THE INVERSE PROBLEM

AND APPLICATIONS TO OPTICAL AND EDDY CURRENT IMAGING

BY

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Abstract

The motivation for this work is the development of the theory of eddy-current imaging in conductors. This is aimed at approaching a full characterisation of defects using either probe impedance responses or electromagnetic field data in order to infer flaw geometry. In chapter V it is shown how the eddy current imaging theory leads to a linear inverse problem.

Frequently no analytic means of solving linear inverse problems is available and we are forced to use numerical methods. But in designing algorithms for numerical solution of such problems one must ask: what do we actually mean by the solution? In chapter I we discuss the general properties of the linear inverse problem and present various definitions of the solution.

Numerical solution of many inverse problems often reduces to a matrix inversion, or more precisely to the solution of a set of linear equations. The algorithms written to provide such solutions must of course be stable and robust and the most suitable algorithm to use in a particular case is well known and depends on the properties of the matrix representation of the operator in question (positive definiteness, symmetry, sparseness etc.). The NAG library, NAG [1], provides a useful flow diagram with which to decide on the best algorithm for a particular case and in chapter II we review the standard algorithms for solving the inverse problem.

However some problems, the ill-conditioned linear inverse problems, do not lend themselves to solution by the usual methods and indeed there is a dearth of algorithms for their solution in
such places as the NAG library. Some optical problems are ill-conditioned and a new technique for matrix inversion which is suitable for such problems, the technique of direct bidiagonalisation (the Jones-Lanczos or JL algorithm), is introduced in chapter II.

In chapter III we discuss the Fourier optical problem and to what extent such problems can be treated as linear. A review of the linearisation methods of Kim and Bose is made and the necessity of using iterative methods in some circumstances is described. In chapter III we also describe the iterative Gerchberg-Papouls algorithm.

In chapter IV we apply software developed to implement the JL algorithm to both one-dimensional and two-dimensional band-limited Fourier optical problems. We also treat the two-dimensional problem using the iterative Gerchberg-Papouls algorithm and compare the results with those obtained using the JL method.

In chapter V we provide an introduction to the subject of eddy current imaging and we show how this leads to a modified complex Laplace transform. Hence we discuss, in chapter VI, the real Laplace transform, the ability to invert this being a prerequisite for successful inversion of the complex eddy current problem.

In chapter VII we apply the JL algorithm to both the real Laplace transform inversion and the eddy current problem. These results show the viability of the techniques presented here for eddy current imaging. In addition the results augment the methods already in use for obtaining consistent and meaningful results to noisy ill-conditioned problems in general.
In chapter VIII we look at the problem of edge detection using *a priori* knowledge before concluding the work and suggesting possible future work in chapter IX.
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Chapter I.

The Linear Inverse Problem.

1.1 Formulation of the Linear Inverse Problem.

A wide variety of problems in all fields of physics can be expressed as, or reduced, to a linear problem. Such a linear one-dimensional problem, can be expressed by the Fredholm equation of the first kind with kernel $L(x,y)$ as

$$b \quad g(x) = \int_a^b L(x,y) \, f(y) \, dy \quad c \leq x \leq d, \quad (1.1)$$

where $f(y)$ is an unknown function in a Hilbert space $U$ and $g(x)$ is a known function in another Hilbert space $W$. In an experimental situation, the observation $g(x)$ will be subject to random noise. Thus, in operator notation, we can express equation (1.1) as

$$g = L \, f + n \quad (1.2)$$

where $f$ is a vector in a Hilbert space $U$, $L$ is a bounded linear transformation of $U$ with image space $L(U) = V$, and $n$ is a random noise vector, so that $g$ is a vector in a Hilbert space $W$ that contains $V$ as a subspace.

The problem is: having made some observation $g$ (including the random noise $n$) to obtain an estimate of $f$. Although we are primarily interested in solution of the problem for imaging applications, where $f$ and $g$ are functions on $\mathbb{R}^2$ or even $\mathbb{R}^3$, it should be born in mind that the techniques used here are equally applicable to many other inverse problems.
The problems we shall examine will be those for which, in the one dimensional case, the kernel $L(x,y)$ is either the Fourier or Laplace transform operator or some closely related transform. However the techniques we apply may be equally valid where the kernel is that of some other integral transform, in particular the modified complex Laplace transform.

1.2 Super-resolution and the Ill-conditioned Inverse Problem.

Of great importance in many branches of imaging is the possibility of restoring images beyond the classical diffraction limit of the imaging instrument; the process of super-resolution (Abbis et al [2]; Gerchberg [3] and Papoulis [4]). It has been shown that for a very poorly resolved image, significant improvement is possible (di Francia [5]).

