ATTRACTORS FOR THE NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AND THEIR PARABOLIC SINGULAR LIMIT

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Abstract. We apply the dynamical approach to the study of the second order semi-linear elliptic boundary value problem in a cylindrical domain with a small parameter ε at the second derivative with respect to the variable t corresponding to the axis of the cylinder. We prove that, under natural assumptions on the nonlinear interaction function f and the external forces $g(t)$, this problem possesses the uniform attractor $\mathcal{A}_\varepsilon$ and that these attractors tend as $\varepsilon \to 0$ to the attractor $\mathcal{A}_0$ of the limit parabolic equation. Moreover, in case where the limit attractor $\mathcal{A}_0$ is regular, we give the detailed description of the structure of the uniform attractor $\mathcal{A}_\varepsilon$, if $\varepsilon > 0$ is small enough, and estimate the symmetric distance between the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$.

Contents

1. Introduction

We consider the following semi-linear elliptic boundary value problem in an infinite cylinder $\Omega := \mathbb{R} \times \omega$:

$$a(\partial_t^2 u + \Delta_x u) - \varepsilon^{-1}\gamma\partial_t u - f(u) = g(t), \quad (t,x) \in \Omega, \quad u\big|_{\partial\omega} = 0, \tag{1.1}$$

where $\omega \subset\subset \mathbb{R}^n$ is a bounded domain of $\mathbb{R}^n$, $u = (u_1, \cdots, u_k)$ is an unknown vector-valued function, a and γ are given constant matrices which satisfy $a + a^* > 0$ and $\gamma = \gamma^* > 0$, f and g are given nonlinear interaction function and the external forces respectively which satisfy some natural assumptions (formulated in Section 2) and $\varepsilon > 0$ is a small parameter.

Elliptic boundary problems of the form (1.1) appear, e.g. under studying the equilibria or the traveling waves for the corresponding evolution equations of mathematical physics.

Date: October 11, 2011.
1991 *Mathematics Subject Classification.* 35B41, 35B45.
Key words and phrases. elliptic boundary value problems, dynamical approach, uniform attractors, regular attractors.

For instance, let us consider the following reaction-diffusion system in the unbounded cylindrical domain Ω:

$$\partial_\eta v = a\Delta_{(t,x)}v - f(v) - g(t-\varepsilon^{-1}\gamma\eta), \quad v\big|_{\partial\Omega} = 0, \tag{1.2}$$

where the variable $t \in \mathbb{R}$ remains to be spatial, the variable η plays the role of physical time and γ is a diagonal matrix. Thus, the external forces $g(\eta, t, x) := g(t - \varepsilon^{-1}\gamma\eta, x)$ in (1.2) have the form of a fast traveling (along the axis of the cylinder) wave (with the wave speed $\varepsilon^{-1}\gamma \gg 1$). Then, the problem of finding the traveling wave solution $v(\eta, t, x) := u(t - \varepsilon^{-1}\gamma\eta, x)$ of equation (1.2) which is modulated by the traveling wave external forcing, obviously, reduces to the study of elliptic problem (1.1). Another natural example is the following reaction-diffusion system in the cylinder Ω:

$$\partial_\eta v = a\Delta_{(t,x)}v - \varepsilon^{-1}\gamma\partial_t v - f(v) - g(t), \quad (t,x) \in \Omega, \quad u\big|_{\partial\Omega} = 0 \tag{1.3}$$

with the strong drift along the axis of the cylinder (which is described by the transport term $\varepsilon^{-1}\gamma\partial_t v$). Then, (1.1) is the equation on equilibria for problem (1.3).

It is convenient to scale from the very beginning the variable t as follows: $t' := \varepsilon^{-1}t$. Then, problem (1.1) reads

$$a(\varepsilon^2\partial_t u + \Delta_x u) - \gamma\partial_t u - f(u) = g_\varepsilon(t), \quad u\big|_{\partial\omega} = 0, \quad g_\varepsilon(t) := g(\varepsilon^{-1}t), \tag{1.4}$$

where we denote the new variable t' by t again for simplicity.

We are interested in the global structure of the set of bounded (with respect to $t \to \pm\infty$) solutions of problem (1.4). To this end, we use the so-called dynamical approach for the study of elliptic boundary value problems in cylindrical domains which has been initiated in [6] and [16], see also [2, 3], [7], [10], [11], [19, 20, 21, 22, 23, 24], [27, 28, 29] and the references therein for its further development. Following this approach, we introduce, for every $\tau \in \mathbb{R}$, the auxiliary elliptic boundary value problem:

$$\begin{cases} a(\varepsilon^2\partial_t u + \Delta_x u) - \gamma\partial_t u = g_\varepsilon(t), \quad (t,x) \in \Omega_+^\tau, \\ u\big|_{\partial\omega} = 0, \quad u\big|_{t=\tau} = u_\tau, \end{cases} \tag{1.5}$$

in the half-cylinder $\Omega_+^\tau := (\tau, +\infty) \times \omega$ equipped by the additional boundary condition $u|_{t=\tau} = u_\tau$ at the origin of the half-cylinder Ω_+^τ and the function u_τ is assumed to belong to the appropriate functional space $V_\varepsilon^p(\omega)$ which will be specified in Section 2. If problem (1.5) possesses a unique (bounded as $t \to +\infty$) solution (in certain functional class), for every $u_\tau \in V_\varepsilon^p(\omega)$, then (1.5) defines a dynamical process $\{U_{g_\varepsilon}^\varepsilon(t,\tau),\ t, \tau \in \mathbb{R},\ t \geq \tau\}$ via

$$U_{g_\varepsilon}^\varepsilon(t,\tau)u_\tau := u(t), \quad \text{where } u(t) \text{ solves (1.5)}, \quad U_{g_\varepsilon}^\varepsilon(t,\tau) : V_\varepsilon^p(\omega) \to V_\varepsilon^p(\omega). \tag{1.6}$$

Moreover, if this dynamical process possesses a global (uniform) attractor $\mathcal{A}_\varepsilon$, then this attractor is generated by all bounded (with respect to $t \to \pm\infty$) solutions of the initial problem (1.4) (and all its shifts along the t axis, together with their closure in the corresponding topology, see Section 4 for the details). Thus, studying of the bounded solutions of (1.4) is, in a sense, equivalent to the study of the attractor $\mathcal{A}_\varepsilon$ of auxiliary dynamical process (1.6).

In the present paper, we give a detailed study of auxiliary problems (1.5) in case ε is small enough and investigate their behavior as $\varepsilon \to 0$. The paper is organized as follows. The existence of a bounded solution $u(t)$ of problem (1.5) and several important estimates are derived in Section 2. The uniqueness of this solution is verified in Section 3 under the assumption that ε is small enough. Moreover, we show there that the dynamical process (1.6) associated with problem (1.5) is uniformly (with respect to ε) Frechet differentiable with respect to the 'initial data' $u_\tau \in V_\varepsilon^p(\omega)$. The existence of the uniform attractor $\mathcal{A}_\varepsilon$

for the process (1.6) is established in Section 4. Moreover, we prove there that, for rather wide class of the external forces g, the attractors $\mathcal{A}_\varepsilon$ converge as $\varepsilon \to 0$ (in the sense of upper semicontinuity) to the attractor $\mathcal{A}_0$ of the limit parabolic problem

$$\gamma\partial_t u - a\Delta_x u + f(u) = g_0(t), \quad u\big|_{\partial\omega} = 0, \quad u\big|_{t=\tau} = u_\tau, \tag{1.7}$$

where the limit external forces $g_0(t)$ average the external forces $g_\varepsilon(t) := g(\varepsilon^{-1}t)$ of problems (1.5). In particular, the class of admissible external forces g contains the autonomous external forces: $g(t) \equiv g_0$, heteroclinic profiles:

$$g(t) \to g_\pm \quad \text{as } t \to \pm\infty \text{ and } g_\pm \text{ are independent of } t, \tag{1.8}$$

solitary waves ($g_+ = g_-$ in (1.8)), periodic, quasiperiodic and almost-periodic with respect to t external forces g and even some classes of non almost-periodic oscillations, see Examples 4.9–4.12.

Furthermore, in Section 5, we prove that dynamical processes (1.6) tend as $\varepsilon \to 0$ to the process $U^0_{g_0}(t,\tau)$ associated with limit parabolic problem (1.7) and obtain the quantitative bounds for that convergence in terms of the parameter ε.

In Sections 6 and 7, we restrict ourselves to consider only the case of almost-periodic external forces $g(t)$ in the right-hand side of equation (1.5). In this case, limit parabolic equation (1.7) is autonomous

$$g_0(t) := \bar{g}, \tag{1.9}$$

where $\bar{g}$ is the mean of almost-periodic function $\bar{g}$. We also assume that the global attractor $\mathcal{A}_0$ of the limit parabolic equation is regular (it will be so if this equation possesses a global Lyapunov function and all of the equilibria are hyperbolic, see Section 7 for the details). Then, using the theory of non-autonomous perturbations of regular attractors developed in [12, 13, 30], we establish the existence of the non-autonomous regular attractor for problems (1.6) if ε is small enough. In this case, the attractors $\mathcal{A}_\varepsilon$ are occurred not only upper semicontinuous, but also lower semicontinuous as $\varepsilon \to 0$ and we give the quantitative bounds for the symmetric distance between them in terms of the perturbation parameter ε. In particular, we prove there that equation (1.4) possesses the finite number of different almost-periodic (with respect to t) solutions and that every other bounded solution of that equation is a heteroclinic orbit between two different almost-periodic solutions. We also recall that the regular attractor for system (1.5) with $\varepsilon = 1$, $\gamma \gg 1$ and *autonomous* external forces g_ε has been considered in our previous paper [28]. Moreover, the estimates for the *nonsymmetric* Hausdorff distance between the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ in terms of the parameter ε have been obtained in [29] for the case where the attractor $\mathcal{A}_0$ of the limit parabolic equation is regular and the external forces $g_\varepsilon(t) = g(\varepsilon^{-1}t)$ are almost-periodic with respect to t.

Finally, several uniform (with respect to ε) estimates for the linear equation of the form (1.4) which are systematically used throughout of the paper are gathered in Appendix.

Acknowledgements:

This research was partially supported by the Russian Foundation of Basic Researches (projects 11-01-00339 and 10-01-00293).

2. Uniform (with respect to $\varepsilon \to 0$) a priori estimates

In this section, we consider the following nonlinear elliptic boundary value problem in a half cylinder $\Omega_+^\tau := [\tau, +\infty) \times \omega$, $\tau \in \mathbb{R}$:

$$
\begin{cases} a(\varepsilon^2 \partial_t^2 u + \Delta_x u) - \gamma \partial_t u - f(u) = g(t), \quad (t,x) \in \Omega_+^\tau, \\ u\big|_{\partial\omega} = 0, \quad u\big|_{t=\tau} = u_\tau, \end{cases} \tag{2.1}
$$

where $\omega \subset\subset \mathbb{R}^n$ is a bounded domain of $\mathbb{R}^n$ with a sufficiently smooth boundary, $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to x, a and γ are given constant $k \times k$-matrices satisfying $a + a^* > 0$ and $\gamma = \gamma^* > 0$, $f(u)$ is a given nonlinear function which satisfies the following assumptions:

$$
\begin{cases} 1. \ f \in C^2(\mathbb{R}^k, \mathbb{R}^k), \\ 2. \ f(v).v \geq -C, \quad f'(v) \geq -K, \quad \forall v \in \mathbb{R}^k, \\ 3. \ |f(v)| \leq C(1 + |v|^q), \ \forall v \in \mathbb{R}^k, \quad q < q_{max}, \end{cases} \tag{2.2}
$$

where $v.w$ stands for the inner product of the vectors $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^k$ and the critical growth exponent q_{max} equals infinity for $n = 1$ or $n = 2$ and $q_{max} := \frac{n+2}{n-2}$ for $n \geq 3$. In order to formulate our assumptions on the solution $u(t)$, the external forces $g(t)$ and the initial data u_0, we need to define the appropriate functional spaces.

Definition 2.1. For every $l \in \mathbb{R}_+$ and $s \in [2, \infty)$, we define the following spaces:

$$
W_b^{l,s}(\Omega_+^\tau) := \{u \in D'(\Omega_+^\tau), \quad \|u\|_{W_b^{l,s}} := \sup_{T \geq \tau} \|u\|_{W^{l,s}(\Omega_T)} < \infty\}, \tag{2.3}
$$

where $\Omega_T := (T, T+1) \times \omega$ and $W^{l,s}$ denotes the ordinary Sobolev space of functions whose derivatives up to order l belong to L^s, see [26]. In particular, we write in the sequel $L_b^s(\Omega_+^\tau)$ instead of $W_b^{0,s}(\Omega_+^\tau)$.

Moreover, we also introduce the following spaces associated with the linear part of equation (2.1):

$$
\begin{aligned} W_\varepsilon^{(1,2),s}(\Omega_T) := \{u \in D'(\Omega_T), \quad \|u\|_{W_\varepsilon^{(1,2),s}} := \\ = \varepsilon^2 \|\partial_t^2 u\|_{L^s(\Omega_T)} + \|\partial_t u\|_{L^s(\Omega_T)} + \|u\|_{L^s([T,T+1], W^{2,s}(\omega))} < \infty, \quad u\big|_{\partial\omega} = 0\} \end{aligned} \tag{2.4}
$$

and, analogously to (2.3)

$$
W_{\varepsilon,b}^{(1,2),s}(\Omega_+^\tau) := \{u \in D'(\Omega_+^\tau), \ \|u\|_{W_{\varepsilon,b}^{(1,2),s}} := \sup_{T \geq \tau} \|u\|_{W_\varepsilon^{(1,2),s}(\Omega_T)} < \infty, \quad u\big|_{\partial\omega} = 0\}. \tag{2.5}
$$

We also introduce the uniform with respect to ε trace space (at $t = \tau$) of functions belonging to the space (2.5):

$$
\begin{aligned} V_\varepsilon^s(\omega) := \{u \in D'(\omega), \\ \|u\|_{V_\varepsilon^p} := \|u\|_{W^{2(1-1/s),s}(\omega)} + \varepsilon^{1/s} \|u\|_{W^{2-1/s,s}(\omega)} < \infty, \ u\big|_{\partial\omega} = 0\}. \end{aligned} \tag{2.6}
$$

We note that, for $\varepsilon > 0$, the space $W_{\varepsilon,b}^{(1,2),s}(\Omega_+^\tau)$ is equivalent to $W_b^{2,s}(\Omega_+^\tau)$ and, for $\varepsilon = 0$ this space coincides with the anisotropic Sobolev-Slobodetskij space $W_b^{(1,2),s}(\Omega_+^\tau)$ which corresponds to a second order parabolic operator. Analogously, for $\varepsilon > 0$, the space $V_\varepsilon^s(\omega)$ coincides with the trace space at $t = \tau$ of of the classical Sobolev space $W^{2,s}(\Omega_\tau)$ and, for $\varepsilon = 0$, we have the trace space at $t = \tau$, for the anisotropic space $W^{(1,2),s}(\Omega_\tau)$, see e.g. [5]. The dependence of the norms (2.4) and (2.6) on the parameter ε is chosen in such way that the constants in the proper maximal regularity estimates and the trace theorems will be independent of $\varepsilon \to 0$, see Appendix below.

Definition 2.2. We restrict ourselves to consider only such solutions $u(t)$ of problem (2.1) which remain bounded as $t \to +\infty$. To be more precise, a function u will be a solution of problem (2.1) if u belongs to the space $W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$ for some $p > p_{min} := \max\{2, (n+2)/2\}$, and satisfies the equation and the boundary conditions in the sense of distributions. Then, we should require the initial data u_0 belongs to the trace space $V^p_\varepsilon(\omega)$ and the external forces $g \in L^p_b(\Omega^\tau_+)$.

Note that the assumption $p > p_{max}$ guarantees that $u \in C_b(\Omega^\tau_+)$ for all ε (including the limit case $\varepsilon = 0$) and, therefore, the non-linearity $f(u)$ is well-defined and belongs to $C_b(\Omega^\tau_+)$ for all $u \in W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$.

The main result of this section is the following theorem.

Theorem 2.3. *Let the above assumptions hold. Then, for every $\varepsilon \in [0,1]$ and $u_\tau \in V^p_\varepsilon(\omega)$, problem* (2.1) *has at least one solution $u \in W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$ and the following estimate hold, for every such solution:*

$$\|u\|_{W^{(1,2),p}_{\varepsilon,b}(\Omega_T)} \le Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)} + Q(\|g\|_{L^p_b(\Omega^\tau_+)}), \tag{2.7}$$

where the constant $\alpha > 0$ and the monotonic function $Q : \mathbb{R}_+ \to \mathbb{R}_+$ are independent of $\varepsilon \in [0,1]$, $u_0 \in V^p_\varepsilon(\omega)$, $\tau \in \mathbb{R}$, $T \ge \tau$ and $g \in L^p_b(\Omega^\tau_+)$.

Proof. We first prove the analogue of estimate (2.7) for $p = 2$.

Lemma 2.4. *Let $u(t)$ be a solution of* (2.1) *(in the sense of Definition 2.2). Then, the following estimate holds:*

$$\|u\|_{W^{(1,2),2}_\varepsilon(\Omega_T)} \le Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)} + Q(\|g\|_{L^2_b(\Omega^\tau_+)}), \tag{2.8}$$

where the constant $\alpha > 0$ and the monotonic function Q are independent of $\tau \in \mathbb{R}$, $T \ge \tau$, $\varepsilon \in [0,1]$, u_0 and g.

Proof. We set $\phi_T(t) := e^{-\alpha|t-T|}$, where $T \in \mathbb{R}$ and $\alpha > 0$ is a small parameter which will be specified below, multiply equation (2.1) by $\phi_T(t)u(t)$ and integrate over Ω^τ_+. Then, after integrating by parts and using that $\gamma = \gamma^*$, we have

$$\begin{gathered}\left\langle \varepsilon^2 a\partial_t u.\partial_t u + a\nabla_x u.\nabla_x u, \phi_T\right\rangle_\tau + \langle f(u).u, \phi_T\rangle_\tau = 1/2\,\langle \gamma u.u, \phi'_T\rangle_\tau - \\ - \varepsilon^2\,\langle a\partial_t u.u, \phi'_T\rangle_\tau - \langle g.u, \phi_T\rangle_\tau + \varepsilon^2(a\partial_t u(\tau).u(\tau), \phi_T(\tau)) - 1/2(\gamma u(\tau).u(\tau), \phi_T(\tau)),\end{gathered} \tag{2.9}$$

where $(u,v) := \int_\omega u(x).v(x)\,dx$ and $\langle v, w\rangle_\tau := \int_\tau^\infty \int_\omega v(t,x).w(t,x)\,dx\,dt$.

