THE USE OF TREATMENT CONCURRENCES TO ASSESS ROBUSTNESS OF BINARY BLOCK DESIGNS AGAINST THE LOSS OF WHOLE BLOCKS

J. D. GODOLPHIN*† AND E. J. GODOLPHIN‡

University of Surrey, and Royal Holloway, University of London.

Summary

Criteria are proposed for assessing the robustness of a binary block design against the loss of whole blocks, based on summing elements of selected upper non-principal sections of the concurrence matrix, which improve on the minimal concurrence concept that has been used previously and provide new conditions for measuring the robustness status of a design. The robustness properties of two-associate partially balanced designs are considered and it is shown that two categories of group divisible designs are maximally robust. These results expand a classic result in the literature, obtained by Ghosh, which established maximal robustness for the class of balanced block designs.

Key words: Block breakdown number; Concurrence; Connectivity; Group divisible design; Maximally robust; Microarray design; Partially balanced design.

1. Introduction

Many experiments involve the use of incomplete block designs. Unforeseen accident, damage, or other random event in the field, can result in observation loss and this has the consequence that the eventual design has properties different from those of the planned design. For some experiments, the nature of the blocking factor means that observation loss tends to correspond to the loss of whole blocks rather than individual observations. A typical example would be an experiment with animals or plants as blocks: the unavailability of an animal or plant, due to death or other reasons, results in the loss of a whole block. If the eventual design is disconnected, the usefulness of the trial is severely compromised since not all treatment contrasts will be estimable. Interest focuses on assessing the robustness of a planned design to give rise to connected eventual designs in the event that some blocks are lost during experimentation.

A planned binary block design in which \( v \) treatments are allocated to \( n_{\text{obs}} \) experimental units, arranged in \( b \) blocks with \( v \times b \) treatment-block incidence matrix \( N \) is denoted by \( D = BD(v, b, n_{\text{obs}}, N) \). If \( b_s \) blocks are lost during experimentation \( D \) is effectively replaced by the eventual design \( D_{\#, b_s} = BD(v, b - b_s, n_{\#}, N_{\#}) \). The problem of concern is to find

*Author to whom correspondence should be addressed.
†Department of Mathematics, University of Surrey, Guildford, Surrey, GU2 7XH, UK.
‡Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, UK.
Acknowledgment. This work was finished while J. D. Godolphin was at Murdoch University, Western Australia, during appointment as the inaugural Frank Hansford-Miller Fellow.
conditions on the parameters of $D$ which ensure that $D_{\#,b_*}$ is a connected design, irrespective of which $b_*$ blocks are lost, so that all treatment contrasts are estimable in $D_{\#,b_*}$. Adapting a term used by Mahbub Latif et al. (2009), who focus on the loss of individual observations, the block breakdown number of $D$ is defined to be the smallest value of $b_*$ for which at least one $D_{\#,b_*}$ is disconnected. $D$ is described as being maximally robust if it has largest possible block breakdown number. Conditions for $D$ to be maximally robust have been been considered in the literature by Ghosh (1982), Baksalary and Tabis (1987), Sathe and Satam (1992), Godolphin and Warren (2011) and others.

Designs with blocks all of size two form a special class: for such designs the loss of a single observation in a block is equivalent to the loss of the whole block. A recent application in which all blocks are size two concerns robustness of designs for two-colour microarray experiments and this has received attention from Landgrebe et al. (2006), Bailey (2007), Mahbub Latif et al. (2009), Bailey et al. (2013) and others. Tsai and Liao (2013) suggest a numerical search procedure which specifies designs based on blocks of size two for $4 \leq \upsilon \leq 7$, and $\upsilon \leq b \leq \frac{1}{2} \upsilon (\upsilon - 1)$ and all 34 designs are shown to be maximally robust.

Most of the available conditions for maximal robustness of an arbitrary design $D$, where block sizes may be bigger than two and indeed need not all be the same, are of two types. The elementary conditions only involve simple design parameters, namely block sizes and treatment replication numbers; useful deductions can be made about the robustness status of $D$ from the elementary conditions, particularly when such design parameters are large, as can be seen from Theorems 3, 4 and Lemma 3 of Godolphin and Warren (2011). On the other hand, the minimal concurrence conditions require further information about the design, notably the smallest concurrence $\lambda_*$ and the smallest weighted concurrence $\kappa_*$ between any pair of the $\upsilon$ treatments. These minimal concurrence conditions take the form of positive bounds that must be exceeded by $\lambda_*$ or by $\kappa_*$ to ensure that $D_{\#,b_*}$ is connected. The conditions are satisfied by several standard design classes, thereby qualifying all designs in these classes as maximally robust; they include all balanced incomplete block designs, thus giving an alternative derivation of the classic result of Ghosh (1982), and all of the equi-replicate variance balanced designs listed in Gupta and Jones (1983).

However, although the minimal concurrence conditions are more sensitive than the elementary conditions, in general, they do suffer from a serious drawback which makes them unsuitable for a large class of designs. Let $\mathcal{D}$ denote the set of binary designs with the simple property that at least two treatments fail to appear together in any block of the design. When $D \in \mathcal{D}$ the smallest concurrence and the smallest weighted concurrence of $D$ will be zero so neither term will exceed a positive bound and the minimal concurrence conditions cannot be satisfied. Thus all designs in $\mathcal{D}$ are effectively excluded from consideration of their robustness properties by using criteria based on minimal concurrence. This is clearly undesirable since $\mathcal{D}$ is a large class which contains many interesting designs. In particular, many designs in which all blocks have size two are members of $\mathcal{D}$ including, for example, 30 of the 34 small designs given by Tsai and Liao (2013). More generally, many partially balanced incomplete block (PBIB) designs belong to $\mathcal{D}$, including the majority of those listed in Clatworthy (1973). Any design with doubtful robustness status or any design which generates disconnected eventual designs through loss of one or two blocks may be contained in $\mathcal{D}$. These considerations suggest that an alternative criterion for assessing the robustness of a design to the loss of
whole blocks is required which makes better use of the information provided by the treatment concurrences and which, in particular, clarifies the robustness status of designs in $D$.