However, for many objects, super-resolution demands impractically high signal to noise ratios. In [5] these limitations are argued to be due to the fact that the number of degrees of freedom of an image is finite. This idea is closely connected with the concept of ill-posed or ill-conditioned systems which is the principal barrier to the solution of noisy inverse problems and was first introduced by Hadamard [6] and [7]. In effect an image produced by some system (be it optical or one dimensional signal data) is ambiguous in the sense that many different objects have the same image within a given accuracy.

There is a large range of problems of this type. Whether our particular problem is in the field of military target identification, geophysical exploration, medical imaging, probing
media (e.g. eddy current imaging) or any other field of remote sensing, the demand for a simple, robust algorithm of general applicability is great.

The ill-conditioned inverse problem is an example of a chaotic system. It can be treated using deterministic chaos techniques which bring together probabilistic and deterministic descriptions. Predicting the path of the falling ball (a chaotic motion) in a game of bagatelle is an ill-conditioned problem. This unifying of deterministic and probabilistic approaches to the chaotic system will be carried out in section (1.4.2)

1.3 Intrinsic Ill-conditioning and Instability due to Noise.

Even in a hypothetical situation where we have made noise free measurements, some of these systems are not usefully tractable using the usual inversion methods such as Gaussian elimination. This is because the finite word length of the computer imposes in effect a small noise level on the observation. Such systems we shall refer to as being ill-conditioned. The inversion of the Laplace transform is one such ill-conditioned system and this will be discussed in chapter VI.

On the other hand if the system is not ill-conditioned then inversion will be less accurate due to the presence of noise in the observation. This is the case in many band-limited Fourier problems such as will be discussed in the two examples of chapter IV in this thesis. In section (2.3) we shall show mathematically the origin of these problems.
Here we shall make a brief review of the method first proposed by Tikhonov [8] for solving Fredholm equations of the first kind which was applied by Miller [9] to solve ill-posed Hilbert space problems. We shall then introduce the Linear Minimum Variance Estimator (LMVE) as the best possible approximation, \( \hat{f} \), to the original signal \( f \), in the sense that \( || g - L\hat{f} || \) is a minimum. We shall then unify the deterministic method of Tikhonov and the probabilistic method of the LMVE.


1.4.1 Deterministic Methods.

The initial data underlying experimental ill-posed problems contain random errors. Depending on the nature of this initial information one can either take a deterministic or a probabilistic approach to the derivation of approximate solutions. The three main deterministic methods are: the Selection Method, the Quasisolution, and the method of replacement of the equation with one close to it.

The selection method is the simplest method of solving ill-posed problems. It consists of calculating the operator \( Lf \) for elements of \( f \) belonging to some given subclass \( H \subset U \) of possible solutions (given by some \textit{a priori} knowledge); in other words we solve the direct problem. The solution is then given by an element \( f \in H \) for which the difference \( \rho_{\gamma}(Lf, g) \) attains its minimum where,
\[ \rho_{\mathcal{V}}(Lf, g) = \| Lf - g \|_B \] (1.3)

That is \( \rho_{\mathcal{V}}(Lf, g) \) gives the minimum distance between \( Lf \) and \( g \).

The two most important points in the stratagem employed by Tikhonov and Arsenin [10] are the use of the Quasisolution of equation (1.2), first employed by Ivanov [11] along with the simple replacement of the equation (1.2) with one close to it.

Suppose now that the operator \( L \) in equation (1.2) is completely continuous. If we seek a solution on the compact subspace \( M \subset U \) and our observation belongs to \( L(M) \) then knowledge of the inverse operator allows inversion using \( \hat{f} = L^{-1}g \). Often though noise renders this expression meaningless since in that case \( g \not\in L(M) \). However a solution \( f \in M \) which minimises the functional \( \rho_{\mathcal{V}}(Lf, g) \) on the set \( M \) is called a quasisolution of equation (1.2) on \( M \).

\[ \rho_{\mathcal{V}}(Lf, g) = \inf_{f \in M} \left[ \rho_{\mathcal{V}}(Lf, g) \right] \] (1.4)

If \( M \) is a compact set, a quasisolution obviously exists for every \( g \in \mathcal{V} \). If, in addition \( g \in L(M) \), then the quasisolution \( \hat{f} \) coincides with the usual (exact) solution of equation (1.2). There may be more than one quasisolution of equation (1.2).

It is possible to exhibit sufficient conditions for a quasisolution to be unique and for it to depend continuously on \( g \) as shown in [10]. Furthermore if \( W \) and \( U \) are Hilbert spaces (with \( V \) a subspace of \( W \), \( M \) is the ball \( \| \hat{f} \| \leq R \) in the space \( U \) and \( L \) a continuous operator, then in this case a quasisolution of equation (1.2) can be represented in the form of a series of eigenfunctions or eigenvectors of the operator \( L^*L \) where the dagger denotes the adjoint operator, Ivanov [12].
1.4.2 Connection between the Deterministic and Probabilistic approaches.