Using now that $a + a^* > 0$, $f(v).v \ge -C$ and the obvious inequality

$$|\partial_t \phi_T(t)| \le \alpha\phi_T(t), \quad t \in \mathbb{R}, \tag{2.10}$$

together with the Cauchy-Schwartz inequality, we have

$$\begin{gathered}\left\langle \varepsilon^2|\partial_t u|^2 + |\nabla_x u|^2, \phi_T\right\rangle_\tau \le C\varepsilon^2\alpha\,\langle |\partial_t u|\cdot|u|, \phi_T\rangle_\tau + C\alpha\left\langle |u|^2, \phi_T\right\rangle_\tau \\ + C\left(1 + \left\langle |g|^2, \phi_T\right\rangle_\tau + \phi_T(\tau)\|u_\tau\|^2_{L^2(\omega)} + \varepsilon^2\phi_T(\tau)\|u_\tau\|_{L^2(\omega)}\|\partial_t u(\tau)\|_{L^2(\omega)}\right),\end{gathered} \tag{2.11}$$

where the constant C is independent of ε, τ, T and α (we also note that all of the integrals in (2.11) have a sense since the solution u is assumed to belong to $W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$). Estimate (2.11) implies that, for sufficiently small (but independent of ε), $\alpha > 0$

$$\begin{gathered}\left\langle \varepsilon^2|\partial_t u|^2 + |\nabla_x u|^2, \phi_T\right\rangle_\tau \le \\ C_1\left(1 + \|g\|^2_{L^2_b(\Omega^\tau_+)} + \phi_T(\tau)\|u_\tau\|^2_{L^2(\omega)} + \varepsilon^2\phi_T(\tau)\|u_\tau\|_{L^2(\omega)}\|\partial_t u(\tau)\|_{L^2(\omega)}\right),\end{gathered} \tag{2.12}$$

where the constant C_1 is independent of ε, τ and T. We now set

$$\mathcal{L}_\varepsilon u := \varepsilon^2\partial_t^2 u + \Delta_x u,\ \ \bar{\mathcal{L}}_\varepsilon u := \varepsilon^2\partial_t(\phi_T\partial_t u) + \phi_T\Delta_x u \equiv \phi_T\mathcal{L}_\varepsilon u + \varepsilon^2\phi_T'(t)\partial_t u,$$

multiply equation (2.1) by $\bar{\mathcal{L}}_\varepsilon u$ and integrate over Ω_+^τ. Then, integrating by parts, using that $\gamma=\gamma^*$, we have

$$\begin{gathered}\text{(2.13)}\quad \langle a\mathcal{L}_\varepsilon u.\mathcal{L}_\varepsilon u, \phi_T\rangle_\tau - \varepsilon^2\langle a\mathcal{L}_\varepsilon u.\partial_t u, \phi_T'\rangle_\tau - \frac{1}{2}\left\langle \varepsilon^2\gamma\partial_t u.\partial_t u + \gamma\nabla_x u.\nabla_x u, \phi_T'\right\rangle_\tau +\\ + \frac{1}{2}\phi_T(\tau)[\varepsilon^2(\gamma\partial_t u(\tau), \partial_t u(\tau)) - (\gamma\nabla_x u(\tau), \nabla_x u(\tau))] + \langle f'(u)\partial_t u.\partial_t u, \phi_T\rangle_\tau +\\ + \langle f'(u)\nabla_x u.\nabla_x u, \phi_T\rangle_\tau - \varepsilon^2\phi_T(\tau)(f(u(\tau)), \partial_t u(\tau)) = \langle h.\mathcal{L}_\varepsilon u, \phi_T\rangle_\tau + \varepsilon^2\langle h.\partial_t u, \phi_T'\rangle_\tau\end{gathered}$$

and using that $a+a^*>0$, $\gamma>0$ and $f'(v)\ge -K$, together with estimate (2.10), after the straightforward estimates, we end up with

$$\begin{gathered}\text{(2.14)}\quad \left\langle|\mathcal{L}_\varepsilon u|^2, \phi_T\right\rangle_\tau + \varepsilon^2\phi_T(\tau)\|\partial_t u(\tau)\|^2_{L^2(\omega)} \le C_2\big(\left\langle\varepsilon^2|\partial_t u|^2 + |\nabla_x u|^2, \phi_T\right\rangle_\tau +\\ + \left\langle|g|^2, \phi_T\right\rangle_\tau + \phi_T(\tau)\|\nabla_x u_\tau\|^2_{L^2(\omega)} + \varepsilon^2\phi_T(\tau)\|f(u_\tau)\|^2_{L^2(\omega)}\big),\end{gathered}$$

where the constant C_2 is independent of ε, τ and T. Applying estimate (2.12) in order to estimate the first term in the right-hand side of (2.14), we obtain, after the obvious estimates that

$$\begin{gathered}\text{(2.15)}\quad \left\langle|\mathcal{L}_\varepsilon u|^2, \phi_T\right\rangle_\tau + \varepsilon^2\phi_T(\tau)\|\partial_t u(\tau)\|^2_{L^2(\omega)} + \left\langle\varepsilon^2|\partial_t u|^2 + |\nabla_x u|^2, \phi_T\right\rangle_\tau \le\\ \le C_3\left(1 + \|g\|^2_{L^2_b(\Omega_+^\tau)} + \phi_T(\tau)\|u_\tau\|^2_{W^{1,2}(\omega)} + \varepsilon^2\phi_T(\tau)\|f(u_\tau)\|^2_{L^2(\omega)}\right),\end{gathered}$$

where the constant C_3 is independent of ε, τ and T. We now claim that

$$\begin{gathered}\text{(2.16)}\quad \varepsilon^4\|\partial_t^2 u\|^2_{L^2(\Omega_T)} + \|\Delta_x u\|^2_{L^2(\Omega_T)} \le\\ \le C_4\left(\left\langle|\mathcal{L}_\varepsilon u|^2, \phi_T\right\rangle_\tau + \varepsilon^2\left\langle|\partial_t u|^2 + |u|^2, \phi_T\right\rangle_\tau + \varepsilon\phi_T(\tau)\|u_\tau\|^2_{W^{3/2,2}(\omega)}\right),\end{gathered}$$

where C_4 is independent of ε, τ and $T\ge\tau$. Indeed, let $\varphi(t)\in C_0^\infty(R)$ be a cut-off function such that $\varphi(t)=1$, for $t\in[0,1]$, and $\varphi(t)=0$, for $t\notin[-1,2]$. For every $T\ge\tau$, we set $\varphi_T(t):=\varphi(t-T)$ and $u_T(t):=\varphi_T(t)u(t)$. Then, the last function satisfies the following equation:

$$\mathcal{L}_\varepsilon u_T(t) = h_u(t) := \varphi_T(t)\mathcal{L}_\varepsilon u(t) + 2\varepsilon^2\varphi_T'(t)\partial_t u(t) + \varepsilon^2\varphi_T''(t)u(t)$$

Applying Lemma 8.2 with $p=2$ (see Appendix) to this equation, we have

$$\begin{aligned}\varepsilon^4\|\partial_t^2 u_T\|^2_{L^2(\Omega_+^\tau)} + \|\Delta_x u_T\|^2_{L^2(\Omega_+^\tau)} &\le C\left(\|h_u\|^2_{L^2(\Omega_+^\tau)} + \varepsilon\|u_T(\tau)\|^2_{W^{3/2,2}(\omega)}\right)\\ &\le C'\left(\left\langle|\mathcal{L}_\varepsilon u|^2, \phi_T\right\rangle_\tau + \varepsilon^2\left\langle|\partial_t u|^2 + |u|^2, \phi_T\right\rangle_\tau + \varepsilon\phi_T(\tau)\|u_\tau\|^2_{W^{3/2,2}(\omega)}\right)\end{aligned}$$

and estimate (2.16) is an immediate corollary of this estimate.

Inserting now estimate (2.15) into the right-hand side of (2.16) and using the embedding $W^{2(1-1/p),p}(\omega)\subset C(\omega)$ (due to our choice of the exponent p), we have

$$\text{(2.17)}\qquad \varepsilon^4\|\partial_t^2 u\|^2_{L^2(\Omega_T)} + \|\Delta_x u\|^2_{L^2(\Omega_T)} \le C_5(1 + \|g\|^2_{L^2_b(\Omega_+^\tau)}) + Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)},$$

where the constant C_5 and the monotonic function Q are independent of ε, τ and $T\ge\tau$.

Thus, there only remains to estimate the L^2-norm of $\partial_t u$. In order to do so, we rewrite elliptic system (2.1) in the following form:

$$\text{(2.18)}\qquad \gamma\partial_t u = a\Delta_x u - f(u) + \tilde{h}_u(t),\ \ u\big|_{\partial\omega} = 0,\ \ \tilde{h}_u(t) := \varepsilon^2 a\partial_t^2 u(t) - g(t).$$

Equation (2.18) has the form of a nonlinear reaction-diffusion system in the bounded domain ω with the non-autonomous external forces $\tilde{h}_u(t)$ belonging to $L^2_b((\tau,\infty)\times\omega)$ (due to estimate (2.17)). Moreover, the nonlinearity $f(u)$ satisfies the quasimonotonicity assumption $f'(v)\ge -K$. Consequently, multiplying (2.18) by $\Delta_x u(t)$, integrating over x and applying the Gronwall's inequality, we derive (in a standard way, see e.g. [9]) that

$$\|u(T)\|^2_{W^{1,2}(\omega)} \le C\|u_\tau\|^2_{W^{1,2}(\omega)}e^{-\alpha(T-\tau)} + C + \int_\tau^T e^{-\alpha(T-t)}\|\tilde{h}_u(t)\|^2_{L^2(\omega)}\,dt, \tag{2.19}$$

where the positive constants α and C are independent of $\tilde{h}_u$. Using estimate (2.17) for estimating the last term in the right-hand side of (2.19), we have

$$\begin{aligned}\varepsilon^4\|\partial_t^2 u\|^2_{L^2(\Omega_T)} + \|u\|^2_{L^2((T,T+1),W^{2,2}(\omega))} + \|u\|^2_{L^\infty((T,T+1),W^{1,2}(\omega))} \le \\ \le C_6(1+\|g\|^2_{L^2_b(\Omega^\tau_+)}) + Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)},\end{aligned} \tag{2.20}$$

where the constant C_6 and the monotonic function Q are independent of ε, τ and T.

We now recall that, according to the embedding theorem, see e.g. [17] and [26]

$$\|u\|_{L^{2q_{max}}(\Omega_T)} \le C\left(\|u\|_{L^\infty((T,T+1),W^{1,2}(\omega))} + \|u\|_{L^2((T,T+1),W^{2,2}(\omega))}\right), \tag{2.21}$$

where the exponent q_{max} is the same as in (2.2). Estimates (2.20) and (2.21), together with the growth restriction (2.2), imply that

$$\|f(u)\|_{L^2(\Omega_T)} \le Q_1(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)} + Q_1(\|g\|_{L^2_b(\Omega^\tau_+)}), \tag{2.22}$$

where the constant $\alpha>0$ and the monotonic function Q_1 are independent of ε, τ and $T\ge\tau$. Expressing now $\partial_t u$ from equation (2.1) and using estimates (2.20) and (2.22), we obtain the desired estimate for $\partial_t u$ and finish the proof of Lemma 2.4. □

We are now ready to prove estimate (2.7), for $p>2$. We consider only the more complicated case $n\ge 3$ and rest the simpler case $n\le 2$ to the reader.

Remind that the nonlinearity $f(u)$ satisfies growth restriction given by the third formula of (2.2) where the exponent q is *strictly* less than q_{max} and, due to the embedding theorem for the anisotropic spaces

$$W^{(1,2),2}_\varepsilon(\Omega_T)\subset W^{(1,2),2}_0(\Omega_T)\subset L^{q_{max}}_b(\Omega_T),$$

see [26], and, consequently, due to (2.8), estimate (2.22) can be improved as follows:

$$\begin{aligned}\|f(u)\|_{L^{2+\delta_0}(\Omega_T)} \le C(1+\|u\|_{L^{q_{max}}(\Omega_T)})^q \le \\ \le C_1(1+\|u\|_{W^{(1,2),2}_\varepsilon(\Omega_T)})^q \le Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)} + Q(\|g\|_{L^2_b(\Omega^\tau_+)}),\end{aligned} \tag{2.23}$$

where $\delta_0 := \frac{2(q_{max}-q)}{q} > 0$ and the constant $\alpha>0$ and the monotone function Q are independent of ε, τ and T. We now rewrite equation (2.1) in the following way:

$$a(\varepsilon^2\partial_t^2 u + \Delta_x u) - \gamma\partial_t u = H_u(t) := g(t) + f(u(t)),\ u\big|_{\partial\omega}=0,\ u\big|_{t=\tau}=u_\tau \tag{2.24}$$

and apply the maximal elliptic regularity estimate of Corollary 8.6 to this linear equation which reads

$$\|u\|^{2+\delta_0}_{W^{(1,2),2+\delta_0}_\varepsilon(\Omega_T)} \le C\|u_\tau\|^{2+\delta_0}_{V^{2+\delta_0}_\varepsilon(\omega)}e^{-\alpha(T-\tau)} + C\int_\tau^\infty e^{-\alpha(T-t)}\|H_u(t)\|^{2+\delta_0}_{L^{2+\delta_0}(\omega)}\,dt,$$

see Corollary 8.6 of Appendix.

Then, using (2.23) in order to estimate the integral into the right-hand side of this formulae, we get

$$\|u\|_{W_\varepsilon^{(1,2),2+\delta_0}(\Omega_T)} \le Q_1(\|u_\tau\|_{V_\varepsilon^p(\omega)})e^{-\alpha(T-\tau)} + Q_1(\|g\|_{L_b^{2+\delta_0}(\Omega_+^\tau)}), \tag{2.25}$$

where the constant C and the function Q_1 are independent of ε, τ and T. We now recall that

$$W_\varepsilon^{(1,2),s}(\Omega_T) \subset W_0^{(1,2),s}(\Omega_T) \equiv (W^{(1,2),s}(\Omega_T) \cap \{u\big|_{\partial\omega} = 0\})$$

and, due to the embedding theorem for anisotropic Sobolev spaces

$$W^{(1,2),s}(\Omega_T) \subset L^{r(s)}(\Omega_T), \quad \text{where} \quad \frac{1}{r(s)} = \frac{1}{s} - \frac{2}{n+2}, \tag{2.26}$$

see [26]. Consequently, analogously to (2.23), we have

$$\|f(u)\|_{L^{2+\delta_1}(\Omega_T)} \le Q_2(\|u_\tau\|_{V_\varepsilon^p(\omega)})e^{-\alpha(T-\tau)} + Q_2(\|g\|_{L_b^{2+\delta_0}(\Omega_+^\tau)}), \tag{2.27}$$

where $\alpha > 0$ and Q_2 are independent of ε, τ and T and

$$2 + \delta_1 := \frac{r(2+\delta_0)}{q} > \frac{r(2+\delta_0)}{q_{max}} = (2+\delta_0)\frac{n-2}{n-2-2\delta_0} > 2 + \delta_0. \tag{2.28}$$

Iterating the above procedure, we finally derive estimates (2.25) and (2.27) with the exponent $2 + \delta_l \equiv p$. Indeed, formulae (2.26) and (2.28) guarantee that the number l of the iterations will be finite. Thus, estimate (2.7) is proved.

In order to finish the proof of Theorem 2.3, there remains to note that the existence of a solution $u \in W_{\varepsilon,b}^{(1,2),p}(\Omega_+^\tau)$ of problem (2.1) can be proved in a standard way based on a priori estimate (2.7). Indeed, for instance, the existence of a solution of the analogue of (2.1) in a finite cylinder $\Omega_{\tau,N} := (\tau, \tau+N) \times \Omega$ can be proved using the Leray-Schauder fixed point principle and the existence of a solution u in the infinite cylinder Ω_+^τ can be obtained then by passing to the limit $N \to \infty$, see e.g. [27, 28] for the details. Theorem 2.3 is proved. □

Remark 2.5. We note that estimate (2.7) is valid for every $p \ge 2$ although we have formally proved it only for $p > p_{min}$. Indeed, we have used the last assumption only in order to estimate the term $\varepsilon^2\phi(\tau)\|f(u(\tau)\|^2_{L^2(\omega)}$ in (2.14) which appears after applying the Schwartz inequality to the term $\varepsilon^2\phi(\tau)\,(\partial_t u(\tau), f(u(\tau)))_{L^2(\omega)}$. But the growth restriction of (2.2), Lemma 8.2 and the appropriate interpolation inequality allow to estimate this term in more accurate way:

$$|\varepsilon^2\,(\partial_t u(\tau), f(u(\tau)))_{L^2(\omega)}| \le \mu\|u\|^2_{W_\varepsilon^{(1,2),2}(\Omega_\tau)} + Q_\mu(\|u_\tau\|_{V_\varepsilon^2(\omega)}),$$

where the parameter μ can be arbitrarily small and a function Q_μ depends on μ, but is independent of ε (see [24] for the details). Inserting this estimate to the right-hand side of (2.14), we can easily derive (2.7) with $p = 2$.

Remark 2.6. If we need not estimate (2.7) to be uniform with respect to $\varepsilon \to 0$, it is possible to relax the growth restriction (2.2)(3) till $q < q'_{max} := \frac{n+1}{n-3}$. Indeed, in this case, it is sufficient to use the embedding $W^{2,2}(\Omega_0) \subset L^{2q'_{max}}(\Omega_0)$ instead of (2.21) in the proof of Theorem 2.3.

Corollary 2.7. *Let the assumptions of Theorem 2.3 hold and let $u \in W_{\varepsilon,b}^{(1,2),p}(\Omega_+^\tau)$ be a solution of* (2.1). *Then, the following estimate holds:*

$$\|u(t)\|_{V_\varepsilon^p(\omega)} \le Q(\|u_\tau\|_{V_\varepsilon^p(\omega)})e^{-\alpha(t-\tau)} + Q(\|g\|_{L_b^p(\Omega_+^\tau)}), \tag{2.29}$$

where the constant $\alpha > 0$ and the function Q are indepndent of ε, τ, $t \ge \tau$ and u.

Indeed, (2.29) is an immediate corollary of (2.7) and the fact that $V^p_\varepsilon(\omega)$ is the uniform (with respect to ε) trace space of functions belonging to $W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$, see Apendix.

Corollary 2.8. *Let the assumptions of Theorem 2.3 hold and let, in addition, the external forces g belong to $L^{p_1}_b(\Omega^\tau_+)$, for some $p_1 > p$. Then, every solution $u \in W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$ of problem* (2.1) *satisfies the following estimate:*

$$\|u\|_{W^{(1,2),p_1}_\varepsilon(\Omega_T)} \le Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)} + Q(\|g\|_{L^{p_1}_b(\Omega^\tau_+)}), \quad T \ge \tau+1, \tag{2.30}$$

where the constant $\alpha > 0$ and the function Q are independent of ε, τ, T and u.

Indeed, since $W^{(1,2),p}(\Omega_T) \subset C(\Omega_T)$ (due to our choice of the exponent p) then, estimate (2.7) implies that

$$\|f(u)\|_{L^\infty(\Omega_T)} \le Q(\|u_\tau\|_{V^p_\varepsilon(\omega)})e^{-\alpha(T-\tau)} + Q(\|g\|_{L^p(\Omega^\tau_+)}), \tag{2.31}$$

where the constant $\alpha > 0$ and the function q are independent of ε, τ, T and u. Rewriting now equation (2.1) in the form of (2.24) and applying the uniform (with respect to ε) interior L^{p_1}-regularity estimate to this equation (see Corollary 8.7 and estimate (8.28)), we derive estimate (2.30).

3. Uniqueness of the solutions

In this section, we prove that the solution $u(t)$ of problem (2.1) which is constructed in Theorem 2.3 is unique if $\varepsilon > 0$ is small enough. Moreover, we also verify the differentiability of that solution with respect to the initial data $u_\tau \in V^p_\varepsilon(\omega)$ in the corresponding functional spaces. We start with the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.3 hold and let, in addition, $\varepsilon \le \varepsilon_0 := \varepsilon_0(a, f, \gamma)$ is small enough. Then, for every two solutions $u_1(t)$ and $u_2(t)$ of problem* (2.1), *the following estimate holds:*

$$\|u_1 - u_2\|_{W^{(1,2),p}_\varepsilon(\Omega_T)} \le Ce^{\Lambda_0(T-\tau)}\|u_1(\tau) - u_2(\tau)\|_{V^p_\varepsilon(\omega)}, \tag{3.1}$$

where the constant Λ_0 is independent of $\varepsilon \le \varepsilon_0$, $\tau \in \mathbb{R}$, $T \ge \tau$, u_1 and u_2 and the constant C depends on $\|u_i(\tau)\|_{V^p_\varepsilon(\omega)}$, but is independent of ε, τ and T. In particular, the solution of (2.1) *is unique if $\varepsilon \le \varepsilon_0$.*

Proof. We set $v(t) := u_1(t) - u_2(t)$. Then, this function satisfies the following equation:

$$a(\varepsilon^2\partial_t^2 v + \Delta_x v) - \gamma\partial_t v - l(t)v = 0, \quad v\big|_{\partial\omega} = 0, \quad v\big|_{t=\tau} = u_1(\tau) - u_2(\tau), \tag{3.2}$$

where $l(t) = l(t,x) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,dt$. Moreover, due to the second assumption of (2.2), estimate (2.7) and the embedding $W^{(1,2),p}(\Omega_T) \subset C(\Omega_T)$, we have

$$l(t,x) \ge -K, \quad \|l(t,x)\|_{L^\infty(\Omega^\tau_+)} \le M, \tag{3.3}$$

where the constant K is defined in (2.2) and the constant M depends on the norms $\|u_i(\tau)\|_{V^p_\varepsilon(\omega)}$, $i = 1, 2$, and $\|g\|_{L^p_b(\Omega^\tau_+)}$, but is independent of ε and τ. It is however convenient to consider more general (than (3.2)) problem

$$a(\varepsilon^2\partial_t^2 w + \Delta_x w) - \gamma\partial_t w - l(t)w = h(t), \ w\big|_{\partial\omega} = 0, \quad w\big|_{t=\tau} = w_\tau, \tag{3.4}$$

where the given matrix-valued function $l(t)$ satisfies (3.3) and $h(t) = h(t,x)$ are given external forces.