In this paper an approach for measuring design robustness is suggested which is based on a rectangular concurrence criterion that involves the concurrences of many pairs of treatments. This approach compares the sum of elements of selected upper non-principal sub-matrices of $NN'$ against a derived upper bound, subject to the provision that, when little is known in advance about $D$, the entries in these sub-matrices are assumed to be the smallest possible. This is an extension of the minimal concurrence criterion and gives a more sensitive measure of robustness, in general, and it implies that a lower bound for the block breakdown number of an arbitrary design with block-size two is the sum of a subset of these concurrence values. These results are derived and described in Section 2 of the paper. For two-associate PBIB designs, even sharper bounds are available. In particular, close bounds are derived for group divisible designs which are improvements on the bounds given by Sathe and Satam (1992) and imply that two of the three categories of group divisible designs cited by Bose and Connor (1952) are maximally robust. These results are established in Section 3 and a number of illustrative examples are given.

2. General Conditions for Robustness

2.1. Preliminary Considerations

Let $k_i$ be the size of the $i$th block of $D$ for $i = 1, \ldots, b$ and let $k[1] \geq \cdots \geq k[b]$ denote the block sizes arranged in decreasing order. Similarly, let $r[1] \geq \cdots \geq r[\upsilon]$ be the treatment replication numbers. Since $r[\upsilon]$ is the largest value possible for the block breakdown number of $D$, it is assumed that the number of missing blocks, $b^*$, is an integer in the range $1 \leq b^* \leq r[\upsilon] - 1$. Let $x_0 = 1$ and, for each $h = 1, 2, \ldots$, define integer-valued sequences $\{x_h\}$ and $\{y_h\}$ by

$$x_h = r[\upsilon - y_h + 1] - b^*, \quad \text{and} \quad y_h = k[b - x_h - 1 + 1].$$

(1)

Both sequences $\{x_h\}$ and $\{y_h\}$ are monotonically nondecreasing and terminate at the stop values $x_{#}, b^*$ and $y_{#}, b^*$, respectively, where $1 \leq x_{#}, b^* \leq r[1] - b^*$ and $k[b] \leq y_{#}, b^* \leq k[1]$. These stop values have the following property. Let $S$ be a nonempty set of the blocks of $D_{#}, b^*$ such that any treatment which occurs in a block belonging to $S$ has all of its replicates occurring in blocks contained in $S$. Then the number of blocks in $S$ is at least as large as $x_{#}, b^*$ and the number of treatments occurring in blocks belonging to $S$ is at least as large as $y_{#}, b^*$. If $D_{#}, b^*$ is disconnected it follows from the $P$-process of Godolphin (2004) that there are two non-empty and non-overlapping sets of blocks $S_1$, $S_2$ with this inclusive treatment property. Hence $D_{#}, b^*$ is connected whenever $y_{#}, b^* > \frac{1}{2} \upsilon$.

Lemma 1. If $y_{#}, b^* > \frac{1}{2} \upsilon$ then the block breakdown number of $D$ exceeds $b^*$.

Lemma 1 generalizes Lemma 3 of Godolphin and Warren (2011) and implies that robustness considerations are only required for ‘small-block’ designs where $y_{#}, b^* \leq \frac{1}{2} \upsilon$. Further, when $k_i = k$ for $i = 1, \ldots, b$, then also $y_{#}, b^* = k$ for $1 \leq b^* \leq r[\upsilon] - 1$ from equation (1). Hence Lemma 1 implies the following basic result.
Theorem 1. If the blocks of a binary design $D$ are of the same size $k$ and $k > \frac{1}{2}v$ then the block breakdown number of $D$ is $r_{[\nu]}$, i.e. $D$ is maximally robust.

2.2. Rectangular Concurrence Conditions

When $D$ is a ‘small-block’ design, as indicated by Lemma 1 or Theorem 1, it is useful to establish conditions for assessing robustness against the loss of whole blocks that improve upon the criteria of minimal concurrence and weighted minimal concurrence that have been used previously. It is assumed that the information on which this assessment is based consists of values of the concurrence for the $\frac{1}{2}v(v - 1)$ treatment pairs. Writing $N = [N_1 \; N_2 \; \ldots \; N_b]$, the concurrence matrix of $D$ is given by

$$NN' = \sum_{i=1}^{b} N_i N_i'$$

(2)

where $N_i N_i'$ is a $v \times v$ matrix such that

$$((N_i N_i'))_{j\ell} = \begin{cases} 1 & \text{if } \ell \neq j \text{ and treatments } j, \ell \text{ appear together in the } i\text{th block;} \\ 1 & \text{if } \ell = j \text{ and treatment } j \text{ appears in the } i\text{th block;} \\ 0 & \text{otherwise.} \end{cases}$$

(3)

For some integer $\theta$, such that $1 \leq \theta \leq \frac{1}{2}v$, let $T_1$, $T_2$ denote subsets of the set of $v$ treatments represented by $T_1 = \{1, 2, \ldots, \theta\}$ and $T_2 = \{\theta + 1, \ldots, v\}$. Put

$$N_i N_i' = \begin{bmatrix} N_{11,i} & N_{12,i} \\ N_{12,i}' & N_{22,i} \\ \end{bmatrix},$$

so that $N_{12,i}$ is the upper non-principal sub-matrix of order $\theta \times (v - \theta)$, and denote

$$\gamma_i = 1_{\theta} N_{12,i} 1_{v-\theta} \quad \text{and} \quad \phi_i = \gamma_i / k_i;$$

(4)

i.e. $\gamma_i$ and $\phi_i$ are, respectively, the concurrence totals and the weighted concurrence totals which are formed in the $i$th block between the treatments in $T_1$ and the treatments in $T_2$. It is straightforward to find upper bounds for $\gamma_i$ and $\phi_i$:

**Lemma 2.** Let the $i$th block of $D$ have size $k_i$. Then

$$\gamma_i \leq q(k_i, \theta) \quad \text{and} \quad \phi_i \leq p(k_i, \theta)$$

(5)

where $\gamma_i, \phi_i$ are given by (4) and $q(k, \theta), p(k, \theta)$ are defined by

$$q(k, \theta) = k \{\theta\} (k - k\{\theta\}), \quad p(k, \theta) = \frac{q(k, \theta)}{k}$$

(6)

such that $k\{\theta\} = \min\left(\lfloor \frac{1}{2}k \rfloor, \theta \right)$, with $\lfloor \frac{1}{2}k \rfloor$ being the integer part of $\frac{1}{2}k$.

