations (PR/NR) and the period 2 oscillation (DO2) when $\gamma = 0.047$. Figure (d) shows the period 3 rotations (PR/NR3) with $\gamma = 0.04$, (e) and (f) show (OSC L) and (OSC R) respectively for $\gamma = 0.05$. The periods may be deduced from the circles which correspond to the Poincaré map.
oscillating support. They noted that rotating attractors which complete more than one
rotation per period of the forcing were not expected to exist. Further comments on this
phenomena are given in Section 6.5.

For $\gamma = 0.003$ the attractors are DFP, DO2, PR/NR, PR/NR1 and PR/NR8 which
attract 88.93%, 3.78% 1.38%, 2.03% and 0.014% of phase space, respectively.
Figure 6.4: Relative sizes of basins of attraction with constant $\gamma$ as per Tables 6.3, 6.5 and 6.6. The plots are labeled as in Table 6.3 and regions in which a bifurcation takes place are marked with a dot. The basins of attraction for the period 3 rotating solutions (PR3 and NR3) and the oscillatory solutions with period 4 (OSC L and OSC R), have not been included due to their small sizes and the short range of $\gamma$ under which the solutions exist.

However a small proportion ($\approx 0.45\%$) of initial conditions have trajectories whose final behaviour could not be determined. Of course one expects problems to arise in numerical simulations with very small coefficients of dissipation, because the trajectories converge very slowly towards the attractors. Similar problems were experienced for $\gamma = 0.004$. For this reason we only include results for $\gamma \geq 0.005$. It would be interesting to study both the pendulum with oscillating support and the pendulum with varying length with smaller coefficients of dissipation utilising schemes designed specifically for small dissipative coefficients such as conformal symplectic schemes, see for example [121, 119].

The calculations in Section 6.2 predict that, with the chosen values of $\alpha$ and $\varepsilon$, taking $\gamma > \bar{\gamma}_1 \approx 0.1508$ ensures the origin to attract a full measure set of phase space. Similarly to the pendulum with oscillating support, when comparing the result of the calculations with those found by numerical simulation (in Table 6.3), we find the numerical value to be much lower. In particular we find $\gamma \geq \bar{\gamma}_1 \approx 0.09023$ to be sufficient.

It is clear from the results in Table 6.3 that the calculations for the threshold values of $\gamma$ in Section 6.3 do not match up with the numerical results as well as they do for
the pendulum with vertically oscillating support. This is a direct consequence of the perturbation in the system (6.14). For threshold calculations at the first order in \( \varepsilon \) the functions

\[
f(\tau) = \frac{\alpha}{1 + \varepsilon \phi(\tau)}, \quad g(\tau) = \frac{2\varepsilon \phi'(\tau)}{1 + \varepsilon \phi(\tau)},
\]

are approximated using equation (6.15), so that at first order we have

\[
f_1(\tau) = \alpha(1 - \varepsilon \cos \tau), \quad g_1(\tau) = -2\varepsilon \sin \tau.
\] (6.25)

Although the approximation to \( f(t) \) is reasonable, the approximation of \( g(t) \) is significantly different. Hence, although the first order in \( \varepsilon \) is often sufficient for threshold calculations [18, 22, 167], the system itself has been changed. Taking \( \varepsilon \) to be very small, we find that the results become more accurate, see Table 6.7. This is to be expected as \( 1 + \varepsilon \cos \tau \rightarrow 1 \) and the approximation \( g_1(\tau) \) becomes comparable to \( g(\tau) \). If we replace \( g(\tau) \) in equation (6.14) with \( g_1(\tau) \) we obtain

\[
p' + (-2\varepsilon \sin \tau + \gamma) p + \frac{\alpha}{1 + \varepsilon \cos t} \sin q = 0, \quad \gamma = \varepsilon C.
\] (6.26)

It is not surprising then that the calculations in Section 6.3 are in agreement with the numerical threshold of dissipation for equation (6.26) see Table 6.8.

### 6.4.1 Bifurcation Structure

As in Sections 5.4.1 and 5.5.1 we begin by fixing \( \gamma \) at a constant value and study the bifurcation structure in \( \alpha - \varepsilon \) space. To the best of my knowledge, an overview of the
bifurcation structure has not been previously conducted for the pendulum with variable length. However, as one may expect, around the first region of instability for the corresponding linearised equation, similarities exist between the bifurcation structures of the two pendulum systems. Further comments on the comparison of the bifurcation structures are given in Section 6.5.

Figure 6.5 shows the bifurcation structure of the period 1 rotating attractors PR/NR with $\gamma = 0.05$ in (a) and 0.2 in (b). At the curve A the rotating solutions appear at a saddle-node bifurcation. The reasoning is the same as in Section 5.4.1, see also [67]. We label a rotating solution which completes $r$ rotations in $n$ cycles of the forcing period as an $(r,n)$ solution. Then an $(r,n)$ solution can only bifurcate with another solution which has the same $r/n$ ratio. Hence rotating solutions can only bifurcate with other rotating solutions; they cannot bifurcate with the fixed point or oscillating solutions which have $r = 0$. At the curve B the rotations undergo a period doubling bifurcation, resulting in $(2,2)$ rotations. This is closely followed by a period doubling cascade starting at the line C, which leads to chaos.

![Figure 6.5](image)

**Figure 6.5:** Bifurcation structure of the rotating solutions PR/NR in $\alpha-\varepsilon$ space. In (a) $\gamma = 0.05$ and in (b) $\gamma = 0.2$. Lines are labeled so as to match with Figure 5.5, see text for details.
Figure 6.6: Bifurcation structure of the oscillating attractor DO2 in $\alpha$-$\varepsilon$ space. In (a) $\gamma = 0.05$ and in (b) $\gamma = 0.2$. Lines are labeled so as to match with Figure 5.6.

Figure 6.6 shows the bifurcation structure of the period 2 oscillation DO2 with $\gamma = 0.05$. The lines are labeled so as to correspond with Figure 5.6. At the curve D the oscillations appear by saddle-node bifurcation, see Section 5.4.1 for further details. At line E the oscillations undergo symmetry breaking, resulting in two asymmetric period 2 oscillations. These asymmetric oscillations undergo a period doubling cascade starting at the line G, which leads to chaos. Note that the asymmetric oscillations created by the symmetry breaking bifurcation at E are different from the attractors OSC L and OSC R shown in Figure 6.3 (e) and (f). Not only are the attractors OSC L and OSC R periodic with period 4, but they also coexist with the period 2 oscillations. Hence OSC L and OSC R appear as the result of another bifurcation. At line F the fixed point $\theta = 0$ loses stability and small perturbations away from $(\theta, \dot{\theta}) = (0,0)$ result in an initial growth. The bifurcations which occur at the line F are the same as in Section 5.4.1, which we refer to for further details.

We now consider $\alpha = 0.5$, $\varepsilon = 0.1$ fixed and study the disappearance of attractors as $\gamma$ is varied. The rotating attractors disappear as $\gamma$ increases past $\gamma \approx 0.06535$. As stated above, their disappearance is similar to rotations of the pendulum system studied.
in Chapter 5, see also [67]. As no other rotations exist for \( \gamma > 0.066 \), by decreasing \( \gamma \) the rotations PR/NR are created by saddle-node bifurcation corresponding to the line A, see Figure 6.7. If we consider the attractors PR1 and NR1, which are \((2, 1)\) rotations, we find that they also appear as a result of a saddle-node bifurcation, as there are no other attractors with the same \( r/n \) ratio.

Figure 6.7: Saddle-node bifurcation leading to the creation of the positive rotation PR as \( \gamma \) decreases. By increasing \( \gamma \) we see that the rotations vanish, in this instance the trajectories are then attracted to the equilibrium point DFP. (a) The amplitude of Poincaré points on the \( \theta \) axis plotted against \( \gamma \). (b) The amplitude of Poincaré points on the \( \theta' \) axis.

Figure 6.8: Saddle-node bifurcation leading to the creation of the oscillating attractor DO2 as \( \gamma \) decreases. By increasing \( \gamma \) we see that the oscillations vanish, in this instance the trajectories are then attracted to the equilibrium DFP. (a) The amplitude of Poincaré points on the \( \theta \) axis plotted against \( \gamma \). (b) The amplitude of Poincaré points on the \( \theta' \) axis.
CHAPTER 6. THE PENDULUM WITH VARIABLE LENGTH

Figure 6.8 shows the disappearance of the period 2 oscillating attractor DO2 as $\gamma$ is increased. If we consider $\gamma$ decreasing then the attractor DO2 is created as $\gamma$ decreases past $\gamma \approx 0.09022$. By considering the location of the parameter values $\alpha, \varepsilon$ in any of Figures 6.1(a), 6.1(b) or 6.6, we see that we are to the right of the first region of instability. This is similar to the case observed for the pendulum with oscillating support in Section 5.4. Here also, the solution DO2 appears at a saddle-node bifurcation as the line D is crossed, see Figure 6.8 for an illustration.

The attractors PR3, NR3, OSC L and OSC R exist only for small windows of $\gamma$. The rotations PR3 and NR3 exist for $\gamma \in [0.038400, 0.042095]$ and the oscillations OSC L and OSC R exist for $\gamma \in [0.0496, 0.05177]$. The appearance and disappearance of these four solutions may be investigated in the same way as above, see also Section 5.5.

We begin with the rotating solutions PR/NR3. As $\gamma$ increases from 0.038 towards 0.0384, the rotating attractors PR and NR gain some of the basin of attraction associated with the attractor DO2. This may be seen in Figure 6.4 and from the results of Table 6.5, where the basin of attraction belonging to DO2 reduces in size for $\gamma$ between 0.038 and 0.039. When $\gamma \approx 0.0384$ the rotations PR3 and NR3 appear, attracting the region of phase space which DO2 had lost. By studying decreasing $\gamma$ and following the solution NR3 we see that the solution undergoes a period doubling cascade. This is expected to lead to chaos which exists in a small region of parameter space before vanishing via a catastrophic bifurcation. An illustration of points on a Poincaré section plotted against $\gamma$ may be seen in Figure 6.9. Chaotic solutions which exist in small windows of parameter values have also been studied in [67, 75]. As $\gamma$ increases further and passes through $\gamma \approx 0.042095$, the period 3 rotations vanish, see Figure 6.10. If we consider $\gamma$ decreasing then the attractors appear at a saddle-node bifurcation. The reasoning is the same as for the attractors PR/NR and PR/NR1, see also [67]. It is worth noting that the rotations PR/NR3 do not bifurcate with the rotations PR/NR. This may be seen
Figure 6.9: Period doubling cascade of the negative rotation NR3 as $\gamma$ decreases. (a) The amplitude of Poincaré points on the $\theta$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

Figure 6.10: Saddle-node bifurcation leading to the creation of the negative rotation NR3 as $\gamma$ decreases. By increasing $\gamma$ we see that the rotations vanish. (a) The amplitude of Poincaré points on the $\theta$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

numerically from the fact that the attractors coexist for these values of the parameters.

The period 4 asymmetric oscillations OSC L and OSC R appear when the coefficient of dissipation increases to $\gamma \approx 0.0496$. Their basins of attraction occupy a region of phase space which, for slightly smaller values of $\gamma$ is inside the basin of attraction of DO2. The reduction in the size of the basin of attraction associated with DO2 may be clearly seen in Figure 6.4 for $\gamma \approx 0.0496$. Starting at $\gamma > 0.0496$ and decreasing, we find that the oscillations OSC L and OSC R undergo a bifurcation leading to chaos, See Figure 6.11.
Figure 6.11: Transition from the unsymmetric oscillation OSC R to chaos as $\gamma$ decreases. By increasing $\gamma$ we see that the oscillations vanish. (a) The amplitude of Poincaré points on the $\theta$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

This is similar to the behaviour of the the oscillation DO2, and also to the solutions DO2 and DO4 for the pendulum with oscillating support in Section 5.5.

As $\gamma$ increases through $\gamma \approx 0.0518$ the period 4 asymmetric oscillations become unstable and the period 2 oscillations recapture the regions of phase space which had been attracted to OSC L and OSC R, this may again be seen in Figure 6.4. Studying the points on the Poincaré section as $\gamma$ increases, see Figures 6.12, we find that OSC L and OSC R disappear as a result of saddle-node bifurcation. This is again similar to the previous attractors, see also the attractors in Chapter 5. Similar behaviour was studied in [79], where the authors witnessed bifurcations in a pendulum system as the forcing amplitude was increased, see the comments in Section 5.5. Again, we note that the oscillations OSC L and OSC R do not bifurcate from the attractor DO2, as here also the two solutions coexist for the values of the parameters. By considering the bifurcation structure of the oscillating attractors studied previously, one expects OSC L and OSC R to result from a symmetry breaking bifurcation. However, numerically it is difficult to see if this is the case as the attractors exist only in a small window of parameter values.
It would be interesting to study the structure of these attractors in more detail, however it is outside the main aims of the thesis.

![Figure 6.12](image)

Figure 6.12: Saddle-node bifurcation leading to the creation of the unsymmetric oscillation OSC L as $\gamma$ decreases. By increasing $\gamma$ we see that the oscillations vanish. (a) The amplitude of Poincaré points on the $\theta$ axis plotted against $\gamma$. (b) The amplitude of Poincaré points on the $\theta'$ axis.

### 6.4.2 Increasing Dissipation

As in Chapter 5 we wish to study the basins of attraction when the coefficient of dissipation initially varies. We again consider a linear variation from an initial value $\gamma_i$ to a final value $\gamma_f$ over a time $[0, T_0]$, as described by equation (5.28). For all $\tau \geq T_0$ the coefficient of dissipation remains constant at $\gamma = \gamma_f$. We give results for a few choices of $\gamma_i$ and $\gamma_f$, which specifically illustrate the hypotheses outlined at the start of Chapter 5. The results from the numerical simulations with $\gamma_i < \gamma_f$ are in agreement with our expectations, see Tables 6.9 to 6.13 along with the corresponding figures. The results in Tables 6.10 and 6.12 illustrate an interesting phenomenon, captured by the case $\mathcal{A}_i \subseteq \mathcal{A}_f$; see the first statement at the beginning of Chapter 5. In this instance, $\gamma_i$ and $\gamma_f$ are chosen such that $\mathcal{A}_i = \mathcal{A}_f$, but as $\gamma(\tau)$ is varied, new attractors appear and then disappear. What happens is that the variation of the basins of attraction follow as
in the first statement in Chapter 5. As $\gamma(\tau)$ is varied over longer durations of time, that is, as $T_0$ is taken larger, the sizes of the basins of attraction trace the curves plotted in Figure 6.4, moving from $\gamma_f$ towards $\gamma_i$. However, the troughs seen in Figure 6.4 for $\gamma$ in the intervals $[0.038, 0.043]$ and $[0.49, 0.52]$ are not observed in Figures 6.14 and 6.16 respectively. The explanation for the phenomenon is as follows: consider an attracting solution $a^*$, which exists for $\gamma$ in some small interval $I_1$ and remains close to another attractor $a_p$ which exists for $\gamma$ in an interval $I_2$ such that $I_1 \subset I_2$. Furthermore for $\gamma \in I_1$ the basin of attraction of $a^*$ occupies a region of phase space which is inside the basin of attraction of $a_p$ when $\gamma \in I_2 \setminus I_1$. Now consider the coefficient of dissipation $\gamma(\tau)$ as a function of time which varies continuously from $\gamma_i \in I_2$ to $\gamma_f \in I_2$ such that $\gamma_i$ and $\gamma_f$ are on opposite sides of the interval $I_1$. As $\gamma(\tau)$ enters the interval $I_1$ the attractor $a^*$ appears and is stable (for example the attractors PR/NR3 for $\gamma(t) \in [0.038, 0.043]$ and OSC L/R for $\gamma(\tau) \in [0.49, 0.52]$). However as $\gamma(\tau)$ increases/decreases toward $\gamma_f$ and exits $I_1$ the attractor $a^*$ becomes vanishes, thus any trajectory which was moving toward or had landed on $a^*$ will be attracted to one of the attractors which exists for $\gamma(\tau) \in I_2 \setminus I_1$. It happens that such trajectories are captured by the attractor $a_p$ due to

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Table 6.9: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.007$ to $\gamma_f = 0.037$. Figure 6.13: Plot of the relative areas of the basins of attraction as per Table 6.9.
CHAPTER 6. THE PENDULUM WITH VARIABLE LENGTH

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Table 6.10: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.037$ to $\gamma_f = 0.045$.

Figure 6.14: Plot of the relative areas of the basins of attraction as per Table 6.10.

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Table 6.11: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.049$ to $\gamma_f = 0.051$.

Figure 6.15: Plot of the relative areas of the basins of attraction as per Table 6.11.

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Table 6.12: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.049$ to $\gamma_f = 0.052$.

Figure 6.16: Plot of the relative areas of the basins of attraction as per Table 6.12.
the closeness of the attractors. Hence the troughs in the size of the basin of attraction corresponding to $a_p$ for constant $\gamma \in I_1$ are not present when $\gamma(\tau)$ crosses over $I_1$ and $\gamma_i, \gamma_f \notin I_1$.

### 6.4.3 Decreasing Dissipation

We now consider decreasing dissipation, that is $\gamma_i > \gamma_f$. Again the results of the simulations show that the variation in the basins of attraction follow the hypotheses stated at the start of Chapter 5. Table 6.15 and the corresponding figure show results with $\gamma_i = 0.045$ and $\gamma_f = 0.037$, these values of $\gamma$ were also chosen in Table 6.10, but the initial and final values have been reversed. We find that exactly the same behaviour occurs in respect to how the basins of attraction vary. Once again the trough in the basins of attraction for the rotations PR/NR seen in Figure 6.4 is not observed. However, with decreasing dissipation we find the sizes of the basins of attraction vary more quickly, that is, smaller values of $T_0$ are required. This is a result of the larger initial dissipation which causes trajectories to move onto the attractors in less time. Hence the sizes of the basins of attraction tend towards those for constant $\gamma = \gamma_i$ for lower values of $T_0$. Indeed we find that for $T_0 = 1000$ the sizes of the basins of attraction are already close to those

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Table 6.13: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.052$ to $\gamma_f = 0.07$.

Figure 6.17: Plot of the relative areas of the basins of attraction as per Table 6.13.
corresponding to constant $\gamma = \gamma_i = 0.045$. This is not the case in Table 6.10 where even $T_0 = 2000$ is not enough. This phenomenon was also observed for the pendulum with oscillating support in Chapter 5.

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Table 6.14: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.041$ to $\gamma_f = 0.037$

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Table 6.15: Relative areas of the basins of attraction with $\gamma$ varying from $\gamma_i = 0.045$ to $\gamma_f = 0.037$

6.5 Comparing the two pendulum systems

Perturbations of the simple pendulum can exhibit a rich variety of dynamics whilst still allowing the user complete control over the unperturbed system. For this reason perturbed pendula are often used as toy models to study new phenomena in dynamics,
see for example [5, 40, 126, 154, 152, 167], or act as simplified models for more complex real world systems [117, 140, 159]. One of the most commonly studied perturbations is the pendulum with vertically oscillating support studied in Chapter 5, see also [15, 27, 28, 65, 167]; often referred to as the parametrically forced pendulum, for example [40, 56, 66, 111, 169]. The pendulum with periodically varying length studied in the previous sections of this chapter is another well known system, which also offers a wide variety of interesting dynamics, see [6, 33, 148, 34].

A motivation for studying the pendulum with variable length in a way analogous to that of the pendulum with oscillating support is to see the similarities and differences between the two systems. In [42, 134] it was noted that the pendulum with variable length is an example of a periodic, forced system whose linearisation is different from Mathieu’s equation, but no further analysis was completed. In 2009 [34] a brief comparison was made between the pendulum with oscillating support and the pendulum with variable length, stating that they are qualitatively different and that further comparison of the two systems deserves a separate study. However there has not been further work in the literature focused on exploring the differences of the two systems. In fact, there are still instances, even as recent as 2014 [53], where the two systems are confused with one another, see also [77, 131, 132]. Therefore to avoid further confusion and misguidance it is important not only to highlight the differences between the two systems but also note where they are similar to one another. In this section we compare the two systems in light of the work completed in the previous sections. For ease we restate the two systems here; the pendulum with oscillating support is described by

\[ \theta'' + \gamma \theta'(\tau) + (\alpha - \varepsilon \cos \tau) \sin \theta(\tau) = 0, \]  

(6.27)

and the pendulum with periodically varying length by

\[ \theta''(\tau) + \left( -\frac{2\varepsilon \sin \tau}{1 + \varepsilon \cos \tau} + \gamma \right) \theta'(\tau) + \frac{\alpha}{1 + \varepsilon \cos \tau} \sin \theta(\tau) = 0, \]  

(6.28)

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where $\gamma > 0$ is the added coefficient of dissipation, which we keep constant throughout this section. On first inspection we see that the parameters $\alpha$ in the two systems correspond to one another, as the systems are perturbations of the simple pendulum, see Appendix A.4. As a consequence, at least within the perturbation regime when $\varepsilon \ll 1$, it is not unreasonable to expect some similarities in the dynamics of the two systems with the same values of $\alpha$. On the other hand, the perturbation parameter $\varepsilon$ appears very differently in the two systems and in general there is no direct link between particular values of $\varepsilon > 0$. Furthermore, as $\varepsilon$ is increased, the dynamics of the two systems become more and more different. For instance, it has long been well known that for $\varepsilon$ large and $\alpha$ chosen accordingly, it is possible to stabilise the upward fixed point $(\theta, \theta') = (\pi, 0)$ of the pendulum with oscillating support [28, 27, 65, 159], whilst stability of the upward fixed point can not be achieved for any values of the parameters in the system of the pendulum with variable length, see [13, 33]. We therefore focus our comparison of the two systems on the case $\varepsilon$ small, and show that even in this scenario the systems can exhibit different dynamics.

In Sections 5.1 and 6.1 we studied the linearised equations of the two pendulum systems. Indeed even here the differences between the two systems are clear. Setting $\gamma = 0$, the linearisation of (6.27) about either one of its fixed points results in Mathieu’s equation, see Appendix A.9, a special case of Hill’s equation which was studied in Chapter 2, see also [116, 12, 96]. Hence the theory outlined in Appendix A.9 may be applied immediately. The same can not be said for the system (6.28) which must first be transformed and then expanded in a Taylor series before one can apply the analysis of Appendix A.9. Furthermore, in the Taylor expansion (6.5) the coefficient of the forcing depends not only on $\varepsilon$ but also on $\alpha$, hence the transition curves are different from the classical transition curves of Mathieu’s equation. In particular the region of instability emanating from $\alpha = 1$ has disappeared completely, see Figure 6.1.
The sufficient conditions for the origin (DFP) to attract a full measure set of phase
space were calculated in Section 5.2 for the pendulum with oscillating support and Section
6.2 for the pendulum with variable length. The calculations show that, in order to ensure
the origin attracts a full measure set of phase space the coefficient of dissipation must
satisfy
\[
\gamma > - \min_{\tau \geq 0} \frac{\varepsilon \sin \tau}{2(\alpha + \varepsilon \cos \tau)},
\]
for the system (6.27) and
\[
\gamma > - \min_{\tau \geq 0} \left( \frac{\varepsilon \sin \tau}{2(1 + \varepsilon \cos \tau)} - \frac{2\varepsilon \sin \tau}{\sqrt{\alpha(1 + \varepsilon \cos \tau)}} \right),
\]
for the system (6.28). As expected, condition (6.30) for the pendulum with variable
length has some similarity to the condition (6.29) for the pendulum with oscillating
support. However for the system of the pendulum with variable length we find that
coefficient of dissipation must be taken larger, so as to exceed the forcing term which
appears in the coefficient of $\theta'(\tau)$. This shows that the periodic attractors of (6.28)
are more robust and exist under larger values of the damping coefficient. This could
be expected by looking at (6.28), since the coefficient of $\theta'(\tau)$ is not of definite sign for
$\gamma = 0$. For both systems, the numerical results showed that global attraction to the
origin was achieved for lower values of $\gamma$, however it was still found that the system
(6.28) required larger damping coefficients and the periodic orbits existed under larger
values of $\gamma$ compared with the system (6.27).

In Sections 5.3 and 6.3 the threshold values of $\gamma$ were calculated for the two systems
perturbation techniques. It was found that the pendulum with variable length is less
suited to study with perturbation theory, as the perturbation parameter $\varepsilon$ appears in
the denominator of (6.28). Therefore when investigating the system (6.28) by means of
perturbation techniques it is imperative to check the validity of results and one may need
to take higher orders of $\varepsilon$ into account. This may be seen when comparing the results of the calculations in Section 6.3 with the numerical results in Section 6.4. For the period two oscillating attractor of (6.28), the calculated value for the threshold of dissipation with $\alpha = 0.5$ is $\gamma = 0.896916\varepsilon$. However, numerically with $\varepsilon = 0.1$ we found the period 2 oscillations exist as far as $\gamma = 0.09022$. Details of the explanation behind this inaccuracy may be found in Section 6.4. This was not the case for the pendulum with oscillating support, where calculated and numerical threshold values matched closely for the period 2 oscillating attractor (at least as far as 5 decimal places), see Sections 5.3 and 5.4.

The numerical results in Sections 5.4 and 6.4 for the two systems centred about the downward fixed point $\theta = 0$, show both similarities and differences in the dynamics of the two systems. For instance, when the solution curves for the system of the pendulum with variable length are mapped into phase plane (see Figure 6.3), we see that the curves of the oscillating attractors clearly cross those of the rotating attractors, the same behaviour is not observed for the system (6.27). This happens as the length of the pendulum varies. Through conservation of momentum, when the pendulum extends (or retracts) the angular velocity $\theta'$ decreases (or increases). Thus an oscillating solution may temporarily move faster than a rotating solution, without escaping the potential well. Another interesting difference are the rotations PR/NR1 shown in Figure 6.3(a). Following the convention in [67] we classify attractors by how many rotations $r$, they complete in $n$ periods of the forcing; that is a period-$n$ solution which completes $r$ rotations is written as an $(r, n)$ rotation. Under this convention the rotations PR/NR1 are classified as $(2, 1)$ rotations. In [67] the authors studied the system (6.27) and noted that they were unable to find a rotating solution with $r > n$ and that such solutions may not exist after the transient phase. However we find that such a solution does exist in the system (6.28) and moreover attracts a nonzero measure set of phase space for some values of the parameters. It may be that a corresponding attractor also exists for the
system (6.27), but exists only for values of $\gamma$ smaller than those which were investigated. It would be interesting to study this phenomenon in more detail.

The system (6.28) also exhibits a wider variety of attractors for the chosen parameter values, some of which only exist in small windows of $\gamma$. Thus solutions may be both created and destroyed by either increasing or decreasing $\gamma$. Similar behaviour was seen in Section 5.5 for the pendulum with oscillating support when the system is far away from the perturbation regime, that is $\varepsilon$ relatively large. This hints that for a given value of $\varepsilon$ the system (6.28) has dynamics which are further away from the unperturbed simple pendulum, than the system (6.27) with the same value of $\varepsilon$. That is, for any particular value of $\varepsilon \ll 1$ the system (6.28) is more perturbed.

However, looking at Figure 6.20 it is apparent that the basins of attraction for the two systems share some common traits. For both systems, when $\gamma$ is large enough, the downward fixed point attracts a full measure set of phase space. Decreasing $\gamma$, the non-fixed point attractors (PR/NR and DO2) appear and attract more of the phase space.

![Figure 6.20: Sizes of the basins of attraction for the systems (6.27) and (6.28) with constant coefficient of dissipation $\gamma$. Figure (a) corresponds to Figure 5.4 for the pendulum with oscillating support and (b) to Figure 6.4 for the pendulum with variable length. For further details see Sections 5.4 and 6.4.](image-url)
Figure 6.21: Basins of attraction for the two pendulum systems. The basins of attraction at the top correspond to the system (6.27) and those on the bottom to the system (6.28). Figures (a) and (c) show basins of attraction for $\gamma = 0.02$ and figures (b) and (d) for $\gamma = 0.04$. The basins are colour coded with the fixed points (DFP) shown in blue, period 2 oscillations in green, period 1 positive and negative rotations in red and yellow respectively and the period 3 positive and negative rotations in cyan and magenta respectively.

space as $\gamma$ continues to decrease. First the size of the basin of attraction corresponding to the oscillating solution peaks and then begins to decrease as the sizes of the basins of attraction corresponding to the rotating solutions peak. Decreasing $\gamma$ further, the non-fixed point attractors attract less of the phase space, and the basin of attraction belonging to the downward fixed point increases again. This was not observed for the cubic oscillator in [22], where decreasing the coefficient of dissipation causes the basin of attraction of the fixed point solution to decrease. This is related to the choice of parameter values $\alpha, \varepsilon$ and the bifurcation structures of the two systems, see the comments.
Although the curves of the sizes of the basins of attraction show similarities in their shape, for any particular value of $\gamma$ they are significantly different. For example, for $\gamma = 0.06$ the downward fixed point attracts a full measure set of phase space for the system (6.27), while in the system (6.28) even the period 1 rotations still exist and the period 2 oscillations attract approximately a quarter of the phase space. In this sense, the basins of attraction for the two systems differ greatly. In Figure 6.21 we show basins of attraction for $\gamma = 0.02$ and 0.04 with the basins of attraction for the pendulum with oscillating support on the top and the pendulum with variable length underneath. It is clear that not only do the sizes of the basins of attraction of similar attractors differ, but the regions of phase space occupied by the corresponding basins of attraction are also different.

