Deterministic amplification of Schrödinger cat states in circuit quantum electrodynamics

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We propose a dynamical scheme for deterministically amplifying photonic Schrödinger cat states based on a set of optimal state-transfers. The scheme can be implemented in strongly coupled superconducting cavity systems and is well suited to the capabilities of state of the art superconducting circuits. The ideal analytical scheme is compared with a full simulation of the open Jaynes-Cummings model with realistic device parameters. This amplification tool can be utilized for practical quantum information processing in non-classical continuous-variable states.

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Superpositions of two large coherent states with opposite phases, called Schrödinger cat states (SCSs) [11], are a canonical example of macroscopic superposed states and have great potential to open up new avenues for quantum technology, including continuous-variable (CV) quantum communication [2], quantum computation [3-5, 23], teleportation [6], and quantum metrology [7, 8]. Fault-tolerant CV quantum computing can be achieved using only linear optics if the size of the SCSs is appropriate to keep the orthogonality of coherent states as well as to prevent excessive decoherence [3-5]. These applications provide significant motivation to engineer SCS amplification.

Realisation of a large enough amplitude SCSs is non-trivial in general, and perfect amplification of SCSs of unknown size is not possible because amplification of coherent states (i.e., $|\alpha\rangle \rightarrow |G\alpha\rangle$ for $G > 1$), which could also amplify SCSs, introduces unavoidable excess noise [9]. However, probabilistic linear amplification of a coherent state with high fidelity is feasible by adding and subtracting single photons [10, 11]. Note that optical amplification operators, for example $\hat{a}\hat{a}^\dagger$ and $(\hat{a}^\dagger)^2$, behave differently to the one proposed here with respect to input $\alpha$ and amplification rate $G$ [12, 13] and a two-photon loss environment could be beneficial to create/stabilize the SCSs [14, 15]. An amplification scheme with two known amplitude SCSs [16, 17] and probabilistic methods of amplifying SCSs have recently been developed in quantum optics [6, 13].

The rapid development of circuit-quantum electrodynamics (QED) technology could provide a new platform for scalable quantum systems. The Josephson non-linearity allows the production of superconducting artificial atoms which can be coupled to a 3D cavity containing microwave photons. Sufficiently large SCSs ($\alpha \approx 2$) [14, 19] and generalized Fock states [20] have very recently been generated in a cavity field with the assistance of a superconducting qubit. The enhanced stabilization of SCSs in a cavity has been recently reported in a specially designed lossy environment [14, 21] and this architecture could be useful for robust quantum memory [22].

Thus, amplification of SCSs would benefit a wide range of CV quantum technologies, particularly a new type of quantum computation in circuit-QED [23].

In this Letter, we propose a deterministic scheme for amplifying an SCS in superconducting circuits. The key difference between optics-only and atom-assisted methods is that the amplification can be performed deterministically. Our scheme is inspired by the fact that applying the two-photon shift operation [24], $(E^\dagger)^2 : |m\rangle \rightarrow |m + 2\rangle$, one or more times on even/odd SCSs preserves their even/odd distribution of amplitudes while $(\hat{a}^\dagger)^2 : |m\rangle \rightarrow \sqrt{(m + 2)!/m!}|m + 2\rangle$ with a renormalization. We analyze and simulate the operation $\hat{E}^\dagger$ on a SCS in a cavity coupled to a transmon qubit in the presence of decoherence [25]. For this scheme, we find fast controls, which perform all the state-transfers required for $\hat{E}^\dagger$ operation on SCS with high fidelity well within realistic decoherence time ($\approx 6\mu s$) in circuit-QED.

We generalize the notion of amplification to the case where an initial SCS $|SC_{\alpha}^{\pm}\rangle$ is transformed by an operation $\hat{A}$ into a state $\hat{A}|SC_{\alpha}^{\pm}\rangle$ that approximates a coherent state superposition $|SC_{\alpha}^{\pm}\rangle$ where $\alpha' > \alpha$,

$$|SC_{\alpha'}^{\pm}\rangle \approx \hat{A}|SC_{\alpha}^{\pm}\rangle = \sum_{k=0}^{\infty} c_k |k + b\rangle \langle k|SC_{\alpha}^{\pm}\rangle, \quad (1)$$

where an even/odd Schrödinger cat state is given by

$$|SC_{\alpha}^{\pm}\rangle = N_{\alpha}^{\pm}(|\alpha\rangle \pm |\alpha\rangle), \quad (2)$$