Lemma 3.2. *Let Λ_0 be a nonnegative number which satisfies the following condition:*

$$\Lambda_0\gamma - \varepsilon^2\Lambda_0^2(a_+ - 2a_-(a_+)^{-1}a_-) - K \ge 0, \tag{3.5}$$

where $a_+ := 1/2(a+a^*)$ and $a_- = 1/2(a-a^*)$. Then, for every $w_\tau \subset V^p_\varepsilon(\omega)$ and every external forces h satisfying

$$e^{-\Lambda_0 t}h(t) \in L^p_b(\Omega^\tau_+), \tag{3.6}$$

problem (3.4) has a unique solution $w(t)$ belonging to the class

$$e^{-\Lambda_0 t}w(t) \in W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+) \tag{3.7}$$

and the following estimate holds:

$$\|w\|^p_{W^{(1,2),p}_\varepsilon(\Omega_T)} \le C\|w_\tau\|^p_{V^p_\varepsilon(\omega)}e^{p(\Lambda_0-\alpha)(T-\tau)}+ \\ + C\int_\tau^\infty e^{-p\alpha|T-t|+p\Lambda_0(T-t)}\|h(t)\|^p_{L^p(\omega)}\,dt, \tag{3.8}$$

where the positive constants α and C depend on M and Λ_0, but are independent of ε, τ and $T \ge \tau$.

Proof. We first note that, due to the fact that $V^p_\varepsilon(\omega)$ is the uniform (with respect to ε) trace space for functions belonging to $W^{(1,2),p}_\varepsilon(\Omega_\tau)$, it is sufficient to verify Lemma 3.2 for the case $w_\tau = 0$ only. In order to do so, we set $\theta(t) := e^{-\Lambda_0 t}w(t)$. Then, this function belongs to $W^{(1,2),p}_{\varepsilon,b}(\Omega^\tau_+)$ (due to assumption (3.7)) and satisfies the following equation:

$$a(\varepsilon^2\partial_t\theta + \Delta_x\theta) - (\gamma - 2\varepsilon^2\Lambda_0 a)\partial_t\theta- \\ - (\Lambda_0\gamma - \varepsilon^2\Lambda_0^2 a + l(t))\theta = \tilde h(t) := e^{-\Lambda_0 t}h(t),\ \theta\big|_{\partial\omega} = 0,\ \theta\big|_{t=\tau} = 0. \tag{3.9}$$

Multiplying now this equation by $\phi_T(t)\theta(t)$ (where the weight function $\phi_T(t) := e^{-\alpha|T-t|}$) and integrating over Ω^τ_+, we obtain after the standard transformations (integrating by parts and using that $\gamma = \gamma^*$, $l(t) \ge -K$ and estimate (2.10)) that

$$\varepsilon^2\left\langle a_+\partial_t\theta.\partial_t\theta, \phi_T\right\rangle_\tau + \left\langle a_+\nabla_x\theta.\nabla_x\theta, \phi_T\right\rangle_\tau + \left\langle(\Lambda_0\gamma - \varepsilon^2\Lambda_0^2 a_+ - K)v.v, \phi_T\right\rangle_\tau \\ \le |\left\langle \tilde h, \phi_T\theta\right\rangle_\tau| + C\varepsilon^2\alpha\left\langle|\partial_t\theta|^2 + |\theta|^2, \phi_T\right\rangle_\tau + 2\varepsilon^2\Lambda_0|\left\langle a_-\partial_t\theta.\theta, \phi_T\right\rangle_\tau|, \tag{3.10}$$

where the constant C depends only on a and γ and $\langle\cdot,\cdot\rangle_\tau$ stands for the inner product in $L^2(\Omega^\tau_+)$. Estimating the last term in the right-hand side of (3.10) as follows:

$$2\varepsilon^2\Lambda_0|a_-\partial_t\theta.\theta| \le 1/2\varepsilon^2\Lambda_0^2 a_+\partial_t\theta.\partial_t\theta - 2\varepsilon^2 a_-(a_+)^{-1}a_-\theta.\theta,$$

fixing the parameter $\alpha > 0$ to be small enough and using the Friedrichs and Schwartz inequalities, we have

$$\varepsilon^2\left\langle a_+\partial_t\theta.\partial_t\theta, \phi_T\right\rangle_\tau + \left\langle a_+\nabla_x\theta.\nabla_x\theta, \phi_T\right\rangle_\tau + \\ + 4\left\langle(\Lambda_0\gamma - \varepsilon^2\Lambda_0^2(a_+ - 2a_-(a_+)^{-1}a_-) - K)v.v, \phi_T\right\rangle_\tau \le C_1\left\langle|\tilde h|^2, \phi_T\right\rangle_\tau.$$

Using assumption (3.5), positivity of a_+ and the obvious inequality $\phi_T(t) \ge e^{-\alpha}$ for $t \in [T, T+1]$, we have

$$\varepsilon^2\|\partial_t\theta\|^2_{L^2(\Omega_T)} + \|\nabla_x w\|^2_{L^2(\Omega_T)} \le C_2\left\langle|\tilde h|^2, \phi_T\right\rangle_\tau$$

Returning to the variable $w(t) = e^{\Lambda_0 t}\theta(t)$, we derive

$$\varepsilon^2\|\partial_t w\|^2_{L^2(\Omega_T)} + \|\nabla_x w\|^2_{L^2(\Omega_T)} \le C_3\int_\tau^\infty e^{-\alpha|T-t|+2\Lambda_0(T-t)}\|h(t)\|^2_{L^2(\omega)}\,dt, \tag{3.11}$$

Estimate (3.8) (with $w_\tau = 0$) can be now derived from (3.11) iterating the maximal elliptic regularity estimate (8.27) exactly as in the end of the proof of Theorem 2.3. The existence of the solution can be then verified in a standard way based on a priori estimate (3.8), see e.g. [27, 28]. Lemma 3.2 is proved. □

We are now ready to finish the proof of Theorem 3.1. To this end, we note that the left-hand side of (3.5) tends to $\Lambda_0\gamma - K$ as $\varepsilon \to 0$ and, consequently, for every sufficiently large $\Lambda_0 > 0$, we may fix (due to positivity of the matrix γ) $\varepsilon_0 = \varepsilon_0(\Lambda_0, K, a, \gamma)$ such that (3.5) is satisfied, for every $\varepsilon \le \varepsilon_0$. Applying then estimate (3.8) (with $h \equiv 0$) to equation (3.2), we finish the proof of Theorem 3.1. □

Let us assume from now on that

$$g \in L^p_b(\Omega), \quad \text{where } \ \Omega := \mathbb{R}\times\omega. \tag{3.12}$$

Then, under the assumptions of Theorem 3.1, problem (2.1) defines a two-parametrical family of solving operators $\{U^\varepsilon_g(t,\tau),\ \tau\in\mathbb{R},\ t\ge\tau\}$ via

$$U^\varepsilon_g(t,\tau) : V^p_\varepsilon(\omega) \to V^p_\varepsilon(\omega), \quad u(t) := U^\varepsilon_g(t,\tau)u_\tau, \tag{3.13}$$

where $u(t)$ solves (2.1) and $u(\tau) = u_\tau$ which, obviously, generates a dynamical process on $V^p_\varepsilon(\omega)$, i.e.

$$U^\varepsilon_g(t,\tau_1)\circ U^\varepsilon_g(\tau_1,\tau) = U^\varepsilon_g(t,\tau), \quad t\ge\tau_1\ge\tau\in\mathbb{R}. \tag{3.14}$$

Moreover, Theorem 3.1 shows that these operators are uniformly (with respect to ε) Lipschitz continuous in $V^p_\varepsilon(\omega)$. Our next task is to prove their Frechet differentiability with respect to the initial data $u_\tau \in V^p_\varepsilon(\omega)$. To this end, we consider the following formal equation of variations associated with a solution $u(t) := U^\varepsilon_g(t,\tau)u_\tau$:

$$a(\varepsilon^2\partial_t^2 v + \Delta_x v) - \gamma\partial_t v - f'(u(t))v = 0, \quad v\big|_{\partial\omega} = 0, \quad v\big|_{t=\tau} = v_\tau. \tag{3.15}$$

Then, due to Lemma 3.2, we have

$$\|v(t)\|_{V^p_\varepsilon(\omega)} \le C\|v_\tau\|_{V^p_\varepsilon(\omega)}e^{(\Lambda_0-\alpha)(t-\tau)}, \tag{3.16}$$

where the solution $v(t)$ satisfies (3.7) and the constants $\alpha > 0$ and C are independent of ε, τ and T. The following theorem shows that (3.15) defines indeed the Frechet derivative of the process $U^\varepsilon_g(t,\tau)$ at u_τ.

Theorem 3.3. *Let the assumptions of Theorem 3.1 hold. Let also $u(t)$ and $u_1(t)$ be two solutions of* (2.1) *and $v(t)$ be a solution of* (3.15) *with $v_\tau := u(\tau) - u_1(\tau)$ (associated with $u(t)$). Then, there exists $\varepsilon_0' = \varepsilon_0'(f, a, \gamma) > 0$ such that $\varepsilon_0' \le \varepsilon_0$ and for every $\varepsilon \le \varepsilon_0'$ the following estimate is valid:*

$$\|u(t) - u_1(t) - v(t)\|_{W^{(1,2),p}_\varepsilon(\Omega_T)} \le Ce^{(2\Lambda_0-\alpha)(T-\tau)}\|u(\tau) - u_1(\tau)\|^2_{V^p_\varepsilon(\omega)}, \tag{3.17}$$

where the constants C and $\alpha > 0$ depend on $\|u(\tau)\|_{V^p_\varepsilon(\omega)}$ and $\|u_1(\tau)\|_{V^p_\varepsilon(\Omega)}$, but are independent of ε, τ and T.

Proof. We set $w(t) := u(t) - u_1(t) - v(t)$. Then, this function satisfies the following equation:

$$a(\varepsilon^2\partial_t^2 w + \Delta_x w) - \gamma\partial_t w - f'(u(t))w = h_{u,u_1}(t), \quad w\big|_{\partial\omega} = 0, \quad w\big|_{t=\tau} = 0, \tag{3.18}$$

where $h_{u,u_1}(t) := \int_0^1 [f'(u(t)) - f'(u(t) + s(u_1(t) - u(t)))]\,ds(u(t) - u_1(t))$. Moreover, since $V_0^p(\omega) \subset C(\omega)$ and $f \in C^2$, then estimates (2.7) and (3.1) implies that

$$\|h_{u,u_1}(t)\|_{L^\infty(\omega)} \le C\|u(t) - u_1(t)\|^2_{L^\infty(\omega)} \le C_1\|u(\tau) - u_1(\tau)\|^2_{V_\varepsilon^p(\omega)} e^{2(\Lambda_0 - \alpha)(t-\tau)}, \tag{3.19}$$

where the constants C_i depend on $\|u(\tau)\|_{V_\varepsilon^p(\omega)}$ and $\|u_1(\tau)\|_{V_\varepsilon^p(\omega)}$, but are independent of ε, τ and $t \ge \tau$. Fixing now $\varepsilon_0' > 0$ small enough that assumption (3.5) holds with Λ_0 replaced by $2\Lambda_0$ and applying Lemma 3.2 (with $2\Lambda_0$ instead of Λ_0) to equation (3.18), we derive estimate (3.17) and finish the proof of Theorem 3.3. $\square$

Corollary 3.4. *Let the assumptions of Theorem 3.3 hold and let $u(t)$, $u_1(t)$ and $v(t)$ be the same as in Theorem 3.3. Then, for every $q \ge p$, $q < \infty$ and every $T \ge \tau + 1$, the following estimate holds:*

$$\|u(t) - u_1(t) - v(t)\|_{W_\varepsilon^{(1,2),q}(\Omega_T)} \le C_q e^{(2\Lambda_0 - \alpha)(T-\tau)} \|u(\tau) - u_1(\tau)\|^2_{V_\varepsilon^p(\omega)}, \tag{3.20}$$

where the constants C_q and $\alpha > 0$ depend on $\|u(\tau)\|_{V_\varepsilon^p(\omega)}$, $\|u_1(\tau)\|_{V_\varepsilon^p(\omega)}$ and q, but are independent of ε, τ and T.

Indeed, rewriting equation (3.18) in the form

$$a(\varepsilon^2\partial_t^2 w + \Delta_x w) - \gamma\partial_t w = f'(u(t))w(t) + h_{u,u_1}(t), \quad w\big|_{\partial\omega} = w\big|_{t=\tau} = 0, \tag{3.21}$$

applying the interior regularity estimate (8.28) (where the exponent p is replaced by q) and using estimates (3.17) and (3.19) for estimating the right-hand side of (3.21), we derive estimate (3.20).

Corollary 3.5. *Let the assumptions of Theorem 3.3 hold. Then, the operators $U_g^\varepsilon(t,\tau)$ are Frechet differentiable with respect to the initial data, their Frechet derivative is defined by $D_u U_g^\varepsilon(t,\tau)(u_\tau)\xi := v(t)$, where $v(t)$ is the solution of (3.15) with $v_\tau = \xi$, and the following estimates hold:*

$$\|U_g^\varepsilon(t,\tau)u_\tau^1 - U_g^\varepsilon(t,\tau)u_\tau^2 - D_u U_g^\varepsilon(t,\tau)(u_\tau^1)(u_\tau^1 - u_\tau^2)\|_{V_\varepsilon^p(\omega)} \le \\ \le Ce^{2\Lambda_0(t-\tau)}\|u_\tau^1 - u_\tau^2\|^2_{V_\varepsilon^p(\omega)} \tag{3.22}$$

for every $u_\tau^1, u_\tau^2 \in V_\varepsilon^p(\omega)$ and, consequently

$$\|D_u U_g^\varepsilon(t,\tau)(u_\tau^1) - D_u U_g^\varepsilon(t,\tau)(u_\tau^2)\|_{\mathcal{L}(V_\varepsilon^p(\omega), V_\varepsilon^p(\omega))} \le Ce^{2\Lambda_0(t-\tau)}\|u_\tau^1 - u_\tau^2\|_{V_\varepsilon^p(\omega)}, \tag{3.23}$$

where the constant C depends on $\|u_\tau^1\|_{V_\varepsilon^p(\omega)}$, $\|u_\tau^2\|_{V_\varepsilon^p(\omega)}$ and $\|g\|_{L_b^p}$, but is independent of ε, τ and t.

Indeed, estimate (3.22) is an immediate corollary of (3.17) and estimate (3.23) is a standard corollary of (3.22).

Arguing analogously, but using estimate (3.20) instead of (3.17), we derive the following result.

Corollary 3.6. *Under the assumptions of Corollary 3.5 the following estimates hold, for every $q \ge p$ and $T \ge \tau + 1$:*

$$\|U_g^\varepsilon(t,\tau)u_\tau^1 - U_g^\varepsilon(t,\tau)u_\tau^2 - D_u U_g^\varepsilon(t,\tau)(u_\tau^1)(u_\tau^1 - u_\tau^2)\|_{V_\varepsilon^q(\omega)} \le \\ \le C_q e^{2\Lambda_0(t-\tau)}\|u_\tau^1 - u_\tau^2\|^2_{V_\varepsilon^p(\omega)} \tag{3.24}$$

for every $u_\tau^1, u_\tau^2 \in V_\varepsilon^p(\omega)$ and, consequently

$$\|D_u U_g^\varepsilon(t,\tau)(u_\tau^1) - D_u U_g^\varepsilon(t,\tau)(u_\tau^2)\|_{\mathcal{L}(V_\varepsilon^p(\omega), V_\varepsilon^q(\omega))} \le C_q e^{2\Lambda_0(t-\tau)}\|u_\tau^1 - u_\tau^2\|_{V_\varepsilon^p(\omega)}, \tag{3.25}$$

where the constant C_q depends on q, $\|u_\tau^1\|_{V_\varepsilon^p(\omega)}$, $\|u_\tau^2\|_{V_\varepsilon^p(\omega)}$ and $\|g\|_{L_b^p}$, but is independent of ε, τ and t.

We now recall that, for $\varepsilon > 0$, operators $U_g^\varepsilon(t,\tau)$ are defined on the space $V_\varepsilon^p(\omega) \sim W^{2-1/p,p}(\omega)$ and, for $\varepsilon = 0$, the limit process $U_g^0(t,\tau)$ is defined on the different space $V_0^p(\omega) \sim W^{2(1-1/p),p}(\omega) \neq V_\varepsilon^p(\omega)$ which is not convenient for the study of the limit $\varepsilon \to 0$. In order to overcome this difficulty, we consider the following discrete analogue of process (3.13):

$$U_g^\varepsilon(l,m) : V_\varepsilon^p(\omega) \to V_\varepsilon^p(\omega), \quad l,m \in \mathbb{Z}, \quad l \geq m. \tag{3.26}$$

Moreover, we assume, in addition, that the exponent p satisfies $p \geq 2p_{min}$ and use the following obvious embeddings:

$$V_\varepsilon^p(\omega) \subset V_0^p(\omega) \subset V_\varepsilon^{p/2}(\omega), \tag{3.27}$$

which are, in a fact, uniform with respect to ε, see Definition 2.1. Then, we have $p/2 > p_{min}$ and, consequently, all previous results remain true if we replace p by $p/2$. In particular, Theorem 2.3, Corollary 2.8 and embeddings (3.27) imply that

$$\begin{aligned} \|U_g^\varepsilon(l,m)u_m\|_{V_0^p(\omega)} \leq \|U_g^\varepsilon(l,m)u_m\|_{V_\varepsilon^p(\omega)} \leq Q(\|u_m\|_{V_\varepsilon^{p/2}(\omega)})e^{-\alpha(l-m)} + \\ + Q(\|g\|_{L_b^p(\Omega)}) \leq Q(2\|u_m\|_{V_0^p(\omega)})e^{-\alpha(l-m)} + Q(\|g\|_{L_b^p(\Omega)}), \end{aligned} \tag{3.28}$$

for every $u_m \in V_0^p(\omega)$, and the constant $\alpha > 0$ and the monotonic function Q are independent of $0 \leq \varepsilon \leq \varepsilon_0'$, $l,m \in \mathbb{Z}$ and $l \geq m$. Moreover, using Corollaries 3.5-3.6 and arguing analogously, we derive the following result.

