SOLVING THE INVERSE RADON TRANSFORM
FOR
VECTOR FIELD TOMOGRAPHIC DATA

by

ARCHONTIS GIANNAKIDIS

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Advanced Technology Institute
Department of Electronic Engineering
Faculty of Engineering and Physical Sciences
University of Surrey
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for Dad

who showed me the way
Abstract

It is widely recognised that the most popular manner of image representation is obtained by using an energy-preserving transform, like the Fourier transform. However, since the advent of computerised tomography in the 70s, another manner of image representation has also entered the center of interest. This new type is the projection space representation, obtained via the Radon transform. Methods to invert the Radon transform have resulted in a wealth of tomographic applications in a wide variety of disciplines.

Functions that are reconstructed by inverting the Radon transform are scalar functions. However, over the last few decades there has been an increasing need for similar techniques that would perform tomographic reconstruction of a vector field when having integral information. Prior work at solving the reconstruction problem of 2-D vector field tomography in the continuous domain showed that projection data alone are insufficient for determining a 2-D vector field entirely and uniquely. This thesis treats the problem in the discrete domain and proposes a direct algebraic reconstruction technique that allows one to recover both components of a 2-D vector field at specific points, finite in number and arranged in a grid, of the 2-D domain by relying only on a finite number of line-integral data. In order to solve the reconstruction problem, the method takes advantage of the redundancy in the projection data, as a form of employing regularisation. Such a regularisation helps to overcome the stability deficiencies of the examined inverse problem. The effects of noise are also examined. The potential of the introduced method is demonstrated by presenting examples of complete reconstruction of static electric fields.

The most practical sensor configuration in tomographic reconstruction problems is the regular positioning along the domain boundary. However, such an arrangement does not result in uniform distribution in the Radon parameter space, which is a necessary
requirement to achieve accurate reconstruction results. On the other hand, sampling the projection space uniformly imposes serious constraints of space or time. In this thesis, motivated by the Radon transform theory, we propose to employ either interpolated data obtained at virtual sensors (that correspond to uniform sampling of the projection space) or probabilistic weights with the purpose of approximating uniformity in the projection space parameters. Simulation results demonstrate that when these two solutions are employed, about 30% decrease in the reconstruction error may be achieved. The proposed methods also increase the resilience to noise. On top of these findings, the method that employs weights offers an attractive solution because it does not increase the reconstruction time, since the weight calculation can be performed off-line.

This thesis also looks at the 2-D vector field reconstruction problem from the aspect of sampling. To address sampling issues, the standard parallel scanning is treated. By using sampling theory, the limits to the sampling steps of the Radon parameters, so that no integral information is lost, are derived. Experiments show that when the proposed sampling bounds are violated, the reconstruction accuracy of the 2-D vector field deteriorates over the case where the proposed sampling criteria are imposed. It is shown that the employment of a scanning geometry that satisfies the proposed sampling requirements also increases the resilience to noise.
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Style Conventions

• References to the bibliography, placed in the end of this thesis, will appear as [48].

• Equation and Figure have been abbreviated to Eq. and Fig. respectively. Likewise, Equations and Figures have been abbreviated to Eqs. and Figs. respectively.

• Equations and Figures are numbered by the chapter, i.e. Eq. (2.10) is the tenth equation in Chapter 2.

• The Fourier transform of a function \( g(x) \) is denoted by \( \text{FT}\{g(x)\} \).

• Scalar variables are denoted by normal letters, while non-scalar variables (vectors and matrices) are denoted by bold-faced letters. Upper case bold letters are used for matrices, whereas lower case bold letters denote vectors. For the vector notation, we also use the symbol \( \hat{\cdot} \) for arbitrary vectors and the symbol \( \hat{\cdot} \) for unit vectors.

• The transpose of \( \mathbf{b} \) is shown as \( \mathbf{b}^T \).

• The determinant of a matrix \( \mathbf{A} \) is denoted by \( |\mathbf{A}| \).

• A matrix \( \mathbf{A} \) with \( I \) rows and \( J \) columns is denoted as \( \mathbf{A} \in \mathbb{R}^{I \times J} \).

• \( \partial D \) denotes the boundary of region \( D \).

• \( * \) denotes the convolution for one dimension.

• The scalar product between two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is denoted by \( \mathbf{a} \cdot \mathbf{b} \).

• Symbol \( \sum \) denotes the sum operator and the symbol \( \prod \) denotes the product operator.
• Symbol $\in$ means “belongs to”.

• Symbol $\forall$ means “for all”.

• Symbol $\cup$ denotes the union in set theory.

• $\lfloor \cdot \rfloor$ is the symbol for the floor operator and $\lceil \cdot \rceil$ denotes the ceiling operator.

• Symbol $\nabla$ denotes the del operator defined in the 2-D Cartesian coordinate system $(x, y)$ as $\nabla = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}$, where $\hat{x}$ and $\hat{y}$ form the basis of the system. It is used as a shorthand form to denote: i) the gradient of a scalar function $f$ ($\nabla f$), ii) the divergence of a vector $\mathbf{v}$ ($\nabla \cdot \mathbf{v}$) and iii) the curl of a vector $\mathbf{v}$ ($\nabla \times \mathbf{v}$).

• The Dirac delta function is denoted by $\delta(\cdot)$.

• The exponential function is denoted by $\exp(\cdot)$.

• The Heaviside step function is denoted by $H(\cdot)$.

• The natural logarithm is denoted by $\log(\cdot)$.

• $\text{rect}(\cdot)$ denotes the rectangular function of value 1 for argument between $-\frac{1}{2}$ and $\frac{1}{2}$, and 0 otherwise.
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Statement of Originality

As far as the author is aware, except where otherwise stated in the text, the work and ideas presented in this thesis are believed to be original contributions.
Abbreviations

ADC: Analogue to Digital Converter
AE: average absolute Angular reconstruction Error
ART: Algebraic Reconstruction Technique
BP: Back-Projection
CT: Computerised Tomography
DFT: Discrete Fourier Transform
EM: Expectation Maximisation
FFT: Fast Fourier Transform
FST: Fourier Slice Theorem
FT: Fourier Transform
IFFT: Inverse Fast Fourier Transform
IFT: Inverse Fourier Transform
IPi: The $i^{th}$ Inter-Polation method
LS: Least Squares
ME: average relative Magnitude reconstruction Error
MLR: Maximum Likelihood Reconstruction
MRI: Magnetic Resonance Imaging
NMR: Nuclear Magnetic Resonance
NW: Non-Weighted measurements
OSEM: Ordered Subset Expectation Maximisation
PET: Positron Emission Tomography
RS: Regularly placed Sensors
SL: Source Location
Abbreviations

SNR: Signal to Noise Ratio
SVD: Singular Value Decomposition
SPECT: Single Photon Emission Computerised Tomography
US: Uniform Sampling
WM: Weighted Measurements
1-D: One-Dimensional
2-D: Two-Dimensional
3-D: Three-Dimensional
Table of Latin Symbols

\( A \): location of a boundary sensor

\( \tilde{A}(k_1, k_2) \): the 2-D FT of \( \tilde{a}(x, y) \)

\( A_c \): the number of elements of \( g \)

\( A_r \): the number of elements of \( b \)

\( A \): the system matrix \( \in \mathcal{R}^{A_r \times A_c} \)

\( b(\rho) \): the ramp filter used in filtered backprojection

\( b_i \): the \( i^{th} \) element of \( b \)

\( b_i^* \): the \( i^{th} \) element of \( b^* \)

\( \mathbf{b} \): the descriptor of directional sensitivity of a Schlieren arrangement

\( \mathbf{b} \): the vector containing the sinogram values \( \tilde{g}(r, t) \) wrapped into a vector

\( \mathbf{b}^* \): the vector containing the unknown mean value of \( \mathbf{b} \)

\( B \): location of a boundary sensor

\( \tilde{B}(v) \): the frequency domain expression of \( b(\rho) \)

\( c(x, y) \): the speed of the sound in a fluid, if the fluid was not flowing [m/s]

\( c_{\text{eff}}(x, y) \): the effective speed of sound [m/s]

\( d_A \): the distance between the sensor at point \( A \) and the foot of the normal [m]

\( d_B \): the distance between the sensor at point \( B \) and the foot of the normal [m]

\( d_{iA} \): the distance between the sensor at \( A \) and its nearest sampling point of the line [m]

\( d_{iB} \): the distance between the sensor at \( B \) and its nearest sampling point of the line [m]

\( D \): the reconstruction region

\( D_i \): the \( i^{th} \) detector

\( E_i(x, y) \): the emission intensity in PET

\( E_x \): the \( x \) component of the examined static electric field [V/m]
Abbreviations

\[ E_y: \] the y component of the examined static electric field \([V/m]\)

\[ f_{||}: \] the component of \(f(x,y)\) along \(L\)

\[ f_{\perp}: \] the component of \(f(x,y)\) transverse to \(L\)

\[ f_x(m,n): \] the \(x\) component of \(f(m,n)\)

\[ f_x(x,y): \] the \(x\) component of \(f(x,y)\)

\[ f_{x\hat{i}}: \] the \(x\) component of \(\hat{f}_i\)

\[ f_{x\hat{A}}: \] the \(x\) component of \(\hat{f}_A\)

\[ f_{x\hat{B}}: \] the \(x\) component of \(\hat{f}_B\)

\[ f_{x_1}(x_1): \] the probability density function of random variable \(x_1\)

\[ f_{x_1x_2}(x_1,x_2): \] the joint probability density function of random variables \(x_1\) and \(x_2\)

\[ f_{x_1y_2}(x_1,y_2): \] the joint probability density function of random variables \(x_1\) and \(y_2\)

\[ f_{x_2}(x_2): \] the probability density function of random variable \(x_2\)

\[ f_y(m,n): \] the \(y\) component of \(\hat{f}(m,n)\)

\[ f_y(x,y): \] the \(y\) component of \(\hat{f}(x,y)\)

\[ f_{y\hat{i}}: \] the \(y\) component of \(\hat{f}_i\)

\[ f_{y\hat{A}}: \] the \(y\) component of \(\hat{f}_A\)

\[ f_{y\hat{B}}: \] the \(y\) component of \(\hat{f}_B\)

\[ f_{y_1}(y_1): \] the probability density function of random variable \(y_1\)

\[ f_{y_1y_2}(y_1,y_2): \] the joint probability density function of random variables \(y_1\) and \(y_2\)

\[ f_{y_2}(y_2): \] the probability density function of random variable \(y_2\)

\[ f_{\rho\theta}(\rho,\theta): \] the joint probability density function of random variables \(\rho\) and \(\theta\)

\[ f_{\rho\theta}^i(\rho,\theta): \] the individual \(f_{\rho\theta}(\rho,\theta)\) density for the \(i\)th set of scanning lines

\[ \hat{f}(m,n): \] the digital vector under investigation

\[ \hat{f}(k_1,k_2): \] the 2-D FT of \(\hat{f}(x,y)\)

\[ \hat{f}(k): \] the same as \(\hat{f}(k_1,k_2)\)

\[ \hat{f}(x,y): \] the planar vector field under investigation

\[ \hat{f}_I(x,y): \] the irrotational (curl-free) component of \(\hat{f}(x,y)\)

\[ \hat{f}_i: \] the examined vector field at sampling point \(l\) of a line

\[ \hat{f}_{IA}: \] the examined vector field at the sampling point that is closest to sensor at \(A\)
Abbreviations

\( \bar{f}_B \): the examined vector field at the sampling point that is closest to sensor at \( B \)

\( \bar{f}_S(x, y) \): the solenoidal (source-free) component of \( \bar{f}(x, y) \)

\( g(m, n) \): the discrete approximation of \( g(x, y) \)

\( g(x, y) \): a two-dimensional function representing an image

\( g(x_m, y_n) \): the same as \( g(m, n) \)

\( \tilde{g}(r, t) \): the discrete approximation of \( \tilde{g}(\rho, \theta) \)

\( \tilde{g}(\rho, \theta) \): the normal Radon transform of \( g(x, y) \)

\( \tilde{g}(\rho, \theta_t) \): the same as \( \tilde{g}(r, t) \)

\( \tilde{g}(\rho, \theta) \): the filtered 2-D sinogram

\( \tilde{g}(\tau) \): the convolution of functions \( g_1(\tau) \) and \( g_2(\tau) \)

\( g_j \): the \( j^{th} \) element of \( \tilde{g} \)

\( g_j^{(k)} \): the \( j^{th} \) element of \( \tilde{g} \) in the \( k^{th} \) iteration

\( g_1(\tau) \): a function of variable \( \tau \) defined as \( \frac{\sin \tau}{\tau} \)

\( g_2(\tau) \): a function of variable \( \tau \) defined as \( \frac{\sin(p\tau+\gamma)}{p\tau+\gamma} \)

\( \tilde{g} \): the vector containing the image values \( g(m, n) \) formed as a vector

\( \tilde{g}^{(k)} \): the vector \( \tilde{g} \) in the \( k^{th} \) iteration

\( G(k_x, k_y) \): the 2-D FT of the image \( g(x, y) \)

\( \tilde{G}(v, \theta) \): the 1-D FT of \( \tilde{g}(\rho, \theta) \) with respect to \( \rho \)

\( \tilde{G}(f) \): the FT of \( \tilde{g}(\tau) \)

\( G_1(f) \): the FT of \( g_1(\tau) \)

\( G_2(f) \): the FT of \( g_2(\tau) \)

\( h(p, \tau) \): a function of \( p \) and \( \tau \) defined as \( \frac{\sin \gamma}{\gamma} \)

\( \tilde{h}(k_p, k_r) \): the 2-D FT of \( h(p, \tau) \)

\( H_s \): the overall number of samples of \( s \) parameter

\( i_c \): the \( 1^{st} \) integer coordinate of a tile of the reconstruction domain

\( I \): the symbol for an integral

\( \tilde{I}(\rho, \theta) \): the received intensity along a line defined by \( (\rho, \theta) \)

\( \tilde{I} \): gas temperature measurements [°C]

\( I_0 \): the intensity of an emitter
Abbreviations

\( j_c \): the 2\( ^{\text{nd}} \) integer coordinate of a tile of the reconstruction domain

\( \tilde{J} \): the Jacobian of a 2 \( \times \) 2 transformation

\( J_i \): the integral measurement of the projection of the examined vector field

\( J_1 \): the vectorial Radon transform of 1\( ^{\text{st}} \) type

\( J_2 \): the vectorial Radon transform of 2\( ^{\text{nd}} \) type

\( k \): the Fourier domain variable when we transform \( \tilde{T}(\rho, \tilde{\rho}) \) with respect to \( \rho \)

\( k_p \): the Fourier domain variable of \( p \) parameter

\( k_x \): the Fourier domain variable of \( x \) parameter

\( k_y \): the Fourier domain variable of \( y \) parameter

\( k_\tau \): the Fourier domain variable of \( \tau \) parameter

\( k_1 \): the same as \( k_x \)

\( k_2 \): the same as \( k_y \)

\( \tilde{k} \): the vector \((k_1, k_2)\)

\( K \): \( 2U/P \)

\( l(g) \): \( \log \tilde{L}(\vec{g}) \)

\( l_A \): the number of \( \Delta s \) between the sensor located at \( A \) and the foot of the normal

\( l_B \): the number of \( \Delta s \) between the sensor located at \( B \) and the foot of the normal

\( L \): the integration (measurement) line

\( L(\rho, \tilde{\rho}) \): the integration line defined by \( \rho \) and \( \tilde{\rho} \)

\( \tilde{L}(\vec{g}) \): the likelihood function of \( b \) for a given \( \vec{g} \) (= \( \hat{P}(b|\vec{g}) \))

\( L_1 \): the line segment between the annihilation site and detector \( D_1 \)

\( L_2 \): the line segment between the annihilation site and detector \( D_2 \)

\( M \): the overall number of samples of \( x \) parameter

\( n(x, y) \): the refractive index of a non-homogeneous medium

\( N \): the overall number of samples of \( y \) parameter

\( p \): the slope of a scanning line

\( p_b \): the probability mass of the \( b^{\text{th}} \) bin

\( P \): the tile size of the digitised reconstruction domain \( D \) [m]

\( \hat{P} \): the probability of an event
Abbreviations

$P_1$: the probability that the photon traveling along $L_1$ will reach detector $D_1$

$P_{1,2}$: the probability that a pair of photons will reach detectors $D_1$ and $D_2$

$P_2$: the probability that the photon traveling along $L_2$ will reach detector $D_2$

$Q$: the foot of the normal from the origin of the axes to a scanning line

$r$: the parameter that is equal to $\frac{1}{p}$

$\mathbf{r}$: the vector describing the position of a point in the 2-D space

$R$: the overall number of samples of $\rho$ parameter

$RL(v)$: the filter used to approximate the ramp filter in filtered backprojection

$R_b$: the number of non-overlapping bins for the radial parameter

$s$: the free parameter in the direction of line $L$

$s_h$: the $h^{th}$ sample of $s$ parameter

$s_{\text{min}}$: the minimum value of $s$ parameter

$\mathbf{s}$: the unit vector along the integration line segment $L$

$S$: the location of a point source of positron emitters

$t$: the variable that is equal to $\frac{\pi}{\Delta x}(x - x_m)$

$T$: the overall number of samples of $\theta$ parameter

$\mathbf{T}(\rho, \hat{\rho})$: the vectorial Radon transform

$\mathbf{T}(k, \hat{\rho})$: the 1-D FT of $\mathbf{T}(\rho, \hat{\rho})$ with respect to $\rho$

$T_b$: the number of non-overlapping bins for the angular parameter

$T_c$: the total recording time in PET [s]

$T_{rs}$: the travel time of an ultrasound pulse from the receiver to the source [s]

$T_{sr}$: the travel time of an ultrasound pulse from the source to the receiver [s]

$U$: half the size of the square reconstruction domain $D$ [m]

$\nu$: the Fourier domain variable when we transform $\hat{g}(\rho, \theta)$ with respect to $\rho$

$\nu$: $-\nu$

$v_f$: the filter’s cut-off frequency [Hz]

$\nu_w$: half the sampling frequency [Hz]

$\mathbf{v}(x,y)$: the velocity of a fluid [m/s]

$w$: the angle between a scanning line and the positive direction of the $x$-axis [°]
$w_i$: the weight, that the $i^{th}$ equation of the system, should be multiplied with

$\mathbf{w}$: the vector that contains the weights, that all equations should be multiplied with

$W(f)$: a function similar to $\text{rect}(\pi f)$ if $|p| \geq 1$, and similar to $\text{rect} \left( \frac{\pi f}{|p|} \right)$ otherwise

$\mathbf{W}(v)$: the weight function multiplied with the Ram-Lak filter to avoid ringing artifacts

$x$: the first coordinate (abscissa) of the Cartesian coordinate system

$x_A$: the $x$ coordinate of the boundary sensor located at $A$

$x_B$: the $x$ coordinate of the boundary sensor located at $B$

$x_{\text{inc}}$: the increment in the $x$ coordinate between successive sampling points of a line

$x_l$: the $x$ coordinate of a sampling point $l$ along a scanning line

$x_m$: the $m^{th}$ sample of $x$ parameter

$x_{\text{max}}$: the maximum value of $x$ parameter

$x_{\text{min}}$: the minimum value of $x$ parameter

$x_Q$: the $x$ coordinate of the foot of the normal

$x_r$: the $x$ coordinate of an ultrasound signal receiver

$x_s$: the $x$ coordinate of an ultrasound signal source

$x_1$: the random variable that corresponds to the $x$-coordinate of the $1^{st}$ sensor

$x_{1a}$: the $1^{st}$ root for $x_1$, by solving $\rho$ and $\theta$, expressed as functions of random variables

$x_{1k}$: the $k^{th}$ root for $x_1$, by solving $\rho$ and $\theta$, expressed as functions of random variables

$x_2$: the random variable that corresponds to the $x$-coordinate of the $2^{nd}$ sensor

$x_{2a}$: the $1^{st}$ root for $x_2$, by solving $\rho$ and $\theta$, expressed as functions of random variables

$\mathbf{x}$: the unit vector in the $x$ direction

$y$: the second coordinate (ordinate) of the Cartesian coordinate system

$y_A$: the $y$ coordinate of the boundary sensor located at $A$

$y_B$: the $y$ coordinate of the boundary sensor located at $B$

$y_{\text{inc}}$: the increment in the $y$ coordinate between successive sampling points of a line

$y_l$: the $y$ coordinate of a sampling point $l$ along a scanning line

$y_{\text{max}}$: the maximum value of $y$ parameter

$y_{\text{min}}$: the minimum value of $y$ parameter

$y_n$: the $n^{th}$ sample of the $y$ parameter
yQ: the y coordinate of the foot of the normal
yR: the y coordinate of an ultrasound signal receiver
ys: the y coordinate of an ultrasound signal source
y1: the random variable that corresponds to the y-coordinate of the 1st sensor
y1a: the 1st root for y1, by solving ρ and θ, expressed as functions of random variables
y2: the random variable that corresponds to the y-coordinate of the 2nd sensor
y2a: the 1st root for y2, by solving ρ and θ, expressed as functions of random variables
y2k: the kth root for y2, by solving ρ and θ, expressed as functions of random variables
\( \hat{y} \): the unit vector in the y direction
z: the third coordinate (applicate) of the Cartesian coordinate system
\( \hat{z} \): the unit vector in the z direction
Table of Greek symbols

\( \hat{\alpha}(x, y) \): a 2-D function

\( \alpha_{i,j} \): the element of system matrix \( A \) located in the \( i^{th} \) row and \( j^{th} \) column

\( \hat{\alpha}_i \): the \( i^{th} \) row of system matrix \( A \)

\( \beta \): the coordinate, where a scanning line intersects the \( y \) axis

\( \gamma \): the variable that is equal to \( \frac{\pi}{\Delta x}(px_m + \tau - y_n) \)

\( \gamma_{GH} \): the sharpness parameter of the generalised Hamming weight

\( \Delta \gamma \): the sampling step of \( \gamma \) parameter

\( \Delta \theta \): the sampling step of \( \theta \) parameter \([^{\circ}]\)

\( \Delta \rho \): the sampling step of \( \rho \) parameter

\( \Delta s \): the sampling step of \( s \) parameter

\( \Delta x \): the sampling step of \( x \) parameter

\( \Delta y \): the sampling step of \( y \) parameter

\( \Delta A \): the integration sub-segment that is closest to sensor at \( A \)

\( \Delta B \): the integration sub-segment that is closest to sensor at \( B \)

\( \mathcal{E}(\rho, \theta) \): the measured emissions in PET along a line defined by \((\rho, \theta)\)

\( \eta \): the parameter that is equal to \(-\frac{\tau}{p}\)

\( \theta \): the angle, that the normal to a line from \((0, 0)\), forms with the positive \( x \) semi-axis \([^{\circ}]\)

\( \hat{\theta} \): \( \theta - \pi \) \([^{\circ}]\)

\( \theta_{b_l} \): the lower value of parameter \( \theta \) in the 2-D region of definition of the \( b^{th} \) bin \([^{\circ}]\)

\( \theta_{b_u} \): the upper value of parameter \( \theta \) in the 2-D region of definition of the \( b^{th} \) bin \([^{\circ}]\)

\( \theta_{\min} \): the minimum value of \( \theta \) parameter \([^{\circ}]\)

\( \theta_t \): the \( t^{th} \) sample of \( \theta \) parameter \([^{\circ}]\)

\( \lambda \): the same as \( p \)
Abbreviations

\( \lambda_k \): the relaxation parameter used in algebraic reconstruction technique
\( \mu \): the attenuation coefficient
\( \rho \): the shortest distance from the origin of the coordinate system to a scanning line
\( \rho_{b_l} \): the lower value of parameter \( \rho \) in the 2-D region of definition of the \( b^{th} \) bin
\( \rho_{b_u} \): the upper value of parameter \( \rho \) in the 2-D region of definition of the \( b^{th} \) bin
\( \rho_{\text{max}} \): the maximum (positive and finite) value of \( \rho \) parameter
\( \rho_{\text{min}} \): the minimum (positive and finite) value of \( \rho \) parameter
\( \rho_r \): the \( r^{th} \) sample of \( \rho \) parameter
\( \hat{\rho} \): the unit vector perpendicular to \( L \) in the investigated plane
\( \mathcal{P}(\rho, \theta) \): the projection being recorded by an X-ray CT scanner along a \((\rho, \theta)\) line
\( \tau \): the same as \( \beta \)
\( \Phi(x, y) \): the scalar function, the gradient of which gives \( \vec{f}_1(x, y) \)
\( \Psi_0(x, y) \): the \( z \) component of \( \Psi(x, y) \)
\( \Psi(x, y) \): the vector field, the rotation of which gives \( \vec{f}_S(x, y) \)
"Nothing that is worth knowing can be taught"

Oscar Wilde
Chapter 1

Introduction

1.1 Background and Motivation

VECTOR field tomography is an area that has received considerable attention during the last decades and deals with the problem of the determination of a vector field distribution from non-invasive integral measurements. When one tries to investigate planar vector fields in bounded domains, two classes of tomographic measurement arise, depending on the interaction between the obtained measurements and the examined vector field. In the first type of measurement, only the component of the investigated vector field along the measurement line is observed (longitudinal measurements), while the second class of tomographic measurement collect information from the component of the investigated vector field perpendicular to the measurement line (transversal measurements). Next, we briefly outline the recent developments and limitations of 2-D vector field tomography.

During the short history of vector field tomography, many investigators attempted to solve the related reconstruction problem. They invariably discussed this inverse problem in continuum terms and used a scalar tomography-based approach. In particular, in order to help their work, the researchers employed the classical Helmholtz decomposition theorem, that decomposes the examined vector field into its irrotational and solenoidal

\footnote{This thesis deals only with vector fields that have two components.}
components, and treated each vector component using a scalar tomography method. The conclusion that was drawn was that, by relying only on line-integral measurements, the reconstruction problem in 2-D vector field tomography was underdetermined [4], [32], [48], [67]. In particular, it was found that only one component of the vector field could be recovered from tomographic measurements. The recovered component was either the curl-free (irrotational) part or the divergence-free (solenoidal) part, depending on the physical principle of the measurements (i.e., the interaction between the obtained set of measurements and the investigated vector field, mentioned above) of the considered application. An algebraic reconstruction method of this type, where the authors considered the problem of only reconstructing the solenoidal component from the tomographic data, was developed in [12].

One possible solution to this problem would be to collect data using both types of relation between the measurements and the examined vector field for every application. Indeed, such an amount of information would be sufficient to allow for a full reconstruction of the vector field, as Braun and Hauck demonstrated in [4]. Unfortunately, there are only very few specialised applications (mainly in Schlieren tomography), where it is physically realisable to have both types of measurement available. Another solution was proposed by Norton [48], who suggested that one may have a full reconstruction based only on longitudinal measurements, as long as, apart from the longitudinal measurements, supplementary information about the investigated vector field, especially boundary conditions or a priori information about its source distribution, is available as well. A study, where the developed algebraic methodology was about fully reconstructing a vector field based on longitudinal measurements and a priori information about the source distribution of the vector field to be imaged, was presented in [61]. Another similar example of using, apart from the projection measurements, also supplementary information about the examined vector field lies in meteorology [31]. The supplementary information, that the authors of [31] employed about the examined wind velocity field, was in the form of angle measurements. In addition, Rouseff and Winters showed in [58] that a complete 2-D vector field reconstruction based on boundary data is possible for scattering geometries. However, the model they used for the available measurements was a scattering model
rather than the integral-geometry transformations, that have been traditionally used in vector field tomography and are based on transmission. Next, we describe the range of issues that this thesis deals with.

1.2 Overall Scope of this Thesis

In this thesis, we look at the application of tomography to the reconstruction of 2-D vector fields. We make an attempt to give an answer to the following questions: “what information about a vector field can be extracted from integral measurements, obtained along lines that go through the vector field’s domain of definition?” and “is it possible to have a complete and unique reconstruction of a vector field based only on projection (line-integral) data, obtained on the boundary of the vector field’s domain of definition?”

An important issue when solving inverse problems is the resilience of the solution to noise. In this thesis, we examine the effect of noise on the reconstruction of 2-D vector fields. The types of noise that we consider are inaccuracies in the sensor measurements, sensor misplacements and both the above effects simultaneously. Methods to improve noise tolerance are also the topic of discussion in this thesis.

Inverse problems, like the reconstruction problem in 2-D vector field tomography, suffer from the notorious ill-posed nature, in the sense of Hadamard [22]. As a result, the solution to these reconstruction problems endures stability deficiencies that are related to the solution’s existence, uniqueness and continuous dependency on the projection data. Stability issues, when solving the 2-D vector field tomography reconstruction problem, are also the subject of consideration in this thesis.

According to the theory of the Radon transform [9], a necessary requirement to produce reconstruction results with the accuracy desired in medical imaging, when using discrete approximations, is to sample uniformly the Radon domain parameter space. However, sampling this space uniformly creates serious impracticalities concerning space or time, that are discussed in Chapter 4. In the light of the above two statements, this thesis attempts to develop methods that approximate uniform sampling in the projection
space with a view to achieving improved 2-D vector field reconstruction quality. Critical issues, like practical sensor configuration and overall reconstruction time, are taken into account by this thesis when designing such methods.

This thesis, also, addresses sampling issues in relation to 2-D vector field tomography. We consider standard parallel scanning and make an attempt to give an answer to questions like “what are the sampling requirements that must be imposed on the distances of the parameters of the projection space, for a given spatial resolution in the sought-for vector field, so as not to lose boundary integral information?” or “given a sampling of the sinogram, what is the maximum acceptable resolution in the reconstruction region?” The influence of the sampling rate of the projection space on the quality of 2-D vector field reconstruction is also studied. Next, we provide an overview of this thesis.

### 1.3 Organisation of this Thesis

The research work and writing up of this PhD thesis were carried out from January 2006 to October 2009 at the Advanced Technology Institute, University of Surrey and the Department of Electrical and Electronic Engineering, Imperial College London, and were supervised by Maria Petrou and S. Ravi P. Silva. This PhD thesis, entitled *Solving the Inverse Radon Transform for Vector Field Tomographic Data*, discusses 2-D vector field tomography and is divided into seven chapters.

The main tools and reconstruction algorithms used in vector field tomography are natural generalisations of those used in conventional (scalar) tomography. Therefore, we find it useful to review the classical tomography in **Chapter 2**. The scalar tomography framework and the applications to X-ray computerised tomography (CT) and positron emission tomography (PET) are presented there. A brief description of some basic scalar tomographic reconstruction algorithms, that rely on Fourier analysis, backprojection, linear algebra and statistics, is also given.

**Chapter 3** on vector field tomography is the central part of this thesis. We give a short account of the application areas of vector field tomography. We define the relevant
integral transforms, namely the two types of vectorial Radon transform. These transforms form the mathematical basis for dealing with the problem of 2-D vector field tomographic mapping. We describe in detail the approaches, that have been employed so far, to solve the 2-D vector field reconstruction problem and discuss their limitations. We introduce a novel direct algebraic reconstruction algorithm that allows one to estimate both components of a 2-D vector field at specific sampling points, finite in number and arranged in a grid of the reconstruction region, by relying only on a finite number of line-integral data obtained on the boundary of this domain. The proposed technique achieves the complete 2-D vector field recovery by exploiting the redundancy in the projection data, as a form of employing regularisation. For the evaluation of the introduced method, we present examples of electric field reconstruction. Chapter 3, also, explains the treatment we employ in order to deal with the stability deficiencies of the 2-D vector field reconstruction problem. It turns out that, by following the proposed reconstruction technique, the ill-posedness of the inverse problem of 2-D vector field tomography is noticeable, but manageable and not serious. The performance of the proposed reconstruction approach in noisy environments is also studied. Experimental results point out that the proposed reconstruction technique is relatively robust to perturbations in the sensor positions.

In Chapter 4, we propose the employment of interpolated integral data as a means of improving the vector field reconstruction quality and maintaining, at the same time, a practical sensor configuration. These data are obtained at “virtual” sensors that correspond to uniform sampling of the projection space. Hence, this method is not limited by physical constraints on sensor placement. We go on to show that the employment of such interpolated data also increases the resilience to noise.

In Chapter 5, we employ probabilistic weights to account for the non-uniformity in the projection space. Simulation results show that this employment leads to significant reduction of reconstruction error without having to resort to impractical sensor positioning or, most importantly, increase the processing time. The reason that the overall reconstruction time does not increase is that the calculation of the proposed weights is based on the known and predetermined sensor configuration. Hence, this calculation can be performed
In advance (off-line).

In Chapter 6, we look at the vector field reconstruction problem from the aspect of sampling. This aspect is crucial for the design of imaging devices. We consider parallel scanning 2-D vector field tomography and derive the sampling bounds, which must be imposed on the sampling of projection space parameters in order to achieve an intended spatial resolution of the investigated 2-D vector field and, at the same time, not to lose boundary integral information. Experimental results demonstrate that when the derived sampling bounds are violated, the reconstruction accuracy of the vector field deteriorates both in noise-free and noisy environments.

In Chapter 7, we conclude this thesis and summarise main contributions and achievements. Possible directions of future research are also outlined.

Finally, references that support statements in this thesis are listed in Bibliography. Next, we list the publications that resulted from the research work, included in this thesis.