for some normalisation $N_{\alpha}^{\pm}$ ($b > 0$). Due to destructive interference between $|\alpha\rangle$ and $|\alpha\rangle$, even SCSs only have even photon numbers while odd SCSs only have odd photon numbers. Note that $c_k$ is determined by the am-
FIG. 1. Fidelities \( F^\pm \) between \((\hat{E}^\dagger)^2|SC_{\alpha'}^\pm\rangle\) and amplified state \(|SC_{\alpha}^\pm\rangle\) for starting amplitudes \(\alpha = 1.0, 1.5, 2.0, 2.5\). (a) \( F_{\max}^{\pm} \) are 0.854, 0.947, 0.974, 0.988 with \( G \approx 1.725, 1.377, 1.229, 1.151 \). (b) \( F_{\max}^{\pm} \) are 0.681, 0.866, 0.960, 0.987 with \( G \approx 1.902, 1.422, 1.235, 1.151 \).

FIG. 2. Energy structure of transmon coupled to a cavity with \( \omega_r = 6\text{GHz} \) and \( \lambda = 0.1\text{GHz} \). Solid lines indicate a set of \( \Lambda \)-type levels \(|+\rangle, |\pm\rangle, |\mp\rangle\) for state-transfers and dashed lines are other eigenstates of the Hamiltonian in Eq. (4). The labels on the right hand side are the product states that approximate the eigenstates for large positive detunings \((|\omega_q - \omega_r|/\lambda \gg 1)\).

In general, the maximum fidelity \( F_{\max}^{\pm} \) approaches 1 for large \( \alpha \) but \( G \) also tends to 1, indicating no amplification at very large SCSs using \((\hat{E}^\dagger)^2\), but stabilisation on the input SCS persists. Interestingly, for small \( \alpha \), \((\hat{E}^\dagger)^2\) works better for even SCSs because \(|SC_{\alpha}^-\rangle\) for \( \alpha \approx 0 \) becomes a one-photon Fock state. This amplified small \(|SC_{\alpha}^\pm\rangle\) maps to a three-photon Fock state, which is very different to any odd SCS (e.g. \( F_{\max}^{\pm} < 0.8 \) for \( \alpha = 1.0 \) in the bottom figure of Fig. 1). This feature disappears for \( \alpha \approx 1.5 \) because \(|\alpha\rangle\) is sufficiently orthogonal to \(|-\alpha\rangle\).

Implementation of \( \hat{E}^\dagger \) in circuit-QED – Here we demonstrate a scheme for deterministically performing \( \hat{E}^\dagger \) on a cavity field with the assistance of an artificial atom. Due to large nonlinearity, circuit-QED provides a suitable parameter range for amplification of SCSs, and we show a state-transfer scheme adapted from the original idea of stimulated Raman adiabatic passage (STIRAP) in a cavity-QED setup [27, 30]. One important advantages of this scheme is that \( \hat{E}^\dagger \) is deterministic while \( \hat{E} \) is probabilistic, which has been proposed to experimentally measure expectation values of \( \hat{E}, \hat{E}^\dagger \) and \( \hat{E}^2 \). Conventional STIRAP has also been demonstrated in circuit-QED [32, 33].

Our amplification operation can be realised with a set of STIRAP-inspired pulses implemented within the qubit-cavity level structure in conjunction with dynamical control via varying local fluxes [34, 35]. In contrast to the conventional cavity-QED setup with a bare atomic \( \Lambda \)-level configuration, we use a set of \( \Lambda \)-type systems in the dressed Jaynes-Cummings (JC) model where the cavity and qubit are on resonance [25]. The key operation is efficient state-transfer from \(|+\rangle, n\rangle \) to \(|-\rangle, n\rangle\). This is compatible with the architecture for creating SCSs in [19].

We model a transmon coupled to a cavity by a gener-
An adiabatic sweep of the qubit frequency \( \omega_q \) into resonance with the cavity \( \omega_c \) transforms an initial state \( |e,n\rangle \) into the dressed state \( |+,n\rangle \). Next a microwave field is first applied between \( |+,n\rangle \) and \( |+,n+1\rangle \) with time dependent amplitude \( \epsilon^2(t) = |\epsilon^2| \exp{[-(t-\tau)^2/T^2]} \) and frequency \( \omega^2 \) (yellow dotted-dashed line), followed by another field driving the second transition \((|+,n\rangle \leftrightarrow |+,n+1\rangle)\) \( \epsilon^2(t) = |\epsilon^2| \exp{[-(t-\tau)^2/T^2]} \), \( \omega^2 \) in purple dashed). For a SCS, the \( |-,n+1\rangle \) state is unpopulated hence does not participate in the dynamics. There is non-zero overlap between the pulses determined by the temporal offset \( \tau \). The microwave frequencies are detuned \( \Delta_n \) from the \( |-,n+1\rangle \) state but satisfy the two-photon transition condition, \( \Delta_n^2 - \omega^2 = 2\lambda\sqrt{n+1} \). For efficient transfer of \( |+,n\rangle \rightarrow |-,n+1\rangle \), we require \( \tau > (\sqrt{2}-1)T \) and \( |\epsilon| T \gg 10^3 \). After the counterintuitive pulse sequence, a further adiabatic sweep of the transmon frequency back up out of resonance results in \( |g,n+1\rangle \), disentangling the atom from the cavity which is now in the state \( \sum c_n|n+1\rangle = \tilde{E}^\dagger \sum c_n|n\rangle \).