Corollary 3.7. *Let the assumptions of Theorem 3.3 hold and let, in addition, $p > 2p_{min}$. Then the following estimates hold:*

$$\begin{aligned} \|U_g^\varepsilon(l,m)u_m^1 - U_g^\varepsilon(l,m)u_m^2 - D_uU_g^\varepsilon(l,m)(u_m^1)(u_m^1 - u_m^2)\|_{V_0^p(\omega)} \leq \\ \leq Ce^{-2\Lambda_0(l-m)}\|u_m^1 - u_m^2\|^2_{V_0^p(\omega)}, \end{aligned} \tag{3.29}$$

for every $u_m^1, u_m^2 \in V_0^p(\omega)$ and $l,m \in \mathbb{Z}$, $l \geq m$, consequently

$$\|D_uU_g^\varepsilon(l,m)(u_m^1) - D_uU_g^\varepsilon(l,m)(u_m^1)\|_{\mathcal{L}(V_0^p(\omega),V_0^p(\omega))} \leq C_qe^{2\Lambda_0(l-m)}\|u_m^1 - u_m^2\|_{V_0^p(\omega)}, \tag{3.30}$$

where the constant C depends on $\|u_m^1\|_{V_0^p(\omega)}$, $\|u_m^2\|_{V_0^p(\omega)}$ and $\|g\|_{L_b^p}$, but is independent of ε, l and m.

Thus, in contrast to the continuous dynamics $\{U_g^\varepsilon(t,\tau),\ \tau \in \mathbb{R},\ t \geq \tau\}$, discrete cascades (3.26) are well defined on the space $V_0^p(\omega)$ which is independent of ε.

To conclude, we formulate the result on injectivity of operators $U_g^\varepsilon(t,\tau)$.

Theorem 3.8. *Let the assumptions of Theorem 3.1 hold and let*

$$U_g^\varepsilon(t,\tau)u_\tau^1 = U_g^\varepsilon(t,\tau)u_\tau^2,$$

for some $\tau \in \mathbb{R}$, $t \geq \tau$ and $u_\tau^1, u_\tau^2 \in V_\varepsilon^p(\omega)$. Then, necessarily, $u_\tau^1 = u_\tau^2$.

The proof of this Theorem is based on the logarithmic convexity results (see [1]) for solutions of (2.1) and can be found, e.g. in [28].

4. Uniform attractors and their convergence as $\varepsilon \to 0$

In this section, we construct the global attractors $\mathcal{A}_\varepsilon$ for problems (2.1) and investigate their behavior as $\varepsilon \to 0$. Since the external forces $g(t)$ in (2.1) (which are assumed from now on to be defined on the whole cylinder Ω and to belong to the space $L^p_b(\Omega)$) depend explicitly on t, then we use below the skew-product technique in order to reduce the nonautonomous dynamical process (3.13) associated with problem (2.1) to the autonomous semigroup on the extended phase space. Following the general procedure (see [9] and [14]), we define a hull $\mathcal{H}(g)$ of the external forces g as follows:

$$\mathcal{H}(g) := \left[T_h g,\, h \in \mathbb{R}\right]_{L^p_{loc,w}(\Omega)}, \quad (T_h g)(t) := g(t+h). \tag{4.1}$$

Here $[\cdot]_{L^p_{loc,w}(\Omega)}$ stands for the closure in the space $L^p_{loc,w}(\Omega)$ which is the space $L^p_{loc}(\Omega)$ endowed by the weak topology. We recall that a sequence $g_k \to g$ in $L^p_{loc,w}(\Omega)$ as $k \to \infty$ if and only if $g_k\big|_{\Omega_T} \to g\big|_{\Omega_T}$ weakly in $L^p(\Omega_T)$, for every $T \in \mathbb{R}$. It is also well-known, that every bounded subset of $L^p_{loc,w}(\Omega)$ is precompact and metrizable and, consequently (due to the assumption $g \in L^p_b(\Omega)$), hull (4.1) is a compact metrizable subset of $L^p_{loc,w}(\Omega)$. Thus, a function $\xi(t)$ belongs to $\mathcal{H}(g)$ if and only if there exists a sequence $\{h_n\}_{n=1}^\infty \in \mathbb{R}$ such that

$$\xi = \lim_{n\to\infty} T_{h_n} g \ \text{ in the space } L^p_{loc,w}(\Omega). \tag{4.2}$$

Moreover, it also obvious that the group $\{T_h,\, h \in \mathbb{R}\}$ of temporal translations acts on $\mathcal{H}(g)$, i.e.

$$T_h : \mathcal{H}(g) \to \mathcal{H}(g), \quad T_h\mathcal{H}(g) = \mathcal{H}(g), \quad h \in \mathbb{R}. \tag{4.3}$$

In order to construct the attractor of (2.1), we consider the following family of equations of type (2.1) which correspond to all external forces $\xi \in \mathcal{H}(g)$:

$$a(\varepsilon^2\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - f(u) = \xi(t), \quad u\big|_{t=\tau} = u_\tau, \quad u\big|_{\partial\omega} = 0, \ \xi \in \mathcal{H}(g) \tag{4.4}$$

which generates the family $\{U^\varepsilon_\xi(t,\tau), \ \ \xi \in \mathcal{H}(g)\}$ of dynamical processes in $V^p_\varepsilon(\omega)$ (under the assumptions of Theorem 3.1). This family of processes generates a semigroup $\{\mathbb{S}^\varepsilon_t,\, t \geq 0\}$ on the extended phase space $\Phi_\varepsilon := V^p_{\varepsilon,w}(\omega) \times \mathcal{H}(g)$ (as usual $V^p_{\varepsilon,w}(\omega)$ denotes the space $V^p_\varepsilon(\omega)$ endowed by the weak topology) by the following expression:

$$\mathbb{S}^\varepsilon_t(u_0,\xi) := (U^\varepsilon_\xi(t,0)u_0, T_t\xi), \quad \mathbb{S}^\varepsilon_t : \Phi_\varepsilon \to \Phi_\varepsilon, \ t \geq 0, \ (u_0,\xi) \in \Phi_\varepsilon \tag{4.5}$$

(see [9] for the details). Thus, we describe the 'longtime' behavior of solutions of (4.4) in terms of the global attractor of semigroup (4.5) in the extended phase space Φ_ε. For the convenience of the reader, we recall below the definition of the attractor adapted to our case, see e.g. [4], [9] and [25] for the detailed exposition.

Definition 4.1. A set $\mathbb{A}_\varepsilon \subset \Phi_\varepsilon$ is a global attractor for the semigroup $\mathbb{S}^\varepsilon_t$ if the following conditions are satisfied:

1. The set $\mathbb{A}_\varepsilon$ is compact in Φ_ε.
2. This set is strictly invariant with respect to $\mathbb{S}_t$, i.e. $\mathbb{S}_t\mathbb{A}_\varepsilon = \mathbb{A}_\varepsilon$.
3. For every bounded subset $\mathbb{B} \subset \Phi_\varepsilon$ and every neighborhood $\mathcal{O}(\mathbb{A}_\varepsilon)$ of the set $\mathbb{A}_\varepsilon$ in the topology of Φ_ε, there exists $T = T(\mathbb{B},\mathcal{O})$ such that

$$\mathbb{S}^\varepsilon_t\mathbb{B} \subset \mathcal{O}(\mathbb{A}_\varepsilon), \quad \text{for} \quad t \geq T. \tag{4.6}$$

A projection $\mathcal{A}_\varepsilon := \Pi_1\mathbb{A}_\varepsilon$ of the global attractor $\mathbb{A}_\varepsilon$ to the first component is called a *uniform* attractor of family (4.4).

The next theorem establishes the existence of the attractor described above.

Theorem 4.2. *Let the assumptions of Theorem 3.1 hold and let $g \in L^p_b(\Omega)$. Then, semigroup* (4.5) *possesses a global attractor $\mathbb{A}_\varepsilon$ in the phase space Φ_ε and, consequently, family of problems* (4.4) *possesses a uniform attractor $\mathcal{A}_\varepsilon$ which can be described as follows:*

$$\mathcal{A}_\varepsilon = \Pi_0 \cup_{\xi \in \mathcal{H}(g)} \mathcal{K}^\varepsilon_\xi, \tag{4.7}$$

where $\mathcal{K}^\varepsilon_\xi$ is a set of all solutions of problem (4.4) *(with the right-hand side $\xi \in \mathcal{H}(g)$) which are defined for all $t \in \mathbb{R}$ and belong to $C_b(\mathbb{R}, V^p_\varepsilon(\omega))$ and $\Pi_0 u := u(0)$.*

Proof. According to the abstract theorem on the global (and uniform) attractors existence (see [4], [9] and [25]), it is sufficient to verify the following conditions:

1. The semigroup $\mathbb{S}^\varepsilon_t$ possesses a compact absorbing set $\mathbb{B}$ in Φ_ε.
2. The operators $\mathbb{S}^\varepsilon_t$ are continuous on $\mathbb{B}$, for every fixed $t \geq 0$.

Let us verify these conditions. It follows from estimate (2.29) that the set

$$\mathbb{B} := \{(u_0, \xi) \in \Phi_\varepsilon,\ \|u_0\|_{V^p_\varepsilon(\omega)} \leq 2Q(\|g\|_{L^p_b(\Omega)}),\ \xi \in \mathcal{H}(g)\} \tag{4.8}$$

is an absorbing set for the semigroup $\mathbb{S}^\varepsilon_t$ (here we have implicitly used the obvious fact that $\|\xi\|_{L^p_b(\Omega)} \leq \|g\|_{L^p_b(\Omega)}$, for every $\xi \in \mathcal{H}(g)$). Moreover, since the space $V^p_\varepsilon(\omega)$ is reflexive, then bounded subsets of it are precompact in a weak topology. Using the fact that $\mathcal{H}(g)$ is also compact, we derive that set (4.8) is compact in Φ_ε. Thus, the first condition is verified.

In order to verify the second one, we first note that the set $\mathbb{B}$ is metrizable, consequently, it is sufficient to verify only the sequential continuity of $\mathbb{S}^\varepsilon_t$ on $\mathbb{B}$. Indeed, let $(u^n_0, \xi_n) \in \mathbb{B}$ be an arbitrary (weakly) convergent sequence in $\mathbb{B}$ and let $(u_0, \xi_0) \in \mathbb{B}$ be its (weak) limit. We set $u_n(t) := U^\varepsilon_{\xi_n}(t,0)u^n_0$. Then, by definition, these functions satisfy the equations:

$$a(\varepsilon^2\partial_t^2 u_n(t) + \Delta_x u_n(t)) - \gamma\partial_t u_n(t) - f(u_n(t)) = \xi_n(t),\ u_n\big|_{t=0} = u^n_0,\ u_n\big|_{\partial\omega} = 0. \tag{4.9}$$

In order to verify the desired continuity, we need to prove that $u_n(t) \to u_0(t)$ weakly in $V^p_\varepsilon(\omega)$, for every $t \geq 0$, where $u_0(t) := U^\varepsilon_\xi(t,0)u_0$ is a solution of the limit (as $n \to \infty$) equation of (4.9). We note that the sequence u^n_0 is uniformly bounded in $V^p_\varepsilon(\omega)$ (since it converges weakly to u_0), consequently, due to Theorem 2.3, we have

$$\|u_n\|_{W^{(1,2),p}_\varepsilon(\Omega_T)} \leq C, \tag{4.10}$$

where C is independent of $T \geq 0$ and $n \in \mathbb{N}$. Therefore, the sequence of the solutions $u_n(t)$ is precompact in a weak topology of the space $W^{(1,2),p}_{\varepsilon,loc}(\Omega^0_+)$ (since this space is reflexive). Let $\bar{u} := \bar{u}(t) \in W^{(1,2),p}_{\varepsilon,loc}(\Omega^0_+)$ be an arbitrary limit point of this sequence. Then, due to estimate (4.10), the function $\bar{u}(t)$ belongs to $W^{(1,2),p}_{\varepsilon,b}(\Omega^0_+)$. Moreover, due to compactness of the embedding $W^{(1,2),p}_\varepsilon(\Omega_T) \subset C(\Omega_T)$, we have

$$u_{n_k} \to \bar{u}, \quad \text{strongly in} \quad C(\Omega_T), \quad T \in \mathbb{R}_+, \tag{4.11}$$

for the appropriate subsequence $\{n_k\}_{k=1}^\infty \in \mathbb{N}$. Passing now to the limit $k \to \infty$ in equations (4.9) and using (4.11) and that $\xi_n \to \xi$ weakly in $L^p_{loc}(\Omega)$, we derive that $\bar{u}$ is a bounded solution of the limit equation of (4.9). Since, due to Theorem 3.1, this solution is unique, then, necessarily, $\bar{u}(t) \equiv u_0(t) := U^\varepsilon_\xi(t,0)u_0$. Moreover, since the limit point $\bar{u}$ is arbitrary, then we have proved that $u_n \to u_0$ weakly in $W^{(1,2),p}_\varepsilon(\Omega_T)$, for every $T \in \mathbb{R}_+$ and, consequently, $u_n(t) \to u_0(t)$ weakly in $V^p_\varepsilon(\omega)$, for every $t \in \mathbb{R}_+$. Thus, the second condition of the abstract theorem on the attractors existence is also verified and, therefore, according to this theorem, the semigroup $\mathbb{S}^\varepsilon_t$ possesses indeed the global attractor $\mathbb{A}_\varepsilon$ in Φ_ε and the family of problems (4.4) possesses the uniform attractor $\mathcal{A}_\varepsilon := \Pi_1\mathbb{A}_\varepsilon \in V^p_\varepsilon(\omega)$.

Description (4.7) is also a standard corollary of that theorem, see [4] and [9]. Theorem 4.2 is proved. □

Remark 4.3. There exists an alternative way to introduce the concept of the uniform attractor of equation (2.1) without using the skew-product flow on the extended phase space Φ_ε. Namely, the set $\mathcal{A}_\varepsilon$ is a uniform attractor for equation (2.1) if the following conditions are satisfied:

1. The set $\mathcal{A}_\varepsilon$ is compact in $V^p_\varepsilon(\omega)$.

2. For every bounded subset $B \subset V^p_\varepsilon(\omega)$ and every neighborhood $\mathcal{O}(\mathcal{A}_\varepsilon)$ of $\mathcal{A}_\varepsilon$ in a weak topology of $V^p_\varepsilon(\omega)$ there exists $T = T(B, \mathcal{O})$ such that

$$U^\varepsilon_g(t+\tau,\tau)B \subset \mathcal{O}(\mathcal{A}_\varepsilon), \quad \text{for every } \tau \in \mathbb{R} \text{ and } t \geq T. \tag{4.12}$$

3. The set $\mathcal{A}_\varepsilon$ is a minimal set which satisfies 1) and 2).

The equivalence of this definition to Definition 4.1 is proved in [9].

Remark 4.4. If the initial external forces g satisfy the additional assumption

$$\mathcal{H}(g) \ \text{ is compact in a } \textit{strong} \text{ topology of } L^p_{loc}(\Omega), \tag{4.13}$$

then, arguing in a standard way (see, e.g. [9] and [27]), we can prove that the attractor $\mathbb{A}_\varepsilon$ attracts the bounded subsets of Φ_ε not only in a *weak* topology, but also in more natural *strong* topology of Φ_ε and $\mathcal{A}_\varepsilon$ is compact in a strong topology of $V^p_\varepsilon(\omega)$. Nevertheless, we prefer to use the weak topology in Definition 4.1, since the choice of the weak topology is more convenient for what follows.

Remark 4.5. Since the embeddings $V^p_\varepsilon(\omega) \subset V^{p-\delta}_\varepsilon(\omega)$, $\delta > 0$ and $V^p_\varepsilon(\omega) \subset C(\omega)$ are compact, then (4.12) implies the following convergence:

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \operatorname{dist}_{V^{p-\delta}_\varepsilon(\omega)\cap C(\omega)} \left(U^\varepsilon_g(t+\tau,\tau)B, \mathcal{A}_\varepsilon\right) = 0, \tag{4.14}$$

for every bounded set $B \subset V^p_\varepsilon(\omega)$ and every $\delta > 0$. Here an below $\operatorname{dist}_V(X,Y)$ denotes the nonsymmetric Hausdorff distance between sets X and Y in the space V:

$$\operatorname{dist}_V(X,Y) := \sup\nolimits_{x\in X} \inf\nolimits_{y\in Y} \|x-y\|_V. \tag{4.15}$$

The rest of this section is devoted to study the behavior of the attractors $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$. To this end, keeping in mind equation (1.4), it is convenient to consider slightly more general family of equations of the form (2.1):

$$a(\varepsilon^2\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - f(u) = g_\varepsilon(t), \ u\big|_{\partial\omega} = 0, \quad u\big|_{t=\tau} = u_\tau, \tag{4.16}$$

where the external forces depend explicitly on ε. We assume (following [8]) that these external forces are uniformly bounded in $L^p_b(\Omega)$:

$$\|g_\varepsilon\|_{L^p_b(\Omega)} \leq C, \tag{4.17}$$

where C is independent of ε, and converge to the limit external forces $g_0 \in L^p_b(\Omega)$ in the following weak sense: for every $\phi \in L^q(\Omega_0)$, $\frac{1}{q} + \frac{1}{p} = 1$, there exists a function $\alpha_\phi : \mathbb{R}_+ \to \mathbb{R}_+$, such that for all $h \in \mathbb{R}$

$$\Big|\int_{\Omega_0} (g_\varepsilon(t+h,x) - g_0(t+h,x)).\phi(t,x)\,dx\,dt\Big| \leq \alpha_\phi(\varepsilon) \ \text{ and } \ \lim_{\varepsilon\to 0^+} \alpha_\phi(\varepsilon) = 0. \tag{4.18}$$

The main result of this section is the following theorem.

Theorem 4.6. *Let the assumptions of Theorem 4.2 hold and let, in addition, the external forces $g_\varepsilon(t)$ satisfy* (4.17) *and* (4.18)*. Let also $\mathcal{A}_\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$, be the uniform attractors of equations* (4.16)*. Then, $\mathcal{A}_\varepsilon$ tends to $\mathcal{A}_0$ in the following sense: for every neighborhood $\mathcal{O}(\mathcal{A}_0)$ of $\mathcal{A}_0$ in a* weak *topology of $V_0^p(\omega)$ there exists $\varepsilon' = \varepsilon'(\mathcal{O})$ such that*

$$\mathcal{A}_\varepsilon \subset \mathcal{O}(\mathcal{A}_0), \quad \text{if} \quad \varepsilon \le \varepsilon'. \tag{4.19}$$

Proof. The proof of this theorem is based on the following lemma which clarifies the nature of convergence (4.18).

Lemma 4.7. *Let functions g_ε, $0 \le \varepsilon \le 1$, belong to $L^p_b(\Omega)$ and satisfy* (4.17) *and* (4.18)*. Then, the following conditions hold:*

1. The functions $g_\varepsilon(t)$ converges to $g_0(t)$ weakly in $L^p_{loc}(\Omega)$ and the set $\cup_\varepsilon \mathcal{H}(g_\varepsilon)$ is weakly precompact in $L^p_{loc}(\Omega)$.

2. For every sequences $\varepsilon_n \to 0$ and $\xi_n \in \mathcal{H}(g_{\varepsilon_n})$ such that $\xi_n \to \xi$ weakly in $L^p_{loc}(\Omega)$, the function ξ necessarily belongs to $\mathcal{H}(g_0)$ and

$$\Big|\int_{\Omega_0} (\xi_n(t+h,x) - \xi(t+h,x)).\phi(t,x)\,dt\,dx\Big| \le \alpha_\phi(\varepsilon_n), \tag{4.20}$$

for every $\phi \in L^{p'}_b(\Omega_0)$ and every $h \in \mathbb{R}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

3. The convergence (4.18) *is uniform with respect to ϕ belonging to compact sets in $L^{p'}(\Omega_0)$, i.e. the function α_ϕ in* (4.18) *can be chosen in such way that*

$$\alpha_V(\varepsilon) := \sup\nolimits_{\phi \in V} \alpha_\phi(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0^+, \tag{4.21}$$

for every compact subset $V \subset L^{p'}(\Omega_0)$.

The assertion of the lemma can be proved in a standard way, using the representation (4.2) for functions belonging to the halls $\mathcal{H}(g_\varepsilon)$ and basic properties of the weak convergence in reflexive spaces, see [8] and [9].