1.4 Relevant Publications by the Author

Many of the outcomes of the research work, presented in this thesis, have been published or submitted to high calibre refereed journals and conference proceedings. A list of publications is given below:

Journals:


- Giannakidis A., Kotoulas L. and Petrou M. “Virtual Sensors for 2-D Vector Field
1.4 Relevant Publications by the Author

Tomography”, *Journal of the Optical Society of America A*, submitted, under review.


**Conference Proceedings:**


Chapter 2

Scalar Tomographic Reconstruction using the Radon Transform

As mentioned in the Introduction, the treatment of scalar and vector fields is similar. Therefore, we find it useful to review scalar tomography in this chapter.

2.1 Integral Geometry and the Radon Transform

Radon transform is an integral transform and it was first described by the Austrian mathematician Johan Radon who published a paper in 1917, "On the determination of functions from their integrals along certain manifolds" [55]. This chapter discusses only the 2-D Radon transform. In particular the Radon transform of an image function is discussed. A special version of the Radon transform applied to binary images is known as the Hough transform.

The Hough transform [25] is suited for line parameter extraction even in the presence of noise. It is able to transform each line into a point in the parametric space of line

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1 The translation of the original paper from German into English may be found in Appendix A of [9].
2 Some of the discussion can be readily generalised to the 3-D Radon transform. For more details see Appendix D of [70].
representation with coordinates corresponding to the parameters of the line. In this way, the Hough transform converts a difficult global detection problem in the image domain into a more easily solved local peak detection problem in the parameter domain, where the actual line parameters can be recovered, e.g. by thresholding the parameter space. This property has led to many line detection algorithms within image processing, computer vision and seismics.

Theoretical ideas found in the early work of Radon apply in many tomographic\textsuperscript{3} techniques: X-ray CT, magnetic resonance imaging (MRI), single photon emission computerised tomography (SPECT), PET and ultrasound imaging. The construction of images in all medical imaging modalities, mentioned above, relies on using the Radon transform. Hence, in the next section, we present the theoretical foundations of the Radon transform.

This chapter is structured as follows. In the rest of Section 2.1, we define the Radon transform, the main tool in tomography, and discuss the sampling properties of its discrete version. In Section 2.2, we present the applications of Radon transform in X-ray CT and PET. In Section 2.3, we present some classical methods to invert the Radon transform, namely Fourier reconstruction and filtered backprojection. In Section 2.4, we present techniques that perform inversion of the scalar Radon transform by relying on linear algebra. Finally, in Section 2.5, we discuss statistical reconstruction methods that are commonly used in emission tomography.

2.1.1 Defining the Radon Transform

The Radon transform can be defined in several different ways, but all of them are related. One of the most popular definitions is the normal Radon transform. This definition is used in many fields of science, e.g. medicine, astronomy and microscopy.

The normal Radon transform \(\tilde{g}(\rho, \theta)\) of a continuous smooth 2-D function \(g(x, y)\) is found by integrating values of \(g\) along lines. A line is defined in terms of parameters \(\rho\) and \(\theta\), defined in Fig. 2.1.

\textsuperscript{3}The word "tomography" origins in the Greek language and consists of the words "τομή" meaning "slice" and "περιστος" meaning "image".
2.1 Integral Geometry and the Radon Transform

Parameter $\rho$ is the shortest distance from the origin of the coordinate system to the line, and $\theta$ is an angle corresponding to the angular orientation of the line. Using the normal parameters $\rho$ and $\theta$, the equation of a line may be put in the form:

$$\rho = x \cos \theta + y \sin \theta$$  \hspace{1cm} (2.1)

Hence the Radon transform of a function $g(x, y)$ is given by

$$\tilde{g}(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \delta(\rho - x \cos \theta - y \sin \theta) \, dx \, dy$$  \hspace{1cm} (2.2)

where $\delta$ is the Dirac delta function. The task of tomographic reconstruction is to find $g(x, y)$ given the knowledge of $\tilde{g}(\rho, \theta)$. An equivalent way of writing Eq. (2.2) is derived if we change the coordinate system of Fig. 2.1, by rotating $(x, y)$ by $\theta$ (as shown in Fig. 2.2) and defining parameters $(\rho, s)$

$$x = \rho \cos \theta - s \sin \theta$$  \hspace{1cm} (2.3)

$$y = \rho \sin \theta + s \cos \theta$$  \hspace{1cm} (2.4)

where the $s$-axis is parallel to the line. Then, Eq. (2.2) is transformed into:

$$\tilde{g}(\rho, \theta) = \int_{-\infty}^{+\infty} g(\rho \cos \theta - s \sin \theta, \rho \sin \theta + s \cos \theta) \, ds$$  \hspace{1cm} (2.5)

The values of $\tilde{g}(\rho, \theta)$ are defined in the 2-D Radon space or parameter domain.
Function $\tilde{g}(\rho, \theta)$ is often referred to as the sinogram because the Radon transform of an off-center point source is a sinusoid (see Eq. (2.1) assuming that $x$ and $y$ are fixed). Thus, the Radon transform of a point source is confined in a finite parameter domain.

Another important property of the normal Radon transform is that all lines can be described by choosing $0 \leq \theta < 2\pi$ and $\rho \geq 0$. However, we frequently introduce negative values of $\rho$ and the parameter domain is bounded by $0 \leq \theta < \pi$ and $-\rho_{\text{max}} \leq \rho \leq \rho_{\text{max}}$ where $\rho_{\text{max}}$ is positive and finite when a discrete implementation is considered. The above boundaries of the parameter domain are very much related to the following nice mathematical property of the sinogram $\tilde{g}(\rho, \theta)$:

$$\tilde{g}(\rho, \theta) = \tilde{g}(-\rho, \theta + \pi) \quad (2.6)$$

Moreover the normal Radon transform is a linear function and, also, rules about translation, rotation and scaling apply.  

### 2.1.2 The Discrete Radon Transform and Sampling Properties

Unfortunately, only a subset of primitive functions can be transformed analytically. These basic functions include the circular disc, the square, the triangle, the pyramid and the Gaussian bell. For all other functions, a discrete approximation to the Radon

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4 More details and derivations about the properties of the 2-D Radon transform may be found in Appendix B of [70].

5 Analytical estimations of the Radon transform of these primitives may be found in Appendix B of [70].
2.1 Integral Geometry and the Radon Transform

Transform applied to a digital image has to be used.

To obtain the discrete approximation, a uniform sampling of the four variables $x, y, \rho$ and $\theta$ with steps $\Delta x, \Delta y, \Delta \theta$ and $\Delta \rho$, respectively, is assumed. Hence a limited set of samples is considered

$$
x = x_m = x_{\min} + m\Delta x, \quad m = 0, 1, \ldots, M - 1
$$

$$
y = y_n = y_{\min} + n\Delta y, \quad n = 0, 1, \ldots, N - 1
$$

$$
\theta = \theta_t = \theta_{\min} + t\Delta \theta, \quad t = 0, 1, \ldots, T - 1
$$

$$
\rho = \rho_r = \rho_{\min} + r\Delta \rho, \quad r = 0, 1, \ldots, R - 1
$$

(2.7)

where $M, N, T$ and $R$ are the total numbers of samples of $x, y, \theta$ and $\rho$, respectively. Sampling function $g(x, y)$ produces a digital image, and likewise the discrete Radon transform can be presented as a digital image:

$$
g(m, n) = g(x_m, y_n)
$$

(2.8)

$$
\tilde{g}(r, t) = \tilde{g}(\rho_r, \theta_t)
$$

(2.9)

Regarding the sampling of $x, y$ and $\rho$, it is optimal to choose symmetrical points around zero:

$$
x_{\min} = -x_{\max} = -\frac{(M - 1)}{2}\Delta x
$$

$$
y_{\min} = -y_{\max} = -\frac{(N - 1)}{2}\Delta y
$$

(2.10)

$$
\rho_{\min} = -\rho_{\max} = -\frac{(R - 1)}{2}\Delta \rho
$$

Considering the angular sampling, the starting point may be chosen to be $\theta_{\min} = 0$ and the sampling interval of $\theta$ should be set to span $\pi$, i.e. $\Delta \theta = \frac{\pi}{T}$.

When implementing a Radon transform algorithm, there are some requirements, relating to the sampling intervals, that must be fulfilled. In particular, for a given digital image, there exist upper limits of sampling steps in the parameter domain. Violating these
limits, by not sampling sufficiently dense the parameters of the Radon domain, can result in aliasing problems for this domain. We usually define bounds on the sampling distances in the parameter domain by demanding that changing either of the Radon parameters with its respective sampling interval, this should not lead to more than a pixel of difference in the image. This is equivalent to a demand that the change of consecutive lines should be below one sample. If we do not meet this criterion, some of the pixels might not add weight to the parameter domain, hence information is lost. The optimal sampling rate depends also on the type of image. The sampling properties of the discrete normal Radon transform are addressed in [70].

A simple and common approach to implement the linear normal Radon transform is to use a sum approximation to Eq. (2.5)

\[
\tilde{g}(r, t) = \int_{-\infty}^{+\infty} g(\rho_r \cos \theta_t - s \sin \theta_t, \rho_r \sin \theta_t + s \cos \theta_t) \, ds
\]

\[
\approx \Delta s \sum_{h=0}^{H-1} g(\rho_r \cos \theta_t - s_h \sin \theta_t, \rho_r \sin \theta_t + s_h \cos \theta_t)
\] (2.11)

where \( s_h \) is a linear sampling of variable \( s \) with step \( \Delta s \)

\[
s = s_h = s_{\text{min}} + h\Delta s, \quad h = 0, 1, \ldots, H_s - 1
\] (2.12)

where \( H_s \) is the total number of samples of \( s \) parameter. This approach gives rise to many problems. The most serious one is that for a given value of \( h \), the image points used to compute the sum on the right-hand side of Eq. (2.11) almost never coincide with samples in the digital image. Hence an interpolation in both variables \( (m, n) \) is needed. This 2-D interpolation should be avoided due to the artifacts it creates. Another question is how densely parameter \( s \) should be sampled.

The Radon transform (and its derivative Hough transform) have been used extensively in image processing for the identification of parametric curves. However, the most significant application of the Radon transform is, by far, in tomographic methods. Non-invasive tomographic methods can be found in almost every branch of science, and an exhaustive review is impossible. In the next section, we describe two applications of the
2.2 Applications of the Radon Transform

2.2.1 Application in X-ray CT

The introduction of X-ray CT in 1979 by Hounsfield and Cormack was perhaps the most revolutionary development in the field of medical imaging since the time of Röntgen. For this work they got the Nobel prize in physiology and medicine. In X-ray CT one probes the part of interest of the human body with non-diffractive radiation (X-rays). With X-ray CT, it was the first time the computer played a central role in the creation of the images. Technical advances in image processing with improved computers and software have, naturally, produced images of much higher quality. Unlike X-ray radiography and ultrasound, X-ray CT produces clear images of the various structures of a human organ based on their ability to block the X-ray beam. Apart from providing structural information, the reconstructed distribution of the attenuation coefficient may, also, be used to discriminate between normal and pathological tissue.

The X-rays are usually arranged in a regular pattern, which is referred to as the scanning geometry. Let us consider the X-ray CT scan configuration shown in Fig. 2.3. This is a 3rd generation scanner called fan-beam “spinning” scanner that leads to rapid data collections. It consists of a ring with one X-ray emitter and a large number of detectors positioned opposite the emitter. Beams of X-rays are passed through the object being imaged. It is assumed that X-rays travel in straight lines. Rotating the emitter and the detector array around the patient makes it possible to cover all body parts of interest.

The travelling X-rays are attenuated at different rates by different tissues. The attenuation takes place due to the photons either being absorbed by the atoms of the material (photoelectric absorption) or being scattered away from their original direction of travel (Compton scattering effect). Finally, the attenuated X-rays are collected by

---

6The preceding description gives no impression of the X-ray CT scanner as it is presented to the patient. A typical scanner is characterised by a smooth framework surrounding a large hole into which the patient's body is inserted. The smooth framework disguises the complex gantry, the X-ray source and the array of X-ray detectors.
2.2 Applications of the Radon Transform

Figure 2.3: The schematic representation of a 3\textsuperscript{rd} generation CT scanner.

the detectors. A typical example of 2-D CT reconstruction is presented in [54], where a 1024 \times 1024 pixel image is recovered by using a 165 detector, 180 view setting that generates 180 data sets (sinograms) of 165 measurements.

Next, we describe the motivation for modelling the measurements, obtained by X-ray CT scanners, by using the Radon transform. In particular, it is shown that the X-ray CT measurements can be converted into samples of the Radon transform of the linear attenuation coefficient, that we want to reconstruct.

The CT scanner (Fig. 2.3) is a 2-D planar scanner with only one slice of the human organ measured, therefore the coordinate system can be chosen so that the slice is the $z = 0$ plane. The scanned organ is non-homogenous, hence the attenuation coefficient is a function of $x$ and $y$, i.e. $\mu(x, y)$. The emitter and each of the detectors define a line $(p, \theta)$ going through the scanned object, where $p$ is the distance from $(0,0)$ to the line and $\theta$ is the angle relative to the first axis (Fig. 2.1). The received intensity is

$$\bar{I}(p, \theta) = I_0 e^{-\int \mu(x, y) \, ds} \quad (2.13)$$

where $I_0$ is the intensity of the emitter, $s$ is the parameter in the normal form of the line and $(x, y)$ lies in the line defined by $(p, \theta)$. Note that the exponential factor in Eq. (2.13) can
be perceived as the probability of a single photon getting through the absorbing medium.

The projection $P(p, \theta)$ is defined as:

$$
P(p, \theta) \equiv \log(\frac{I_0}{I(\rho, \theta)}) = \int \mu(x, y) \, ds =
$$

$$
= \int_{-\infty}^{+\infty} \mu(\rho \cos \theta - s \sin \theta, \rho \sin \theta + s \cos \theta) \, ds
$$

Hence, the projections, that are being recorded by an X-ray CT scanner, consist of line integrals of the attenuation coefficient and it can be recognised that $P(p, \theta)$ is the Radon transform of $\mu(x, y)$. Using the delta Dirac function the projections can be written as:

$$
P(p, \theta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mu(x, y) \delta(\rho \cos \theta - x \cos \theta - y \sin \theta) \, dx \, dy
$$

Unfortunately, the actual data collected by an X-ray CT scanner do not correspond exactly to the Radon transform of the "true" attenuated coefficient. In any imaging system data will be corrupted by noise. Also the projections are measured with only limited resolution. Unless we make some assumptions about the object being imaged, no finite number of projections defines the original image uniquely and exactly. Furthermore, the geometry of the scanner may differ from the ideal, especially in cases of fan-beam imaging systems.

The collected measurements, that are modelled by the Radon transform of the attenuation coefficient, are converted to digital data by the analogue to digital converters (ADCs). Finally, the digital data are fed to a computer system. The computer reconstructs the distribution of the attenuation coefficient of the examined human organ by inverting the Radon transform following one of the methods that will be discussed in Sections 2.3 and 2.4.

### 2.2.2 Application in PET

Another example of human organ imaging is PET, which is an interesting example of the joint effort of many disciplines including chemistry and computer science. In PET
one monitors the distribution of a radiopharmaceutical in a desired cross-section of the human body, by measuring the radiation outside the body.

PET requires short-lived cyclotron-produced radionucleides. These radionucleides are suitable for radiopharmaceuticals\(^7\) that can be administered into the patient by either inhalation or injection. The decay of radionucleides results in the emission of positrons. Within a few millimetres each positron interacts with a nearby electron and annihilates with the emission of two 511 keV photons. These two annihilation photons are generated simultaneously and travel in opposite directions, nearly 180° back to back (collinearity). This near collinearity of the two annihilation photons makes it possible to identify the annihilation event (or the existence of positron emitters) through the detection of the two photons by two detectors posed exactly in opposite sides within a short time (i.e. within \(10^{-8}\) sec or less). The detectors are placed in a ring in the PET scanner, as illustrated in Fig. 2.4. Note that here, contrary to what happens in X-ray CT, only detectors are needed, as the emitter is placed within the patient.

![Figure 2.4: Emission of two photons from the place of decay in a PET scanner.](image)

The PET hardware sorts the arrival times of the photons, so only two photons that arrived (almost) at the same time at the detector ring are taken into account. The

\(^7\)If the radionucleide is attached to glucose, then the interesting possibility of measuring the activity of brain arises. In particular, in regions of the brain with high activity, the glucose metabolism will be high as well, and a corresponding number of radionucleides will decay under emission of positrons.
photons travel with the speed of light, and the small difference in arrival time (< 2ns) is neglected.

In practical PET scanners several problems arise such as scatter and random coincidences, geometrical factors, penetration of photons through several detectors before detection etc. Next, the motivation for modelling the measurements, obtained by PET scanners, by using the Radon transform is presented.

Let us assume that we have the PET scanner shown in Fig. 2.4, where the detectors are placed in a ring. The patient is administered with radionuclides and pairs of photons, that travel in opposite directions, are generated within the human organ according to the procedure described above.

A two dimensional matrix of the possible detector versus detector combinations is created, and all values are initialised to zero. The number of possible detector combinations corresponds to a finite set of possible line parameters \((\rho, \theta)\). Assuming that two photons have been detected and the line between the two detectors has line parameters \((\rho_0, \theta_0)\), then, the array is incremented by one at position \((\rho_0, \theta_0)\). This is because the only obtainable information from the two photons is that the photon emission took place somewhere along that line. Depending on the radioactive dosage given to the patient, many decays take place each second in each domain element. When the recording is terminated, an array of emissions has been recorded. However, we must note that the majority of photon pairs are never detected due to the limited size of the detector ring. A typical example of 2-D PET reconstruction is presented in [66], where 128 x 128 pixel images are recovered by relying on partitioning the data space in 8932 sinogram bins.

The obtained array of emissions is approximately proportional to the total emission intensity along that particular line, times the total recording time, denoted by \(T_e\). If the measured array of emissions is \(E(\rho, \theta)\), then

\[
E(\rho, \theta) \simeq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_i(x, y) \delta(\rho - x \cos \theta - y \sin \theta) \, dx \, dy \, T_e
\]  

(2.16)

where \(E_i(x, y)\) is the emission intensity.

\(^8\)For more details regarding practical problems in PET scanners one can see [70].
It can be recognised from Eq. (2.16) that the recorded array $\mathcal{E}(\rho, \theta)$ is the Radon transform of the emission intensity. Hence, the emission intensity (which is proportional to the concentration of the radionucleides) can be reconstructed by inverting the Radon transform following one of the methods that will be discussed in Section 2.5. This intensity will not be a constant in time for any cross section. Therefore, all data for one cross-sectional image must be collected in a short time interval.

So far, the absorption of photons in the tissue has been neglected. To study the effect of attenuation in PET, consider the geometry of Fig. 2.5.

![Figure 2.5: Attenuation compensation for PET.](image)

Let us assume that a point source of positron emitters is located at point S. Suppose for a particular positron annihilation, the two annihilation gamma-ray photons are released towards detectors $D_1$ and $D_2$. The photon travelling along line segment $L_1$ will reach detector $D_1$ with probability $P_1$, given by

$$P_1 = e^{-\int_{L_1} \mu(x,y) \, ds}$$

(2.17)

where $\mu(x,y)$ is the attenuation coefficient. Similarly, the probability of the other photon
2.2 Applications of the Radon Transform

travelling along line segment $L_2$ and reaching detector $D_2$ is given by:

$$P_2 = e^{-\int_{L_2} \mu(x,y) \, ds}$$  \hspace{1cm} (2.18)

The two photons travel independently, hence, the probability that this particular annihilation will be recorded by the detectors is given by the product of the above two probabilities:

$$P = e^{-\int_{L_1} \mu(x,y) \, ds} e^{-\int_{L_2} \mu(x,y) \, ds} = e^{-\int_\mathcal{L} \mu(x,y) \, ds}$$  \hspace{1cm} (2.19)

This is a most remarkable result, because, first, this attenuation factor is the same no matter where positron annihilation occurs on the line joining $D_1$ and $D_2$, and, second, the factor above is exactly the attenuation that a beam of photons at 511 keV would undergo in propagating from $D_1$ to $D_2$. Therefore, one can easily compensate for attenuation by first doing a transmission study to record total transmission loss for each ray in each projection. Then, in the PET study, the data for each ray can, simply, be attenuation compensated when corrected (by division) by this transmission factor.

It was shown above that the attenuation coefficient in PET becomes a constant along every line, and only produces a multiplicative factor in the Radon domain. Thus, the essential part of producing human organ images can, still, be based on the inversion of the Radon transform. In a SPECT scanner, however, which operates by measuring emission of a single photon only, the attenuation correction also depends on the annihilation position and is not as simple as in PET.

Apart from the clinical applications, the Radon transform is also used to model the measurements in industrial process tomography. An example is shown in [50], where $20 \times 20$ pixel images of the formation of caverns inside mixing tanks are recovered by relying on 104 boundary measurements.

It was shown in this section that the measurements made in X-ray CT are samples of the Radon transform of the attenuation coefficient to be reconstructed. Also, in PET the recorded array is the Radon transform of the emission intensity (which is identical to the concentration of the radionuclides to be reconstructed). Hence, for both the modalities
mentioned above, our instruments give us \( \hat{g}(r, t) \) and we wish to recover the original cross-sectional images \( g(m, n) \). To achieve this, we must invert the Radon transform \( \hat{g}(r, t) \). The methods used for this inversion are discussed in the next section. These methods are broadly divided into three categories: inversion via some transformation, inversion based on linear algebra and inversion based on statistics. The development of the various CT and PET scanners has been based, more or less, on these methods.

2.3 Inverting the Radon Transform via Transformation

The inversion via transformation methods rely on the Fourier slice theorem or on filtered backprojection. In the derivation of the image reconstruction via some transformation methods, the continuous versions of the original image, i.e. \( g(x, y) \), and its Radon transform, i.e. \( \hat{g}(\rho, \theta) \), are used.

2.3.1 Fourier Reconstruction - The Fourier Slice Theorem

In order to reconstruct the image, we can use the Fourier slice theorem (FST), also known as the central slice theorem [14] and [41]. This theorem makes it possible to invert the Radon transform by relating the 2-D Fourier transform (FT) \( G(k_x, k_y) \) of image \( g(x, y) \) along a radial line with the 1-D FT \( \tilde{G}(v, \theta)^9 \) of \( \hat{g}(\rho, \theta) \) for a specific angle \( \theta \). The derivation of the FST is given next.

The 1-D FT of \( \hat{g}(\rho, \theta) \) for a specific angle \( \theta \) is:

\[
\tilde{G}(v, \theta) = \int_{-\infty}^{+\infty} \hat{g}(\rho, \theta) e^{-j2\pi\rho v} d\rho
\]

\[
= \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) \delta(\rho - x \cos \theta - y \sin \theta) \, dx \, dy \right] e^{-j2\pi\rho v} d\rho
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) \left[ \int_{-\infty}^{+\infty} \delta(\rho - x \cos \theta - y \sin \theta) e^{-j2\pi\rho v} d\rho \right] \, dx \, dy \quad (2.20)
\]

\(^9\)Variable \( v \) is the Fourier-domain variable when we transform \( \hat{g}(\rho, \theta) \) with respect to \( \rho \) while \( \theta \) takes a specific value.
One important property of the delta function is:

\[
\int_{-\infty}^{+\infty} g(x) \delta(\alpha x + b) \, dx = \frac{1}{|\alpha|} g\left(\frac{-b}{\alpha}\right)
\]  

(2.21)

By applying Eq. (2.21) to Eq. (2.20) we obtain:

\[
\hat{G}(v, \theta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) e^{-j2\pi v(x \cos \theta + y \sin \theta)} \, dx \, dy
\]  

(2.22)

The 2-D FT of \(g(x, y)\) is given by:

\[
G(k_x, k_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) e^{-j2\pi (k_x x + k_y y)} \, dx \, dy
\]  

(2.23)

By comparing Eq. (2.22) and Eq. (2.23), the mathematical expression of the FST is obtained:

\[
\hat{G}(v, \theta) = G(v \cos \theta, v \sin \theta)
\]  

(2.24)

The above equation states that the values of the 2-D FT of \(g(x, y)\) along a line with orientation \(\theta\) are given by the 1-D FT of \(\hat{g}(\rho, \theta)\), the projection of the sinogram acquired at angle \(\theta\) (Fig. 2.6).

Figure 2.6: The Fourier slice theorem: the FTs of projections of the sinogram for various angles \(\theta\) correspond to the FTs of the original function along lines with the same orientations \(\theta\) in the frequency space.

Hence with enough projections (angles), \(\hat{G}(v, \theta)\) can fill the \((k_x, k_y)\) space to generate \(G(k_x, k_y)\). Once the 2-D Fourier domain data are available, the estimated image...
function can be obtained simply by use of the 2-D inverse FT (IFT). The numerical im-
plementation of the FST leads to the Fourier reconstruction methods. The flow chart of
such methods is given in Fig. 2.7.

\[
g(x,y) \xleftarrow{\text{2D IFT}} \quad \overline{G(k_x,k_y)}
\]

\[
\overline{g}(\rho,\theta) \xrightarrow{\text{1D FT}} \quad \overline{G(v,\theta)}
\]

Figure 2.7: Flow chart of the inversion of the Radon transform via the Fourier slice theorem.

In the implementation of the FST, the forward 1-D fast Fourier transform (FFT) is
used to calculate the discrete spectrum of the sinogram for each of the angular samples. Each such spectrum is considered as the sequence of polar samples of the 2-D spectrum of
the image along the same angle, and must be mapped onto a rectangular frequency grid
in order to use the inverse FFT (IFFT) to get the reconstructed image.

This mapping requires 2-D interpolation in the frequency domain. The standard
nearest neighbour interpolation can be used which is very fast, but the cost is that arti-
facts are produced in the recovered image. One common solution is the slower but more
stable bilinear interpolation. Also we can distribute each of the polar samples onto the
rectangular map using proper weights. Note that higher-order interpolation and use of a
non-linear grid in the Radon domain can also provide better numerical results. However,
all these methods increase the computational cost to an unacceptable level.

The 2-D interpolation in the frequency domain, described above, is considered as
the major problem in the implementation of the FST, and the Fourier reconstruction
methods have apparently found limited success because of this.

\[10\]Before using the FFT algorithm, we wrap the signal (sinogram) to get the phase of the spectrum
correctly. Also, we usually zeropad, i.e. fill with zeros, the sinogram so that its length becomes a power of
two and we can use one of the fast radix-2 FFT algorithms. This zero-padding also affects the spectrum,
which is now smoother. This is desired if spectrum interpolation is required.
2.3 Inverting the Radon Transform via Transformation

2.3.2 Filtered Backprojection

An attempt to reconstruct an image from a large number of projections obtained at different angles, by projecting each one backwards, results in an unacceptable level of general blurring of the details of the image. Therefore, simple backprojection cannot provide a generally satisfactory method of reconstruction. We do circumvent this blurring by convolving the projections \( g(\rho, \theta) \) for various angles) with a suitable filter. The individual filtered projections are then combined to produce the filtered sinogram \( \tilde{g}(\rho, \theta) \), which is finally backprojected into \((x, y)\) to create the image.

The process described above is called the filtered backprojection algorithm and is the most important reconstruction algorithm in tomography. It is the most widely used algorithm in clinics and uses the FST. The filtered backprojection algorithm is extremely accurate and amenable to fast implementation. Its derivation is done by introducing polar coordinates to the 2-D IFT of \( g(x, y) \):

\[
g(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_x \ dk_y \ G(k_x, k_y) e^{j2\pi k_x x} e^{j2\pi k_y y}
= \int_{0}^{2\pi} \int_{0}^{+\infty} vG(v \cos \theta, v \sin \theta) e^{j2\pi v(x \cos \theta + y \sin \theta)} \ dv \ d\theta \\
= \int_{0}^{\pi} \int_{0}^{+\infty} vG(v \cos \theta, v \sin \theta) e^{j2\pi v(x \cos \theta + y \sin \theta)} \ dv \ d\theta \\
+ \int_{\pi}^{2\pi} \int_{0}^{+\infty} vG(v \cos \theta, v \sin \theta) e^{j2\pi v(x \cos \theta + y \sin \theta)} \ dv \ d\theta
\]

In the part of the right-hand side of Eq. (2.25), that is underbraced by \( B' \), we set \( \tilde{\theta} = \theta - \pi \Rightarrow d\tilde{\theta} = d\theta, \sin \tilde{\theta} = -\sin \theta, \cos \tilde{\theta} = -\cos \theta \). By substituting in Eq. (2.25), we obtain:

\[
B' = \int_{0}^{\pi} \int_{0}^{+\infty} vG(-v \cos \tilde{\theta}, -v \sin \tilde{\theta}) e^{-j2\pi v(x \cos \tilde{\theta} + y \sin \tilde{\theta})} \ dv \ d\tilde{\theta}
\]
By setting in Eq. (2.26) \( \tilde{v} = -v \Rightarrow d\tilde{v} = -dv \) and dropping the \( \sim \) from \( \theta \) we obtain:

\[
B' = \int_{0}^{\pi} \int_{-\infty}^{\infty} (-\tilde{v}) G(\tilde{v} \cos \theta, \tilde{v} \sin \theta) e^{j2\pi \tilde{v}(x \cos \theta + y \sin \theta)} (-1) d\tilde{v} \ d\theta
\]

\[
= \int_{0}^{\pi} \int_{-\infty}^{0} |\tilde{v}| G(\tilde{v} \cos \theta, \tilde{v} \sin \theta) e^{j2\pi \tilde{v}(x \cos \theta + y \sin \theta)} d\tilde{v} \ d\theta \tag{2.27}
\]

If we drop the \( \sim \) from \( v \) in Eq. (2.27) and also use the FST (Eq. (2.24)), then, Eq. (2.25) is transformed into

\[
g(x, y) = \int_{0}^{\pi} \int_{-\infty}^{+\infty} |v| G(v, \theta) e^{j2\pi v(x \cos \theta + y \sin \theta)} d\theta \ dv \tag{2.28}
\]

where \( G(v, \theta) \) is the 1-D FT of the Radon transform \( \tilde{g}(p, \theta) \) for a specific angle \( \theta \). Eq. (2.28) can be written as

\[
g(x, y) = \int_{0}^{\pi} \tilde{g}(x \cos \theta + y \sin \theta, \theta) \ d\theta \tag{2.29}
\]

where

\[
\tilde{g}(p, \theta) = \int_{-\infty}^{+\infty} |v| G(v, \theta) e^{j2\pi v p} \ dv \tag{2.30}
\]

Eq. (2.30) gives the IFT of the product of two functions \( |v| \) and \( G(v, \theta) \). It also implies that the FT of \( \tilde{g}(p, \theta) \) is the product of these two functions. However, multiplication in the \( v \)-domain is equivalent to convolution in the \( p \)-domain. Hence Eq. (2.30) becomes

\[
\tilde{g}(p, \theta) = \tilde{g}(p, \theta) \star b(p) \tag{2.31}
\]

where the symbol \( \star \) means convolution in 1-D and the FT of \( b(p) \) is \( |v| \).

The product \( |v| G(v, \theta) \) in Eq. (2.30) represents a filtering operation, where the frequency response of the filter is given by \( |v| \). Hence, in order to reconstruct the image we first obtain the filtered sinogram \( \tilde{g}(p, \theta) \) by applying the ramp filter \( b(p) \), defined in the frequency space as \( \tilde{B}(v) = |v| \), to \( \tilde{g}(p, \theta) \) for each angle (high-pass filtering). Then, the obtained profile \( \tilde{g}(p, \theta) \) is backprojected according to Eq. (2.29) to give the image \( g(x, y) \).

The flow chart of this method is given in Fig. 2.8, where BP stands for backprojection. The above described reconstruction process can be considered as an inverse Radon transform algorithm. Thus, this inverse Radon transform method involves both filtering and
backprojection. Next, we discuss some important implementation issues.

\[ g(x,y) \xrightarrow{\text{BP}} g(\rho,\theta) \xrightarrow{\text{FT}} \overline{G}(v,\theta) \xrightarrow{\text{ramp filter + IFT}} \hat{g}(\rho,\theta) \]

Figure 2.8: Flow chart of the filtered backprojection algorithm.

When implementing the filtered backprojection based reconstruction algorithm, we approximate the backprojection operator using a sum. This, however, requires a 1-D interpolation in the \( \rho \)-direction. Also, we can see by examining Eq. (2.30) that the polar spectrum of the sinogram at zero frequency \( (v = 0) \) is multiplied with zero. This implies that a non-zero mean value of the sinogram is set to zero at all times. Hence, the mean value of the reconstructed image, in the same way, is not correct.