It is interesting that in the proof of the previous statement [12] it is pointed out that with a small perturbation of the functional in equation (1.4).

\[ \rho^2_{\nu} (L_f, \hat{g}) = \langle L\hat{f} - g|L\hat{f} - g\rangle + \epsilon \langle \hat{f} | \hat{f} \rangle \]  

(1.5)

the minimisation of which provides the quasisolution, has an Euler equation of the form

\[ (L^\dagger L^\epsilon I) f = L^\dagger g \]  

(1.6)

as proved elsewhere, Smirnov [13]. Note how, for \( g = Lf + n \), as the noise \( n \rightarrow 0 \) and consequently \( \epsilon \rightarrow 0 \) then the equality is maintained. In other words what we have done is to replace the original equation with one close to it. Alternatively, following Blanchard, Travis and Jones (to be published) [14], minimisation of the functional (1.5) can be done without differentiating.

The form of this Euler equation is important as it enables us to make the best choice of the regularisation parameter \( \epsilon \). Many workers seem unsure of how to choose \( \epsilon \) to get the best solution and furthermore, the literature is full of examples (e.g. Rushforth, Crawford and Zhou [15]) where only individual estimates are displayed and it is left up to the reader to imagine how consistent the results would have been for a set of observations \( g \) scattered around the true value \( Lf \). This problem has again been addressed most recently by Bertero, de Mol and Pike [16].

Tikhonov and Arsenin [10] state that "it is usually difficult to actually find the regularisation parameter \( \epsilon \) as a function \( \epsilon(\delta) \)"
(where \( \delta \) is the error in the initial data)''. On the contrary one can show (see section (1.4.3)) that the best choice of the regularisation parameter is simply related to the noise in the system and to the original function \( f \) which we are trying to reconstruct. Thus an estimate of the noise present along with some a priori knowledge about the original signal and the consistency of the reconstruction over several observations will enable us to obtain a consistent reconstruction with different noise samples of the same variance.

1.4.3 The Probabilistic Linear Minimum Variance Estimator.

The Linear Minimum Variance Estimator (LMVE), Sage [17], (also known as the the Markov estimator by Deutsch [18] and the stochastic estimator elsewhere) for equation (1.2) is given by

\[
\hat{f}_{LMV} = f_0 + P_A L^+ AB (R_B + L P L^+ AB)^{-1} (g - L f_0)
\]

(1.7)

where \( f_0 \) is the signal mean, \( E(f) \); \( P_A \) is the signal covariance, \( E[\delta f \delta f^* A] \), a linear operator in \( U \); and \( R_B \) is the noise covariance, \( E[n n^* B] \), a linear operator in \( W \). (Here \( E[.] \) denotes an expectation value, and \( \delta f = f - f_0 \)). By using the matrix inversion lemma, Sage [19], one can rewrite (1.7) giving

\[
\hat{f}_{LMV} = f_0 + (P^{-1}_A + L^+ AB R_B^{-1} L)^{-1} L^+ AB R_B^{-1} (g - L f_0)
\]

(1.8)

Notice that (1.8) reduces to the Euler Equation (1.6) when

\[
f_0 = 0, \quad P_A = \sigma^2_s I, \quad R_B = \sigma^2_n I \quad \text{and} \quad \epsilon = \sigma^2_n / \sigma^2_s.
\]

(1.9)

The LMVE gives the best possible estimation of the unknown signal \( f \) in the sense that the variance about a mean specified by a priori knowledge is minimised.
Chapter II.

Inversion Methods: Solution of Sets of Linear Algebraic Equations.

Introduction.

Work on image restoration can be classified into two broad areas according to the restoration method:
1. Linear, such as inverse filtering, Wiener filtering and discrete convolution, Frieden [20].
2. Non-linear, such as algebraic methods and maximum entropy: Frieden and Wells [21], Frieden [22], Frieden and Burke [23]. Bayes method: Hunt [24], Froehlich et al [25], Habbibi [26], Richardson [27], and maximum information, Nahi and Assefi [28].

Here we shall be primarily interested in linear inverse filtering though bilinear problems are considered in chapter III and we look ahead to other non-linear problems in chapter VIII.

2.1 Frames of Reference.

We shall deal here with algorithms for the inversion of the matrix representation of the operator $L$. For a detailed comparison of the previously used methods of solving systems of linear equations one should refer to Fox [29].

All the previously used algorithms for direct inversion depend on the invariance of matrix equations under orthogonal transformation. Coordinates have no absolute significance and the ones in which the problem is initially formulated are merely a
consequence of the theoretician's initial perception of the problem. Thus one coordinate system can be discarded in favour of another if one so wishes and the new system will be equally admissible. We can view a matrix from a certain frame of reference but we may introduce another frame of reference which might be more suited to the nature of the problem given.