In Section 5.3, it was shown that the threshold of dissipation for the period 2 oscillations of the system (6.28) is given by, at first order, $\gamma = 0.896916\varepsilon$. Choosing $\varepsilon = 0.0667$ we have $\gamma = 0.896916\varepsilon \approx 0.0598$; this value matches with the threshold of dissipation for the pendulum with oscillating support when $\varepsilon = 0.1$. In Table 6.16 we give the sizes of the basins of attraction for the system (6.28) with $\varepsilon = 0.0667$. Note that for $\gamma = 0.025$ we also find rotations with 4:4 resonance, which attract 0.57% of phase space each. When the 4:4 rotations appear, the size of the fixed points basin of attraction reduces in size. When the 4:4 rotations disappear their basin of attraction is absorbed by the 1:1 rotations; resulting in the increase in size shown in Table 6.16 for $\gamma = 0.027$. It is evident that the sizes of the basins of attraction and the values at which attractors disappear is similar to the pendulum with oscillating support with $\varepsilon = 0.1$, see Table 5.3 in Section 5.4. This confirms further the hypothesis that the pendulum with variable length is perturbed to a greater extent for the same value of the perturbation parameter.
Table 6.16: Relative areas of the basins of attraction for the system (6.28) with $\alpha = 0.5, \varepsilon = 0.0667$. The solutions are named as per Table 6.3.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>DFP</th>
<th>PR/NR</th>
<th>DO2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00500</td>
<td>79.53</td>
<td>4.44</td>
<td>11.60</td>
</tr>
<tr>
<td>0.01000</td>
<td>72.02</td>
<td>5.94</td>
<td>16.11</td>
</tr>
<tr>
<td>0.01500</td>
<td>70.55</td>
<td>3.93</td>
<td>21.59</td>
</tr>
<tr>
<td>0.02000</td>
<td>72.33</td>
<td>3.68</td>
<td>20.32</td>
</tr>
<tr>
<td>0.02500</td>
<td>68.48</td>
<td>3.90</td>
<td>22.61</td>
</tr>
<tr>
<td>0.02700</td>
<td>68.78</td>
<td>4.45</td>
<td>22.32</td>
</tr>
<tr>
<td>0.03000</td>
<td>68.40</td>
<td>4.25</td>
<td>23.11</td>
</tr>
<tr>
<td>0.03500</td>
<td>68.20</td>
<td>3.13</td>
<td>25.54</td>
</tr>
<tr>
<td>0.04000</td>
<td>73.40</td>
<td>1.39</td>
<td>23.82</td>
</tr>
<tr>
<td>0.04300</td>
<td>77.50</td>
<td>0.24</td>
<td>22.01</td>
</tr>
<tr>
<td>0.04500</td>
<td>80.39</td>
<td>0.00</td>
<td>19.61</td>
</tr>
<tr>
<td>0.05000</td>
<td>85.42</td>
<td>0.00</td>
<td>14.58</td>
</tr>
<tr>
<td>0.05500</td>
<td>91.27</td>
<td>0.00</td>
<td>8.73</td>
</tr>
<tr>
<td>0.05900</td>
<td>96.78</td>
<td>0.00</td>
<td>3.22</td>
</tr>
<tr>
<td>0.05970</td>
<td>98.34</td>
<td>0.00</td>
<td>1.67</td>
</tr>
<tr>
<td>0.05979</td>
<td>98.64</td>
<td>0.00</td>
<td>1.36</td>
</tr>
<tr>
<td>0.06000</td>
<td>100.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Figure 6.22: Plot of the relative areas of the basins of attraction for the system (6.28) with $\alpha = 0.5, \varepsilon = 0.1$, for different values of $\gamma$ as per Table 6.16.

In Sections 5.4.1 and 6.4.1 the bifurcation structures of the two pendulum systems were studied. Figures 6.23 and 6.24 show the bifurcation curves for the systems (6.27) and (6.28), respectively, with $\gamma = 0.05$. For both systems we focused on a region of parameter space around the first tongue of instability for the corresponding linearised equations. The bifurcations which occur for the main attractors (DFP, DO2 and PR/NR) are the same in both systems, in this sense the two pendula are very similar, although of course the regions of parameter space in which each bifurcation curve lies and the shape of the curve is different. In particular for the pendulum with variable length, the line at which the first period doubling bifurcation for the rotating solutions occurs, does not appear to fold back over itself. Therefore, by increasing $\varepsilon$ no period halving bifurcation is observed for any of the values of $\alpha$ and $\varepsilon$ investigated. Moreover, as $\gamma$ is taken larger, the line A moves up much more quickly for the pendulum with variable length than for the pendulum with oscillating support, see Sections 5.4.1 and 6.4.1. Hence, for larger $\gamma$ the region of existence of the rotating attractors PR/NR in $\alpha - \varepsilon$ space becomes...
very thin. This is particularly clear when compared with the pendulum with oscillating support with the same parameter values. Furthermore if we were to focus on a region of parameter space around $\alpha = 1$ the bifurcation structure would look quite different. This is because the pendulum with variable length is missing this instability tongue for the corresponding linearised equation, see previous comments as well as Section 6.1 for further details.

Figure 6.23: Bifurcation curves for the system (6.27) with fixed $\gamma = 0.05$. Figure (a) shows the bifurcations of the rotations PR/NR and (b) the bifurcation structure of the oscillation DO2. Details of the bifurcation structure are given in Section 5.4.1.

Figure 6.24: Bifurcation curves for the system (6.28) with fixed $\gamma = 0.05$. Figure (a) shows the bifurcations of the rotations PR/NR and (b) the bifurcation structure of the oscillation DO2. Details of the bifurcation structure are given in Section 6.4.1.
The bifurcation curves in Figures 6.23 and 6.24 are linked to the sizes of the basins of attraction shown in Figure 6.20. In particular, we focus on the increase in the fixed points basin of attraction as $\gamma$ is decreased below $\gamma \approx 0.35$ for the system (6.27) and $\gamma \approx 0.5$ for the system (6.28). We outline the scenario for the system (6.27), which also carries over to the system (6.28). As previously mentioned, increasing (decreasing) the value of $\gamma$ causes the bifurcation curves for the rotating solutions shown in Figure 6.23(a) to move up and to the right (down and to the left). Similarly, by increasing (decreasing) the value of $\gamma$, the curves in Figure 6.23(b) move up (down). Hence a point corresponding to fixed values of $\alpha$ and $\varepsilon$ in the system (6.27) may be found inside a given region of existence rather than another, depending on the value of $\gamma$.

In particular, the point $P$ corresponding to the values $\alpha = 0.5$, $\varepsilon = 0.1$ is in a region of parameter space for which the fixed point is (Lyapunov) stable even when $\gamma = 0$, see Figure 5.1. Hence the fixed point remains asymptotically stable however small $\gamma$ may be, as far as it is positive. As $\gamma \to 0$ the bifurcation curves move downward: the point $c$ gets closer and closer to the $\alpha$-axis and the point $P$ turns out to be situated close to the curves $B$ and $E$ in Figure 6.23(a) and (b), respectively. By moving toward the edge of the region of existence of an attractor, the size of the corresponding basin of attraction decreases. As there are no new attractors in this region of parameter space, the basin of attraction of the fixed point increases, in agreement with Figure 6.20(a). Hence the increase is related to the location of parameters $\alpha$ and $\varepsilon$ in respect to the regions of existence of the attractors at fixed $\gamma$, shown in Figure 6.23. In particular, by choosing $(\alpha, \varepsilon)$ either to the left of $c$ in Figure 6.23(b) or in a region where the fixed point is (Lyapunov) unstable for $\gamma = 0$, this phenomenon would not be observed.
Chapter 7

Utilising basins of attraction with constant dissipation

In Chapters 5 and 6 we showed that it is important to consider the entire time evolution of dissipation to obtain an accurate measure of the sizes of the basins of attraction. However, in much of the literature basins of attraction are calculated using a constant coefficient of dissipation. As such, there is already a large amount of data which has been obtained for the case of constant dissipation. In this chapter we outline and apply a methodology which utilises the data already obtained for basins of attraction in the constant coefficient scenario to acquire basins of attraction in the scenario where dissipation initially varies with time. In the process we are able to obtain information about a phenomenon observed in Chapter 5, where the basins of attraction can suddenly drastically change with small variations in $T_0$.

7.1 Outline of the problem

Consider the ordinary differential equation

$$\theta''(\tau) + f(\theta, \tau) + \gamma(\tau) \theta'(\tau) = 0,$$

(7.1)

where $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the dashes denote derivatives with respect to time $\tau$, the driving force $f$ is smooth and $2\pi$-periodic with respect to its arguments and the damping
coefficient $\gamma(\tau) \geq 0$ depends on time. For concreteness we shall consider explicitly the case of a pendulum with oscillating support, as studied in Chapter 5, which is described by the ordinary differential equation

$$\theta''(\tau) + (\alpha - \beta \cos \tau) \sin \theta(\tau) + \gamma(\tau) \theta(\tau) = 0, \quad \alpha, \beta \in \mathbb{R}, \quad (7.2)$$

but our results apply to systems (7.1) with more general forces, or even systems of the form $x'' + g(x, \tau) + \gamma(\tau)x' = 0$, with $x \in \mathbb{R}$, such as those considered in [23].

With a few exceptions, forced systems of the form (7.1) have been studied only in the case of constant dissipation. Despite the simplicity of the model, not much is known analytically. For instance only a finite set of attractors are expected to exist [133, 76, 145], but no proof exists for this. Furthermore, the corresponding basins of attraction are usually calculated only numerically; see for example [109, 155, 7, 97, 115, 33], see also [39, 150]. To add complication, in many physical systems the damping coefficient is not constant throughout its entire evolution. Time-dependent dissipation has been studied in the previous chapters (Chapters 5 and 6) as well as in the literature [22], where the focus was mainly on the case of $\gamma(\tau)$ varying linearly over some initial time span, say

\[ \begin{align*}
\gamma_i &\quad \text{(a)} \\
\gamma_f &\quad \text{(b)}
\end{align*} \]

**Figure 7.1:** Pictorial representations showing how time can be split into two intervals. Interval 1 where dissipation is allowed to vary with time and interval 2 where dissipation remains constant. The initial value of dissipation is represented by $\gamma_i$ and $\gamma_f$ represents the final value of dissipation.
CHAPTER 7. UTILISING BASINS OF ATTRACTION WITH CONSTANT DISSIPATION

\( \tau \in [0, T_0] \), after which it remains constant. In that case considering only the final value of the damping coefficient does not give a correct representation of the basins of attraction and in fact the entire time evolution of \( \gamma(\tau) \) must be taken into account.

Nevertheless, the analysis of a system with constant damping coefficient \( \gamma \) may provide information for the same system with damping coefficient varying in time. For fixed constant \( \gamma \), consider an attractor and denote by \( A(\gamma) \) the relative area of the corresponding basin of attraction, that is the percentage of initial data in a given sample region whose trajectories go to that attractor. It has already been pointed out in Chapter 5 and [22, 167] that, if one knows the profiles \( \gamma \mapsto A(\gamma) \) (i.e. the sizes of the basins of attraction for each value of constant \( \gamma \)) describing how the relative areas of the basins of attraction depend on \( \gamma \) in the system with constant dissipation, then, when \( \gamma(t) \) varies quasi-statically (that is very slowly) from an initial value \( \gamma_i \) towards a final value \( \gamma_f \) over a time \( T_0 \), one may be able to predict the relative areas of the basins by looking only at the profiles, and the slower the variation of the damping coefficient, the better the prediction. More precisely, at least for systems which have small enough forcing and damping coefficients that they may be considered as a perturbation of an integrable system, if an attractor exists for \( \gamma = \gamma_f \), then, when \( \gamma(\tau) \) increases slowly from \( \gamma_i \) towards the value \( \gamma_f \), the relative area of the corresponding basin of attraction is close to the value that the function \( A(\gamma) \) attains for \( \gamma = \gamma_i \). Moreover, the larger \( T_0 \), the closer the relative area will be to the value \( A(\gamma_i) \). For more details see Chapter 5.

In this chapter, we wish to discuss other cases, where knowledge of the behaviour of the system for constant \( \gamma \) may be utilised in the case in which dissipation changes in time. In particular, we shall discuss the following:

1. Suppose that one is interested in investigating systems with \( \gamma(\tau) \) varying linearly from an initial value \( \gamma_i \) to a final value \( \gamma_f \), with \( \gamma_i \) and \( \gamma_f \) fixed, over a time interval
[0, T_0], for many values of T_0. In Section 7.3 we outline a method to speed up the computation of the basins of attraction. The method utilises basins of attraction calculated for the system with constant damping coefficient \( \gamma_f \), and it reduces the length of time over which it is necessary for the equations of motion to be numerically integrated. In effect, our method is to extrapolate from an observation time \( T_1 \geq T_0 \) to the full evolution time \( T_f \), by using pre-computed data. We will refer to this as ‘the method of fast numerical computation’. This is first stated as a general method and later numerically implemented for the pendulum with oscillating support.

2. In Section 7.4, we come back to a phenomenon observed in Section 5.5. When the damping coefficient increases linearly over a time \( T_0 \) towards a final value \( \gamma_f \), the relative areas of the basins of attraction of the surviving attractors generally change smoothly as functions of \( T_0 \), without abrupt variations; however, in a few cases we saw sharp jumps, concentrated in small intervals of values of \( T_0 \) — see Figure 7.2. By using the same ideas as in the method of fast numerical computation, we give an explanation as to why this happens.

Figure 7.2: Relative areas of the basins of attraction for (7.2), with \( \alpha = -0.1 \), \( \beta = 0.545 \) and \( \gamma(t) \) varying linearly over time \( T_0 \) from (a) \( \gamma_i = 0.2 \) and (b) \( \gamma_i = 0.23 \) to \( \gamma_f = 0.2725 \). UFP, DO2 and DO4 denote the attractors: upward fixed point, period-2 oscillation and period-4 oscillation — see Section 7.4 for details.
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The method of fast numerical computation can be described as follows. Assume that the basins of attraction of a system of the form (7.1) with constant \( \gamma = \gamma_f \) are known. Of course, this information can only be obtained numerically and requires integration of the equations of motion for a time \( T_f \) large enough for the attractors to be closely approached. Then, if one is interested in investigating a system of the form (7.1) with \( \gamma(\tau) \) varying in time from an initial value \( \gamma_i \) to the final value \( \gamma_f \) over a time \( T_0 \) much smaller than \( T_f \), the corresponding basins of attraction can be obtained by integrating the differential equation over a time interval \([0, T_1] \), with \( T_1 \geq T_0 \) still much smaller than the entire time evolution \( T_f \). As a result the computation time is reduced by a factor \( T_f/T_1 \); for example, with \( T_1 = 48\pi \) and \( T_f = 3000 \), simulations take under 12 hours to complete, compared with nearly 10 days for the simulations using full time \( T_f \).

Roughly, \( T_1 \) is the time needed for the trajectories to fall inside the basins of attraction for constant \( \gamma = \gamma_f \), so that their evolution from that instant onwards is known — we refer to Section 7.3 for more details.

This method of calculating the basins of attraction can significantly reduce computation time when one has to investigate a large number of values of \( T_0 \) or different functions of \( \gamma(\tau) \), particularly for systems which require long time integration or where computationally heavy integration methods are needed. From a physical point of view, it is natural to assume that dissipation increases with time, as an effect of wear, aging, cooling, the build up of deposits and so on. A linear increase, in addition to being particularly well-suited for numerical investigation, also simulates situations where dissipation tends to settle around some asymptotic value, which again are of physical interest; we refer to [22] for further comments. For these reasons, as well as to fit with the investigations carried out in Chapters 5 and 6, in this chapter we have focused on the linear case. Nevertheless we expect our results to apply in more general contexts, where the variation of the damping coefficient is not linear (or not even monotonic), or
the final value is reached only asymptotically.

The results produced by the method of fast numerical computation show good agreement with the results obtained by integrating the system over the full time $T_f$ required for trajectories to settle numerically on the attractive solutions. We shall find not only that there are small differences between the results obtained with the method of fast numerical computation and those of the full time integration, but also that these differences are less than the width of the 95% confidence interval on the results as computed by Monte Carlo simulation, see Chapter 4. However, as we shall see, the choice of the time $T_1$ may be a delicate matter and need a few caveats.

### 7.2 Some required definitions

In order to study the basins of attraction of a dissipative dynamical system $\mathbf{x}' = F(\mathbf{x}, \tau)$ in $\mathbb{T} \times \mathbb{R}$, one takes initial conditions in a sample region of phase space, say $S \subset \mathbb{T} \times \mathbb{R}$, and lets them evolve in time. As a result of dissipation, the sample region $S$ will contract with time as trajectories move towards the attractors. This leads to the following definition.

**Definition 7.1 (Contracted phase space)** Given a dissipative dynamical system and a sample region $S$, the region of phase space still occupied by trajectories starting in $S$ at time $\tau = 0$, after time $T$ has elapsed, will be called the contracted phase space at time $T$ and denoted by $C_T$.

In principle $S$ is arbitrary, but in practice it is convenient to take $S$ in such a way that (a) it contains the (relevant) attractors and (b) one has $C_T \subset S$ for $T$ large enough. Thus, if the total time over which the system evolves consists of two intervals, a first interval $[0, T_0]$ in which $\gamma = \gamma(\tau)$ varies and a second interval $[T_0, T_f]$ in which $\gamma = \gamma_f$ is constant, then whilst the first interval takes initial conditions from the whole of $S$, the second interval only receives initial conditions from $C_{T_0}$, see Figure 7.1. It is also useful
to know where points in $S$ are mapped to in $C_T$. This information is captured in the movement map, which is defined as follows.

**Definition 7.2 (Movement map)** Fix a set $X_0$ of points $x_0 \in \mathbb{T} \times \mathbb{R}$ at time $\tau = 0$ and let $X_1$ be the set of corresponding points $x_1$ that the trajectories with such initial conditions arrive at for $\tau = T$. The continuous, bijective map $M_T$ defined by $M_T(X_0) = X_1$ is called a movement map from time 0 to time $T$.

Definition 7.2 may be easily generalised so that the movement map acts on a set $X_0$ of points $x_0$ at time $\tau = \tau_0$ and maps them to the corresponding set $X_1$ of points $x_1$ at time $\tau = \tau_1$. However, as such a case is not considered in the following sections and would require extra notation, we continue to use the definition as stated.

The points in the contracted phase space need not (and in most cases will not) be uniformly distributed. To specify how “dense” regions of the contracted phase space are...
we introduce the following definition.

**Definition 7.3 (Density map)** Let $r > 0$ be small enough. Given a dissipative dynamical system and a sample region $S$, let $C_T$ be the contracted phase space at time $T$. Consider a cover of $C_T$ with cubes of side $r$ and denote by $X_1$ the set of centres $x_1$ of the cubes. Fix $X_0 \subset S$ and for $x_1 \in X_1$ define $\rho(x_1)$ as the fraction of trajectories starting in $X_0$ which evolve into the cube of side $r$ and centre $x_1$ at time $T$. We define the density map from time $t = 0$ to time $T$ as the map which associates the value $\rho(x_1)$ with each $x_1 \in X_1$.

If $X_0 = S$, the function $\rho(x_1)$ is expected to become $r$-independent for $r$ small enough; for this reason we have omitted its dependence on $r$. On the other hand, if $X_0$ is a discrete set (as in any numerical implementation), $r$ cannot be too small for the definition to make sense.

The definitions above are given explicitly in $\mathbb{T} \times \mathbb{R}$, but could be easily extended to $\mathbb{R}^2$, or even to higher dimensional systems. Throughout the rest of the chapter we focus on systems of the form (7.1), so that $x = (\theta, \theta')$ and the driving force is $2\pi$-periodic.

![Figure 7.4: Contracted phase space for (7.2), with $\alpha = 0.5$, $\beta = 0.1$ and $\gamma(\tau)$ decreasing linearly from $\gamma_i = 0.05$ to $\gamma_f = 0.02$ over a time (a) $T_0 = 32\pi$ and (b) $T_0 = 48\pi$. The densities are coded with red being the most dense, then yellow, green, blue and black.](image)
in \( t \). Moreover we consider damping coefficients \( \gamma(\tau) \) which vary linearly between two values and become constant after a time \( T_0 \). If \( X_0 \subset S \) is the set of initial conditions, we take \( T_1 \geq T_0 \) such that \( X_1 = M_{T_1}(X_0) \subset C_{T_1} \subset S \). For the system (7.2), a convenient choice for the sample region \( S \) is \( S = [-\pi, \pi] \times [-4, 4] \). When considering a trajectory \((\theta(t), \dot{\theta}(t))\), the points \( \mathbf{x}_0 = (\theta_0, \theta'_0; \tau = 0) \) and \( \mathbf{x}_1 = (\theta_1, \theta'_1; \tau = T_1) \) are just different points on the same trajectory and share the same path as \( \tau \to \infty \). Furthermore, if \( T_1 = 2N\pi \), where \( N \in \mathbb{N} \), since the forcing is \( 2\pi \)-periodic then the solutions with initial conditions \((\theta_1, \theta'_1; \tau = T_1)\) and \((\theta_1, \theta'_1; \tau = 0)\) move towards the same attractor if we set \( \gamma = \gamma_f \) from \( \tau = 0 \). This observation will be used when choosing \( T_1 \) to be a multiple of \( 2\pi \) in the following — see Section 7.3.1.

A discrete approximation to the movement map will be represented by a \( p \times m \) matrix, in which the indices \( i = 0, \ldots, p \) and \( j = 0, \ldots, m \) correspond to the coordinates of the initial condition \((\theta_0 = -3.14 + i\Delta \theta, \theta'_0 = -4 + j\Delta \theta')\), with \( p\Delta \theta = 6.28 \) and \( m\Delta \theta' = 8 \), and in each \( i,j \)-th entry is a vector of the coordinates \((\theta_1, \theta'_1)\) for the corresponding trajectory at time \( T_1 \). Colour coding the phase space we can represent the action of the movement map as shown in Figure 7.3. A numerical representation of a density map can be achieved by averaging trajectories onto a grid, as done for the movement map. Different colours may be used to express how dense regions of phase space are. In particular a density map allows us to see how the most dense regions of the contracted phase space change with variations in \( T_0 \) — see Figure 7.4.

### 7.3 Fast numerical computation of the basins of attraction

#### 7.3.1 General setting

A movement map can be used to compute quickly the relative areas of basins of attraction for systems in the form of (7.1) with \( \gamma(\tau) \) varying over a time \( T_0 \) before reaching a final...
value $\gamma_f$, with only the need to integrate over a time $T_1 \geq T_0$, provided that the basins of attraction for constant damping coefficient $\gamma = \gamma_f$ are known. This can be achieved as follows.

First, one covers the sample region $S$ with a mesh of points, with the requirement that the limiting solution of each point on the mesh is known for constant $\gamma = \gamma_f$. Of course, the basins of attraction for constant dissipation are obtained numerically, so that which attractor any point of the mesh converges to is known only within numerical tolerance; hence, high accuracy is required at this preliminary stage. In particular for initial conditions in the mesh, the equations of motion have to be integrated over a time interval $[0, T_f]$ sufficiently large for the attractors to be reached numerically. Next, a set of initial conditions in $S$ (not necessarily the same as those on the mesh) is chosen and for each, the equations with time-dependent $\gamma(\tau)$ are integrated for $\tau \in [0, T_1]$. The point in phase space which the trajectory occupies at time $T_1$ is then rounded to the nearest point on the mesh. Provided $T_1$ is chosen to be a multiple of the forcing period, we know towards which attractor the rounded point moves asymptotically (as $T_1 > T_0$, thus $\gamma = \gamma_f$). Repeating this for all initial conditions allows one to estimate the relative areas of the basins of attraction.

From the discussion above, it may seem natural to take $T_1$ to be the smallest multiple of the forcing period $2\pi$ larger than $T_0$, because then a trajectory which is inside a given basin of attraction at time $T_1$ will move towards the corresponding attractor. However, error is inevitably introduced when approximating the coordinates of the trajectories at time $T_1$ to the nearest point in the mesh. In systems where the basins of attraction are sparse, the slight change of co-ordinate during the approximation may cause the results to be inaccurate. As a consequence it is found that in cases where $T_0$ is relatively small, or the basins of attraction are increasingly broken and sparse, $T_1$ must be taken larger than this. Using a finer mesh reduces the error in the approximation and should
because the results obtained with the method of fast numerical computation described above to tend towards those when the system is fully integrated. In practice this is not always true, especially for systems with high sensitivity to initial conditions. Indeed, it is possible that with a coarse mesh the approximation to the point at time $T_1$ has the same attractor, whilst with a finer mesh, the new, more accurate approximation moves to a different attractor. Moreover there can be compensations which are destroyed by refining the mesh; this is best explained by a simple example. Imagine there are two initial conditions, one of which tends to attractor $a_1$ and the other to attractor $a_2$, but which, on the contrary, the method of fast numerical computation predicts them to tend to attractors $a_2$ and $a_1$, respectively, in this case, the two errors cancel each other out and the relative areas of the basins of attraction match with those of the full integration. Suppose we then use a finer mesh and one of the two points is predicted correctly, while the other prediction remains incorrect, now the errors do not cancel and the predictions of the relative areas of the basins of attraction will be in error. However, provided that this is a rare occurrence, increasing the number of points on the mesh will tend to increase the accuracy in general.

The basins of attraction are computed by taking a large but finite set of initial conditions. The statistical analysis in Chapter 4 may be used to compute the confidence interval for a given sample size. In the forthcoming analysis, the error in the results obtained with the method of fast numerical computation relative to the fully integrated results will be deemed acceptable if it is less than the estimated 95% confidence interval for the relative areas of the basins of attraction, see Table 4.1.

### 7.3.2 Application to the pendulum with oscillating support

In this section we investigate, in a concrete model, which types of basin of attraction are suitable for our method of fast numerical computation, how to reduce the error
with respect to the full integration and how to increase the accuracy. We consider the pendulum with oscillating support, see (7.2), with $\gamma(\tau)$ varying linearly over a time $T_0$ from the initial value $\gamma_i$ to the final value $\gamma_f$. Of course, the reduction in computational time compared with the full time-span $[0, T_f]$ is a result of the smaller integration time $T_1$. Since $T_1 \geq T_0$, in systems where $T_0$ is large and comparable with $T_f$ this advantage is lost. However, we saw in Chapters 5 and 6 that the most significant changes to the relative areas of the basins of attraction happen over a short initial time $T_0$, where the method is particularly effective.

The main numerical integration scheme used to test the method is MATLAB’s ODE113, which is a variable order Adams-Bashforth-Moulton scheme – chosen simply because MATLAB offers ease in programming compared to using a low-level language such as C. Although it is found that the integrator ODE113 is not always reliable for this system, our aim is mainly to compare the method of fast numerical computation with the full time integration, rather than the accuracy of the full simulations relative to the true dynamics. For the same reason, throughout, the relative areas of the basins of attraction are given to 4 decimal places, despite the number of initial conditions for the full

Figure 7.5: Basins of attraction for (7.2) with constant damping coefficient $\gamma(\tau) = \gamma_f$. In Figure (a) $\alpha = -0.1$, $\beta = 0.545$ and $\gamma_f = 0.2$ (for clarity purposes the origin is centred at the upward position of the pendulum $\xi = \theta + \pi$). In Figures (b) and (c) $\alpha = 0.5$, $\beta = 0.1$ and $\gamma_f = 0.05$ and $0.02$, respectively. The colours black, blue, red, yellow and green represent the basins of attraction for the attractors UFP, DFP, PR, NR and DO2, respectively – see Chapter 5 for further details.
simulations producing uncertainty in the first or second decimal place — see Table 4.1. As well as providing the results obtained for both fast and full simulations with the chosen method of integration, we also give the results obtained with two different, more efficient integration methods, a standard Runge-Kutta integrator and a scheme based on series expansion, see Section 4.3.1, both of which were implemented in C. The reason for this is to check whether the error produced by the method of fast numerical computation is within the difference produced by simply choosing a different numerical method and a different selection of initial conditions.