The terms $q(k_i, \theta)$ and $p(k_i, \theta)$ are due to Sathe and Satam (1992) and represent best possible upper bounds for $\gamma_i$ and $\phi_i$ in the absence of further information about the
configuration of $D$. Clearly $q(k; \theta_1) \leq q(k; \theta_2)$ when $1 \leq \theta_1 \leq \theta_2 \leq \frac{1}{2}v$ and $q(k_{[i]}, \theta) \leq q(k_{[\ell]}, \theta)$ when $h \geq \ell$. Further, from Result 2 of Sathe and Satam (1992, p97) it follows that

$$\frac{q(k_{[i]}, \theta_2)}{\theta_2(v - \theta_2)} \leq \frac{q(k_{[i]}, \theta_1)}{\theta_1(v - \theta_1)}. \tag{7}$$

Similar monotonic properties of ordering apply to $p(k, \theta)$.

The notion of rectangular concurrences and rectangular weighted concurrences is based on an observation of Eccleston and Hedayat (1974) that the concurrence matrix of a disconnected design can be made block diagonal, which raises the following question. Let $b_s$ blocks be lost from $D$: after suitable rearrangement of the rows and columns of the concurrence matrix of $D_{\#, b_s}$, can there exist an upper non-principal sub-matrix of order $\theta \times (v - \theta)$ which consists entirely of zeros, where $1 \leq \theta \leq \frac{1}{2}v$? If there exists a sub-matrix of this kind then the corresponding components $N_i N'_i$ are similarly block diagonal. To examine this question it is helpful to define $\Lambda_{\theta}$ to be the sum of the $\theta(v - \theta)$ smallest terms from the set of $\frac{1}{2}v(v - 1)$ concurrence values for pairs of treatments of $D$. Similarly, define $\Omega_{\theta}$ to be the sum of the $\theta(v - \theta)$ smallest terms from the set of weighted concurrences. Also let the corresponding average values be denoted by the terms

$$\bar{\Lambda}_\theta = \frac{\Lambda_{\theta}}{\theta(v - \theta)} \text{ and } \bar{\Omega}_{\theta} = \frac{\Omega_{\theta}}{\theta(v - \theta)}. \tag{8}$$

The following inequalities are derived immediately from the definitions of $\Lambda_{\theta}$ and $\Omega_{\theta}$.

**Lemma 3.** Let $1 \leq \theta_1 \leq \theta_2 \leq \frac{1}{2}v$. Then

$$\Lambda_{\theta_1} \leq \Lambda_{\theta_2} \text{ and } \lambda_s \leq \bar{\Lambda}_{\theta_1} \leq \bar{\Lambda}_{\theta_2} \tag{9}$$

$$\Omega_{\theta_1} \leq \Omega_{\theta_2} \text{ and } \kappa_\ast \leq \bar{\Omega}_{\theta_1} \leq \bar{\Omega}_{\theta_2}. \tag{10}$$

For convenience $\Lambda_{\#}$ and $\Omega_{\#}$ are used in place of $\Lambda_{\theta}$ and $\Omega_{\theta}$ whenever $\theta$ is the stop value $\theta = \theta_{\#}$. The focus of the theorems presented in this paper is the provision of bounds for the concurrence subtotal $\Lambda_{\#}$ and the weighted concurrence subtotal $\Omega_{\#}$, rather than for the minimal concurrence $\lambda_s$ and minimal weighted concurrence $\kappa_s$, respectively, which formed the focus of robustness theorems given previously in the literature.

**Theorem 2.** Suppose that $b_s$ blocks are lost from a binary block design $D$, where $b_s$ is a fixed integer satisfying $1 \leq b_s \leq r_{[v]} - 1$. Let $x_0 = 1$ and for each $h = 1, 2, \ldots$ define $x_h$ and $y_h$ by (1) and assume that the stop values $x_{\#, b_s}$ and $y_{\#, b_s}$ are such that $y_{\#, b_s} \leq \frac{1}{2}v$. If either of the two conditions

$$(i) \quad \Lambda_{\#} > \sum_{j=1}^{b_s} q(k_{[j]}, y_{\#, b_s}) \quad \text{or} \quad (ii) \quad \Omega_{\#} > \sum_{j=1}^{b_s} p(k_{[j]}, y_{\#, b_s}), \tag{11}$$

is satisfied, where $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$ are defined by (6), then $D_{\#, b_s}$ is a connected design, i.e. the block breakdown number of $D$ exceeds $b_s$.

**Proof:** Suppose that $b_s$ blocks are removed from $D$ which, without loss of generality, can be taken to be the first $b_s$ blocks of $D$. It is assumed that the eventual design $D_{\#, b_s}$ is disconnected; as a consequence, conditions (11) will be obtained by contradiction.

© 2014 Australian Statistical Publishing Association Inc.
Prepared using anzsauth.cls
The \( P \)-process of Godolphin (2004) implies that the \( v \) treatments can be divided into two non-empty and non-overlapping subsets \( T_1 \) and \( T_2 \) such that all replicates of \( T_1 \) occur in a proper subset of the blocks of \( D_{\#,b_*} \) and all replicates of \( T_2 \) occur in the remaining blocks of \( D_{\#,b_*} \). Without loss of generality it can be supposed that treatments are labelled such that \( T_1 = \{1, 2, \ldots, \theta\} \) and \( T_2 = \{\theta + 1, \ldots, v\} \), where now \( \theta \) is interpreted as an integer which is unknown and has a value in the range \( 1 \leq \theta \leq \frac{1}{2} v \). From (2) the concurrence matrix of \( D \) can be expressed as

\[
NN' = \sum_{i=1}^{b_*} N_iN_i' + \sum_{i=b_*+1}^{b} N_iN_i'.
\]  

(12)