Lanczos [30] has pointed out that matrix algebra is an example of the operation of the principle of relativity. In Newtonian physics 'scaffold and building' were so strongly cemented together that the entire building collapsed if the scaffold was removed. The fundamental difference between scaffold and building was the departure point of Einstein's theory of relativity. In matrix algebra the laws expressed by the matrix equations we wish to solve are not altered by a coordinate transformation.

The most commonly used algorithms for the solution of sets of linear equations are Gaussian elimination, Triangularisation, Householder transformation, Givens Transformation and the Lanczos method, Wilkinson [31]. All these algorithms start by transforming the matrix $L$ into upper triangular form. At that point the solution of the set of equations is straightforward since for the $n$th equation in

$$g = Lf$$

we only have $f_n$, whilst the $(n-1)$th equation only contains $f_n$ and $f_{n-1}$ and so on. This of course is the classic method of back substitution.

As already suggested, straightforward inversion of the equation (2.1), using for example Gaussian elimination, will fail if the system is at all ill-conditioned. On the other hand
Wilkinson states that *if the system is invertible* then the method of Gaussian elimination is often as good, if not better, than any in terms of stability, accuracy and of course speed. Indeed in [31] Wilkinson states that "Gaussian elimination with interchanging or pivoting for size cannot break down under any circumstances and will also give a unique result".

In general though, which algorithm works best depends on the particular matrix we are decomposing. For example Wilkinson states that for a positive definite symmetric matrix the symmetric Cholesky decomposition has all the virtues. Use of the flow diagram in the NAG library [32] will enable one to make the best choice of the previously existing algorithms for any particular (well-conditioned) case.

In any case, before employing any other scheme of inversion one might well consider using the classic method of solution by elimination.

However if one wishes to obtain insight into the nature of the system being looked at, one might well profit considerably by looking at the problem from another frame of reference. On the other hand whilst a well-conditioned system can be inverted quite adequately by direct Gaussian elimination we may well require to invert the same system many times with different sets of input data. Whether the system is an 'optical' one, as we shall shortly examine in chapter III, or one in medical tomographic imaging or indeed in geophysical imaging or some other field of remote sensing, once we have found the most suitable frame of reference for our particular problem this can be stored and used many times; the more times we use it then the more justified will be any expense we incurred in setting up the frame to start with.
In such circumstances where further simplification of the matrix is required, triangularisation is merely the first step; this is the case in the full singular value decomposition (SVD) which enables us to obtain the eigenvalues and eigenvectors of that matrix. Hitherto, SVD has been the most popular method for analysing and inverting ill-conditioned problems, Stewart [33].

For a full description of the techniques used to obtain the SVD one must refer to either [29] or to Wilkinson [34]. Here we shall explain the origin of the SVD and outline the methods for obtaining it.

2.2 The Singular Value Decomposition: A useful Orthonormal Coordinate Transformation.

The adjoint operator, $L_{AB}^*$, $W \rightarrow U$, of $L$ is defined by

$$\langle L_{AB}^* w | u \rangle_A = \langle w | Lu \rangle_B$$  \hspace{1cm} (2.2)

where the left-hand and right-hand terms are scalar products in $U$ and $W$ respectively and $w$ and $u$ are matrices whose columns are the vectors spanning $W$ and $U$. Thus $L_{AB}^*(V)$ is the orthogonal complement in $U$ of the null space of $L$; the subscripts indicate that $L$ is dependent on the choice of scalar products in equation (2.2) - this choice will be used in section (2.4.6).

Bases of orthonormal vectors can be used to solve the linear inverse problem. Such bases in $U$ and $V$ are denoted by $\{a_k\}$ and $\{b_k\}$ respectively reflecting the choice of scalar product where

$$\langle a_k | a_j \rangle_A = \delta_{k,j} \text{ and } \langle b_m | b_n \rangle_B = \delta_{m,n}$$  \hspace{1cm} (2.3)

Bases in the null space of $L$ and the orthogonal complement of $V$ in $W$ are denoted by $\{s_k\}$ and $\{t_k\}$, respectively.

The SVD of $L$ produces a particularly useful pair of bases. In
In the absence of noise, using equations (2.1) and (2.4)

\[ \langle b_k | g \rangle_B = \langle b_k | L f \rangle_B = \langle L^*_{AB} b_k | f \rangle_A = \langle \chi_k a_k | f \rangle_A = \chi_k \langle a_k | f \rangle_A \]

and we obtain

\[ f = \sum_k \chi_k^{-1} \langle b_k | g \rangle_B a_k + s \]

where \( s \) is any vector in the null space of \( L \).