To test the method, we consider cases where the basins of attraction for constant values of $\gamma = \gamma_f$ become increasingly intertwined. We study (7.2) with $\alpha = 0.5$, $\beta = 0.1$ and $\gamma_f = 0.05$ and $0.02$; the corresponding basins of attraction are shown in Figure 7.5(b) and Figure 7.5(c), respectively. For $\gamma = 0.02$ the system exhibits four attractors, namely the fixed point $(\theta, \theta') = (0, 0)$, two rotating attractors (one positively rotating and one negatively rotating) and an oscillatory attractor; we shall refer to them as DFP, PR, NR and DO2, respectively. For $\gamma = 0.05$, only DFP and DO2 exist. Further details on these attractors can be found in Section 5.4. Similar tests have also been conducted for $\alpha = -0.1$, $\beta = 0.545$ and $\gamma_f = 0.2$, with the basins of attraction as in Figure 7.5(a). However, owing to the simple geometry of these basins of attraction, the results obtained were of little use for studying to what extent the numerical method may be applied. As such the results have not been included and hence we shall use the parameters $\alpha = 0.5$, $\beta = 0.1$ throughout the rest of this section.

In order to implement the method of fast numerical computation, first a mesh of initial conditions must be set up and for each point in the mesh the equations must be integrated for constant damping coefficient $\gamma_f$. This is computationally expensive as the system is integrated over the full time span $[0, T_f]$, required for solutions to move sufficiently close to the corresponding attractors that they can be identified. For constant
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γ, this happens over times $O(1/\gamma)$ and values of $T_f$ between 10 and 100 times $1/\gamma$ turn out to be sufficient; for the chosen values of $\gamma$, this yields $T_f$ larger than $10^3$. Using the contracted phase space — see Definition 7.1 and Figure 7.6 — for our smallest value of $T_0$ (that is $8\pi$) we can see that it is unnecessary to cover the entire region $S$ with a mesh of initial conditions, as roughly half of them will never be used. Only covering the region $C_{8\pi}$ would reduce both the computation time and memory required by a factor of roughly 2; see Figure 7.6.

Figure 7.6: Contracted phase space for (7.2) $\alpha = 0.5$, $\beta = 0.1$ and $\gamma(\tau)$ varying linearly over a time $T_0 = 8\pi$ (a) from $\gamma_i = 0.02$ to $\gamma_f = 0.05$ and (b) from $\gamma_i = 0.05$ to $\gamma_f = 0.02$. The rectangle containing the shaded region marks the area $S$, from which initial conditions were taken.

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>$\theta$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>8π</td>
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</tr>
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</tr>
<tr>
<td>128π</td>
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<tr>
<td>160π</td>
<td>77.9221</td>
<td>22.0779</td>
</tr>
</tbody>
</table>

Table 7.1: Relative areas of the basins of attraction for (7.2) with $\alpha = 0.5$, $\beta = 0.1$, $\gamma_i = 0.02$, $\gamma_f = 0.05$. Initial conditions for ODE113 were taken from the same mesh (with $\Delta \theta = \Delta \phi = 0.01$) as used to approximate trajectories at time $T_1$, totalling 503829 points. The Runge-Kutta and Series integrators each used a different set of 500001 random initial conditions in $S$. The “ODE113 Fast” and “Series Fast” results used the same integrator and the same initial conditions as “ODE113” and “Series”, respectively.
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We first consider $\gamma(\tau)$ linearly evolving from 0.02 to 0.05 over a time $T_0 = 2\pi N$, with $N \in \mathbb{N}$, and fix the integration time at $T_1 = T_0$. A mesh of roughly 500,000 points is considered in $S$ and each trajectory at time $T_1$ is rounded to the nearest point of the mesh. The results in Table 7.1 show that the method performs well in this instance. Indeed, the error produced by estimating the basins of attraction with the method of fast numerical computation rather than integrating over the full time $T_f$ is less than the difference in results obtained by choosing a different integrator with different initial conditions — see Figure 7.7. Note that the integrator ODE113 disagrees with the other integrators for $T_0 = 128\pi$ and $T_0 = 160\pi$; however, for each integrator the results obtained with the method of fast numerical computation are still close to the full integrations.

Of course Table 7.1 allows comparison only of the relative areas of the basins of attraction, but does not reveal to what extent the method correctly determines the asymptotic dynamics of each initial condition. As pointed out in Section 7.3.1, there could be substantial compensations and, in principle, the distribution of the basins of attraction in phase space could be wrongly described by the method of fast numerical computation.

Figure 7.7: Error in the relative areas of the basins of attraction in Table 7.1. The full line represents the error of the method of fast numerical computation with respect to the full integration: (a) difference between “ODE113 Fast” and “ODE113”, (b) difference between “Series Fast” and “Series”. The dashed line represents the difference between the estimates of “Series” and “Runge-Kutta” and the dotted line represents the 95% confidence interval, as calculated in Chapter 4.
CHAPTER 7. UTILISING BASINS OF ATTRACTION WITH CONSTANT DISSIPATION

Accuracy of the method %

<table>
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<th>Series Fast</th>
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<td>16π</td>
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<td>99.1206</td>
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<td>32π</td>
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</tr>
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Table 7.2: The percentages of initial conditions for which the limiting behaviour is correctly described for the simulations in Table 7.1 using the method of fast numerical computation instead of the full integration.

<table>
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<th>Relative area %</th>
<th>Attractor</th>
<th>Fast 1</th>
<th>Fast 2</th>
<th>Fast 3</th>
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<th>Series</th>
<th>Runge-Kutta</th>
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<td></td>
<td></td>
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<tr>
<td>0</td>
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<td>N/A</td>
<td>N/A</td>
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<td>N/A</td>
<td>N/A</td>
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</tr>
<tr>
<td>160π</td>
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</tr>
<tr>
<td></td>
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<td>18.1508</td>
<td>18.1508</td>
<td>18.1509</td>
<td>18.0683</td>
<td>18.0614</td>
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</tbody>
</table>

Table 7.3: Results for $\gamma(\tau)$ varying from $\gamma_i = 0.05$ to $\gamma_f = 0.02$ over times $T_0$. The Runge-Kutta and Series integrators used different sets of 1 000 001 random initial conditions in $S$, while ODE113 used a mesh of initial conditions in $S$ with increments $(\Delta \theta, \Delta \theta') = (0.01, 0.005)$. The meshes in “Fast 1”, “Fast 2” and “Fast 3” have increments $(\Delta \theta, \Delta \theta') = (0.01, 0.01), (0.01, 0.005)$ and (0.005, 0.005), respectively.

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computation despite the good agreement between the corresponding relative areas. However, Table 7.2 shows that this is not the case: it is true that the percentage of points assigned by the method of fast numerical computation to the wrong basin is larger than suggested by Table 7.1, but still is not large, and also tends to decrease with increasing $T_0$ and hence $T_1$.

This may still be regarded as a very simple case, as only two attractors coexist and the boundaries of their basins of attraction for constant $\gamma_f$ still have relatively simple geometry. We now consider (7.2), with the same values for $\alpha$ and $\beta$ as in the previous case, but with $\gamma(\tau)$ linearly decreasing from $\gamma_i = 0.05$ to $\gamma_f = 0.02$. We first ran simulations with roughly $500,000$ initial conditions and two meshes with increments $(\Delta \theta, \Delta \theta') = (0.01, 0.01)$ and $(0.005, 0.005)$, respectively; at time $T_1 = T_0$ each trajectory was rounded to the nearest point of the mesh. Afterwards we considered roughly $1,000,000$ initial conditions and three meshes with increments $(\Delta \theta, \Delta \theta') = (0.01, 0.01), (0.01, 0.005)$ and $(0.005, 0.005)$, respectively; once more we chose $T_1 = T_0$. The corresponding results for $1,000,000$ initial conditions are reported in Table 7.3. Figure 7.8 illustrates that, as previously, the difference between the fast numerical computations and the full time integrations is comparable to — if not smaller than — both the difference between the results obtained with the other two integrators and the 95% confidence interval.

As expected, the difference between the results obtained using different integrators reduces significantly when increasing the number of initial conditions. Contrary to this, neither increasing the number of initial conditions nor using a finer mesh in general significantly improves the error relative to the full time integration — see Table 7.4. Slight improvements are only obtained for the attractors with smaller basins of attraction (that is the two rotating attractors), where the error becomes comparable to that created by using different numerical integrators and sets of initial conditions.
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Figure 7.8: Error in the relative areas of the basins of attraction in Table 7.3: Figures (a) through to (d) show the differences for DFP, PR, NR and DO2, respectively, with (b) and (c) only going as far as $T_0 = 64\pi$ since the rotating solutions disappear after this. The plain, starred and crossed full lines show the difference between “Fast 1”, “Fast 2” and “Fast 3”, respectively, and “ODE113”. The dashed line shows the difference between the estimates using “Series” and “Runge-Kutta”. The dotted line shows the 95% confidence interval.

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>Accuracy of the method %</th>
<th>$N \approx 500000$</th>
<th>$N \approx 1000000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8$\pi$</td>
<td>Mesh 1</td>
<td>Mesh 3</td>
<td>Mesh 1</td>
</tr>
<tr>
<td>87.9880</td>
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<td>97.7471</td>
<td>97.5068</td>
<td>96.3252</td>
</tr>
</tbody>
</table>

Table 7.4: The percentages of initial conditions for which the limiting behaviour is correctly described for the simulations in Table 7.3 and the analogous ones with 500000 initial conditions. Mesh 1, Mesh 2 and Mesh 3 correspond to increments $(\Delta \theta, \Delta \theta') = (0.01, 0.01), (0.01, 0.005)$ and $(0.005, 0.005)$, respectively.

It is also apparent that the larger the value of $T_0$ (and thus also $T_1$), the more reliable the method. Table 7.4 shows that for $T_0 \geq 48\pi$ the results are correct for more than
99% of initial conditions. The reason behind this can be seen by plotting the contracted phase space $C_{T_1}$ for different values of $T_1$ and superimposing it on the basins of attraction for constant $\gamma = \gamma_f = 0.02$; see Figure 7.9. When $T_1$ is larger, $C_{T_1}$ occupies regions of the basins of attraction which are deep inside the cores surrounding the attractors; thus, they are distant from the boundaries and hence less sensitive to slight variations in initial conditions. In turn, the error created by the process of approximation onto the mesh is less significant. For the same reason, the results in Table 7.3 corresponding to $T_0 = 128\pi$ and 160$\pi$ show that decreasing the spacing of points in the mesh does not improve the results for $T_0$ large.

All this suggests that setting a minimum value for the integration time $T_1$ increases the accuracy of the method of fast numerical computation. Doing so also results in an increase in the computational time if $T_1 \gg T_0$. However it is still quicker than integrating over the full time $T_f$. Moreover the larger $T_1$, the smaller is $C_{T_1}$ and hence the mesh is required to cover less of the phase space, so reducing the points in the mesh. Tables 7.5 and 7.6 give the results obtained by integrating solutions over a time $T_1 = 48\pi$ instead of $T_1 = T_0$, to be compared with those in Table 7.3 and 7.4, respectively. Figure 7.10 shows that the error has dramatically decreased, falling below the differences between

![Figure 7.9](image_url)

Figure 7.9: Images of the contracted phase space $C_{T_1}$ superimposed on top of the basins of attraction for constant $\gamma_f = 0.02$. The values of $T_1$ are (a) $8\pi$, (b) $32\pi$ and (c) $160\pi$. 
CHAPTER 7. UTILISING BASINS OF ATTRACTION WITH CONSTANT DISSIPATION

<table>
<thead>
<tr>
<th>$T_0$</th>
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<th>Fast 3</th>
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<tbody>
<tr>
<td>$8\pi$</td>
<td>72.1750</td>
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</tr>
<tr>
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<tr>
<td>32$\pi$</td>
<td>76.9679</td>
<td>76.9765</td>
</tr>
</tbody>
</table>

Table 7.5: Results for $\gamma(\tau)$ varying from $\gamma_i = 0.05$ to $\gamma_f = 0.02$ over times $T_0$, with integration time $T_1 = 48\pi$. The meshes for “Fast 1” and “Fast 3” have increments of $(0.01, 0.01)$ and $(0.005, 0.005)$, respectively.

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>Accuracy of the method %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8\pi$</td>
<td>94.8805</td>
</tr>
<tr>
<td>16$\pi$</td>
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<tr>
<td>24$\pi$</td>
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<td>32$\pi$</td>
<td>98.7308</td>
</tr>
</tbody>
</table>

Table 7.6: The percentages of initial conditions for which the limiting behaviour is correctly described for the simulations in Table 7.5 using an integration time $T_1 = 48\pi$ instead of $T_1 = T_0$. The results are less than 99% accurate because $\gamma_i > \gamma_f$ and hence $S$ contracts more slowly at $\gamma = \gamma_f$, i.e. in the region $[T_0, T_1]$.

Figure 7.10: The same as Figure 7.8 with an integration time $T_1 = \max\{48\pi, T_0\}$. Figures (a) through to (d) show the differences for DFP, PR, NR and DO2, respectively, with (b) and (c) only going as far as $T_0 = 64\pi$ since the rotating solutions disappear after this. The plain and crossed full lines show the differences between “Fast 1” and “Fast 3”, respectively, and “ODE113”. The dashed line shows the difference between the estimates using “Series” and “Runge-Kutta”. The dotted line shows the 95% confidence interval.
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the other two integrators, even when a coarse mesh is used (to reduce further the overall integration time).

7.4 Jumps in the relative areas of the basins of attraction

We consider (7.2) with $\alpha = -0.1$, $\beta = 0.545$ and $\gamma(\tau)$ varying from $\gamma_i = 0.23$ to $\gamma_f = 0.2725$. For $\gamma = \gamma_i$ the system admits four attractors: the upward fixed point (UFP), the period-1 positively and negatively rotating solutions (PR/NR) and the period-2 oscillations (DO2) — we refer to Section 5.5 for more details. For $\gamma = \gamma_f$ only UFP and DO2 survive and a new attractor appears: the period-4 oscillation (DO4). The corresponding basins of attraction are shown in Figure 7.11; in Figure 7.12 the basins of attraction for $\gamma(\tau)$ varying in time from $\gamma_i = 0.23$ to $\gamma_f = 0.2725$ are shown for some values of $T_0$.

In Section 5.5, for $\gamma(t)$ varying from $\gamma_i = 0.23$ to $\gamma_f = 0.2725$ over a time $T_0$, we noticed large transitions in the basins of attraction for $T_0 = 99, 100, 500$; see Figure 7.2(b). To understand why a jump appears, say, for $T_0 = 99$, we can reason as follows: we increase $T_0$ in steps of one from 92 to 99. For all such values the appropriate density map shows that, after a time $T_1 = 32\pi > T_0$, over 90% of the trajectories are clustered

![Figure 7.11: Basins of attraction for (7.2) with $\alpha = -0.1$, $\beta = 0.545$ and (a) $\gamma = 0.23$ and (b) 0.2725. The basins of attraction are as follows: Blue = UFP, Green = DO2, Red=PR, Yellow=NR and Orange = DO4. The origin is centred at the upward vertical position of the pendulum, with $\xi = \theta + \pi$.](image)
in very small regions; this is an effect of the dissipation being rather large. Indeed, initially the trajectories are attracted to the attractors which exist at $\gamma = \gamma_i = 0.23$. The attractors UFP and DO2 still exist at $\gamma = 0.2725$. Most of the trajectories moving towards them form clusters at time $T_1$ close to the points at which the attractors cross the plane at $\tau = T_1$; we denote by UFP and DO2 such clusters, according to the attractor they are approaching. Instead, the attractors PR and NR disappear when the damping coefficient reaches the value $\gamma \approx 0.269$, see Section 5.5. The trajectories that were moving towards them form clusters which we denote by U. Thus, at time $T_1$, there are seven clusters of points: two clusters $U$, one cluster UFP and four clusters DO2; since up to $\gamma \approx 0.27$ there are two period-2 solutions — see Figure 7.13. The cluster UFP and each cluster DO2 correspond roughly to 20% and 11-13% of initial conditions, respectively, while the two clusters U each represent roughly 12% of the initial conditions (approximately the relative areas of the basins of attraction for the rotating solutions at $\gamma = \gamma_i = 0.23$).

Since both attractors UFP and DO2 remain (asymptotically) stable for all values of $\gamma \in [\gamma_i, \gamma_f]$, it is unlikely that the clusters labelled as UFP and DO2 will change their limiting solutions. This is confirmed by the fact that the clusters UFP and DO2 are
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Figure 7.13: Clusters in phase space superimposed onto basins of attraction for constant $\gamma = 0.2725$: (a) whole sample region with clusters UFP, DO2 and U in Blue, Green and Red, respectively; and (b) magnification of the region containing the clusters U in the upper half-plane (here the colour code is: Blue=92, Cyan=93, Green=94, Magenta=95, Yellow=96, Orange=97, Red=98, Brown=99). The basins are colour coded with Black for UFP, Dark Grey for DO2 and Light Grey for DO4. The snapshots in time are taken at time $T_1 = 32\pi$. The coloured dots have been drawn larger than the actual sizes of the clusters for clarity purposes. Where $\xi = \theta + \pi$ so that the origin is centred at the upward vertical position of the pendulum.

well inside the basins of attraction of the corresponding attractors UFP and DO2, respectively, for all values of $T_0$. On the contrary the clusters U move slightly from right to left in the upper half-plane and from left to right in the lower half-plane as $T_0$ increases. In doing so, they cross the boundaries of the basins of attraction for constant $\gamma = 0.2725$.

We interpret the results above as follows: after some transient behaviour, the original sample phase space has contracted enough and been re-organised so that some regions of phase space become more dense than others; in particular most of the trajectories end up inside some small, well separated clusters. The clusters U were initially converging towards the attractors PR/NR, however, they occupy slightly different positions at time $T_1$ as $T_0$ is varied due to the different evolution of $\gamma(\tau)$. Since they fall in a region where the basins of attraction for $\gamma = 0.2725$ are sparse and formed of very thin bands, changing the value of $T_0$ causes the clusters to cross the boundaries separating the bands belonging to different basins — see Figure 7.13(b). This results in a large jump in the relative area of the basin of attraction of UFP, as shown in Figure 7.2; see also Figure 7.12. According to the results reported in Figures 7.13(b) and 7.14, the clusters U fall
inside the basin of attraction of UFP for $T_0 = 93$ and 99, inside the basin of attraction of DO2 for $T_0 = 92, 95, 96, 97$ and 98, and inside the basin of attraction of DO4 for $T_0 = 94$.

Of course, the jumps occur for many other values of $T_0$. Furthermore, it may be seen in Figure 7.13(b) that the structure of the basins of attraction for $\gamma = \gamma_f$ in the region which the clusters $U$ traverse, is such that it forms periodically spaced bands. Therefore, as the clusters move linearly with changes in $T_0$, the jumps in the sizes of the basins of attraction are locally periodic. For much larger values of $T_0$ this may not be the case, as the clusters may move into very different regions of phase space.

Figure 7.14 shows that jumps upward take place for UFP, for instance, at $T_0 \approx 72.6, 79.2, 85.8, 92.4, 99.0$ and 105.6. When a peak appears, it survives for a very narrow range of $T_0$ (proportional to the width of the bands of the basins of attraction), after which a jump downward follows. For this reason, in practice it is difficult to predict exactly the values of $T_0$ where jumps occur, even though apparently they follow a periodic pattern — at least in the range of values we have investigated. Indeed, every value of $T_0$ at which a jump appears is obtained by adding the same quantity $\Delta T_0 \approx 6.6$ to the previous one. For some values of $T_0$ the basin of attraction of the period-4 oscillation has grown, which implies that the basin of attraction of the new attractor DO4 has formed exactly where the dense region $U$ has arrived at that time. Also in the case of DO2 and DO4 the jumps seem to have some periodicity in $T_0$, however with a more intricate structure. It would be interesting to investigate further such a phenomenon.

Note that, if one applies the method of fast numerical computation described in Section 7.3 to compute the relative areas of the basins of attraction in these cases, one needs to take a much larger value for the time $T_1$ at which the trajectories are approximated to the closest points of the mesh. More generally, this happens every time (i) an attractor disappears with increasing dissipation and (ii) the trajectories which
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Figure 7.14: Relative areas of the basins of attraction for (7.2) with $\alpha = -0.1$, $\beta = 0.545$ and $\gamma(\tau)$ varying from $\gamma_i = 0.02$ to $\gamma_f = 0.05$ over a time $T_0$. The circles represent results from numerical simulation, while the dashed lines are extrapolated on the basis of the observed periodicity — see the text for details.

were moving towards that attractor cluster into a small set in a region with tiny bands of the basins of attraction for $\gamma = \gamma_f$. Then $T_1$ must be large enough for the clusters to have reached the core surrounding one of the attractors. When $T_0$ is such that at time $t = 2\pi N$ the clusters (which have finite size) touch or even spread across the boundaries of the bands, time $T_1$ must be very large for the method of fast numerical integration to work, in principle as large as $T_f$ itself. In this case the method may have no advantage over full time integration of the equations.

7.5 Comparison with cell-to-cell mapping

On first inspection the methodology outlined in Section 7.3 may appear similar to that of cell-to-cell mapping, see Section 4.4 and [92, 93]. However there are some important differences separating the two methods.

Cell-to-cell mapping involves integrating (mapping) departure points in cells over
some interval of time to obtain the image of the cell. In doing so, one is able to quickly build up an estimate to the basins of attraction of a continuous system. In practice this can result in trajectories being repeatedly averaged to nearby points in phase space, see Section 4.4. Furthermore, as the method is used to obtain a picture of the basins of attraction the user is (in general) unaware of the topology of the basins and cannot make well informed judgements about suitable cell sizes.

Contrary to this, the technique outlined in Section 7.3 utilises basins of attraction already obtained for the scenario in which the damping coefficient is constant, to quickly estimate the basins of attraction in the case where damping varies with time. This methodology is of no use to construct basins of attraction for the constant damping scenario, since it requires these basins of attraction to already be known to a relatively high degree of accuracy. Using this technique, the process of averaging trajectories to nearby points in phase space happens one time only. Furthermore, the user is already aware of the topology of the basins of attraction for the constant case and is able to make informed judgements to reduce the errors induced by the process of averaging trajectories. Typically this is done by ensuring that the trajectories are only averaged once they are well inside an attractors corresponding basin of attraction i.e. the trajectories are not near the basin boundary. Another important difference is that, for the method outlined in Section 7.3, the attractive solutions are already known, so it is impossible for the method to introduce “fake solutions”.

Further to this, using the density map (see Definition 7.3) we are able to follow the most dense “clusters” of trajectories in phase space. This was utilised in Section 7.4 to explain a phenomenon noticed previously in Chapter 5, where the sizes of the basins of attraction undergo large jumps concentrated in small intervals of the parameter $T_0$. It is not possible to utilise cell-to-cell mapping in this way, since the user has no information about the “density” of trajectories in phase space.
Chapter 8

Conclusions and further research

In Chapter 2 we studied Hill’s equation, focusing mainly on the boundedness of the solutions. We also expanded the literature [25], finding explicit solutions to a class of Hill’s equation. As an example we found a solution to a particular case of the linearised system of a pendulum with variable length, as well as a particular case of the pendulum with variable length and oscillating support. We also utilised a classical transformation which transforms second order differential equations into their corresponding Hill form. By applying the transformation, and then an inverse transformation to get back to the original form, we linked the solutions of second order differential equations which have the same corresponding Hill’s equation. As a consequence, by considering systems for which the solutions can be found, we are able to construct a class of systems with corresponding solutions. Furthermore, we are able to obtain solutions to systems with time-dependent coefficients using the solutions to systems with constant coefficients.

At the end of Section 2.1 we noted that all of the solutions to a class of Hill’s equation are bounded, provided the frequency of the forcing is sufficiently high. By means of example, it was shown that this approach cannot be applied to obtain information about the boundedness of solutions to nonlinear systems. In Chapter 3 we returned to the problem for nonlinear systems and applied KAM theory to show that here also, even outside of the perturbation regime, the solutions are bounded if the frequency of the
forcing is sufficiently high. In fact, we were able to show that, provided the forcing frequency is high enough, not only are the solutions bounded, but they remain close to the corresponding solutions of the unperturbed system. We also showed the same is true for low frequency forcing, although in this instance no analytical result was obtained when the coefficient of forcing is greater than the coefficient of the unperturbed system. The phenomena was illustrated with the use of numerical simulations for a forced cubic oscillator. Numerically, we were able to show that even in cases where the coefficient of forcing is larger than that of the unperturbed system, the dynamics of the perturbed system remain close to those of the unperturbed one, provided the forcing frequency is sufficiently low. It would be interesting to investigate this further and extend the analysis to cover this scenario.

Throughout the remainder of the thesis we considered systems with the inclusion of dissipation. In Chapter 4 we gave details of the numerical simulations in the chapters which followed. In the process we illustrated the application of analytic continuation to systems with trigonometric nonlinearities. We also calculated the confidence intervals for the numerical simulations which estimate the sizes of the basins of attraction. This was shown to depend on the square root of the number of initial conditions. As such, we are able see the accuracy of the results in Chapters 5, 6 and 7. Furthermore, it makes clear the importance of using enough initial conditions when numerically estimating the sizes of basins of attraction, a detail which is often left out in the literature. In particular, we emphasised the importance of using enough initial conditions when one or more attractors has a small basin of attraction.

In Chapters 5 and 6 we studied two pendulum systems, focusing on the attractive solutions and their corresponding basins of attraction. For both systems we computed the sufficient conditions on the coefficient of dissipation $\gamma$, such that global attraction to the stable fixed point $(\theta, \theta') = (0, 0)$ is achieved; however in both instances we numerically
found a lower value of $\gamma$ was sufficient. We applied perturbation theory to calculate the threshold values of $\gamma$ under which a given periodic solution exists, noting also the limitations of the approach, due to its roots in perturbation theory and as a result of the separatrices in the systems. In Sections 5.4.1, 5.5.1 and 6.4.1 we looked at the bifurcations which occur as the parameter $\gamma$ is varied. This work was related to bifurcation analysis already in the literature [67, 154, 65, 159, 79, 36, 66, 49, 99]. We then numerically investigated the basins of attraction of both systems with different values of $\gamma$. It was seen that, in the perturbation regime (that is $\varepsilon$ small) as $\gamma$ is decreased towards $\gamma = 0$, for both systems the basin of attraction belonging to the downward fixed point increases significantly. This is not observed for the cubic system in [22]. A brief explanation for this behaviour was given at the end of Section 6.5. It would be interesting to investigate this phenomenon further and find out if such behaviour may be observed in other systems.

In our study of initially varying dissipation, we extended the ideas put forward in [22] and outlined four specific cases regarding the sets of attractors at the initial value of the coefficient of dissipation $\gamma_i$ and the final value $\gamma_f$. These four cases were illustrated using numerical simulations for the two pendulum systems, the pendulum with vertically oscillating support and the pendulum with variable length, although we expect the results to apply to a wider class of systems. Moreover, even though we specifically focused on the coefficient of dissipation, we expect the results to apply when varying any parameter of a system which effects which attractors exist and their basins of attraction. It would be interesting to extend this work further, to include non-monotonic and nonlinear variations of parameter values. Another interesting focal point is to consider coupled systems of ODEs, to check if the results carry over and what, if any, extra conditions must be satisfied by the system and its parameters. It would also be interesting to take into consideration the speed of attraction to different attractors. Attractors which attract trajectories more quickly, can capture more unstable solutions as $\gamma(\tau)$ varies and are
more likely to obtain a larger basin of attraction. Finally, we saw that it is possible for an attractors basin of attraction to tend towards a full measure set of phase space as $\gamma(\tau)$ varies over longer periods of time $T_0$, despite not attaining 100% of phase space for any intermediate values of constant $\gamma$. This happens when only one attractive solution exists at both $\gamma_i$ and $\gamma_f$, see for example the basin of attraction corresponding to the fixed point in Table 5.19. It would be interesting to study this further, as it creates the opportunity to maximise the basin of attraction of a solution by suitably controlling the parameters in the system. This can be useful in systems where a particular solution or outcome is desired.