Another important implementation issue of this algorithm relates to the 1-D filtering of the sinogram that is needed, where the filter's frequency response is the absolute value of the frequency parameter. This filter is implemented in many ways. The simplest one is the Ram-Lak filter which is a simple windowed ramp filter function

\[ RL(v) = |v|, \quad \text{for } v < v_u \]  

(2.32)

where \( v_u \) is equal to half the sampling frequency. The Ram-Lak filter has sharp boundaries, which create a filter with long fluctuating tails in the real domain. Therefore, it introduces a ringing artifact in the reconstructed image. In order to overcome this problem, we usually multiply the Ram-Lak filter with a weight function, and, as a result, the influence of the long fluctuating tails is suppressed and a better signal-to-noise ratio (SNR) is achieved. Some of these weight functions, also known as apodising windows, are the Shepp-Logan
weight, the generalised Hamming weight and the Hann weight \cite{29} and \cite{70}.

\begin{equation}
\text{The Shepp-Logan weight: } \tilde{W}(v) = \frac{\sin \left( \frac{\pi v}{v_l} \right)}{\frac{\pi v}{v_l}}
\end{equation}

(2.33)

\begin{equation}
\text{The Hann weight: } \tilde{W}(v) = \frac{1}{2} \left( 1 + \cos \left( \frac{\pi v}{v_l} \right) \right) = \cos^2 \left( \frac{\pi v}{2v_l} \right)
\end{equation}

(2.34)

\begin{equation}
\text{The Generalised Hamming weight: } \tilde{W}(v) = \gamma_{GH} + (1 - \gamma_{GH}) \cos \left( \frac{\pi v}{v_l} \right)
\end{equation}

(2.35)

where typical values of $\gamma_{GH}$ are $0.5 - 0.54$ and $v_l$ is the weight’s cut-off frequency. We must note that Eqs. (2.33)-(2.35) give just the weight functions. In order to obtain the desired filters, we multiply these weights with $RL(v) = |v|$.

Fig. 2.9 shows the amplitudes of these three weight functions, while Fig. 2.10 displays the filters that result from multiplying these weights with the Ram-Lak filter, $RL(v) = |v|$. These figures were obtained using Matlab and the following parameter values: $\gamma_{GH} = 0.52$, $v_l = 150$, $v_u = 150$. It is obvious that the multiplication described above results in filters with less sharp boundaries. Also, it should be noted that there is a trade-off between SNR and resolution. Hence, the filter’s cut-off frequency should be chosen with special care.

When implementing the reconstruction techniques, described in this section, we employ the FFT/IFFT algorithm either for filtering (filtered backprojection) or for remapping the spectra (FST). The use of FFT/IFFT algorithm calls for proper discretisation of the continuous formulas and careful selection of the sampling parameters. In order to avoid aliasing problems, sampling must be adequate in all parameters and this implies bounds in the sampling intervals.

Another property that should be fulfilled, so as to ensure good performance for the reconstruction algorithm, is that the fundamental function should have compact support. This means that it should be $g(x, y) = 0$ for $\sqrt{x^2 + y^2} > \rho_{\text{max}}$. The demand described above ensures that $\tilde{g}(\rho, \theta) = 0$ if $\rho > \rho_{\text{max}}$. Otherwise, the numerical implementation of the inverse Radon transform algorithm would not have all the non-zero information required for reconstructing function $g(x, y)$. 

2.3 Inverting the Radon Transform via Transformation

![Plot of the weight functions](image1)

Figure 2.9: The amplitudes of three common weight functions.

![Plot of the resulting filters](image2)

Figure 2.10: The amplitudes of the Ram-Lak filter and three other filters, that are obtained by multiplying the three weights of Fig. 2.9 with the Ram-Lak filter, as a function of frequency.
It is also possible to perform the backprojection before filtering [70]. According to this method the backprojection is succeeded by a 2-D high-pass filtering. The filtering after backprojection algorithm is rarely used because the resultant images are usually poor in comparison with those obtained by filtered backprojection.

Other problems arise due to the cyclical behaviour of the discrete Fourier transform (DFT). In particular, if we assume that we want to reconstruct a function $g(x, y)$ with compact support, i.e. $g(x, y) = 0$ for $\sqrt{x^2 + y^2} > \rho_{\text{max}}$, then, the backprojected sinogram will have large non-zero values due to filters used in this method. Hence, the cyclical behaviour of DFT will create edge problems during filtering. These can be solved by backprojecting onto a larger image than it is necessary. At the final stage, the image must be truncated to match the original image and this might result in loss of information.

2.4 Inverting the Radon Transform via Linear Algebra

In some situations, it is not possible to measure a large number of projections or the projections are not uniformly distributed over 180 (or 360) degrees, i.e. we have problems with missing data. The transform-based techniques, described in Section 2.3, cannot produce results with the accuracy desired in medical imaging because they require a large number and uniform distribution of data. It is more suitable to perform image reconstruction in such situations by using techniques based on linear algebra, as an alternative to frequency domain reconstruction.

2.4.1 Direct Algebraic Algorithms

In direct algebraic algorithms, one considers the measurements as bounded linear functionals. Hence, the reconstruction problem may be written in a matrix vector formulation

$$\mathbf{b} = \mathbf{A} \mathbf{g}$$  \hspace{1cm} (2.36)

where vector $\mathbf{b}$ contains the sinogram values $\mathbf{g}(r, t)$ wrapped into a vector (the vector length is $A_r = RT$), and $\mathbf{g}$ is the unknown set of reconstructed pixels in the image $g(m, n)$ formed
as a vector (the vector length is $A_c = MN$). The transformation matrix $A \in \mathbb{R}^{A_r \times A_c}$ is called the system matrix. It contains the weight factors between each of the image pixels and each of the line orientations from the sinogram, as illustrated in Fig. 2.11.

Matrix $A$ can be estimated in several ways. One commonly used approach is to use the nearest neighbour interpolation. This approach sets the matrix elements to 1 if the line with parameters $(r, t)$ crosses the pixel-square and to 0 if it does not. Another approach for the estimation of $A$ uses the sinc function.

In medicine, the system of equations (2.36) is usually an underdetermined system. Also, system matrix $A$ is near singular, i.e. it has very small singular values. This means that reconstruction by using a direct algebraic algorithm is an ill-conditioned problem. Then, in order to determine a solution to this ill-conditioned problem, one may use the Moore-Penrose generalised method [46] or the singular value decomposition (SVD). Other possibilities to solve the reconstruction problem, include the Bayesian estimation [46] and the Tikhonov regularisation method [46]. Some examples of constraints, that could be imposed in order to improve the stability of the algorithm, are the non-negativity and the upper-limit constraints. Furthermore, a simple way to include a regularisation term in the algorithm is by expanding the set of equations in system (2.36) with a set of regularisation...
In all the cases, mentioned above, one has to invert matrix $A$, which is huge and does not have a simple structure. On the other hand, the system matrix is sparse due to the fact that only approximately $\sqrt{A_c}$ out of the $\sqrt{A_c} \times \sqrt{A_c} = A_c$ (for square images) image pixels add weight to a certain bin in the sinogram. This property can be exploited to produce a much faster algorithm using hybrid solutions.

### 2.4.2 The Algebraic Reconstruction Technique

A well-known way to solve Eq. (2.36) is the algebraic reconstruction technique (ART). ART was published in the biomedical literature in 1970 [19], and Cormack and Hounsfield used ART for reconstructing the very first tomographic images.

The main idea of ART is to fulfil the following condition: The scalar product between a certain row $\alpha_i$ of the system matrix and a solution vector $\mathbf{g}$ has to be equal to the $i^{th}$ element of the known vector $b$. ART is formulated as an iterative reconstruction algorithm, where the solution vector in iteration $k$ is updated by adding a scaled version of the row $i$ of the system matrix and also, a relaxation parameter $\lambda_k$ is introduced in the form of a weight factor

$$
\mathbf{g}^{(k)} = \mathbf{g}^{(k-1)} + \lambda_k \frac{b_i - \alpha_i^T \cdot \mathbf{g}^{(k-1)}}{\alpha_i^T \cdot \alpha_i} \alpha_i^T
$$

where $\cdot$ is the symbol for the dot product. The choice of row $i$ at each iteration step can be made at random. Initial solution values $\mathbf{g}^{(0)}$ may be chosen to be equal to zero or to a constant. A solution from a fast algorithm based on the FST can also be used for initialisation. The selected value of $\lambda_k$ is a function of $k$, the sinogram values, and the sampling parameters of the reconstructed image.

ART presents better convergence properties than the Landweber iteration method [46]. However, ART is computationally inefficient. It is only used in some specific applications such as the case of limited view reconstruction and, therefore, has lost popularity. On the other hand, iteration techniques of various types based on statistical properties,
like the expectation maximisation algorithm, are more useful and are commonly used.

2.5 Statistical Reconstruction Methods

In emission tomography, with limited counts in each sinogram bin, the statistical noise can dominate the reconstructed images when using transform-based reconstruction methods. Therefore, many statistical approaches have been considered to derive reconstruction algorithms from incomplete data.

One prominent iterative reconstruction method is the maximum likelihood reconstruction (MLR) using the expectation maximisation (EM) algorithm [39], [68] and [71]. EM algorithm assumes that the measurements originate from uncorrelated Poisson generators. This is an ideal model in PET and requires no justification, since radioactive emissions occur according to a spatial Poisson point process.\(^{11}\) Then, the measured data elements \(b_i, i = 1, \ldots, A_r\) will also constitute \(A_r\) independent Poisson random variables, since classifying emissions according to the detector pairs that detect them is a stochastic thinning of the Poisson point process [6]. The key idea of the algorithm is to maximise the likelihood function (or the probability) of the observed data

\[
\tilde{L}(\mathbf{g}) = \tilde{P}(\mathbf{b}|\mathbf{g}) = \prod_{i=1}^{A_r} \frac{(b_i^*)^{b_i}}{b_i!} e^{-b_i^*} \quad (2.38)
\]

where \(b_i^*\) is the \(i^{th}\) element of \(\mathbf{b}^*\), \(\mathbf{b}^*\) contains the unknown mean value of \(\hat{b}\) (i.e. the true values of the sinogram without noise) and it is assumed that \(\mathbf{b}^*\) is the perfect solution:

\[
\mathbf{b}^* = \mathbf{A}_r \mathbf{g} \quad (2.39)
\]

Also, the elements of the system matrix have been normalised

\[
1 = \sum_{i=1}^{A_r} \alpha_{i,j} \quad (2.40)
\]

so that the weights with which lines \(i = 1, 2, \ldots, A_r\) contribute to the value of pixel \(j\) sum

\(^{11}\)However, problems like attenuation correction are not modelled in this framework.
The log\(^{12}\) likelihood is defined as:

\[
\ln(l) = \ln(\tilde{L}) = \ln\left(\prod_{i=1}^{A_c} \frac{(b_i^*)^{b_i}}{b_i!} e^{-b_i^*}\right)
\]  

(2.41)

The likelihood function \(\tilde{L}(g)\) is maximised for the same \(g\) that maximises the log likelihood \(l(g)\). Eq. (2.39) implies that \(b_i^* = \sum_{j=1}^{A_c} a_{i,j} g_j\). Hence, by substituting in Eq. (2.41) we obtain:

\[
l(g) = -\sum_{i=1}^{A_r} \log b_i! - \sum_{i=1}^{A_r} \sum_{j=1}^{A_c} a_{i,j} g_j + \sum_{i=1}^{A_r} b_i \log \left(\sum_{j=1}^{A_c} a_{i,j} g_j\right)
\]

(2.42)

The derivatives of \(l(g)\) are:

\[
\frac{\partial l(g)}{\partial g_j} = -\sum_{i=1}^{A_r} \alpha_{i,j} + \sum_{i=1}^{A_r} \frac{\alpha_{i,j} b_i}{\sum_{j'=1}^{A_c} \alpha_{i,j'} g_j'}
\]

\[
= -1 + \sum_{i=1}^{A_r} \frac{\alpha_{i,j} b_i}{\sum_{j'=1}^{A_c} \alpha_{i,j'} g_j'} \quad \text{for} \quad j = 1, 2, \ldots, A_c
\]

(2.43)

It is shown in the literature [71] that the matrix of second derivatives of \(l(g)\) is negative semidefinite, hence \(l(g)\) is concave and all its maxima are global maxima. Therefore, it follows from [75] that sufficient conditions for \(g\) to be a maximiser of \(l(g)\) (or, equivalently, \(\tilde{L}(g)\)) are the following Kuhn-Tucker conditions for each \(j = 1, \ldots, A_c\):

\[
g_j \frac{\partial l(g)}{\partial g_j} = 0 \quad \forall j \quad \text{where} \quad g_j > 0
\]

(2.44)

\[
\frac{\partial l(g)}{\partial g_j} \leq 0 \quad \forall j \quad \text{where} \quad g_j = 0
\]

(2.45)

The condition in Eq. (2.44) when combined with Eq. (2.43) results in:

\[
-g_j + g_j \sum_{i=1}^{A_r} \frac{\alpha_{i,j} b_i}{\sum_{j'=1}^{A_c} \alpha_{i,j'} g_j'} = 0
\]

(2.46)

With such a relatively simple expression for the right-hand side of Eq. (2.46), one can think of many iterative schemes that would converge to a maximum of \(l(g)\). Of particular

\(^{12}\)Symbol \(\log\) denotes the natural logarithm \(\ln\).
appeal is the following scheme that is obtained using the EM algorithm [71]:

\[
\mathbf{g}^{(k)} = \mathbf{g}^{(k-1)} \sum_{i=1}^{A_c} \frac{\alpha_{i,j} b_i}{\sum_{j'=1}^{A_c} \alpha_{i,j'} g_{j'}^{(k-1)}}
\]  

(2.47)

The above iterative scheme is an instance of the EM algorithm and, therefore, converges monotonically to a global maximum of \( l(\mathbf{g}) \). Other versions of the EM algorithm that do not require normalisation also exist [7].

The EM algorithm is computationally demanding. For the initialisation of the solution vector \( \mathbf{g} \), a fast direct algorithm, such as an FST based method, can be employed and the initial values have to be positive.

The EM algorithm also suffers from the notorious problems of slow convergence and lack of smoothness [46]. To correct these problems, the ordered subset EM (OSEM) algorithm has been proposed [28]. According to OSEM, system matrix \( \mathbf{A} \) and measurement vector \( \mathbf{b} \) are split into submatrices and subvectors, respectively, and Eq. (2.47) is applied to each submatrix individually.
Chapter 3

Complete Tomographic Reconstruction of 2-D Vector Fields using Discrete Integral Data

3.1 Introduction

The Radon transform and its application in conventional tomographic reconstruction were discussed in the previous chapter. Functions that are reconstructed by using traditional tomography are scalar functions, e.g. absorption or scattering coefficients. However, over the last few decades there has been a growing demand for similar techniques that would perform tomographic reconstruction of a vector field, rather than a scalar one, when having integral information. Primary driving force for this has been the awareness that there are certain applications which have measurements that are inherently line integrals of the inner product of the examined vector field with a fixed vector.

In this chapter, the application of tomography to vector field reconstruction is discussed. The problem of recovering a vector field from its projections has received far less attention than the scalar one, not for lack of potential applications, but because it has generally been regarded as an underdetermined problem. This seems to be clear from the fact that a scalar function is determined uniquely from its Radon transform (which is a
scalar function), whereas a vector field requires two (in 2-D) or three (in 3-D) component functions to be determined.\(^1\) Also, it is important to note that when trying to determine a scalar function, the state at a particular point is considered to contribute equally to all lines passing through it. However, the situation for vector fields is far more complicated in that the contribution also depends on the direction of the line.

This chapter is organised as follows. In Section 3.2, we present the applications of vector field tomography. In Section 3.3, we provide the framework for 2-D vector field tomography, focusing on the two types of vectorial Radon transform and the limitations of the approaches that have been employed so far to solve the 2-D vector field reconstruction problem. In Section 3.4, we introduce our direct algebraic reconstruction methodology. In Section 3.5, we present an example application, where a static electric field is reconstructed by relying only on projection measurements, obtained at the boundary of the reconstruction region. This example was chosen because, from Coulomb's law, we can compute exactly the ground truth and, thus, evaluate the proposed methodology. Stability issues and the effect of noise on the reconstruction of the electric field are also examined in Section 3.5. We conclude in Section 3.6.

### 3.2 Applications of Vector Field Tomography

Several applications of vector field tomography have been considered in the literature that are able to acquire data in the form of a line integral of the inner product of the investigated vector field and a fixed vector. These include:

- blood flow imaging in vessels [33] and [69];
- fluid mesoscale velocity imaging in ocean acoustic tomography [26], [44] and [59];
- fluid-flow imaging by using: \(i\) acoustic time-of-flight measurements [4], [30], [47], [48], [49], [72] and [73] and \(ii\) nuclear magnetic resonance (NMR) [37];
- electric field imaging in Kerr materials by measuring the polarisation of light passing

\(^1\)This chapter deals only with vector fields that have two components.
through the sample from many directions [1], [23] and [74];

- imaging of the component of the gradient of the refractive index field, which is transversal to the beam, in Schlieren tomography when temperature measurements in gases are used [4];

- velocity field imaging of heavy particles in plasma physics by using the first moment of the velocity distribution measured by Doppler shifts of the spectral lines [15];

- density imaging in supersonic expansions and flames in beam deflection optical tomography [16];

- non-destructive stress distribution imaging of transparent specimens in photoelasticity by using measurements of the change in polarisation of light passing through a birefringent medium [2] and [67];

- determination of temperature distributions and velocity vector fields in furnaces [62];

- magnetic field imaging in tokomak [65] and hot plasmas [13] in polarimetric tomography and

- wind velocity imaging in meteorology [31].

3.3 Vector Field Tomography Framework

3.3.1 Vectorial Radon Transform

The measurements that we obtain in the applications discussed in Section 3.2 have been the main motivation for introducing the vectorial Radon transform. Using the physics of these applications, it can be shown that the acquired data, in each case, reduce to an integral transform of the examined vector field along integration lines, the vectorial Radon transform. When we try to investigate planar vector fields in bounded domains, two classes of the vectorial Radon transform, that model the tomographic measurements, arise, depending on the interaction between the obtained measurements and the vector
field that we want to image. The two types of vectorial Radon transform for planar vector fields are a natural generalisation of the classical (scalar) Radon transform to vector fields.

In order to help the definition of the two types of vectorial Radon transform, let us assume that the domain of the vector field, that we want to image, is $D$ and its boundary, where we obtain the tomographic measurements, is $\partial D$ (Fig. 3.1)$^2$. Then, the first type of the line integral transform, $J_1$, is

$$J_1 = \int_A^B \mathbf{f}(x, y) \cdot \hat{s} \, ds$$

(3.1)

where $\mathbf{f}(x, y)$ is the planar vector field under investigation, $A$ and $B$ are points that range over the boundary $\partial D$ of $D$ and define the integration line section (see Fig. 3.1), $\hat{s}$ is the unit vector along the integration line section (see Fig. 3.1), $ds$ is an element of path length along this line section and $\cdot$ is the symbol for the dot product. By setting $f(x, y) = 0$

---

$^2\partial D$ may either be a physical boundary or, simply, the locus of points between which the integration paths are defined.

---

Figure 3.1: A line $L$ in the $x - y$ plane that is defined by the two points $A$ and $B$ that lie on the boundary $\partial D$ of the domain $D$. Also, the unit vectors $\hat{s}$ and $\hat{\rho}$ which are parallel and perpendicular, respectively, to this line. $L$ goes through $D$. 
outside the domain $D^3$, it is mathematically permissible to extend the integration path in Eq. (3.1) along the whole line $L$ from $-\infty$ to $\infty$, where $L$ is the line defined by the two points $A$ and $B$ (see Fig. 3.1)

$$J_1 = \int_L \tilde{f}(x,y) \cdot \hat{s} \, ds$$

$$= \int_L f_\parallel \, ds$$

with $f_\parallel$ being the component of $\tilde{f}(x,y)$ along $L$. Eq. (3.3) states that $J_1$ is the line integral of $f_\parallel$ along line $L$, so only this component of the vector field is observed by the measurement. That is the reason why Braun and Hauck in [4] called the tomographic measurements, modelled by this type of integral transform, as "longitudinal" measurements. Next, we show why this type of vectorial Radon transform may be employed to model ultrasound time-of-flight measurements, when we investigate velocity fields in fluids.

Consider a moving fluid within a finite region, and let $\mathbf{v}(x,y)$ denote the velocity of the fluid (which is the investigated vector field) and $c(x,y)$ denote the spatially varying sound speed, i.e. the speed of the sound in the medium if the fluid was not flowing. Suppose that an ultrasound signal propagates along the line$^4$ between a source at point $(x_s, y_s)$ and a receiver at point $(x_r, y_r)$. We also assume that $|\mathbf{v}(x,y)| \ll c(x,y)$ everywhere in the domain. Hence, it is reasonable to approximate the total speed of the sound (also called effective speed), $c_{eff}(x,y)$, by the following linear formula

$$c_{eff}(x,y) = c(x,y) + \mathbf{v}(x,y) \cdot \hat{s}$$

where $\hat{s}$ is the unit vector along the propagation line section. Hence, the travel time, $T_{sr}$, of the ultrasound pulse from the source to the receiver can be expressed as:

$$T_{sr} = \int_{(x_s,y_s)}^{(x_r,y_r)} \frac{ds}{c(x,y) + \mathbf{v}(x,y) \cdot \hat{s}}$$

$^3$This is permitted since the line integrals are only obtained through the interior of $D$.

$^4$We assume that the variation in $c(x,y)$ is sufficiently small and/or that the path lengths are sufficiently short. Hence, ray refraction over all paths may be neglected and the ultrasound signal travels along straight lines.
If we interchange source and receiver, the travel time is

$$T_{rs} = \int_{(x_s,y_s)}^{(x_r,y_r)} \frac{ds}{c(x,y) - \mathbf{v}(x,y) \cdot \hat{s}}$$  \tag{3.6}$$

We assumed that $|\mathbf{v}(x,y)| \ll c(x,y)$. Hence, by neglecting terms of second order, we obtain

$$T_{sr} + T_{rs} = 2 \int_{(x_s,y_s)}^{(x_r,y_r)} \frac{ds}{c(x,y)}$$  \tag{3.7}$$

$$T_{sr} - T_{rs} = 2 \int_{(x_s,y_s)}^{(x_r,y_r)} \frac{\mathbf{v}(x,y) \cdot \hat{s}}{c^2(x,y)} \, ds$$  \tag{3.8}$$

Eq. (3.7) gives the scalar Radon transform of $\frac{1}{c(x,y)}$. Hence, sound speed $c(x,y)$ can be recovered by means of conventional (scalar) tomography, discussed in previous chapter. However, the differential time (Eq. (3.8)) is of the form (3.1) when we identify $f(x,y)$ with $\frac{2\mathbf{v}(x,y)}{c^2(x,y)}$. Therefore, the differential time-of-flight measurements in acoustics are modelled by the (first type of) vectorial Radon transform of the investigated fluid velocity field.

The second class of vectorial Radon transform, $J_2$, is used to model tomographic measurements that collect information from the component of the investigated vector field perpendicular to the measurement line:

$$J_2 = \int_A^B \mathbf{f}(x,y) \cdot \hat{\rho} \, ds$$  \tag{3.9}$$

$$= \int_L \tilde{\mathbf{f}}(x,y) \cdot \hat{\rho} \, ds$$  \tag{3.10}$$

$$= \int_L f_\perp \, ds$$  \tag{3.11}$$

Here $\hat{\rho}$ is the unit vector perpendicular to the line of integration $L$ (see Fig. 3.1), $f_\perp$ is the component of $\mathbf{f}(x,y)$ transverse to $L$ and the rest of the notation is as in Eqs. (3.1)-(3.3). Moreover, it was assumed, again, that $\mathbf{f}(x,y) = 0$ outside the domain $D$. Eq. (3.11) states that $J_2$ is the line integral of $f_\perp$ along line $L$, and therefore measurements that are modelled by this type of integrals are called in [4] “transversal” measurements. Next, we show why such measurements arise in Schlieren tomography.

Consider a non-homogeneous medium with refractive index $n(x,y)$, within which
propagation of light rays takes place. By employing an optical Schlieren arrangement [4], the differences in the propagation direction are converted into intensity variations. One, then, can make the following gas temperature measurements [46], $\tilde{I}$, between a source at point $(x_s, y_s)$ and a receiver at point $(x_r, y_r)$

$$\tilde{I} = \int_{(x_s,y_s)}^{(x_r,y_r)} (\hat{b} \times \hat{s}) \cdot \nabla n(x,y) \, ds$$

(3.12)

where unit vector $\hat{b}$ describes the directional sensitivity of the arrangement, unit vector $\hat{s}$ is the tangent vector to a light ray and $\nabla$ denotes the nabla operator. If $\hat{b}$ is chosen perpendicular to the measurement plane, we obtain from Eq. (3.12)

$$\tilde{I} = \int_{(x_s,y_s)}^{(x_r,y_r)} \hat{\rho} \cdot \nabla n(x,y) \, ds$$

(3.13)

where $\hat{\rho}$ denotes the unit vector normal to the ray in the investigated plane. Eq. (3.13) is of the form (3.9) when we identify $\tilde{f}(x,y)$ with $\nabla n(x,y)$. Therefore, the temperature measurements in gases by Schlieren tomography are modelled by the (second type of) vectorial Radon transform of the investigated gradient of the refractive index.

The two types of projection transform (Eqs. (3.2) and (3.10)), discussed in this section, form the mathematical basis to deal with the problem of vector field tomographic mapping from line-integral data. This problem is discussed in the next section, where it is shown that the classification of the vectorial integral transforms (or, equivalently, of the interactions between the acquired measurements and the investigated vector fields) in two types turns out to be essential for the reconstruction of the examined vector field.

3.3.2 The Reconstruction Problem

The introduction of the vectorial Radon transform in the previous section gives rise to some natural questions: What information about a vector field $\tilde{f}(x,y)$ within a bounded domain can be extracted from the vectorial Radon transform ($J_1$ or $J_2$) when its value is known for all lines that go through this domain? Is it possible to have a complete and unique reconstruction of $\tilde{f}(x,y)$ within a bounded domain based only on projection
3.3 Vector Field Tomography Framework

(line-integral) data obtained on the boundary of this domain?

During the short history of 2-D vector field tomography, many investigators attempted to give an answer to these questions and solve the reconstruction problem [4], [32], [48] and [67]. All of them discussed the inverse problem of 2-D vector field tomography in continuum terms. Motivated by the FST theorem, the researchers invariably adopted a conventional (scalar) tomography theory-based approach to the problem. Next, we shortly present the treatment they followed and the conclusions they drew.

To set up our notation and help the problem formulation, we consider a quasi-stationary planar vector field \( \vec{f}(x, y) \) that lies on the \( x - y \) plane and belongs to the Schwartz class \( S \) consisting of rapidly decreasing functions [45]. In the analysis that follows we consider the case where the interaction between the measurement and \( \vec{f}(x, y) \) is longitudinal. Hence, the reconstruction problem is mathematically described as the task of solving Eq. (3.2) for the 2-D vector field \( \vec{f}(x, y) \) in \( D \), given a complete set of integrals \( J_I \) through \( D \).

It was shown in Fig. 2.1 how to parameterise every line by its distance \( \rho \) from the origin and angle \( \theta \) that determines the angular orientation of the line. To simplify matters, we will swap parameter \( \theta \) with the unit vector \( \hat{p} \), normal to the line (see Fig. 3.1), which also determines uniquely the angular orientation of the line.

Then, every line is denoted by \( L(\rho, \hat{p}) \) and Eq. (3.2) may be rewritten as

\[
\hat{T}(\rho, \hat{p}) = \int_{L(\rho, \hat{p})} \vec{f}(x, y) \cdot \hat{s} \, ds
\]

\[
\hat{T}(\rho, \hat{p}) = \hat{s} \cdot \int_{L(\rho, \hat{p})} \vec{f}(x, y) \, ds
\]

where \( \hat{T}(\rho, \hat{p}) \) is the vectorial Radon transform and the unit vector \( \hat{s} \), parallel to the line, is fixed along the whole line, for every line. Using parameters \( \rho \) and \( \theta \) of Fig. 2.1, the equation of a line \( L \) may be put in the form

\[
\rho = x \cos \theta + y \sin \theta
\]

\[
\rho = r \cdot \hat{p}
\]
where \( \mathbf{r} = (x, y) \) and the unit vector \( \mathbf{\hat{r}} \), perpendicular to \( L \), may also be written as \( \mathbf{\hat{r}} = (\cos \theta, \sin \theta) \). This is obtained by comparing Fig. 2.1 and Fig. 3.1. By using Eq. (3.17), Eq. (3.15) becomes

\[
\tilde{T}(\rho, \mathbf{\hat{r}}) = \hat{s} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x, y) \delta(\rho - \mathbf{r} \cdot \mathbf{\hat{r}}) \, dx \, dy
\]

where \( \delta \) is the Dirac delta function and all the remaining quantities have been defined earlier.

Taking the 1-D FT of Eq. (3.18) with respect to \( \rho \) for a specific unit vector \( \mathbf{\hat{r}} \), gives

\[
\tilde{T}(k, \mathbf{\hat{r}}) = \hat{s} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x, y) \delta(\rho - \mathbf{r} \cdot \mathbf{\hat{r}}) \, dx \, dy \, e^{-j\rho \mathbf{\hat{r}} \cdot \mathbf{k}} \, d\rho
\]

where variable \( k \) is the Fourier-domain variable when we transform \( \tilde{T}(\rho, \mathbf{\hat{r}}) \) with respect to \( \rho \) while \( \mathbf{\hat{r}} \) takes a specific value, and \( \tilde{T}(k, \mathbf{\hat{r}}) \) is the corresponding 1-D FT. One important property of the delta function is:

\[
\int_{-\infty}^{\infty} g(x) \delta(ax + b) \, dx = \frac{1}{|a|} g \left( \frac{-b}{a} \right)
\]

By applying Eq. (3.20) to Eq. (3.19) we obtain:

\[
\tilde{T}(k, \mathbf{\hat{r}}) = \hat{s} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x, y) e^{-j(k \mathbf{\hat{r}} \cdot \mathbf{x})} \, dx \, dy
\]

The 2-D FT of \( \tilde{f}(x, y) \), \( \tilde{f}(k_1, k_2) \), is given by

\[
\tilde{f}(k_1, k_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x, y) e^{-j(k_1 x + k_2 y)} \, dx \, dy
\]

where \( k_1 \) and \( k_2 \) are the Fourier-domain variables of \( x \) and \( y \), respectively. By introducing vector \( \mathbf{k} \equiv (k_1, k_2) \), Eq. (3.22) is simplified to:

\[
\tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x, y) e^{-j\mathbf{k} \cdot \mathbf{r}} \, dx \, dy
\]
3.3 Vector Field Tomography Framework

By comparing Eq. (3.21) and Eq. (3.23), the mathematical expression of the vectorial version of the Fourier slice theorem is obtained:

$$\tilde{T}(k, \hat{p}) = \hat{s} \cdot \hat{f}(k\hat{p}) \tag{3.24}$$

To proceed with the reconstruction task, we apply the classical Helmholtz decomposition theorem [42] to vector field $\tilde{f}(x, y)$ that we want to reconstruct. Its importance in determining what information can be extracted by the vectorial Radon transform (measurements) becomes clear next. This theorem allows us to uniquely⁵ decompose $\tilde{f}(x, y)$ into an irrotational (or equivalently curl-free) vector field component, $\tilde{f}_I(x, y)$, and a solenoidal (or equivalently source-free) vector field component, $\tilde{f}_S(x, y)$:

$$\tilde{f}(x, y) = \tilde{f}_I(x, y) + \tilde{f}_S(x, y) \tag{3.25}$$

The two components of $\tilde{f}(x, y)$ may be written as

$$\tilde{f}_I(x, y) = \nabla \Phi(x, y) \tag{3.26}$$

$$\tilde{f}_S(x, y) = \nabla \times \tilde{\Psi}(x, y) \tag{3.27}$$

where $\Phi(x, y)$ and $\tilde{\Psi}(x, y)$ are some functions. These expressions come about because it is known that $\nabla \times (\nabla \Phi(x, y)) = 0$ and $\nabla \cdot (\nabla \times \tilde{\Psi}(x, y)) = 0$. In this chapter we only deal with 2-D planar vectors $\tilde{f}(x, y)$ that lie on the $x - y$ plane, so from Eq. (3.27) we may deduce that function $\tilde{\Psi}(x, y)$ is of the form

$$\tilde{\Psi}(x, y) = \Psi_0(x, y) \hat{z} \tag{3.28}$$

where $\hat{z}$ is the unit vector normal to the investigated $x - y$ plane.

By examining Eqs. (3.25)-(3.28), it is easy to see that the objective now becomes to recover both functions $\Phi(x, y)$ and $\Psi_0(x, y)$ which together determine uniquely $\tilde{f}(x, y)$.