Furthermore the second term on the right of (12) is the concurrence matrix of \( D_{\#,b_*} \) and is therefore a block diagonal matrix since \( D_{\#,b_*} \) is disconnected. It follows from (4) that \( \gamma_i = 0 \) for each \( i = b_* + 1, \ldots, b \), so that, from (5),

\[
\lambda_{\theta} \leq \sum_{i=1}^{b_*} q(k_i, \theta) \leq \sum_{i=1}^{b_*} q(k_i, \theta)
\]

where the final inequality on the right is required because, under normal circumstances, it is not possible to anticipate the missing \( b_* \) blocks in advance so their block sizes are unknown and are replaced by the \( b_* \) largest block sizes. Therefore,

\[
\overline{\lambda}_{\theta} \leq \frac{\sum_{i=1}^{b_*} q(k_i, \theta)}{\theta(v - \theta)}.
\]

(13)

However, neither side of the inequality (13) is identifiable because \( \theta \) is unknown. Using (7) a suitable upper bound for the right hand side of (13) is obtained by replacing \( \theta \) by \( y_{\#,b_*} \). Furthermore, since \( y_{\#,b_*} \leq \theta \leq \frac{1}{2} v \) then \( \overline{\lambda}_{\#} \leq \overline{\lambda}_{\theta} \) from (9). This shows that a sufficient condition for \( D_{\#,b_*} \) to be connected after the loss of any \( b_* \) blocks is given by

\[
\overline{\lambda}_{\#} > \frac{\sum_{i=1}^{b_*} q(k_i, y_{\#,b_*})}{y_{\#,b_*}(v - y_{\#,b_*})}
\]

which is condition (i) of the theorem. Condition (ii) is derived in a similar way. \( \square \)

**Example 1.** Consider an equi-replicate cyclic design in which four replicates of ten treatments are arranged in ten blocks of size four:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 \\
5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

where columns show the blocks. Ten pairs of treatments have concurrence zero, ten pairs have concurrence unity and 25 pairs have concurrence two. Hence \( \lambda_* = \kappa_* = 0 \) and this design is a member of the class \( D \). Let \( b_* = 3 \). Then \( y_{\#,3} = 4 \) and \( y_{\#,3}(v - y_{\#,3}) = 24 \) so the concurrence subtotal \( \Lambda_{\#} = \Lambda_4 = 10 \times 0 + 10 \times 1 + 4 \times 2 = 18 \); since each of the three terms on the right side of (11) (i) has value 4, condition (11) (i) becomes \( 18 > 12 \) which is
valid and shows that the block breakdown number is 4, i.e. the design is maximally robust. The same conclusion is reached using condition (11) (ii).

The following corollary of Theorem 2 is a remarkably simple result which shows the central role held by treatment concurrences when determining lower bounds for the block breakdown numbers of arbitrary designs in which all blocks have size two.

**Corollary.** Let $D = BD(v, b, 2b, N)$ be a binary block design such that all blocks have size two. Then the breakdown number of $D$ has lower bound $\min \{ \Lambda_2, r[v] \}$.

**Proof:** Since all blocks have the same size then conditions (i) and (ii) in (11) are equivalent. Furthermore condition (1) gives $y_{\#.,.b.} = 2$ so if $v \geq 4$ then (11) (i) becomes $\Lambda_2 > \sum_{i=1}^{b/2} q(2, 2)$. But $q(2, 2) = 1$ from (6) and the result follows.

The corollary provides a sharp bound for the breakdown number of some designs with block-size two, as demonstrated by the following example.

**Example 2.** Clatworthy (1973) gives four regular group divisible designs, cited as $R_{14}$, $R_{15}$, $R_{16}$ and $R_{17}$, which all have ten replicates of four treatments arranged in twenty blocks of size two. Their average efficiency factors are 0.42, 0.60, 0.66 and 0.65 respectively and it is interesting to see if their block breakdown numbers rank them in a similar way. For instance, the design $R_{14}$ is displayed as follows:

```
1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3 4 3 4
```

where columns show the blocks. Of the six treatment pairs, four have concurrence unity and two have concurrence 8, so $\Lambda_2 = 4$, giving a bound of 4 for the block breakdown number. This bound is achieved: if the last four blocks of $R_{14}$, as displayed above, are removed then eight of the remaining blocks contain treatments 1 and 3 and eight contain treatments 2 and 4, i.e. the eventual design is disconnected. Similarly, the corollary gives a bound of 8 for the block breakdown number for $R_{15}$, which is also achieved, and it gives a common bound of 10 for the other designs, i.e. $R_{16}$ and $R_{17}$ are maximally robust.

Theorem 2 gives the rectangular concurrence conditions for any binary design. The information needed for the method is the same as that required for the method of minimal concurrence; however the rectangular concurrence criterion is superior, as Example 1 demonstrates clearly. Nevertheless further improvements to Theorem 2 and its corollary can be made if knowledge of the design configuration makes it permissible to assume that upper non-principal sub-matrices of $NN'$ include terms that are not necessarily the smallest possible. This idea is investigated for some PBIB designs in Section 3.

## 3. Conditions for Partially Balanced Two-Associate Designs

### 3.1. PBIB[2] Designs

A PBIB design $D$ with two associate classes, denoted PBIB[2], can be represented by the parameter set $(v, b, r, k; (\lambda_1, \lambda_2))$, where $v$ equally replicated treatments with replication number $r$ are arranged in $b$ blocks of size $k$ such that any two treatments that are first associates occur together in $\lambda_1$ blocks whilst any two treatments that are second associates...
occur together in \( \lambda_2 \) blocks (\( \lambda_1 \neq \lambda_2 \)). Each treatment has \( n_1 \) first associates and \( n_2 \) second associates, and (Bose and Connor, 1952)

\[
vr = bk, \ n_1 + n_2 = v - 1 \text{ and } n_1 \lambda_1 + n_2 \lambda_2 = r(k - 1),
\]