In Section 6.5 we directly compared the two pendulum systems in respect to the work contained in the previous sections. We found that in general, the dynamics of the two systems are different and that their behaviour becomes increasingly disparate as the perturbation parameter is taken larger. Even when considering the linearised equations for the two systems the differences are clear, see Section 6.5. In our analysis of the nonlinear systems, we found the periodic orbits of the pendulum with variable length exist for larger values of $\gamma$; this is apparent from the conditions for global attraction to the origin, the thresholds of dissipation and the numerical analysis. Furthermore, we found that the pendulum with variable length is more difficult to study with perturbation theory. In particular, for any specific value of the perturbation parameter, when compared with the pendulum with oscillating support, the pendulum with variable length exhibits the characteristics of a system which is perturbed to a greater extent. Moreover, for the parameter values chosen in Chapters 5 and 6, we found that not only do more attractors exist for the pendulum with variable length, but even when the two systems share similar attractors, the regions of phase space occupied by the corresponding basins of attraction are different.

In Chapter 7 we introduced a method for the fast numerical computation of basins of
attraction. We showed that the method can be used to estimate the basins of attraction in systems with damping coefficient $\gamma(t)$ varying between two given values $\gamma_i$ and $\gamma_f$ over a time $T_0$. Although we only tested explicitly the scenario in which dissipation varies linearly, there is no reason why the method cannot be applied to nonlinear functions of $\gamma(t)$. The method was found to work well for systems which have at least some regions surrounding the attractors which are not sensitive to slight perturbations in initial conditions. It becomes more accurate as the integration time $T_1$ increases, a result of the phase space contracting into regions distant from the boundaries of the basins of attraction. Thus the results can be improved by setting a minimum time $T_{\text{min}}$, and taking $T_1 = 2\pi N \geq \max\{T_0, T_{\text{min}}\}$. This seems to be necessary when $T_0$ is rather small or such that an attractor disappears in a region where the basins of attraction for $\gamma = \gamma_f$ have a multi-band structure. Provided $T_1$ is chosen appropriately, the method was shown to correctly identify the asymptotic behaviour of initial conditions in more than 99% of cases and significantly reduce the required computation time – up to a factor 20 in some cases we considered. We found that using a finer mesh to approximate trajectories at the time $T_1$ does not necessarily improve the results. However, smaller basins of attraction benefit from a finer mesh, as they intersect with fewer points in the mesh. The accuracy could be improved further by using a mesh with non-uniform spacing and better fitting the mesh to the region $C_{T_1}$ to which the sample phase space has contracted at time $T_1$. This was not implemented in Chapter 7, as it would add extra numerical complexities when rounding a trajectory at time $T_1$ to the nearest point on the mesh. Both measures would reduce unnecessary full time integration of initial conditions on the mesh. Further work could be carried out to remove the use of a mesh and instead have a set of random initial conditions. Similarly the density of the random initial conditions in particular regions could be chosen so as to optimise the results.

In the last section of Chapter 7 we returned to a phenomenon observed in Section
5.5.2, that is, for particular values of $T_0$ the sizes of the basins of attraction undergo large jumps. Using Density maps we showed that when an attractor becomes unstable, it leaves areas of phase space which are densely occupied by trajectories. Furthermore, the position of these dense regions in phase space at times $\tau = 2n\pi$ varies as a function of $T_0$. If the basins of attraction at constant $\gamma = \gamma_f$ are sufficiently intertwined in bands, these dense regions can cross the boundaries of the basins of attraction, hence varying which attractor they move towards. The result is that the basins of attraction of the remaining attractors experience large variations in size, as they gain or lose the dense clusters of trajectories. It would be interesting to study this further for other systems which also have “banded” basins of attraction, for example the cubic oscillator [22] and the Duffing equation.
Appendix A

Background knowledge

The following sections are devoted to introducing the main techniques and ideas used throughout the previous chapters, as well as providing further details to some of the calculations in the main body of the thesis.

A.1 An introduction to Lagrangian and Hamiltonian dynamics

Classical mechanics is an area of applied mathematics in which the motion of macroscopic objects is studied, encompassing everything from the famous examples of the pendulum, harmonic oscillator and rigid body to the motion of heavenly bodies in celestial mechanics. The foundations for classical mechanics were laid in the 17th century, largely by Sir Isaac Newton who proposed three basic laws,

1. Inertia: If an object experiences no net force then its velocity is constant.

2. Acceleration: The acceleration of a body is proportional and parallel to the net force acting upon it and inversely proportional to the mass of the body. i.e $F = ma$.

3. Action and reaction: When a body exerts a force on a second body, the second body simultaneously exerts an equal and opposite force on the original body.
It is also well known that Newton was the first to discover a mathematical formulation of gravity\(^1\), which he showed to be applicable to all objects around us as well as celestial objects. Due to his considerable contribution, the early classical mechanics is now referred to as Newtonian mechanics. Another way of studying such systems is through the methods of Lagrangian and Hamiltonian mechanics, which fundamentally are reformulations and extensions of Newtonian mechanics. Whereas Newtonian mechanics is based on using the vectorised forces in the system, both Lagrangian and Hamiltonian mechanics use the total energy of the system. The prior, Lagrangian mechanics, applies to both conservative and non-conservative systems where as the latter only applies to systems in which the total energy is conserved with respect to time. The energy of a system consists of two types, kinetic energy \(T\), and potential energy \(V\), the sum of which is the total energy of the system \(E\). In Lagrangian mechanics the entire evolution of the system is described by a function \(L\), known as the Lagrangian of the system. The Lagrangian \(L\) is a continuous and differentiable function in all of its arguments. The integral of the function \(L\) between two points in time is known as the action, often denoted by \(S\). A fundamental point of Lagrangian mechanics is that the action of a Lagrangian for any system satisfies *Hamilton’s principle of stationary action*. That is

\[
\delta S = \delta \int_{t_1}^{t_2} L \, dt = 0, \tag{A.1.1}
\]

i.e. the variation of the action is zero. We shall later use this property to calculate the equations of motion of a general system. We begin by defining the general Lagrangian

\(^1\)In the 4th century BC, the Greek philosopher Aristotle believed there is no effect or motion without a cause. The downward motion of heavy bodies such as the element “earth”, was related to where they naturally belonged, which caused them to move downward toward the centre of the universe. Conversely lighter elements such as fire moved upward toward the moon. The Roman engineer and architect Vitruvius contended that gravity does not depend on the weight of a substance but on its “nature”, commenting that a small amount of gold would sink when placed on top of quicksilver, whereas a heavy stone would not. During the 17th century Galileo found that all objects accelerated equally when falling and Robert Hook came close to an experimental proof that gravity follows an inverse square law, which he hypothesised to govern the motions of the planets. In the late 17th century these ideas were developed by Sir Isaac Newton who derived a mathematical formulation for gravity and was able to derive Kepler’s three kinematic laws of planetary motion.
for a system of \( n \) particles moving in three-dimensional space with \( k \) degrees of freedom. The position of the \( i \)-th particle can be described in Cartesian co-ordinates by a vector \( \mathbf{x}_i = (x_i, y_i, z_i) \) which depends on time \( t \). As the position of the particle is measured by \( \mathbf{x}_i(t) \) the velocity can be measured by \( \dot{\mathbf{x}}_i(t) \) where the dot denotes derivative with respect to time \( t \). In this formulation the total kinetic energy of the system, \( T \) is

\[
T = \frac{1}{2} \sum_{i=1}^{n} m_i \mathbf{x}_i \cdot \dot{\mathbf{x}}_i, \tag{A.1.2}
\]

where \( m_i \) is the mass of the \( i \)-th particle. The potential energy is a function of the forces which appear in the system and does not have a general formula. The Lagrangian in Cartesian co-ordinates is then defined as

\[
\mathcal{L}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_n; t) = T(\dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_n; t) - V(\mathbf{x}_1, \ldots, \mathbf{x}_n; t), \tag{A.1.3}
\]

An alternative approach to describe the system is via the number of degrees of freedom it possesses. For each degree of freedom we require a co-ordinate, known as a generalised co-ordinate which we shall denote \( q_j \) for \( j = 1, \ldots, k \) and the derivatives \( \dot{q}_j \) are known as the generalised velocities. The Cartesian co-ordinates are then functions of the generalised co-ordinates, namely

\[
x_i = x_i(q_1, \ldots q_k),
\]

\[
y_i = y_i(q_1, \ldots q_k),
\]

\[
z_i = z_i(q_1, \ldots q_k).
\]

This approach is advantageous as it incorporates the restrictions of the system while using no more co-ordinates than are necessary to describe the position of the particles. Using

\[
\dot{x}_i = \sum_{j=1}^{k} \frac{\partial x_i}{\partial q_j} \dot{q}_j,
\]
we may write the total kinetic energy of the system as a function of the generalised velocities, giving

\[ T = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} a_{i,j} \dot{q}_i \dot{q}_j, \tag{A.1.4} \]

for some constants \( a_{i,j} \). The Lagrangian, \( \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}; t) \), of the system thus becomes

\[ \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n; t) = T(\dot{q}_1, \ldots, \dot{q}_n; t) - V(q_1, \ldots, q_n; t). \tag{A.1.5} \]

As previously mentioned, we can use the principle of stationary action to calculate the Lagrangian equations of motion. We have

\[
\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}; t) \, dt
= \int_{t_1}^{t_2} \delta \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}; t) \, dt
= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \delta \mathbf{q} + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \delta \dot{\mathbf{q}} \right) \, dt
= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \delta \mathbf{q} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \cdot \delta \mathbf{q} \right) \, dt
= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \right) \cdot \delta \mathbf{q} \, dt,
\]

where the multiplication is via the dot product and both the derivatives and variations are applied component wise. The step between the third and fourth line uses integration by parts of the second term and the fact that the variations vanish on the boundary. Referring back to equation (A.1.1), and as the variation \( \delta \mathbf{q} \) is arbitrary we have

\[ \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) = 0, \quad j = 1, \ldots, k, \tag{A.1.6} \]

which are the Lagrangian equations of motion also known as the Euler Lagrangian equations. Similarly to Lagrangian mechanics, in Hamiltonian mechanics the dynamics...
of a system are described by a function called the Hamiltonian which is a function of the generalised co-ordinates $q$, $\dot{q}$. We proceed to define the Hamiltonian starting with the Lagrangian function, this approach highlights the connection between the two functions and allows us to see the properties of the Hamiltonian via simple comparisons.

Differentiating the Lagrangian with respect to time, using the chain rule we have

$$\frac{dL}{dt} = \sum_{j=1}^{k} \left( \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j + \frac{\partial L}{\partial \ddot{q}_j} \ddot{q}_j \right) = \sum_{j=1}^{k} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \frac{\partial L}{\partial \ddot{q}_j} \ddot{q}_j \right)$$

$$= \frac{d}{dt} \left( \sum_{j=1}^{k} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right),$$

where the second equality uses equation (A.1.6). It follows that

$$\frac{d}{dt} \left( \sum_{j=1}^{k} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \right) = 0,$$

and hence

$$\sum_{j=1}^{k} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = \text{constant w.r.t. time.}$$

This constant is defined as the Hamiltonian, $\mathcal{H}$. Furthermore, we can show that generally for standard physical systems the Hamiltonian is equal to the total energy of the system. Noticing from (A.1.5) that $L$ is separable, that is $L(q, \dot{q}) = T(\dot{q}) - V(q)$, so

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s} = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \left( a_{i,j} \frac{\partial \dot{q}_i}{\partial \dot{q}_s} \dot{q}_j + a_{i,j} \dot{q}_i \frac{\partial \dot{q}_j}{\partial \dot{q}_s} \right)$$

$$= \sum_{i=1}^{k} a_{i,s} \dot{q}_i,$$
we have

\[ H = \sum_{j=1}^{k} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = \sum_{j=1}^{k} \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j - L = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{i,j} \dot{q}_i \dot{q}_j - L = 2T - (T - V) = T + V, \]

which of course is the total energy of the system, \( E \). This shows that for Hamiltonian systems the energy is constant with respect to time, i.e. the systems are energy conserving. To derive the equations of motion from the Hamiltonian, we introduce \( n \) new variables \( p_j(t) \) which are defined as \( \dot{p}_j = \partial L / \partial q_j \) (and hence \( p_j = \partial L / \partial \dot{q}_j \)). This yields

\[ dH = d \sum_{j=1}^{k} p_j \dot{q}_j - dL = \sum_{j=1}^{k} (\dot{p}_j dp_j + p_j dq_j) - \sum_{j=1}^{k} \left( \frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt = \sum_{j=1}^{k} (\dot{p}_j dp_j + p_j dq_j) - \sum_{j=1}^{k} (\dot{p}_j dq_j + p_j d\dot{q}_j) - \frac{\partial L}{\partial t} dt = \sum_{j=1}^{k} (\dot{p}_j dp_j - \dot{q}_j dq_j) - \frac{\partial L}{\partial t}. \]

We also have

\[ dH = \sum_{j=1}^{k} \left( \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial \dot{q}_j} d\dot{q}_j \right) + \frac{\partial H}{\partial t} dt, \]
which upon comparison we see that

\[
\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}.
\] (A.1.7)

The above equations are what we call Hamilton’s equations of motion.

### A.2 Canonical transformations and action-angle variables

When transforming from one set of co-ordinates to another it is often required that certain properties of the system are preserved. A canonical transformation is one which is area preserving and hence preserves the form of the Hamilton’s equation, i.e. preserves the energy of the system. In this section we primarily follow the discussion in [135], see also [74]. If we start with a system with one degree of freedom which has trajectories in 2-dimensional phase space measured by a set of co-ordinates \( p, q \) and wish to transform to co-ordinates \( P, Q \), then the areas of a region \( R \) in each representation are defined as

\[
A_{p,q} = \int \int_{R_{p,q}} dq dp,
\]

\[
A_{P,Q} = \int \int_{R_{P,Q}} dP dQ,
\]

where \( A_{p,q}, A_{P,Q}, R_{p,q} \) and \( R_{P,Q} \) are the areas and regions in each co-ordinate space respectively. In general, the two integrals above are related by

\[
\int \int_{R_{P,Q}} dP dQ = \int \int_{R_{p,q}} \frac{\partial(P,Q)}{\partial(p,q)} dq dp.
\]

Thus if the co-ordinate change is canonical, that is we have \( A_{p,q} = A_{P,Q} \), then the transformation must satisfy

\[
\frac{\partial(P,Q)}{\partial(p,q)} = \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} = 1,
\] (A.2.1)
or of course equivalently

\[ \frac{\partial(p, q)}{\partial(P, Q)} = \frac{\partial q}{\partial Q} \frac{\partial p}{\partial P} - \frac{\partial q}{\partial P} \frac{\partial p}{\partial Q} = 1. \]  

(A.2.2)

It is clear that the two equations above are equivalent to requiring that the determinant of the Jacobian transformation matrix is unity. Furthermore, if the Jacobian determinant of a transformation from co-ordinates \( p, q \) to \( P, Q \) is equal to some constant, \( C \neq 1 \), then we can take a new set of co-ordinates \( \tilde{Q} = Q/C \), \( \tilde{P} = P \) for which the equations (A.2.1) and (A.2.2) are satisfied, [135]. Of course, the reason for transforming from one set of co-ordinates to another set is that the new set of co-ordinates make the dynamics easier to understand. For the bounded motion of conservative Hamiltonian systems there exists a set of co-ordinates in which the dynamics can be expressed in the simplest possible way, these co-ordinates are known as action-angle variables (sometimes called angle-action variables). More precisely, the action-angle variables \( (I, \varphi) \), are such that the action \( I \), depends solely on the energy and as such is not only constant along each phase curve but each of the phase curves is uniquely labeled by the action. The angle variable \( \varphi \), is an angle in the true sense and increases by \( 2\pi \) each time the curve is traced, thus it is linear with respect to time and each point on the phase curve is labeled by a single valued function of \( \varphi \). From this description it is apparent that

\[ \dot{I} = \frac{\partial H}{\partial \varphi} = 0, \quad \dot{\varphi} = \frac{\partial H}{\partial I} = \text{constant}, \]

from which it follows that the Hamiltonian is a function of \( I \) only. One benefit of action-angle variables is that the frequency of the system is immediately obvious without the need to solve any equations of motion, that is, as \( \dot{\varphi} \) is constant, this constant must be the frequency.

We now continue to outline a method for calculating the action-angle variables, which
Figure A.1: A pictorial representation of the transformation from $p,q$ co-ordinates to action-angle variables. Note, the picture on the right could also be drawn as concentric circles each with a fixed radius $\sqrt{2I}$ and $\varphi$ increasing by $2\pi$ each time the curve is traced.

can be found in [74, 135]. We only consider Hamiltonians of the form

$$E = \mathcal{H}(p, q) = \frac{p^2}{2m} + V(q), \quad (A.2.3)$$

where $V(q)$ is some periodic function of $q$. The area is then defined by

$$A(E) = \oint p(q, E)\, dq = \oint \sqrt{2m(E - V(q))} \, dq. \quad (A.2.4)$$

Clearly, there are three different cases that can occur, $E$ is less than the maximum of $V(q)$, $E$ is greater than the maximum of $V(q)$ or $E$ is equal to the maximum of $V(q)$. When $E$ is less than the maximum of $V(q)$ the system is in libration and the area can be written as

$$A(E) = \oint p(q, E)\, dq = 2 \int_{q_0}^{q_1} \sqrt{2m(E - V(q))} \, dq, \quad (A.2.4)$$

where $V(q_0) = V(q_1) = E$, see Figure A.2. In action-angle variables the area is given by

$$A(E) = \int_0^{2\pi} I \, dq = 2\pi I, \quad (A.2.5)$$

which upon comparison with equation (A.2.4) we see that

$$I(E) = \frac{1}{\pi} \int_{q_0}^{q_1} \sqrt{2m(E - V(q))} \, dq. \quad (A.2.6)$$

When $E$ is greater than the maximum of $V(q)$ the system rotates, although the basic idea remains the same, now the phase curves are not closed and are either increasing or decreasing. The area in action-angle variables has the same form as in (A.2.5) however the area in $p, q$ co-ordinates is now

$$A(E) = \oint p(q, E) \, dq = \int_0^{2\pi} \sqrt{2m(E - V(q))} \, dq,$$

and thus

$$I(E) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2m(E - V(q))} \, dq. \quad (A.2.7)$$

The oscillating and rotating dynamics are separated by a separatrix, which occurs when $E$ equals the maximum of $V(q)$, on which action-angle variables are not defined. Variables separated by a separatrix are in general unrelated.

The angle variable can then be found in one of two ways, both of which, unlike finding the action can be applied in exactly the same manner for both the rotational and librational dynamics. The first method, which holds true for all systems, is to consider two nearby phase curves of the action, say $I$ and $I + \varepsilon I$, and measure the area between them, $A_\varepsilon$, in both co-ordinate representations, which we then compare. The second
method is most easily seen by example, and will be demonstrated later for the simple pendulum in Appendix A.4. We begin the first method by finding the area between the two phase curves in \( p, q \) co-ordinates,

\[
A_\varepsilon = \int_0^q p \, dq = \int_0^q p(q, I + \varepsilon I) - p(q, I) \, dq = \varepsilon I \int_0^q \frac{\partial}{\partial I} p(q, I) \, dq + h.o.t,
\]

and in the \( I, \varphi \) co-ordinates we have

\[
A_\varepsilon = \int_0^\varphi I + \varepsilon I - I \, d\varphi = \varepsilon I \int_0^\varphi d\varphi = \varepsilon I \varphi. 
\]

Then by simple comparison we find

\[
\varphi = \int_0^q \frac{\partial}{\partial I} p(q, I) \, dq. \quad (A.2.8)
\]

In many cases, including those of the pendulum system and cubic oscillator in Appendices A.4 and A.5, the integrals involved in finding the action-angle variables are elliptic, and a knowledge of elliptic integrals is required in order to evaluate the integrals and calculate the co-ordinates. This brings us on to Appendix A.3

### A.3 Elliptic integrals and the Jacobi elliptic functions

Here we review some basic knowledge about elliptic integrals and the Jacobi elliptic functions, all of which may be found in the literature [44, 54, 107, 165]. In general any integral of the form

\[
\int \frac{A(x) + B(x)\sqrt{S(x)}}{C(x) + D(x)\sqrt{S(x)}} \, dx,
\]

where \( A(x), B(x), C(x) \) and \( D(x) \) are polynomials in \( x \) and \( S(x) \) is a polynomial of third or fourth degree is an elliptic integral. More simply put, an elliptic integral is any integral \( \int R \left( x, \sqrt{P(x)} \right) \, dx \) where \( P(x) \) is a cubic or quartic polynomial in \( x \) with
no repeated roots and $R$ is a rational function of its arguments. Furthermore, without loss of generality we may restrict ourselves to consider only the cubic polynomials, as any polynomial $P(x)$ of degree $2n$ can be written as a polynomial of degree $2n - 1$, using the transformation $x = a + 1/y$ where $a$ is a root of $P(x)$, [83]. Conversely, we may restrict ourselves to only consider polynomials of fourth degree, as any polynomial $P(x)$ of degree $2n - 1$ may be transformed into a polynomial of degree $2n$ using the transformation $x = a + 1/y$ where $a$ is not a root of $P(x)$. Then

$$\sqrt{P(x)} = \frac{1}{y^n} \sqrt{P_1(y)},$$

and hence

$$\int R \left(x, \sqrt{P(x)}\right) \, dx = \int R_1 \left(y, \sqrt{P_1(y)}\right) \, dy.$$

In the literature it is more common to consider $P(x)$ as a polynomial of degree four. Some early work on elliptic integrals was done by Fagnano [73] in relation to the rectification of the arc of the lemniscate. This was later expanded by Euler, who derived an addition theorem. Further work was done by Lagrange and Landen; however it was Legendre who first systematically investigated elliptic integrals and showed that any elliptic integral can, by linear transformations and reduction formula, be expressed as a finite sum of elementary integrals and three fundamental elliptic integrals. These three fundamental elliptic integrals, $F(\phi, k)$, $E(\phi, k)$, $\Pi(n; \phi, k)$, are now known as the incomplete elliptic integrals of the First, Second and Third kinds respectively. As there is not a definite set of notation used for elliptic integrals, which can create unnecessary confusion, we shall make clear the terminology used throughout, which follows closely to that of Whittaker and Watson [165]. The first through to third kinds of incomplete elliptic integrals are defined as follows

$$F(\phi, k) = \int_0^\phi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta,$$  \hspace{1cm} (A.3.1)
It is clear from the left-hand sides of the equations that incomplete elliptic integrals take two arguments with the exception of the Third kind, which takes three. The argument 
$k$ is known as the elliptic modulus and in theory can take any complex value, however in application it is usually confined to take values on the real line in the open region $(0,1)$. When confined, the elliptic modulus can be thought of as a measure of the eccentricity of the ellipse, taking $k = 0$ gives an arc of a perfect circle while $k = 1$ causes the ellipse to break. The complimentary modulus, $k'$ is defined by its relation to $k$ as $k' = \sqrt{1-k^2}$ and the complimentary elliptic integrals are defined as the elliptic integrals of the complimentary elliptic modulus. The argument $\phi$ represents the Jacobi amplitude also known as the amplitude angle and is normally restricted to take values in $[0, \pi/2]$. Finally, the parameter $n$ in the Third elliptic integral is known as the elliptic characteristic, however as knowledge of this is not required in the main body of the thesis, the details of are not included here. Defining the change of co-ordinate $y = \sin \theta$ the three incomplete elliptic integrals can be expressed in another common form. Often when expressed in this form, many books use the variable $t = \sin \theta$, however we have chosen $y$ so as to avoid confusion both here and throughout this thesis with time $t$. Under this transformation, the three incomplete elliptic integrals are

\begin{align}
\mathbf{F}(\phi, k) & = \int_0^{\sin \phi} \frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}} \, dy, \\
\mathbf{E}(\phi, k) & = \int_0^{\sin \phi} \frac{\sqrt{1-k^2y^2}}{\sqrt{1-y^2}} \, dy, \\
\Pi(n; \phi, k) & = \int_0^{\sin \phi} \frac{1}{(1-ny^2)(1-y^2)(1-k^2y^2)} \, dy.
\end{align}
APPENDIX A. BACKGROUND KNOWLEDGE

As well as the incomplete elliptic integrals, we also have the complete elliptic integrals which are found by fixing $\phi$ at its upper limit, that is $\phi = \pi/2$. The three types of complete elliptic integrals are given by

\[ K(k) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta = \int_{0}^{1} \frac{1}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} \, dy, \quad (A.3.7) \]
\[ E(k) = \int_{0}^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta = \int_{0}^{1} \frac{1}{\sqrt{1 - k^2 y^2}} \, dy, \quad (A.3.8) \]
\[ \Pi(n; k) = \int_{0}^{\pi/2} \frac{1}{(1 - n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta \]
\[ = \int_{0}^{1} \frac{1}{(1 - ny^2)\sqrt{(1 - y^2)(1 - k^2 y^2)}} \, dy, \quad (A.3.9) \]

where we now use $K(k)$ to denote the complete elliptic integral of the First kind out of convention. The First and Second complete elliptic integrals can also be expanded simply as an infinite power series as

\[ K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[ \frac{(2n)!}{2^{2n}(n!)^2} \right]^2 k^{2n}, \]
\[ E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[ \frac{(2n)!}{2^{2n}(n!)^2} \right]^2 \frac{k^{2n}}{1 - 2n}. \]

The derivatives of the complete elliptic integrals with respect to their arguments can be found in [107] and are stated below without proof.

\[ \frac{dK(k)}{dk} = \frac{1}{kk'/(E(k) - k^2K(k))}, \]
\[ \frac{dE(k)}{dk} = \frac{1}{k}(E(k) - K(k)), \]
\[ \frac{\partial \Pi(n; k)}{\partial n} = \frac{1}{2(k^2 - n)(n - 1)} \left( E(k) + \frac{1}{n}(k^2 - n)K(k) + \frac{1}{n}(n^2 - k^2)\Pi(n, k) \right), \]
\[ \frac{\partial \Pi(n; k)}{\partial k} = \frac{k}{n - k^2} \left( \frac{E(k)}{k^2 - 1} + \Pi(n, k) \right). \quad (A.3.10) \]
APPENDIX A. BACKGROUND KNOWLEDGE

The idea of inverting an elliptic integral, to obtain what is known as an elliptic function was first pioneered by Abel [1], Jacobi [94, 95] and Gauss independently of one another. Later work was also conducted by Weierstrass, however the most commonly used elliptic functions are those of Jacobi, known as the Jacobi elliptic functions. In general, an elliptic function is any function \( f(u) \), which is analytic except at poles, has no singularities other than poles for any finite value of \( u \) and is doubly periodic with periods \( T_1, T_2 \) which are such that \( T_1/T_2 \notin \mathbb{R} \).