---

⁵The components of the Helmholtz decomposition are unique as long as vector field $\tilde{f}(x, y)$ rapidly decreases in $\mathbb{R}^2$ and vanishes at infinity [42]. This condition is satisfied here, since this analysis treats only vector fields that belong to the Schwartz class $S$ [45].
The combination of Eqs. (3.25)-(3.28) yields:

\[
\tilde{f}(x, y) = \nabla \Phi(x, y) + \nabla \times \tilde{\Phi}(x, y) = \left( \frac{\partial \Phi(x, y)}{\partial x} \hat{x} + \frac{\partial \Phi(x, y)}{\partial y} \hat{y} \right) + \left( \frac{\partial \Psi_0(x, y)}{\partial y} \hat{x} - \frac{\partial \Psi_0(x, y)}{\partial x} \hat{y} \right)
\]  

(3.29)

A fundamental property of Fourier transform states that if a 2-D function \( \hat{\alpha}(x, y) \) has FT \( \hat{A}(k_1, k_2) \), then, the following equations are valid

\[
\text{FT} \left\{ \frac{\partial \hat{\alpha}(x, y)}{\partial x} \right\} = jk_1 \hat{A}(k_1, k_2) \tag{3.30}
\]

\[
\text{FT} \left\{ \frac{\partial \hat{\alpha}(x, y)}{\partial y} \right\} = jk_2 \hat{A}(k_1, k_2) \tag{3.31}
\]

where \( j \) is the imaginary unit. By taking the 2-D FT of Eq. (3.29) and also using the property mentioned above, we obtain

\[
\tilde{\Phi}(k_1, k_2) = jk_1 \hat{\Phi}(k_1, k_2) \hat{x} + jk_2 \hat{\Phi}(k_1, k_2) \hat{y} + jk_2 \hat{\Psi}_0(k_1, k_2) \hat{x} - jk_1 \hat{\Psi}_0(k_1, k_2) \hat{y}
\]

\[
= j\hat{\Phi}(k_1, k_2)(k_1 \hat{x} + k_2 \hat{y}) + j\hat{\Psi}_0(k_1, k_2)(k_2 \hat{x} - k_1 \hat{y})
\]  

(3.32)

where \( \hat{\Phi}(k_1, k_2) \) and \( \hat{\Psi}_0(k_1, k_2) \) are the 2-D FTs of the scalar functions \( \Phi(x, y) \) and \( \Psi_0(x, y) \), respectively. Since \( \hat{k} = (k_1, k_2) \) and also,

\[
\hat{k} \times \hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_1 & k_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = k_2 \hat{x} - k_1 \hat{y}
\]  

(3.33)

Eq. (3.32) becomes:

\[
\tilde{f}(\hat{k}) = j\hat{\Phi}(\hat{k}) \hat{k} + j(\hat{k} \times \hat{z}) \hat{\Psi}_0(\hat{k})
\]  

(3.34)

By substituting Eq. (3.34) into Eq. (3.24) and also noting (see Fig. 3.1) that \( \hat{s} \cdot \hat{p} = 0 \) and \( \hat{p} \times \hat{z} = -\hat{s} \), Eq. (3.24) reduces to:

\[
\tilde{T}(k, \hat{p}) = -jk \hat{\Psi}_0(k \hat{p})
\]  

(3.35)
We note that $\tilde{\Psi}(k)$ drops out of Eq. (3.35). Hence, the Fourier transform of $\tilde{\Psi}_0(x, y)$ (or, equivalently, the solenoidal component $\nabla \times \tilde{\Psi}(x, y)$ of $\tilde{f}(x, y)$) may only be recovered from the 1-D Fourier transform of the vectorial Radon transform $\tilde{T}(k, \rho)$, independently of the irrotational part $\nabla \Phi(x, y)$. Consequently, in applications where only longitudinal measurements are available, the irrotational part cannot be imaged and information only about the curl of the vector field (or, equivalently, the solenoidal part) may be recovered. Likewise, it can be shown that if we had considered an application where the measurements were the path integrals of the component of the vector field normal to the line (transversal measurements), then, only information about the divergence of the vector field (or, equivalently, the irrotational component) would be recovered and the solenoidal component of the vector field would not be reconstructed. In both cases the kernel of the vectorial Radon transform is non-empty making it impossible to achieve a complete reconstruction.

Hence, the above treatment proves that by following the FST-based approach and by relying only on projection data from one type of measurement, only one component of the vector field can be recovered. The recovered component will be either the curl-free (irrotational) part or the divergence-free (solenoidal) part, depending on the physical principle of the measurements, namely the relation between the obtained set of measurements and the investigated vector field. An algebraic reconstruction method of this type, where the authors considered the problem of only reconstructing the solenoidal component from the tomographic data, was developed in [12]. Possible solutions to this problem were discussed in Section 1.1. However, these solutions involved either different type of modelling for the available measurements or the incorporation of supplementary information, apart from the projection measurements.

All the conclusions about 2-D vector field tomography described above were drawn from work that was based on a scalar tomography theory approach (FST). The work was carried out in the continuous domain and the solution to the reconstruction problem was helped by the classical Helmholtz decomposition theorem, that decomposes the examined vector field into its irrotational and solenoidal components. In this chapter, we employ a
3.4 The Proposed Reconstruction Methodology

different approach to achieve complete recovery of a vector field based only on a limited number of projection measurements. The whole treatment is performed in the discrete domain. The proposed reconstruction methodology is a direct algebraic reconstruction technique. We consider the acquired projection measurements as linear functionals on the space of 2-integrable functions in the reconstruction region $D$. Hence, we cast the tomographic reconstruction problem as the solution of a system of linear equations. The unknowns of the system are the Cartesian components of the examined vector field in specific sampling points, finite in number and arranged in a grid, of the 2-D reconstruction region.

In order to solve this system of linear equations, we take advantage of the redundancy in the projection data, as a form of employing regularisation to deal with the ill-posed nature of the vector field reconstruction problem. The regularisation lies in the fact that by using many line orientations passing through every sampling point, and, then, viewing the related recordings as weighted sums of the local vector field's Cartesian components, we achieve, in this way, to include additional information about the investigated vector field itself in the system of equations. Hence, the regularisation term consists of the extra set of regularisation rows, added to the system matrix. Next, we present the proposed reconstruction methodology.

3.4 The Proposed Reconstruction Methodology

The whole treatment is performed in the digital domain. Let us assume that we have the digitised square 2-D domain that is shown in Fig. 3.2, within which we want to recover the vector field $\mathbf{f}(x, y) = f_x(x, y)\hat{x} + f_y(x, y)\hat{y}$. The length of each side of the square reconstruction domain is taken to be equal to $2U$ and the origin of the axes of the coordinate system is chosen to be at the centre of the domain. The digitised square domain consists of tiles of finite size, $P \times P$, so that $\frac{2U}{P}$ is an integer. The goal is to recover vector field $\mathbf{f}(x, y)$ at the centre of every tile of this space, namely the sampling points of the domain.
3.4 The Proposed Reconstruction Methodology

Figure 3.2: A line segment between two boundary sensors, that reside at points $A$ and $B$. The line segment goes through the digital square reconstruction domain of size $2U \times 2U$. $AB$ is inclined at an angle $w$ to the positive direction of the $x$-axis. The size of the tiles, with which we sample the 2-D space, is $P \times P$. Point $Q$ is the foot of the normal from the origin of the axes to the line segment.

Regarding the data acquisition, we assume that ideal point sensors, that integrate only the component of the field projected on the line, reside on predetermined and regularly placed positions of the whole border of the 2-D square domain. These positions are the middle points of the boundary edges of all boundary tiles. Hence, there are $2U/P$ ideal point sensors on each side of the boundary of the domain of Fig. 3.2. The solution to the reconstruction problem is based only on projection data along lines defined by the finite number of measurement points.

Let us consider a scanning line segment $AB$ between two such sensors, chosen arbitrarily, crossing this domain as shown in Fig. 3.2. The scanning line segment $AB$ yields a line-integral measurement (collected by sensors at points $A$ and $B$) of the projection of the vector field along the line's direction. Since we assumed that each pair of sensors measures only the integral of the component of the vector field along the scanning segment, the integral transform that models the process of data acquisition is given by:

$$J_1 = \int_{AB} \tilde{f}(x,y) \cdot \hat{s} \, ds = \int_{AB} f_1 \, ds \quad (3.36)$$

Here $\hat{s} = \cos w \hat{x} + \sin w \hat{y}$ is the unit vector along the integration (measurement) segment $AB$, where $w$ is the angle at which the scanning line is inclined to the positive direction of
3.4 The Proposed Reconstruction Methodology

the $x$-axis (see Fig. 3.2). In addition, $ds$ is an element of path length along this segment and $f_{||}$ is the component of $F(x, y)$ along $AB$. In order to translate into the digital domain, the integration expressed by Eq. (3.36) along a continuous line, the integral of the vector field along the scanning line has to be expressed in terms of the components of the field at the sampling points of the 2-D grid. To do that we follow the methodology used in [53] for the implementation of the trace transform. Next, we show how the available line-integral measurement $J_1$ between $A$ and $B$, that is described by Eq. (3.36), may be approximated by a linear equation.

The known coordinates of points $A$ and $B$ are $(x_A, y_A)$ and $(x_B, y_B)$, respectively. Therefore, the equation of line $AB$ is

$$\frac{y - y_A}{y_B - y_A} = \frac{x - x_A}{x_B - x_A}$$  \hspace{1cm} (3.37)

or

$$y = \lambda x + \beta$$  \hspace{1cm} (3.38)

where

$$\lambda \equiv \frac{y_B - y_A}{x_B - x_A}$$  \hspace{1cm} (3.39)

and

$$\beta \equiv y_A - \frac{y_B - y_A}{x_B - x_A} x_A$$  \hspace{1cm} (3.40)

Parameter $\lambda$ is called the angular coefficient or slope of the line; it is equal to the tangent of the angle $w$:

$$\lambda = \tan w \Rightarrow w = \arctan \lambda$$  \hspace{1cm} (3.41)

The next step is to perform a sampling of the line segment. The starting point of this sampling will be the foot of the normal of this line from the origin of the axes (point $Q$ in Fig. 3.2). For two orthogonal lines with slopes $\lambda_1$ and $\lambda_2$, we have $\lambda_1 \lambda_2 = -1$. Therefore, the equation that describes the normal from the origin, is:

$$y = \frac{1}{\lambda} x$$  \hspace{1cm} (3.42)
By combining Eqs. (3.38)-(3.40) with Eq. (3.42), the coordinates of the starting point $Q$ for the calculations along the line are:

$$x_Q = -\frac{\beta}{(\lambda + \frac{1}{\lambda})}$$  \hspace{1cm} (3.43)

$$y_Q = -\frac{1}{\lambda} x_Q$$  \hspace{1cm} (3.44)

The sampling along the line segment will be performed on either side of $Q$ and we assume that the sampling step is $\Delta s$. The maximum number of sampling intervals that we can fit in this line segment is determined by the intersection points between the line and the border of the 2-D domain. The distance $d_A$ between the starting point $Q$ and point $A$, at which the line intersects the bottom edge of the domain, is:

$$d_A = \sqrt{(x_Q - x_A)^2 + (y_Q - y_A)^2}$$  \hspace{1cm} (3.45)

Similarly, the distance $d_B$ between the starting point $Q$ and point $B$, at which the line intersects the top edge of the domain, is:

$$d_B = \sqrt{(x_Q - x_B)^2 + (y_Q - y_B)^2}$$  \hspace{1cm} (3.46)

Consequently, the numbers of $\Delta s$, $l_A$ and $l_B$, that we may move away along the line segment from the foot of the normal, $Q$, towards boundary points $A$ and $B$, respectively, are:

$$l_A = \left\lfloor \frac{d_A}{\Delta s} \right\rfloor$$  \hspace{1cm} (3.47)

$$l_B = \left\lfloor \frac{d_B}{\Delta s} \right\rfloor$$  \hspace{1cm} (3.48)

where $\lfloor \cdot \rfloor$ is the symbol for the floor operator. Therefore, the sampling points we shall consider along line segment $AB$ will have as coordinates

$$x_l = x_Q + lx_{inc} \hspace{1cm} \text{for } l \in [-l_A, l_B]$$  \hspace{1cm} (3.49)

$$y_l = y_Q + ly_{inc} \hspace{1cm} \text{for } l \in [-l_A, l_B]$$  \hspace{1cm} (3.50)
where the increments $x_{inc}$ and $y_{inc}$ of the coordinates, between successive sampling points, are given by:

\[
x_{inc} = \Delta s \cos w \\
y_{inc} = \Delta s \sin w
\]

The total number of sampling points along the line segment is $l_A + l_B + 1$.

An example of a sampling of line segment $AB$, with sampling step $\Delta s = P$ (=tile size), is shown in Fig. 3.3, where the estimated sampling points are marked with $\odot$.

![Figure 3.3: An example of sampling a scanning line segment. The sampling step was taken to be equal to the tile size $P$. The sampling points that were identified are marked with $\odot$.](image)

After having worked out the coordinates of the sampling points of the line, we must assign them values from the vector field. To achieve this, we employ nearest neighbour interpolation. Hence, the value of the vector field, assigned to each sampling point of the line, is the unknown value of the vector field at the nearest neighbour sampling point of the reconstruction domain. Next, we describe the process we follow to determine for each sampling point of the line, the tile, the centre point of which, is its nearest neighbour.

Consider the integer coordinates $(i_c, j_c)$ with $i_c, j_c = 1, \ldots, \frac{2U}{P}$, of each tile of the
2-D domain as shown in Fig. 3.4.

![Figure 3.4: Integer coordinates \((i_c, j_c)\) with \(i_c, j_c = 1, \ldots, \frac{2U}{P}\) of the tiles of the 2-D reconstruction domain.](image)

Then, the tile \((i_c, j_c)\) that corresponds to a sampling point \((x_l, y_l)\) is identified by using the formulae:

\[
i_c = \left\lceil \frac{x_l + U}{P} \right\rceil \quad (3.53)
\]

\[
j_c = \left\lceil \frac{y_l + U}{P} \right\rceil \quad (3.54)
\]

where \([·]\) is the ceiling operator. The application of Eqs. (3.53) and (3.54) to the sampling points that were obtained at the example of Fig. 3.3 (with \(U = 5.5, P = 1\) and \(\Delta s = 1\)) results in the integer tile coordinates, that are listed in Table 3.4.

In the case that the line segment is parallel to the \(x\)-axis \((y = y_Q)\), then, the tile with centre point nearest to a line sampling point has coordinates \((i_c, j_c)\) where:

\[
j_c = \left\lceil \frac{y_Q + U}{P} \right\rceil \quad \text{and} \quad i_c \in \{1, 2, \ldots, \frac{2U}{P}\}.
\]

Similarly, if the examined line segment is parallel to the \(y\)-axis \((x = x_Q)\), the nearest tile centre to a line sampling point will be \((i_c, j_c)\) where:

\[
i_c = \left\lceil \frac{x_Q + U}{P} \right\rceil \quad \text{and} \quad j_c \in \{1, 2, \ldots, \frac{2U}{P}\}.
\]

In order to form the equation that corresponds to the line-integral measurement \(J_1\),
3.4 The Proposed Reconstruction Methodology

Table 3.1: Integer coordinates of tiles, the centre points of which are the nearest neighbours of the sampling points of the scanning line.

<table>
<thead>
<tr>
<th>l</th>
<th>((x_l, y_l))</th>
<th>(\frac{x_l + U}{p})</th>
<th>(\frac{y_l + U}{p})</th>
<th>(i_c)</th>
<th>(j_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7</td>
<td>(-4.7990, -5.3122)</td>
<td>0.7010</td>
<td>0.1878</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-6</td>
<td>(-4.2990, -4.4462)</td>
<td>1.2010</td>
<td>1.0538</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>-5</td>
<td>(-3.7990, -3.5801)</td>
<td>1.7010</td>
<td>1.9199</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>-4</td>
<td>(-3.2990, -2.7141)</td>
<td>2.2010</td>
<td>2.7859</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-3</td>
<td>(-2.7990, -1.8481)</td>
<td>2.7010</td>
<td>3.6519</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>-2</td>
<td>(-2.2990, -0.9821)</td>
<td>3.2010</td>
<td>4.5179</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>(-1.7990, -0.1160)</td>
<td>3.7010</td>
<td>5.3840</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>(-1.2990, 0.7500)</td>
<td>4.2010</td>
<td>6.2500</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>(-0.7990, 1.6160)</td>
<td>4.7010</td>
<td>7.1160</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>(-0.2990, 2.4821)</td>
<td>5.2010</td>
<td>7.9821</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>(0.2010, 3.3481)</td>
<td>5.7010</td>
<td>8.8481</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>(0.7010, 4.2141)</td>
<td>6.2010</td>
<td>9.7141</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>(1.2010, 5.0801)</td>
<td>6.7010</td>
<td>10.5801</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

collected by sensors placed at points \(A\) and \(B\), we consider the sampling points of the line, that we obtained, as the centres of linear segments of length \(\Delta s\), apart from the sampling points with \(l = -l_A\) and \(l = l_B\) which are special cases. Along each of these segments of length \(\Delta s\), the vector field is assumed constant, equal to the assigned (unknown) value of the vector field at the corresponding sampling point of the line. It is possible, then, to approximate the integral of Eq. (3.36) by a sum, by projecting the value of the field at each sampling point \(l\) of the line onto the vector that represents the direction of the line:

\[
J_l = \sum_{l=-l_A+1}^{l_{l_B-1}} \hat{f}_l \cdot \Delta s + \hat{f}_{l_A} \cdot \Delta s_A + \hat{f}_{l_B} \cdot \Delta s_B
\]

(3.55)

Here \(\hat{f}_l = (f_{xl}, f_{yl})\), \(\hat{f}_{l_A} = (f_{x1A}, f_{y1A})\) and \(\hat{f}_{l_B} = (f_{x1B}, f_{y1B})\) are the (unknown) assigned vector field values at sampling points \(l, -l_A\) and \(l_B\), respectively, \(\Delta s = \Delta s\hat{s} = \Delta s(\cos w\hat{x} + \sin w\hat{y})\), and

\[
\Delta A = \frac{\Delta s}{2} + d_{lA}
\]

(3.56)

\[
\Delta B = \frac{\Delta s}{2} + d_{lB}
\]

(3.57)

where \(d_{lA}\) is the distance between the sampling point with \(l = -l_A\) and the boundary
3.4 The Proposed Reconstruction Methodology

point \( A \), whereas \( d_{lB} \) is the distance between the sampling point with \( l = l_B \) and the boundary point \( B \):

\[
d_{lA} = \sqrt{(x_Q - l_A x_{inc} - x_A)^2 + (y_Q - l_A y_{inc} - y_A)^2} \tag{3.58}
\]
\[
d_{lB} = \sqrt{(x_Q + l_B x_{inc} - x_B)^2 + (y_Q + l_B y_{inc} - y_B)^2} \tag{3.59}
\]

In order to obtain the system of linear equations, the solution of which will give the components of the examined vector field \( \mathbf{f}(x, y) \) at all sampling points of the 2-D domain, according to the proposed methodology, we repeat the procedure described above for all possible pairs of boundary point sensors. According to the sensor configuration, employed in this study, there are \( \frac{2l}{P} \) ideal point sensors on each side of the boundary of the reconstruction domain. Since, projection data obtained from pairs of sensors that reside in the same side of the boundary of the square are not useful, the number of equations is

\[
A_r = \frac{2l^2}{P^2} \left[ = \frac{2l}{P} \left( 3 \times \frac{2l}{P} + 2 \times \frac{2l}{P} + 1 \times \frac{2l}{P} \right) \right].
\]

In addition, the sampling of the 2-D space, that we selected, resulted in having \( \frac{4l^2}{P^2} \) as the total number of tiles of the digitised reconstruction domain, and \( A_c = \frac{8l^2}{P^2} \) as the overall number of the unknowns of the system, since, we have two unknowns per sampling point, namely the components \( (f_x(x, y), f_y(x, y)) \) of the investigated vector field. From these two selections (i.e., the data acquisition geometry and the sampling of the reconstruction domain) we made, it is obvious that the number of the equations is far larger than the number of the unknowns \( (A_r > A_c) \), in accordance with our intention to take advantage of the redundancy in the line-integral data, as a form of employing regularisation to deal with the ill-posed nature of the vector field reconstruction problem. Therefore, we have to deal with an overdetermined system of linear equations.

To summarise, our formulation of the vector field reconstruction problem may be written in matrix formalism as

\[
\mathbf{b} = A \mathbf{g} \tag{3.60}
\]

where \( \mathbf{b} \in \mathbb{R}^{A_r \times 1} \) is the vector that contains the available projection measurements wrapped into a vector, \( \mathbf{g} \in \mathbb{R}^{A_c \times 1} \) is the set of the components of the vector field to
be reconstructed at all sampling points of the 2-D digitised domain written as a vector, and $A \in \mathbb{R}^{A \times A}$ is the system matrix, containing the weight factors between each of the components of the vector field at every reconstruction point and each of the corresponding scanning line orientations from the set of measurements. System matrix $A$ is obtained from the analysis described above. Next, we demonstrate the potential of the reconstruction methodology, proposed in this section, by presenting an example of vector field recovery.

3.5 An Example of 2-D Vector Field Imaging

3.5.1 Simulations

In this section, we consider the case where the vector field that we want to recover is the electric field created by a static charge. Four different cases for the location of the source of the vector field are reported. There are many ways to recover the electric field from boundary data. However, here we use the electric field only to demonstrate our method. In order to avoid problems with singularities, this section only treats the case where the source of the vector field that we aim to recover is outside the bounded 2-D area. In a real physical system, we do not expect to have to deal with real singularities anyway. We would like to stress that the problem we solve is intentionally kept simple in order to demonstrate the method. So, instead of avoiding singularities by using a realistic version of Coulomb's law for sources of finite size, we place the source outside the domain of interest and make it infinitesimally small.

For a static electric field, every voltage difference between any two points is the line-integral of the field projected along the line that connects these two points. Therefore, we assume that the boundary sensors measure the potential, so that the difference in the measurements between any two such sensors gives the vectorial Radon transform of the investigated electric field. For the simulations we present here, the potential in all these sensors is obtained by using Coulomb's law. It must be noted that the electric field is irrotational, so according to [48], only transversal measurements would be helpful
3.5 An Example of 2-D Vector Field Imaging

to recover this field. However, the only realisable measurements for this application are longitudinal.

For our experiments, we employ the digital square reconstruction domain of Fig. 3.2 and choose $2U = 11$, as domain size, and $P = 1$, as tile size. Hence, the domain consists of 121 tiles. Regarding the data acquisition geometry, according to the proposed methodology, ideal point sensors are regularly placed along the whole border of the domain. These regular positions are the middle points of the boundary edges of all boundary tiles. Hence, the above selection of parameters $U$ and $P$ results in having 11 sensors in every side of the boundary of the square domain. Therefore, by considering all possible voltage differences between pairs of these sensors, apart from sensors lying on the same border line, and by sampling the line segments joining these pairs of sensors with a step equal to $P$ ($\Delta s = 1$), we form the system of linear equations (3.60), according to the analysis presented in Section 3.4. The number of the linear equations of the system is $A_r = 726$, whereas the number of the unknowns (the $E_x$ and $E_y$ components of the field at the centre of every tile of the domain) is 242, hence, it is an overdetermined system. Then, in order to obtain the reconstruction results, we have to solve Eq. (3.60).

3.5.2 Stability Considerations

Inverse problems, like the one described by Eq. (3.60), suffer from the notorious ill-posed nature, in the sense of Hadamard [22]. As a result, the solution to these reconstruction problems endures stability deficiencies that are related to the solution’s existence, uniqueness and continuous dependency on the projection data. Next, we give a short account of the treatment we followed to deal with these stability deficiencies, when solving system (3.60), that we obtained following the proposed methodology.

In order to deal with the ill-posed nature of the vector field reconstruction problem, we exploited the redundancy in the line-integral data, as a form of regularisation. Next, we describe how this regularisation helped us to do away with the matters of existence and uniqueness of the solution. The systems of equations, that we obtained in our simulations, were overdetermined. As all columns of matrix $A$ were found to be linearly independent,
it is only possible to have a solution in the least squares (LS) sense. The rank of matrix \( A \) was found in each case to be equal to the number of the unknowns of the system. So, the LS error solution exists and is unique. In this study, we obtained the LS solution (or, else, the reconstruction results) by applying the Gauss-Newton LS method [17], the most efficient numerical technique to perform LS estimations. The fact that the Gauss-Newton LS method may, also, return negative solutions, is not a problem for the case of vector field tomography, as it is for conventional scalar tomography. In addition, the simulations, we carried out in this study, were limited to the 2-D case. Therefore, the size of the associated system matrices was not prohibitively large to prevent us from using the Gauss-Newton LS method. Moreover, it must be noted that since the residual we computed by using the LS Gauss-Newton method was not large when compared with the solution vector, there was no need to use the Cholesky method [5].

Next, we describe how the employed regularisation, by using redundant projection data, helped us to restore the solution’s continuous dependency on the projection data. A good measure of the degree of ill-posedness of system (3.60) is the condition number, i.e. the ratio of the maximal to minimal eigenvalue of matrix \( A \). This measure gives us all the information we require about the ill-posedness, because the larger the value of the condition number, the more pronounced is the ill-posedness of the inverse problem. For the simulations we carried out in this study, the range of values of the condition number showed that the ill-posedness is noticeable but manageable and not serious. Hence, the exploitation of the redundancy in the line-integral data, as proposed by the methodology, led to the ill-posedness being not much of a problem. To confirm these findings, we also tested the Householder orthogonalisation method [63], which is a numerically useful procedure in order to solve LS value problems for cases where the condition number of the matrix of coefficients is large [64]. However, the results we obtained were identical with the results we obtained using the Gauss-Newton LS method.
3.5.3 Reconstruction Results

The reconstruction results, namely the solution of the overdetermined system of linear equations, were obtained. These results are shown in Figs. 3.5a-3.8a for four different cases of the location of the source of the vector field. For the sake of comparison, Figs. 3.5b-3.8b depict the respective electric fields that are obtained by using directly the theoretical Coulomb's law. In Figs. 3.5c-3.8c, the relative differences between the magnitudes of the two vector fields (i.e. the absolute values of the differences between the magnitudes of the reconstructed field and the theoretical field as acquired by Coulomb’s law, divided by the theoretical magnitude) are shown. The absolute values of the angular differences (in degrees) between the reconstructed vector field values and the theoretical ones are illustrated for each case in Figs. 3.5d-3.8d. Finally, Figs. 3.5e,f-3.8e,f display the distributions (histograms) of relative magnitude and absolute angular errors, respectively, in all reconstruction points of the bounded domain.

By careful inspection of Figs. 3.5a,b-3.8a,b we may say that the directions of the vectors that were reconstructed, based on the boundary voltages, are almost identical with the directions of the vectors that were obtained by using Coulomb's law, since in both cases the vectors are oriented towards the source of the field. Furthermore, vectors in both fields reduce in magnitude with the distance from the source, as expected, even though the recovered vectors seem to reduce a bit more slowly than those computed by the application of Coulomb's law.

It must be noted that in the magnitude error plot, shown in Fig. 3.5c, there is an area at the left top corner of the plot where the error appears to be larger than in the rest of the reconstruction region, whereas similar observations can be made in Figs. 3.6c-3.8c. This discontinuity in error occurs because the measurement content obtained from scanning lines crossing areas that are at a great distance from the source of the vector field is very small, when compared with the information collected from scanning lines going through other areas of the reconstruction region. In addition, it must be said that by increasing the resolution of the reconstruction domain, then, the size of the area where reconstruction results are more inexact will also increase. This is due to the fact
3.5 An Example of 2-D Vector Field Imaging

Figure 3.5: Simulation results when the sensors are uniformly placed along the boundary and the location of the source of the electric field is at \((19, -19)\): (a) the recovered vector field (solution of the system of linear equations); (b) the theoretical electric field as computed from Coulomb's law; (c) the relative magnitude difference between the above two fields (%); (d) the absolute angular difference (in degrees) between the above two fields; (e) the histogram of relative magnitude errors and (f) the histogram of absolute angular errors.

that increased resolution will result in the boundary measurements being entangled in even more reconstruction pixels. Hence, the 2-D vector field reconstruction problem will become more ill-conditioned and, therefore, larger areas of reconstruction pixels in error will affect the line-integral measurements the same (as before increasing the resolution).

To summarise, we may say that by following the direct algebraic reconstruction methodology, proposed in this chapter, the problem of the recovery of both components of a 2-D electric field at a finite number of sampling points of its domain, based only on a limited number of line-integral data, is tractable. Next, we discuss the effect of noise on
3.5 An Example of 2-D Vector Field Imaging

Figure 3.6: As in Fig. 3.5 but here the location of the source of the field is at (-16, 21).

3.5.4 Effect of Noise on Reconstruction

An important issue when solving inverse problems is the resilience of the solution to noise. In this section, we investigate the effect of noise on the reconstruction of the vector field. In all experiments reported in the previous section, the sensors were placed exactly in the positions we had decided, and the measurement taken by each sensor was exactly the value computed by Coulomb’s law. In a practical system, however, some of the sensor measurements are expected to have inaccuracies and some of the sensor positions are also expected to be somehow inaccurate. To emulate these effects, we performed the following series of experiments.
Figure 3.7: As in Fig. 3.5 but here the location of the source of the field is at (12.5, 30).

(i) We added a noise value to a measurement as a fraction of the true value, with random sign. For example, 2% noise means that the sensor measurement was changed by 2% of the value dictated by Coulomb's law. The change was either incremental or decremental, the choice made at random for each sensor.

(ii) We moved a sensor away from its true position by a fraction of the true position. For example, if according to the theory, a sensor should be placed at position \((x, y)\), and we consider a 2% error, then, the coordinates of this sensor were shifted by 2% the corresponding correct values, with a positive or negative sign chosen at random.

(iii) We considered both the above errors simultaneously.

We performed four series of experiments: (a) we perturbed only 25% of the sensors; (b) we perturbed 50% of the sensors; (c) we perturbed 75% of the sensors and (d) all sensors
3.6 Discussion and Conclusions

Figure 3.8: As in Fig. 3.5 but here the location of the source of the field is at \((-19, -40)\).

were perturbed. The source of the vector field for all simulations was located at \((19, -19)\).

The results of these experiments are shown in Figs. 3.9-3.12.

We observe that the results are relatively robust to perturbations in the position of the sensors, but much more sensitive to perturbations in the sensor measurements.

3.6 Discussion and Conclusions

In this chapter, the vector field tomography problem was discussed. In previous attempts to map integral measurements, obtained along scanning lines, onto a vector field, conventional (scalar) tomography theory and the FST had invariably been applied [4], [44] and [48]: this had led to an underdetermined problem. However, in this chapter, a new direct algebraic reconstruction technique was presented that aimed at the recovery of all
Figure 3.9: (a) and (b) Errors in vector field orientation and magnitude, respectively, when noise was added to the measurements of sensors, as a percentage of the true value. (c) and (d) Errors in vector field orientation and magnitude, respectively, when small perturbations in the sensor positions were added. Position perturbations were a percentage of the true positions. (e) and (f) Errors in vector field orientation and magnitude, respectively, when both sensors' measurements and positions were changed by a percentage of their true values. In all cases, 25% of the sensors were perturbed.

3.6 Discussion and Conclusions

components of a vector field at the sampling points of a 2-D digitised bounded domain. The reconstruction was based only on a limited number of boundary integral measurements. To achieve the recovery, the method takes advantage of the redundancy in the projection data, as a form of employing regularisation, since these data may be seen as weighted sums of the local vector field's Cartesian components. The results demonstrate that the tomographic reconstruction of such type of vector field in the discrete domain, by relying only on redundant projection data, is tractable.

The noise model assumed in the experiments, that were carried out in this chapter, was signal strength dependent. Noise processes of this type are inherent in many fields such as optics [56], kinematics [60] and magnetic resonance imaging [38]. However, in many cases, like for example in telecommunications, the noise that corrupts the data is signal independent. The implication of employing signal independent (additive or multiplicative)
3.6 Discussion and Conclusions

Figure 3.10: As in Fig. 3.9, but here 50% of the sensors were perturbed.

Figure 3.11: As in Fig. 3.9, but here 75% of the sensors were perturbed.

noise for the proposed algorithm is that the quality of reconstruction deteriorates slightly with the regions, where the field is weak, being the worst affected areas by this change. In general, when one tries to develop a noise removal technique, by having made false
3.6 Discussion and Conclusions

An important issue when solving inverse problems is the sensitivity of the solution to noise. In the case of this problem, there were two possible sources of noise: inaccuracies in the sensor measurements and inaccuracies in the positions of the sensors. In a practical application, one may hope that one may use very accurate sensors and that even more accurate sensors may be developed in the future. The inaccuracies, however, in the sensor positions are rather intrinsic to the problem: the domain over which the vector field is to be reconstructed may not have a shape that helps the correct placement of the sensors. It was very encouraging, therefore, that the solution of the problem was relatively stable to perturbations in the sensor positions.

The solution was rather sensitive to the sensor measurements. For example, if only 25% of the sensors yielded measurements that were only 4% wrong, the orientation angle of the reconstructed field was recovered with an average error of about 15°, while the magnitude of the reconstructed field was recovered with an average relative error of about 23%. Such sensitivity to errors in the measurements may be overcome with the help of

Figure 3.12: As in Fig. 3.9, but here all sensors were perturbed.
robust reconstruction methods. There are two ways to go about this.

(i) One may solve the system of linear equations in a robust way. For example, instead of working out a solution that minimises the sum of the squares of the errors with which individual equations are satisfied, one may use a robust redescending kernel [24] that will reduce the effect of outliers. The problem contains enough redundancy to permit such an approach.