### 2.3 Origin of Ill-conditioning.

In the presence of noise the values of our observation \( \langle b_k | g \rangle \) will in fact be \( \langle b_k | L f + n \rangle \) and when we divide by any small value of the singular value \( \chi_k \), as in (2.6), large errors accrue. Whether \( n \) is due to the dominance of rounding errors or due to observation noise this is the origin of the ill-posed problem.

Expressing the inversion operation mathematically we have

\[ f_{\text{approx}} = L_{\text{approx}}^{-1} g_{\text{noisy}} \]

and this is in error due to (i) inaccuracy in \( L_{\text{approx}}^{-1} \) and (ii) due to noise on \( g \). In singular value representation

\[ f_{\text{approx}}^{(i)} = (\chi^{-1}_{\text{approx}}) f_{\text{actual}}^{(i)} + n \]

The error in the first term is due to inaccuracy in \( (\chi^{-1}_{\text{approx}}) \), i.e. inversion. The error in the second term is unavoidable unless \( n_i = 0 \), even if \( (\chi^{-1}_{\text{approx}}) \) has no error, but it is compounded if it does.
2.4 Algorithms for the Singular Value Decomposition.

In the standard SVD, either the Householder or Givens transformation is used in order to first tridiagonalise the matrix (which transformation is used depends on the relative dimensions of the matrix). This tridiagonalisation can also be achieved by using the standard Lanczos algorithm with comparable stability Wilkinson [31] and about three times the number of multiplications (Table 2.1). See chapter III for an explicit explanation of this process.

Table 2.1 Number of Multiplications Involved in Each Algorithm for Condensation of a Matrix

<table>
<thead>
<tr>
<th>Algorithm Used</th>
<th>Number of Multiplications For Square Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Householder Tridiagonalisation</td>
<td>$5/3 \ n^3$</td>
</tr>
<tr>
<td>Givens Tridiagonalisation</td>
<td>$10/3 \ n^3$</td>
</tr>
<tr>
<td>Standard Lanczos Tridiagonalisation</td>
<td>$4n^3$</td>
</tr>
<tr>
<td>Jones-Lanczos Bidiagonalisation</td>
<td>$5n^3 + n(n-1)$</td>
</tr>
</tbody>
</table>

*The three tridiagonalisation methods are described in [31]*

The bidiagonal form is then obtained by using a sequence of plane rotations, and finally the full SVD is obtained by using the Golub version of the QR algorithm [31]. The computational cost of this last part of the SVD is unpredictable, since the number of iterations of the QR algorithm required in order to extract each singular value and the corresponding left-hand and right-hand singular vectors depends on the operator itself.
As already stated, we do not need to fully decompose the matrix in order to invert it successfully. Therefore it seems unnecessary to risk the possibility of many iterations of the QR algorithm, as despite continued increases in computer speed, fast inversion is still at a premium. In any case, the SVD does seem a rather inelegant process.

A method of inversion which retains all the useful attributes of the SVD and yet is much simpler to code is the method of direct bidiagonalisation used by Jones and Travis [35]. The method is faster than the SVD though it is not possible to say by how much in general. This is because although direct bidiagonalisation will always have an upper limit on the number of multiplications for a given matrix dimension, the number in the SVD depends on the number of iterations of the QR algorithm as stated above.

The algorithm used by Jones and Travis is a modification of the Lanczos method. Hereinafter it will be referred to as the Jones-Lanczos or JL algorithm after Jones and Lanczos. Here we shall explain and compare the two methods before introducing an example which will be solved by application of the JL algorithm to $L$. 
2.5 Jones-Lanczos Direct Bidiagonalisation
(The JL Algorithm).

As mentioned above the standard SVD algorithm produces, at one stage, a bidiagonal form, though this is not produced directly. Algorithms for the regularised inversion of ill-conditioned least squares problems starting from a bidiagonalisation produced in this way have been given by Elden [36].

A more general formulation is given here in which a modified Lanczos method (The Jones-Lanczos or JL method) is used to obtain the bidiagonal form directly. Firstly though we shall describe the standard Lanczos method.