The Jacobi elliptic function \( \text{sn}(u, k) \) is defined as the inverse of the incomplete elliptic integral of the First kind. A very good and easy to read derivation of the elliptic functions can be found in [44], here we will give a summary of the approach. We begin by defining \( u \) as

\[
u = \int_0^x \frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}} \, dy, \tag{A.3.11}
\]

where \(-1 \leq x \leq 1 \) and real. This characterises \( u \) as an odd function of \( x \) which increases from 0 to \( K(k) \) as defined in (A.3.7) as \( x \) increases from 0 to 1 (or indeed decreases from 0 to \(-K(k) \) as \( x \) decreases from 0 to \(-1 \)). By inverting equation (A.3.11) it is possible to define \( x \) as a function of \( u \), which increases from 0 to 1 as \( u \) increases from 0 to \( K(k) \), this function we shall denote as \( \text{sn}(u, k) \).

\[
x = \text{sn}(u, k), \quad \text{sn}^{-1}(x, k) = u.
\]

From this derivation we can see that \( \text{sn}(u, 0) = \sin u \), which is perhaps made more clear by comparing equations (A.3.4) and (A.3.11), where it may be seen that \( x = \sin \phi \) and hence \( \text{sn}^{-1}(\sin \phi) = u, \sin \phi = \text{sn}(u, k) \). Setting \( k = 0 \) we have

\[
u = \int_0^{\sin \phi} \frac{1}{\sqrt{1-y^2}} \, dy = \arcsin (\sin \phi) - \arcsin 0 = \phi,
\]

hence \( \text{sn}(u, 0) = \sin u \). The other two main Jacobi elliptic functions are then defined
using sn(u, k) as follows

\[ cn(u, k) = \sqrt{1 - sn^2(u, k)}, \quad dn(u, k) = \sqrt{1 - k^2 sn^2(u, k)}. \] (A.3.12)

Similar to the circular functions (trigonometric functions) there are also composite functions, which are

\[ ns(u) = \frac{1}{sn(u)}, \quad nc(u) = \frac{1}{cn(u)}, \quad nd(u) = \frac{1}{dn(u)}, \]

\[ sc(u) = \frac{sn(u)}{cn(u)}, \quad sd(u) = \frac{sn(u)}{dn(u)}, \]

\[ cs(u) = \frac{cn(u)}{sn(u)}, \quad cd(u) = \frac{cn(u)}{dn(u)}, \]

\[ ds(u) = \frac{dn(u)}{sn(u)}, \quad dc(u) = \frac{dn(u)}{cn(u)}, \]

where we have admitted \( k \) to save on clutter. By simple rearrangement of (A.3.12) we can derive the following identities

\[ sn^2(u) + cn^2(u) = 1, \]

\[ dn^2(u) + k^2 sn^2(u) = 1, \]

\[ k^2 cn^2(u) + k'^2 = dn^2(u), \]

\[ cn^2(u) + k'^2 = dn^2(u) \] (A.3.13)

To find the derivatives of the Jacobi elliptic functions with respect to \( u \) we use equation (A.3.11) coupled with the knowledge that \( x = sn(u, k) \). Thus

\[ \frac{\partial}{\partial u} sn(u, k) = \frac{\partial x}{\partial u} = \sqrt{(1 - x^2)(1 - k^2 x'^2)} = cn(u, k) dn(u, k). \] (A.3.14)

From which, with the use of (A.3.12) we obtain

\[ \frac{\partial}{\partial u} cn(u, k) = -sn(u, k) dn(u, k), \]

\[ \frac{\partial}{\partial u} dn(u, k) = -k^2 sn(u, k) cn(u, k). \] (A.3.15)
The derivatives of the Jacobi elliptic functions with respect to the parameter $k$ can be found in [107] as follows

$$
\frac{\partial}{\partial k} \text{sn}(u, k) = \frac{1}{k} u \text{cn}(u, k) \text{dn}(u, k) + \frac{k}{k^2} \text{sn}(u, k) \text{cn}^2(u, k) - \frac{1}{kk'2} E(u, k) \text{cn}(u, k) \text{dn}(u, k),
$$

$$
\frac{\partial}{\partial k} \text{cn}(u, k) = -\frac{1}{k} u \text{sn}(u, k) \text{dn}(u, k) - \frac{k}{k^2} \text{sn}^2(u, k) \text{cn}(u, k) + \frac{1}{kk'2} E(u, k) \text{sn}(u, k) \text{dn}(u, k),
$$

$$
\frac{\partial}{\partial k} \text{dn}(u, k) = -ku \text{sn}(u, k) \text{cn}(u, k) - \frac{k}{k^2} \text{sn}^2(u, k) \text{dn}(u, k) + \frac{k}{k^2} E(u, k) \text{sn}(u, k) \text{cn}(u, k).
$$

Until now, we have restricted $u$ to the interval $[-K(k), K(k)]$. We now show that $\text{sn}(u, k)$ can be extended for all $u \in \mathbb{R}$ and in fact $u \in \mathbb{C}$. We give without proof, the addition formulae for the Jacobi elliptic functions $\text{sn}(u)$, $\text{cn}(u)$ and $\text{dn}(u)$

$$
\text{sn}(u + v) = \frac{\text{sn}(u) \text{cn}(v) \text{dn}(v) + \text{sn}(v) \text{cn}(u) \text{dn}(u)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)},
$$

$$
\text{cn}(u + v) = \frac{\text{cn}(u) \text{cn}(v) - \text{sn}(u) \text{sn}(v) \text{dn}(u) \text{dn}(v)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)}, \quad (A.3.16)
$$

$$
\text{dn}(u + v) = \frac{\text{dn}(u) \text{dn}(v) - k^2 \text{sn}(u) \text{sn}(v) \text{cn}(u) \text{cn}(v)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)}.
$$

Substituting $v = K(k)$ into the above relations and using that $\text{sn}(K) = 1$, $\text{cn}(K) = 0$ and $\text{dn}(K) = k'$ (which can be seen from equation (A.3.12) taking the values $u = 0$ and
\( u = K \), yields

\[
\begin{align*}
\text{sn}(u + K) &= \frac{\text{sn}(u) \text{cn}(K) \text{dn}(K) + \text{sn}(K) \text{cn}(u) \text{dn}(u)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(K)} = \frac{\text{cn}(u)}{\text{dn}(u)}, \\
\text{cn}(u + K) &= \frac{\text{cn}(u) \text{cn}(K) - \text{sn}(u) \text{sn}(K) \text{dn}(u) \text{dn}(K)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(K)} = -\frac{k' \text{sn}(u)}{\text{dn}(u)}, \\
\text{dn}(u + K) &= \frac{\text{dn}(u) \text{dn}(K) - k^2 \text{sn}(u) \text{sn}(K) \text{cn}(u) \text{cn}(K)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(K)} = \frac{k'}{\text{dn}(u)}.
\end{align*}
\]

Further substituting in \( u = K \) gives

\( \text{sn}(2K) = 0, \quad \text{cn}(2K) = -1, \quad \text{dn}(2K) = 1. \)

Similarly, by putting \( v = 2K \) and \( v = 4K \) into (A.3.16) and using our result above we have

\( \text{sn}(u + 2K) = -\text{sn}(u), \quad \text{sn}(u + 4K) = \text{sn}(u), \)

\( \text{cn}(u + 2K) = -\text{cn}(u), \quad \text{cn}(u + 4K) = \text{cn}(u), \)

\( \text{dn}(u + 2K) = \text{dn}(u), \)

which shows \( \text{sn}(u) \) and \( \text{cn}(u) \) to be periodic with period \( 4K \) and \( \text{dn}(u) \) to be periodic with period \( 2K \), allowing us to take any real value of \( u \). In Figure A.3 the three Jacobi elliptic functions are shown for the elliptic modulus \( k^2 = 0.5 \). As mentioned previously, elliptic functions are doubly periodic and by an analogous method we can also find the complex periods. Doing so we find \( \text{sn}(u) \) to be periodic with period \( 2iK' \), \( \text{cn}(u) \) to be periodic with period \( 2K + 2iK' \) and the period of \( \text{dn}(u) \) as \( 4iK' \).
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Figure A.3: Plot of the Jacobi elliptic functions with $k^2 = 0.5$ over the real axis. The function $\text{cn}(u, k)$ is in blue, $\text{sn}(u, k)$ in red and $\text{dn}(u, k)$ in black.

Finally, as it shall later be required in Chapters 5 and 6, we give the integral for the square of the Jacobi elliptic function $\text{dn}(u, k)$. This is found as

$$\int_0^{x_1} \text{dn}^2(x, k) \, dx = \int_0^{\text{sn}(x_1, k)} \frac{\sqrt{1 - k^2 \hat{x}^2}}{\sqrt{1 - \hat{x}^2}} \, d\hat{x} = E\left(\text{sn}(x_1, k), k\right),$$

from which using the identities (A.3.13) the squares of $\text{sn}(u, k)$ and $\text{cn}(u, k)$ can also be simply computed. We also state the Fourier series expansions of the Jacobi elliptic functions as

$$\text{sn}(u, k) = \frac{2\pi}{kK(k)} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{1 - q^{2n-1}} \sin \left(\frac{(2n - 1)\pi u}{2K(k)}\right),$$

$$\text{cn}(u, k) = \frac{2\pi}{kK(k)} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{1 + q^{2n-1}} \cos \left(\frac{(2n - 1)\pi u}{2K(k)}\right),$$

$$\text{dn}(u, k) = \frac{\pi}{2K(k)K(k)} \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \cos \left(\frac{2n\pi u}{2K(k)}\right),$$

where $q$ is the nome, defined by

$$q = \exp \left(-\frac{\pi K(k')}{K(k)}\right).$$

A more in depth analysis of elliptic integrals and elliptic functions can be found in [44, 54, 107, 165] including details of Theta functions and alternate derivations, however the
above is sufficient for the purpose of understanding the work contained in this thesis and provides all the necessary tools, without unnecessary complication. In Chapters 5 and 6, dashes are used to represent differentiation, hence, to avoid confusion the complimentary elliptic modulus will be written as \( \sqrt{1-k^2} \) and not \( k' \).

### A.4 The simple pendulum

We consider the simple pendulum as an example to illustrate the techniques described in the previous sections of this appendix, as well as carrying out calculations which are referred to in the main body of the thesis. The simple pendulum is one of the fundamental systems in classical mechanics and has attracted the interest of mathematicians throughout history. The system is an example of an integrable system and while itself is fully analytically understood, it acts as a gateway to perturbed systems which we still could not hope to find general closed form solutions to. Two of the most famous perturbations of the pendulum system are those of the pendulum with vertically oscillating support and the pendulum with periodically varying length. In Chapters 5 and 6 we study both of these perturbations, however here we shall concentrate our analysis on the system of the simple pendulum. The Lagrangian of the simple pendulum can be found by considering the mechanics of the system and using the approach described in Appendix A.1. The Lagrangian may then be written as

\[
\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2} ml^2 \dot{\theta}^2(t) + mgl \cos \theta(t),
\]

where the dots represent derivative with respect to the unscaled time \( t \). From the Lagrangian we can find the Hamiltonian

\[
E = \mathcal{H}(\theta, \dot{\theta}) = \frac{1}{2} ml^2 \dot{\theta}^2(t) - mgl \cos \theta(t),
\]
and from either of which we can obtain the equation of motion

\[ \ddot{\theta}(t) + \frac{g}{l} \sin \theta(t) = 0. \]  (A.4.1)

An equivalent equation of motion for the simple pendulum is

\[ \theta''(\tau) + \alpha \sin \theta(\tau) = 0, \]  (A.4.2)

where the dashes represent derivative with respect to the scaled time \( \tau \). We shall proceed using the system (A.4.2) in time \( \tau \) as it will be advantageous when studying the perturbed systems in Chapters 5 and 6. Here and henceforth unless otherwise stated, \( \theta = \theta(\tau) \).

The Hamiltonian for the simple pendulum thus becomes

\[ E = H(\theta, \theta') = \frac{1}{2} (\theta')^2 - \alpha \cos \theta. \]  (A.4.3)

Although using the angle \( \theta \) and its time derivatives as the co-ordinates is perhaps the most natural approach when choosing a co-ordinate system and calculating the equations of motion for the pendulum, they do not describe the dynamics in the simplest possible way. As mentioned in Appendix A.2, for the bounded motion of conservative Hamiltonian systems it is possible to change into a canonical set of co-ordinates known as action-angle variables. Rewriting the Hamiltonian (A.4.3) in standard Hamiltonian co-ordinates, substituting \( q(\tau) = \theta, p(\tau) = q' = \theta' \), we have

\[ E = \mathcal{H}(p, q) = \frac{1}{2} p^2 - \alpha \cos q. \]  (A.4.4)

which is in the form of equation (A.2.3), as required for transforming into action-angle variables. Rearranging equation (A.4.4) for \( p \) we obtain \( p = \pm p(E, q) \), with

\[ p(E, q) = \sqrt{2(E + \alpha \cos q)} = \sqrt{2\alpha(E_0 + \cos q)}, \]

where \( E_0 = E/\alpha \). It is clear that there are two types of dynamics, oscillatory dynamics when \( E_0 < 1 \) and rotational dynamics when \( E_0 > 1 \), separated by a separatrix where
$E_0 = 1$, for which no action-angle variables exist, see Appendix A.2. We treat the two cases separately.

**Librations**

We first consider the case $E_0 < 1$. The action variable is given by

$$I = \frac{1}{2\pi} \int p dq = \frac{2}{\pi \alpha} \int_0^{q_1} \sqrt{E_0 + \cos q} \, dq = \frac{8}{\pi \sqrt{\alpha}} \left[ (k_1^2 - 1)K(k_1) + E(k_1) \right], \quad (A.4.5)$$

where $k_1^2 = (E_0 + 1)/2$ and $q_1 = \arccos(-E_0)$. The functions $K(k)$ and $E(k)$ are the complete elliptic integrals of the First and Second kinds respectively as described in Appendix A.3.

The angle variable $\varphi$ is found as

$$\varphi' = \frac{\partial H}{\partial I} = \frac{dE}{dI} = \left( \frac{dI}{dE} \right)^{-1}, \quad (A.4.6)$$

where

$$\frac{dI}{dE} = \frac{d}{dE} \frac{2}{\pi} \int_0^{q_1} \sqrt{E + \alpha \cos q} \, dq = \frac{2}{\pi \sqrt{\alpha}} K(k_1).$$

Hence we have

$$\varphi(\tau) = \frac{\pi}{2K(k_1)} \sqrt{\alpha} (\tau - \tau_0). \quad (A.4.7)$$

Taking $s = \sin(q/2)$; then using equation (A.4.4) it is easy to show that

$$(s')^2 = \alpha(1 - s^2) \left( k_1^2 - s^2 \right). \quad (A.4.8)$$

Integrating using the Jacobi elliptic functions

$$s(\tau) = k_1 \text{sn} \left( \sqrt{\alpha} (\tau - \tau_0), k_1 \right),$$

the expression can then be rearranged to achieve the following result:

$$q = 2 \arcsin \left[ k_1 \text{sn} \left( \frac{2K(k_1)}{\pi} \varphi, k_1 \right) \right], \quad p = 2k_1 \sqrt{\alpha} \text{cn} \left( \frac{2K(k_1)}{\pi} \varphi, k_1 \right),$$
Using equations (A.3.10), one obtains from (A.4.5)

\[
\frac{\partial I}{\partial k_1} = \frac{8}{\pi} k_1 K(k_1) \sqrt{\alpha}, \tag{A.4.9}
\]

a relation which is later used in Chapters 5 and 6.

**Rotations**

When the pendulum rotates we have

\[
I = \frac{1}{2} \pi \int_{0}^{2\pi} p \, dq = \frac{1}{2} \pi \sqrt{\alpha} \int_{0}^{2\pi} \sqrt{E_0 + \cos q} \, dq = \frac{4}{k_2 \pi} \sqrt{\alpha} E(k_2), \tag{A.4.10}
\]

where we now let \( k_2^2 = \frac{2}{E_0 + 1} = \frac{1}{k_1^2} \). The angle variable \( \varphi \) can similarly be found using (A.4.6), where \( dI/dE \) is calculated as

\[
\frac{dI}{dE} = \frac{1}{E_0 + \cos q} \sqrt{E + \alpha \cos q} \, dq = \frac{k_2}{\pi \sqrt{\alpha}} K(k_2),
\]

which hence gives

\[
\varphi(\tau) = \frac{\pi}{K(k_2)} \sqrt{\alpha} \frac{(\tau - \tau_0)}{k_2}. \tag{A.4.11}
\]

Using (A.4.8) and the definition of \( k_2 \), for the rotating solutions we find that

\[
s(\tau) = \text{sn} \left( \sqrt{\alpha} \frac{(\tau - \tau_0)}{k_2}, k_2 \right),
\]

and similarly, by simple rearrangement, we arrive at

\[
q = 2 \arcsin \left[ \text{sn} \left( \frac{K(k_2)}{\pi} \varphi, k_2 \right) \right], \quad p = \frac{2}{k_2} \sqrt{\alpha} \, \text{dn} \left( \frac{K(k_2)}{\pi} \varphi, k_2 \right).
\]

Using equations (A.3.10), in conjunction with (A.4.10) we obtain

\[
\frac{\partial I}{\partial k_2} = -\frac{4}{\pi k_2^2} K(k_2) \sqrt{\alpha}. \tag{A.4.12}
\]

which has been used to derive relations in Chapters 5 and 6.
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A result of calculating \( q(t) \) and \( p(t) \) in terms of the action-angle variables is that we have found explicit solutions to the simple pendulum equation in the regions of phase space for which the action-angle variables are valid. These solutions are extended for the perturbed systems in Chapters 5 and 6.

**Jacobian determinant**

We now compute the entries of the Jacobian matrix, \( J \), of the transformation to action-angle variables, which will be used when considering the perturbed systems in Chapters 5 and 6. As a by-product we check that the determinant of \( J \) is equal to 1, that is

\[
\frac{\partial q}{\partial \phi} \frac{\partial p}{\partial I} - \frac{\partial q}{\partial I} \frac{\partial p}{\partial \phi} = 1. 
\]

(A.4.13)

The derivative with respect to \( \phi \) is straightforward in both the libration and rotation case, however the dependence of \( p \) and \( q \) on the action \( I \) is less obvious. That said, the dependence of \( p \) and \( q \) on \( k_1 \) in the oscillating case and \( k_2 \) in the rotating case is clear and we know the relationship between \( I \) and \( k \) in both cases, hence the derivative of the Jacobi elliptic functions can be calculated by using, see also [45]

\[
\frac{\partial}{\partial I} = \frac{\partial k}{\partial I} \frac{\partial}{\partial k} + \frac{\partial u}{\partial I} \frac{\partial}{\partial u} = \frac{\partial k}{\partial I} \left( \frac{\partial}{\partial k} + \frac{\partial u}{\partial k} \frac{\partial}{\partial u} \right), 
\]

(A.4.14)

where \( u \) is the first argument of the functions, i.e \( \text{sn}(u,k) \), etc. Then for the oscillations we have

\[
\frac{\partial q}{\partial I} = \frac{\pi}{4k_1K(k_1)\sqrt{\alpha}} \left[ \frac{\text{sn}(-)}{\text{dn}(-)} + \frac{2E(k_1)\phi \text{cn}(-)}{\pi(1 - k_1^2)} + \frac{k_1^2 \text{sn}(-) \text{cn}^2(-)}{(1 - k_1^2) \text{dn}(-)} - \frac{E(-) \text{cn}(-)}{1 - k_1^2} \right],
\]

\[
\frac{\partial p}{\partial I} = \frac{\pi}{4k_1K(k_1)} \left[ \text{cn}(-) - \frac{2E(k_1)\phi \text{sn}(-) \text{dn}(-)}{\pi(1 - k_1^2)} - \frac{k_1^2 \text{sn}^2(-) \text{cn}(-)}{1 - k_1^2} + \frac{E(-) \text{sn}(-) \text{dn}(-)}{1 - k_1^2} \right],
\]

\[
\frac{\partial q}{\partial \phi} = \frac{4k_1K(k_1) \text{cn}(-)}{\pi},
\]

\[
\frac{\partial p}{\partial \phi} = -\sqrt{\alpha} \frac{4k_1K(k_1) \text{sn}(-) \text{dn}(-)}{\pi}.
\]
where \((\cdot) = \left(\frac{2K(k_1)\varphi}{\pi}, k_1\right)\). From the above it is easy to check that equation (A.4.13) is satisfied. Similarly, for the rotations we have

\[
\frac{\partial q}{\partial I} = -\frac{\pi k_2^2}{2K(k_2)^2 \sqrt{\alpha}} \left[ \frac{\varphi E(k_2) \, dn(\cdot)}{k_2 \, 1 - k_2^2} + k_2 \frac{sn(\cdot) \, cn(\cdot)}{\pi} - E(\cdot) \frac{dn(\cdot)}{k_2 \, 1 - k_2^2} \right],
\]

\[
\frac{\partial p}{\partial I} = \frac{\pi k_2^2}{2K(k_2)} \left[ \frac{dn(\cdot)}{k_2^2} + \varphi \left( E(k_2) \frac{sn(\cdot) \, cn(\cdot)}{1 - k_2^2} \right) + \frac{sn^2(\cdot) \, dn(\cdot)}{1 - k_2^2} - \frac{E(\cdot) \, sn(\cdot) \, cn(\cdot)}{1 - k_2^2} \right],
\]

\[
\frac{\partial q}{\partial \varphi} = \frac{2K(k_2)}{\pi} \, dn(\cdot),
\]

\[
\frac{\partial p}{\partial \varphi} = -\sqrt{\alpha} \frac{2k_2 K(k_2)}{\pi} \, sn(\cdot) \, cn(\cdot),
\]

where \((\cdot) = \left(\frac{K(k_2)}{\pi} \varphi, k_2\right)\). It is once again easily checked from the above that (A.4.13) is satisfied.

### A.5 The forced cubic oscillator

In this section we calculate the action-angle variables for the forced cubic oscillator which are used in Chapter 3. The system has Hamiltonian

\[
H(y, x, t) = \frac{y^2}{2} + (1 + \mu f(\omega t)) \frac{x^4}{4}, \quad \text{(A.5.1)}
\]

where \((y, x) \in \mathbb{R}^2\), \(\mu \in \mathbb{R}\), \(\omega \in \mathbb{R}\) is the frequency of the drive and \(f\) is a 2\(\pi\)-periodic analytic function with zero average and \(\|f\|_\infty = 1\). The phase space for the system is \(\mathbb{R}^2 \times T_\omega\), with \(T_\omega = \mathbb{R} / (2\pi/\omega)\mathbb{Z}\) and the equations of motion for the system are

\[
\begin{cases}
    \dot{x} = y, \\
    \dot{y} = -a(\omega t) \, x^3, \quad a(t) := 1 + \mu f(t),
\end{cases}
\]

where the dots denote derivative with respect to the time \(t\). For the unperturbed system we focus explicitly on the case of low frequency forcing with \(\omega = \varepsilon\) as in Section 3.2 and calculate the Hamiltonian \(H_0(I)\) in equation (3.12). For the high frequency regime it is
assumed that the frequency is high enough that the system only experiences the average
of the forcing \( f \) at first order, thus the Hamiltonian \( \mathcal{H}_0(I) \) in (3.4) can be obtained by
setting \( a(t) = 1 \) in the following calculations. Using the methodology in Appendix A.2
we may write the action variable as

\[
I = \frac{1}{\pi} \int_{-x_1}^{x_1} \sqrt{2E - a(\varepsilon t) \frac{x^4}{2}} \, dx,
\]

where \( \pm x_1 \) are the points where the integrand equals zero, hence \( 4E = a(\varepsilon t)x_1^4 \) and we
may write

\[
I = \frac{\sqrt{a(\varepsilon t)}}{\sqrt{2\pi}} \int_{-x_1}^{x_1} \sqrt{x_1^4 - x^4} \, dx = \frac{x_1^2 \sqrt{a(\varepsilon t)}}{\sqrt{2\pi}} \int_{-x_1}^{x_1} \sqrt{1 - \left( \frac{x}{x_1} \right)^4} \, dx.
\]

Making the co-ordinate change \( x = x_1 \hat{x} \) we get

\[
I = \frac{x_1^3 \sqrt{a(\varepsilon t)}}{\sqrt{2\pi}} \int_{-1}^{1} \sqrt{1 - \hat{x}^4} \, d\hat{x},
\]

then setting \( \hat{x} = \text{cn}(x_1 s, k) \) we arrive at

\[
I = \frac{x_1^4 \sqrt{a(\varepsilon t)}}{\sqrt{2\pi}} \int_0^{2K(k)/x_1} \text{sn} (x_1 s, k) \, dn (x_1 s, k) \sqrt{1 - \text{cn}^4(x_1 s, k)} \, ds.
\]

If we fix \( k = \sqrt{1/2} \) then the above may be written as

\[
I = \frac{x_1^4 \sqrt{a(\varepsilon t)}}{\sqrt{2\pi}} \int_0^{2K}(s) \text{sn}(x_1 s) \, dn(x_1 s) \sqrt{2 \text{sn}^2(x_1 s) \, dn^2(x_1 s)} \, ds
\]

\[
= \left( \frac{x_1^4 \sqrt{a(\varepsilon t)}}{\sqrt{2\pi}} \right) \left( \frac{2\sqrt{2K}}{x_1} \right) \langle \text{sn}^2(x_1 s) \, dn^2(x_1 s) \rangle,
\]
where \( \langle \cdot \rangle \) represents the average of the function. The average \( \langle \text{sn}^2(\hat{s}) \text{dn}^2(\hat{s}) \rangle \) can be calculated using the relations in Appendix A.3, see also [18].

\[
\langle \text{sn}^2(\hat{s}) \text{dn}^2(\hat{s}) \rangle = -\langle \text{cn}'(\hat{s}) \text{sn}(\hat{s}) \text{dn}(\hat{s}) \rangle = \langle \text{cn}(\hat{s}) (\text{sn}(\hat{s}) \text{dn}(\hat{s}))' \rangle
\]

\[= \langle \text{cn}^2(\hat{s}) (\text{dn}^2(\hat{s}) - \frac{1}{2} \text{sn}^2(\hat{s})) \rangle = \langle \text{cn}^2(\hat{s})(1 - \text{sn}^2(\hat{s})) \rangle \]

\[= \langle \text{cn}^2(\hat{s}) - (2 \text{dn}^2(\hat{s}) - 1) \text{sn}^2(\hat{s}) \rangle
\]

\[= \langle \text{cn}^2(\hat{s}) + \text{sn}^2(\hat{s})\rangle - 2\langle \text{sn}^2(\hat{s}) \text{dn}^2(\hat{s}) \rangle = 1 - 2\langle \text{sn}^2(\hat{s}) \text{dn}^2(\hat{s}) \rangle,
\]

where the dashes represent derivative with respect to \( \hat{s} \). Hence \( \langle \text{sn}^2(\hat{s}) \text{dn}^2(\hat{s}) \rangle = 1/3 \) and we have

\[
I = \frac{1}{3} \left(\frac{x_1^4 \sqrt{a(\varepsilon t)}}{\sqrt{2\pi}}\right) \left(\frac{2\sqrt{2K}}{x_1}\right) = \frac{Tx_1^3 \sqrt{a(\varepsilon t)}}{3} = \frac{T(4E)^{3/4}a^{-1/4}(\varepsilon t)}{3},
\]

where \( T = 2K/\pi \). Rearranging we have

\[
\mathcal{H}_0(I, \varphi, t) = E = \frac{1}{4} \left(\frac{3I}{T}\right)^{4/3} a^{1/3}(\varepsilon t).
\]

and

\[
x = x_1 \tilde{x} = \left(\frac{4E}{a(\varepsilon t)}\right)^{1/4} \text{cn} \left(\left(\frac{4E}{a(\varepsilon t)}\right)^{1/4} s\right) = \left(\frac{3I}{T}\right)^{1/3} a^{-1/6}(\varepsilon t) \text{cn}(T\varphi),
\]

where \( \varphi = (3I/T)^{1/3}a^{-1/6}(\varepsilon t)s/T \) so \( x \) is periodic in \( \varphi \) with period \( 2\pi \).

**High frequency perturbed Hamiltonian**

We now give details of the calculation for the perturbed Hamiltonian \( \mathcal{H}_1 \) in equation (3.4). We first calculate the time derivative of the unperturbed angle from (A.5.3) with \( a(t) = 1 \) which gives

\[
\dot{\varphi} = \left(\frac{dI}{dE}\right)^{-1} = \frac{1}{T} \left(\frac{3I}{T}\right)^{1/3}.
\]
Then using equations (A.5.1), (A.5.4) and (A.5.5) we may calculate $\dot{x}$, $\dot{y}$ as

$$
\begin{align*}
\dot{y} &= \dot{x} = -\left(\frac{3I}{T}\right)^{2/3} \frac{\text{sn}(T\varphi)}{\text{dn}(T\varphi)}, \\
\dot{y} &= -(1 + \mu f) \left(\frac{3I}{T}\right) \text{cn}^3(T\varphi).
\end{align*}
$$

(A.5.6)

Then, the angle variable of the perturbed system satisfies

$$
\dot{\varphi} = \frac{\partial y}{\partial I} \dot{x} - \frac{\partial x}{\partial I} \dot{y} \\
= \frac{2}{T} \left(\frac{3I}{T}\right)^{1/3} \text{sn}^2(T\varphi) \text{dn}^2(T\varphi) + \frac{1}{T} (1 + \mu f(t)) \left(\frac{3I}{T}\right)^{1/3} \text{cn}^4(T\varphi) \\
= \frac{1}{T} \left(\frac{3I}{T}\right)^{1/3} + \mu f(t) \left(\frac{3I}{T}\right)^{1/3} \text{cn}^4(T\varphi).
$$

(A.5.7)

where we have used $\text{cn}^4(\cdot) = 1 - 2 \text{sn}^2(\cdot) \text{dn}^2(\cdot)$, see Appendix A.3. Integrating (A.5.7) with respect to $I$ we obtain

$$
\mathcal{H}(I, \varphi, t) = \mathcal{H}_0(I) + \mu \mathcal{H}_1(I, \varphi, t) = \frac{1}{4} \left(\frac{3I}{T}\right)^{4/3} + \frac{\mu}{4} f(t) \left(\frac{3I}{T}\right)^{3/3} \text{cn}^4(T\varphi),
$$

which is as in equation (3.4).