(ii) The problem may be formulated as a Bayesian reconstruction problem [43], where a regularisation term is added to a global cost function that expresses the adherence of the values of the reconstructed field to the obtained measurements. The regularisation term may be such that it encourages the smooth variation of the field inside the domain. Expecting smooth field variation between neighbouring sampling positions is compatible with the assumption that there are no singularities inside the domain. Indeed, we consider this assumption pretty realistic as singularities usually arise due to poor mathematical modelling rather than being present in a physical system. Once a cost function of the solution has been formulated, it can be solved using Bayesian methods [3], [18] and a global optimisation approach, like, for example, simulated annealing [36].

The analysis in this chapter treated 2-D vector field tomography. In the case where the vector field that one wants to recover is 3-D, then, a set of parallel planes, that cover the whole volume of interest, have to be considered. Hence, by applying the reconstruction methodology, proposed in this chapter, to each of the stacked parallel planes separately, the 2-D solutions that one obtains are the projections of the 3-D vector field onto these planes. Finally, in order to determine uniquely the out-of-plane component of the 3-D vector field, this process has to be repeated over a second set of parallel planes inclined at some angle with respect to the first set of planes.

In many practical situations of interest in image reconstruction, it is not possible to collect data over a complete angular range [9]. This situation is referred to as the limited-view problem. The reason that causes this problem to arise depends on the application. Limited data collection time, geometric constraints on the structure of the measurement apparatus and the size and structure of the object to be imaged are some
of the root causes for preventing one from traversing completely around the investigated object. Regarding the model presented in this chapter, it would not be possible in a practical situation to measure all lines connecting sensors and crossing the 2-D domain. If we consider, for example, the case where the geometry of the scanning system dictates that a measurement is collected only if the related measurement line makes an angle of at least 20° with each of the associated boundary edges\(^6\), then, it would not be possible for about 16% of the total line-integral measurements \(\frac{24\pi^2}{27}\) to be acquired. The limited angular coverage, discussed above, may cause several difficulties with the most typical one being the increase of instability [46]. Hence, the computed inversions become more sensitive to noise. Also, due to the fact that the information provided to the reconstruction problem is not complete, problems of non-uniqueness of the solution may arise. To make up for this lack of information, one may employ adequate \textit{a priori} knowledge about the investigated vector field. In general, because of the importance of the limited-angle problem, many specialised algorithms have been introduced [11].

The sensor configuration that we employed in the simulations of this chapter assumes that sensors reside in the middle points of the boundary edges of all boundary pixels. However, in most tomographic applications, especially in the medical field, images of very high resolution are required and it is not possible to have so dense sensor positioning by relying on current sensor technology. As a result, the systems of equations, that one has to deal with in practical image reconstruction, are underdetermined. The last decade, nevertheless, has witnessed [40] a rapid surge of interest in manufacturing techniques of miniaturised sensors for healthcare and industry. Therefore, the odds are that a rapid expansion in development of sensors of smaller size will take place over the next ten years. Such advances in sensor technology will facilitate the implementation of sensor arrangements, that are in the direction of the sensor arrangements we proposed in this chapter. This development will make it possible for the reconstruction algorithm, that we introduced in this chapter, to meet the desired standards of most tomographic applications.

\(^6\)A missing angle of 40° (= 20° + 20°) out of 180° is a case that one often comes across in limited view tomographic reconstruction [11] and [57].
Chapter 4

Virtual Sensors for 2-D Vector Field Tomography

4.1 Introduction and Motivation

In the previous chapter, the tomographic mapping of a 2-D vector field from projection data was discussed. It was shown that by following the direct algebraic reconstruction methodology, proposed there, the recovery of both components of a 2-D vector field at a finite number of sampling points of its domain, based only on a limited number of line-integral data, may be achieved. The proposed technique assumed that the measurements were collected by sensors that were regularly placed along the whole border of the reconstruction domain, since, such a sensor placement is the most convenient.

The approach, described in Chapter 3, formulated the tomographic reconstruction problem in terms of a system of linear equations. However, there is a duality between this matrix formalism and the Radon transform scheme. Hence, solving the system of linear equations, obtained by following the description in Chapter 3, is equivalent to inverting the vectorial Radon transform. According to the theory of Radon transform [9], necessary requirements to produce results with the accuracy desired in medical imaging, when using discrete approximations, are to have a large number of projections (i.e. adequately dense sampling of the Radon domain parameters) and, also, substantially uniform distribution.
of projection data, as functions of the two Radon domain variables, normally designated as the radial ($\rho$) and angular ($\theta$) coordinates (see Fig. 2.1).

However, sampling the Radon parameter domain uniformly has the following major drawbacks.

(i) It dictates a prohibitively large number of sensors.

(ii) It results in impractical sensor positioning. In particular, the uniform sampling of the $(\rho, \theta)$ space dictates that the sensors that have to be placed at the ends of a scanning line may be impractically close to the sensors of another scanning line.

In the case where the sensors may be mounted on a common rotating frame, the problems described in (i) and (ii), regarding the uniform sampling of the Radon parameter domain, are no longer present. However, in this case, each scan of the domain corresponds to only one value of the angular parameter. Hence, in order to cover all angular orientations, the scanning process needs to be repeated many times. This leads to prohibitively large total scanning times and it cannot be applied to the medical field, where scanning time is crucial.

In this chapter, we show how these problems may be overcome by using virtual sensors. In particular, we propose to maintain the convenient sensor configuration of the previous chapter, that corresponds to uniform sampling in the space of the intersection coordinates with the boundary of the reconstruction domain, and we also introduce the concept of "virtual sensors". The data values at these virtual sensors, that correspond to uniform sampling in the $(\rho, \theta)$ domain, are obtained from the known values of the true sensors, that are placed at regular points in relation to the Cartesian intersection coordinates with the boundary of the reconstruction domain, by using some interpolation method. This approach allows one to use as many scanning lines as one can afford, taking into consideration the computational cost of solving the corresponding system of linear equations. However, the increase of the number of the available line-integral data in such a way, is not limited by physical constraints on sensor placement or total scanning time constraints.
This chapter is organised as follows. In Section 4.2, we formulate the problem and set up our notation. In Section 4.3, we present an example of static electric field reconstruction and demonstrate the effect of the use of interpolated data on the quality of reconstruction. In Section 4.4, we examine the effect of the employment of interpolated measurements on resilience to noise. We conclude in Section 4.5.

4.2 The Reconstruction Methodology

The treatment in this section is similar to the one in Section 3.4. We perform the analysis in the digital domain. The same digitised reconstruction region (Fig. 3.2) is employed, that is repeated here as Fig. 4.1 for the sake of convenience. The goal is to recover vector field \( \mathbf{f}(x, y) = f_x(x, y)\hat{x} + f_y(x, y)\hat{y} \) at the sampling points of this domain.

![Figure 4.1: A tracing line segment AB that unites two virtual sensors that reside at points A and B. The tracing line is defined by the two parameters \( \rho \) and \( \theta \) (Radon domain coordinates) and goes through the square digitised reconstruction region of size \( 2U \). The line segment is sampled with sampling step \( \Delta s \). AB is inclined at an angle \( \omega \) to the positive direction of the x-axis. The size of the tiles, with which we sample the 2-D space, is \( P \times P \). Also shown is the unit vector \( \hat{s} \) which is parallel to line segment AB.](image)

Moreover, we assume, in line with Section 3.4, that ideal point sensors, that integrate only the component of the planar field projected onto the line, reside in predetermined and regularly placed positions of the whole border of the 2-D square domain. These positions are the middle points of the boundary edges of all boundary tiles. However, as argued in the previous section, in order to achieve the best vector field reconstruction,
4.2 The Reconstruction Methodology

the data should not be collected by these regularly placed sensors, but by sensors that correspond to uniform sampling of parameters $\rho$ and $\theta$. Therefore, we propose to use interpolated measurements that correspond to uniform sampling of the $(\rho, \theta)$ space. It is assumed that the interpolated measurements are observed at virtual sensors.

Let us consider a tracing line $AB$ (see Fig. 4.1) that connects two virtual sensors. In terms of parameters $\rho$ and $\theta$, the equation of the line is:

$$\rho = x \cos \theta + y \sin \theta \quad (4.1)$$

Sampling $\rho$ and $\theta$ parameters uniformly results in a set of such lines. We make the assumption that each tracing line $(\rho, \theta)$ of the set yields a line-integral measurement $J_i$. The value of this measurement is obtained from the available measurements of the true sensors by using some interpolation method. Since we assumed for our analysis that each pair of sensors measures only the integral of the component of the vector field projected onto the integration line, the integral transform that models the interpolated measurement $J_i$, collected by virtual sensors at points $A$ and $B$, is given by:

$$J_i = \int_{AB} \tilde{f}(x, y) \cdot \hat{s} \, ds = \int_{AB} f_\parallel \, ds \quad (4.2)$$

Here $\hat{s} = \cos wx \hat{x} + \sin wy \hat{y}$ is the unit vector along integration line segment $AB$, where $w$ is the angle at which the tracing line is inclined to the positive direction of the $x$-axis (see Fig. 4.1). In addition, $ds$ is an element of path length along this line segment and $f_\parallel$ is the component of $\tilde{f}(x, y)$ along $AB$. In order to translate into the digital domain, the integration expressed by Eq. (4.2) in the continuous domain, the analysis of Section 3.4 is applied. Then, the integral of Eq. (4.2) can be approximated by the following sum:

$$J_i = \sum_l \tilde{f}_l \cdot \Delta \hat{s} \quad (4.3)$$

Here $\tilde{f}_l = (f_{xl}, f_{yl})$ are the unknown vector field values at these sampling points of the reconstruction domain, that are nearest neighbours to the sampling points $l$ of the line. Also, $\Delta \hat{s} = \Delta s \hat{s}$, where $\Delta s$ is the sampling step along the line segment (see Fig. 4.1). The
number of equations (4.3) we have, depends on the number of tracing lines between virtual sensors we consider. In general, we consider overdetermined systems of linear equations, and we obtain the solution in the LS error sense.

In the next section, we demonstrate that the reconstruction results we obtain using interpolated line-integral measurements observed at virtual sensors, as described above, are more accurate than the ones obtained in Chapter 3, where reconstruction was based on line-integral measurements collected by sensors that were placed uniformly in the space of the Cartesian intersection coordinates with the boundary of the reconstruction domain.

4.3 An Example: Electric Field Imaging

We considered the same case as in Section 3.5, where the vector field that we want to recover is the electric field created by a static charge. Four different cases for the location of the source of the electric field are reported. We assumed that the boundary sensors measured the potential, so that the difference in the measurements between any two such sensors gave the vectorial Radon transform of the examined electric field.

We employed the digital square reconstruction domain of Fig. 4.1 and chose $2U = 11$ as domain size and $P = 1$ as tile size. Hence, the domain consisted of 121 tiles and the number of the unknowns (the $E_x$ and $E_y$ components of the field at the centre of every tile of the domain) was 242. To exemplify the theory of the study described in this chapter, we performed five sets of experiments for each source location.

The first set of experiments was performed following the analysis described in Section 3.4. Hence, we considered the practical case where ideal point sensors are regularly placed (RS), in relation to the Cartesian intersection coordinates with the boundary of the reconstruction domain, in known and predetermined positions of the whole border of the domain. These positions were the middle points of the boundary edges of all boundary tiles. Therefore, we used 11 sensors in every side of the boundary of the square domain. The potential in all these sensors was obtained by using Coulomb's law. We considered all possible voltage differences between pairs of boundary sensors, apart from sensors lying
on the same border line, and we formed the system of linear equations according to the
description of Section 3.4. The line segments joining sensors were sampled with a step
equal to 1 (\(\Delta s = 1\)). The number of linear equations, that we obtained, was 726.

In the second set of experiments, we used the same sensor placement as in the
first set of simulations. However, we performed the vector field reconstruction by relying
only on interpolated line-integral data observed at virtual sensors, that corresponded to
uniform sampling of the \((\rho, \theta)\) Radon space, as proposed in this chapter. To obtain the
positions where the virtual sensors had to be inserted, we considered for the Radon domain
parameters the sampling steps recommended in [34] and [52]: \(\Delta \rho = 1\) and \(\Delta \theta = 3^\circ\). The
data values of the virtual sensors were obtained from the (Coulomb's law) data of the true
sensors, that were regularly placed in relation to the Cartesian intersection coordinates
with the boundary of the reconstruction domain, by using some interpolation method. In
this study, we examined the following methods: 1-D linear interpolation (IP1) [35], 1-D
piecewise cubic spline interpolation (IP2) [35], 1-D piecewise cubic Hermite interpolation
(IP3) [35], bilinear interpolation (IP4) [35], bicubic interpolation (IP5) [35] and 2-D spline
interpolation (IP6) [35]. The line segments joining virtual sensors were sampled with a step
equal to 1 (\(\Delta s = 1\)). The selected sampling steps of parameters \(\rho\) and \(\theta\) resulted in having
6 samples for the radial parameter and 120 samples for the angular parameter, so that the
region of interest (Fig. 4.1) was fully covered. Consequently, the overdetermined system
of linear equations, the solution of which gave the reconstructed field, had 720 (= 6 \times 120)
equations, almost the same number as in the first set of experiments.

In the third set of experiments, we used uniform sampling (US) in the parameter
space, the same as in the second set of experiments. However, the sensor placement was
different. In particular, the vector field recovery was not based on interpolated measure-
ments, but we assumed that there are true ideal point sensors at the ends of all lines
that cross the domain and that are uniformly distributed in the \((\rho, \theta)\) space. Then, the
potential in all these sensors was obtained by using Coulomb's law. To implement this
requirement, we had to use about 1440 (= 2 \times 720) ideal point sensors, i.e. a thirty-
fold increase when compared with the first two sets of experiments. Alternatively, if it
was possible to employ a rotating sensor configuration, then, the number of the required sensors would not increase by using actual measurements and uniform sampling in \((\rho, \theta)\). However, the employment of rotating sensor arrangement would result in an unwanted one-hundred-and-twentyfold increase in the total scanning time, when compared with the first two sets of experiments.

In the fourth and fifth sets of experiments, the vector field reconstruction was performed as in the second and third sets of experiments, respectively, apart from the fact that the employed sampling rates in the Radon space were increased twofold: \(\Delta \rho = 0.5\) and \(\Delta \theta = 1.5^\circ\). This resulted in having 2640 (= 11 x 240) linear equations. Hence, to implement the fifth set of experiments, we had either to use about 5280 (= 2 x 2640) ideal point sensors (i.e. an one-hundred-and-twentyfold increase, when compared with the first and fourth sets of experiments) or to increase the total scanning time two-hundred-and-fortyfold by employing a rotating sensor configuration.

We must note that for the second and fourth sets of experiments, where interpolated measurements were used for the reconstruction, the increase of the available line-integral data was not limited by the physical limitations that the sensor placement imposes. In addition, this increase was made taking into consideration that the resulting system of equations would not be prohibitively large and its solution would not increase the processing time significantly.

The reconstruction results, namely the solution of the overdetermined systems of linear equations for the five sets of experiments and the four source locations were obtained by applying the Gauss-Newton LS method. The Householder orthogonalisation method, which is a numerically useful procedure in order to solve LS problems that suffer from ill-posedness, was also tested for our reconstruction problem. However, the results we obtained were identical with the results we obtained using the Gauss-Newton LS method. Moreover, it must be noted that since the residue we computed by using the Gauss-Newton LS method was not large, when compared with the solution vector, there was no need to use the Cholesky method.

The relative magnitude reconstruction error values (i.e. the absolute values of the
4.3 An Example: Electric Field Imaging

differences between the magnitudes of the reconstructed fields and the theoretical ones, as obtained by using directly the governing Coulomb's law, divided by the theoretical magnitude) and the absolute angular reconstruction error values (i.e. the absolute angular differences (in degrees) between the reconstructed vector field values and the theoretical ones) for the five sets of experiments and for the four locations of the source were calculated. The means of these errors per reconstruction tile are shown in Figs. 4.2-4.5.

Figure 4.2: The comparison of the reconstruction performance for the cases when reconstruction was based on: (i) line-integral data from regularly placed sensors (RS) in relation to \((x, y)\) coordinates; (ii) interpolated line-integral data obtained at virtual sensors that corresponded to uniform sampling of the Radon space and the employed interpolation method was the 1-D linear (IP1), the 1-D piecewise cubic spline (IP2), the piecewise cubic Hermite (IP3), the bilinear (IP4), the bicubic (IP5) and the 2-D spline (IP6); (iii) uniform sampling (US) of the parameter space using the actual measurements. The location of the source of the electric field was at \((19, -19)\).

We notice from these figures that the cases where we used interpolated measurements obtained at virtual sensors, that corresponded to uniform sampling in the \((\rho, \theta)\) space, outperform the case where reconstruction was based on line-integral data obtained at sensors that were regularly placed in relation to the Cartesian intersection coordinates with the boundary of the reconstruction domain. In addition, the higher the sampling rate of parameters \(\rho\) and \(\theta\), the more accurate the obtained reconstruction. By careful inspection of Figs. 4.2-4.5, we may also see that the interpolation method that led to the most accurate reconstruction was the the 1-D piecewise cubic spline interpolation [8] and [35]. In particular, it was found that the average difference in vector field orientation measured
4.3 An Example: Electric Field Imaging

Figure 4.3: As in Fig. 4.2, but here the location of the source of the electric field was at (-16, 21).

Figure 4.4: As in Fig. 4.2, but here the location of the source of the electric field was at (-21, -12).

in degrees was 34% lower when we employed interpolated data (using the 1-D piecewise cubic spline method) that corresponded to uniform sampling in the Radon domain with \( \Delta \rho = 0.5 \) and \( \Delta \theta = 1.5^\circ \), as opposed to the regular positioning of sensors in the space of the Cartesian intersection coordinates with the boundary of the reconstruction domain, whereas the average error in magnitude was lower by 30%. The reconstructed vector fields for the case where we used interpolated data (1-D piecewise cubic spline method) that corresponded to uniform sampling in the Radon domain with \( \Delta \rho = 0.5 \) and \( \Delta \theta = 1.5^\circ \) are
shown in Fig. 4.6a. For the sake of comparison, Fig. 4.6b depicts the respective theoretical electric fields that were obtained by using directly the governing Coulomb's law.

From Figs. 4.2-4.5, we may also see that, as expected, when uniform sampling of the parameter space was employed, the use of actual measurements resulted in more accurate reconstructions than when interpolated measurements were used. In particular, it was found that for sampling steps $\Delta \rho = 0.5$ and $\Delta \theta = 1.5^\circ$, the case where actual measurements were used led to 8% and 14% lower average angular and magnitude errors, respectively, as opposed to using interpolated measurements and the 1-D piecewise cubic spline method. However, by relying on interpolated measurements, the number of the overall sensors required is about 120 times lower than the respective number when actual measurements and uniform sampling in $(\rho, \theta)$ are used. Alternatively, if it was possible to employ a rotating sensor arrangement, the number of the required sensors by using actual measurements and uniform sampling in $(\rho, \theta)$ would not need to increase. However, the employment of rotating sensor arrangement would result in an inevitable two-hundred-and-fortyfold increase in the total scanning time, when compared with the case of using interpolated measurements obtained by virtual sensors. Hence, the employment of interpolated measurements, that is proposed in this chapter, relies on a much more practical and efficient sensor configuration.
Figure 4.6: Simulation results when the location of the source of the electric field was (from top to bottom) at (19, -19), (-16, 21), (-21, -12), and (24, 14.5): (a) the recovered vector field when reconstruction was based on interpolated line-integral data (1-D piecewise cubic spline method) obtained at virtual sensors that corresponded to uniform sampling of the Radon space with $\Delta \rho = 0.5$ and $\Delta \theta = 1.5^\circ$; (b) the theoretical electric field as computed from Coulomb's law.

4.4 Virtual Sensors and Noise

As mentioned in Section 3.5.4, an important issue when solving inverse problems is the sensitivity of the solution to noise. In this section, we investigate the effects of noise on the use of interpolated measurements obtained at virtual sensors, that correspond to
uniform sampling of the \((\rho, \theta)\) space. In all experiments reported in the previous section, the sensors were placed exactly at the positions we had decided, and the measurement taken by each sensor was exactly the value predicted by Coulomb's law. In a practical system, however, some of the sensor measurements are expected to have inaccuracies and some of the sensors are also expected to be somehow misplaced. To emulate these effects, we considered the following.

(i) A noise value was added to a measurement as a fraction of the true value, with random sign. For example, 2% noise means that the sensor measurement was changed by 2% of the value dictated by Coulomb's law. The change was either incremental or decremental, the choice made at random for each sensor.

(ii) A sensor was moved away from its true position by a fraction of the true position. For example, if according to the theory, a sensor should be placed at position \((x, y)\), and we considered a 2% error, then, the coordinates of this sensor were shifted by 2% the corresponding correct values, with a positive or negative sign chosen at random.

(iii) Both the above errors were considered simultaneously.

We performed four series of experiments by perturbing, by the three types of noise described above, (a) 25% of the sensors; (b) 50% of the sensors; (c) 75% of the sensors; (d) all sensors. In order to evaluate the robustness of the employment of interpolated data, proposed in this chapter, against noise, we examined for each series of experiments the following three cases: (I) when integral data from regularly placed sensors, in relation to the Cartesian intersection coordinates \((x, y)\) with the boundary of the reconstruction domain, were used; (II) when interpolated measurements (1-D piecewise cubic spline method), that corresponded to uniform sampling of the \((\rho, \theta)\) space with \(\Delta \rho = 0.5\) and \(\Delta \theta = 1.5^\circ\), were used and (III) when actual measurements, that corresponded to uniform sampling of the \((\rho, \theta)\) space with \(\Delta \rho = 0.5\) and \(\Delta \theta = 1.5^\circ\), were used. For every noise value (of each noise type, each reconstruction approach and each percentage of perturbed sensors), ten simulations were performed and the average reconstruction errors in relative magnitude and absolute vector field orientation were obtained. The source of the vector field for all the simulations was located at \((19, -19)\).
4.5 Discussion and Conclusions

The results of these experiments are shown in Figs. 4.7-4.10. We observe that the employment of interpolated measurements collected by virtual sensors that correspond to uniform sampling of the \((\rho, \theta)\) space, that is proposed in this chapter, increases the resilience to all three discussed types of noise, when compared with the case of the regular sensor positioning in the space of the Cartesian intersection coordinates with the boundary of the reconstruction domain, discussed in Chapter 3. Another interesting observation that we can make by inspecting Figs. 4.7-4.10, is that, when uniform sampling of the \((\rho, \theta)\) space was employed in a noisy environment, the use of interpolated measurements often provides even higher quality in the reconstruction than by relying on the actual measurements. This phenomenon occurs because the employment of interpolation arrays results in the error (caused by additive noise or misplacement) in one sensor being, somehow, counterbalanced by the possibly correct measurements (or placements) of its neighbour sensors that are also included in the interpolation array.

4.5 Discussion and Conclusions

In this chapter, we employed interpolated boundary data obtained at virtual sensors that corresponded to uniform sampling of the \((\rho, \theta)\) space. The simulation results pointed out that this employment led to about 30% reduction of the reconstruction error, when compared with the case where data from sensors, that were regularly placed in relation to the Cartesian intersection coordinates \((x, y)\) with the boundary of the reconstruction domain, were used. If we had opted to use actual measurements that corresponded to uniform sampling of the \((\rho, \theta)\) space, then, a further 10% decrease in the reconstruction error would have been achieved, but at the expense of an one-hundred-and-twentyfold increase in the required sensors or a two-hundred-and-fortyfold increase in the total scanning time (by employing a rotating sensor configuration).

The adoption of data that are collected at virtual sensors that correspond to uniform sampling in the \((\rho, \theta)\) domain, allows us to use as many line-integral data as we can afford, taking into consideration the computational cost of solving the corresponding system of linear equations. However, most importantly, the increase of the number of the
Figure 4.7: Comparison of the reconstruction performance in noisy environments for the cases (i) when integral data from regularly placed sensors, in relation to the Cartesian intersection coordinates \((x, y)\) with the boundary of the reconstruction domain, were used; (ii) when interpolated (1-D piecewise cubic spline method) measurements, that corresponded to uniform sampling of the \((\rho, \theta)\) space with \(\Delta \rho = 0.5\) and \(\Delta \theta = 1.5^\circ\), were used and (iii) when actual measurements, that corresponded to uniform sampling of the \((\rho, \theta)\) space with \(\Delta \rho = 0.5\) and \(\Delta \theta = 1.5^\circ\), were used: (a) and (b) Errors in vector field orientation and magnitude, respectively, when noise was added to the measurements of 25% of the sensors, as a percentage of the true value. (c) and (d) Errors in vector field orientation and magnitude, respectively, when small perturbations in the sensor positions were added. Position perturbations were a percentage of the true positions. (e) and (f) Errors in vector field orientation and magnitude, respectively, when both sensors’ measurements and positions were changed by a percentage of their true values. In all cases, 25% of the sensors were perturbed.
4.5 Discussion and Conclusions

available line-integral data in such a way, is not limited by neither physical constraints on sensor placement nor total scanning time constraints. Hence, contrary to the case where the uniform sampling in the \((\rho, \theta)\) domain is combined with actual measurements, the employment of interpolated measurements, as proposed in this chapter, achieves reconstruction of higher quality by maintaining, at the same time, a practical and efficient sensor configuration.

Another significant outcome of the study presented in this chapter was that the

Figure 4.8: As in Fig. 4.7, but here 50\% of the sensors were perturbed.
use of interpolated projection data, obtained on virtual sensors, resulted also in improved noise tolerance. This result is of great importance, especially in clinical situations, where dealing with noise is a major issue.

As mentioned above, the employment of interpolated data, that correspond to uniform sampling of the \((\rho, \theta)\) space, offers the possibility to increase the number of the available line-integral data without being limited by physical constraints on sensor placement or total scanning time constraints. To take advantage of this statement, we carried
out a series of experiments in Section 4.3 of this chapter, where the number of the available interpolated line-integral data, that we provided the reconstruction algorithm with, was nearly 3.5 times the respective number used in the base case of Chapter 3. However, it is generally known that generating additional data points through interpolation does not increase the amount of information. Hence, the amount of information contained in the set of interpolated data was the same as in the base case. The explanation for the results presented in this chapter lies in Radon transform theory [9]. In particular, the method we developed is a direct algebraic reconstruction technique that performs inver-
sion of the vectorial Radon transform. According to the theory of Radon transform [9], a necessary requirement to produce reconstruction results of great accuracy, when using discrete approximations of Radon transform, is to have uniform distribution of projection data as functions of the two Radon domain variables, normally designated as the radial and angular coordinates. Hence, employing interpolated data collected at virtual sensors which correspond to uniform sampling of the projection space, as proposed in this chapter, results in feeding our reconstruction algorithm with data that are more favourable towards reconstruction accuracy. However, it must be noted that by increasing further the overall number of the available interpolated measurements, the benefit of achieving uniformity in the \((\rho, \theta)\) space is counterbalanced by the error of the numerical LS method. Also, the time it takes to do the interpolation grows exponentially. Finally, a basic presupposition in order the approach described in this chapter to be effective is to have a smooth variation in the measured data.
Chapter 5

Improved 2-D Vector Field Reconstruction using Probabilistic Weights

5.1 Introduction and Motivation

The reconstruction method, that was presented in Chapter 3, is a direct algebraic reconstruction technique. This technique treats the discretised available measurements as bounded linear functionals on the space of two-integrable functions in the reconstruction region. Hence, the 2-D vector field reconstruction problem is cast as the solution of a system of linear equations, where the unknowns of the system are the Cartesian components of the examined vector field in specific sampling points, finite in number and arranged in a grid, of the 2-D reconstruction region. However, there is a duality between this matrix formalism and the vectorial Radon transform scheme. Hence, solving the system of linear equations, obtained by following the methodology of Chapter 3, is equivalent to inverting the vectorial Radon transform.

The motivation for this chapter is similar to the one of Chapter 4 and lies in Radon transform theory [9]. As mentioned in Chapter 4, a necessary requirement to produce reconstruction results with the accuracy desired in medical imaging is to sample
uniformly the Radon domain parameter space, defined by the length $\rho$ of the normal to a scanning line and the anticlockwise angle $\theta$ this normal forms with the positive $x$ semi-axis (see Fig. 2.1). The scanning geometry, that was employed in Chapter 3, assumed that the measurements were collected by sensors that followed uniform distribution in the space of the Cartesian intersection coordinates with the boundary of the reconstruction domain. Such a sensor placement might be the most practical, however, it does not result in scanning lines that follow uniform distribution in the $(\rho, \theta)$ projection space. On the other hand, sampling the Radon parameter domain uniformly imposes serious constraints of space or time that were discussed in Chapter 4.

In this chapter, we achieve approximate uniformity of sampling in the $(\rho, \theta)$ projection space by employing weights. These weights are obtained by relying on random variables theory and calculating the resulting joint probability density function of $\rho$ and $\theta$ that the practical sensor arrangement of Chapter 3 generates. Hence, the proposed modification to the direct algebraic reconstruction technique, presented in Chapter 3, accounts for the non-uniform density of the projection space by inversely weighing every equation (line-integral measurement) according to the local $(\rho, \theta)$ density of the scanning line associated with this equation, and, also, multiplying with the (uniform) probability mass that the pair $(\rho, \theta)$ should have. It must be noted that, due to the fact that the calculation of the proposed weights is based on the known and predetermined sensor arrangement, this calculation can be performed in advance (off-line).

This chapter is structured as follows. In Section 5.2, we formulate the problem and set up our notation. In Section 5.3, we work out the weights that should be employed in the reconstruction process, so as to approximate flatness in the $(\rho, \theta)$ density of the scanning lines. In Section 5.4, we present an example of static electric field reconstruction to demonstrate the effect of the proposed probabilistic weights on the quality of the reconstruction by presenting an example of static electric field reconstruction. We conclude in Section 5.5.
5.2 Problem Formulation

The whole treatment in this section is performed in the digital domain. Let us assume that we have the digitised square 2-D domain, shown in Fig. 5.1, within which vector field $\mathbf{f}(x, y) = f_x(x, y)\hat{x} + f_y(x, y)\hat{y}$ is defined. The length of each side of the square domain is taken to be equal to $2U$ and the origin of the axes of the coordinate system is chosen to be at the centre of the domain. The square domain is divided into tiles of finite size, $P \times P$, so that $K = 2U/P$ is an integer. The goal is to recover vector field $\mathbf{f}(x, y)$ at the centre of every tile of this space, namely the sampling points of the domain.

![Figure 5.1: The digitised reconstruction region is a square of size $2U$. The size of the tiles, with which we sample the 2-D space, is $P \times P$. Marked with $\circ$ are the known and predetermined sensor positions from which we obtain the line-integral data. These positions are the middle points of the boundary edges of all boundary tiles. A scanning line segment $AB$ is sampled with sampling step $\Delta s$. $AB$ is inclined at an angle $\theta$ to the positive direction of the $x$-axis. Also shown are the two parameters $\rho$ and $\theta$ used to define the scanning line (projection space coordinates) and the unit vectors $\hat{s}$ and $\hat{p}$ which are parallel and perpendicular, respectively, to line segment $AB$.](image)

Regarding the data acquisition, we assume that ideal point sensors, that integrate only the component of the field projected on the line, reside on predetermined and regularly placed positions of the whole border of the 2-D square domain. These positions are the middle points of the boundary edges of all boundary tiles (see Fig. 5.1). Hence, there are $\frac{2U}{P}$ ideal point sensors on each side of the boundary of the domain. The solution to the reconstruction problem is based on projection data along lines defined by the finite number of measurement points. Next, as a part of the problem formulation, we present
a short summary of the direct algebraic reconstruction technique, that was discussed in Section 3.4.

The employed sensor arrangement of Fig. 5.1 yields a set of scanning lines. Let us consider a scanning line $AB$ that belongs to this set and connects two boundary sensors located at points $A$ and $B$, chosen arbitrarily (see Fig. 5.1). Then, scanning line $AB$ yields a line-integral measurement $J_i$. Since we assumed that a pair of sensors measures only the integral of the component of the investigated vector field along the scanning line, then, the integral-geometry transform that models the measurement is given by

$$J_i = \int_{AB} \tilde{f}(x, y) \cdot \hat{s} \, ds \quad (5.1)$$

Here $\hat{s} = \cos w\hat{x} + \sin w\hat{y}$ is the unit vector along the integration (measurement) line $AB$, where $w$ is the angle at which the scanning line is inclined to the positive direction of the $x$-axis (see Fig. 5.1). In addition, $ds$ is an element of path length along this line.

In order to translate into the digital domain, the integration expressed by Eq. (5.1) in the continuous domain, the analysis of Section 3.4, is applied. This analysis also involves a sampling along the line segment with sampling step $\Delta s$ (see Fig. 5.1). Then, the integral of Eq. (5.1) can be approximated by the following sum:

$$J_i = \sum_l \tilde{f}_l \cdot \Delta s \quad (5.2)$$

Here $\tilde{f}_l = (f_{xl}, f_{yl})$ are the unknown vector field values at these sampling points of the reconstruction domain that are nearest neighbours to the sampling points $l$ of the line. Also, $\Delta s = \Delta s\hat{s}$.