We are now ready to prove Theorem 4.6. We first note that, due to Theorem 2.3, Corollary 2.3 and estimate (4.17), we have

$$\|\mathcal{A}_\varepsilon\|_{V^p_\varepsilon(\omega)} + \sup_{\xi \in \mathcal{H}(g_\varepsilon)} \|\mathcal{K}^\varepsilon_\xi\|_{W^{(1,2),p}_{\varepsilon,b}(\Omega)} \le C, \tag{4.22}$$

where the constant C is independent of ε. Thus, in order to prove the theorem, it is sufficient to verify that, if $u^0_n \in \mathcal{A}_{\varepsilon_n}$, $\varepsilon_n \to 0$ as $n \to \infty$, be an arbitrary sequence which converges weakly in $V^p_0(\omega)$ to some $u_0 \in V^p_0(\omega)$, then $u_0 \in \mathcal{A}_0$ (see [9]). Taking into account description (4.7), estimates (4.17) and (4.22) and the weak compactness of bounded sets in reflexive spaces, this assertion can be reformulated as follows: if $\varepsilon_n \to 0$, $\xi_n \in \mathcal{H}(g_{\varepsilon_n})$ and $u_n \in \mathcal{K}^{\varepsilon_n}_{\xi_n}$ be arbitrary sequences such that $u_n \to u$ weakly in $W^{(1,2),p}_{0,loc}(\Omega)$ and $\xi_n \to \xi$ weakly in $L^p_{loc}(\Omega)$, then $\xi \in \mathcal{H}(g_0)$ and $u \in \mathcal{K}^0_\xi$. Let us verify this assertion. Indeed, the fact that $\xi \in \mathcal{H}(g_0)$ is an immediate corollary of Lemma 4.7. Thus, there only remains to pass to the weak limit in the following equations:

$$a(\varepsilon_n^2 u_n(t) + \Delta_x u_n(t)) - \gamma\partial_t u_n(t) - f(u_n(t)) = \xi_n(t),\ t \in \mathbb{R}, \quad u_n\big|_{\partial\omega} = 0. \tag{4.23}$$

We recall that the embedding $W^{(1,2),p}_0(\Omega_T) \subset C(\Omega_T)$ is compact, consequently, the weak convergence $u_n \to u$ in $W^{(1,2),p}_{loc}(\Omega)$ implies the strong convergence $u_n \to u$ in $C_{loc}(\Omega)$. Passing now to the limit $n \to \infty$ in (4.23), we derive that the function $u \in W^{(1,2),p}_{0,b}(\Omega)$ and satisfies

$$a\Delta_x u(t) - \gamma\partial_t u(t) - f(u(t)) = \xi(t), \quad t \in \mathbb{R}$$

and, consequently $u \in \mathcal{K}^0_\xi$ and Theorem 4.2 is proved. $\square$

Remark 4.8. Since the embedding $V_0^p(\omega) \subset V_0^{p-\delta}(\omega) \cap C(\omega)$ is compact, then (4.19) implies that

$$\lim_{\varepsilon\to 0} \operatorname{dist}_{V_0^{p-\delta}(\omega)\cap C(\omega)} (\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0, \tag{4.24}$$

for every $\delta > 0$.

To conclude this section, we consider the applications of Theorem 4.6 to equation (1.4) and, consequently, we assume from now on that

$$g_\varepsilon(t) := g(\varepsilon^{-1}t), \quad \text{for some } g \in L_b^p(\Omega). \tag{4.25}$$

Example 4.9. Let the assumptions of Theorem 4.2 hold, (4.25) be satisfied and the function $g \in L_b^p(\Omega)$ have the following heteroclinic profile structure: there exist $g_\pm := g_\pm(x) \in L^p(\omega)$ such that

$$\lim_{h\to\pm\infty} \|T_h g - g_\pm\|_{L^p(\Omega_0)} = 0. \tag{4.26}$$

Then, obviously, $g_\varepsilon \to g_0$ as $\varepsilon \to 0$ in $L_b^p(\Omega)$, where

$$g_0(t) := \begin{cases} g_+, & \text{for } t \geq 0, \\ g_-, & \text{for } t < 0 \end{cases} \tag{4.27}$$

and, consequently, (4.17) and (4.18) are also satisfied. Thus, due to Theorem 4.6, the uniform attractors $\mathcal{A}_\varepsilon$ of equations (4.16) (or, which is the same, the attractors of (1.4)) tend as $\varepsilon \to 0$ (in the sense of (4.19)) to the uniform attractor of the limit parabolic equation with the external forces (4.27). In particular, if $g_+ = g_-$ then the limit parabolic problem is autonomous.

In order to consider the next examples, we need the following proposition which is adopted to the study of oscillating in time external forces g_ε in (4.16).

Proposition 4.10. *Let $g \in L_b^p(\Omega)$ and g_ε be defined by* (4.25). *We also assume that there exists $\bar g = \bar g(x) \in L^p(\omega)$ such that*

$$\frac{1}{T}\int_t^{t+T} g(s)\,ds \to \bar g \ \text{ in } L^p(\omega) \ \text{ as } T \to \infty, \tag{4.28}$$

uniformly with respect to $t \in \mathbb{R}$. Then, the functions $g_\varepsilon(t) := g(\varepsilon^{-1}t)$, $\varepsilon \neq 0$ and $g_0(t) \equiv \bar g$ satisfy conditions (4.17) *and* (4.18).

For the proof of Proposition 4.10, see [8] or [9].

Example 4.11. Let the assumptions of Theorem 4.2 hold, (4.25) be satisfied and the function g belong to $C_b(\mathbb{R}, L^p(\omega))$ and be almost-periodic with respect to t with values in $L^p(\omega)$ (the latter means that the hull $\mathcal{H}(g)$ is compact in $C_b(\mathbb{R}, L^p(\omega))$, according to the Bochner-Amerio criterium, see [18]). Then, assumption (4.28) is satisfied, due to the Kronecker-Weyl theorem, see [18]. Thus, the uniform attractors $\mathcal{A}_\varepsilon$ of elliptic problems (4.16) with the rapidly oscillating external forces $g_\varepsilon(t) := g(\varepsilon^{-1}t)$ (g is now almost-periodic) converge as $\varepsilon \to 0$ to the global attractor $\mathcal{A}_0$ of the limit autonomous parabolic equation with the averaged external forces $g_0 \equiv \bar g$.

In conclusion, we give an example of oscillating external forces $g \in L_b^p(\Omega)$ which are not almost-periodic with respect to time, but satisfy the assumptions of Proposition 4.10 (see [9] for further examples).

Example 4.12. Let $g_1(t)$ and $g_2(t)$ be two *different* 1-periodic functions with respect to t which belong to $L^p_b(\Omega)$ and have zero mean. We set

$$g(t) := \begin{cases} g_1(t), & ;\text{for } t \in [4k^2, (2k+1)^2) \text{ and } k \in \mathbb{Z}, \\ g_2(t), & ;\text{for } t \in [(2k-1)^2, 4k^2) \text{ and } k \in \mathbb{Z}. \end{cases} \tag{4.29}$$

Then, obviously, this function is not almost-periodic with respect to t (even in the case where g_1 and g_2 are smooth), but condition (4.28) is obviously satisfied with $\bar g = 0$, since the periodic functions g_1 and g_2 have zero mean. Thus, in this case, the attractors $\mathcal{A}_\varepsilon$ of equations (4.16) with non almost-periodic rapidly oscillating external forces $g_\varepsilon(t) := g(\varepsilon^{-1}t)$ converge as $\varepsilon \to$ to the attractor $\mathcal{A}_0$ of the limit parabolic equation with zero external forces.

5. Local convergence as $\varepsilon \to 0$ of the individual solutions

In this section, we obtain several auxiliary results on the convergence of the solution $u_\varepsilon(t) := U^\varepsilon_{g_\varepsilon}(t,\tau)u_\tau$ as $\varepsilon \to 0$ to the corresponding solution $u_0(t) := U^0_{g_0}(t,\tau)u_\tau$ of the limit parabolic problem which will be essentially used in the next sections. We also assume (for simplicity) that condition (4.18) is satisfied with the *autonomous* limit function $g_0 \equiv \bar g \in L^p(\omega)$. Then, equations (4.16) converge as $\varepsilon \to 0$ to the following autonomous reaction-diffusion problem:

$$\gamma\partial_t u_0 = a\Delta_x u_0 - f(u_0) + \bar g, \quad u_0\big|_{\partial\omega} = 0, \quad u_0\big|_{t=\tau}, \tag{5.1}$$

which generates a dissipative semigroup $S_t := U^0_{\bar g}(t,0)$ in the phase space $V^p_0(\omega)$ and possesses the global attractor $\mathcal{A}_0 \subset V^p_0(\omega)$, see Theorems 2.3, 3.1 and 4.2. The following theorem gives the estimate for the $L^2(\omega)$-norm of distance between $U^\varepsilon_{g_\varepsilon}(t,\tau)$ and $S_{t-\tau}$.

Theorem 5.1. *Let the assumptions of Theorem 4.6 hold, $p > 2p_{min}$ and $g_0(t) \equiv \bar g \in L^p(\omega)$. Then, for every $R > 0$, there exist a function $\alpha_R : \mathbb{R}_+ \to \mathbb{R}_+$, $\lim_{\varepsilon\to0}\alpha_R(\varepsilon) = 0$ and a positive constant K such that, for every $\varepsilon \le \varepsilon_0'$, $h_\varepsilon \in \mathcal{H}(g_\varepsilon)$, $\tau \in \mathbb{R}$, $t \ge \tau$ and $u_\tau \in V^p_\varepsilon(\omega)$ satisfying $\|u_\tau\|_{V^p_0(\omega)} \le R$, the following estimate holds:*

$$\|U^\varepsilon_{h_\varepsilon}(t,\tau)u_\tau - S_{t-\tau}u_\tau\|_{L^2(\omega)} \le \alpha_R(\varepsilon)e^{K(t-\tau)}. \tag{5.2}$$

Proof. We set $u_\varepsilon(t) := U^\varepsilon_{h_\varepsilon}(t,\tau)u_\tau$, $u_0(t) := S_{t-\tau}u_\tau$ and $v_\varepsilon(t) := u_\varepsilon(t) - u_0(t)$. Then, the last function satisfies

$$\gamma\partial_t v_\varepsilon - a\Delta_x v_\varepsilon + l_\varepsilon(t)v_\varepsilon = a\varepsilon^2\partial_t^2 u_\varepsilon(t) + (h_\varepsilon(t) - \bar g), \quad v_\varepsilon\big|_{\partial\omega} = 0, \ v_\varepsilon\big|_{t=\tau} = 0, \tag{5.3}$$

where $l_\varepsilon(t) := \int_0^1 f'(su_\varepsilon(t) + (1-s)u_0(t))\,dt$. Multiplying equation (5.3) by $v_\varepsilon(t)$, integrating over $(\tau,t)\times\Omega$ and using that $l_\varepsilon(t) \ge -K$ and $\gamma = \gamma^* > 0$, we have

$$\begin{aligned} &\|v(t)\|^2_{L^2(\omega)} - K_1\int_\tau^t \|v(s)\|^2_{L^2(\omega)}\,ds \le \\ &\le C\varepsilon^2\int_\tau^t \Big(\|\partial_t u_\varepsilon(s)\|^2_{L^2(\omega)} + \|\partial_t u_0(s)\|^2_{L^2(\omega)}\Big)\,ds + C\varepsilon^4\|\partial_t u_\varepsilon(t)\|^2_{L^2(\omega)} + \\ &\qquad + C\Big|\int_\tau^t\int_\omega (h_\varepsilon(s) - \bar g).v(s)\,ds\,dx\Big|, \end{aligned} \tag{5.4}$$

where the constants C and K_1 depend only on γ, K and a. We now note that, due to the assumption $p > 2p_{min}$, we may apply estimate (2.7) with the exponent $p/2 > 2$ instead of p to equations (4.16). Then, using Lemma 8.2, estimate (4.17) and embedding (3.27), we have

$$(5.5)\quad \|u_\varepsilon\|^2_{W^{1,2}(\Omega_T)}+\|u_0\|^2_{W^{1,2}(\Omega_T)}+\varepsilon^3\|\partial_t u_\varepsilon(T)\|^2_{L^2(\omega)}+\|u_\varepsilon\|^2_{L^\infty(\Omega_T)}+\|u_0\|^2_{L^\infty(\Omega_T)}\le C_R,$$

where the constant C_R is independent on $\|u_\tau\|_{V_0^p(\omega)}$ (which satisfies $\|u_\tau\|_{V_0^p(\omega)}\le R$), $\varepsilon\le\varepsilon_0$, $h_\varepsilon\in\mathcal{H}(g_\varepsilon)$, $\tau\in\mathbb{R}$ and $t\ge\tau$. Thus, there only remains to estimate the last term in (5.4). To this end, we recall that the limit function $g_0(t)\equiv\bar g$ in (4.18) is now independent of t. Consequently, Lemma 4.7 implies that every sequence $h_{\varepsilon_k}\in\mathcal{H}(g_{\varepsilon_k})$, $\varepsilon_k\to 0$ converges weakly in $L^p_{loc}(\Omega)$ to the mean value $\bar g$ and (due to (4.20)), for every $\phi\in L^{p'}(\Omega_0)$, there exists $\alpha_\phi(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that

$$(5.6)\qquad \Big|\int_0^1\int_\omega (h_\varepsilon(t+s,x)-\bar g(x)).\phi(t,x)\,dx\,dt\Big|\le\alpha_\phi(\varepsilon),\quad \forall s\in\mathbb{R}\text{ and }h_\varepsilon\in\mathcal{H}(g_\varepsilon).$$

Moreover, this convergence is uniform with respect to ϕ belonging to compact sets in $L^{p'}(\Omega_0)$. We now note that, due to (5.5), the set of functions $\{T_s v_\varepsilon,\ s\ge\tau,\ \|u_\tau\|_{V_0^p(\omega)}\le R,\ h_\varepsilon\in\mathcal{H}(g_\varepsilon),\varepsilon\le\varepsilon_0'\}$ is bounded in $W^{1,2}(\Omega_0)$ and, consequently, it is compact in $L^{p'}(\Omega_0)$ (we recall that $\frac1p+\frac1{p'}=1$ and $p>2$, consequently, $p'<2$). Therefore, (5.6) implies that

$$(5.7)\qquad \Big|\int_{\tau+s}^{\tau+s+1}\int_\omega (h_\varepsilon(t,x)-\bar g).v_\varepsilon(t,x)\,dx\,dt\Big|\le\tilde\alpha_R(\varepsilon),$$

where $\tilde\alpha_R(\varepsilon)\to 0$ as $\varepsilon\to 0$ uniformly with respect to $h_\varepsilon\in\mathcal{H}(g_\varepsilon)$, $s\in\mathbb{R}_+$, $\varepsilon\le\varepsilon_0'$ and $\|u_\tau\|_{V_0^p(\omega)}\le R$. Inserting estimates (5.5) and (5.7) to (5.4), we have

$$(5.8)\qquad \|v(t)\|^2_{L^2(\omega)}-K_1\int_\tau^t\|v(s)\|^2_{L^2(\omega)}\,ds\le C_R(t-\tau)\varepsilon+C(t-\tau)\tilde\alpha_R(\varepsilon).$$

Applying the Gronwall inequality to estimate (5.8), we finish the proof of Theorem 5.1. □

The following corollary reformulates estimate (5.2) in terms of discrete cascades (3.26) acting on the phase space $V_0^p(\omega)$.

Corollary 5.2. *Let the assumptions of Theorem 5.1 hold and let, in addition, the external forces g_ε be uniformly bounded in $L^{p+\delta}_b(\Omega)$, for some $\delta>0$, i.e.*

$$(5.9)\qquad \|g_\varepsilon\|_{L_b^{p+\delta}(\Omega)}\le C,$$

where C is independent of ε. Then, the following estimate is valid:

$$(5.10)\qquad \|U^\varepsilon_{h_\varepsilon}(l,m)u_m-S_{l-m}u_m\|_{V_0^p(\omega)}\le e^{K'(l-m)}\bar\alpha_R(\varepsilon),\quad \|v_m\|_{V_0^p(\omega)}\le R,$$

where the constant K' and the function $\bar\alpha_R$ ($\bar\alpha_R(\varepsilon)\to 0$ as $\varepsilon\to 0^+$) are independent of $\varepsilon\le\varepsilon_0$, $h_\varepsilon\in\mathcal{H}(g_\varepsilon)$, $u_m\in V_0^p(\omega)$ and $l,m\in\mathbb{Z}$ (with $l\ge m$).

Proof. Since the functions g_ε are assumed to be uniformly bounded in $L_b^{p+\delta}(\omega)$, then (replacing the exponent p by $p+\delta$) we derive from Theorem 2.3 and Corollary 2.8 (analogously to (3.28)) that

$$(5.11)\qquad \|U^\varepsilon_{h_\varepsilon}(l,m)u_m-S_{l-m}u_m\|_{V_0^{p+\delta}(\omega)}\le C_R,$$

where the constant C_R is independent of ε, h_ε, l,m and u_m. Estimate (5.10) is an immediate corollary of (5.2), (5.11) and the following interpolation inequality:

$$(5.12)\qquad \|w\|_{V_0^p(\omega)}\le C\|w\|^{\kappa_\delta}_{L^2(\omega)}\cdot\|w\|^{1-\kappa_\delta}_{V_0^{p+\delta}(\omega)},$$

for the appropriate $0<\kappa_\delta<1$ (see, e.g. [26]) and Corollary 5.2 is proved. □

Our next task is to obtain the analogue of Theorem 5.1 and Corollary 5.2 for the Frechet derivatives of the processes $U^\varepsilon_{g_\varepsilon}(t,\tau)$.

Theorem 5.3. *Let the assumptions of Theorem 5.1 hold. Then, for every $R\in\mathbb{R}_+$ and u_τ, $\|u_\tau\|_{V_0^p(\omega)}\le R$, the following estimate is valid:*

$$\|D_uU^\varepsilon_{h_\varepsilon}(t,\tau)(u_\tau)-D_uS_{t-\tau}(u_\tau)\|_{\mathcal{L}(V_0^p(\omega),L^2(\omega))}\le e^{K''(t-\tau)}\alpha_R(\varepsilon), \tag{5.13}$$

where the constant K'' and the function α_R ($\alpha_R(\varepsilon)\to 0$ as $\varepsilon\to 0^+$) are independent of $\varepsilon\le\varepsilon_0$, $h_\varepsilon\in\mathcal{H}(g_\varepsilon)$, $u_\tau\in V_0^p$ and $t,\tau\in\mathbb{R}$ (with $t\ge\tau$).

Proof. We set $w_\varepsilon(t):=D_uU^\varepsilon_{h_\varepsilon}(t,\tau)(u_\tau)\xi$ and $w_0(t):=D_uS_{t-\tau}(u_\tau)\xi$, where $\xi\in V_0^p(\omega)$ is an arbitrary vector. Then, according to Theorem 3.3, these functions satisfy the equations

$$\begin{cases} a(\varepsilon^2\partial_t^2w_\varepsilon+\Delta_xw_\varepsilon)-\gamma\partial_tw_\varepsilon-f'(u_\varepsilon(t))w_\varepsilon=0,\ w_\varepsilon\big|_{\partial\omega}=0,\ w_\varepsilon\big|_{t=\tau}=\xi,\\ a\Delta_xw_0-\gamma\partial_tw_0-f'(u_0(t))w_0=0,\ \ w_0\big|_{\partial\omega}=0,\ \ w_0\big|_{t=\tau}=\xi,\end{cases} \tag{5.14}$$

where $u_0(t)$ and $u_\varepsilon(t)$ are the same as in the proof of Theorem 5.1. Then, according to Lemma 3.2, embeddings (3.27) and Corollary 8.7, analogously to (5.5), we have

$$\|w_\varepsilon\|^2_{W^{1,2}(\Omega_T)}+\varepsilon^3\|\partial_tw_\varepsilon(T)\|^2_{L^2(\omega)}+\|w_0\|^2_{W^{1,2}(\Omega_T)}+\\ +\|w_0\|^2_{L^\infty(\Omega_T)}+\|w_\varepsilon\|^2_{L^\infty(\Omega_T)}\le C_Re^{2\Lambda_0(T-\tau)}\|\xi\|^2_{V_0^p(\omega)}, \tag{5.15}$$

where the constant C_R is independent of $\varepsilon\le\varepsilon_0'$, $h_\varepsilon\in\mathcal{H}(g_\varepsilon)$, $\xi\in V_0^p(\omega)$, $u_\tau\in V_0^p(\omega)$ (which satisfies $\|u_\tau\|_{V_0^p(\omega)}\le R$), $\tau\in\mathbb{R}$ and $T\ge\tau$.