**Low frequency perturbed Hamiltonian**

The low frequency regime follows a similar approach to the high frequency regime, with $a := a(\varepsilon t)$. The unperturbed Hamiltonian $\mathcal{H}_0$ is given by equation (A.5.3) so that the unperturbed angle variable satisfies

$$
\dot{\varphi} = \frac{1}{T} \left(\frac{3I}{T}\right)^{1/3} a^{1/3}(\varepsilon t).
$$

(A.5.8)

Using equations (A.5.1), (A.5.4) and (A.5.8) we have

$$
\begin{align*}
\dot{y} &= \dot{x} = -\left(\frac{3I}{T}\right)^{2/3} a^{1/6}(\varepsilon t) \text{sn}(T\varphi) \text{dn}(T\varphi) - \frac{\mu}{6} \left(\frac{3I}{T}\right)^{1/3} a(\varepsilon t)a^{-7/6}(\varepsilon t) \text{cn}(T\varphi), \\
\dot{y} &= -(\frac{3I}{T})^{1/4} a^{1/2}(\varepsilon t) \text{cn}^3(T\varphi).
\end{align*}
$$
Hence the perturbed angle variable satisfies

\[ \dot{\phi} = \frac{1}{T} \left( \frac{3I}{T} \right)^{1/3} a^{1/3}(\varepsilon t) \cn^4(T\phi) + \frac{3\varepsilon}{6T} \dot{a}(\varepsilon t) a^{-1}(\varepsilon t) \sn(T\phi) \cn(T\phi) \dn(T\phi) + O(\varepsilon^2), \]

which may be integrated with respect to \( I \) to obtain

\[ \mathcal{H}(I, \phi, \varepsilon t) = \mathcal{H}_0(I) + \mathcal{H}_1(I, \phi, t) \]

\[ = \frac{1}{4} \left( \frac{3I}{T} \right)^{4/3} a^{1/3}(\varepsilon t) + \frac{\varepsilon}{6} \left( \frac{3I}{T} \right)^{-1/3} \dot{a}(\varepsilon t) a^{-1}(\varepsilon t) \sn(T\phi) \cn(T\phi) \dn(T\phi) + O(\varepsilon^2), \]

as in equation (3.12).

**Co-ordinate transformation (3.13)**

We now wish to transform the co-ordinates to those in (3.13), so that the Hamiltonian (3.12) becomes \( \mathcal{A}(p, q, s) \) as in equation (3.14). We begin by setting

\[ p = \mathcal{H}_0(I, \phi, t) + O(\varepsilon), \quad q = t, \quad s = \phi \]

so that \( \dot{q} = 1 \) and \( \dot{s} = \dot{\phi} = \partial_I \mathcal{H} = \partial_I \mathcal{H}_0 + O(\varepsilon) \). In the new co-ordinates we have

\[ \partial_p \mathcal{A} = \frac{dq}{ds} = \frac{\dot{q}}{\dot{s}} = T \left( \frac{3I}{T} \right)^{-1/3} a^{-1/3}(\varepsilon q) + O(\varepsilon). \]

Using the relation

\[ I = \frac{T}{3} (4p)^{3/4} a^{-1/4}(\varepsilon q) + O(\varepsilon), \]

we have

\[ \partial_p \mathcal{A} = T (4p)^{-1/4} a^{-1/4}(\varepsilon q) + O(\varepsilon), \]

which we integrate with respect to \( p \) to obtain

\[ \mathcal{A}(p, q, s) = \mathcal{A}_0(p, q, s) + \varepsilon \mathcal{A}_1(p, q, s), \]

\[ \mathcal{A}_0 = \frac{T(4p)^{3/4}}{3(a(\varepsilon q))^{1/4}}, \]
which is as in equation (3.14). The final change of co-ordinates to obtain the Hamiltonian in (3.15) is defined by the usual action-angle variable procedure, using equation (A.2.7) in Appendix A.2.

A.6 Introduction to KAM theory

In 1954 the Russian mathematician Andrey N. Kolmogorov wrote a letter to the International Congress of Mathematicians concerning tori of integrable Hamiltonian systems which still exist when the Hamiltonian is perturbed away from an integrable one. The point of the letter was that the motion of perturbed Hamiltonian systems may be shown to remain bounded under certain criteria. Later the same year Kolmogorov sketched a proof of his theory in [101]. Since then a considerable amount of work has been carried out extending and sharpening Kolmogorov’s ideas. Much of the early work was completed by Kolmogorov’s student Vladimir I. Arnold [8, 9, 10] and Jürgen K. Moser [124, 125], to which we now owe the name KAM theory. In this section we introduce the basic ideas of KAM theory, which are used in Chapter 3 where it is applied to Hamiltonian systems which have either a high or low frequency perturbation. The book by H. Scott Dumas [72] provides a very accessible look at KAM theory, explaining the ideas, history and application. Another pedagogical approach is given in Jürgen Pöschel [137]. More mathematical (yet still readable) approaches may be found in the lecture notes by Rafael de la Llave [69] and C. Eugene Wayne [166], see also [47, 138].

Consider a smooth (or analytic) completely integrable $n$-degree-of-freedom Hamiltonian system $h$, in action-angle variables $(I, \varphi)$ valid in a region $U$ of phase space. We saw in Appendix A.2 that the Hamiltonian only depends on the action and we may write $h = h(I)$. We then wish to add a perturbation to our integrable system $h$ in the form of a smooth (or analytic) function of both actions and angles which we write as $\varepsilon f(I, \varphi, \varepsilon)$, where $\varepsilon > 0$ represents a smallness factor. This gives us a perturbed Hamiltonian system
(also called nearly integrable Hamiltonian or quasi-integrable Hamiltonian)

\[ \mathcal{H}(I, \varphi, \varepsilon) = h(I) + \varepsilon f(I, \varphi, \varepsilon), \quad (A.6.1) \]

with \((I, \varphi, \varepsilon) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}\). Prior to KAM theory, the case \(\varepsilon = 0\) was understood but as soon as \(\varepsilon\) became positive very little was known and such systems were conjectured to be ergodic\(^2\). KAM theory shows that, although positive \(\varepsilon\) immediately breaks integrability in the classical (19th century) sense\(^3\), there is a gradual disruption of integrability as \(\varepsilon\) increases away from zero.

One method to find out if a Hamiltonian is integrable is to look for a smooth canonical change of co-ordinates \((I, \varphi) \rightarrow (J, \phi)\) such that in the new action-angle variables \((J, \phi)\) the Hamiltonian depends only on \(J\), that is \(\mathcal{H}(I, \varphi, \varepsilon) \rightarrow \mathcal{K}(J)\). This approach was developed in the 19th century, largely by \textit{C.G.J Jacobi}. As the perturbation in (A.6.1) is \(O(\varepsilon)\), we expect the change of co-ordinates to be \(O(\varepsilon)\)-close to the identity transformation. The change to the new action-angle variables may be achieved using the Lie series method. The method involves looking for a generating function \(\chi = \chi(J, \phi)\) which is inserted into the left of the Poisson bracket\(^4\) \{,\} to make a ‘Liouville operator’ denoted by \(L_\chi\). The Liouville operator acts on sufficiently smooth phase-space functions as \(L_\chi(\mathcal{H}) = \{\chi, \mathcal{H}\}\). To create a near-identity change of co-ordinates we define the operator

\[ e^{\varepsilon L_\chi} = I + \varepsilon L_\chi + O(\varepsilon^2). \]

This co-ordinate change acts on the Hamiltonian \(\mathcal{H} = h + \varepsilon f\) to give a transformed

---

\(^2\)All measurable sets that are invariant under the flow are either negligable or of full measure. Intuitively, an ergodic flow moves almost all sets all over the place. Any system that is ergodic is non-integrable, but the converse is not necessarily true.

\(^3\)A Hamiltonian system with \(n\) degrees of freedom is completely integrable if it has \(n\) constants of motion \(f_1, \ldots, f_n\) which are independent and the Poisson bracket \(\{f_i, f_j\} \equiv 0\) for all \(i, j\). An older meaning of the word integrable applied to systems which could be solved explicitly.

\(^4\)In Hamiltonian mechanics the Poisson bracket is a binary operation. Given a phase space \(M\) with canonical co-ordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) and two functions \(f, g\) of class \(C^1\) which map \(M \to \mathbb{R}\), the Poisson bracket is defined as \(\{f, g\} = \sum_{k=1}^n \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right)\). The Poisson bracket characterises canonical transformations which map canonical co-ordinate systems into canonical co-ordinate systems.
Hamiltonian $K(J, \phi, \varepsilon)$ given by

$$K = e^{\varepsilon L_\chi} \mathcal{H} = \mathcal{H} + \varepsilon L_\chi(\mathcal{H}) + \mathcal{O}(\varepsilon^2).$$

Expanding terms explicitly we can re-write the above as

$$K(J, \phi, \varepsilon) = h(J) + \varepsilon f(J, \phi) + \varepsilon \{\chi, h\}(J, \phi) + \mathcal{O}(\varepsilon^2), \quad (A.6.2)$$

from which it is clear that, in order for the transformed Hamiltonian $K$ to depend only on the action $J$, at first order in $\varepsilon$ we require $f + \{\chi, h\} = 0$ and hence

$$f(J, \phi) = -\sum_{k=1}^n \frac{\partial \chi}{\partial \phi_k} \frac{\partial h}{\partial J_k}. \quad (A.6.3)$$

Note that the second half of the Poisson bracket is missing as $h$ does not depend on the angles $\phi$. The aim is to solve equation (A.6.3) to find the generating function $\chi$.

Expanding both $f$ and $\chi$ in Fourier series, we may write

$$f(J, \phi) = \sum_{k \in \mathbb{Z}^n} \hat{f}_k(J)e^{2\pi i k \cdot \phi}, \quad \chi(J, \phi) = \sum_{k \in \mathbb{Z}^n} \hat{\chi}_k(J)e^{2\pi i k \cdot \phi}, \quad (A.6.4)$$

which upon substituting into (A.6.3) and using the fact that the frequency $\omega(J) = \frac{\partial h}{\partial J}$, see Appendix A.2, we arrive at

$$\sum_{k \in \mathbb{Z}^n} \hat{f}_k(J)e^{2\pi i k \cdot \phi} = -2\pi i \sum_{k \in \mathbb{Z}^n} \hat{\chi}_k(J)k \cdot \omega(J)e^{2\pi i k \cdot \phi}.$$  

Formally solving for the Fourier coefficients $\hat{\chi}$ we get

$$\hat{\chi}_k(J) = \frac{-\hat{f}_k(J)}{2\pi ik \cdot \omega(J)}. \quad (A.6.5)$$

It does not require one to look closely at the equation (A.6.5) to notice that when $k = 0$ the denominator of the right-hand side is zero. By itself this does not pose too much of a problem, since $f_0(J)e^{2\pi i 0 \cdot \phi} = f_0(J)$ is the only term in the Fourier series of $f$ that never
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has any dependence on $\phi$, thus we may redefine $\dot{\chi}_0 \equiv 0$. However there could be other ‘zero divisors’ in the right-hand side where $k \cdot \omega(J) = 0$. This brings us on to resonant and nonresonant tori.

**Resonant and nonresonant tori**

To understand the idea of resonant and nonresonant tori it is easiest to pick a single 2-dimensional torus (i.e. consider a system with 2-degrees of freedom described by variables $(I_1, I_2, \varphi_1, \varphi_2)$ and fix the actions $I_1$ and $I_2$ to be constant) on which the flow is described by the angle variables $(\varphi_1, \varphi_2)$. The frequency of the flow on the torus is then $\omega = (\omega_1, \omega_2)$. A 2-dimensional torus may be cut open and unwrapped to form a topological square, as shown in Figure A.4. When the flow crosses over the edge of the square, it appears on the opposite side. It is clear that the flow in the interior of the square is linear with fixed gradient $\omega_2/\omega_1$. There are two cases which may arise:

1. $\omega_2/\omega_1$ is a rational number, in which case the flow eventually returns to the point where it started; that is the flow is periodic.

2. $\omega_2/\omega_1$ is an irrational number and the flow never returns to the same point, but instead fills the torus more and more densely as time increases.

In the first instance where $\omega_2/\omega_1$ is a rational number it is easy to see that $a\omega_1 + b\omega_2 = 0$ is satisfied for some nonzero integers $a, b$. In the case where $\omega_2/\omega_1$ is an irrational number the equation $a\omega_1 + b\omega_2 = 0$ is only satisfied when $a = b = 0$. This generalises to flows on $n$-dimensional tori with frequency $\omega = (\omega_1, \ldots, \omega_n)$ as follows: Consider the equation

$$k \cdot \omega = 0,$$  \hspace{1cm} (A.6.6)

where $k = (k_1, \ldots, k_n)$ is an n-dimensional vector with integer components, i.e. $k \in \mathbb{Z}^n$. Either equation (A.6.6) has at least one nonzero solution $k$ and we say that the frequency
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Figure A.4: A representation of a 2-dimensional torus unfolded as a topological square. Opposite edges of the square correspond to one another with the arrows pointing in the same direction. In the square, the flow on the torus appears linear with direction vector $\omega = (\omega_1, \omega_2)$ and gradient $\omega_2/\omega_1$.

$\omega$ is resonant or the only solution is $k = (0, 0, \ldots, 0)$ and we say it is nonresonant. As the frequency $\omega = \omega(I)$ depends solely on the action variables (which uniquely define the tori), the terms resonant and nonresonant also apply to the flow and the torus on which it resides.

Returning to equation (A.6.5), we find that the right-hand side has zero divisors for every resonant torus, which are dense in phase space for nondegenerate systems (the resonant and nonresonant tori occupy the phase space in the same way that the rational and irrational numbers occupy the real line). The result is that we are not able to define a change of co-ordinates using (A.6.5) on any domain which is open or contains any nonempty open subsets. Removing the portions of phase space for which zero-divisors occur it is still possible that the divisors become arbitrarily small. Hence even on the part of phase space from which all zero-divisors have been removed, we are unable to ensure the convergence of the Fourier series for $\chi$ using the coefficients defined in equation (A.6.5). This brings us on to the problem of small divisors.
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The small divisor problem and diophantine conditions

In 1942, Carl Ludwig Siegel was working on a problem left over from the research of E. Schrödinger and H. Poincaré, which involved a small divisor problem. Siegel solved this problem in [151] and became the first researcher to successfully overcome small divisors. The techniques that Siegel used may be adapted and applied to KAM theory in the following way: in order for the Fourier series (A.6.4) for $\chi$ to converge, we require the Fourier coefficients $\hat{\chi}_k$ in equation (A.6.5) to decay rapidly enough. Siegel knew that the Fourier coefficients $\hat{f}_k$ of a smooth or analytic function $f$ decay rapidly (faster than polynomially) as their index is increased. More specifically we may write\(^5\)

$$||\hat{f}_k||_B \leq \frac{C}{|k|^b},$$

where $C > 0$ and the exponent $b > 0$ can be made as large as desired by taking sufficiently smooth $f$. The notation $||\hat{f}_k||_B$ represents the uniform norm (also called the supremum norm) of $\hat{f}_k$ over a closed, bounded set of $J$ values, $B$, i.e. $||\hat{f}_k||_B = \max_{J \in B} |\hat{f}_k(J)|$, and $|k| = \sum_{i=1}^{n} k_i$. It therefore remains to be shown that the divisor on the right-hand side of equation (A.6.5) remains sufficiently large on some nonempty set of $J$ values which intersects with $B$. The following definition is given in [69], see also [46, 47] for more details.

**Definition A.1** A number $\alpha$ is called Diophantine of type $(\sigma, \nu)$ for $\sigma > 0$ and $\nu \geq 1$ if

$$|\alpha - \frac{p}{q}| \geq \frac{\sigma}{|q|^{1+\nu}} \quad (A.6.7)$$

for all $p/q \in \mathbb{Q}$. We denote the set of numbers which satisfy (A.6.7) by $D^\alpha(\sigma, \nu)$.

When considering a frequency vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, we slightly modify Definition A.1 and say that $\omega$ is Diophantine if equation (A.6.7) is satisfied with $\alpha$ replaced by

\(^5\)In fact the Fourier coefficients of an analytic function $f$ may be shown to decrease even more quickly, satisfying $||\hat{f}_k||_B \leq Ce^{-b|k|}$ for suitable $C, b > 0$.\n
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ω₂/ω₁. We may generalise for frequencies of higher dimensional flows ω ∈ ℝⁿ, with n ≥ 2 in which case the Diophantine condition (A.6.7) becomes

\[ |k \cdot \omega| \geq \frac{\sigma}{|k|^m} \quad \forall k \in \mathbb{Z}\setminus\{0\}, \quad (A.6.8) \]

where again σ > 0 and m > 0. The set of Diophantine frequencies is then

\[ D^ω(σ, m) = \{ ω ∈ ℝⁿ | \text{for each nonzero } k ∈ \mathbb{Z}ⁿ, |k \cdot ω| \geq \frac{σ}{|k|^m} \}. \]

The set \( D^ω(σ, m) \) is a union of closed half-lines⁶ in the sense that if \( ω ∈ D^ω(σ, m) \), then for all s ≥ 1 we have \( sω ∈ D^ω(σ, m) \), [46]. In 2-dimensions, we are able to visualise \( D^ω(σ, ν) \) as the plane \( \mathbb{R}² \) with linear strips removed. For each inequality \( |k \cdot ω| \geq σ|k|⁻^m \) a linear strip is removed between the two parallel lines \( k \cdot ω = ±σ|k|⁻^m \) (i.e the strip where \( |k \cdot ω| < σ|k|⁻^m \)), lying on opposite sides and equidistant from the origin. Once all the strips have been removed, the remaining set is a union of closed half-lines pointing towards the origin. An illustration of this is shown in [46]. For \( m > n⁻¹ \), the set \( D^ω(σ, m) \) is nonempty provided \( σ > 0 \) is small enough⁷. In fact we have that the set of invariant tori in phase space leaves out a set of measure \( O(\sqrt{ε}) \), [138]. Furthermore, provided the unperturbed Hamiltonian \( h(J) \) is nondegenerate⁸, the frequency map \( ω(J) = \partial h/\partial J \) preserves the basic structure of the frequencies when mapping back to the actions, thus the corresponding set of \( J \) values

\[ D^J(σ, m) = \{ J ∈ ℝⁿ | \text{for each nonzero } k ∈ \mathbb{Z}ⁿ, |k \cdot ω(J)| \geq \frac{σ}{|k|^m} \}. \]

---

⁶A half-line is given by \( A = \{ x ∈ ℝⁿ | \forall t > 0, x = x₀ + t(a - x₀) \} \) for some starting point \( x₀ \) and constant \( a \).

⁷\( D^ω(σ, m) = \emptyset \) for \( m < n⁻¹ \) and is nonempty but has zero measure for \( m = n⁻¹ \), see [137]. Although almost all frequencies are strongly nonresonant provided \( m > n⁻¹ \), it is not true that almost all tori survive a perturbation εf, however small ε is chosen. This is because the parameter \( σ \) in (A.6.8) restricts the perturbation through the condition \( ε ≪ σ² \), hence \( σ \gg \sqrt{ε} \), [137]. A consequence of this is that \( σ \) may not be varied, but must instead be fixed in advance.

⁸Nondegeneracy is a condition on the frequency map \( I → ω \). Kolmogorov’s nondegeneracy condition is: \( \det \partial ω/\partial I = \partial²h/\partial I² ≠ 0 \), i.e. the map \( I → ω \) is a local diffeomorphism.
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is also nonempty. The sets $D^\omega(\sigma, m)$ and $D^J(\sigma, m)$ are Cantor-like in the sense that they have empty interior, i.e. they contain no nonempty open sets. We now consider $J$ values in the set $S = D^J(\sigma, m) \cap B$, on which the coefficients $\hat{\chi}_k$ for $k > 0$ satisfy

$$||\hat{\chi}_k||_S = \left|\left|\frac{\hat{f}_k(J)}{2\pi ik \cdot \omega(J)}\right|\right|_S \leq \frac{C}{2\pi \sigma |k|^{b-m}}.$$  

If $f$ is chosen smooth enough that $b - m > n$, then on the set $S$ the Fourier coefficients $\hat{\chi}_k$ decay fast enough to get absolute convergence of the Fourier series for $\chi$ in equation (A.6.4), i.e.

$$\sum_{k \in \mathbb{Z}^n \setminus \{0\}} ||\hat{\chi}_k||_S \leq \frac{C}{2\pi \sigma} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{m-b} < \infty.$$  

As a result, on the set $S$ the transformation $e^{iL\chi}$ is defined and we eliminate the $\phi$ dependence of equation (A.6.2) at order $\varepsilon$.

Iterating the process

Having shown that it is possible to construct a transformation of the co-ordinates on a nonempty set, so that the Hamiltonian is of the form $H_1 = h_1(J_1) + \varepsilon^2 f_1(J_1, \phi_1, \varepsilon)$, we may iterate the process so that the dependence on the angle variable is removed at every order. If we iterate in a regular manner, solving the homological equation at order $\varepsilon^l$ at the $l$-th step, we may expect that after $n$ steps the Hamiltonian will have been transformed to integrable form through $O(\varepsilon^n)$, i.e.

$$H_n(J_n, \phi_n, \varepsilon) = h_n(J_n) + \varepsilon^{n+1} f_n(J_n, \phi_n, \varepsilon).$$  

However this is problematic as the distance of the actions $J_n$ from the original actions $I$, is $|I - J_n|$ which may be up to $O(\varepsilon^{n+1})$. Similarly the frequency $\omega(J_n)$ will be comparatively close to the frequency $\omega(I)$. The consequence is that the frequency can move too close to resonance, causing the Diophantine condition (A.6.8) to eventually fail.
to be satisfied. To solve the problem Kolmogorov adapted Newton's method [102, 9], so that the transformed Hamiltonian after the $n$-th step is of the form

$$H_n(J_n, \phi_n, \varepsilon) = h_n(J_n) + (\varepsilon^s)^n f_n(J_n, \phi_n, \varepsilon),$$

with $s > 1$. In turn $|I - J_n|$ is $O(\varepsilon^\alpha)$. This rapid convergence coupled with the Diophantine condition allows us to overcome the small divisors at all orders of $\varepsilon$ and transform the Hamiltonian into the integrable form

$$H_\infty = h_\infty(J_\infty, \varepsilon),$$

for any $J \in \mathcal{D}^I(\sigma, m)$. We may then state the following result, see also [72, 137]:

**Theorem A.1 KAM Theorem.** Let $(I, \varphi) = (I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$ be action-angle variables for the smooth completely integrable Hamiltonian $h : M \to \mathbb{R}$ with $n \geq 2$ degrees of freedom. Assume $h$ is nondegenerate, and let $\varepsilon \in \mathbb{R}^+$ be a small parameter. Then for any smooth perturbation $\varepsilon f(I, \varphi, \varepsilon)$ there exists a threshold value $\varepsilon_0$ such that whenever $\|\varepsilon f\|_M < \varepsilon_0$, the perturbed Hamiltonian $H(I, \varphi, \varepsilon) = h(I) + \varepsilon f(I, \varphi, \varepsilon)$ has a nonempty set of invariant $n$-dimensional KAM tori in its phase space. On each invariant torus the flow of $H$ is quasiperiodic with highly nonresonant frequency. Furthermore the set is large in that its measure becomes full as $\varepsilon \to 0$.

There are several different versions of the KAM theorem which are suited to different applications, see for example [47, 166, 8, 10, 138]. In Chapter 3, we also utilise Moser’s twist theorem which is applicable to maps. A map can be created by integrating the equations over one cycle of the forcing; in the work in Chapter 3 this is $2\pi$. Doing so we obtain an unperturbed system of the form

$$\varphi_1 = \varphi + 2\pi \Omega(I), \quad I_1 = I$$

(A.6.9)

where $\varphi = \varphi(0), \varphi_1 = \varphi(2\pi), I = I(0)$ and $I_1 = I(2\pi)$. The parameter $\Omega(I)$ corresponds to the amount of twist in one application of the map, i.e. the frequency, and we require
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dΩ/dI ≡ Ω' > 0 in the region a ≤ I ≤ b. We may assume a ≥ 0 since if a is negative we can translate the variable I. This map, M₀, on the annulus D = {0 ≤ a ≤ I ≤ b, 0 ≤ Ω ≤ 2π}, keeps each circle invariant, simply rotating the points of the circle round in a counter clockwise direction, through the angle 2πΩ(I). Clearly M₀ is area preserving, since the map is derived from a Hamiltonian system.

If we now consider the perturbed system, then our perturbed map Mᵋ, can be written as

\[
\begin{align*}
\varphi_1 &= \varphi + 2\pi\Omega(I) + \varepsilon f(I, \varphi, \varepsilon), \\
I_1 &= I + \varepsilon g(I, \varphi, \varepsilon),
\end{align*}
\]

(A.6.10)

where \( f(I, \varphi, 0) = g(I, \varphi, 0) = 0, \Omega' > 0, I \in \mathcal{D}, \Omega(I) \) is smooth, and \( f, g \) are smooth in all variables and periodic in \( \varphi \) with period 2π. Again \( M_\varepsilon \) is area preserving and we have the following theorem due to Moser, see [30, 52]

**Theorem A.2 Moser’s Twist Theorem** Assume \( f, g \) and \( \Omega \) are \( C^j, j \geq 5 \), and \( |\Omega'(I)| \geq \nu > 0 \) on \( \mathcal{A} \). Then there exists a \( \delta(\mu, \Omega(I)) > 0 \) such that if

\[
\sup_{(I, \varphi) \in \mathcal{D}} \{||f||_j + ||g||_j\} < \nu \delta,
\]

then \( M_\mu \) possesses an invariant curve \( \gamma_\mu \) of the form

\[
I = c + u(\psi), \varphi = \psi + v(\psi)
\]

in \( \mathcal{D} \), where \( u, v \in C^1 \), are periodic of period 2π, and \( ||u||_1 + ||v||_1 < \varepsilon, a < c < b \). The induced map on \( \gamma_\varepsilon \) is given by

\[
\varphi \to \varphi + 2\pi \Omega,
\]

where \( \Omega \) is irrational, and satisfies the infinite set of inequalities

\[
|\Omega - \frac{n}{m}| \geq \beta m^{-\alpha}
\]
for some $\beta, \alpha > 0$, and all integers $m, n > 0$. Each choice of $\Omega$ satisfying the above (diophantine) condition in the range of $\Omega(I)$ gives rise to such an invariant curve.

A.7 Averaging systems

In this section we give a brief introduction to an averaging theorem for ODE systems, which is applied in Chapter 3. There is more than one variation on the theorem, see for example [87, 129, 130]. Consider systems of the form

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \quad (A.7.1)$$

where $x \in U \subseteq \mathbb{R}^n$, $0 \leq \varepsilon \ll 1$, the dot denotes derivative with respect to time $t$ and the function $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ is $C^m$ smooth for $m \geq 2$, bounded on bounded sets and periodic in $t$ with period $T > 0$. The corresponding averaged system is defined as

$$\dot{y} = \frac{\varepsilon}{T} \int_0^T f(y, t, 0) \, dt = \varepsilon \bar{f}(y). \quad (A.7.2)$$

For the systems (A.7.1) and (A.7.2) we state the following theorem, as given in [85];

**Theorem A.3 (The averaging theorem)** There exists a $C^m$ change of co-ordinates $x = y + \varepsilon w(y, t, \varepsilon)$ under which equation (A.7.1) becomes

$$\dot{y} = \varepsilon \bar{f}(y) + \varepsilon^2 f_1(y, t, \varepsilon),$$

where $f_1$ is of period $T$ in $t$. Moreover

1. If $x(t)$ and $y(t)$ are solutions of (A.7.1) and (A.7.2) with initial conditions $x_0$ and $y_0$ respectively, at $t = 0$, and $|x_0 - y_0| = O(\varepsilon)$, then $|x(t) - y(t)| = O(\varepsilon)$ on a time scale $t \sim 1/\varepsilon$.

2. If $y_p$ is hyperbolic fixed point of (A.7.2) then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ equation (A.7.1) possesses a unique periodic orbit $x_\varepsilon(t) = y_p + O(\varepsilon)$ of the same stability type as $y_p$. 

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3. If \( x^s \in W^s(x_\varepsilon) \) is a solution of (A.7.1) lying in the stable manifold of the hyperbolic periodic orbit \( x(\varepsilon) = y_p + O(\varepsilon) \), \( y^s \in W^s(y_p) \) is a solution of (A.7.2) lying in the stable manifold of the hyperbolic fixed point \( y_p \) and \( |x^s(0) - y^s(0)| = O(\varepsilon) \), then \( |x^s(t) - y^s(t)| = O(\varepsilon) \) for \( t \in [0, \infty) \). Similar results apply to solutions lying in the unstable manifolds on the time interval \( t \in (-\infty, 0] \).