In order to obtain the system of linear equations, the solution of which will give the components of the examined vector field $f(x, y)$ at all sampling points of the 2-D domain, the procedure described above is repeated for all possible pairs of boundary point sensors, that yield integral measurements along scanning lines, apart from pairs of sensors where both sensors reside in the same side of the boundary of the square and are not useful. Hence, the number of the available equations (5.2) depends on the selection of the data...
acquisition geometry. In general, in this analysis we have to deal with overdetermined systems of linear equations. Hence, the solution is obtained in the LS error sense.

In Eq. (5.2), $J_i$ is the measurement obtained by integrating along scanning line $AB$. This line is defined, in terms of projection space parameters $\rho$ and $\theta$, by using the Hessian normal form of Eq. (2.1), that we repeat here as Eq. (5.3) for the sake of convenience:

$$\rho = x \cos \theta + y \sin \theta \quad (5.3)$$

Parameters $\rho$ and $\theta$ have been defined in Fig. 2.1 and, also, it is $\rho \geq 0$ and $-\pi < \theta \leq \pi$. Hence, associated with each line-integral measurement, is a pair of $(\rho, \theta)$ parameter values. However, it is important to note that by placing the sensors uniformly distributed in the space of the Cartesian intersection coordinates with the boundary of the reconstruction domain, as proposed in Section 3.4, the joint distribution of $\rho$ and $\theta$ parameters of the resulting scanning lines is not uniform. According to the theory of Radon transform [9], failure to achieve uniformity in the projection space parameters results in loss of accuracy in the reconstruction results.

In this chapter, we propose to achieve approximate uniformity in the Radon domain parameters of the scanning lines by employing weights. In particular, to account for the non-uniform $(\rho, \theta)$ density of the set of scanning lines, every equation obtained with the analysis described above is inversely weighed, according to the local $(\rho, \theta)$ probability mass of the scanning line associated with this equation, and multiplied with the (uniform) probability that the pair $(\rho, \theta)$ should have. In the next section, we work out the weights that should be employed in the reconstruction process, so as to approximate flatness in the $(\rho, \theta)$ density of the set of scanning lines.

5.3 The Weighted Reconstruction Methodology

In this chapter, we propose to modify the direct algebraic reconstruction technique, that we introduced in Section 3.4, by employing probabilistic weights. These weights should be multiplied with the system's equations, obtained following the analysis in Sec-
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tion 3.4, so as to account for the non-uniformity in the \((\rho, \theta)\) space. Before we calculate the proposed weights, we first make an attempt to answer the following question: Given the employed scanning geometry of Fig. 5.1, where the sensors are uniformly distributed in the space of the Cartesian intersection coordinates with the boundary of the reconstruction domain, what is the \((\rho, \theta)\) distribution of the resulting scanning lines?

Let us call \((x_1, y_1)\) and \((x_2, y_2)\) the end points of an arbitrary scanning line segment that goes through the reconstruction region of Fig. 5.1. Our first task is to express scanning line parameters, \(\rho\) and \(\theta\), in terms of the intersection parameters (sensor Cartesian coordinates). We, then, go on to work out the joint distribution of parameters \(\rho\) and \(\theta\).

Intersection parameters \(x_1, y_1, x_2\) and \(y_2\) are not independent, as they are constrained to refer to points on the domain border. For this reason, we have the following possibilities for a scanning line.

1. A scanning line where the two sensors lie on the domain borders \(y = -U\) and \(x = U\) (Fig. 5.2). The coordinates of the two sensors are \((x_1, -U)\) and \((U, y_2)\). Both sensors lie on the scanning line \((\rho, \theta)\). Hence, Eq. (5.3) yields:

\[
\rho = x_1 \cos \theta - U \sin \theta \tag{5.4}
\]

\[
\rho = U \cos \theta + y_2 \sin \theta \tag{5.5}
\]

In order to determine the joint probability density function of parameters \(\rho\) and \(\theta\), the Cartesian sensor coordinates \(x_1\) and \(y_2\) are treated as random variables. By making
the assumption that there are infinite many sensors\(^1\), then, these sensor coordinates may take any value in the range \((-U, U)\) with the same probability. Hence, \(x_1\) and \(y_2\) are uniformly distributed random variables with corresponding density functions, respectively:

\[
\begin{align*}
    f_{x_1}(x_1) &= \frac{1}{2U} [H(x_1 + U) - H(x_1 - U)] \\
    f_{y_2}(y_2) &= \frac{1}{2U} [H(y_2 + U) - H(y_2 - U)]
\end{align*}
\]

(5.6)

(5.7)

In the above formulae, \(H(\ )\) is the Heaviside step function, the value of which is zero for a negative argument and one for a positive argument.

Since the value of the coordinate \(x_1\) is independent of the value of the coordinate \(y_2\), these two variables are statistically independent. Hence, the joint probability density function of \(x_1\) and \(y_2\), \(f_{x_1y_2}(x_1, y_2)\), is given by:

\[
f_{x_1y_2}(x_1, y_2) = f_{x_1}(x_1)f_{y_2}(y_2) \Rightarrow f_{x_1y_2}(x_1, y_2) = \frac{1}{4U^2} [H(x_1 + U) - H(x_1 - U)][H(y_2 + U) - H(y_2 - U)]
\]

(5.8)

From Eqs. (5.4) and (5.5) it follows that:

\[
\begin{align*}
    \theta &= \arctan \frac{x_1 - U}{y_2 + U} \\
    \rho &= x_1 \cos \left( \arctan \frac{x_1 - U}{y_2 + U} \right) - U \sin \left( \arctan \frac{x_1 - U}{y_2 + U} \right)
\end{align*}
\]

(5.9)

(5.10)

In this chapter, we restrict the inverse function \(\arctan\) to take only its principal values. That is, values in the range \((-\frac{\pi}{2}, \frac{\pi}{2})\).

The following fundamental theorem is valid [51]: If \(\rho\) and \(\theta\) are two functions of two random variables \(x_1\) and \(y_2\)

\[
\begin{align*}
    \rho &= \rho(x_1, y_2) \\
    \theta &= \theta(x_1, y_2)
\end{align*}
\]

(5.11)

(5.12)

---

\(^1\)We make this assumption in order to reduce the computational complexity.
then, we may express the joint probability density function of $\rho$ and $\theta$, $f_{\rho\theta}(\rho, \theta)$, in terms of the joint probability density function of $x_1$ and $y_2$, $f_{x_1y_2}(x_1, y_2)$, as:

$$f_{\rho\theta}(\rho, \theta) = \frac{f_{x_1y_2}(x_{1a}, y_{2a})}{|\mathcal{J}(x_{1a}, y_{2a})|} + \cdots + \frac{f_{x_1y_2}(x_{1k}, y_{2k})}{|\mathcal{J}(x_{1k}, y_{2k})|}$$  (5.13)

where

$$\mathcal{J}(x_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}^{-1}$$  (5.14)

is the Jacobian determinant of the transformation of Eqs. (5.11) and (5.12), and $(x_{1a}, y_{2a}), \ldots, (x_{1k}, y_{2k})$ are the $k$ real roots of the system of the same equations.

For the considered set of scanning lines, we have $\rho > 0$ and $\theta \in (-\frac{\pi}{2}, 0)$. Hence, the system of Eqs. (5.9) and (5.10) has a single solution:

$$(x_{1a}, y_{2a}) = \left( \frac{\rho}{\cos \theta} + U \tan \theta, \frac{\rho}{\sin \theta} - U \cot \theta \right)$$  (5.15)

This solution is obtained by solving Eqs. (5.4) and (5.5), with respect to $x_1$ and $y_2$, and also taking into account that it is $\cos \theta \neq 0$ and $\sin \theta \neq 0$ for the examined set of scanning lines. Considering Eq. (5.15), we obtain from Eq. (5.14) that

$$\mathcal{J}(x_{1a}, y_{2a}) = \begin{vmatrix} \frac{1}{\cos \theta} & \frac{\rho \sin \theta}{\cos^2 \theta} + \frac{U}{\cos \theta} \\ \frac{1}{\sin \theta} & -\frac{\rho \cos \theta}{\sin^2 \theta} + \frac{U}{\sin \theta} \end{vmatrix}^{-1} = \left[ \frac{U - \rho \cos \theta}{\cos \theta \sin^2 \theta} - \frac{U + \rho \sin \theta}{\sin \theta \cos^2 \theta} \right]^{-1}$$  (5.16)

For the examined scanning lines $(\rho, \theta)$, it is $U(\cos \theta - \sin \theta) - \rho \neq 0$. Hence, Eq. (5.16) yields:

$$\mathcal{J}(x_{1a}, y_{2a}) = \frac{\cos^2 \theta \sin^2 \theta}{U(\cos \theta - \sin \theta) - \rho}$$  (5.17)

Taking into account Eqs. (5.15) and (5.17), we conclude from Eq. (5.13) that the joint probability density function of $\rho$ and $\theta$ for the $1^{st}$ set of scanning lines is given
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by:

\[ f^1_{\rho\theta}(\rho, \theta) = \left| \frac{U(\cos \theta - \sin \theta) - \rho}{\cos^2 \theta \sin^2 \theta} \right| f_{x_1y_2} \left( \frac{\rho}{\cos \theta} + U \tan \theta, \frac{\rho}{\sin \theta} - U \cot \theta \right) \]  

(5.18)

Finally, by substituting Eq. (5.8) into Eq. (5.18), we obtain that:

\[ f^1_{\rho\theta}(\rho, \theta) = \frac{U(\cos \theta - \sin \theta) - \rho}{4U^2} \left[ H \left( \frac{\rho}{\cos \theta} + U \tan \theta + U \right) - H \left( \frac{\rho}{\cos \theta} + U \tan \theta - U \right) \right] \times \]

\[ \left[ H \left( \frac{\rho}{\sin \theta} - U \cot \theta + U \right) - H \left( \frac{\rho}{\sin \theta} - U \cot \theta - U \right) \right] \]  

(5.19)

2. A scanning line where the two sensors lie on the domain borders \( y = -U \) and \( x = -U \) (Fig. 5.3). The coordinates of the two sensors are \((x_1, -U)\) and \((-U, y_2)\).

![Figure 5.3: The 2nd case of scanning lines.](image)

Both sensors lie on the scanning line \((\rho, \theta)\). Hence, Eq. (5.3) yields:

\[ \rho = x_1 \cos \theta - U \sin \theta \]  

(5.20)

\[ \rho = -U \cos \theta + y_2 \sin \theta \]  

(5.21)

Following the same line of thinking as above, the probability density functions of random variables \(x_1\) and \(y_2\) are given by Eqs. (5.6) and (5.7), respectively. Moreover, the joint probability density function of \(x_1\) and \(y_2\), \(f_{x_1y_2}(x_1, y_2)\), is determined by
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Eq. (5.8). From Eqs. (5.20) and (5.21) it follows that

$$\begin{align*}
\theta &= \arctan \frac{x_1 + U}{y_2 + U} - \pi \\
\rho &= x_1 \cos \left( \arctan \frac{x_1 + U}{y_2 + U} - \pi \right) - U \sin \left( \arctan \frac{x_1 + U}{y_2 + U} - \pi \right)
\end{align*}$$

(5.22)

(5.23)

where it was taken into account that, for the considered set of scanning lines, we have $\rho > 0$ and $\theta \in (-\pi, -\frac{\pi}{2})$. The system of Eqs. (5.22) and (5.23) has a single solution:

$$(x_{1a}, y_{2a}) = \left( \frac{\rho}{\cos \theta} + U \tan \theta, \frac{\rho}{\sin \theta} + U \cot \theta \right)$$

(5.24)

This solution is obtained by solving Eqs. (5.20) and (5.21), with respect to $x_1$ and $y_2$, and also taking into account that it is $\cos \theta \neq 0$ and $\sin \theta \neq 0$ for the examined set of lines. Considering Eq. (5.24), we obtain from Eq. (5.14) that

$$\begin{align*}
\tilde{J}(x_{1a}, y_{2a}) &= \begin{vmatrix}
\frac{1}{\cos \theta} & \frac{\rho \sin \theta + U}{\cos^2 \theta} \\
\frac{1}{\sin \theta} & \frac{-\rho \cos \theta + U}{\sin^2 \theta}
\end{vmatrix}^{-1} \\
&= \begin{vmatrix}
-U - \rho \cos \theta & U + \rho \sin \theta \\
\cos \theta \sin \theta & \sin \theta \cos \theta
\end{vmatrix}^{-1} \\
&= \begin{vmatrix}
-U(\cos \theta + \sin \theta) - \rho
\end{vmatrix}^{-1}
\end{align*}$$

(5.25)

For the examined scanning lines $(\rho, \theta)$, it is $-U(\cos \theta + \sin \theta) - \rho \neq 0$. Hence, Eq. (5.25) yields:

$$\tilde{J}(x_{1a}, y_{2a}) = \frac{\cos^2 \theta \sin^2 \theta}{-U(\cos \theta + \sin \theta) - \rho}$$

(5.26)

Taking into account Eqs. (5.24) and (5.26), we conclude from Eq. (5.13) that the joint probability density function of $\rho$ and $\theta$ for the $2^{nd}$ set of scanning lines is given by:

$$f^2_{\rho \theta}(\rho, \theta) = \left| \frac{U(\cos \theta + \sin \theta) + \rho}{\cos^2 \theta \sin^2 \theta} \right| f_{x_1y_2} \left( \frac{\rho}{\cos \theta} + U \tan \theta, \frac{\rho}{\sin \theta} + U \cot \theta \right)$$

(5.27)
Finally, by substituting Eq. (5.8) into Eq. (5.27), we obtain that:

\[
f_{\rho\theta}^2(\rho, \theta) = \frac{U(\cos \theta + \sin \theta) + \rho}{4U^2} \left[ H \left( \frac{\rho}{\cos \theta} + U \tan \theta + U \right) - H \left( \frac{\rho}{\cos \theta} + U \tan \theta - U \right) \right] \times \left[ H \left( \frac{\rho}{\sin \theta} + U \cot \theta + U \right) - H \left( \frac{\rho}{\sin \theta} + U \cot \theta - U \right) \right]
\]  

(5.28)

3. A scanning line where the two sensors lie on the domain borders \( x = U \) and \( y = U \) (Fig. 5.4). The coordinates of the two sensors are \((U, Y_2)\) and \((X_1, U)\). Both sensors lie on the scanning line \((\rho, \theta)\). Hence, Eq. (5.3) yields:

\[
\rho = U \cos \theta + y_2 \sin \theta \quad (5.29)
\]

\[
\rho = x_1 \cos \theta + U \sin \theta \quad (5.30)
\]

The probability density functions of random variables \(x_1\) and \(y_2\) and their joint probability density function, \(f_{x_1y_2}(x_1, y_2)\), are determined by Eqs. (5.6)-(5.8), respectively, the same as in the previous two cases. From Eqs. (5.29) and (5.30) it follows that:

\[
\theta = \arctan \frac{x_1 - U}{y_2 - U} \quad (5.31)
\]

\[
\rho = U \cos \left( \arctan \frac{x_1 - U}{y_2 - U} \right) + y_2 \sin \left( \arctan \frac{x_1 - U}{y_2 - U} \right) \quad (5.32)
\]

For the considered set of scanning lines, we have \(\rho > 0\) and \(\theta \in (0, \frac{\pi}{2})\). The system
of Eqs. (5.31) and (5.32) has a single solution:

\[ (x_{1a}, y_{2a}) = \left( \frac{\rho}{\cos \theta} - U \tan \theta, \frac{\rho}{\sin \theta} - U \cot \theta \right) \]  \hspace{1cm} (5.33)

This solution is obtained by solving Eqs. (5.29) and (5.30), with respect to \( x_1 \) and \( y_2 \), and also taking into account that it is \( \cos \theta \neq 0 \) and \( \sin \theta \neq 0 \) for the examined set of lines. Considering Eq. (5.33), we obtain from Eq. (5.14) that

\[ \begin{bmatrix} \frac{1}{\cos \theta} & \frac{\rho \sin \theta}{\cos^2 \theta} - \frac{U}{\cos \theta} \\ \frac{1}{\sin \theta} & -\frac{\rho \cos \theta}{\sin^2 \theta} + \frac{U}{\sin \theta} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\rho \sin \theta}{\cos^2 \theta} - \frac{U}{\cos \theta} \\ \frac{\rho \cos \theta}{\sin^2 \theta} - \frac{U}{\sin \theta} \end{bmatrix}^{-1} = \begin{bmatrix} U - \rho \cos \theta \cos^2 \theta - \rho \sin \theta - U \\ \cos \theta \sin^2 \theta \end{bmatrix}^{-1} \]

\[ \bar{J}(x_{1a}, y_{2a}) = \frac{\cos^2 \theta \sin^2 \theta}{U(\cos \theta + \sin \theta) - \rho} \]  \hspace{1cm} (5.35)

For the examined scanning lines \((\rho, \theta)\), it is \( U(\cos \theta + \sin \theta) - \rho \neq 0 \). Hence, Eq. (5.34) yields:

\[ f_\rho^3(\rho, \theta) = \frac{U(\cos \theta + \sin \theta) - \rho}{\cos^2 \theta \sin^2 \theta} \left( \frac{\rho}{\cos \theta} - U \tan \theta, \frac{\rho}{\sin \theta} - U \cot \theta \right) \]  \hspace{1cm} (5.36)

Finally, by substituting Eq. (5.8) into Eq. (5.36), we obtain that:

\[ f_\rho^3(\rho, \theta) = \frac{U(\cos \theta + \sin \theta) - \rho}{4U^2} \left[ H \left( \frac{\rho}{\cos \theta} - U \tan \theta + U \right) - H \left( \frac{\rho}{\cos \theta} - U \tan \theta - U \right) \right] \times \left[ H \left( \frac{\rho}{\sin \theta} - U \cot \theta + U \right) - H \left( \frac{\rho}{\sin \theta} - U \cot \theta - U \right) \right] \]  \hspace{1cm} (5.37)

4. A scanning line where the two sensors lie on the domain borders \( y = U \) and \( x = -U \) (Fig. 5.5). The coordinates of the two sensors are \((x_1, U)\) and \((-U, y_2)\). Both sensors
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Figure 5.5: The 4th case of scanning lines.

lie on the scanning line \((\rho, \theta)\). Hence, Eq. (5.3) yields:

\[
\rho = x_1 \cos \theta + U \sin \theta \tag{5.38}
\]

\[
\rho = -U \cos \theta + y_2 \sin \theta \tag{5.39}
\]

The probability density functions of random variables \(x_1\) and \(y_2\) and their joint probability density function, \(f_{x_1y_2}(x_1, y_2)\), are determined by Eqs. (5.6)-(5.8), respectively, the same as in the previous three cases. From Eqs. (5.38) and (5.39) it follows that

\[
\theta = \arctan \frac{x_1 + U}{y_2 - U} + \pi \tag{5.40}
\]

\[
\rho = x_1 \cos \left( \arctan \frac{x_1 + U}{y_2 - U} + \pi \right) + U \sin \left( \arctan \frac{x_1 + U}{y_2 - U} + \pi \right) \tag{5.41}
\]

where it was taken into account that, for the considered set of scanning lines, we have \(\rho > 0\) and \(\theta \in \left(\frac{\pi}{2}, \pi\right)\). The system of Eqs. (5.40) and (5.41) has a single solution:

\[
(x_{1a}, y_{2a}) = \left( \frac{\rho}{\cos \theta} - U \tan \theta, \frac{\rho}{\sin \theta} + U \cot \theta \right) \tag{5.42}
\]

This solution is obtained by solving Eqs. (5.38) and (5.39), with respect to \(x_1\) and \(y_2\), and also taking into account that it is \(\cos \theta \neq 0\) and \(\sin \theta \neq 0\) for the examined
set of lines. Considering Eq. (5.42), we obtain from Eq. (5.14) that

$$\tilde{J}(x_{1a}, y_{2a}) = \begin{vmatrix} \frac{1}{\cos \theta} & \frac{\rho \sin \theta}{\cos^2 \theta} - \frac{U}{\cos \theta} \\ \frac{1}{\sin \theta} & -\frac{\rho \cos \theta}{\sin^2 \theta} - \frac{U}{\sin \theta} \end{vmatrix}^{-1} = \left[ \frac{-U - \rho \cos \theta}{\cos \theta \sin^2 \theta} - \frac{\rho \sin \theta - U}{\sin \theta \cos^2 \theta} \right]^{-1}$$

$$= \left[ \frac{U(\sin \theta - \cos \theta) - \rho}{\cos^2 \theta \sin^2 \theta} \right]^{-1} \quad (5.43)$$

For the examined scanning lines \((\rho, \theta)\), it is \(U(\sin \theta - \cos \theta) - \rho \neq 0\). Hence, Eq. (5.43) yields:

$$\tilde{J}(x_{1a}, y_{2a}) = \frac{\cos^2 \theta \sin^2 \theta}{U(\sin \theta - \cos \theta) - \rho} \quad (5.44)$$

Taking into account Eqs. (5.42) and (5.44), we conclude from Eq. (5.13) that the joint probability density function of \(\rho\) and \(\theta\) for the 4th set of scanning lines is given by:

$$f_{\rho\theta}^{4}(\rho, \theta) = \frac{U(\sin \theta - \cos \theta) - \rho}{\cos^2 \theta \sin^2 \theta} f_{x_{1y_{2}}} \left( \frac{\rho}{\cos \theta} - U \tan \theta, \frac{\rho}{\sin \theta} + U \cot \theta \right) \quad (5.45)$$

Finally, by substituting Eq. (5.8) into Eq. (5.45), we obtain that:

$$f_{\rho\theta}^{4}(\rho, \theta) = \frac{U(\sin \theta - \cos \theta) - \rho}{4U^2} \left[ H \left( \frac{\rho}{\cos \theta} - U \tan \theta + U \right) - H \left( \frac{\rho}{\cos \theta} - U \tan \theta - U \right) \right] \times \left[ H \left( \frac{\rho}{\sin \theta} + U \cot \theta + U \right) - H \left( \frac{\rho}{\sin \theta} + U \cot \theta - U \right) \right] \quad (5.46)$$

5. A scanning line where the two sensors lie on the domain borders \(y = -U\) and \(y = U\) (Fig. 5.6). The coordinates of the two sensors are \((x_1, -U)\) and \((x_2, U)\). Both sensors lie on the scanning line \((\rho, \theta)\). Hence, Eq. (5.3) yields:

$$\rho = x_1 \cos \theta - U \sin \theta \quad (5.47)$$

$$\rho = x_2 \cos \theta + U \sin \theta \quad (5.48)$$

Sensor coordinates \(x_1\) and \(x_2\) are treated as uniformly distributed random variables in the range \((-U, U)\). Hence, the probability density function of \(x_1\) is determined
by Eq. (5.6), whereas the corresponding density function of random variable $x_2$ is given by:

$$f_{x_2}(x_2) = \frac{1}{2U} [H(x_2 + U) - H(x_2 - U)] \quad (5.49)$$

The two random variables are statistically independent. Hence, their joint probability density function is:

$$f_{x_1x_2}(x_1, x_2) = f_{x_1}(x_1) f_{x_2}(x_2) \Rightarrow$$

$$f_{x_1x_2}(x_1, x_2) = \frac{1}{4U^2} [H(x_1 + U) - H(x_1 - U)] [H(x_2 + U) - H(x_2 - U)] \quad (5.50)$$

For the examined set of scanning lines, we have $ho \geq 0$ and $	heta \in (-\pi, -\frac{3\pi}{4}) \cup (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$. The following cases have to be distinguished:

- When $\theta \in (-\pi, -\frac{3\pi}{4})$, it follows from Eqs. (5.47) and (5.48) that:

  $$\theta = \arctan \frac{x_1 - x_2}{2U} - \pi$$

  $$\rho = x_1 \cos \left( \arctan \frac{x_1 - x_2}{2U} - \pi \right) - U \sin \left( \arctan \frac{x_1 - x_2}{2U} - \pi \right) \quad (5.51)\quad (5.52)$$

- When $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, it follows from Eqs. (5.47) and (5.48) that:

  $$\theta = \arctan \frac{x_1 - x_2}{2U}$$

  $$\rho = x_1 \cos \left( \arctan \frac{x_1 - x_2}{2U} \right) - U \sin \left( \arctan \frac{x_1 - x_2}{2U} \right) \quad (5.53)\quad (5.54)$$
5.3 The Weighted Reconstruction Methodology

- When \( \theta \in \left( \frac{3\pi}{4}, \pi \right] \), we get from Eqs. (5.47) and (5.48):

\[
\theta = \arctan \frac{x_1 - x_2}{2U} + \pi \quad (5.55)
\]

\[
\rho = x_1 \cos \left( \arctan \frac{x_1 - x_2}{2U} + \pi \right) - U \sin \left( \arctan \frac{x_1 - x_2}{2U} + \pi \right) \quad (5.56)
\]

The three \( 2 \times 2 \) systems of Eqs. (5.51)-(5.56) have all a single solution in the corresponding intervals of variable \( \theta \). This solution is given for all three systems by the same formula

\[
(x_{1a}, x_{2a}) = \left( -\frac{\rho}{\cos \theta} + U \tan \theta, \frac{\rho}{\cos \theta} - U \tan \theta \right) \quad (5.57)
\]

It is obtained by solving Eqs. (5.47) and (5.48), with respect to \( x_1 \) and \( x_2 \), and also by taking into account that it is \( \cos \theta \neq 0 \) for the examined set of scanning lines. As a result of this and the theorem, expressed by Eqs. (5.11)-(5.14), the formula that determines the joint density \( f_{\rho \theta}^{5}(\rho, \theta) \) for this set of scanning lines will be common for all intervals of variable \( \theta \). Considering Eq. (5.57), the application of the theorem of Eqs. (5.11)-(5.14), for random variables \( x_1 \) and \( x_2 \) and all three systems of Eqs. (5.51)-(5.56), yields:

\[
\tilde{J}(x_{1a}, x_{2a}) = \left| \begin{array}{cc}
\frac{\partial \rho}{\partial x_{1a}} & \frac{\partial \rho}{\partial x_{2a}} \\
\frac{\partial \theta}{\partial x_{1a}} & \frac{\partial \theta}{\partial x_{2a}}
\end{array} \right|^{-1} = \left| \begin{array}{cc}
\frac{\partial x_{1a}}{\partial \rho} & \frac{\partial x_{1a}}{\partial \theta} \\
\frac{\partial x_{2a}}{\partial \rho} & \frac{\partial x_{2a}}{\partial \theta}
\end{array} \right|^{-1}
\]

\[
= \left| \begin{array}{cc}
\frac{1}{\cos \theta} & \frac{\rho \sin \theta}{\cos^2 \theta} + \frac{U}{\cos^2 \theta} \\
\frac{1}{\cos \theta} & \frac{\rho \sin \theta}{\cos^2 \theta} - \frac{U}{\cos^2 \theta}
\end{array} \right|^{-1} = \left[ \frac{\rho \sin \theta - U}{\cos^3 \theta} - \frac{\rho \sin \theta + U}{\cos^3 \theta} \right]^{-1}
\]

\[
= \left[ -\frac{2U}{\cos^3 \theta} \right]^{-1} = -\frac{\cos^3 \theta}{2U} \quad (5.58)
\]

Taking into account Eqs. (5.57) and (5.58), we conclude from the theorem that the joint probability density function of \( \rho \) and \( \theta \) for the 5\textsuperscript{th} set of scanning lines is given by:

\[
f_{\rho \theta}^{5}(\rho, \theta) = \left| \frac{2U}{\cos^3 \theta} \right| f_{x_1x_2} \left( \frac{\rho}{\cos \theta} + U \tan \theta, \frac{\rho}{\cos \theta} - U \tan \theta \right) \quad (5.59)
\]
Finally, by substituting Eq. (5.50) into Eq. (5.59), we obtain that:

\[ f^5_{\rho \theta} (\rho, \theta) = \frac{2U}{4U^2} \left[ H \left( \frac{\rho}{\cos \theta} + U \tan \theta + U \right) - H \left( \frac{\rho}{\cos \theta} + U \tan \theta - U \right) \right] \times \left[ H \left( \frac{\rho}{\cos \theta} - U \tan \theta + U \right) - H \left( \frac{\rho}{\cos \theta} - U \tan \theta - U \right) \right] \]  

(5.60)

6. A scanning line where the two sensors lie on the domain borders \( x = U \) and \( x = -U \) (Fig. 5.7). The coordinates of the two sensors are \((U, y_1)\) and \((-U, y_2)\). Both sensors lie on the scanning line \((\rho, 0)\). Hence, Eq. (5.3) yields:

\[ \rho = U \cos \theta + y_1 \sin \theta \]  

(5.61)

\[ \rho = -U \cos \theta + y_2 \sin \theta \]  

(5.62)

Sensor coordinates \( y_1 \) and \( y_2 \) are treated as uniformly distributed random variables in the range \((-U, U)\). Hence, the probability density function of \( y_2 \) is determined by Eq. (5.7), whereas the corresponding density function of random variable \( y_1 \) is given by:

\[ f_{y_1} (y_1) = \frac{1}{2U} [H(y_1 + U) - H(y_1 - U)] \]  

(5.63)

The two random variables are statistically independent. Hence, their joint probabil-
5.3 The Weighted Reconstruction Methodology

The joint probability density function is:

\[ f_{y_1y_2}(y_1, y_2) = f_{y_1}(y_1)f_{y_2}(y_2) \Rightarrow \]

\[ f_{y_1y_2}(y_1, y_2) = \frac{1}{4U^2} [H(y_1 + U) - H(y_1 - U)] [H(y_2 + U) - H(y_2 - U)] \quad (5.64) \]

For the examined set of scanning lines, we have \( \rho \geq 0 \) and \( \theta \in (-\frac{3\pi}{4}, -\frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{3\pi}{4}) \).

The following cases have to be distinguished:

- When \((-\frac{3\pi}{4}, -\frac{\pi}{4})\), it follows from Eqs. (5.61) and (5.62) that:

  \[
  \theta = \arccot \frac{y_2 - y_1}{2U} - \pi \\
  \rho = U \cos \left( \arccot \frac{y_2 - y_1}{2U} - \pi \right) + y_1 \sin \left( \arccot \frac{y_2 - y_1}{2U} - \pi \right) 
  \]

  (5.65) \quad (5.66)

  In this chapter, we restrict the inverse function \( \arccot \) to take only its principal values. That is, values in the range \((0, \pi)\).

- When \(\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})\), we obtain from Eqs. (5.61) and (5.62):

  \[
  \theta = \arccot \frac{y_2 - y_1}{2U} \\
  \rho = U \cos \left( \arccot \frac{y_2 - y_1}{2U} \right) + y_1 \sin \left( \arccot \frac{y_2 - y_1}{2U} \right) 
  \]

  (5.67) \quad (5.68)

The two \(2 \times 2\) systems of Eqs. (5.65)-(5.68) have both a single solution in the corresponding intervals of variable \( \theta \). This solution is given for both systems by the same formula

\[
(y_{1_{a}}, y_{2_{a}}) = \left( \frac{\rho}{\sin \theta} - U \cot \theta, \frac{\rho}{\sin \theta} + U \cot \theta \right) 
\]

(5.69)

It is obtained by solving Eqs. (5.61) and (5.62), with respect to \( y_1 \) and \( y_2 \), and also by taking into account that it is \( \sin \theta \neq 0 \) for the examined set of scanning lines. As a result of this and the theorem, expressed by Eqs. (5.11)-(5.14), the formula that determines the joint probability density function \( f_{y_1y_2}(\rho, \theta) \) for this set of scanning lines is common for all intervals of variable \( \theta \). Considering Eq. (5.69), the application of the theorem of Eqs. (5.11)-(5.14), for random variables \( y_1 \) and \( y_2 \)
and the two systems of Eqs. (5.65)-(5.68), yields:

\[
\begin{vmatrix}
\frac{\partial \rho}{\partial y_{1a}} & \frac{\partial \rho}{\partial y_{2a}} \\
\frac{\partial \theta}{\partial y_{1a}} & \frac{\partial \theta}{\partial y_{2a}}
\end{vmatrix}
= \begin{vmatrix}
\frac{\partial y_{1a}}{\partial \rho} & \frac{\partial y_{1a}}{\partial \theta} \\
\frac{\partial y_{2a}}{\partial \rho} & \frac{\partial y_{2a}}{\partial \theta}
\end{vmatrix}^{-1}
\]

\[
\begin{vmatrix}
\frac{1}{\sin \theta} & -\frac{\rho \cos \theta}{\sin^2 \theta} + \frac{U}{\sin^2 \theta} \\
\frac{1}{\sin \theta} & -\frac{\rho \cos \theta}{\sin^2 \theta} - \frac{U}{\sin^2 \theta}
\end{vmatrix}^{-1} = \begin{vmatrix}
-\frac{\rho \cos \theta - U}{\sin^3 \theta} & -\frac{\rho \cos \theta + U}{\sin^3 \theta}
\end{vmatrix}^{-1} = \begin{vmatrix}
-2U
\end{vmatrix}^{-1} = -\frac{\sin^3 \theta}{2U}
\]

(5.70)

Taking into account Eqs. (5.69) and (5.70), we conclude from the theorem that the joint probability density function of \(\rho\) and \(\theta\) for the 6\(^{th}\) set of scanning lines is given by:

\[
f^{6}_{\rho \theta}(\rho, \theta) = \begin{vmatrix}
2U \\
\sin^3 \theta
\end{vmatrix} f_{y_{1}y_{2}} \left( \frac{\rho}{\sin \theta} - U \cot \theta, \frac{\rho}{\sin \theta} + U \cot \theta \right)
\]

(5.71)

Finally, by substituting Eq. (5.64) into Eq. (5.71), we obtain that:

\[
f^{6}_{\rho \theta}(\rho, \theta) = \frac{2U}{\sin^3 \theta} \left\{ H \left( \frac{\rho}{\sin \theta} - U \cot \theta + U \right) - H \left( \frac{\rho}{\sin \theta} - U \cot \theta - U \right) \right\} \times \left\{ H \left( \frac{\rho}{\sin \theta} + U \cot \theta + U \right) - H \left( \frac{\rho}{\sin \theta} + U \cot \theta - U \right) \right\}
\]

(5.72)

Fig. 5.8 shows the regions of \((\rho, \theta)\) space that each of the six individual densities (obtained above for the six cases of scanning line) cover. It can be easily seen there that \(f^{1}_{\rho \theta}(\rho, \theta), f^{2}_{\rho \theta}(\rho, \theta), f^{3}_{\rho \theta}(\rho, \theta)\) and \(f^{4}_{\rho \theta}(\rho, \theta)\) cover the same area. Also, it was found that the \((\rho, \theta)\) areas of \(f^{5}_{\rho \theta}(\rho, \theta)\) and \(f^{6}_{\rho \theta}(\rho, \theta)\) (that are of similar size) are \(\sqrt{2}\) times the respective \((\rho, \theta)\) area of the previous four cases. Next, we show how to determine the overall probability density function \(f_{\rho \theta}(\rho, \theta)\), that the employed sensor arrangement of Fig. 5.1 generates in the Radon space, by making use of the six individual densities.