We now set $\theta_\varepsilon(t):=w_\varepsilon(t)-w_0(t)$. Then, this function satisfies

$$\gamma\partial_t\theta_\varepsilon-a\Delta_x\theta_\varepsilon+f'(u_\varepsilon(t))\theta_\varepsilon=a\varepsilon^2\partial_t^2w_\varepsilon(t)-[f(u_\varepsilon(t))-f(u_0(t))]w_\varepsilon(t),\ \ \theta_\varepsilon\big|_{t=\tau}=0.$$

Multiplying this equation by $\theta_\varepsilon(t)$, integrating over $(T,\tau)\times\omega$ and using that $f'\ge -K$ and the functions $u_\varepsilon(t)$ and $u_0(t)$ are uniformly bounded in the L^∞-norm, we have (analogously to (5.4))

$$\|\theta_\varepsilon(t)\|^2_{L^2(\omega)}-K_3\int_\tau^T\|\theta_\varepsilon(t)\|^2_{L^2(\omega)}\,dt\le\\ \le C\varepsilon^2\int_\tau^T\left(\|\partial_tw_\varepsilon(t)\|^2_{L^2(\omega)}+\|\partial_tw_0(t)\|^2_{L^2(\omega)}\right)dt+C\varepsilon^4\|\partial_tw_\varepsilon(T)\|^2_{L^2(\omega)}+\\ +C_R\int_\tau^T\|u_\varepsilon(t)-u_0(t)\|^2_{L^2(\omega)}\|w_\varepsilon(t)\|^2_{L^\infty(\omega)}\,dt, \tag{5.16}$$

where the constants C, K_3 and C_R are independent of ε and T. Inserting estimates (5.2) and (5.15) to the right-hand side of (5.16), we have

$$\|\theta_\varepsilon(T)\|^2_{L^2(\omega)}-K_3\int_\tau^T\|\theta_\varepsilon(t)\|^2_{L^2(\omega)}\,dt\le C_R'(T-\tau)(\varepsilon+\alpha_R(\varepsilon)^2)e^{4\Lambda_0(T-\tau)}\|\xi\|^2_{V_0^p(\omega)}. \tag{5.17}$$

Applying the Gronwall inequality to this estimate, we derive estimate (5.13) (with the appropriate new function α_R) and finish the proof of Theorem 5.3. □

The following corollary is the analogue of Corollary 5.2 for the Frechet derivatives.

Corollary 5.4. *Let the assumptions of Theorem 5.1 hold and let, in addition, the external forces g_ε be uniformly bounded in $L_b^{p+\delta}(\Omega)$, for some $\delta>0$ (i.e., (5.9) be satisfied) Then, the following estimate is valid, for every $R\in\mathbb{R}_+$ and $u_m\in V_0^p(\omega)$ with $\|u_m\|_{V_0^p(\omega)}\le R$:*

$$\|D_uU^\varepsilon_{h_\varepsilon}(l,m)(u_m)-D_uS_{l-m}(u_m)\|_{\mathcal{L}(V_0^p(\omega),V_0^p(\omega))}\le e^{K''(l-m)}\bar\alpha_R(\varepsilon), \tag{5.18}$$

where the constant K'' and the function $\bar\alpha_R$ ($\bar\alpha_R(\varepsilon)\to 0$ as $\varepsilon\to 0^+$) are independent of $\varepsilon\le\varepsilon_0$, $h_\varepsilon\in\mathcal{H}(g_\varepsilon)$, $u_m\in V_0^p$ and $l,m\in\mathbb{Z}$ (with $l\ge m$).

Indeed, it follows from (5.15) and Corollary 8.7 (where the exponent p is replaced by $p+\delta$) that

$$\|D_uU^\varepsilon_{h_\varepsilon}(l,m)(u_m)-D_uS_{l-m}(u_m)\|_{\mathcal{L}(V_0^p(\omega),V_0^{p+\delta}(\omega))}\le C_R'e^{\Lambda_0(l-m)},\tag{5.19}$$

where $l\ge m+1$ and the constant C_R' is independent of ε, l and m. Estimate (5.18) is an immediate corollary of (5.19), (5.13) and (5.12).

To conclude this section, we investigate the dependence of functions $\alpha_R(\varepsilon)$ and $\bar\alpha_R(\varepsilon)$ which are introduced in Theorems 5.1-5.3 and Corollaries 5.2-5.4 respectively on $\varepsilon\to 0$. For simplicity, we assume that the external forces $g_\varepsilon(t)$ satisfy (4.25) (i.e., $g_\varepsilon(t):=g(\varepsilon^{-1}t)$) and and $g\in L_b^{p+\delta}(\Omega)$.

Theorem 5.5. *Let the assumptions of Theorem 5.1 hold and* (4.25) *be satisfied for some $g\in L_b^{p+\delta}(\Omega)$, $\delta>0$. Assume also that the function $g(t)-\bar g$ has a bounded primitive, i.e.*

$$g(t)-\bar g=\partial_tG(t),\tag{5.20}$$

for some $G\in L_b^2(\omega)$. Then, the functions $\alpha_R(\varepsilon)$ in Theorems 5.1 and 5.3 and the functions $\bar\alpha_R(\varepsilon)$ in Corollaries 5.2 and 5.4 have the following structure:

$$\alpha_R(\varepsilon):=C_R\varepsilon^{1/2},\quad \bar\alpha_R(\varepsilon):=\bar C_R\varepsilon^{\kappa_\delta/2},\tag{5.21}$$

where κ_δ is the same as in (5.12) *and the constants C_R and $\bar C_R$ depend on R, but are independent of ε.*

Proof. We first note that, due to representation (4.2) and the weak compactness of bounded subsets of $L^p_{loc}(\Omega)$, it follows from (5.20) that, for every $h\in\mathcal{H}(g)$, there exists $H\in\mathcal{H}(G)$ such that

$$h(t)-\bar g=\partial_tH(t).\tag{5.22}$$

We also note that, according to (5.4)-(5.5) and (5.17), it is sufficient to verify that

$$I_\varepsilon(T):=\Big|\int_\tau^T\int_\omega(h_\varepsilon(t,x)-\bar g(x)).v_\varepsilon(t,x)\,dt\,dx\Big|\le C_R''(T-\tau+1)\varepsilon,\tag{5.23}$$

where $h\in\mathcal{H}(g)$, $h_\varepsilon(t):=h(\varepsilon^{-1}t)$ and the constant C_R is independent of ε. Let us verify this inequality. Indeed, it follows from (5.22) that

$$h_\varepsilon(t)-\bar g=\varepsilon\partial_tH_\varepsilon(t),\quad H_\varepsilon(t):=H(\varepsilon^{-1}t),\tag{5.24}$$

consequently, integrating by parts in (5.23), we have

$$\begin{aligned}I_\varepsilon(T)\le\varepsilon\Big|\int_\tau^T\int_\omega H_\varepsilon(t).\partial_tv_\varepsilon(t)\,dx\,dt\Big|+\varepsilon\Big|\int_\omega H_\varepsilon(T).v_\varepsilon(T)\,dx\Big|\le\\ \le C\varepsilon(T-\tau+1)\left(\|G\|_{L^2_b(\Omega)}+\|\partial_tG\|_{L^2_b(\Omega)}\right)\|v_\varepsilon\|_{W^{1,2}_b(\Omega)},\end{aligned}\tag{5.25}$$

where the constant C is independent of ε (here we implicitly used that $\|H_\varepsilon\|_{L^2_b(\Omega)}\le\|H\|_{L^2_b(\Omega)}\le\|G\|_{L^2_b(\Omega)}$). Estimate (5.23) is an immediate corollary of (5.25) and (5.5). Theorem 5.5 is proved. ∎

6. Nonautonomous unstable manifolds of the nonlinear elliptic equation

In this section, using the standard perturbation technique, we define, for a sufficiently small $\varepsilon > 0$, the nonautonomous unstable manifold of nonautonomous elliptic system (2.1) which corresponds to the hyperbolic equilibrium of the limit parabolic equation (5.1). This result will be used in the next section in order to construct the nonautonomous regular attractor for system (2.1). For simplicity, we restrict ourselves to consider only the case of rapidly oscillating external forces $g_\varepsilon(t) := g(\varepsilon^{-1}t)$ where g is an almost-periodic function with respect to t with values in $L^{p+\delta}(\omega)$:

$$g \in AP(\mathbb{R}, L^{p+\delta}(\omega)), \quad p > 2p_{min}, \quad \delta > 0, \tag{6.1}$$

although all of the results formulated below remain true (after minor changing) under assumptions of previous section. Moreover, since the results of this section are more or less standard, we give below only schematic proofs resting the details to the reader.

Our main assumption is that equation (5.1) possesses a hyperbolic equilibrium $z_0 \in V_0^p(\omega)$, i.e.

$$a\Delta_x z_0 - f(z_0) + \bar{g} = 0 \ \text{ and } \ \sigma(L_{z_0}) \cap \{\operatorname{Re}\lambda = 0\} = \varnothing \tag{6.2}$$

where $L_{z_0} := \gamma^{-1}(a\Delta_x - f'(z_0))$ and $\sigma(L)$ denotes the spectrum of the operator L. It is well known that hyperbolicity assumption (6.2) implies existence of the spectral decomposition

$$V_0^p(\omega) = V_+ + V_-, \quad V_+ \cap V_- = \{0\}, \tag{6.3}$$

where the linear subspaces $V_\pm$ are invariant with respect to the operator L_{z_0} and satisfy the following properties:

$$\sigma(L_{z_0}\big|_{V_+}) \subset \{\operatorname{Re}\lambda \geq \nu\}, \quad \sigma(L_{z_0}\big|_{V_-}) \subset \{\operatorname{Re}\lambda \leq -\nu\},$$

for a sufficiently small positive ν. Moreover, the dimension of the unstable subspace V_+ is finite and is called the instability index of the equilibrium z_0:

$$\operatorname{ind}_{z_0} := \dim V_+ < \infty, \tag{6.4}$$

see e.g. [4] and [26]. It is also well known (see [4], [25]) that spectral decomposition (6.3) generates two invariant manifolds $\mathcal{M}^+_{z_0}$ and $\mathcal{M}^-_{z_0}$ for *nonlinear* problem (5.1) in a sufficiently small neighborhood of z_0 which correspond to linear subspaces V_+ and V_- respectively. Since only unstable manifolds are important for the attractors theory, we formulate below the rigorous result on the unstable manifold $\mathcal{M}^+_{z_0}$ only.

Theorem 6.1. *Let the assumptions of Theorem 5.1 hold and let, in addition,* (6.2) *be satisfied. Then, there exists a small neighborhood $\mathcal{W}_{z_0}$ of the equilibrium z_0 (in the V_0^p-metric) such that the set*

$$\mathcal{M}^{+,loc}_{z_0} := \big\{u_0 \in V_0^p(\omega)\,,\ \exists u \in C_b(\mathbb{R}, V_0^p(\omega)) \text{ which solves } (5.1),\ u(0) = u_0,\\ \lim_{t\to-\infty} u(t) = z_0 \ \text{ and } \ u(t) \in \mathcal{W}_{z_0} \ \ \forall t \leq 0\big\} \tag{6.5}$$

is a finite dimensional C^1-submanifold of $V_0^p(\omega)$ which is diffeomorphed to V_+.

The main task of this Section is to construct the analogue of unstable manifold (6.5) for nonlinear *elliptic* equation (4.16) if $\varepsilon > 0$ is small enough. Since equation (4.16) is 'nonautonomous' then we first need to find the analogue of the hyperbolic equilibrium z_0.

Proposition 6.2. *Let the assumptions of Theorem 5.1 hold and* (6.1)-(6.2) *be satisfied. Then there exist $\tilde\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for every $\varepsilon \le \tilde\varepsilon_0$, $h \in \mathcal{H}(g)$ there exists a unique solution $u = u^\varepsilon_{h_\varepsilon,z_0}(t)$ of the problem*

$$a(\varepsilon^2\partial_t^2 u + \Delta_x u) - \gamma\partial_t u - f(u) = h_\varepsilon(t),\ t \in \mathbb{R}, \quad h_\varepsilon(t) := h(\varepsilon^{-1}t), \tag{6.6}$$

which satisfies the condition $\|u^\varepsilon_{h_\varepsilon,z_0} - z_0\|_{C_b(\mathbb{R},V_0^p(\omega))} \le \delta_0$. Moreover, this solution is almost-periodic with respect to t:

$$u^\varepsilon_{h_\varepsilon,z_0} \in AP(\mathbb{R}, V^p_\varepsilon(\omega)) \text{ and } \quad \|u^\varepsilon_{h_\varepsilon,z_0} - z_0\|_{C_b(\mathbb{R},V_0^p(\omega))} \le C\bar\alpha_{R_0}(\varepsilon), \tag{6.7}$$

where the constants R_0 and C are independent of ε and h and the function $\bar\alpha_{R_0}(\varepsilon)$ is the same as in Corollaries 5.2 and 5.4. In particular, $u^\varepsilon_{h_\varepsilon,z_0} \to z_0$ as $\varepsilon \to 0$.

Sketch of proof. Instead of solving (6.6), we first solve the following discrete analogue of this equation:

$$u(n+1) = U^\varepsilon_{h_\varepsilon}(n+1,n)u(n) \tag{6.8}$$

in the space of sequences $u \in l^\infty(\mathbb{Z}, V_0^p(\omega))$. We are going to solve (6.8) near the constant sequence $u(n) \equiv z_0$ using the implicit function theorem. Indeed, due Corollaries 3.7, 5.2 and 5.4, the operators $U^\varepsilon_{h_\varepsilon}(n+1,n)$ are close to S_1 together with their Frechet derivatives as $\varepsilon \to 0$ (uniformly with respect to n and h). Moreover, the linearized problem (which corresponds to (6.8) at $\varepsilon = 0$ and $u = z_0$)

$$w(n) - D_uS_1(z_i)w(n) = \tilde h(n) \tag{6.9}$$

is uniquely solvable in $l^\infty(\mathbb{Z}, V_0^p(\omega))$ for every $\tilde h \in l^\infty(\mathbb{Z}, V_0^p(\omega))$ (due to hyperbolicity of the equilibrium z_i. Thus, the implicit function theorem is indeed applicable to equation (6.8) and gives the existence and uniqueness of the solution $\bar u^\varepsilon_{h_\varepsilon,z_0} \in l^\infty(\mathbb{Z}, V_0^p(\omega))$ of (6.8), for sufficiently small $\varepsilon > 0$, which belongs to a small neighborhood of the equilibrium z_0. Moreover, it follows in a standard way from (3.29), (5.10) and (5.18) that this solution satisfies

$$\|\bar u^\varepsilon_{h_\varepsilon,z_0}(n) - z_0\|_{V_0^p(\omega)} \le C\bar\alpha_{R_0}(\varepsilon),\ \forall n \in \mathbb{Z}, \tag{6.10}$$

where C is independent of h, i and ε and R_0 is the radius of the uniform (with respect to ε and h) absorbing ball in $V_0^p(\omega)$ for the discrete processes $U^\varepsilon_{h_\varepsilon}(l,m)$, $l, m \in \mathbb{Z}$, $l \ge m$, and $h \in \mathcal{H}(g)$ (which exists due to estimate (3.28)). The desired continuous function $u^\varepsilon_{h_\varepsilon,z_0}(t)$, $t \in \mathbb{R}$, can be now defined as follows:

$$u^\varepsilon_{h_\varepsilon,z_0}(t) := u^\varepsilon_{T_th_\varepsilon,z_0}(0). \tag{6.11}$$

Obviously, (6.11) is a solution of (6.6) which belongs to the space $C_b(\mathbb{R}, V_\varepsilon^{p+\delta}(\omega))$ (due to Corollary 2.8) and satisfies

$$\|u^\varepsilon_{h_\varepsilon,z_0}\|_{C_b(\mathbb{R},V_\varepsilon^{p+\delta}(\omega))} \le C, \tag{6.12}$$

where the constant C is independent of ε, i and h. The uniqueness of this solutions (in a small neighborhood of z_0) is an immediate corollary of the uniqueness of the discrete solution $\bar u^\varepsilon_{h_\varepsilon,z_0}(m)$. The almost-periodicity of this function is a standard corollary of that uniqueness, see e.g. [18]). Proposition 6.2 is proved.

$\square$

Now we are ready to define the analogues of the unstable sets $\mathcal{M}^+_{z_0}$ for problem (6.6). Since this problem is 'nonautonomous' then these manifolds also depend on t.

Definition 6.3. Let the assumptions of Proposition 6.2 hold. For every $\varepsilon \le \widehat{\varepsilon}_0$, $h \in \mathcal{H}(g)$ and $t \in \mathbb{R}$, we define the set $\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(\tau)$ as follows:

$$\begin{aligned} \mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(\tau) := \big\{ & u_\tau \in V^p_\varepsilon(\omega),\ \ \exists u \in C_b(\mathbb{R}, V^p_\varepsilon(\omega)),\ \text{which solves (6.6)}, \\ & u(\tau) = u_\tau,\ \ \lim_{t\to-\infty} \|u(t) - u^\varepsilon_{h_\varepsilon,z_0}(t)\|_{V^p_\varepsilon(\omega)} = 0,\ \ u(t) \in \mathcal{W}_{z_0}\ \ \forall t \le \tau \big\}, \end{aligned} \tag{6.13}$$

where $u^\varepsilon_{h_\varepsilon,z_0}(t)$ is the solution of (6.6) constructed in Proposition 6.2 and $\mathcal{W}_{z_0}$ is the same as in Theorem 6.1. Thus, set (6.13) consists of the values $u(\tau)$ at moment τ of all solutions $u \in C_b(\mathbb{R}, V^p_\varepsilon(\omega))$ of (6.6) which tend to $u^\varepsilon_{h_\varepsilon,z_0}(t)$ as $t \to -\infty$ and belong to $\mathcal{W}_{z_0}$ for $t \le \tau$.

The following theorem is the 'nonautonomous' analogue of Theorem 6.1.