(14) where it is assumed that \( k \leq \frac{1}{2}v \). The design \( D \) has the number of first-associate treatment pairs given by \( \frac{1}{2}vn_1 \) and the number of second-associate treatment pairs given by \( \frac{1}{2}vn_2 \). Extend these terms to a proper subset, \( T \) say, of \( \omega \) treatments of \( D \), where \( \omega \) satisfies \( k \leq \omega \leq \frac{1}{2}v \). Let \( \pi_T \) be the number of first-associate treatment pairs from treatments belonging to \( T \) so that the number of second-associate treatment pairs from treatments belonging to \( T \) is \( \frac{1}{2}\omega(\omega - 1) - \pi_T \); then define

\[
\Pi_\omega = \begin{cases} 
\text{maximum value of } \pi_T \text{ over all sets of } \omega \text{ treatments when } \lambda_1 > \lambda_2; \\
\text{minimum value of } \pi_T \text{ over all sets of } \omega \text{ treatments when } \lambda_1 < \lambda_2.
\end{cases}
\]

(15)

**Theorem 3.** Let \( D \) be a PBIB\([2]\) design with parameter set \((v, b, r, k; (\lambda_1, \lambda_2))\) and let \( W = \{k, k + 1, \ldots, \lfloor \frac{1}{2}v \rfloor\} \). A lower bound for the block breakdown number of \( D \) is given by

\[
\min_{\omega \in W} \left\{ \frac{8[\omega r(k - 1) - \omega(\omega - 1)\lambda_2 + 2\Pi_\omega(\lambda_2 - \lambda_1)]}{2k^2 - 1 + (-1)^k} \right\},
\]

(16)
or \( r \), whichever is the smaller, where \( \Pi_\omega \) is defined by (15).

**Proof:** Let \( b_* \) blocks be removed from \( D \), where \( 1 \leq b_* \leq r - 1 \), and suppose that the eventual design \( D_{\#, b_*} \) is disconnected. Then the \( v \) treatments can be placed into two non-empty and non-overlapping subsets \( T_1 \) and \( T_2 \) of sizes \( \theta \) and \( v - \theta \), where \( k \leq \theta \leq \frac{1}{2}v \), such that all replicates of treatments in \( T_1 \) occur in some blocks of \( D_{\#, b_*} \) and all replicates of treatments in \( T_2 \) occur in the remaining blocks of \( D_{\#, b_*} \). Without loss of generality, the treatments are labelled such that \( T_1 = \{1, 2, \ldots, \theta\} \) and \( T_2 = \{\theta + 1, \ldots, v\} \).

As in Theorem 2 it is required to find a lower bound for \( \sum_{i=1}^{b_*} \gamma_i \), the sum of entries in the upper non-principal \( \theta \times (v - \theta) \) sub-matrix of \( NN' \), and this is achieved by first evaluating the vector \( \sum_{i=1}^{b} N_{12,i} 1_{v - \theta} \). Let treatment \( j \in T_1 \) and let \( \xi_j \) be the number of first associates of treatment \( j \) confined to \( T_1 \), so the number of second associates of \( j \) confined to \( T_1 \) is \( \theta - \xi_j - 1 \). The \( j \)th element of \( \sum_{i=1}^{b} N_{12,i} 1_{v - \theta} \) is the sum of the entries in the \( j \)th row of the sub-matrix, which is given by

\[
(n_1 - \xi_j)\lambda_1 + (n_2 - \theta + \xi_j + 1)\lambda_2 = r(k - 1) - (\theta - 1)\lambda_2 + \xi_j(\lambda_2 - \lambda_1), \quad (1 \leq j \leq \theta)
\]

(17) using (14). The sum of the terms in the upper \( \theta \times (v - \theta) \) component of \( NN' \) is

\[
\sum_{i=1}^{b} \gamma_i = \sum_{i=1}^{b} N_{12,i} 1_{v - \theta} = \theta \{ r(k - 1) - (\theta - 1)\lambda_2 \} + 2\pi_{T_1}(\lambda_2 - \lambda_1),
\]

noting that \( \pi_{T_1} = \frac{1}{2} \sum_{j=1}^{\theta} \xi_j \) represents the number of first associate treatment pairs from treatments confined to \( T_1 \); therefore from (15) it follows that

\[
\sum_{i=1}^{b_*} \gamma_i \geq \Phi_\theta \geq \min_{\omega \in W} \Phi_\omega,
\]

(18)
where $\Phi_\omega$ is defined by

$$
\Phi_\omega = \omega r (k - 1) - \omega (\omega - 1) \lambda_2 + 2 \Pi_\omega (\lambda_2 - \lambda_1),
$$

(19)

noting that the term $\min_{\omega \in W} \Phi_\omega$ is required on the right of (18) because $\theta$ is unknown.

Furthermore, $\sum_{i=1}^{b_*} \gamma_i \leq \sum_{i=1}^{b_*} q(k, \theta) = b_* q(k, \theta)$ using inequality (5). But since $\theta \geq k$, a straightforward algebraic argument gives $q(k, \theta) = \frac{1}{8} \{2k^2 - 1 + (-1)^k\}$, hence

$$
\sum_{i=1}^{b_*} \gamma_i \leq b_* \frac{1}{8} \{2k^2 - 1 + (-1)^k\}.
$$

(20)

It follows from (18) and (20) that the number of missing blocks, $b_*$, must be at least as large as the quantity (16). This implies that a lower bound for the block breakdown number of $D$ is (16) or $r$, whichever is the smaller.

**Example 3.** Bose (1963) has given a PBIB[2] design based on a cyclic association scheme in which three replications of thirteen treatments are arranged in thirteen blocks of size three, with $n_1 = n_2 = 6$ and $\lambda_1 = 1, \lambda_2 = 0$

\begin{center}
\begin{tabular}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 1 & 2 \\
9 & 10 & 11 & 12 & 13 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{tabular}
\end{center}

where columns show the blocks. The Bose design is in $D$; in fact the number of second-associate treatment pairs is $\frac{1}{2} \nu n_2 = \frac{1}{2} \times 13 \times 6 = 39$, i.e. 39 of the 78 treatment pairs have zero concurrence, from which it follows that the concurrence subtotals $\Lambda_\# = \Omega_\# = 0$ so the conditions (11) of Theorem 2 are not satisfied. Nevertheless the robustness status of the design is ascertained easily using the approach of Theorem 3. The set $W$ for the Bose design consists of values $\{3, 4, 5, 6\}$, and the corresponding values of $\Pi_\omega$ and $\Phi_\omega$ are $\{3, 5, 7, 10\}$ and $\{12, 14, 16, 16\}$ respectively. Therefore expression (16) for this design has value $\frac{8 \times 12}{16} = 6$. Also the replication number is $r = 3$, consequently the block breakdown number is 3. It follows that the Bose design is maximally robust.