In this analysis, we are dealing with scanning lines connecting any two sensors, that reside in the boundary edges of the square reconstruction region, apart from pairs where
5.3 The Weighted Reconstruction Methodology

both sensors lie in the same square edge. We considered six cases of scanning line because
the number of combinations of two boundary edges (where the two sensors are located at)
from a set of four boundary edges is \( \frac{4!}{2!(4-2)!} = 6 \). A basic assumption of our analysis is
that a sensor lies in each of the boundary edges of the square region with equal probability.
This is obvious given the shape, spatial arrangement and (the same) length of the border
edges and, also, the fact that we employed regular positioning of sensors. In addition,
we assumed that the two placements of the sensors are independent. Based on these two
fundamental assumptions, that we made in our problem formulation, it is obtained that
each of the six cases of scanning line has the same probability. This probability is equal
to \( 2 \times \frac{1}{4} \times \frac{1}{3} = \frac{1}{6} \), where factor \( \frac{1}{4} \) gives the probability of the arbitrary placement of the
1st sensor of the pair (that defines the scanning line) in one of the four boundary edges.

Figure 5.8: The areas that the six individual and the overall probability densities
cover in the projection space.
of the square region and factor $\frac{1}{3}$ gives the probability of the placement of the 2nd sensor in a random boundary edge, having already placed the 1st sensor in a different edge. In addition, factor 2 is employed to account for the case of reverse placement of sensors to the same boundary edges. Also, we use multiplication due to the independency of the events. Considering the above and also taking into account the fact that the six cases, over which we partitioned the problem, are mutually exclusive and exhaustive, then, the application of the law of total probability for densities (see [51])

$$f_{p\theta}(\rho, \theta) = \sum_{i=1}^{6} f_{i}^{p\theta}(\rho, \theta | i^{th} \text{case}) \text{Prob}(i^{th} \text{case})$$

(5.73)

yields (by substituting):

$$f_{p\theta}(\rho, \theta) = \frac{1}{6} (f_{1}^{p\theta}(\rho, \theta) + f_{2}^{p\theta}(\rho, \theta) + f_{3}^{p\theta}(\rho, \theta) + f_{4}^{p\theta}(\rho, \theta) + f_{5}^{p\theta}(\rho, \theta) + f_{6}^{p\theta}(\rho, \theta))$$

(5.74)

The area that the overall probability density function $f_{p\theta}(\rho, \theta)$ covers in the $(\rho, \theta)$ space is shown in Fig. 5.8.

Having obtained the probability density function of parameters $\rho$ and $\theta$, that the sensor arrangement of Fig. 5.1 generates, we next describe the method of calculating the weight, that each equation of system (5.2) should be multiplied with. In order to obtain these weights, the $(\rho, \theta)$ space is divided into $R_b \times T_b$ non-overlapping 2-D bins of the same size, namely $R_b$ bins for the $\rho$ parameter and $T_b$ bins for the $\theta$ parameter. Then, each of the $R_b \times T_b$ bins has a probability mass

$$p_b = \int_{\theta_{bi}}^{\theta_{bu}} \int_{\rho_{bi}}^{\rho_{bu}} f_{p\theta}(\rho, \theta) \, d\rho \, d\theta \quad b = 1, 2, \ldots, R_b \times T_b$$

(5.75)

where $(\theta_{bi}, \theta_{bu})$ and $(\rho_{bi}, \rho_{bu})$, with $\theta_{bi} < \theta_{bu}$ and $\rho_{bi} < \rho_{bu}$, determine the 2-D region of definition of the $b^{th}$ bin. The mass in the entire $(\rho, \theta)$ plane (over the $R_b \times T_b$ bins) equals 1:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{p\theta}(\rho, \theta) \, d\rho \, d\theta = 1$$

(5.76)

Hence, for any scanning line defined by parameter values $(\rho_i, \theta_i)$ that lies in the $b^{th}$
5.4 Simulations

bin which has probability mass $p_b$, the weight, that the corresponding equation (projection measurement) should be multiplied with, is

$$w_i = \frac{1}{p_b R_b \times T_b} \quad i = 1, 2, \ldots, A_r$$

(5.77)

where $\frac{1}{R_b \times T_b}$ is the probability mass of the $b^{th}$ bin, if the density were flat and $A_r$ is the total number of the system's linear equations. The reasoning behind using these weights is to make the histogram of the Radon domain variables approximately flat.

After multiplying all equations (for all scanning lines) with the corresponding weights, the overdetermined system of equations is solved to obtain the reconstruction results. It must be noted that the linear equations are obviously not affected by the multiplication described above, since $\tilde{b} = A\mathbf{g}$ is equivalent to $\tilde{w}\tilde{b} = wA\mathbf{g}$, where vector $\tilde{w}$ contains the weights. However, since the system is solved in a LS error sense, this weight vector does affect the final solution. In the next section, we present an example of static electric field reconstruction with the purpose of demonstrating the improvement in reconstruction quality gained by employing the probabilistic weights, as proposed in this chapter, over the case of Section 3.4, where the measurements were not weighed.

5.4 Simulations

We considered the same case as in Section 3.5, where the vector field under investigation was the electric field created by a static charge. Four different cases for the location of the source of the electric field are reported. We assumed that the boundary sensors measured the potential, so that the difference in the measurements between any two such sensors gave the vectorial Radon transform of the examined electric field.

We employed the digital square reconstruction domain of Fig. 5.1 and chose $2U = 11$ as domain size and $P = 1$ as tile size. Hence, the domain consisted of 121 tiles and the number of the unknowns (the $E_x$ and $E_y$ components of the field at the centre of every tile of the domain) was 242. Regarding the data acquisition geometry, the above selection of values for parameters $U$ and $P$ resulted in having 11 sensors in every side of
the boundary of the square domain. For the simulations we present here, the potential in all these sensors was obtained by using Coulomb's law. We considered all possible voltage differences between pairs of sensors, apart from pairs where both sensors resided in the same border line. For the electric field recovery, we relied only on these line-integral data.

We first formed the system of linear equations according to the analysis that was presented in Section 3.4. The scanning line segments joining sensors were sampled with a step equal to 1 ($\Delta s = 1$). The number of linear equations was 726, whereas the number of the unknowns was 242. Hence, we obtained an overdetermined system of linear equations. Subsequently, these equations were weighed, according to the methodology analysed in Section 5.3, in order to approximate uniform sampling in the Radon space. For the weight computation, we used $R_b = 5$ bins for the radial parameter and $T_b = 7$ bins for the angular parameter. The choice of these parameter values for the binning of the projection space was made experimentally and, also, by taking into account the fact that all resulting bins must have non-zero probability mass. This is necessary for the proposed weighted reconstruction approach, where each equation must be divided with the probability mass of the associated bin.

Then, in order to obtain the reconstruction results, we had to solve the overdetermined system of weighted linear equations. For the experiments of this chapter, the LS error solution was obtained by applying the Gauss-Newton LS method. Stability issues similar to those of Section 3.5.2 were addressed.

The reconstruction results (or, else, the solution of the overdetermined systems of the weighted linear equations) are shown in Fig. 5.9a for four different source locations. For the sake of comparison, Fig. 5.9b depicts the respective theoretical electric fields that were obtained by using directly the governing Coulomb's law, while Fig. 5.9c shows the respective recovered fields when we applied direct uniform sampling in the Radon domain parameters, using the sampling steps recommended in [34] and [52], namely $\Delta \theta = 2^\circ$ and $\Delta \rho = 0.5$. To achieve such sampling we had to use ninety times more sensors than in the case of employing probabilistic weights. Alternatively, the actual uniform sampling of the projection space could have been achieved by employing a rotating acquisition system.
However, this would result in a one-hundred-and-eightyfold increase in the total scanning time, when compared with the case of employing probabilistic weights.

Figure 5.9: Simulation results when the location of the source of the electric field was (from top to bottom) at (19, -19), (-16, 21), (24, 11.5) and (-21, -12): (a) the recovered vector field when reconstruction was based on weighted linear equations that approximate uniform sampling of the Radon space; (b) the theoretical electric field as computed from Coulomb's law and (c) the recovered vector field when reconstruction was based on linear equations that correspond to actual uniform sampling of the Radon space.
5.4 Simulations

By careful inspection of Fig. 5.9, we may say that the directions of the vectors that were reconstructed, based on the boundary voltages, are almost identical with the directions of the vectors that were obtained by using Coulomb's law, since in all three cases the vectors are oriented towards the source of the field. Furthermore, vectors in all three cases reduce in magnitude when moving away from the source, as expected, even though the recovered vectors seem to reduce a bit more slowly than those computed by the application of Coulomb's law.

In order to demonstrate the improvement in reconstruction accuracy gained by using probabilistic weights, as proposed in this chapter, over the case of Section 3.4, where the measurements were not weighed, we present in Fig. 5.10 the histograms of the errors for these two cases. Fig. 5.10 also shows the respective histograms of the errors that were obtained when actual uniform sampling of the projection space was used. We may see in Fig. 5.10 that, as expected, the employment of actual uniform sampling in \((\rho, \theta)\) space resulted in the most accurate reconstruction. However, the difference in the reconstruction quality between the two cases, where probabilistic weights were used and actual uniform sampling was employed, is insignificant and in order to be achieved, we have either to overcome sensor placement impracticalities or to use a rotating acquisition system at the expense of temporal efficiency.

To obtain a quantitative idea of the observations made in Fig. 5.10, in Table 5.1, we tabulate the average values per pixel of the relative magnitude and absolute angular reconstruction errors for the base method (introduced in Section 3.4), the modified weighted reconstruction technique (proposed in this chapter) and the reconstruction method that employs actual uniform sampling of the projection space.

By inspecting Table 5.1, we observe the effectiveness of the probabilistic weights, proposed in this chapter, in suppressing the reconstruction error. In particular, it was found that the average error in vector field orientation was 31% lower when we employed probabilistic weights that approximate uniform sampling in the Radon domain, as opposed to the case where the measurements were not weighed, whereas the average error in magnitude was lower by 22%. This improvement in reconstruction accuracy took place without,
Figure 5.10: Left three columns: the histograms of the relative error in magnitude for the cases (i) regular sensor placement along the boundary of the domain, as proposed in Section 3.4; (ii) the same sensor arrangement as in (i), but, also, using weights that approximate uniform sampling in $(\rho, \theta)$ space, as proposed in this chapter and (iii) actual uniform sampling in $(\rho, \theta)$ space. Right three columns: the histograms of the error in vector field orientation for the same cases. The location of the source of the electric field was (from top to bottom) at $(19, -19)$, $(-16, 21)$, $(24, 11.5)$ and $(-21, -12)$. We note that the histograms of the first column have heavier tails towards higher values, when compared with the respective histograms of the second and third columns. We also note that the histograms of the fourth column have heavier tails towards higher values, when compared with the respective histograms of the fifth and sixth columns.
5.5 Discussion and Conclusions

Table 5.1: The average relative magnitude reconstruction error (%) per pixel (ME) and the average absolute angular reconstruction error (in degrees) per pixel (AE) for the three cases: (i) when data were not weighed (NW); (ii) when the reconstruction method employed weighted measurements (WM) to approximate uniform sampling in the Radon domain and (iii) when actual uniform sampling (US) in the \((\rho, \theta)\) Radon space was used. Four different source locations (SL) are reported.

<table>
<thead>
<tr>
<th>SL</th>
<th>ME (NW)</th>
<th>ME (WM)</th>
<th>ME (US)</th>
<th>AE (NW)</th>
<th>AE (WM)</th>
<th>AE (US)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(19, -19)</td>
<td>3.6791</td>
<td>2.8006</td>
<td>2.3790</td>
<td>2.2882</td>
<td>1.5400</td>
<td>1.4206</td>
</tr>
<tr>
<td>(-16, 21)</td>
<td>3.8114</td>
<td>2.8273</td>
<td>2.5339</td>
<td>2.4093</td>
<td>1.6217</td>
<td>1.5426</td>
</tr>
<tr>
<td>(24.11.5)</td>
<td>4.0363</td>
<td>3.3109</td>
<td>3.1747</td>
<td>2.5210</td>
<td>1.8122</td>
<td>1.8720</td>
</tr>
<tr>
<td>(-21, -12)</td>
<td>4.3803</td>
<td>3.5199</td>
<td>3.2126</td>
<td>2.7013</td>
<td>1.8702</td>
<td>1.8594</td>
</tr>
</tbody>
</table>

at the same time, having to increase the algorithm processing time and/or the number of the required sensors. It was also found that a further 2% decrease in the angular and 7% decrease in the magnitude reconstruction errors can be achieved by employing a scanning geometry that corresponds to the actual sampling of the \((\rho, \theta)\) space. However, to realise this further improvement, we would have to either use ninety times more sensors placed at very specific locations along the boundary of the reconstruction domain or, for rotating data acquisition systems, increase the total scanning time one-hundred-and-eighty times.

5.5 Discussion and Conclusions

In this chapter, we achieved approximate uniformity in the \((\rho, \theta)\) projection space by employing probabilistic weights. Simulation results indicated that this resulted in a significant (about 27%) reduction of both the angular and magnitude reconstruction error, as compared with the case where unweighed data from sensors were used, and insignificant (about 4.5%) difference from the reconstructions obtained when the \((\rho, \theta)\) space was sampled uniformly by either using ninety times more sensors or increasing the total scanning time one-hundred-and-eightyfold. One could also think about improving the reconstruction accuracy by applying weighting functions to each reconstruction pixel with the view to compensating for the non-uniform \(\theta\) distribution of the scanning lines that go through this reconstruction point.

The proposed method decreases the reconstruction error without increasing either
the number of sensors or the processing time, while maintaining a practical sensor placement configuration. The reason that the overall processing time does not increase is that the calculation of the weights is based on the known and predetermined sensor configuration. Hence, this calculation can be performed in advance (off-line).

The results of this study can be explained, since, according to the theory of Radon transform, a necessary requirement to achieve reconstruction results of great accuracy is to sample uniformly the Radon domain parameter space.
Chapter 6

Resolution Considerations for 2-D Vector Field Tomography

6.1 Introduction

In this chapter, we look at the 2-D vector field reconstruction problem from the aspect of sampling. This aspect is crucial for the design of imaging devices. We make an attempt to give an answer to questions like “what are the sampling requirements that must be imposed on the distances of the parameters of the projection space, for a given spatial resolution in the sought-for vector field, so as not to lose boundary integral information?” or “given a sampling of the sinogram, what is the maximum acceptable resolution in the reconstruction region?”. The influence of the sampling rate of the vectorial Radon transform on the quality of reconstruction is also studied.

To address sampling issues, we rely on Fourier theory of sampling as used in communication theory and image processing. By using the frequency properties of the vectorial Radon transform, we derive the lower bounds that must be imposed on the sampling rates of the variables in the projection space, for a given spatial resolution in the reconstruction region, so that no measurement information is lost.

Sampling issues in relation to vector field tomography were also discussed in [12]. However, the authors of [12] considered the problem of reconstructing only one of the two
components of the examined vector field from tomographic data, in line with the conclusions drawn in [4], [32], [48] and [67]. Their key insight was to extend efficient sampling schemes of scalar tomography into vector field tomography. They did not attempt to perform complete reconstruction of the examined vector field, but seemed merely interested in recovering only one component. In this study, we deal with the problem of reconstructing both components of a 2-D vector field based only on line-integral data and, therefore, we investigate sampling issues of the scanning geometry with a view to solving this problem.

This chapter is organised as follows. In Section 6.2, we set up our scanning geometry and formulate the problem. In Section 6.3, we derive the minimum adequate sampling rates of the parameters in the projection space, so as not to lose boundary information and, at the same time, to achieve an intended spatial resolution of the investigated vector field. For the derivation, we rely on sampling theory for deterministic bandlimited signals and the sinc-expansion procedure. In Section 6.4, we demonstrate the effectiveness of the proposed sampling bounds of the vectorial Radon transform by presenting some examples of complete reconstruction of static electric fields. In Section 6.5, we examine the behaviour of the derived sampling criteria of the sinogram in noisy environments. We conclude in Section 6.6.

6.2 The Scanning Geometry

Let us assume that we have the digitised square 2-D domain that is shown in Fig. 6.1, within which the investigated vector field \( \mathbf{f}(x, y) = f_x(x, y)\hat{x} + f_y(x, y)\hat{y} \) is defined. The length of each side of the square domain is taken to be equal to \( 2U \) and the coordinate system is chosen so that the origin of the axes is at the centre of the domain. The square domain consists of tiles of finite size, \( P \times P \), so that \( \frac{2U}{P} \) is an integer. Also, it is assumed that line-integral data are collected by sensors that reside in the boundary of this domain.

Consider a scanning line segment \( AB \) connecting two such sensors, located at points \( A \) and \( B \) (see Fig. 6.1). Then, this scanning line yields a line-integral measurement. By assuming that any pair of sensors measure only the integral of the component of
6.2 The Scanning Geometry

![Diagram of scanning geometry](image)

Figure 6.1: A square digitised reconstruction region of size $2U$. The size of the tiles, with which we sample the 2-D domain, is $P \times P$. A scanning line segment $AB$ goes through this region. $AB$ is sampled with sampling step $\Delta s$. The angle, at which the line segment is inclined to the positive direction of the $x$-axis, is $\omega$. Also shown are the two parameters $\rho$ and $\theta$ used to define the scanning line (Radon domain coordinates) and the unit vectors $\hat{s}$ and $\hat{p}$ which are parallel and perpendicular, respectively, to line segment $AB$.

The examined vector field along the associated scanning line, then, the integral-geometry transform $J_i$ that models the measurement, collected by sensors at points $A$ and $B$, is described by Eq. (3.1) that we repeat here as Eq. (6.1) for the sake of convenience:

$$J_i = \int_{A}^{B} f(x, y) \cdot \hat{s} \, ds$$

Here $\hat{s}$ is the unit vector along scanning line segment $AB$ (see Fig. 6.1) and $ds$ is an element of path length along this line segment.

The goal of this analysis is to derive sampling requirements for the vectorial Radon transform $J_i$. To achieve this goal, we must first determine the scanning geometry. In this treatment, we study the simplest case of scanning geometry, namely standard parallel scanning.\(^1\) This scheme results in scanning rays that are arranged in parallel bunches. Therefore, the projection space is most conveniently parameterised, for this type of scanning geometry, by using parameters $\rho$ and $\theta$, where $\rho$ is the length of the normal from the origin of the axes to the scanning line and $\theta$ is the angle at which this normal is inclined to the positive $x$ semi-axis (see Fig. 6.1). Next, we express available measurements $J_i$ as

\(^1\)For the fan beam scanning geometry, the treatment is similar, even though a bit more complicated.
a function of variables $\rho$ and $\theta$.

In order to express integral transform $J_i$, in terms of parameters $\rho$ and $\theta$, let us first define a coordinate system $(\hat{\rho}, \hat{s})$, such that $\hat{\rho}$ is the unit vector along the direction of the normal to the scanning line and $\hat{s}$ is the unit vector orthogonal to that, forming a right-handed coordinate system (see Fig. 6.1). The transformation relationships, between the system we defined and the $(\bar{x}, \bar{y})$ coordinate system, are:

\begin{align}
    x &= -s \sin \theta + \rho \cos \theta \quad (6.2) \\
    y &= s \cos \theta + \rho \sin \theta \quad (6.3)
\end{align}

By examining Fig. 6.1, we may see that unit vector $\hat{s}$, parallel to scanning line $AB$, may be written:

$$\hat{s} = -\sin \theta \bar{x} + \cos \theta \bar{y} \quad (6.4)$$

By combining the $(\rho, \theta)$ line parameterisation for scanning line $AB$ with Eqs. $(6.2)-(6.4)$, and, also, by assuming that $\tilde{f}(x, y) = 0$ outside the square reconstruction region, Eq. (6.1), that describes the available data, becomes:

$$J_i(\rho, \theta) = \int_{-\infty}^{+\infty} \tilde{f}(\rho \cos \theta - s \sin \theta, \rho \sin \theta + s \cos \theta) \cdot (-\sin \theta \bar{x} + \cos \theta \bar{y}) \, ds \quad (6.5)$$

In practice, the measured projections are discretised. Hence, function $J_i(\rho, \theta)$ needs to be sampled. In the next section, we derive the minimum sampling rates that should be used for parameters $\rho$ and $\theta$, so as to avoid losing measurement information and, at the same time, to achieve an intended spatial resolution in the reconstruction domain.

### 6.3 Sampling the Vectorial Radon Transform

In order to impose upper bounds on sampling intervals $\Delta \rho$ and $\Delta \theta$, we use sampling theory for deterministic bandlimited signals [51]. The derivation we provide is based on the sinc-expansion procedure [51] and the study of the 2-D frequency content of the available integral measurements in 2-D vector field tomography.
Let us assume that the spatial frequency content of the investigated vector field \( f(x, y) \) has an upper bound that we know. Such a piece of information about the vector field under investigation is expected to be known, see for example [69] about blood flow imaging. We also assume that, based on this knowledge, the sampling of the reconstruction region was made according to the Whittaker-Shannon theorem [51], i.e. half the sampling frequency in the reconstruction region is equal to or larger than the spatial frequency upper limit. Also, we note that \( J_t(p, \theta) \) is a function that belongs in \( \mathbb{R}^2 \) and is \( 2\pi \) periodic in the second argument. Hence, according to the sinc-expansion procedure [51], vector field \( \tilde{f}(x, y) \) may be recovered from the digital vector \( \tilde{f}(m, n) = f_x(m, n)x + f_y(m, n)y \) by convolution with a sinc function

\[
\tilde{f}(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \tilde{f}(m, n) \frac{\sin \left( \frac{x}{\Delta x} (x - x_m) \right)}{\frac{x}{\Delta x} (x - x_m)} \frac{\sin \left( \frac{y}{\Delta y} (y - y_n) \right)}{\frac{y}{\Delta y} (y - y_n)}
\] (6.6)

where \( M \) and \( N \) are the total numbers of samples in the \( x \) and \( y \) directions, respectively, \( \Delta x \) and \( \Delta y \) are the sampling steps of the reconstruction domain in the same directions and \( (x_m, y_n) \) are the coordinates of the vector field reconstruction points. Eq. (6.6) represents convolution, since the samples are delta functions at the sample locations. The reason that it is possible to have the vector field recovery from its equally spaced samples, described above, is that convolution with a sinc function in the spatial domain is equivalent to multiplication with a rectangle in the spatial frequency domain. Therefore, the operation of Eq. (6.6) represents an ideal reconstruction filter that reproduces vector field \( \tilde{f}(x, y) \) from its samples \( \tilde{f}(m, n) \) without distortion.\(^2\)

In many practical situations, a vector field is defined in a spatial domain of finite size. Hence, its spatial frequency content, as obtained by the Fourier transform, has no upper bound. To prevent spatial aliasing problems in the sampling process of the reconstruction region, a filtering, that removes the components of the investigated vector field that are of higher frequency, has to be applied. In this case, the resolution in the reconstruction region has to meet the condition that half the sampling frequency is equal to or larger than the filter's cut-off frequency. The implication of using an anti-aliasing

\(^2\)It is not quite true that \( \tilde{f}(x, y) \) is exactly recovered from \( \tilde{f}(m, n) \) because the summations in Eq. (6.6), in principle, should be infinitely long, but these extra \( \tilde{f}(m, n) \) are assumed to have zero value.
filter is that the examined vector field becomes blurred.

By substituting Eq. (6.6) into Eq. (6.5), we obtain:

\[
J_i(p, \theta) =
\]

\[
= \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_x(m, n) \frac{\sin \left( \frac{\pi}{\Delta x} (p \cos \theta - s \sin \theta - x_m) \right)}{\Delta x} \frac{\sin \left( \frac{\pi}{\Delta y} (p \sin \theta + s \cos \theta - y_n) \right)}{\Delta y} (-\sin \theta) \, ds
\]

\[
+ \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_y(m, n) \frac{\sin \left( \frac{\pi}{\Delta x} (p \cos \theta - s \sin \theta - x_m) \right)}{\Delta x} \frac{\sin \left( \frac{\pi}{\Delta y} (p \sin \theta + s \cos \theta - y_n) \right)}{\Delta y} (\cos \theta) \, ds
\]

We want to study the sampling properties in the \((p, \theta)\) parameter domain. These are determined by the upper limit frequency of \(J_i(p, \theta)\), as it is expressed by Eq. (6.7). However, some issues are easier derived by using the \((p, r)\) line parameterisation, where lines are defined by slope \(p\) and intersection \(r\) (see Fig. 6.2) as:

\[
y = px + r
\]

The conclusions drawn in the \((p, r)\) domain can be easily translated, afterwards, in the \((p, \theta)\) domain. By using the \((p, r)\) line parameterisation, described by Eq. (6.8), and also by

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.2.png}
\caption{The two parameters used to define a scanning line: slope \(p\) and intersection \(r\).}
\end{figure}

taking into account that \(dx = -\sin \theta \, ds\), \(dy = \cos \theta \, ds\) (see Eqs. (6.2)-(6.3)) and \(dy = p \, dx\)
6.3 Sampling the Vectorial Radon Transform

(see Eq. (6.8)), Eq. (6.7) may be put in the form

\[ J_i(p, \tau) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_x(m, n) I(p, \tau, x_m, y_n) + \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_y(m, n) p I(p, \tau, x_m, y_n) \]  

(6.9)

where:

\[ I(p, \tau, x_m, y_n) = \int_{-\infty}^{+\infty} \frac{\sin \left( \frac{\pi}{\Delta x} (x - x_m) \right) \sin \left( \frac{\pi}{\Delta y} (px + \tau - y_n) \right)}{\frac{\pi}{\Delta x} (x - x_m) \frac{\pi}{\Delta y} (px + \tau - y_n)} \, dx \]  

(6.10)

In this analysis, we discuss vector fields only inside a rectangle. Hence, we choose to have \( \Delta x = \Delta y \), so that all reconstruction points lie in symmetrical intervals around the origin of the coordinate system. By taking this into account and, also, introducing variables

\[ t \equiv \frac{\pi}{\Delta x} (x - x_m) \]  

(6.11)

\[ \gamma \equiv \frac{\pi}{\Delta x} (px_m + \tau - y_n) \]  

(6.12)

integral \( I(p, \tau, x_m, y_n) \) is simplified to:

\[ I(p, \tau, x_m, y_n) = \frac{\Delta x}{\pi} \int_{-\infty}^{+\infty} t \sin t \sin (pt + \gamma) \, dt \]  

(6.13)

In order to calculate integral \( I(p, \tau, x_m, y_n) \), we define two functions \( g_1(\tau), g_2(\tau) \) as:

\[ g_1(\tau) \equiv \frac{\sin \tau}{\tau} \]  

(6.14)

\[ g_2(\tau) \equiv \frac{\sin (p \tau + \gamma)}{p \tau + \gamma} \]  

(6.15)

The convolution of these two functions yields:

\[ \tilde{g}(\tau) \equiv g_2(\tau) \ast g_1(\tau) = \int_{-\infty}^{+\infty} g_2(t) g_1(\tau - t) \, dt \]  

\[ \Rightarrow \]

\[ \tilde{g}(\tau) = \int_{-\infty}^{+\infty} \frac{\sin(\tau - t) \sin(pt + \gamma)}{\tau - t} \, dt \]  

(6.16)
By substituting \( \tau = 0 \) in Eq. (6.16), we obtain:

\[
\begin{align*}
\check{g}(0) &= \int_{-\infty}^{+\infty} \frac{\sin(-t) \sin(pt + \gamma)}{-t} \ dt \\
\check{g}(0) &= \int_{-\infty}^{+\infty} \frac{\sin(t) \sin(pt + \gamma)}{t} \ dt
\end{align*}
\]

Hence, by Eqs. (6.13) and (6.17), in order to compute integral \( I(p, \tau, x_m, y_n) \), it is enough to compute the convolution, described by Eq. (6.16), at \( \tau = 0 \). However, convolution in the \( \tau \)-domain results in multiplication in the frequency domain. Therefore, the Fourier transform of function \( \check{g}(\tau) \) is given by

\[
\check{G}(f) = G_1(f)G_2(f)
\]  

where \( G_1(f) \) and \( G_2(f) \) are the Fourier transforms of \( g_1(\tau) \) and \( g_2(\tau) \), respectively. These are both sinc functions. Hence, their Fourier transforms are

\[
\begin{align*}
G_1(f) &= \pi \text{rect}(\pi f) \\
G_2(f) &= \frac{\pi}{|p|} \text{rect} \left( \frac{\pi f}{|p|} \right) \exp \left( j2\pi f \frac{\gamma}{p} \right)
\end{align*}
\]

where \( \text{rect}(f) \) is a rectangular function of value 1, for argument between \(-\frac{1}{2}\) and \(\frac{1}{2}\), and zero otherwise. The combination of Eqs. (6.18)-(6.20) yields:

\[
\check{G}(f) = \frac{\pi^2}{|p|} W(f) \exp \left( j2\pi f \frac{\gamma}{p} \right)
\]

where \( W(f) \) is similar to \( \text{rect}(\pi f) \), if \(|p| \geq 1\) and similar to \( \text{rect} \left( \frac{\pi f}{|p|} \right) \) otherwise. The inverse Fourier transform of function \( \check{G}(f) \) yields convolution function \( \check{g}(\tau) \):

\[
\check{g}(\tau) = \int_{-\infty}^{+\infty} \check{G}(f) \exp(j2\pi f \tau) \, df
\]

Two cases have to be distinguished: \(|p| \geq 1\) and \(|p| < 1\).

1. Case \(|p| \geq 1\)
6.3 Sampling the Vectorial Radon Transform

In this case, \( W(f) \) is similar to \( \text{rect}(\pi f) \) and Eq. (6.22) becomes:

\[
\tilde{g}(\tau) = \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} \frac{\pi^2}{|p|} \exp \left( j2\pi f \frac{\gamma}{p} \right) \exp(j2\pi f \tau) \, df = \\
= \frac{\pi^2}{|p|} \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} \exp \left( j2\pi f \left( \tau + \frac{\gamma}{p} \right) \right) \, df = \frac{\pi}{|p|} \frac{\sin \left( \frac{\tau + \frac{\gamma}{p}}{p} \right)}{\tau + \frac{\gamma}{p}} 
\]

(6.23)

For \( \tau = 0 \), we obtain:

\[
\tilde{g}(0) = \frac{\pi}{|p|} \frac{\sin \left( \frac{\gamma}{p} \right)}{\frac{\gamma}{p}} = \frac{\pi}{|p|} \frac{\sin \left( \frac{\gamma}{|p|} \right)}{\gamma} 
\]

(6.24)

By comparing Eqs. (6.13), (6.17) and (6.24), Eq. (6.13) becomes:

\[
I(p, \tau, x_m, y_n) = \Delta x \frac{\sin \left( \frac{\gamma}{|p|} \right)}{\gamma} 
\]

(6.25)

2. Case \(|p| < 1\)

In this case, \( W(f) \) is similar to \( \text{rect} \left( \frac{\pi f}{|p|} \right) \) and Eq. (6.22) becomes:

\[
\tilde{g}(\tau) = \int_{-\frac{|p|}{2\pi}}^{\frac{|p|}{2\pi}} \frac{\pi^2}{|p|} \exp \left( j2\pi f \frac{\gamma}{|p|} \right) \exp(j2\pi f \tau) \, df = \\
= \frac{\pi^2}{|p|} \int_{-\frac{|p|}{2\pi}}^{\frac{|p|}{2\pi}} \exp \left( j2\pi f \left( \tau + \frac{\gamma}{|p|} \right) \right) \, df = \frac{\pi}{|p|} \frac{\sin \left( |p| \left( \tau + \frac{\gamma}{|p|} \right) \right)}{\tau + \frac{\gamma}{|p|}} 
\]

(6.26)

For \( \tau = 0 \), we obtain:

\[
\tilde{g}(0) = \frac{\pi}{|p|} \frac{\sin \left( \frac{|p| \gamma}{|p|} \right)}{\frac{\gamma}{p}} = \frac{\pi}{|p|} \frac{\sin \gamma}{\gamma} 
\]

(6.27)

By comparing Eqs. (6.13), (6.17) and (6.27), Eq. (6.13) becomes:

\[
I(p, \tau, x_m, y_n) = \Delta x \frac{\sin \gamma}{\gamma} 
\]

(6.28)
By taking Eqs. (6.25) and (6.28) into account, Eq. (6.9) may be rewritten as:

\[
J_i(p, \tau) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_x(m, n) \Delta x \frac{\sin \left( \gamma \min \left\{ 1, \frac{1}{|p|} \right\} \right)}{\gamma} + C'
\]

\[
+ \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_y(m, n) \Delta x \frac{p \sin \left( \gamma \min \left\{ 1, \frac{1}{|p|} \right\} \right)}{\gamma} \tag{6.29}
\]

This expression of the available projection data in 2-D vector field tomography, obtained above, will be now used to establish the sampling requirements in the \((p, \theta)\) domain. The result of Eq. (6.29) shows that the continuous line-integral data are given by the sum of two quantities, namely \(C'\) and \(D'\).