\[
\tilde{H}^\dagger = \omega_q \hat{a} \hat{a}^\dagger + \sum_j \frac{\omega_j^2}{2} |j\rangle \langle j| + \sum_{j,k} \lambda_{j,k} (\hat{a} \hat{a}^\dagger |j\rangle \langle k| + \hat{a}^\dagger \hat{a} |k\rangle \langle j|),
\]

for transmon energy levels \( j,k = \{ g,e,f,h,... \} \) and transmon-cavity couplings \( \lambda_{j,k} \). As shown in Fig. 2 when the transmon frequencies are far from resonance with the cavity, the bare states are given by \( |j,n\rangle \) with transmon state \( j \) and photon number \( n \), while they become dressed states at near resonance. Considering only two transmon levels, \( \{|g\rangle, |e\rangle\} \), the eigenstates are

\[
\begin{align*}
|+,n\rangle &= \cos \theta_n |e\rangle |n\rangle + \sin \theta_n |g\rangle |n+1\rangle, \\
|-,n\rangle &= -\sin \theta_n |e\rangle |n\rangle + \cos \theta_n |g\rangle |n+1\rangle,
\end{align*}
\]

where \( \omega_q = \omega_{qg} \) is the \( |g\rangle \leftrightarrow |e\rangle \) transition frequency, \( \lambda = \lambda_{g,e} \) is the qubit-cavity coupling, and \( 2\theta_n = \tan^{-1}(2\lambda\sqrt{n+1}/\Delta) \), with \( \delta = \omega_q - \omega_c \). Note that \( |+,n\rangle \approx |e,n\rangle \) and \( |-,n\rangle \approx |g,n+1\rangle \) for large \( \delta \), so if we start in \( |e,n\rangle \) far from resonance, the state adiabatically becomes \( |+,n\rangle \) near resonance \( \delta \approx 0 \).

The protocol for performing \( \tilde{E}^\dagger \), illustrated in Fig. 3, is as follows. (1) An SCS \( |SC_{\alpha^{(n)}c}\rangle \) is initialized in the cavity, with the qubit in \( |e\rangle_q \) and far detuned from the cavity frequency \( \omega_c \). (2) We adiabatically sweep the qubit frequency \( \omega_q \) into resonance with the cavity, slowly transferring system into a superposition of dressed states \( \sum_{n=0}^\infty c_n|+,n\rangle_{qc} \), where the \( c_n \) vanish for odd (even) \( n \) for even (odd) cat states. (3) A set of STIRAP-type pulses are performed in the manifolds of dressed states \( \{|+,n\rangle, |-,n\rangle, |-,n+1\rangle, \} \), in order, from the \( n \)-th to \( 0 \)-th manifold. This produces a final state \( \sum_{n=0}^\infty c_n|-,n\rangle \). (4) Finally, we blue detune the qubit frequency away from \( \omega_q \), thereby disentangling the qubit from the cavity and the final cavity state becomes \( \sum_{n=0}^\infty c_n|n+1\rangle_{qc} \) with the qubit in the ground state. In contrast to the original cavity QED proposals [28], \( \pi \) pulses can reset the qubit states \( |g\rangle_q \rightarrow |e\rangle_q \) directly without affecting the cavity state [19], and hence \( \tilde{E}^\dagger k \) can be performed by repeating the protocol.