Theorem 6.4. *Let the assumptions of Proposition 6.2 hold. Then, for sufficiently small $\varepsilon \le \varepsilon_0 \ll 1$ and every $\tau \in \mathbb{R}$, the sets* (6.18) *are C^1-submanifolds of $V^p_\varepsilon(\omega)$ which are diffeomorphed to V_+.*

Sketch of the proof. We first note that

$$\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(\tau) = \mathcal{M}^{+,loc}_{\varepsilon,T_\tau h_\varepsilon,z_0}(0) \tag{6.14}$$

and, consequently, it is sufficient to prove the theorem for $\tau = 0$ only. In order to do so, we seek for the solution $u(n)$, $n \le 0$, of the following discrete problem

$$u(n+1) = U^\varepsilon_{h_\varepsilon}(n+1,n)u(n),\ \ n \in \mathbb{Z}_-,\ \ u(n) \in \mathcal{W}_{z_0} \tag{6.15}$$

in the space $\Theta_- := l^\infty(\mathbb{Z}_-, V^p_0(\omega))$ of V^p_0-valued sequences which remain bounded as $n \to -\infty$. Then, as known (see e.g. [12] and [13, 30]) the union of all initial values $u(0)$ for such sequences gives the desired unstable set $\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(0)$. Thus, there remains to prove that the set of all solutions $u(n)$ of (6.15) belonging to Θ_- is a submanifold of Θ_-. To this end, we introduce a new sequence $w(n) := u(n) - u^\varepsilon_{h_\varepsilon,z_0}(n)$ where $u^\varepsilon_{h_\varepsilon,z_0}(t)$ is defined in Proposition 6.2 and consider the following problem:

$$\begin{aligned} & w(n+1) = \mathbb{T}_{n,h_\varepsilon,\varepsilon} w(n),\ \ \Pi_+ w(0) = w_0 \in V_+, \\ & \mathbb{T}_{n,h_\varepsilon,\varepsilon} w := U^\varepsilon_{h_\varepsilon}(n+1,n)(u^\varepsilon_{h_\varepsilon,z_0}(n) + w) - U^\varepsilon_{h_\varepsilon}(n+1,n)u^\varepsilon_{h_\varepsilon,z_0}(n), \end{aligned} \tag{6.16}$$

where Π_+ is a spectral projector to the spectral space V_+. For every w_0, we find the solution $w \in \Theta_-$ using the implicit function theorem. Indeed, due to Corollaries 3.7, 5.2 and 5.4 and Proposition 6.2, the operators $\mathbb{T}_{n,h_\varepsilon,\varepsilon}(w)$ tend together with their Frechet derivative to $S_1(w + z_0) - z_0$ (where S_t is a solving semigroup of the limit parabolic equation (5.1)) as $\varepsilon \to 0$ uniformly with respect to $h \in \mathcal{H}(g)$ and $n \in \mathbb{Z}$. Moreover, the linearized (at $w = 0$) limit equation

$$v(n+1) = D_{u_0} S_1(z_0) v(n),\ \ \Pi_+ v(0) = v_0,\ \ n \in \mathbb{Z}_- \tag{6.17}$$

is uniquely solvable in Θ_-, for every $v_0 \in V_+$ (due to the hyperbolicity of the equilibrium z_0). Applying the implicit function theorem to equation (6.16), we derive that, for every $\varepsilon \le \varepsilon_0$ and every w_0 belonging to a sufficiently small neighborhood of zero in V_+, there exists a unique solution $w \in \Theta_-$ of (6.16) and that the set of all such solutions is a C^1-submanifold in Θ_- diffeomorphed to V_+. Thus, we have verified that $\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(0)$ is a submanifold in $V^p_0(\omega)$. Using now the smoothing property for the operators $U^\varepsilon_{h_\varepsilon}(t,\tau)$ and embeddings (3.27), it is easy to show that $\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(0)$ is a submanifold not only in $V^p_0(\omega)$, but also in $V^p_\varepsilon(\omega)$. Theorem 6.4 is proved.

□

Remark 6.5. It is not difficult to verify that the manifolds $\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(\tau)$ are almost periodic with respect to τ and tend to the unstable manifold $\mathcal{M}^{+,loc}_{z_0}$ of the limit parabolic problem (5.1) as $\varepsilon\to 0$ uniformly with respect to $h\in\mathcal{H}(g)$ and $\tau\in\mathbb{R}$, [12, 13] and [15] for the details.

Finally, we define the *global* unstable manifolds $\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(\tau):=\mathcal{M}^{+,gl}_{\varepsilon,h_\varepsilon,z_0}(\tau)$ as a union of all values $u(\tau)$ for all solutions $u\in C_b(\mathbb{R},V^p_\varepsilon(\omega))$ of problem (6.6) which tend to $u^\varepsilon_{h_\varepsilon,z_0}(t)$ as $t\to-\infty$. Then, these sets can be expressed in terms of the local unstable manifolds via

$$\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(\tau)=\cup_{n=1}^\infty U^\varepsilon_{h_\varepsilon}(\tau,\tau-n)\mathcal{M}^{+,loc}_{\varepsilon,h_\varepsilon,z_0}(\tau-n). \tag{6.18}$$

Moreover, these sets are, obviously, strictly invariant with respect to the dynamical process $U^\varepsilon_{h_\varepsilon}(t,\tau)$:

$$U^\varepsilon_{h_\varepsilon}(t,\tau)\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(\tau)=\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(t) \tag{6.19}$$

and satisfy the following translation property:

$$\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(\tau)=\mathcal{M}^{+}_{\varepsilon,T_\tau h_\varepsilon,z_0}(0), \tag{6.20}$$

see e.g. [12] for the details.

Remark 6.6. Arguing in a standard way, it is not difficult to verify that expression (6.18) together with the backward uniqueness proved in Theorem 3.8 allows indeed to endow the sets $\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(\tau)$ by the structure of a C^1-manifold diffeomorphed to V_+, see [4]. We however note that, in contrast to local unstable manifolds, the global ones are not, in general, *submanifolds* of $V^p_\varepsilon(\omega)$ (even in the limit autonomous case $\varepsilon=0$). Indeed, if the limit equation (5.1) possesses a homoclinic orbit to the equilibrium z_0, then the corresponding global unstable manifold $\mathcal{M}^+_{z_0}$ cannot be a submanifold of $V^p_0(\omega)$. Nevertheless, in particular case where the limit parabolic equation possesses a global Lyapunov function, the sets $\mathcal{M}^{+}_{\varepsilon,h_\varepsilon,z_0}(\tau)$ are occurred to be *submanifolds* of the phase space V^p_ε for a sufficiently small ε, see [4] for the details.

7. The nonautonomous regular attractor

In this section, using the theory of nonautonomous perturbations of regular attractors (see [12]), we obtain the detailed description of the structure of attractors $\mathcal{A}_\varepsilon$, $\varepsilon\ll 1$, of equations (4.16) in case where the limit parabolic equation (5.1) is autonomous and possesses a global Lyapunov function. In order to have the explicit expression for the Lyapunov function, we assume that

$$a=a^* \quad\text{and}\quad f(u):=\nabla_u F(u),\quad\text{for some}\quad F\in C^1(\mathbb{R}^k,\mathbb{R}). \tag{7.1}$$

Then, system (5.1) possesses the following global Lyapunov function:

$$\mathcal{L}(u_0):=\int_\omega a\nabla_x u_0.\nabla_x u_0+2F(u_0)+2\bar g.u_0\,dx. \tag{7.2}$$

Let $\mathcal{R}\subset V^p_0(\omega)$ be the set of equilibria of problem (5.1), i.e.

$$\mathcal{R}:=\{z\in V^p_0(\omega),\ \ a\Delta_x z-f(z)+\bar g=0\}. \tag{7.3}$$

Our main assumption is that all of the equilibria of $\mathcal{R}$ are hyperbolic, i.e.

$$\mathcal{R}=\{z_i\}_{i=1}^N \text{ and all of } z_i \text{ are hyperbolic, see (6.2)} \tag{7.4}$$

As known, see [4], assumption (7.4) is satisfied for generic $\bar{g} \in L^p(\omega)$ (belonging to some open and dense subset of $L^p(\omega)$)). It is also well-known that, under above assumptions, the global attractor $\mathcal{A}_0$ of problem (5.1) possesses the following description.

Theorem 7.1. *Let the assumptions of Theorem 5.1 hold and let, in addition,* (6.1), (7.1) *and* (7.4) *be satisfied. Then,*

1. Every solution $u(t)$, $t \in \mathbb{R}$, of (5.1) *belonging to the attractor $\mathcal{A}_0$ stabilizes as $t \to \pm\infty$ to different equilibria $z_\pm \in \mathcal{R}$:*

$$\lim_{t\to\pm\infty} \|u(t) - z_\pm\|_{V_0^p(\omega)} = 0, \tag{7.5}$$

where $z_+ \neq z_-$.

2. The attractor $\mathcal{A}_0$ possesses the following description:

$$\mathcal{A}_0 = \cup_{i=1}^N \mathcal{M}_{z_i}^+, \tag{7.6}$$

where $\mathcal{M}_{z_i}^+$ are finite-dimensional unstable manifold of the equilibrium $z_i \in \mathcal{R}$. Moreover, $\mathcal{M}_{z_i}^+$ is a C^1-submanifold of $V_0^p(\omega)$ which is diffeomorphic to $\mathbb{R}^{\kappa_i^+}$, where κ_i^+ is the instability index of the equilibrium z_i.

3. The set $\mathcal{A}_0$ is an exponential attractor of the semigroup S_t associated with equation (5.1)*, i.e. there exist a positive constant α and a monotonic function Q such that, for every bounded subset $B \subset V_0^p(\omega)$, the following estimate holds:*

$$\operatorname{dist}_{V_0^p(\omega)}(S_tB, \mathcal{A}_\varepsilon) \le Q(\|B\|_{V_0^p(\omega)})e^{-\alpha t}. \tag{7.7}$$

The proof of this theorem can be found, e.g. in [4].

The main task of this section is to construct the analogue of the regular attractor $\mathcal{A}_0$ for the 'nonautonomous' equation (4.16) if $\varepsilon > 0$ is small enough. In this case, the equilibria $z_i \in \mathcal{R}$ should be replaced by the almost-periodic solutions $u^\varepsilon_{h_\varepsilon,z_i}(t)$, $i = 1, \cdots, N$, constructed in Proposition 6.2 and, instead of the unstable manifolds $\mathcal{M}^+_{z_i}$, we should use the 'nonautonomous' unstable manifolds $\mathcal{M}^+_{\varepsilon,h_\varepsilon,z_i}(\tau)$, $\tau \in \mathbb{R}$, defined in (6.18). Analogously to (7.6), for every $\tau \in \mathbb{R}$, $h \in \mathcal{H}(g)$ and every sufficiently small $\varepsilon > 0$, we define the attractor $\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau)$ by the following expression:

$$\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau) := \cup_{i=1}^N \mathcal{M}^+_{\varepsilon,h_\varepsilon,z_i}(\tau), \tag{7.8}$$

(we recall that, by definition, $h_\varepsilon(t) = h(\varepsilon^{-1}t)$). Then, due to (6.19), the family of attractors $\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau)$, $\tau \in \mathbb{R}$, is also strictly invariant with respect to the dynamical process $U^\varepsilon_{h_\varepsilon}(t,\tau)$:

$$\mathcal{A}_{\varepsilon,h_\varepsilon}(t) = U^\varepsilon_{h_\varepsilon}(t,\tau)\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau). \tag{7.9}$$

Moreover, the following theorem shows that this family is indeed a nonautonomous regular attractor for the dynamical process $U^\varepsilon_{h_\varepsilon}(t,\tau)$ if $\varepsilon > 0$ is small enough.

Theorem 7.2. *Let the assumptions of Theorems 5.1 and 7.1 hold. Then, there exists $\widehat{\varepsilon}_0' > 0$, $0 < \widehat{\varepsilon}_0' \le \widehat{\varepsilon}_0 \ll 1$ such that, for every $\varepsilon \le \widehat{\varepsilon}_0'$ and $h \in \mathcal{H}(g)$, the following conditions are satisfied:*

1. Every bounded solution $u \in C_b(\mathbb{R}, V^p_\varepsilon(\omega))$ of problem (6.6) *stabilizes as $t \to \pm\infty$ to different almost-periodic 'equilibria' of* (6.6) *constructed in Proposition 6.2:*

$$\lim_{t\to\pm\infty} \|u(t) - u^\varepsilon_{h_\varepsilon,z_{i_\pm}}(t)\|_{V^p_\varepsilon(\omega)} = 0, \quad i_\pm \in \{1, \cdots, N\}, \ i_- \neq i_+. \tag{7.10}$$

In particular, $u(\tau) \in \mathcal{A}_{\varepsilon,h_\varepsilon}(\tau)$, for every $\tau \in \mathbb{R}$.

2. For every fixed $\tau \in \mathbb{R}$, the sets $\mathcal{M}^+_{\varepsilon,h_\varepsilon,z_i}(\tau)$ are C^1-submanifolds of $V^p_\varepsilon(\omega)$ C^1-diffeomorphic to the unstable manifolds $\mathcal{M}^+_{z_i}$ of the limit autonomous parabolic problem (5.1) *(which are independent of τ and h).*

3. The sets $\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau)$, $\tau \in \mathbb{R}$, attract exponentially the images of all bounded subsets of $V^p_0(\varepsilon)$, i.e, there exist a positive number α and a monotonic function Q (which are independent of ε, h, t and τ) such that

$$\operatorname{dist}_{V^p_\varepsilon(\omega)}\left(U^\varepsilon_{h_\varepsilon}(t,\tau)B, \mathcal{A}_{\varepsilon,h_\varepsilon}(t)\right) \le Q(\|B\|_{V^p_\varepsilon(\omega)})e^{-\alpha(t-\tau)}, \tag{7.11}$$

for every bounded subset $B \subset V^p_\varepsilon(\omega)$, $h \in \mathcal{H}(g)$ and $t, \tau \in \mathbb{R}$, $t \ge \tau$.

4. The attractors $\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau)$ tend to $\mathcal{A}_0$ as $\varepsilon \to 0$ in the following sense:

$$\sup_{\tau\in\mathbb{R}} \sup_{h\in\mathcal{H}(g)} \operatorname{dist}^{sym}_{V^p_0(\omega)}\left(\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau), \mathcal{A}_0\right) \le \bar{C}\big[\bar{\alpha}_{R_0}(\varepsilon)\big]^\kappa, \tag{7.12}$$

where the constants $\bar{C}$, R_0 and $0 < \kappa < 1$ are independent of ε, $\bar{\alpha}_{R_0}(\varepsilon)$ is the same as in Corollaries 5.2 and 5.4 and

$$\operatorname{dist}^{sym}_V(X,Y) := \max\{\operatorname{dist}_V(X,Y), \operatorname{dist}_V(Y,X)\} \tag{7.13}$$

is the symmetric Hausdorff distance between sets X and Y in V.

Sketch of proof. The result of Theorem 7.2 is a corollary of the nonautonomous perturbation theory of regular attractors developed in [12, 30]. In order to apply this theory to equation (6.6), we consider the discrete processes

$$U^\varepsilon_{h_\varepsilon}(l,m) : V^p_0(\omega) \to V^p_0(\omega) \tag{7.14}$$

associated with this problem in phase space $V^p_0(\omega)$ which is independent of ε. Then, it follows Corollaries 3.7, 5.2 and 5.4 then these processes tend (together with their Frechet derivative) as $\varepsilon \to 0$ to the semigroup S_{l-m} associated with the limit parabolic equation (5.1) in the phase space $V^p_0(\omega)$ and this convergence is uniform with respect to $h \in \mathcal{H}(g)$. Moreover, estimate (3.28) guarantees the uniform (with respect to ε and $h \in \mathcal{H}(g)$) dissipativity of these processes and Theorem 3.8 gives the injectivity of all the operators (7.14). We also recall that the limit discrete semigroup S_n, $n \in \mathbb{N}$, possesses the *regular* attractor $\mathcal{A}_0$ (due to Theorem 7.1). Then, arguing in a standard way (see [12]), we derive that discrete processes (7.14) possess the nonautonomous regular attractors $\mathcal{A}_{\varepsilon,h_\varepsilon}(l)$, $l \in \mathbb{Z}$ (if $\varepsilon > 0$ is small enough) in $V^p_0(\omega)$ which satisfy the discrete analogue of Theorem 7.2 (since all of the estimates formulated in Corollaries 3.7, 5.2 and 5.4 are uniform with respect to $h \in \mathcal{H}(g)$ then estimates (7.11) and (7.12) also hold for $\mathcal{A}_{\varepsilon,h_\varepsilon}(l)$ uniformly with respect to $h \in \mathcal{H}(g)$).

When the discrete nonautonomous attractors $\mathcal{A}_{\varepsilon,h_\varepsilon}(l)$, $l \in \mathbb{Z}$, for processes (7.14) (which satisfy all of the assertions of Theorem 7.2) are already constructed, we can extend in a standard way this result to the continuous case by the following expression:

$$\mathcal{A}_{\varepsilon,h_\varepsilon}(\tau) := \mathcal{A}_{\varepsilon,T_\tau h_\varepsilon}(0) \tag{7.15}$$

which is an immediate corollary of (7.20), see [12] for the details. Theorem 7.2 is proved. □

Corollary 7.3. *Let the assumptions of Theorem 7.2 hold. Assume also that the almost-periodic function $g(t) - \bar{g}$ (where $\bar{g}$ is a mean value of $g(t)$) has a bounded primitive*

$$g(t) - \bar{g} = \partial_t G(t), \quad G \in L^2_b(\Omega). \tag{7.16}$$

Then, estimate (7.12) *can be improved as follows:*

$$\sup_{\tau\in\mathbb{R}} \sup_{h\in\mathcal{H}(g)} \operatorname{dist}^{sym}_{V^p_0(\omega)}\left(\mathcal{A}_{\varepsilon,h}(\tau), \mathcal{A}_0\right) \le \bar{C}_1\varepsilon^{\kappa_1}, \tag{7.17}$$

where $\bar C_1$ is independent of ε, $\kappa_1 := \kappa_\delta\cdot\kappa/2$ and κ_δ is defined in (5.12).

Indeed, (7.17) is an immediate corollary of (7.12) and Theorem 5.5.

Corollary 7.4. *Let the assumptions of Theorem 7.2 hold. Then the nonautonomous regular attractor $\mathcal{A}_{\varepsilon,g}(t)$, $t\in\mathbb{R}$, of* (6.6) *and its uniform attractor $\mathcal{A}_\varepsilon$ (constructed in Theorem 4.2) satisfy the following relation:*

$$\mathcal{A}_\varepsilon=\cup_{h\in\mathcal{H}(g)}\mathcal{A}_{\varepsilon,h_\varepsilon}(t)=\left[\cup_{t\in\mathbb{R}}\mathcal{A}_{\varepsilon,g_\varepsilon}(t)\right]_{V_0^p(\omega)}\tag{7.18}$$

and, consequently

$$\operatorname{dist}^{sym}_{V_0^p(\omega)}\left(\mathcal{A}_\varepsilon,\mathcal{A}_0\right)\le\bar C\big[\alpha_{R_0}(\varepsilon)\big]^\kappa.\tag{7.19}$$

In particular, the uniform attractors $\mathcal{A}_\varepsilon$ tend to $\mathcal{A}_0$ (upper and lower semicontinuous) as $\varepsilon\to0$.

Indeed, the first equality in (7.18) is an immediate corollary the first assertion of Theorem 7.2 and description (4.7) of uniform attractor $\mathcal{A}_\varepsilon$. The second inequality in (7.18) can be easily verified using the exponential attraction property (7.11) and the alternative definition of the uniform attractor $\mathcal{A}_\varepsilon$ which is formulated in Remark 4.3. Estimate (7.19) follows immediately from (7.17) and (7.18).

Remark 7.5. The first assertion of Theorem 7.2 can be reformulated as follows: problem (6.6) has exactly N almost-periodic solutions $u^\varepsilon_{h_\varepsilon,z_i}(t)$ which are localized near the equilibria $z_i\in\mathcal{R}$, $i=1,\cdots,N$, and every other bounded solution $u\in C_b(\mathbb{R},V_0^p(\omega))$ is a heteroclinic connection between two different almost periodic solutions of this problem.

Remark 7.6. We note that condition (7.16) is, obviously, always satisfied if the external force $g(t)$ is *periodic* with respect to t. Thus, in case of periodic g, we have estimate (7.17) for the symmetric distance between the perturbed ($\mathcal{A}_{\varepsilon,h}(t)$) and nonperturbed ($\mathcal{A}_0$) regular attractors without any additional assumptions and (as a corollary) the following estimate is satisfied for the uniform attractors:

$$\operatorname{dist}^{sym}_{V_0^p(\omega)}\left(\mathcal{A}_\varepsilon,\mathcal{A}_0\right)\le\bar C_1\varepsilon^{\kappa_1}.\tag{7.20}$$

Unfortunately, in more general case of quasiperiodic or almost-periodic external forces, condition (7.16) is not satisfied automatically and should be verified, see e.g. [9], and [18] for various sufficient conditions.