The term $\Phi_\omega$ is defined entirely in terms of elements of $NN'$ and $r, k, \lambda_1, \lambda_2$ and is obtained from (19) in a straightforward way when the design is not large. Moreover, $\min_{\omega \in W} \Phi_\omega$ gives a realizable lower bound through (18) for the sum of the entries in any upper non-principal sub-matrix of $NN'$ of order $\omega \times (\nu - \omega)$. This realizable bound is likely to be much sharper than that obtained from the sum of the $\omega \times (\nu - \omega)$ smallest possible values; for instance, $\min_{\omega \in W} \Phi_\omega = 12$ in Example 3 although the concurrence subtotal $\Lambda_\# = 0$. Hence Theorem 3 gives an improved lower bound for the block breakdown number of a partially balanced design, compared to that of Theorem 2. It is shown in what follows that this is particularly useful for the class of group divisible designs, since it is characteristic of these designs that general expressions for $\Pi_\omega$ can be found. This enables several general conclusions to be made about group divisible designs.

### 3.2. Group Divisible Designs

Group divisible designs are a useful class of designs for incomplete block experiments when the number of treatments $\nu = mn$ is composite. Many group divisible designs are
highly efficient; see Cheng (1978) and Cheng and Bailey (1991). The treatments are divided into \( m \) disjoint groups of size \( n \) such that all pairs of treatments belonging to the same group occur in \( \lambda_1 \) blocks and pairs of treatments from different groups occur together in \( \lambda_2 \) blocks. A group divisible design is a PBIB[2] design with parameter set \((v, b, r, k; (\lambda_1, \lambda_2))\), satisfying (14) such that \( n_1 = n - 1 \) and \( n_2 = n(m - 1) \), with the requirement that \( \lambda_2 > 0 \) to ensure that \( D \) is connected. Bose and Connor (1952) classified the group divisible designs into three categories, depending on the values of \( r - \lambda_1 \) and \( rk - v\lambda_2 \); the design is singular if \( r - \lambda_1 = 0 \); semi-regular if \( r - \lambda_1 > 0 \) and \( rk - v\lambda_2 = 0 \); regular if \( r - \lambda_1 > 0 \) and \( rk - v\lambda_2 > 0 \).

The hypothesis that all group divisible designs are maximally robust is not valid since the regular designs \( R14 \) and \( R15 \) of Clatworthy (1973) provide counter examples, as shown in Example 2. It is interesting to note that, in an investigation of the robustness of PBIB[2] designs against the loss of individual observations, Ghosh et al. (1983) provide a necessary and sufficient condition for group divisible designs to be robust against the loss of any \( r - 1 \) observations, namely \( n(n - 1)(m - 1)\lambda_2 \geq (n - 1)\lambda_1 - (k - 2) \). For group divisible designs with blocks of size two, the property of being robust against the loss of any \( r - 1 \) observations is equivalent to the property of being maximally robust and thus, for \( R14 - R17 \), the condition of Ghosh et al. confirms the results of Example 2. To cater for designs with \( k > 2 \), Theorem 3 implies the following useful result.

**Theorem 4.** Every group divisible design such that \( \lambda_1 < \lambda_2 \) is maximally robust.

**Proof:** It follows from conditions (14) with \( n_1 = n - 1 \) and \( n_2 = n(m - 1) \) that

\[
\lambda_2 \leq \frac{r(k - 1)}{n(m - 1)},
\]

with equality holding if and only if \( \lambda_1 = 0 \). To establish the theorem consider the set \( W = \{ k, k + 1, \ldots, \lfloor \frac{1}{2}m \rfloor \} \), let \( \omega \in W \) and define non-negative integers \( \alpha, \beta \) by \( \omega = \alpha m + \beta \), where \( 0 \leq \beta < m \). Since \( \lambda_1 < \lambda_2 \) it follows from (15) that \( \Pi_\omega \) is the smallest number of pairs of first associates amongst the \( \frac{1}{2}\omega(\omega - 1) \) treatment pairs in any set of \( \omega \) treatments. This arises when treatments are distributed as evenly as possible between the \( m \) groups, i.e. \( \alpha + 1 \) treatments in each of \( \beta \) groups and \( \alpha \) treatments in each of the other \( m - \beta \) groups, which implies the following explicit expression for \( \Pi_\omega \):

\[
\Pi_\omega = \frac{\alpha(\alpha + 1)}{2} \beta + \frac{\alpha(\alpha - 1)}{2}(m - \beta) = \frac{\alpha(\omega - (m - \beta))}{2}.
\]

Substitution of this formula for \( \Pi_\omega \) given by (21) in (16) yields an improved lower bound for the breakdown number of the group divisible design. This is given by

\[
\min_{\omega \in W} \left\{ \frac{8[\omega(v - \omega)\{(m - 1)\lambda_2 + \lambda_1\} + (\lambda_2 - \lambda_1)\beta(m - \beta)]}{m\{2k^2 - 1 + (-1)^k\}} \right\}.
\]

But in expression (22), the term \( \omega(v - \omega) \) increases monotonically for \( \omega \in W \) and the term \( (\lambda_2 - \lambda_1)\beta(m - \beta) \) is non-negative when \( \lambda_1 < \lambda_2 \); hence the lower bound (22) is greater

© 2014 Australian Statistical Publishing Association Inc.
Prepared using anzauth.cls
than or equal to the quantity
\[
\frac{4(v - k)}{mk} \{ (m - 1) \lambda_2 + \lambda_1 \} = \frac{4(v - k)}{nmk} \left\{ r(k - 1) + \lambda_1 \right\} 
\geq \frac{4r(v - k)(k - 1)}{v k} \geq r,
\]
using (14) and identity \(mn = v\), also noting that \(2 \leq k \leq \frac{1}{2} v\) so that \((v - k)/v \geq \frac{1}{2}\) and \((k - 1)/k \geq \frac{1}{2}\). Thus the block breakdown number is \(r\), which proves the theorem. \(\square\)

It is well known that the group divisible designs with \(m = 2\) and \(\lambda_2 = \lambda_1 + 1\) have been shown to be optimal by Conniffe and Stone (1975) and Cheng (1978) with respect to a large class of optimality criteria. Theorem 4 shows that the same designs are maximally robust. Two results which are obtained directly from Theorem 4 are cited for completeness in the following corollary.