2.5.1 The Generalised Hessenberg Process and the Lanczos method for a non-symmetric matrix.

Suppose we are given a set of $\{a_i\}$ in a Hilbert space $U$, the naming of the vectors reflecting the choice of inner product which gives $\langle a_i, a_i \rangle = 1$. Suppose also that an operator $M:U\rightarrow U$ is defined. If we operate on an arbitrary normalised vector $b$ with $M$ then the resulting vector will have a component in the direction of $b$ and can be written

$$
\beta b = Mb - \alpha b
$$

where $\beta$ is a normalising factor and $\alpha$ allows us choice in how much of $b$ is in the direction of $b$. Similarly

$$
\beta b = Mb - \alpha b
$$

(2.9)
and, in general, at the rth step we can write the result as any combination of all the previous vectors

$$\beta_r b_{r+1} = Mb_r - \sum_{i=1}^{r} \alpha_i b_i$$  \hspace{1cm} (2.11)

We can assemble equation (2.11) into one matrix equation

$$[H^b] = [\langle b_j | Mb_k \rangle] = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \ldots & \alpha_{1n} \\
\beta_1 & \alpha_{22} & \alpha_{23} & \ldots & \alpha_{2n} \\
\beta_2 & \alpha_{33} & \ldots & \ldots & \alpha_{3n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\alpha_{nn}
\end{bmatrix}$$  \hspace{1cm} (2.12)

So now $[H^b]_{jk}$ is the representation of the matrix operator $M$ in the new frame of reference.

In the general Hessenberg process [37] the $\alpha_i$ are chosen so that $b_{r+1}$ is orthogonal to $a_1, \ldots, a_r$. By contrast, in the method of Lanczos [38] the vectors $a_i$ are generated simultaneously with the $b_i$. In fact the $a_i$ are determined from $H^T$ in exactly the same way as the $b_i$ are generated from $H$. Since the $a_i$ and the $b_i$ play such closely connected roles, Wilkinson [31], following Lanczos [38], denotes them by $c_i$ and $c_i^*$ where the asterisk does not indicate complex conjugate but that the $c_i^*$ derive from the $H^T$ in exactly the same way as the $c_i$ derive from $H$. Here we use a different notation, that is $a_i$ and $b_i$, in anticipation of the next section where the definition of an $A$-orthonormal vector space and a $B$-orthonormal vector space opens the way for a mixed vector space representation. For the time being though both the sets $a_i$ and $b_i$ are in the same space.

We have therefore
\[ \beta_{r+1} b_{r+1} = M b_r - \sum_{i=1}^{r} \alpha_{i} b_i \]  
\[ \text{and} \quad \beta_{r+1}' a_{r+1} = M^\top a_r - \sum_{i=1}^{r} \alpha_{i} a_i \]

or as before

\[ [N_j^b] = [\langle b_j | M b \rangle] \quad \text{and} \quad [M_j^\top a] = [\langle a_j | M^\top a \rangle] \]

From here we can show, following Wilkinson [31], or see (for the proof in abstract vector notation) Appendix I, that from equations (2.13) and (2.14), we have

\[ M b_r - \alpha_{r} b_r - \beta_{r-1} b_{r-1} = \beta_{r+1} b_{r+1} \]  
\[ \text{and} \quad M^\top a_r - \alpha_{r} a_r - \beta_{r-1}' a_{r-1} = \beta_{r+1}' a_{r+1} \]

So the Lanczos algorithm starts by choosing arbitrary \( b_1 \) and \( a_1 \) and step one is

\[ M b_1 = \alpha_1 b_1 + \beta_1 b_2 \]  
\[ \text{and} \quad M^\top a_1 = \alpha_1 a_1 + \beta_1' a_2 \]

with

\[ \langle b_1 | M a_1 \rangle = \alpha_1 = \alpha_1' \]

\( \beta_1, \beta_1', b_2 \) and \( a_2 \) are obtained by normalising the term \( \langle M a_1 - \alpha_1' a_1 | M b_1 - \alpha_1 b_1 \rangle \) so the next step is

\[ M b_2 = \alpha_2 b_2 + \beta_1 b_1 + \beta_2 b_3 \]

and

\[ M^\top a_2 = \alpha_2 a_2 + \beta_1' a_1 + \beta_2' a_3 \]

with

\[ \langle a_2 | M c_2 \rangle = \alpha_2 = \alpha_2' \]

and we obtain \( \beta_2, \beta_2', b_3 \) and \( a_3 \) by normalising the term \( \langle M a_2 - \alpha_2' a_2 | M b_2 - \alpha_2 b_2 - \beta_1 b_1 \rangle \) and at the rth step

\[ M b_r = \alpha_{r} b_r + \beta_{r-1} b_{r-1} + \beta_{r+1} b_{r+1} \]  
\[ M^\top a_r = \alpha_{r} a_r + \beta_{r-1}' a_{r-1} + \beta_{r+1}' a_{r+1} \]

\[ \alpha_r = \langle a_r | M b_r \rangle \]
and we obtain $\beta_{r-1}', \beta_{r-1}, b_r$ and $a_r$ by normalising the term

$$< M^T a_r - \alpha_r a_r - \beta_{r-1}' a_{r-1}, M b_r - \alpha_r b_r - \beta_{r-1}' b_{r-1} > .$$

Hence we obtain the tridiagonal matrix $[M_{jk}]$:

$$[M_{jk}] = 
\begin{bmatrix}
\alpha_1 \beta_1 & & \\
\beta_1 \alpha_2 \beta_2 & \ddots & \\
& \ddots & \ddots & \ddots \\
& & \beta_{N-1} & \\
& & & \beta_{N-1} \alpha_N \\
\end{bmatrix}$$

(2.27)

and similarly for $[M_{jk}^T]$ where instead of $\beta_n$ we have $\beta_n'$.