By considering lines with slope \(|p| \leq 1\), the frequency content of quantity \(C'\) is determined by function \(\frac{\sin \gamma}{\gamma}\), which, as a function of \(\gamma\), has an upper limit frequency of \(\frac{1}{2\pi}\).

In a similar way, the frequency content of quantity \(D'\) is determined by function \(\frac{p \sin \gamma}{\gamma}\). It is \(\frac{\sin \gamma}{\gamma}\) a function of variables \(p\) and \(\tau\) (see Eq. (6.12)). Therefore, we may write:

\[h(p, \tau) \equiv \frac{\sin \gamma}{\gamma}\]

In order to determine the frequency content of the product \(\frac{p \sin \gamma}{\gamma} = ph(p, \tau)\), we consider the 2-D FT \(\tilde{h}(k_p, k_\tau)\) of \(h(p, \tau)\)

\[
\tilde{h}(k_p, k_\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(p, \tau) e^{-j(k_p p + k_\tau \tau)} \, dp \, d\tau \tag{6.30}
\]

where \(k_p\) and \(k_\tau\) are the Fourier domain variables of \(p\) and \(\tau\), respectively. By differentiating Eq. (6.30), with respect to \(k_p\), we obtain:

\[
\frac{\partial \tilde{h}(k_p, k_\tau)}{\partial k_p} = -j \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ph(p, \tau) e^{-j(k_p p + k_\tau \tau)} \, dp \, d\tau \Rightarrow
\]

\[
\frac{j \partial \tilde{h}(k_p, k_\tau)}{\partial k_p} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ph(p, \tau) e^{-j(k_p p + k_\tau \tau)} \, dp \, d\tau \tag{6.31}
\]
By examining Eq. (6.31), we notice that the second part of this equation gives the Fourier transform of the product \( ph(p, \tau) \). Therefore, we may deduce that the product \( ph(p, \tau) \) is bandlimited to the upper limit frequency of function \( h(p, \tau) \). Hence, we conclude that the frequency content of \( D' \) is also determined by function \( \frac{\sin \gamma}{\gamma} \), and the overall tomographic data have, as a function of \( \gamma \), an upper limit frequency of \( \frac{1}{2\pi} \). Thus, if \( \gamma \) should be sampled, this should be done with a rate faster than \( \pi \):

\[
\Delta \gamma \leq \pi \tag{6.32}
\]

If we use the transformation relationships between the \((p, \tau)\) and \((\rho, \theta)\) domains (obtained by solving system of Eqs. (3.16) and (6.8))

\[
p = -\cot \theta \tag{6.33}
\]
\[
\tau = \frac{\rho}{\sin \theta} \tag{6.34}
\]

then, Eq. (6.12) becomes:

\[
\gamma = \frac{\pi}{\Delta x} \left( -\cot \theta x_m + \frac{\rho}{\sin \theta} - y_n \right) \tag{6.35}
\]

From Eqs. (6.32) and (6.35), we obtain:

\[
\Delta \gamma = \left| \frac{\partial \gamma}{\partial \rho} \right| \Delta \rho = \frac{\pi}{\Delta x |\sin \theta|} \Delta \rho \leq \pi \quad \Rightarrow \quad \Delta \rho \leq \Delta x |\sin \theta| \tag{6.36}
\]

We are examining lines where \(|p| \leq 1\). Hence, \(|\cot \theta| \leq 1\). Therefore, \( \theta \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right] \cup \left[ \frac{-3\pi}{4}, \frac{-\pi}{4} \right] \) and \(|\sin \theta|\), as a function of \( \theta \), takes values between 1 and \( \frac{1}{\sqrt{2}} \). By using the minimum value of \( \frac{1}{\sqrt{2}} \), Eq. (6.36) yields:

\[
\Delta \rho \leq \frac{\Delta x}{\sqrt{2}} \tag{6.37}
\]
Similarly, Eqs. (6.32) and (6.35) give for the angular parameter:

\[
\Delta \gamma = \left| \frac{\partial \gamma}{\partial \theta} \right| \Delta \theta = \frac{\pi}{\Delta x} \left| \frac{x_m - \rho \cos \theta}{\sin^2 \theta} \right| \Delta \theta \leq \pi \Rightarrow \frac{\pi}{\Delta x} \left| \frac{x_m - \rho \cos \theta}{\sin^2 \theta} \right| \Delta \theta \leq \Delta x \left( \frac{\sin^2 \theta}{x_m - \rho \cos \theta} \right)
\]  

(6.38)

We want the sampling criterion that we shall derive to be valid for all values of \(x_m\). Hence, we must find the minimum value of quantity \(F'\) in Eq. (6.38). \(F'\) reduces to the minimum when the numerator becomes minimum and the denominator becomes maximum.

For the denominator we have

\[
|x_m - \rho \cos \theta| \leq |x_m| + |\rho \cos \theta| \leq |x_{max}| + |\rho_{max} \cos \theta|
\]  

(6.39)

where \(x_{max}\) and \(\rho_{max}\) are the maximum values of parameters \(x\) and \(\rho\), respectively. We consider uniform sampling at \((\rho, \theta)\) for the employed square domain of Fig. 6.1. Therefore, it must be \(\rho_{max} \leq x_{max}\). Otherwise, for \(\rho_{max} > x_{max}\) and \(\theta = k \frac{\pi}{2}\), where \(k\) is an integer, the resulting scanning lines do not lie within the region of interest. So, Eq. (6.39) becomes:

\[
|x_m - \rho \cos \theta| \leq |x_{max}|(1 + |\cos \theta|)
\]  

(6.40)

By taking into account Eq. (6.38), Eq. (6.40) and the considered area of values of \(\theta\), then, we may deduce that \(F'\) reduces to the minimum when \(|\sin \theta| = |\cos \theta| = \frac{1}{\sqrt{2}}\), i.e. when \(\theta = k \frac{\pi}{4}\) with \(k = -3, -1, 1\) and 3. Hence, Eq. (6.38) may be written as:

\[
\Delta \theta \leq \Delta x \frac{\left( \frac{1}{\sqrt{2}} \right)^2}{|x_{max}|(1 + \frac{1}{\sqrt{2}})} \Rightarrow \Delta \theta \leq \frac{\Delta x}{\sqrt{2}(1 + \sqrt{2})|x_{max}|}
\]  

(6.41)

It must be noted that the evaluation of the upper bounds of Eqs. (6.37) and (6.41) was very flexible in order to make sure that these expressions are valid for all values of \((x_m, y_n)\).

It can be easily proven that the same bounds for the sampling steps in the \((\rho, \theta)\)
domain, as these are expressed by Eqs. (6.37) and (6.41), are obtained when using lines with slope $|p| > 1$. In this case, it is easier to describe lines as $x = ry + \eta$ with $r = \frac{1}{p}$ and $\eta = -\frac{\xi}{p}$ and, then, obtain the sampling steps in the $(\rho, \theta)$ domain, based on the frequency properties of the parameter domain $(r, \eta)$.

In summary, if $\Delta x$ is the sampling interval that describes the spatial resolution that we want to achieve for the recovered vector field and $x_{\text{max}}$ is the maximum value of parameter $x$, the steps one should use to sample parameters $\rho$ and $\theta$ should be:

\[
\Delta \rho \leq \frac{\Delta x}{\sqrt{2}} \quad \text{(6.42)}
\]
\[
\Delta \theta \leq \frac{\Delta x}{\sqrt{2}(1 + \sqrt{2})|x_{\text{max}}|} \quad \text{(6.43)}
\]

In an equivalent manner, Eqs. (6.42) and (6.43) give the minimum acceptable sampling steps in the reconstruction region for a given sampling of the sinogram.

In practice, one wants to use as few data as possible for obtaining the intended resolution. This means that one usually chooses values for sampling steps $\Delta \rho$ and $\Delta \theta$ close to equality in Eqs. (6.42) and (6.43), respectively. Next, we provide evidence that shows the favourable behaviour of the sampling bounds, derived in this section, towards vector field reconstruction accuracy by presenting an example.

## 6.4 Sampling Bounds and Quality of Reconstruction: An Example

In this section, we conducted some experiments in order to test the effectiveness of the sampling bounds, derived in Section 6.3. The check was performed by studying the influence of various sampling rates of the vectorial Radon transform on the quality of the complete reconstruction of 2-D vector fields. We treated, as investigated vector field, the electric field created by a static charge. Four different cases of the location of the source of the electric field are reported. We assumed that the boundary sensors measured the potential, so that the difference in the measurements between any two such sensors gave
the vectorial Radon transform of the investigated electric field. For the simulations we present here, the potential in all these sensors was obtained by using Coulomb's law.

We employed the digitised square reconstruction domain of Fig. 6.1 and chose $2U = 11$ as domain size and $P = 1$ as tile size. Hence, the domain consisted of 121 tiles and the resolution in the reconstruction region was $\Delta x = 1$. In addition, in the sampling process along line segments connecting sensors (with a view to approximating the integral measurements by sums), we selected sampling step $\Delta s$ to be equal to 1 (=tile size) in all cases.

In this study, we discuss sampling considerations for standard parallel scanning schemes, where the projection space is most conveniently parameterised by variables $\rho$ and $\theta$. Therefore, we obtained the discretised measured projections, for our experiments, by performing a sampling of these two variables. Each selected combination of sampling steps $\Delta \rho$ and $\Delta \theta$ gave rise to a set of scanning lines that were uniformly distributed in the $(\rho, \theta)$ space. By applying the direct algebraic reconstruction technique, introduced in Section 3.4, to every such set of scanning lines, we obtained the system of equations, the solution of which gave the components of the investigated electric field at all sampling points of the reconstruction domain. To demonstrate the favourable behaviour of the sampling bounds, derived in this chapter, towards vector field reconstruction accuracy, we performed four sets of experiments for each source location.

In the first set of experiments, we used a parallel scanning geometry that corresponded to uniform sampling of parameters $\rho$ and $\theta$, where the sampling criteria that we derived in Section 6.3 were satisfied. For the employed rectangle of interest, we had $\Delta x = \Delta y = P = 1$ and $x_{\text{max}} = U = 5.5$. Hence, the sampling criteria of Eqs. (6.42) and (6.43) yielded:

$$\Delta \rho \leq 0.7071 \quad \text{and} \quad \Delta \theta \leq 3.0512^\circ$$

In order to meet these requirements, we selected as sampling step values: $\Delta \rho = 0.7$ and $\Delta \theta = 3^\circ$. With the purpose of covering fully the region of interest (Fig. 6.1), the chosen steps resulted in having 8 samples of the radial parameter and 120 samples of the angular parameter. As a result, the systems of linear equations, the solution of which gave the
reconstructed fields for the four source locations, had 960 ($= 8 \times 120$) equations, whereas the number of the unknowns (the $E_x$ and $E_y$ components of the field at the centre of every tile of the domain) was 242. Hence, we had to deal with overdetermined systems of linear equations. We must note that we could have chosen smaller values for sampling steps $\Delta \rho$ and $\Delta \theta$. However, we opted out of such a selection because, in practice, one uses as few data as possible. Moreover, by choosing much smaller values for sampling steps $\Delta \rho$ and $\Delta \theta$, the number of equations would increase too much and we would have to solve a prohibitively large system of linear equations. The solution to the systems of equations, that we formed, was obtained by applying the LS Gauss-Newton method. To test the ill-conditioning of the system, stability issues similar to those of Section 3.5.2 were addressed. The reconstruction results are shown in Fig. 6.3a for the four source locations. For the sake of comparison, Fig. 6.3b depicts the respective electric fields that were obtained by using directly the governing Coulomb's law.

In order to test the effectiveness of the sampling bounds, that we derived in this chapter, we carried out three more sets of experiments without imposing the derived upper bounds for $\Delta \rho$ and $\Delta \theta$ on the sampling of the projection space. More specifically, in the second set of experiments, we chose $\Delta \rho = 1$ and $\Delta \theta = 2^\circ$. It is obvious, from Eq. (6.44), that such a selection for sampling step $\Delta \rho$ was a clear violation of the criterion we derived in Section 6.3 about the sampling rate of the radial parameter. The above choice of parameter values resulted in having 6 samples of the radial parameter and 180 samples of the angular parameter. Hence, the system of equations consisted of 1080 ($= 6 \times 180$) equations. In the third set of experiments, vector field recovery was carried out by using uniform sampling in the Radon domain and selecting sampling step values: $\Delta \rho = 0.5$ and $\Delta \theta = 4^\circ$. Hence, it was sampling step $\Delta \theta$, this time, the one that did not fulfill the sampling requirements proposed in this chapter (Eq. (6.44)). This selection of parameter values resulted in having 11 samples of the radial parameter and 90 samples of the angular parameter. Hence, the system of equations consisted of 990 ($= 11 \times 90$) equations. Finally, in the last set of experiments, we chose $\Delta \rho = 1$ and $\Delta \theta = 4^\circ$. Hence, both sampling criteria, that we derived in this chapter, were not satisfied (Eq. (6.44)). The last choice of sampling steps resulted in having 6 samples of the radial parameter and 90 samples of the
Figure 6.3: Simulation results for the case when the proposed sampling criteria were met ($\Delta \rho = 0.7$ and $\Delta \theta = 3^\circ$) and the location of the source of the electric field was (from top to bottom) at (19, -19), (-16, 21), (12.5, 30) and (-19, -40): (a) the recovered vector field and (b) the theoretical electric field as computed from Coulomb's law.
angular parameter. Hence, the system of equations consisted of 540 (= 6 x 90) equations. It must be noted that the number of linear equations was about the same for the four sets of experiments, apart from the last one, where it was inevitable to have a reduced number of linear equations.

The systems of equations, that we obtained in the last three sets of experiments (for all four source locations), were also solved by using the LS Gauss-Newton method. The reconstruction results were obtained. The relative magnitude reconstruction error plots (i.e. the plots of the absolute values of the differences between the magnitudes of the reconstructed fields and the theoretical ones divided by the theoretical magnitude) and the absolute angular reconstruction error plots (i.e. the plot of the absolute angular differences (in degrees) between the reconstructed vector field values and the theoretical ones) for all four sets of experiments and four source locations can be seen in Fig. 6.4 and Fig. 6.5, respectively. We notice from these figures that the case where the derived sampling criteria were met outperforms the other three cases where we had a violation of at least one of these criteria. This observation was made even in the case where the number of equations, by having violation of a sampling criterion, was larger than the respective number by satisfying both sampling criteria.

To appreciate better the degradation in the performance of the direct algebraic reconstruction method of Chapter 3, by not imposing the upper sampling bounds on $\Delta \rho$ and $\Delta \theta$, in Fig. 6.6 and Fig. 6.7, we present the histograms of the errors in each case. By close examination of these figures, we may see that the violation of the lower bounds to the sampling rates of the radial and/or angular parameters resulted in having vector field reconstructions of lower quality. In particular, it was found that the average error in the vector field orientation was 35% higher, when the upper bound on sampling interval $\Delta \theta$ was not imposed, as opposed to the case where both sampling criteria were met, whereas the average error in the magnitude was higher by 24%. Similarly, it was found that the average error in the vector field orientation was 24% higher, when the upper bound on sampling interval $\Delta \rho$ was not imposed, as opposed to the case where both sampling criteria were met, whereas the average error in the magnitude was higher by 10%. The
corresponding differences in the angular and magnitude errors for the case where both the lower bounds on sampling rates of parameters $\rho$ and $\theta$ were not imposed, over the case of fulfilling the derived sampling requirements, were 38% and 26%, respectively.
6.5 Effect of Sampling Rate on Resilience to Noise

In this section, we investigate the effect of the sampling rate of the Radon domain parameters on robustness against noise. In all experiments, reported in the previous section, the sensors were placed exactly in the positions we had decided, and the measurement taken by each sensor was exactly the value predicted by Coulomb's law. In a practical system, however, some of the sensor measurements are expected to have inaccuracies and some of the sensors are also expected to be somehow misplaced. To emulate these effects, we considered the following.

(i) A noise value was added to a measurement, as a fraction of the true value, with
6.5 Effect of Sampling Rate on Resilience to Noise

Figure 6.6: The histograms of the relative magnitude errors for the cases when: (a) the proposed sampling criteria were met ($\Delta \rho = 0.7$ and $\Delta \theta = 3^\circ$); (b) the proposed sampling criterion about the radial parameter was not fulfilled ($\Delta \rho = 1$ and $\Delta \theta = 2^\circ$); (c) the proposed sampling criterion about the angular parameter was not fulfilled ($\Delta \rho = 0.5$ and $\Delta \theta = 4^\circ$) and (d) both proposed sampling criteria about the radial and angular parameters were violated ($\Delta \rho = 1$ and $\Delta \theta = 4^\circ$). The location of the source of the electric field was (from top to bottom) at (19, -19), (-16, 21), (12.5, 30) and (-19, -40). We note that the histograms of the last three columns have heavier tails towards higher values, when compared with the histograms of the first column.
Figure 6.7: As in Fig. 6.6, but here the histograms of the errors in vector field orientation are plotted. Again, we note that the histograms of the last three columns have heavier tails towards higher values, when compared with the histograms of the first column.

random sign. For example, 2% noise meant that the sensor measurement was changed by 2% of the value dictated by Coulomb's law. The change was either incremental or decremental, the choice made at random for each sensor.

(ii) A sensor was moved away from its correct position by a fraction of the correct
position. For example, if according to the theory, a sensor should be placed at position \((x, y)\), and we considered a 2% error, then, the coordinates of this sensor were shifted by 2% the corresponding correct values, with a positive or negative sign chosen at random.

(iii) Both the above errors took place simultaneously.

We performed four series of experiments by perturbing (by the three types of noise described above) (a) 25% of the sensors; (b) 50% of the sensors; (c) 75% of the sensors and (d) all sensors. In order to evaluate the robustness of the proposed sampling bounds against noise, we examined for each series of experiments, the four cases of Section 6.4: (I) when both derived sampling criteria were met \((\Delta \rho = 0.7 \text{ and } \Delta \theta = 3^\circ)\); (II) when the proposed sampling criterion about the radial parameter was not fulfilled \((\Delta \rho = 1 \text{ and } \Delta \theta = 2^\circ)\); (III) when the derived sampling criterion about the angular parameter was not fulfilled \((\Delta \rho = 0.5 \text{ and } \Delta \theta = 4^\circ)\) and (IV) when both proposed sampling criteria about the radial and angular parameters were violated. For every noise value (of each noise type, sampling rate and percentage of perturbed sensors), ten simulations were performed and the average reconstruction errors in relative magnitude and absolute vector field orientation were obtained. The source of the vector field for all simulations was located at \((19, -19)\).

The results of these experiments are shown in Figs. 6.8-6.11. We observe there that the employment of a scanning geometry that satisfied the sampling bounds, that had been derived in Section 6.3, increased the resilience to all three types of noise, when compared with the cases where at least one of the derived sampling criteria were not imposed.

6.6 Discussion and Conclusions

In this chapter, we addressed resolution issues in the context of 2-D vector field tomography. Such issues are crucial for the design of imaging devices. For our treatment, we relied on sampling theory for deterministic bandlimited signals. Since, in this research we dealt with the problem of reconstructing both components of a 2-D vector field based only on line-integral data, therefore, we investigated sampling issues of the scanning geometry
6.6 Discussion and Conclusions

- Both criteria satisfied
- Not satisfying criterion about p
- Not satisfying criterion about \( e \)
- Not satisfying both criteria

Figure 6.8: Comparison of the reconstruction performance in noisy environments for the four cases of sampling rates of Figs. 6.4-6.7: (a) and (b) Errors in vector field orientation and magnitude, respectively, when noise was added to the measurements of the sensors, as a percentage of the true value. (c) and (d) Errors in vector field orientation and magnitude, respectively, when small perturbations in the sensor positions were added. Sensor misplacements were a percentage of the true positions. (e) and (f) Errors in vector field orientation and magnitude, respectively, when both sensors' measurements and positions were changed by a percentage of their true values. In all cases, 25% of the sensors were perturbed.

with a view to solving this problem.

We discussed sampling issues about parallel scanning 2-D vector field tomography. Therefore, the projection space was most conveniently parameterised by using parameters \( \rho \) and \( \theta \), where \( \rho \) was the length of the normal from the origin of the axes to the scanning
line, and \( \theta \) was the angle at which this normal is inclined to the positive \( x \) semi-axis. Hence, we investigated the sampling requirements about these two Radon domain variables. For fan beam scanning 2-D vector field tomography, the treatment would be similar. However, the projection space, for this case, would be most conveniently parameterised by a pair of angles, where the one angle would define the source position and the other angle would determine, for a specific source position, the angle that the considered scanning line would make with the central ray. Hence, one would have to find the sampling requirements about these two parameters.

Figure 6.9: As in Fig. 6.8, but here 50\% of the sensors were perturbed.
The sampling bounds, which must be imposed on the sampling of parameters \( \rho \) and \( \theta \) in order not to lose boundary integral information and, at the same time, to achieve an intended spatial resolution of the investigated 2-D vector field, were derived. Equivalently, it may be said that the derived criteria also described the maximum acceptable resolution in the reconstruction region, given the sampling of the sinogram.

Evidence that showed the favourable behaviour of the proposed sampling bounds towards the accuracy of the complete 2-D vector field reconstruction was provided by presenting examples. It was also shown that the implication of using a scanning geometry...
Figure 6.11: As in Fig. 6.8, but here all sensors were perturbed.

that violated the derived lower bounds to sampling rates in the sinogram was a degradation in the performance of the direct algebraic reconstruction technique of Chapter 3. This was expected, since by not sampling the Radon parameters densely enough, the information content of the line-integral measurements was inadequate and the aliasing problems that occurred had an adverse effect on the reconstruction quality.

An important issue when solving inverse problems is the sensitivity of the solution to noise. In the case of this problem, there were two possible sources of noise: inaccuracies in the sensor measurements and misplacements of the sensors. It is very encouraging,
therefore, that more resilience to noise was observed when the sampling bounds, proposed in this chapter, were imposed.
Chapter 7

Conclusions and Future Work

7.1 Discussion and Main Contributions of this Thesis

In this thesis, we focused on the reconstruction problem of 2-D vector field tomography by relying only on line-integral data. In previous attempts to map integral measurements obtained along scanning lines onto a vector field, conventional (scalar) tomography theory and the FST had invariably been applied [4], [44] and [48]: this had led to an underdetermined problem. Possible solutions to this problem, that have been proposed in the literature, were discussed in Section 1.1. However, these solutions involve either different type of modelling of the available measurements or the incorporation of supplementary information, apart from the projection measurements. Next, we briefly outline the main contributions of this thesis.

The main contribution of this thesis is that it demonstrated that in the discrete domain, the reconstruction problem of 2-D vector field tomography, based only on a finite number of line-integral data, is tractable. The proposed direct algebraic reconstruction technique treated the discretised available measurements as bounded linear functionals on the space of two-integrable functions in the reconstruction region. Hence, the 2-D vector field reconstruction problem was cast as the solution of a system of linear equations, where the unknowns of the system were the Cartesian components of the examined vector field in specific sampling points, finite in number and arranged in a grid, of the 2-D reconstruction
region. This contribution may open new possibilities in a wide variety of disciplines, a short account of which was given in Section 3.2 of this thesis.

Another finding in this thesis was that the solution of the inverse problem of 2-D vector field tomography, by following the introduced direct algebraic reconstruction techniques, was relatively robust to perturbations in the sensor positions. This result is very encouraging since such inaccuracies in the sensor positions are rather intrinsic to inverse problems. Therefore, any tomographic application, where the domain (over which the vector field is to be reconstructed) does not have a shape that helps the firm and stable placement of the sensors, may benefit from this property.

Other offering of this thesis is that it provided methods to improve the reconstruction quality of 2-D vector field tomography and also to increase the resilience to noise, namely inaccuracies in the integral measurements and/or sensor misplacements. The proposed methods were based on Radon transform theory. They employed either interpolated boundary data, obtained at "virtual sensors", or probabilistic weights with the view to approximating uniformity in the projection space. Experimental results pointed out that about 30% reduction of the reconstruction error may be achieved by employing either of these two solutions. Most importantly, the two proposed methods improved the reconstruction quality and also increased the noise tolerance, without being limited by the applicability constraints (i.e. physical constraints on sensor placement and total scanning time constraints) that are imposed when one employs actual uniform sampling in the Radon space. On top of these enhancements, it must be noted that the method that employs probabilistic weights is also very time-efficient, since the weight calculation can be performed in advance. These outcomes may be of benefit in related applications, where noise and time are crucial factors.

Another contribution of this thesis is that it addressed resolution issues in the context of 2-D vector field tomography. Since, the topic of research in this thesis is the problem of reconstructing both components of a 2-D vector field by relying only on line-integral information, sampling issues of the scanning geometry with a view to solving this problem were investigated. The treatment employed the simplest case of scanning geome-
try, namely standard parallel scanning. The novelty introduced in this thesis is that lower sampling bounds were derived. These bounds must be imposed on the sampling rates of the variables in the projection space, for a given spatial resolution in the reconstruction region, so that no measurement information is lost. Equivalently, it may be said that the derived criteria also describe the maximum acceptable resolution in the reconstruction region, given the sampling of the sinogram. The derived limits demonstrated favourable behaviour towards vector field reconstruction accuracy. It was also shown that the implication of using a scanning geometry that violated the derived lower bounds of sampling rates in the sinogram was a degradation in the performance of the direct algebraic reconstruction technique. This was expected, since by not sampling the Radon parameters densely enough, the information content of the line-integral measurements was inadequate and the aliasing problems that occurred had an adverse effect on the reconstruction quality. In addition, it was reported that by imposing the proposed sampling bounds, also the resilience to noise increased. The bounds to the sampling rates of the sinogram, derived in this thesis, may provide the mathematical analysis tools that are necessary to understand the computational data acquisition systems’ design. One may, then, implement the derived bounds on current hardware by manufacturing a measurement geometry such that the data set it accommodates will satisfy these bounds. Finally, the derived sampling bounds may be integrated (together with the proposed reconstruction algorithms and the available hardware) into the design of tomographic imaging systems.

Finally, another achievement of this thesis is that it proposed a method to handle the stability deficiencies of the 2-D vector field reconstruction problem. In particular, in order to do away with the matters of existence and uniqueness of the solution and, also, to restore the solution’s continuous dependency on the projection data, this thesis proposed to take advantage of the redundancy in the line-integral data, as a form of employing regularisation. The regularisation lies in the fact that by using many line orientations passing through every sampling point, and, then, viewing the related recordings as weighted sums of the local vector field’s Cartesian components, one manages to include additional information about the investigated vector field itself in the problem formulation. Hence, the regularisation term consisted of the extra set of regularisation rows, added to the system.
matrix. This approach may be advantageous to the solution of various ill-posed problems in engineering, physical sciences, medicine and finance.

However, we must note that due to the physical limitations of current sensing systems, it is not possible, at the moment, to achieve redundancy in the line-integral data (as required by the proposed regularisation) for most tomographic applications where images of very high resolution are required. The last decade, nevertheless, has witnessed [40] a rapid surge of interest in manufacturing techniques of miniaturised sensors for healthcare and industry. Therefore, the odds are that a rapid expansion in development of sensors of smaller size will take place over the next ten years. Such advances in sensor technology will facilitate the implementation of the regularisation proposed in this thesis by taking advantage of the redundancy in the projection data. Moreover, this type of developments will make it possible for the reconstruction algorithms, that we introduced in this thesis, to meet the desired standards of most tomographic applications.

For all the contributions of this thesis, mentioned above, evidence was provided by presenting examples of complete reconstruction. The vector field under investigation, in all the simulations reported in this thesis, was the static electric field. This example application was chosen because one can compute the ground truth very easily and with great accuracy (using Coulomb's law) and, thus, evaluate the proposed methodologies. Future work could also take the algorithms, proposed in this thesis, and apply them successfully to the complete reconstruction of vector fields, other than the static electric field presented in this thesis. Next, we discuss possible directions of future research.

7.2 Topics for Further Research

Vector field tomography has substantial potential and this thesis only scratches the surface of a very interesting and promising problem. Next, we outline some topics that are worthy of investigation in the future.

As a first step, it would be of interest to take the direct algebraic reconstruction technique, introduced in Chapter 3 of this thesis, and extend it into 3-D. Such an extension
is straightforward and it involves the appropriate parameterisation of lines in 3-D. The only limitation of this extension is the number of simultaneous linear equations one can solve.

Simulations, where the topic of investigation will be the reconstruction of vector fields, other than the electric field reported in this thesis, by relying only on projection measurements is also of interest. Further work could also take the algorithms, proposed in this thesis, and apply them successfully to the reconstruction of actual vector fields, such as MRI flow velocity fields. This, of course, would involve one obtaining actual projection data.

The solution to the 2-D vector field reconstruction problem, by following the proposed reconstruction techniques of this thesis, was found to be rather sensitive to the sensor measurements’ errors. Methods to overcome such sensitivity may also be a possible future direction of the research work in this thesis. Robust reconstruction methods that employ a redescending kernel [24] or formulation of the inverse problem in 2-D vector field tomography as a Bayesian reconstruction problem [43] might be the way to tackle this problem. The development of very accurate sensors may also provide another solution to this problem.

In Chapter 5, it was proposed to achieve approximate uniformity in the \((\rho, \theta)\) projection space by employing probabilistic weights. A modification of this heuristic, that may result in further enhancement of the vector field reconstruction quality, is to apply weighting functions to each reconstruction pixel with the view to compensating for the non-uniform \(\theta\) distribution of the scanning lines that go through this reconstruction point.

In Chapter 6, resolution issues in the context of 2-D vector field tomography, and with the purpose of achieving the complete reconstruction of the examined vector field, were addressed. The scanning geometry that was studied was standard parallel scanning. Therefore, the projection space was most conveniently parameterised by using parameters \(\rho\) and \(\theta\), where \(\rho\) was the length of the normal from the origin of the axes to the scanning line, and \(\theta\) was the angle at which this normal in inclined to the positive \(x\) semi-axis. Hence, sampling requirements about these two Radon domain variables were under invest-
7.2 Topics for Further Research

tigation. For future work, it would be of particular interest to address sampling issues, also for fan beam 2-D vector field tomography and with the view to achieving the complete reconstruction of the examined vector field. The treatment for this case would be similar to parallel scanning, although a bit more complicated. Additionally, the projection space for the fan beam scanning geometry would be most conveniently parameterised by a pair of angles, where the one angle would define the source position and the other angle would determine, for a specific source position, the angle that the considered scanning line would form with the central ray. Hence, one would have to find the sampling requirements about these two parameters.

This thesis has neglected a number of interesting things at lower layers, such as incorporating the sensor modelling into the formulation of the 2-D vector field reconstruction problem. The sensor effects are out of the scope of this thesis, but how to incorporate accurately these effects into the problem formulation is, easily, a research area by itself.

Finally, it would be worthy to explore the possibility of extending the vector field reconstruction algorithms, developed in this thesis, to the reconstruction of tensor fields. This would be a really interesting path to follow, since tensor tomography builds on much of the work accomplished in vector field tomography [10], [20], [21] and [27]. As a starting step, simulation studies in 2-D could be performed aiming at fully reconstructing a $2 \times 2$ tensor field in a 2-D domain, based only on a few directional projections of the examined tensor field.

In the same context, it would be of particular interest to try and apply the tensor field reconstruction techniques, that would be obtained, to diffusion tensor MRI. The goal, then, would be to recover the entire diffusion tensor field, under investigation, based only on its MRI projections. The characterisation of the structure of myocardium and brain white matter could benefit from such an application. However, this application is challenged at the moment by severe eddy currents caused by the rotating diffusion gradients.

This thesis should finish at some point and this is the best place for this to happen.
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