**Proposition 1.** (i) Semi-regular group divisible designs are maximally robust.

(ii) Regular group divisible designs with \(\lambda_1 < \lambda_2\) are maximally robust.

**Proof:** (i) If \(D\) is a semi-regular group divisible design then, by definition, it has the property that \(\lambda_2 = rk/(mn)\); hence from (14) and noting that \(v > k\)
\[
\lambda_1 = \frac{r(k - m)}{m(n - 1)} < \frac{r k}{mn} = \lambda_2,
\]
giving Proposition 1 (i). Result (ii) follows immediately from Theorem 4.

**Example 4.** Tsai and Liao (2013) give a group divisible design with parameter set \((v, b, r, k; (\lambda_1, \lambda_2)) = (12, 36, 6, 2; (0, 1))\) as their Example 2. The design has 30 treatment pairs with concurrence \(\lambda_1 = 0\) and 36 pairs with concurrence \(\lambda_2 = 1\), so \(\Lambda_2 = 0\) and the corollary to Theorem 2 does not apply in this case. However the design is semi-regular so is maximally robust from Proposition 1, which is the conclusion that is also reached using the computational search procedure of Tsai and Liao (2013) and the condition of Ghosh et al. (1983).

Group divisible designs with \(\lambda_1 > \lambda_2\) have the property that both concurrences are positive and regular designs \(R14, R15\) of Clatworthy (1973) show that not all designs of this kind are maximally robust. However Theorem 3 implies the following result.

**Theorem 5.** Let \(D\) be a group divisible design such that \(\lambda_1 < \lambda_2\), let \(W\) be the integer set defined by \(W = \{ k, k + 1, \ldots, \left\lfloor \frac{1}{2} v \right\rfloor \}\) and define non-negative integers \(\gamma\) and \(\delta\) by \(\omega = \gamma n + \delta\) where \(0 \leq \delta < n\) and \(\omega \in W\). A lower bound for the block breakdown number of \(D\) is given by
\[
\min_{\omega \in W} \left\{ \frac{8[\lambda_2 \omega(v - \omega) + (\lambda_1 - \lambda_2)\delta(n - \delta)]}{2k^2 - 1 + (-1)^k} \right\},
\]
or \(r\), whichever is the smaller.

**Proof:** Let \(\lambda_1 > \lambda_2\). From (15), \(\Pi_\omega\) is the largest number of pairs of first associates amongst the \(\frac{1}{2} \omega(\omega - 1)\) treatment pairs in any set of \(\omega\) treatments, which arises when these treatments are arranged in \(\gamma\) complete groups of \(n\) treatments and the remaining \(\delta\) treatments form a
common group. Such an arrangement gives:

$$
\Pi_\omega = \frac{n(n-1)}{2} \gamma + \frac{\delta(\delta - 1)}{2} = \frac{\omega(n-1) - \delta(n-\delta)}{2}.
$$

(24)

Substitution of (24) in (16) gives the block breakdown number lower bound (23).

Proposition 2. Singular group divisible designs are maximally robust.

By definition a group divisible design \(D\) is singular if, in addition to the usual conditions, \(\lambda_1 = r\) and it follows from (14) that

$$
\lambda_2 = \frac{r(k-n)}{n(m-1)} < r = \lambda_1;
$$

(25)

furthermore, \(k\) is a multiple of \(n\) which implies that \(\delta = 0\), so it is only necessary to consider the case \(\omega = k\) when applying (23) to a singular design. It also follows from (25) that \(k \geq 2n\) because \(\lambda_2 > 0\). Since

$$
\frac{8\lambda_2(k(v-k))}{2k^2 - 1 + (-1)^k} \geq \frac{4r(k-n)(v-k)}{kn(m-1)};
$$

then from condition (23) it is clear that the block breakdown number exceeds \(r - 1\); i.e. \(D\) is maximally robust, thus establishing the result.

An alternative intuitive argument leading to Proposition 2 is suggested by a result of Bose and Connor (1952, Theorem 2) which shows that every singular design is obtained by substituting each treatment from a balanced incomplete block design by a group of \(n\) distinct treatments. In view of this result it seems reasonable to conjecture that Proposition 2 should follow as a logical consequence of Ghosh’s theorem, which asserts that all balanced incomplete block designs are maximally robust.

The condition (23) gives an improved bound compared to the bounds for assessing maximal robustness of group divisible designs that are given by the conditions of Sathe and Satam (1992; Theorem 2). For example, the Sathe-Satam bound for the case \(\lambda_1 > \lambda_2\) has the form of (23) except that it does not include the term \((\lambda_1 - \lambda_2)\delta(n-\delta)\) which may be relatively large. The point is well demonstrated by the following example.

Example 5. Consider the regular group divisible design \(R207\) of Clatworthy (1973, p.228) in which ten replications of 27 treatments are arranged in 27 blocks of size ten, with \(m = 3\), \(n = 9\) and \(\lambda_1 = 9\), \(\lambda_2 = 1\); the first block of \(R207\) is \((1\ 2\ 4\ 7\ 10\ 13\ 16\ 22\ 25)') and the remaining blocks are developed cyclically, modulo 27. Sathe and Satam’s conditions are not satisfied by this design and these authors concluded, incorrectly, that \(R207\) is not maximally robust. The expression (23) is, when \(\omega = 10\),

$$
\frac{8[\lambda_2\omega(mn-\omega) + (\lambda_1 - \lambda_2)\delta(n-\delta)]}{2k^2 - 1 + (-1)^k} = \frac{8}{200}[1 \times 10 \times 17 + 8 \times 1 \times 8] = 9.36;
$$