2.5.2 Solution in the absence of noise (standard Lanczos algorithm).

In the new frame of reference obtained in section (2.41) the explicit expression for equation (2.1) becomes:

$$
\begin{bmatrix}
<c_1|g> \\
<c_2|g> \\
<c_3|g> \\
\vdots \\
<c_N|g>
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \beta_1 & & & \\
\beta_1 \alpha_2 \beta_2 & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & \beta_{N-1} & & \\
& & & \beta_{N-1} \alpha_N
\end{bmatrix}
\begin{bmatrix}
c_1 |f> \\
c_2 |f> \\
c_3 |f> \\
\vdots \\
c_N |f>
\end{bmatrix}
$$

(2.28)

From here one can proceed to obtain the full singular value decomposition by first bidiagonalising using a series of plane rotations as described in [31] followed by application of the QR algorithm.
2.5.3 The Jones-Lanczos Algorithm (JL algorithm).

It is noted that the operator $M$ of section 2.4.1 mapped a vector into the same space. Consider an operator such as that which was introduced in Chapter I for deriving the SVD. In that case $L(U)$ was not a subset of $U$ as in the standard Lanczos method. For example a matrix representation of $L$, $[L_{jk}]$, which is not square will, when operating on a vector in one space, produce a vector of different dimension. In this case, unless the new vector is embedded in the original space it will belong to a new space. Furthermore if the inner product is defined to be different for members of $L(U)$ then we certainly have a different space.

Therefore let us reconsider the operator $L$ where $L(U) \subset V$ as originally stated in the previous chapter. If we choose an arbitrary vector $a_0$ in $U$ and operate on it with $L$ then we obtain a vector in $V$. The result can then be normalised by taking the inner product of $L a_0$ with itself to give $b_1$. That is:

$$L a_0 = \beta_0 b_1$$

(2.29)

($a_0$ is not used subsequently; its only purpose being to ensure that $b_1$ is in $L(U)$). Bases $\{a_k\}$ and $\{b_k\}$ are now obtained to satisfy
\[ \begin{align*}
\langle b_1 | b_1 \rangle_B &= 1 \\
L^*_B b_1 &= \tilde{a}_1 a_1 \quad \text{with} \quad \langle a_1 | a_1 \rangle_A = 1 \\
L a_1 &= \alpha_1 b_1 + \beta_1 b_2 \quad \text{with} \quad \langle b_2 | b_2 \rangle_B = 1, \langle b_2 | b_1 \rangle_B = 0, \\
L^*_B b_2 &= \tilde{\alpha}_1 a_1 + \alpha_2 a_2 \quad \text{with} \quad \langle a_2 | a_2 \rangle_A = 1, \langle a_2 | a_1 \rangle_A = 0, \\
\vdots \\
L a_k &= \alpha_k b_k + \beta_k b_{k+1} \quad \text{with} \quad \langle b_{k+1} | b_{k+1} \rangle_B, \langle b_{k+1} | b_k \rangle_B = 0, \\
L^*_B b_{k+1} &= \tilde{\beta}_k a_k + \tilde{\alpha}_{k+1} a_{k+1} \quad \text{with} \quad \langle a_{k+1} | a_{k+1} \rangle_A = 1, \\
\vdots \\
\text{with} \quad \langle a_{k+1} | a_k \rangle_A = 0.
\end{align*} \]

Notice how at the kth step, operation with either \( L \) or its adjoint produces a vector with two components. One of these lies in the direction of the previously generated vector in that space while the other lies in the direction of the new vector. This is in contrast with the standard Lanczos method where there is also a component left in the direction of the vector operated upon, all vectors lying in the same space.

Thus, at the kth step, \( L a_k \) has a component in the direction \( b_k \) with a coefficient
\[ \langle b_k | L a_k \rangle_B = \langle L^*_B b_k | a_k \rangle_B = (\tilde{\beta}_{k-1} a_{k-1} + \tilde{\alpha}_k a_k | a_k \rangle_A = \alpha_k \quad (2.36) \]
The component orthogonal to \( b_k \), \( L a_k - \alpha_k b_k \), is normalised to give \( b_{k+1} \) and the normalisation factor \( \beta_k \). Similarly, \( L^*_B b_{k+1} \) has a component \( \tilde{\beta}_k a_k \) along \( a_k \), and \( \tilde{\alpha}_{k+1} \) and \( a_{k+1} \) are obtained by normalisation of \( L^*_B b_{k+1} - \tilde{\beta}_k a_k \).