8. Appendix. Uniform elliptic regularity in L^p-spaces

In this Appendix, we consider the following singular perturbed elliptic boundary value problem in a half-cylinder $\Omega_+:=\mathbb{R}_+\times\omega$:

$$a(\varepsilon^2\partial_t^2u+\Delta_xu)-\gamma\partial_tu=h(t),\quad u\big|_{\partial\omega}=0,\quad u\big|_{t=0}=u_0,\tag{8.1}$$

where $u=(u^1,\cdots,u^k)$ is a vector-valued function, a and γ are given constant matrices such that $a+a^*>0$ and $\gamma=\gamma^*>0$ and the right-hand side h belongs to $L^p(\Omega_+)$, $2\le p<\infty$.

The main result of this appendix is the following uniform (with respect to ε) maximal regularity estimate for the solutions of (8.1).

Theorem 8.1. *Let $u\in W^{(1,2),p}_\varepsilon(\Omega_+)$ be a solution of* (8.1)*. Then, the following estimate holds:*

$$\|u\|_{W^{(1,2),p}_\varepsilon(\Omega_+)}\le C\left(\|u_0\|_{V^p_\varepsilon(\omega)}+\|h\|_{L^p(\Omega_+)}\right),\tag{8.2}$$

where the constant C is independent of $\varepsilon\in[0,\varepsilon_0]$. In particular, $V^p_\varepsilon(\omega)$ is a uniform (with respect to ε) trace space for functions belonging to $W^{(1,2),p}_\varepsilon(\Omega_+)$.

Proof. The proof of estimate (8.2) is based on the classical localization technique and on the multiplicators theorems in Fourier spaces and is more or less standard (see e.g. [17], [26]). That is the reason why, in order to show that constant C is indeed independent of ε, we discuss below only the principal points of this proof resting the details to the reader. We start with the most simple case $\gamma=0$.

Lemma 8.2. *Let u be a solution of* (8.1) *with $\gamma=0$. Then, the following estimate holds:*

$$C'\varepsilon^{1/p}\left(\|u_0\|_{W^{2-1/p,p}(\omega)}+\varepsilon\|\partial_t u(0)\|_{W^{1-1/p,p}(\omega)}\right)\le\\ \le\varepsilon^2\|\partial_t^2u\|_{L^p(\Omega_+)}+\|u\|_{L^p(\mathbb{R}_+,W^{2,p}(\omega))}\le C\left(\varepsilon^{1/p}\|u_0\|_{W^{2-1/p,p}(\omega)}+\|h\|_{L^p(\Omega_+)}\right),\tag{8.3}$$

where the constants C and C' are independent of ε.

Indeed, scaling the time $t=\varepsilon t'$ and introducing the functions $\tilde u(t'):=u(t/\varepsilon)$ and $\tilde h(t'):=h(t/\varepsilon)$, we deduce that the function $\tilde u$ satisfies equation (8.1) with $\gamma=0$, $\varepsilon=1$ and with the right-hand side $\tilde h$. Applying the standard elliptic regularity theorem to this equation, see e.g. [26], we infer

$$C'\left(\|u_0\|_{W^{2-1/p,p}(\omega)}+\|\partial_{t'}\tilde u(0)\|_{W^{1-1/p,p}(\omega)}\right)\le\\ \le\|\tilde u\|_{L^p(\mathbb{R}_+,W^{2,p}(\omega))}+\|\partial^2_{t'}\tilde u\|_{L^p(\Omega_+)}\le C\left(\|\tilde h\|_{L^p(\Omega_+)}+\|u_0\|_{W^{2-1/p,p}(\omega)}\right).\tag{8.4}$$

Returning to the time variable t, we derive estimate (8.3).

In the next step, we consider the Hilbert case $p=2$.

Lemma 8.3. *Let $p=2$ and $u\in W^{(1,2),2}_\varepsilon(\Omega_+)$. Then, estimate* (8.2) *holds.*

Proof. Indeed, multiplying equation (8.1) by $\varepsilon^2\partial_t^2u+\Delta_xu$, integrating over Ω_+, integrating by parts and using that $\gamma=\gamma^*$, we have

$$\|a(\partial_t^2u+\Delta_xu)\|^2_{L^2(\Omega_+)}+\frac{\varepsilon^2}{2}(\gamma\partial_tu(0),\partial_tu(0))-\frac12(\gamma\nabla_xu(0),\nabla_xu(0))=\left\langle h,\varepsilon^2\partial_t^2u+\Delta_xu\right\rangle_0$$

and, therefore, since a is non-degenerate and $\gamma>0$,

$$\|\varepsilon^2\partial_t^2u+\Delta_xu\|^2_{L^2(\Omega_+)}+\varepsilon^2\|\partial_tu(0)\|^2_{L^2(\omega)}\le C\left(\|h\|^2_{L^2(\Omega_+)}+\|u_0\|^2_{W^{1,2}(\omega)}\right),\tag{8.5}$$

where C is independent of ε. Estimate (8.5), together with (8.3), imply estimate (8.2) with $p=2$ and Lemma 8.3 is proved. ☐

We are now ready to consider the general case $p>2$. We first note that, due to the classical localization technique and estimate (8.2) for $p=2$ (which is necessary in order to estimate the subordinated terms appearing under the localization technique), it is sufficient to verify estimate (8.2) only for equation

$$a(\varepsilon^2\partial_t^2u+\Delta_xu-u)-\gamma\partial_tu=h,\quad u\big|_{t=0}=u_0,\quad u_0\big|_{\partial\omega}=0\tag{8.6}$$

and only for two choices of the domain Ω, namely, for 1) $\omega=\mathbb{R}^n$ and 2) $\omega_+=\mathbb{R}^{x_1}_+\times\mathbb{R}^{n-1}_{x_2,\cdots,x_n}$ (see e.g. [17] and [26]). Moreover, we also note that the second case of semi-space ω_+ can be easily reduced to the first one of the the whole space $\omega=\mathbb{R}^n$ by considering the odd (with respect to x_1) solutions of (8.6) in $\omega=\mathbb{R}^n$. Thus, there only remains to verify estimate (8.2) for solutions of (8.6) in $\omega=\mathbb{R}^n$.

In the next step, we reduce the problem of studying the elliptic system of equations (8.6) to the analogous problem for the scalar equation. In order to do so, it is convenient to extend the class of admissible solutions of (8.6) and consider also the *complex-valued* solutions $u(t,x) = \operatorname{Re} u(t,x) + i \operatorname{Im} u(t,x) \in \mathbb{C}^k$, for every $(t,x) \in \Omega_+$. Then, equation (8.6) is equivalent to the following one:

$$\varepsilon^2\partial_t^2 u + \Delta_x u - u - \gamma'\partial_t u = h, \quad u\big|_{\partial\omega} = 0, \quad u\big|_{t=0} = u_0, \tag{8.7}$$

where $\gamma' := a^{-1}\gamma$. Moreover, without loss of generality we may assume that the matrix γ' is reduced to its Jordan normal form. Then, our conditions on matrices a and γ guaranties that the real parts of all eigenvalues of γ' are strictly positive:

$$\sigma(\gamma') \subset \{\lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda > 0\}. \tag{8.8}$$

Thus, (8.7) is a cascade system of scalar elliptic equations coupled by the terms $\gamma'\partial_t u$ and γ' is in Jordan normal form. That is why, it is sufficient to verify estimate (8.2) only for scalar complex-valued elliptic equations of the form

$$\varepsilon^2\partial_t^2 u + \Delta_x u - u - 2(\alpha + i\beta)\partial_t u = h, \quad u\big|_{\partial\omega} = 0, \quad u\big|_{t=0} = u_0, \tag{8.9}$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$. We start with the case $h = 0$.

Lemma 8.4. *Let $u_0 \in V^p_\varepsilon(\mathbb{R}^n)$ and let u be a solution of* (8.9) *with $h = 0$. Then, it satisfies uniform estimate* (8.2)*.*

Proof. Indeed, factorizing equation (8.9) (with $h \equiv 0$), we obtain that the function $u(t)$ satisfies the following pseudodifferential equation:

$$\partial_t u = -A_\varepsilon(1 - \Delta_x)u, \quad u\big|_{t=0} = u_0, \tag{8.10}$$

where

$$A_\varepsilon(z) := -\frac{\alpha + i\beta - \sqrt{(\alpha+i\beta)^2 + \varepsilon^2 z}}{\varepsilon^2} \equiv \frac{z}{\alpha + i\beta + \sqrt{(\alpha+i\beta)^2 + \varepsilon^2 z}} \tag{8.11}$$

and we take the branch of $\sqrt{\cdot}$ which is positive on $\mathbb{R}_+$. Let us study equation (8.10).

Proposition 8.5. *The solution of* (8.10) *satisfies*

$$\|\partial_t u\|_{L^p(\Omega_+)} + \|A_\varepsilon(1-\Delta_x)u\|_{L^p(\Omega_+)} \le C\|u_0\|_{W^{2(1-1/p),p}(\mathbb{R}^n)}, \tag{8.12}$$

where C is independent of ε.

Proof. We first consider the following nonhomogeneous analogue of equation (8.10):

$$\partial_t w + A_\varepsilon(1-\Delta_x)w = h(t), \quad w\big|_{t=0} = 0, \ w\big|_{\partial\omega} = 0, \quad h \in L^p(\Omega_+) \tag{8.13}$$

and verify that

$$\|\partial_t w\|_{L^p(\Omega_+)} \le C_3\|h\|_{L^p(\Omega_+)}, \tag{8.14}$$

where C_3 is independent of ε. Indeed, let us extend functions $w(t)$ and $h(t)$ by zero for $t < 0$ and apply the Fourier transform $((t,x) \to \xi := (\lambda, \xi') \in \mathbb{R}\times\mathbb{R}^n)$ to equation (8.13). Then, we have

$$\widehat{(\partial_t w)}(\xi) = K_\varepsilon(\xi)\widehat{h}(\xi), \quad K_\varepsilon(\xi) := \frac{i\lambda}{i\lambda + A_\varepsilon(|\xi'|^2 + 1)}. \tag{8.15}$$

According to the multiplicators theorem (see e.g. [26]), in order to verify estimate (8.14), it is sufficient to prove that

$$\sup_{1\le i_1<\dots<i_k\le n+1} \sup\nolimits_{\xi\in\mathbb{R}^{n+1}} \left|\xi_{i_1}\cdots\xi_{i_k}\partial^k_{\xi_{i_1},\dots,\xi_{i_k}} K_\varepsilon(\xi)\right| \le C < \infty, \tag{8.16}$$

where C is independent of ε. So, we need to verify (8.16). To this end, we note that, due to the assumption $\alpha > 0$, the following estimates hold:

$$\left|\operatorname{Im}\sqrt{(\alpha+i\beta)^2+\varepsilon^2(|\xi'|^2+1)}\right| \le \kappa_1\sqrt{1+\varepsilon^2(1+|\xi'|^2)} \le \\ \le \kappa_2 \operatorname{Re}\sqrt{(\alpha+i\beta)^2+\varepsilon^2(|\xi'|^2+1)} \le \kappa_3\sqrt{1+\varepsilon^2(1+|\xi'|^2)} \tag{8.17}$$

where $\kappa_i > 0$, $i = 1,2,3$, are independent of ε (indeed, these estimates can be easily verified by direct computations based on the fact that $\alpha > 0$). Estimates (8.17), the fact that $\alpha > 0$ and definition (8.11) immediately imply that

$$\kappa_1'\left|\operatorname{Im} A_\varepsilon(|\xi'|^2+1)\right| \le \frac{|\xi'|^2+1}{\sqrt{1+\varepsilon^2(|\xi'|^2+1)}} \le \\ \le \kappa_2' \operatorname{Re} A_\varepsilon(|\xi'|^2+1) \le \kappa_3' \frac{|\xi'|^2+1}{\sqrt{1+\varepsilon^2(|\xi'|^2+1)}} \tag{8.18}$$

and, consequently

$$\kappa_1''\left(|\lambda|+|A_\varepsilon(|\xi'|^2+1)|\right) \le \left|i\lambda + A_\varepsilon(|\xi'|^2+1)\right| \le \kappa_2''\left(|\lambda|+|A_\varepsilon(|\xi'|^2+1)|\right), \tag{8.19}$$

where the positive constants κ_i' and κ_i'' are independent of ε. Moreover, due to (8.17) and (8.18)

$$\left|\xi_{i_1}\cdots\xi_{i_k}\partial^k_{\xi_{i_1}\cdots\xi_{i_k}} A_\varepsilon(|\xi'|^2+1)\right| = \\ \frac{C_k(\varepsilon^2|\xi_{i_1}|^2)\cdots(\varepsilon^2|\xi_{i_{k-1}}|^2)}{|(\alpha+i\beta)^2+\varepsilon^2(|\xi'|^2+1)|^{k-1}} \cdot \frac{|\xi_{i_k}|^2}{\sqrt{|(\alpha+i\beta)^2+\varepsilon^2(|\xi'|^2+1)|}} \le C_k'\left|A_\varepsilon(|\xi'|^2+1)\right| \tag{8.20}$$

holds, for every $2 \le i_1 < \cdots < i_k \le n+1$, where the constants C_k and C_k' are independent of ε. There remains to note that estimates (8.19) and (8.20) imply (8.16). Indeed, differentiating the kernel $K_\varepsilon(\xi)$ with respect to $\xi_{i_1}, \cdots, \xi_{i_k}$ and using (8.20), we see that, for $2 \le i_1 < \cdots < i_k \le n+1$,

$$|\xi_{i_1}\cdots\xi_{i_k}\partial^k_{\xi_{i_1},\cdots\xi_{i_k}} K_\varepsilon(\xi)| \le C_k\left(\sum_{l=1}^k \frac{|\lambda|\cdot|A_\varepsilon(|\xi'|^2+1)|^l}{(|\lambda|+|A_\varepsilon(|\xi'|^2+1))^{l+1}}\right) \le C$$

and the analogous uniform (with respect to $\varepsilon \to 0$) estimate for the case where $i_1 = 1$ also holds. Thus, estimate (8.16) holds and, therefore, (8.14) is also verified.

Let us now prove estimate (8.12). To this end, we fix an extension $v(t)$ of the initial data u_0 inside of Ω_+ in such way that

$$\|\partial_t v\|_{L^p(\Omega_+)} + \|v\|_{L^p(\mathbb{R}_+,W^{2,p}(\omega))} \le C_1\|u_0\|_{W^{2(1-1/p),p}(\mathbb{R}^n)}, \tag{8.21}$$

where C_1 is independent of u_0 (such an extension exists due to the classical trace theorems, see [26]) and introduce a function $w(t) := u(t) - v(t)$ which, obviously, satisfies equation (8.13) with $h(t) := \partial_t v(t) + A_\varepsilon(1-\Delta_x)v(t)$. Thus, thanks to (8.14) and (8.21), it is sufficient to verify that

$$\|A_\varepsilon(1-\Delta_x)v(t)\|_{L^p(\mathbb{R}^n)} \le C_2\|v(t)\|_{W^{2,p}(\mathbb{R}^n)}, \tag{8.22}$$

where C_2 is independent of ε and t. But this estimate can be easily verified using the multiplicators theorem and estimates (8.18) and (8.20) (in the same way as it was done in the proof of estimate (8.14)). Proposition 8.5 is proved. □

We are now able to finish the proof of Lemma 8.4. Indeed, according to Proposition 8.5, every solution $u(t)$ of (8.9) with $h = 0$ satisfies estimate (8.12). Interpreting now the term $2(\alpha+i\beta)\partial_t u$ in equation (8.9) as the right-hand side and using Lemma 8.2, we derive that $u(t)$ satisfies indeed estimate (8.2) with $h = 0$ which finishes the proof of Lemma 8.4. $\square$

In particular, Lemma 8.4 implies that $V^p_\varepsilon(\mathbb{R}^n)$ is a uniform trace space for functions from $W^{(1,2),p}_\varepsilon(\Omega_+)$ at $t = 0$. Indeed, the solving operator $T_+ : u_0 \to u$ for (8.9) with $h = 0$ can be considered as uniform (with respect to ε) extension operator for functions from $V^p_\varepsilon(\mathbb{R}^n)$ to $W^{(1,2),p}_\varepsilon(\Omega_+)$ and the inverse estimate

$$\|u(0)\|_{V^p_\varepsilon(\mathbb{R}^n)} \le C\|u\|_{W^{(1,2),p}_\varepsilon(\Omega_+)} \tag{8.23}$$

is an immediate of Lemma 8.2 and the standard trace theorem for the 'parabolic' space $W^{(1,2),p}_0(\Omega_+)$.

We are now ready to finish the proof of Theorem 8.1. As it was shown before, in order to do so, it is sufficient to verify estimate (8.2) for equation (8.9) in $\Omega_+ := \mathbb{R}_+ \times \mathbb{R}^n$. Moreover, due to Lemma 8.4 and due to the fact that V^p_ε is a uniform trace space for functions from $W^{(1,2),p}_\varepsilon$, it is sufficient to verify that every solution $u \in W^{(1,2),p}_\varepsilon(\mathbb{R}^{n+1})$ of

$$\varepsilon^2\partial_t u + \Delta_x u - u - 2(\alpha + i\beta)\partial_t u = h(t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \tag{8.24}$$

satisfies the estimate

$$\|u\|_{W^{(1,2),p}_\varepsilon(\mathbb{R}\times\mathbb{R}^n)} \le C\|h\|_{L^p(\mathbb{R}^{n+1})}, \tag{8.25}$$

where C is independent of ε. Applying the Fourier transform to (8.24), we infer

$$\widehat{u}(\lambda, \xi') = \left(\varepsilon^2\lambda^2 + |\xi'|^2 + 1 - (\alpha + i\beta)i\lambda\right)^{-1}\widehat{h}(\lambda, \xi') \tag{8.26}$$

Applying the multiplicators theorem to (8.26) (as we did in the proof of Proposition 8.5), we derive estimate (8.25) which finishes the proof of Theorem 8.1. $\square$

To conclude, we formulate several standard corollaries of the proved theorem the rigorous proof of which is left to the reader.

Corollary 8.6. *Let $h \in L^p_b(\Omega_+)$ and let $u \in W^{(1,2),p}_{\varepsilon,b}(\Omega_+)$ be a solution of* (8.1). *Then, the following estimate holds for every $T \ge 0$:*

$$\|u\|^p_{W^{(1,2),p}_\varepsilon(\Omega_T)} \le C\|u_0\|^p_{V^p_\varepsilon(\omega)}e^{-\alpha T} + C\int_0^\infty e^{-\alpha|T-t|}\|h(t)\|^p_{L^p(\omega)}\,dt, \tag{8.27}$$

where positive constants C and α are independent of ε, u_0, T and u.

Indeed, multiplying equation (8.1) by $\phi_{T,\alpha}(t) := 1/\cosh(\alpha(T-t))$, where $\alpha > 0$ is a sufficiently small number, and applying Theorem 8.1 to the function $w_{T,\alpha}(t) := \phi_{T,\alpha}(t)u(t)$, we obtain (8.27) after the standard estimations.

The next corollary gives the standard interior (with respect to t) estimate for solutions of (8.1).

Corollary 8.7. *Let $h \in L^p_b(\Omega_+)$ and let $u \in W^{(1,2),p}_{\varepsilon,b}(\Omega_+)$ be a solution of* (8.1). *Then, the following estimate holds for every $T \ge 0$:*

$$\begin{aligned}\|u\|_{W^{(1,2),p}_\varepsilon(\Omega_T)} &\le \\ &\le C(\|h\|_{L^p(\Omega_{T-1/2,T+3/2})} + \|u\|_{L^2(\Omega_{T-1/2,T+3/2})} + \chi(1-2T)\|u_0\|_{V^p_\varepsilon(\omega)}),\end{aligned} \tag{8.28}$$

where $\Omega_{T_1,T_2} := [\max\{T_1, 0\}, T_2] \times \omega$, $\chi(z)$ *is the Heaviside function and the constant* C *is independent of* ε, T *and* u.

Indeed, the prove of (8.28) is based on multiplication of equation (8.1) by the special cut-off function $\psi_T(t)$ which vanishes for $t \notin [T-1/2, T+3/2]$ and equals one for $t \in [T, T+1]$ and on application of Theorem 8.1 to the function $u_T(t) := \psi_T(t)u(t)$ and can be derived in a standard way.

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