and larger values of 11.52, 12.96, 13.68 are obtained when \(\omega = 11, 12, 13\) respectively. Therefore the block breakdown number is 10 so the design \(R207\) is maximally robust.
For a more general comparison of the application of these two sets of conditions, Sathe and Satam found that five singular designs and 22 regular group divisible designs listed by Clatworthy (1973) failed to satisfy their conditions. However, all five singular designs and seven regular designs have sufficiently large block sizes and are maximally robust because of Theorem 1. Theorem 5 shows three other regular designs, i.e. $R_{188}$, $R_{198}$, $R_{207}$, are maximally robust. The remaining twelve regular designs are not found to be maximally robust, since Theorem 5 gives a lower bound for the block breakdown number which is less than $r$ in each case. However the bounds for the block breakdown number given by condition (23) of the theorem are: $r - 1$ for 5 designs ($R_2$, $R_{12}$, $R_{26}$, $R_{53}$, $R_{66}$), $r - 2$ for 3 designs ($R_4$, $R_{15}$, $R_{28}$) and, for four designs, $r - 3$ ($R_5$), $r - 4$ ($R_8$), $r - 5$ ($R_{11}$) and $r - 6$ ($R_{14}$). These results are the best possible since on examination it transpires that, in every one of the 12 cases, the lower bound for the breakdown number is realised.

Following publication of the Clatworthy catalogue, some additional group divisible designs have been proposed in the literature. Freeman (1976), John and Turner (1977), Dey (1977) and Sinha (1987) give, in total, 37 regular designs, all of which are maximally robust by Theorems 4 or 5, and one semi-regular design which is maximally robust by Proposition 1. Sinha and Kageyama (1989) and Arasu and Harris (1996) give constructions for families of semi-regular and regular group divisible designs, however it is difficult to draw general conclusions about the robustness status of designs belonging to these families.

Constructions of new families of group divisible designs with $m = 2$ and $k = 4$ are given by Hurd and Sarvate (2008), some of which are not maximally robust as demonstrated in Example 6. Rodger and Rogers (2010) generalize the three Clatworthy designs $S_2$, $S_4$ and $R_{96}$ to provide necessary and sufficient conditions for the existence of families of regular designs with the parameter sets $(3n, b, r; 4; (4, 2))$, $(3n, b, r; 4; (8, 4))$ and $(3n, b, r; 4; (4, 5))$ for certain values of $n$; all designs in the family generalizing $R_{96}$ are maximally robust by Theorem 4 and all designs in the other two families are maximally robust by Theorem 5, as shown in Example 7 for the case of the family generalizing $S_2$.

**Example 6.** A family of regular designs is given by a construction of Lemma 3 of Hurd and Sarvate (2008). Ten treatments are allocated to $10t + 15$ blocks, which consist of the blocks from $2t + 1$ small designs with block-size $k = 4$. These designs comprise: $t$ copies of a balanced incomplete block design (BIBD) with treatments labelled $1, \ldots, 5$ arranged in five blocks; $t$ copies of a BIBD with treatments $6, \ldots, 10$ in five blocks and a ‘link’ BIBD with treatments $1, \ldots, 10$ in fifteen blocks; thus the parameters of these designs are $m = 2$, $n = 5$, $\lambda_1 = 3t + 2$, $\lambda_2 = 2$ and $r = 4t + 6$. In particular, the designs which correspond to $t = 1$ or $t = 2$ are maximally robust. For designs with $t \geq 3$, Theorem 5 gives 13 as a lower bound for the breakdown number, but this is an underestimate; block losses confined to the ‘link’ BIBD show that the breakdown number is either 14 or 15, depending on allocation of treatments to this ‘link’. Since $r \geq 18$ for $t \geq 3$ it is immediately clear that none of these designs is maximally robust.

**Example 7.** A family of group divisible designs generalizing the Clatworthy design $S_2$ is given by Rodger and Rogers (2010). The design parameters for this family are $m = 3$, $n = 3s + 2$, $k = 4$, $\lambda_1 = 4$, $\lambda_2 = 2$, where $s$ is a non-negative integer, possibly not taking the value 3, so $r = 8s + 4$ from (14). The singular design $S_2$ has $s = 1$ and is maximally robust from Proposition 2, whilst all other designs in the family are regular. When $s > 1$ a lower
bound for the left hand side of inequality (23) is
\[
\frac{8[\lambda_2 k(v-k)]}{2k^2 - 1 + (-1)^k} = 2(v - 4) = 18s + 4 > r - 1.
\]
Thus, all designs in the family are shown to be maximally robust.


The bounds on the block breakdown number established in Section 2 are particularly useful in practice, since they can be applied to any binary incomplete block design and are not restricted to specific design types, or to designs with common block size or treatment replication. It should be noted that whilst the results focus on the loss of whole blocks, any design which is found to be robust against the loss of a number of blocks, \( b_s \) say, will also be robust against the loss of any \( b_s \) individual observations. Thus, for a design which is demonstrated to be maximally robust by the results of the paper, in the event that any set of \( r_{[u]} - 1 \) individual observations are lost, the eventual design is guaranteed to be connected. Whether observation loss tends to affect individual observations or whole blocks depends on the nature of the experimental units and blocks. For example, where blocks are batches of raw material, a contaminated batch would mean the loss of an entire block. Conversely, for example where blocks are forests and experimental units are individual trees, it is perhaps more likely that individual observations will be lost. For designs which are not identified as being maximally robust by Theorem 2, a useful extension of the results of Section 2 would be to investigate whether improved conditions can be developed to cover the situation of the loss of individual observations.

The results of Section 3 provide bounds with additional sensitivity which take account of the structure of PBIB[2] designs. For the group divisible designs, in Section 3.2 it is established that designs categorised as semi-regular and singular are all maximally robust. For the remaining category, namely regular designs, members of the class with \( \lambda_1 < \lambda_2 \) are established as being maximally robust. For regular designs with \( \lambda_1 > \lambda_2 \), a lower bound for the block breakdown number is easily obtained from Theorem 5. Work is in progress to establish results corresponding to those in Section 3.2 for PBIB[2]s which are not group divisible. This would avoid the need to identify the \( \Pi_\omega \) of Theorem 3 for these designs which can be tedious if the design is large.

References


© 2014 Australian Statistical Publishing Association Inc.
Prepared using anzauth.cls