For simulation, we examine a simplified driven JC Hamiltonian with two atomic levels [36, 37]

\[
\hat{H}_{tot} = \omega_r \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \omega_q \hat{\sigma}^z + \lambda (\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+) + \sum_{n} \sum_{j=1}^2 \epsilon_j^n(t) \left( e^{-i\omega_j^n t} \hat{a} + e^{i\omega_j^n t} \hat{a}^\dagger \right),
\]

where \( \omega_j^n \) are the frequencies the microwaves driving the cavity, and \( \epsilon_j^n \) their strengths and Pauli operators are \( \hat{\sigma}^- \) and \( \hat{\sigma}^+ \). We briefly note that this procedure in the driven JC system has a slightly different character to conventional STIRAP on a bared Λ-level atom with directly driven transitions (see Fig. 6 in the Supplementary Material). While the microwave driving terms in Eq. 9 couple all of the excitation subspaces of the undriven JC Hamiltonian, the Hamiltonian is only slightly perturbed...
of the procedure and hence to reduce decoherence to practical levels, we perform all the transfers simultaneously and find that this produces almost same fidelities as three independent state-transfer sets. We use a single \( \omega_1 \) which is shared between all transfers, adjusting \( \Delta_1 \) for each \( \Lambda \)-level system to find the appropriate value of \( \omega_2 \). Our simulation parameters are \( \lambda = 0.1 \text{ GHz}, \omega_0 = 6.0 \text{ GHz}, \tau = 0.57 \text{ \mu s}, T = 1 \text{ \mu s}, |\epsilon_1| = 10 \text{ MHz}, |\epsilon_2| = 35 \text{ MHz}, |\epsilon_2'| = 38 \text{ MHz}, |\epsilon_2''| = 50 \text{ MHz}, \omega_1 = 5.949 \text{ GHz}, \omega_0' = 5.749 \text{ GHz}, \omega_0'' = 5.603 \text{ GHz}, \omega_0''' = 5.501 \text{ GHz}, \Delta_0 = 10 \text{ MHz}, \Delta_2 = 14 \text{ MHz}, \Delta_4 = 32 \text{ MHz}. \) With these parameters, the total state-transfer time is approximately 6 \( \mu s \).

As shown in Figs. 4 and 5, the full simulation of the three sets without decoherence (\( \gamma_- = \gamma_\phi = \kappa = 0 \)) is performed with maximum fidelity above 0.96 for both \( \alpha = 0.7, 1.0 \). This clearly shows the shift operation \( \tilde{E}^\dagger \) has been performed on \( |SC_\alpha^+ \rangle \) and the components of the density matrix of SCSs are shifted by one Fock-basis element (see Fig. 8 in the Supplementary Material for details). Even without decoherence, a gap exists between the analytical and ideal cases, caused by the imperfections in the transfer method, and partly due to a small population of higher dressed states over \( |+, 0 \rangle \), which are not transferred. In Fig. 5 we present the Wigner functions of the final cavity state compared to an analytic SCS for \( \alpha = 0.7, 1.0 \) and a fringe pattern clearly appears with negative values. We see that the smaller SCS is amplified almost perfectly, while for \( \alpha = 1.0 \), phase shifts between the different number states causes distortion of the fringes in the Wigner functions. To model decoherence, we use \( \gamma_- = \gamma_\phi = 10 \kappa = 5.10 \text{ kHz} \) using realistic parameters from Ref. [39] and the dotted points in Fig. 4 show that decoherence almost linearly reduces \( F^{+ \rightarrow -} \) to 0.84 for our worst case (\( \alpha = 1.0 \) and \( \kappa = 1 \text{ kHz} \)).

**Summary and remarks** – We have demonstrated a scheme for deterministic amplification of microwave SCSs using \( (\tilde{E}^\dagger)^2 \) in circuit-QED. Based on a STIRAP-inspired state-transfer method, this amplification can be performed with a high fidelity under realistic decoherence. The deterministic and noisless method overcomes the barrier to probabilistic amplification tools in optics-only methods by utilising artificial atomic states. In CV quantum information processing using SCSs, \( \tilde{E}^\dagger \) can be used as a bit-flip operation, by switching the state parity with minimal amplification for \( \alpha > 2 \), while \( (\tilde{E}^\dagger)^2 \) can act as a stabilizer operation on SCSs. If one can perform either \( \tilde{E}^\dagger \) or \( (\tilde{E}^\dagger)^2 \) depending on the outcome of a parity measurement of the cavity state, it can be used for a discretized purification of SCSs. Taking an advantage of well-separated lower energy levels, fluxonium or flux qubits can be also used for \( \tilde{E}^\dagger \). The full analysis of driven JC system in the dressed-state representation for state-transfers will be investigated in the future.

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SUPPLEMENTARY NOTE

Evidence of STIRAP-type operations

Although this STIRAP-type operation behaves well-enough for our desired state-transfer (|+⟩, n → |−⟩, n), the details of the proposed state-transfer cannot be explained by conventional STIRAP in a bare Λ atomic system. In STIRAP, the overlap of the two pulse envelopes is a crucial parameter to determine the state-transfer efficiency [1]. In particular, efficient state-transfer only occurs for the counter-intuitive sequence of the two pulses (c1 first and c2 second).