A proof of Theorem A.3 is given in [85], see also [87].

Example A.1 Cubic Oscillator

As an example we consider the system of a parametrically forced cubic oscillator, described by

\[
\begin{align*}
\dot{x}_1(t) &= \varepsilon x_2(t), \\
\dot{x}_2(t) &= -\varepsilon(1 + \sin(t))x_1^3(t),
\end{align*}
\]

which is in the form \( \dot{x} = \varepsilon f(x, t, \varepsilon) = \varepsilon \tilde{f}(x) + \varepsilon \hat{f}(x, t, \varepsilon) \) where \( x = (x_1, x_2) \). We transform to the co-ordinates \( u = (u_1, u_2) \) with \( u_2 = \dot{u}_1 \) defined by the transformation \( x = u + \varepsilon w(u, t, \varepsilon) \), and hence

\[
\dot{u} = \dot{x} - \varepsilon \dot{w} - \varepsilon D_u w.
\]

Expanding the above we have

\[
\begin{align*}
\dot{u}_1 &= \ddot{x}_1 - \varepsilon \dot{w}_1 - \varepsilon D_u w_1 = \varepsilon(u_2 + \varepsilon w_2) - \varepsilon \dot{w}_1 - \varepsilon D_u w_1, \\
\dot{u}_2 &= \ddot{x}_2 - \varepsilon \dot{w}_2 - \varepsilon D_u w_2, \\
&= -\varepsilon(u_1 + \varepsilon w_1)^3 - \varepsilon(u_1 + \varepsilon w_1)^3 \sin(t) - \varepsilon \dot{w}_2 - \varepsilon D_u w_2.
\end{align*}
\]

We then choose \( \dot{w} \) such that the forcing terms are of \( O(\varepsilon^2) \) or higher, that is, we wish to cancel out the terms of \( O(\varepsilon) \). This motivates the choice

\[
\begin{align*}
\dot{w} &= \begin{pmatrix} 0 \\ -u_1^3 \sin(t) \end{pmatrix}, \\
w &= \begin{pmatrix} 0 \\ u_1^3 \cos(t) \end{pmatrix} + \begin{pmatrix} \hat{w}_1(u, \varepsilon) \\ \hat{w}_2(u, \varepsilon) \end{pmatrix},
\end{align*}
\]

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and setting $\dot{w} = 0$ we obtain the system

\[
\begin{align*}
\dot{u}_1 &= \varepsilon u_2 + \varepsilon^2 u_1^3 \cos(t) + O(\varepsilon^3), \\
\dot{u}_2 &= -\varepsilon u_1^3 - \varepsilon^2 3u_1^2 u_2 \cos(t) + O(\varepsilon^3).
\end{align*}
\]

To push the forcing term to $O(\varepsilon^3)$, we repeat the averaging process, defining a new coordinate transformation $u = q + \varepsilon s(q, t, \varepsilon)$. The process follows the same structure and we arrive at the system given by

\[
\begin{align*}
\dot{q}_1 &= \dot{u}_1 - \varepsilon \dot{s}_1 - \varepsilon D_q s_1 \dot{q} \\
&= \varepsilon (q_2 + \varepsilon s_2) + \varepsilon^2 (q_1 + \varepsilon s_1)^3 \cos(t) - \varepsilon \dot{s}_1 - \varepsilon D_q s_1 \dot{q}, \\
\dot{q}_2 &= \dot{u}_2 - \varepsilon \dot{s}_2 - \varepsilon D_q s_2 \dot{q} \\
&= -\varepsilon (q_1 + \varepsilon s_1)^3 - 3\varepsilon^2 (q_1 + \varepsilon s_1)^2 (q_2 + \varepsilon s_2) \cos(t) - \varepsilon \dot{s}_2 - \varepsilon D_q s_2 \dot{q}.
\end{align*}
\]

Again we wish to find $s$ so as to eliminate the forcing terms at $O(\varepsilon^2)$. It may be checked that taking

\[
s = \varepsilon \begin{pmatrix} q_1^3 \sin(t) \\ -3q_1^2 q_2 \sin(t) \end{pmatrix},
\]

we eliminate the $O(\varepsilon^2)$ terms and arrive at the equations of motion

\[
\begin{align*}
\dot{q}_1 &= \varepsilon q_2 - \varepsilon^3 6q_1^2 q_2 \sin(t) + O(\varepsilon^4), \\
\dot{q}_2 &= -\varepsilon q_1^3 - \varepsilon^3 \left[6q_1^5 \sin(t) + 6q_1 q_2^2 \sin(t)\right] + O(\varepsilon^4).
\end{align*}
\]

It is possible to repeat the process up to $n$ times where $n \sim 1/\varepsilon$ so that the perturbation terms only appear at $O(\varepsilon^{n+1})$, [130]. Furthermore, in [130] A.I. Neishtadt showed that there exists a real analytic change of co-ordinates such that the perturbation terms decay exponentially, in particular as $C \exp(-c/\varepsilon)$ for positive constants $c$ and $C$. 

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A.8 A discussion of stability

Here we introduce some definitions of stability and discuss their relevance in context of the work throughout the thesis. Consider the system

\[ \dot{x} = f(x, t), \]  

(A.8.1)

where \( x = (x, \dot{x}) \). We begin by discussing the stability of solutions in the phase plane, that is, to perturbations in initial conditions. Later the ideas are extended to structural stability, which is concerned with perturbations of the system itself. One of the most basic definitions of stability in the phase plane is due to Aleksandr Lyapunov, and is so named Lyapunov stability [141].

**Definition A.2 (Lyapunov stability)** A solution \( x_1(t) \) of (A.8.1) is Lyapunov stable if for each \( \varepsilon > 0 \) and \( t_0 \in \mathbb{R} \) there exists \( \delta = \delta(\varepsilon, t_0) > 0 \) such that if \( x_2(t) \) is a solution of (A.8.1) and \( |x_1(t_0) - x_2(t_0)| < \delta \) then \( |x_1(t) - x_2(t)| < \varepsilon \) for all \( t \geq t_0 \).

In particular, consider a fixed point \( p_0 \) in the \((x, \dot{x})\) plane (also called equilibrium point) of the system (A.8.1). The fixed point is said to be Lyapunov stable if by choosing any initial conditions sufficiently close to \( p_0 \) (inside a \( \delta \)-ball) the resulting trajectory remains inside some small neighbourhood of \( p_0 \) (inside a \( \varepsilon \)-ball) as \( t \to \infty \). For linear systems there is an equivalent definition of stability, which is given in Appendix A.9. If a solution is not Lyapunov stable, it is said to be Lyapunov unstable. In the definition A.2 of Lyapunov stability, the parameter \( \delta \) depends not only on \( \varepsilon \), but also on the initial time \( t_0 \). If \( \delta \) depends only on \( \varepsilon \) and not on the initial time \( t_0 \), then the solution is uniformly stable. More precisely we have

**Definition A.3 (uniform stability)** A solution \( x_1(t) \) of (A.8.1) is uniformly stable if for each \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that if \( x_2(t) \) is a solution of (A.8.1) and \( |x_1(t_0) - x_2(t_0)| < \delta \) for some \( t_0 \in \mathbb{R} \) then \( |x_1(t) - x_2(t)| < \varepsilon \) for all \( t \geq t_0 \).
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From the definition A.2, it may seem that if a solution is not Lyapunov stable, then nearby solutions diverge as time increases. If the solution \( x_1 \) is a fixed point then the curves of nearby solutions diverge when mapped into the phase plane \((x, \dot{x})\). However if \( x_1 \) is a periodic orbit, then it may not be the case that the curves diverge, only the location of the solution at particular points in time. This prompts the following definition, which becomes equivalent to Lyapunov stability if the solution \( x_1 \) is an equilibrium point.

**Definition A.4** A solution \( x_1(t) \) of (A.8.1) is orbitally stable if for every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that if \( x_2(t) \) is a solution of (A.8.1) and \(|x_1(t_0) - x_2(t_1)| < \delta \) for some \( t_0, t_1 \in \mathbb{R} \), then

\[
\bigcup_{t \geq t_1} x_2(t) \subseteq \bigcup_{t \geq t_0} B(x_1(t), \varepsilon).
\]

The following example illustrates the difference between orbital stability and Lyapunov stability.

**Example A.2**

Consider the system

\[
\dot{\phi} = \Omega(r), \quad \dot{r} = 0,
\]

where \( \Omega(r) \) is a function of the radius \( r \) which may be thought of as the frequency. If we pick any circle of the system we find that it is not Lyapunov stable, since nearby solutions move with different angular velocities. The solutions are however orbitally stable, since nearby orbits remain close to one another in phase space (as concentric circles).

To relate this to the work in Section 2.1 and Appendix A.9 we consider a linearisation of the system (A.8.1) so that we may rewrite the system as \( \dot{x} = P(t)x \), where \( P(t) \) is periodic with minimal period \( T \). If the system is autonomous (\( P \) does not depend on time) then we may simply look at the eigenvalues of \( P \) to determine the stability. If all of the eigenvalues are nonpositive, the solution is Lyapunov stable. If at least one eigenvalue
is positive, then the solution is Lyapunov unstable. For nonautonomous systems we consider the Poincaré map of the system, see Floquet’s Theorem 2.1 in Chapter 2, also Theorem A.6 in Appendix A.9. Then the Lyapunov stability of the linear system may be checked using Lyapunov exponents (the real parts of the $\rho_j$ defined in Theorem A.7 of Appendix A.9, see also Definition 2.1 in Section 2.1). If all of the Lyapunov exponents are nonpositive (all the Floquet multipliers lie on or inside the unit circle), then the solution is Lyapunov stable. If at least one of the exponents is positive, then the solution is unstable. In dissipative systems, it is possible that all the Lyapunov exponents are negative, which means the solutions decay, this prompts the following definition.

**Definition A.5 (Asymptotic stability)** A solution $x_1(t)$ of (A.8.1) is asymptotically stable if it is Lyapunov stable and if for every $t_0 \in \mathbb{R}$ there exists $\delta = \delta(t_0) > 0$ such that if $x_2(t)$ is a solution of (A.8.1) and $|x_1(t_0) - x_2(t_0)| < \delta$ then $|x_1(t) - x_2(t)| \to 0$ as $t \to \infty$.

Asymptotic stability is the definition prominently used in dissipative systems, i.e. the solution is attractive. In this instance the solution $x_1(t)$ is known as an attractor and has an associated basin of attraction, see Chapters 5 and 6 and Appendix A.10.

**Linear systems and nonlinear systems**

What can we deduce about the nonlinear system, using stability analysis for the linear system? Even when the linearised system is stable, this does not necessarily imply stability of the nonlinear system. We consider autonomous systems of the form

$$\dot{x} = f(x), \quad (A.8.2)$$

where $x = (x, \dot{x})$ and $f$ is continuous. Then we have the following theorem [141].
Theorem A.4 (Lyapunov’s Theorem)

1. An equilibrium point in a nonlinear system is asymptotically stable if all the eigenvalues of the linearised system have negative real parts.

2. An equilibrium point in a nonlinear system is Lyapunov unstable if there exists at least one eigenvalue of the linearised system which has positive real part.

We define a hyperbolic equilibrium point of the system (A.8.2) as follows.

Definition A.6 An equilibrium point of the system (A.8.2) is said to be hyperbolic if all eigenvalues of its linear variational equations have non-zero real parts.

Then by Lyapunov’s Theorem, if an equilibrium is hyperbolic the Lyapunov stability analysis of the linear system correctly predicts the Lyapunov stability of the equilibrium in the nonlinear system. This is extended by the Hartman-Grobman theorem, see [170, 141], which says that, in some neighbourhood of a hyperbolic equilibrium point, the flow of the nonlinear system is homeomorphic to the flow of the linearised system.

Structural stability

We now continue the idea of stability to that of structural stability. The discussion presented here closely follows that found in [141]. Structural stability concerns the relationship between the dynamics of a given system, and those of a neighbouring system i.e. a system which is a slight perturbation of the original one. Here we give the results for autonomous equations, however the ideas may be extended to nonautonomous equations, see for example [105]. Consider the system (A.8.2) and a perturbed system given by

\[ \dot{x} = f(x) + \varepsilon f_1(x) \quad (A.8.3) \]

where \( \varepsilon \ll 1 \) and \( f_1 \) is continuous. A system is structurally stable if all nearby systems are topologically equivalent to it. In particular, the system (A.8.2) is structurally stable
if there exists a homeomorphism taking the motions of (A.8.2) to motions of (A.8.3) for some \( \varepsilon > 0 \). If a system in the form (A.8.2) has an equilibrium point which is not hyperbolic, then the system is not structurally stable. However it is possible that all of the equilibrium points of a system are hyperbolic, and still the system may not be structurally stable. We state the following theorem due to Mauricio Peixoto, see [139, 85] for details.

**Theorem A.5 (Peixoto’s Theorem)** Let \( \dot{x} = f(x) \) be a \( C^1 \) dynamical system on a smooth closed surface \( M \). Then the dynamical system is \( C^1 \)-structurally stable if and only if it satisfies the following properties.

- All recurrent behaviour consists of only finitely many equilibrium points and periodic orbits, all of which are hyperbolic.
- There are no saddle-saddle connections (no separatrices).

For example, we see that the simple pendulum discussed in Appendix A.4 is not structurally stable. Thus when dissipation is added to the pendulum system, the topology of the dynamics changes. However, the pendulum with dissipation already added is structurally stable, since all equilibrium points are hyperbolic and the separatrix is broken.

### A.9 Mathieu’s equation

Mathieu’s equation is a special case of Hill’s equation, studied in Chapter 2, and has become one of the predominant examples of a system with parametric excitation [118, 96, 141, 142]. The equation may be written as

\[
\ddot{x}(t) + (\alpha + \beta \cos(t))x(t) = 0,
\]  

(A.9.1)
where the dots denote derivative with respect to time $t$ and $(x, \dot{x}) \in \mathbb{R}^2$. Often the period of the forcing is taken as $T = \pi$, however here we choose $T = 2\pi$ so as to match with the linearisation of the pendulum system studied in Chapter 5. In general the solutions of equation (A.9.1) cannot be found in an explicit form, however the stability of the solutions under different parameter values can be studied.

For undamped linear systems such as Mathieu’s equation (A.9.1) and Hill’s equation (2.4) we use the following definition of stability, known as linear stability, see [116, 118].

**Definition A.7** If all the solutions are bounded, they are said the be stable. If at least one solution is unbounded, they are said to be unstable.

The above definition is equivalent to Lyapunov stability for linear systems. More details on different types of stability are given in Appendix A.8. The regions of parameter space in which solutions are stable or unstable are called the regions of stability and instability, respectively. These regions are separated by “transition curves”, also known as “stability curves”, see [118, 96, 141]. The method used to calculate the transition curves requires a number of theorems from Chapter 2, which are restated below in a more applicable manner. All of the following theorems and definitions can also be found in [96].

**Theorem A.6 (Floquet’s Theorem)** The regular system $\dot{x} = P(t)x$ where $P$ is an $n \times n$ matrix function with minimal period $T$, has at least one non-trivial solution $x = \chi(t)$ such that,

$$\chi(t + T) = \mu \chi(t), \quad -\infty < t < \infty,$$

where $\mu$ is a constant.

**Definition A.8** Take the regular system $\dot{x} = P(t)x$ where $P$ is an $n \times n$ matrix function with minimal period $T$. Let $X(t)$ be the fundamental matrix for the system and define
the nonsingular matrix \( \Xi \) by

\[
X(t + T) = X(t)\Xi,
\]

so that

\[
\Xi = X^{-1}(t_0)X(t + T),
\]

for a convenient value of time \( t_0 \).

**Theorem A.7** Take the regular system \( \dot{x} = P(t)x \) where \( P \) is an \( n \times n \) matrix function with minimal period \( T \). Let \( X(t) \) be the fundamental matrix for the system and \( \Xi \) as per definition A.8. Suppose that \( \Xi \) has \( n \) distinct eigenvalues, \( \mu_j, j = 1, 2, \ldots, n \) then the system \( \dot{x} = P(t)x \) has \( n \) linearly independent normal solutions of the form

\[
x_j = p_j(t)e^{\rho_j t},
\]

where the \( p_j(t) \) are vector functions with period \( T \). The eigenvalues \( \mu \) are known as the characteristic numbers (also called characteristic multipliers or Floquet multipliers) of the system and \( \rho_j \) are the characteristic exponents (also called Floquet exponents) corresponding to \( \mu_j \) defined by

\[
e^{\rho_j T} = \mu_j
\]

**Theorem A.8 (A variation of Abel’s identity)** For any \( t_0 \), the Wronskian of the system \( \dot{x} = P(t)x \) is,

\[
W(t) = W(t_0)\exp\left(\int_{t_0}^{t} tr(P(s)) \, ds\right),
\]

where \( tr(P(s)) \) is the trace of \( P(s) \).

**Theorem A.9** For the system \( \dot{x} = P(t)x \) where \( P \) has minimal period \( T \), let the characteristic numbers of the system be \( \mu_1, \mu_2, \ldots, \mu_n \). Then

\[
\mu_1\mu_2\ldots\mu_n = \exp\left(\int_{t_0}^{t} tr(P(s)) \, ds\right),
\]
a repeated characteristic number being counted according to its multiplicity.

The system (A.9.1) can be written in the form $\dot{x} = P(t)x$ where

$$x = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0 & 1 \\ -\alpha + \beta \cos t & 0 \end{pmatrix},$$

so that $P(t)$ is periodic with minimal period $2\pi$. The structure of the solution can thus be determined by Theorem A.7. Clearly $tr(P(t)) = 0$, hence by Theorem A.9 we have

$$\mu_1 \mu_2 = e^0 = 1,$$

(A.9.2)

where $\mu_1, \mu_2$ are the solutions of the equation

$$\det(\Xi - \mu I) = 0.$$

There are five possible cases which can arise for the values of $\mu_1, \mu_2$, which correspond to the intervals mentioned in Theorem 2.3.

(i) $\mu_1, \mu_2$ are real, different and positive, hence by equation (A.9.2) one of them is greater than one. The corresponding characteristic exponents have the property $\rho_1 = -\rho_2 > 0$ and by Theorem A.7 the solution is of the form

$$x(t) = c_1 e^{\rho_1 t} p_1(t) + c_2 e^{-\rho_1 t} p_2(t),$$

where $c_1, c_2$ are constants and $p_1, p_2$ have minimal period $2\pi$.

(ii) $\mu_1 = \mu_2 = 1$, hence $\rho_1 = \rho_2 = 0$. By Theorem A.7 there is one bounded solution of period $2\pi$.

(iii) $\mu_1, \mu_2$ are complex conjugates with $|\mu_j| = 1$. Thus $\rho_1 = -\rho_2 = i\nu$ for some real value $\nu$ and the solution is of the form

$$x(t) = c_1 e^{i\nu t} p_1(t) + c_2 e^{-i\nu t} p_2(t),$$

for $c_1, c_2$ constant and $p_1, p_2$ have minimal period $2\pi$.  

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(iv) \( \mu_1 = \mu_2 = -1 \), hence \( \rho_1 = \rho_2 = -\frac{1}{2}i \), there is one bounded solution with period \( 4\pi \).

(v) \( \mu_1, \mu_2 \) are real, different and negative, hence by equation (A.9.2) one of them is less than minus one. The general solution is then of the form

\[ x(t) = c_1 e^{(c_3 + \frac{1}{2})t} p_1(t) + c_2 e^{(-c_3 + \frac{1}{2})t} p_2(t), \]

for constants \( c_1, c_2 \) and \( c_3 > 0 \) with \( p_1, p_2 \) again having minimal period \( 2\pi \). This can also be written as,

\[ x(t) = c_1 e^{c_3 t} q_1(t) + c_2 e^{-c_3 t} q_2(t), \]

where \( q_1 \) and \( q_2 \) have minimal period \( 4\pi \).

From the above five cases, see also Theorem 2.3, it is apparent that the curves on which period \( 2\pi \) and period \( 4\pi \) solutions occur separate the parameter regions where unbounded solutions exist from those where all solutions are bounded. Therefore, if we find the parameter values for which these solutions exist, we have also found the boundaries between the stable and unstable regions of parameter space i.e the transition curves. Further details on the method used for finding the transition curves can be found in [96, 118]

We begin by finding the \( 2\pi \) periodic solutions, which we represent by the complex Fourier series

\[ x(t) = \sum_{n=\infty}^{\infty} c_n e^{int}. \]

Substituting the above series into equation (5.4) and writing \( \cos t \) as \( (e^{it} + e^{-it})/2 \) yields

\[ \sum_{n=-\infty}^{\infty} \left[-\beta c_{n+1} + 2(\alpha - n^2)c_n - \beta c_{n-1}\right] e^{int} = 0. \]

The equality can only be satisfied for all \( \tau \) if

\[ -\beta c_{n+1} + 2(\alpha - n^2)c_n - \beta c_{n-1} = 0, \]
for \( n = 0, \pm 1, \pm 2, \ldots, \pm \infty \). Defining

\[
\zeta_n = -\frac{\beta}{2(\alpha - n^2)},
\]  
(A.9.3)

the equation in \( \alpha \) and \( \beta \) may be written as

\[
\zeta_n c_{n+1} + c_n + \zeta_n c_{n-1} = 0,
\]

which can be represented by the infinite determinant

\[
\begin{vmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \zeta_1 & 1 & \zeta_1 & 0 & \cdots \\
\cdots & 0 & \zeta_0 & 1 & \zeta_0 & 0 & \cdots \\
\cdots & 0 & 0 & \zeta_1 & 1 & \zeta_1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix} = 0.
\]

Note that \( \zeta_n = \zeta_{-n} \). Solving this determinant for \( \alpha \) and \( \beta \) with \( n \) large enough gives accurate estimates to curves of the \( 2\pi \) periodic solutions. The curves for the \( 4\pi \) periodic solutions are calculated in the same way, this time substituting

\[
x(t) = \sum_{n=-\infty}^{\infty} d_n e^{\frac{1}{2}int}. 
\]

Following the same steps as before, we arrive at

\[
-\frac{1}{2}\beta d_{n+2} + (\alpha - \frac{1}{4}n^2) d_n - \frac{1}{2}d_{n-2} = 0,
\]

for \( n = 0, \pm 1, \pm 2, \cdots, \pm \infty \). Substituting

\[
\eta_n = -\frac{\beta}{2(\alpha - \frac{1}{4}n^2)},
\]  
(A.9.4)

the equation in \( \alpha \) and \( \beta \) becomes

\[
\eta_n d_{n+2} + d_n + \eta_n d_{n-2} = 0,
\]
which again we can write as an infinite determinant as

\[
\begin{vmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \eta_2 & 0 & 1 & 0 & \eta_2 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \eta_1 & 0 & 1 & 0 & \eta_1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \eta_0 & 0 & 1 & 0 & \eta_0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \eta_1 & 0 & 1 & 0 & \eta_1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \eta_2 & 0 & 1 & 0 & \eta_2 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{vmatrix} = 0.
\]

However, solving the above determinant would give us the stability curves for both $2\pi$ and $4\pi$ periodic solutions. Only considering the solutions for odd $n$, we can eliminate the terms involving $d_{2m} = 0$, resulting in the new infinite determinant

\[
\begin{vmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \eta_2 & 1 & \eta_2 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \eta_1 & 1 & \eta_1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \eta_1 & 1 & \eta_1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \eta_2 & 1 & \eta_2 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{vmatrix} = 0,
\]

where we have redefined $\eta_m = -\beta/2 \left( \alpha - \frac{1}{4}(2m - 1)^2 \right)$. Choosing $m$ large enough and solving the $m$ by $m$ determinant for $\alpha$ and $\beta$ gives an estimate to the stability curves for the $4\pi$ periodic solutions. We should note that the convergence of the two determinants are ensured by both $\zeta_n$ and $\eta_n$ having order $O(n^{-2})$. Figure 5.1 shows both the $2\pi$ and $4\pi$ periodic solution curves. Regions of instability have been shaded and the regions which are stable (also known as the stability tongues) are white.

The values of $\alpha$ for which the transition curves start, that is where they touch the axis at $\beta = 0$, may easily be found without having to deal with large matrix determinants. For $\beta = 0$ the system reduces to the simple harmonic oscillator and the solutions have natural frequency $\omega_0 = \sqrt{\alpha}$ (and hence natural period $T_0 = 2\pi/\sqrt{\alpha}$). However, $T_0$ is the minimal period and the solutions are of course also periodic with periods $nT_0$ for $n \geq 0 \in \mathbb{Z}$. Therefore resonance occurs when the period of the forcing (that is $2\pi$) is
equal to $nT_0$, hence

$$\frac{2n\pi}{\sqrt{\alpha}} = 2\pi.$$ 

Solving for $\alpha$ we find the transition curves touch the axis at $\alpha = n^2$ for $n \geq 0 \in \mathbb{Z}$. However, this only corresponds to the tongues of the period $2\pi$ solutions. To find where the period $4\pi$ tongues touch the axis we set

$$\frac{2n\pi}{\sqrt{\alpha}} = 4\pi.$$ 

and hence $\alpha = n^2/4$. Notice that again to avoid repeating the period $2\pi$ solutions we must replace $n$ with $2m + 1$ for $m > 0 \in \mathbb{Z}$. This may also be seen by substituting $\beta = 0$ into the equations (A.9.3) and (A.9.4) then rearranging. Therefore the stability tongues touch the $\beta = 0$ axis at

$$\alpha = 0, \frac{1}{4}, 1, \frac{9}{4}, 4, \frac{25}{4}, \ldots, \frac{n^2}{4}, n \in \mathbb{Z}.$$ 

Applying the analysis in Section 2.3 with $Q(t) = \cos t$ we can calculate the instability
domains of Mathieu’s equation. In particular, the $n$-th instability region occurs where

$$
\left( \sqrt{\alpha} - \frac{n}{2} \right)^2 \leq \frac{r_n^2 \beta^2}{4n^2} \tag{A.9.5}
$$

is satisfied, where $r_n$ is defined in equation (2.37). Then for $\beta$ small we can rearrange the above relation and set $n = 1$ to obtain

$$
\frac{1}{4} - \frac{\beta}{2} + \mathcal{O}(\beta^2) < \alpha < \frac{1}{4} + \frac{\beta}{2} + \mathcal{O}(\beta^2),
$$

which is a well known result for Mathieu’s equation, see [118, 147, 96, 141, 142]. The above result may also be obtained using the method of two variable expansion, writing $s = t$, $\hat{s} = \beta t$, see [141, 142].

To put the results concerning high frequency forcing in Chapters 2 and 3 in context of the analysis here, we instead consider Mathieu’s equation with forcing frequency $\omega$. Then the tongues touch the axis at $\alpha = n^2 \omega^2 / 4$. Therefore, for large $\omega$ the first interval of stability occupies a wide interval of $\alpha$. To look at this in a different way we can

![Diagram of the first region of instability using the analysis in Section 2.3.](263)
renormalise time to $\tau = \omega t$ to obtain the equation

$$\ddot{x} + \frac{1}{\omega^2}(\alpha + \beta \cos \tau)x = 0.$$ 

Then clearly for $\omega$ large enough, the coefficients $\alpha/\omega^2$, $\beta/\omega^2$ are situated in the first region of stability. In particular for any values of the parameters $\alpha$ and $\beta$, it is possible to suitably choose $\omega$ such that this happens. Conversely for small $\omega$, the stability intervals bunch up towards the $\alpha = 0$ axis. Thus choices of $\alpha$ which are not too small put the system in the $n$-th interval of stability where $n$ is very large and by equation (A.9.5) the intervals of instability are very thin, see also [147]. Then the system is stable for parameters $\beta < \alpha$.

**The damped Mathieu’s equation**

We now study the damped Mathieu’s equation, given by

$$\ddot{x}(t) + (\alpha + \beta \cos(t))x(t) + \gamma \dot{x} = 0,$$  \hspace{1cm} (A.9.6)

where $\gamma > 0$ is the coefficient of dissipation. The equation (A.9.6) can be transformed back into the form of equation (A.9.1) using the transformation (2.3) in Chapter 2, by which $\alpha$ is mapped to $\alpha - \gamma^2/4$. Studying the system (A.9.6) and applying the analysis of Section 2.3 we find that the $n$-th region of instability occurs where

$$4 \left(\sqrt{\alpha} - \frac{n}{2}\right)^2 + \gamma^2 < \frac{r_n^2\beta^2}{n^2}$$  \hspace{1cm} (A.9.7)

is satisfied, where again $r_n$ is defined in equation (2.37). One immediately apparent result is that if $\gamma > r_n\beta/n$ the $n$-th region of instability vanishes; this was also noted in [147]. Therefore, for fixed parameters $\alpha$ and $\beta$, as $\gamma$ is increased the instability regions vanish in the order of decreasing values of $n$. Another way of saying this is, as $\gamma$ increases the instability regions move away from the axis $\beta = 0$. Furthermore, the distance between
the \( n \)-th instability region and the axis \( \beta = 0 \) increases with increasing \( n \). For \( n = 1 \) equation (A.9.7) reduces to

\[
4 \left( \sqrt{\alpha} - \frac{1}{2} \right)^2 + \gamma^2 < \beta^2
\]

which can be rearranged to obtain

\[
\frac{1}{4} - \frac{1}{2} \sqrt{\beta^2 - \gamma^2} + \mathcal{O}(\beta^2) < \alpha < \frac{1}{4} + \frac{1}{2} \sqrt{\beta^2 - \gamma^2} + \mathcal{O}(\beta^2),
\]

which is a well known result for the damped Mathieu’s equation, often calculated using the two variable expansion method, see [141].

The stability tongues for the damped equation (A.9.6) may be calculated using the method in [96, 118], as used above for the undamped Mathieu’s equation. The system may be written in matrix form as

\[
\dot{x} = P(t)x, \quad P(t) = \begin{pmatrix} 0 & 1 \\ -\alpha + \beta \cos t & -\gamma \end{pmatrix}.
\]

(A.9.8)

Now \( \text{tr}(P(t)) = -\gamma \) and by Theorem A.9 we find \( \mu_1 \mu_2 = \exp(-2\pi\gamma) \). If \( \mu_1 > \mu_2 > 0 \), the system is stable if \( \mu_1 \leq 1 \) and for \( \mu_1 = 1 \) we have \( \mu_2 = \exp(-2\pi\gamma) \). If \( \mu_2 < \mu_1 < 0 \), the system is stable if \( \mu_2 \geq -1 \) and if \( \mu_2 = 1 \) we find \( \mu_1 = -\exp(-2\pi\gamma) \). These correspond to the \( 2\pi \) and \( 4\pi \) periodic solutions, respectively, see [96]. In this instance, the equations (A.9.3) and (A.9.4) become

\[
\zeta_n = \frac{\beta}{2(\alpha - n^2 + i\gamma n)}, \quad \eta_m = \frac{\beta}{2 \left[ \alpha - \frac{1}{4}(2m-1)^2 + \frac{1}{2}(2m-1)i\gamma \right]},
\]

where \( n, m \in \mathbb{Z} \). For \( n \) and \( m \) large enough we can solve the matrix determinates to obtain the stability tongues, see Figure A.7. Note that, as the system (A.9.6) is damped, in the stable regions of Figure A.7 one of the eigenvalues has absolute value less than one. Hence the trajectories decay to the fixed point \( (x, \dot{x}) = (0, 0) \) and asymptotic stability
Figure A.7: Stability tongues for damped Mathieu’s equation (A.9.6). The values of $\gamma$ are $\gamma = 0, 0.1$ and $0.2$ with the grey regions increasingly dark for larger values of $\gamma$. As $\gamma$ increases the stability tongues move further away from the $\beta = 0$ axis.

is predicted. In the unstable regions, one of the eigenvalues has absolute value greater than one, and the trajectories are unstable and diverge [96].

The effects of nonlinearity can be explained as follows: as the resonance causes the amplitude of the motion to increase, the relation between the period and amplitude causes the resonance to detune, decreasing the tendency to produce large motions. Therefore in the unstable tongues of Figures A.5 and A.7 we find stable solutions which are oscillatory and rotating, see also [79, 141].

### A.10 Basins of attraction

The corresponding basin of attraction of a particular attractive solution (a so-called attractor) is the set of initial conditions which move towards that solution in the limit $t \to \infty$. In Chapters 5, 6 and 7 basins of attraction were numerically estimated using a large number of random initial conditions which uniformly populate a desired region of phase space. This method is computationally heavy, but is well suited to estimating the sizes of the basins of attraction relative to the region investigated, see [22, 155, 115, 33, 58].
APPENDIX A. BACKGROUND KNOWLEDGE

Further details of the approach are given in Chapter 4. However when investigating basins of attraction, the size of a basin of attraction is not always the feature of interest [39, 7, 161]. Here we give a brief discussion of how the boundaries of basins of attraction may be calculated using stable manifolds. This has particular application when one is interested in studying the topology of basins of attraction or the location in phase space [39, 7, 161], without wishing to know approximations to the size.

We consider systems of the form

\[ \ddot{x} + f(x, \dot{x}, t) = 0, \]  

(A.10.1)

where \( f \) is a nonautonomous function which is periodic in time \( t \). We restrict our attention to systems with a linear dissipative term with constant coefficient, and so may rewrite the system as

\[ \ddot{x} + \gamma \dot{x} + h(x, t) = 0, \quad \gamma > 0. \]  

(A.10.2)

If \( h \) is linear in \( x \) then only one attractive solution can exist [39], and the corresponding basin of attraction occupies the whole domain of phase space up to a zero measure set. If \( h \) is nonlinear then multiple attractive solutions can coexist, see for example [27, 28, 22, 76, 58], each with their own basin of attraction. Here we are interested in studying fixed points and periodic orbits of such systems, which relates to the work in Chapters 5, 6 and 7. If we study the Poincaré map of the system (A.10.2), the periodic attractors are represented by points. An attractor which is periodic with period \( n \) times the of the forcing period is represented by a series of \( n \) points, which the trajectory jumps between on each iteration of the map. Of course, the location of the Poincaré points in phase space is dependent both on the attractor and the phase at which the Poincaré section taken.
The trajectories of a dissipative system are governed by characterizing points such as stable nodes, saddle points or a spiral in to a stable point, see Figure A.8. Saddle points can be used to determine the boundaries between the basins of attraction for the various coexisting attractors. The stable manifold (also called the inset) of a saddle point, that is the pair of trajectories which tend toward the saddle point as time goes to infinity, form a separatrix between two basins of attraction. The unstable manifold (also called the outset) of the saddle point does not play a role in determining the basins boundaries, however it is useful when determining the location of the saddle point. The positions of both the stable and unstable manifolds can be found using a numerical technique, which essentially involves integrating a set of initial conditions over one cycle of the forcing period. By joining up successive Poincaré points, a set of vectors is obtained. A stable manifold separates vectors which point in opposite directions. An unstable manifold may be found by completing the same process with time running backward. Having located both the stable and unstable manifolds to an acceptable degree of accuracy, the saddle point may be found where the two cross one another.

Once a saddle point has been located, if we give the system an initial condition

---

9This is only true if the saddle is not an inverting saddle, which is always true for systems of the form (A.10.1), see [39, 85] for further details
which is very close to the saddle point and allow it to evolve under backward time, the resulting trajectory will trace out a curve which corresponds to the stable manifold. Using a Poincaré map, we instead have a series of points which are positioned along the curve. Depending on the eigenvalues of the map the distance between successive points may be large, and thus the path of the stable manifold can be unclear. This problem can be overcome by using the “ladder method” [39], which results in a pseudo-continuous curve. By considering suitably chosen initial conditions close to the saddle point, the distance between successive Poincaré points can be subdivided (into “rungs” of the ladder). Using enough initial conditions we obtain a curve which is nearly continuous. By following the path of the stable manifold we can determine the boundary of the basin of attraction for a particular attractor, see Figure A.9.

Figure A.9: An illustration of a typical case. The region enclosed by the stable manifold of the saddle point is creates the basin of attraction of the attractor to which the unstable manifold moves toward.

Studying the basins of attraction in this way, it is clear that, if the stable and unstable manifolds cross one another the basins of attraction become sensitive to slight perturbations in initial conditions, see Appendix A.11 for details. Of course, as we change the phase of the Poincaré section, $t_0$, at which we take a snapshot of the system, the location of the stable and unstable manifolds change, see Appendix A.11. The movement
of the manifolds is periodic in $t_0$ with the same period as the forcing. Therefore, by changing $t_0$ we find the location and shape of the basins of the attraction also varies periodically. In Chapters 5, 6 and 7 we studied the sizes of the basins of attraction, taking our initial conditions in phase space at time $t_0 = 0$. A possible question to ask is: as we change the initial time $t_0$, the shape and location of the basins of attraction vary, does the size of the basin of attraction also change?

The answer is no, see Table A.1. One possible explanation is as follows. We focus on the interval $t_0 \in [0, 2\pi]$, since the forcing of the systems investigated in Chapters 5, 6 and 7 are periodic with period $2\pi$; hence also the phase $t_0$ repeats every $2\pi$. If we take a set of pseudo-random initial conditions in phase space at a time $t = t_0$ and integrate forward we find that the region of phase space still occupied by the trajectories contracts as $t$ increases, see Chapter 7. By taking initial conditions $x_0$ at time $t_0 = 0$ and letting them evolve up to a time $t_1 \in [0, 2\pi]$, the trajectories have moved slightly away from their initial conditions. Of course, the positions of each trajectory at time $t_1$ could also be used as a set of initial conditions for the system, starting at time $t_1$. Furthermore, each initial condition $(x_1, t_1)$ continues toward the same attractor as its corresponding initial condition $(x_0, t_0)$. As $t_1$ is small, provided $\gamma$ is not too large, much of the original phase space is still occupied and approximately uniformly populated by the set $x_1$, see Figure A.10. Therefore the initial conditions $(x_1, t_1)$ are a reasonable approximation to a uniform, pseudo-random set of initial conditions $(y_1, t_1)$, which covers the whole of the original region of phase space. Therefore, we only need to consider what happens outside of the contracted phase space $C_{t_1}$, in particular whether or not this region changes the sizes of the basins of attraction. Again, provided $\gamma$ is not too large, the region of the initial phase space which is outside of $C_{t_1}$ should be distant from the attractive solutions, see $C_{2\pi}$ in Figure A.10. Therefore it is expected that, provided $C_{t_1}$ is not too small, initial conditions outside of $C_{t_1}$ move towards the attractors in similar proportions as initial
conditions inside of $C_{t_1}$ [22].

Table A.1 gives the sizes of the basins of attraction for the system (5.2), see Chapter 5, with initial conditions taken at times $t_0 = n\pi/4$, for $n = 0, 1, \ldots, 7$. The results were obtained using 100 000 pseudo-random initial conditions, thus we expect an error in the first decimal place, see Table 4.1 in Chapter 4. We find that the sizes of the basins of attraction do not change significantly. Any small variation in the sizes is due to the choice of initial conditions and by taking a larger set of initial conditions we expect this difference to decrease.

<table>
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<tr>
<td>7</td>
<td>70.91</td>
<td>5.16</td>
<td>18.77</td>
</tr>
</tbody>
</table>

Table A.1: Relative areas of the basins of attraction for the pendulum with oscillating support (see Chapter 5) with constant damping coefficient $\gamma = 0.023$. The initial conditions for the system were taken at times $t = n/\pi$, for $n = 0, 1, \ldots, 7$.  

Figure A.10: An image of $C_{2\pi}$ for the pendulum with oscillating support as studied in Chapter 5. The parameter values are chosen as $\alpha = 0.5$, $\beta = 0.1$ and $\gamma = 0.023$. The initial region of phase space is $[-\pi, \pi] \times [-4, 4]$ and the black region is $C_{2\pi}$. 
A.11 Melnikov theory

Melnikov’s method is a procedure which can be applied to a class of systems of ordinary differential equations, and gives a bound on the parameter values where transverse intersections of the stable and unstable manifolds occur. The method may be applied to perturbations of conservative systems, where the unperturbed system has one degree of freedom and a separatrix (homoclinic orbit). The separatrix of the unperturbed system will in general be broken when the perturbation is applied. The idea of Melnikov’s method is to utilise perturbation expansions and show that there exists an intersection of the stable and unstable manifolds of the saddle point in the Poincaré map when the separatrix is broken. This implies the existence of a horseshoe [153] in the Poincaré map, which in turn implies that there exist periodic motions of all periods, as well as motions which are not periodic. In particular there is an infinite number of unstable periodic orbits.

As well as this, the horseshoe map exhibits sensitive dependence on initial conditions, and is hence of interest when studying the topology of basins of attraction, see Appendix A.10. Melnikov’s method has previously been applied to the pendulum system in [66, 35] where the authors focused on the existence of a horseshoe with three sections (so called a 3-shoe). As a result they were able to show the existence of high period orbits which become stable only inside small regions of the parameter values. Here we only intend to give an overview of the method and discuss the relevance that it has to the work contained in the previous chapters, for a more in depth discussion of the method see for example [85].

We consider systems of the form

\[ \dot{x} = f(x) + \varepsilon g(x, t), \]  
(A.11.1)

where \( x \in \mathbb{R}^2 \), \( f, g \) are \( C^r \) smooth with \( r \geq 2 \) and \( g \) is periodic in time \( t \) with period \( T \).
APPENDIX A. BACKGROUND KNOWLEDGE

The equation (A.11.1) may be rewritten as

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
 f_1(x) \\
 f_2(x)
\end{pmatrix} + \varepsilon \begin{pmatrix}
 g_1(x,t) \\
 g_2(x,t)
\end{pmatrix},
\]

(A.11.2)

We then make the following assumptions:

1. For \( \varepsilon = 0 \) the system (A.11.1) possesses a homoclinic (or heteroclinic) orbit \( q^0(t) \), to a hyperbolic saddle point \( p_0 \).

2. Let \( C^0 = \{ q^0(t) \mid t \in \mathbb{R} \} \cup \{ p \} \). The interior of \( C_0 \) is filled with a continuous family of periodic orbits \( q^a, a \in (-1,0) \). Let \( d(x,C^0) = \inf_{q \in C^0} |x - q| \) then we have

\[
\lim_{a \to 0} \sup_{t \in \mathbb{R}} d(q^a(t),C^0) = 0.
\]

3. The period \( T_a \) of the orbit \( q^a \) tends to \( \infty \) monotonically as \( a \to 0 \).

The first assumption implies that the Poincaré map (starting at time \( t_0 \)) \( P^0_{t_0} \) of the unperturbed system has a hyperbolic saddle point \( p_0 \) and that the closed curve \( C^0 = W^u(p_0) \cap W^s(p_0) \) is filled with nontransverse homoclinic points on \( P^0_{t_0} \). This structure is expected to break for \( \varepsilon > 0 \), resulting in either transverse homoclinic orbits, or no homoclinic points at all. Note that the second assumption implies that \( q^{-1} \) is a fixed point of the system.

Figure A.11: An illustration of the unperturbed system with homoclinic orbit \( q^0(t) \).
Lemma A.1 Under the above assumptions, for \( \varepsilon \) sufficiently small, the system (A.11.1) has a unique hyperbolic periodic orbit \( x_\varepsilon = p_0 + \mathcal{O}(\varepsilon) \). The corresponding Poincaré map \( P_{\varepsilon}^{t_0} \) has a unique hyperbolic saddle point \( p_{\varepsilon}^{t_0} = p_0 + \mathcal{O}(\varepsilon) \).

Note that the orbit \( x_\varepsilon \) may indeed be the equilibrium \( p_0 \), as it is for the inverted position of the pendulum systems studied in the previous chapters. We wish to obtain the distance between the stable and unstable manifolds of \( x_\varepsilon \) in the perturbed system, \( W^s_{\varepsilon}(x_\varepsilon) \) and \( W^u_{\varepsilon}(x_\varepsilon) \), respectively. To do so we study the Poincaré map of the perturbed system \( P_{\varepsilon}^{t_0} \), where again \( t_0 \) is the initial time. The points of \( P_{\varepsilon}^{t_0} \) on \( W^s_{\varepsilon}(x_\varepsilon) \) and \( W^u_{\varepsilon}(x_\varepsilon) \) are given by \( q^s_{\varepsilon}(t, t_0) \in W^s_{\varepsilon}(p_{\varepsilon}^{t_0}) \) and \( q^u_{\varepsilon}(t, t_0) \in W^u_{\varepsilon}(p_{\varepsilon}^{t_0}) \), respectively. The points \( q^s_{\varepsilon}(t, t_0), q^u_{\varepsilon}(t, t_0) \) may be estimated by, see [85]

\[
q^s_{\varepsilon}(t, t_0) = q^0(t - t_0) + \varepsilon q^s_{1}(t, t_0) + \mathcal{O}(\varepsilon^2), \quad t \in [t_0, \infty),
\]

\[
q^u_{\varepsilon}(t, t_0) = q^0(t - t_0) + \varepsilon q^u_{1}(t, t_0) + \mathcal{O}(\varepsilon^2), \quad t \in (-\infty, t_0].
\]  

Then the separation of the manifolds \( W^u_{\varepsilon}(p_{\varepsilon}^{t_0}), W^s_{\varepsilon}(p_{\varepsilon}^{t_0}) \) on \( P_{\varepsilon}^{t_0} \) at \( q^0(0) \) is defined as

\[
d(t_0) = q^u_{\varepsilon}(t_0) - q^s_{\varepsilon}(t_0).
\]

Figure A.12: The stable and unstable manifolds of the perturbed system. Figure (b) shows a zoomed image of the region around \( q^s_{\varepsilon}(t_0) \) and \( q^u_{\varepsilon}(t_0) \).
APPENDIX A. BACKGROUND KNOWLEDGE

The above may be rewritten as, see [85]

\[ d(t_0) = \varepsilon \frac{f(q^0(0)) \wedge (q^u_1(t, t_0) - q^s_1(t, t_0))}{|f(q^0(0))|} + O(\varepsilon^2). \]  

(A.11.4)

where the wedge product is defined by \(a \wedge b = a_1b_2 - a_2b_1\). It may be shown that, see [85]

\[ d(t_0) = \frac{\varepsilon M(t_0)}{|f(q^0(0))|} + O(\varepsilon^2), \]  

(A.11.5)

where

\[ M(t_0) = \int_{-\infty}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t) \, dt, \]  

(A.11.6)

is the Melnikov function. The integral spans the time \(-\infty\) to \(+\infty\) as the period on the homoclinic orbit \(q^0\) is infinite. In particular we integrate over the unstable manifold from \(-\infty\) to \(t_0\) then over the stable manifold from \(t_0\) to \(+\infty\), which gives the overall interval of \(-\infty\) to \(+\infty\). We are now able to state the following theorem, see [85] for a detailed proof.

**Theorem A.10** If \(M(t_0)\) has simple zeros and is independent of \(\varepsilon\), then, for \(\varepsilon > 0\) sufficiently small \(W^u_\varepsilon(p^0_\varepsilon)\) and \(W^s_\varepsilon(p^0_\varepsilon)\) intersect transversely. If \(M(t_0)\) remains away from zero then \(W^u_\varepsilon(p^0_\varepsilon) \cap W^s_\varepsilon(p^0_\varepsilon) = \emptyset\)

Theorem A.10 allows us to test for the existence of a transverse homoclinic orbit in a system of the form (A.11.2). The presence of such an orbit implies that the Poincaré map has a Smale horseshoe and thus contains a countable infinity of unstable periodic orbits, an uncountable set of bounded nonperiodic orbits and a dense orbit. As previously mentioned, in [66, 35] the authors showed that, for particular values of the parameters the pendulum with vertically oscillating support contains a 3-shoe and therefore the system also has these properties.
Figure A.13: An illustration of the stable and unstable manifolds crossing. As they cross, they partition the phase space and separate basins of attraction.

We can think of computing the Melnikov function as standing at a point \( q^0(0) \) on a moving cross section of the flow and watching the stable and unstable manifolds of the perturbed system oscillate as a periodic function of \( t_0 \). Note that the periodicity of \( M(t_0) \) in \( t_0 \) is implied since the maps \( P^{t_0}_\varepsilon \) and \( P^{t_0+T}_\varepsilon \) are the same. Therefore if \( M(t_0) = 0 \) for one value of \( t_0 \), then there are infinitely many values of \( t_0 \in (-\infty, \infty) \) for which \( M(t_0) = 0 \). Furthermore, as the orientation of the manifolds is the same for \( t = t_0 \) and \( t = t_0 + nT \), where \( n \in \mathbb{Z} \), the manifolds must also cross for values of \( t \in (t_0 + (n - 1)T, t_0 + nT) \). In particular near the hyperbolic equilibrium \( p^{t_0}_\varepsilon \) the stable and unstable manifolds intersect transversely an infinite amount of times, see Figure A.13. This is known as a homoclinic tangle. One result of this is that the manifolds are stretched and squashed. As the stable manifold separates basins of attraction, see Appendix A.10, the boundaries of the basins of attraction are fractal and the basins of attraction in this region exhibit sensitivity to initial conditions.

For the pendulum system where the fixed point \( \theta = 0 \) is stable, there is a saddle point at \( \theta = \pi \), with a separatrix. Hence when the system is perturbed parametrically, provided the forcing amplitude and coefficient of dissipation are sufficiently small and chosen such that \( M(t_0) \) has a simple zero for some \( t_0 \in [0, 2\pi] \), the basins of attraction...
around \( \theta = \pi \) are sensitive to slight perturbations in the initial conditions. We now calculate the conditions such that this happens. First we find the separatrix of the unperturbed system \( \ddot{\theta} + \alpha \sin \theta = 0 \). The energy of the unperturbed system may be written as

\[
E = \mathcal{H}(x, y, \alpha) = \frac{y^2}{2} + \alpha(1 - \cos x),
\]

where \( x = \theta \) and \( y = \dot{x} \). As mentioned above, we know that the separatrix originates from the saddle point at \((x, y) = (\pi, 0)\). Therefore, the energy on the separatrix is given by \( E(\alpha) = \mathcal{H}(\pi, 0, \alpha) = 2\alpha \), and hence on the separatrix we have

\[
\frac{(y^0)^2}{2} + \alpha \left(1 - \cos x^0\right) = 2\alpha,
\]

\[
\Rightarrow \frac{(y^0)^2}{2} - \alpha \left(1 + \cos x^0\right) = 0,
\]

\[
\Rightarrow \frac{(y^0)^2}{2} = \alpha \left(1 + \cos x^0\right) = 2\alpha \cos^2 \left(\frac{x^0}{2}\right).
\]

The above may be rearranged to obtain \( y^0 = \pm 2\sqrt{\alpha} \cos(x^0/2) \). Setting \( z = x^0/2 \) we have \( \dot{z} = \pm \sqrt{\alpha} \cos(z) \). This first order differential equation is separable and may be solved by

\[
\int \sec z \, dz = \pm \sqrt{\alpha}(t - t_0),
\]

from which we obtain

\[
\sec z + \tan z = e^{\pm \sqrt{\alpha}(t-t_0)}.
\]

Using the identity \( \sec z = \sqrt{1 + \tan^2 z} \), and writing \( s = \exp(\pm \sqrt{\alpha}(t-t_0)) \) the above may be rewritten as

\[
\tan z = \frac{s^2 - 1}{2s} = \frac{1}{2}(s - s^{-1}) = \pm \sinh(\sqrt{\alpha}(t-t_0)).
\]

Going back to our co-ordinates \( q, p \) we have

\[
x^0 = 2z = \pm 2 \arctan \left( \sinh(\sqrt{\alpha}(t-t_0)) \right),
\]

\[
y^0 = \dot{x} = \pm 2\sqrt{\alpha} \operatorname{sech}(\sqrt{\alpha}(t-t_0)),
\]
on the separatrix. If we consider the perturbed system studied in Chapter 5, given by

\[ \dot{x} = y, \quad \dot{y} + (\alpha - \epsilon \cos(t)) \sin x + \epsilon Cy = 0, \]

then we have, from equation (A.11.2),

\[ f_1 = y, \quad g_1 = 0, \]
\[ f_2 = -\alpha \sin x, \quad g_2 = \cos(t) \sin(x) - Cy, \]

so that the Melnikov function may be written as

\[ M_\pm(t_0, \alpha, C) = \int_{-\infty}^{\infty} f_0(q_\pm(t)) \wedge g_0(q_\pm(t), t + t_0) \, dt \]
\[ = \int_{-\infty}^{\infty} y_\pm^0 \left( \cos(t + t_0) \sin x_\pm^0 - Cy_\pm^0 \right) \, dt. \]

Calculating \( M_+ \) we have

\[ M_+(t_0, \alpha, C) = \int_{-\infty}^{\infty} 2\sqrt{\alpha} \, \text{sech}(\sqrt{\alpha} t) \sin \left( 2 \arctan(\sinh(\sqrt{\alpha} t)) \right) \cos(t + t_0) \, dt \]
\[ - \int_{-\infty}^{\infty} 4\alpha C \, \text{sech}^2(\sqrt{\alpha} t) \]
\[ = I_1 + I_2. \]

The integral \( I_2 \) is evaluated as \(-8\sqrt{\alpha}C\). The integral \( I_1 \) may be calculated by the method of residues, see [123].

\[ I_1 = 2\sqrt{\alpha} \int_{-\infty}^{\infty} \text{sech}(\sqrt{\alpha} t) \sin \left( 2 \arctan(\sinh(\sqrt{\alpha} t)) \right) \cos(t + t_0) \, dt \]
\[ = 2 \int_{-\infty}^{\infty} \text{sech}(\tau) \sin \left( 2 \arctan(\sinh(\tau)) \right) \cos(\left( \tau + \tau_0 \right) / \sqrt{\alpha}) \, d\tau \]
\[ = -\frac{4\pi}{\alpha} \frac{\sinh(\pi/2\sqrt{\alpha})}{1 - \cosh(\pi/\sqrt{\alpha})} \sin(\tau_0 / \sqrt{\alpha}). \]

Therefore we have

\[ M_+(t_0, \alpha, C) = -\frac{4\pi}{\alpha} \frac{\sinh(\pi/2\sqrt{\alpha})}{1 - \cosh(\pi/\sqrt{\alpha})} \sin(t_0) - 8\sqrt{\alpha}C. \]
In order for there to exist a value of $t_0$ such that $M_+(t_0) = 0$, the values of $\alpha$ and $C$ must satisfy

$$C \leq C_{\text{max}}(\alpha) = -\frac{\pi \sinh(\pi/2\sqrt{\alpha})}{2\alpha \sqrt{\alpha}(1 - \cosh(\pi/\sqrt{\alpha}))}.$$  \hspace{1cm} (A.11.7)

For $\alpha = 1/2$ we find that $C_{\text{max}}(1/2) \approx 0.12189$. It is easily checked that $M_-$ results in the same condition. An image of the basins of attraction for the pendulum with oscillating support, as studied in Chapter 5, is shown in Figure A.14. The parameters are chosen as $\alpha = 0.5$, $\beta = \varepsilon = 0.01$. In (a) $\gamma = 0.002 > \gamma_{\text{max}} = \varepsilon C_{\text{max}}$ and in (b) $\gamma = 0.001 < \gamma_{\text{max}}$. The basins of attraction are shown in a region around the unstable fixed point $\theta = \pi$, where there is a homoclinic tangle for $\gamma < \gamma_{\text{max}}$. It is clear from Figure A.14(b) that in a neighbourhood around $(\theta, \dot{\theta}) = (\pi, 0)$ the basins are fragmented and sparse. Furthermore, the regions shown in blue, cyan, magenta and orange correspond to attractors which have high periods; this is in keeping with the presence of a horseshoe. In Figure A.14(a), where $\gamma > \gamma_{\text{max}}$, we see that there are no attractors with high periods.

Figure A.14: Basins of attraction for the pendulum with oscillating support given by equation (5.2) with $\alpha = 0.5$, $\beta = \varepsilon = 0.01$. In (a) $\gamma = 0.002 > \gamma_{\text{max}}$ and in (b) $\gamma = 0.001 < \gamma_{\text{max}}$. The black region corresponds to the fixed point $\theta = 0$, green to the period two oscillations, red and yellow to the period one positive and negative rotations, respectively, see Chapter 5 for further details of these attractive solutions. The regions coloured blue, cyan, magenta and orange correspond to chaotic attractors.
However, the basin of attraction corresponding to the oscillating solution (shown in green) still appears fragmented, albeit less so. In fact, this is the case even for $\gamma$ taken large enough that the rotating solutions no longer exist. Therefore it is possible for the basins of attraction to be fragmented even when the stable and unstable manifolds of the unstable equilibrium do not cross. This was also observed in Section 5.4, see Figure 5.3.

The Melnikov function for the pendulum with variable length has been calculated in [21], to which we refer the interested reader. Further applications to similar systems may be found in [19, 20] Melnikov theory is a much more in depth topic than what is presented here. Including for example, results pertaining to perturbations of the periodic orbits $q^a$ of the unperturbed system, as well as the location of saddle points at which periodic orbits occur, see [85]. However the results are outside the scope of the thesis and so are not presented here.
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