We have examined the transfer efficiency of our scheme for the simplest transfer from |+, 0⟩ to |−, 0⟩ with detuning δ0 in Fig. 4. For positive τ, the behaviour is similar to the normal STIRAP counter-intuitive pulse sequence, with transfer efficiency rapidly increasing as τ increases, nearly reaching 1 plateauing. The efficiency then drops with decreasing overlap area. However, the suggested STIRAP-type operation also shows excellent state-transfer for the reverse pulse sequence.

In our parameter region, and without decoherence, the transfer efficiency is symmetric about τ = 0 (fully overlapped pulses). However, the transfer efficiency for reversed pulses is more sensitive to changes in δ0 and the length of pulse envelopes. Oscillations are seen in the transfer efficiency, indicating that the process may not be ‘as adiabatic’ as conventional STIRAP. These phenomena might be better understood in adiabatic Floquet theory [2] and we believe they are caused by the existence of energy levels outside the Λ-system [3]. Further detailed investigation of the driven JC system in the dressed-state representation for this STIRAP-type operation will be presented in later work [4].

Amplitude distribution for even SCSs

In order to perform $\hat{E}^\dagger$ efficiently and practically, the minimum number of STIRAP-type sets can be decided by the plot of amplitudes of SCSs. Fig. 7 shows that $|SC^+_{0,7}\rangle$ and $|SC^+_{1,0}\rangle$ have most of their populations in three Fock states, $\{|0\rangle, |2\rangle, |4\rangle\}$, with the population in $|6\rangle$ starting to contribute significantly for $|SC^+_{1,2}\rangle$. Thus, four or more sets of STIRAP-type operations are required for even SCSs at $\alpha > 1.2$ to obtain a high fidelity $\hat{E}^\dagger$. 

FIG. 6. Transfer efficiency of the STIRAP-like pulses as a function of the overlap between the two pulses.

FIG. 7. Photon number amplitudes for $|SC^+_{\alpha}\rangle$ for $\alpha = 0.7, 1.0, 1.2$.

FIG. 8. Simulated density matrix plots for (a) $|SC^+_{0,7}\rangle$, (b) $\hat{E}^\dagger|SC^+_{0,7}\rangle$, (c) $|SC^+_{1,0}\rangle$, and (d) $\hat{E}^\dagger|SC^+_{1,0}\rangle$ without decoherence. They indicate that the shift operation $\hat{E}^\dagger$ is performed with high fidelity on $|SC^+_{\alpha}\rangle$. 


Evidence of $\hat{E}^\dagger$ operation on $|SC^+_\alpha\rangle$

It is relatively straightforward to show the performance of $\hat{E}^\dagger$ with a density matrix of the final state. For example, we can write the density matrix of the initial even SCS
\[
\rho_{\text{int}} = |SC^+_\alpha\rangle\langle SC^+_\alpha| = \sum_{n,m=0}^{\infty} c_{nm}|2n\rangle\langle 2m|.
\tag{12}
\]
Then, if the one-photon shift operation has been performed,
\[
\rho_{\text{out}} = \hat{E}^\dagger |SC^+_\alpha\rangle\langle SC^+_\alpha|\hat{E} = \sum_{n,m=0}^{\infty} c_{nm}|2n+1\rangle\langle 2m+1|,
\tag{13}
\]
As shown in Fig. [8] the coefficient $c_{nm}$ have been preserved while the Fock basis has been shifted for $|SC^+_\alpha\rangle$ and $|SC^+_1\rangle$. This type of quantum process tomography has already been performed experimentally and SCSs in a high-Q cavity field can be measured through a low-Q cavity via a superconducting qubit [5]. One can assume that the initial SCS is prepared using the method of Ref. [5], and that the $\hat{E}^\dagger$ operation can be performed twice with the assistance of the sandwiched superconducting qubit. As explained in the protocol, the outcome state is ideally expected to be $|g\rangle_q|SC^+_\alpha\rangle_c$ with high fidelity. Using Ramsey interferometry, one can measure the qubit state-dependent phase shift of the cavity state, with detailed methods explained in the Supplementary Materials of Ref. [5], then one can perform tomography on the state in the high-Q cavity through the low-Q cavity. Alternatively, a parity measurement experiment can also be used to show the parity difference between initial state $|SC^+_\alpha\rangle$ and $\hat{E}^\dagger |SC^+_\alpha\rangle$ [6].

References