Using the matrix representation with the bases \( \{a_k\} \) and \( \{b_k\} \), the operator \( L \) has the matrix representation
and the adjoint operator \( L^\dagger_{AB} \) has the representation

\[
\begin{bmatrix}
L^\dagger_{jk}^{ab}
\end{bmatrix} \triangleq \begin{bmatrix}
\langle a | L^\dagger_{AB} b \rangle
\end{bmatrix}
= \begin{bmatrix}
\alpha_1 & 0 & \ldots & \ldots \\
\beta_1 & \alpha_2 & 0 & \ldots \\
0 & \beta_2 & \alpha_3 & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \beta_{N-1} & \alpha_N
\end{bmatrix}
\] (2.37)

where \([.]^H\) is the Hermitian conjugate matrix. Other choices of the adjoint operator \( L^\dagger_{AB} \) are available depending on our choice of the scalar product in equation (2.2). Use of this will be made in section 2.4.6.

Note that whatever the choice of the operators \( A \) and \( B \) the \( \alpha_n \) and \( \beta_n \) may always be chosen real. The only restriction is that

\[
|\beta_k|^2 = \langle L a_k - \alpha_k b_k | L a_k - \alpha_k b_k \rangle
\] (2.39)

and

\[
|\alpha_k|^2 = \langle L^\dagger_{AB} b_k - \beta_k a_k | L^\dagger_{AB} b_k - \beta_k a_k \rangle
\] (2.40)

which enables us to choose the normalisation coefficients to be real and positive so that the bidiagonalised matrix is then rendered positive definite by the regularisation factor \( \epsilon \) (equation (2.51)). In the more general case they need not be real; hence complex conjugate bars are on the appropriate normalisation coefficients in equations (2.30-2.35).

Also note that equation (2.38) is a mixed basis representation in contrast to that in the case of the standard Lanczos method of section 2.4.1.
2.5.4 Solution in the Absence of Noise (JL algorithm).

In the absence of noise equation (2.1) has the solution

\[ f = \sum_k |a_k \rangle \langle a_k | f \rangle_A \]

(2.41)

where

\[ \sum_k L_{jk}^b a_k \langle a_k | f \rangle_A = \langle b_k | g \rangle_B \]

(2.42)

Thus

\[ \langle a_1 | f \rangle_A = \frac{1}{a_1} \langle b_1 | g \rangle_B \]

(2.43)

\[ \langle a_2 | f \rangle_A = \frac{1}{a_2} \left[ \langle b_2 | g \rangle_B - \beta_1 \langle a_1 | f \rangle_B \right] \]

(2.44)

This solution enables us to associate possible errors arising in the noisy case with small values of \( a_k \) as is the case in the SVD method.

2.5.5 Solution in the Absence of Noise (Singular Case).

Suppose the matrix representation of the operator \( L \) is singular, that is the rank of \([L^b_{jk}]\) is \( r \) where \( r < L \), \( L \) being the smaller dimension of \([L^b_{jk}]\). In that case, by definition, there must be no more than \( r \) non-zero singular values in the SVD of \([L^b_{jk}]\). Therefore the space \( U \) becomes a truncated singular space and the linear system can only be solved for, at most, \( r \) variables.

Thus suppose we are solving the linear system by the JL method using the procedure described by equations 2.30-2.35. If at
the rth stage we should find that the unnormalised vector
\[ \beta_r b_{r+1} = L a_r - \alpha_r b_r \]  \hspace{1cm} (2.45)

goes to zero (within machine accuracy) then we may interpret this as a consequence of the singular (rank r) quality of the matrix $[L_{jk}]$. We can then invert the system in just the same way except that in equation (2.39) $f$ will consist of a sum over $r$ terms only.

We shall encounter an example of such a singular problem in chapter IV.

However, one should note that if some term $\beta_r b_{r+1}$ goes to zero, this does not necessarily mean the matrix is singular. In a similar manner to that described in [31] for the standard Lanczos method, it is quite possible that for some choice of the starting vector $u_0$, $\beta_r b_{r+1}$ may go to zero and yet the vectors already generated do not form a complete set.

If there is doubt about the rank of the matrix another $u'_0 \neq u_0$ should be generated and another $b'_r = b_{r+1}$ obtained by operating with $L$. If, after orthogonalising $b'_r$ with respect to the vectors $b_0 - b_r$, the result is not zero the process can be continued setting $\beta_r$ to zero and using $b'_r$ instead of $b_{r+1}$ in equation (2.43). When it is found that reorthogonalisation provides a null vector whatever the value of $b'_{r+1}$ a complete set of vectors has been obtained and the space filled. An example of such an occurrence will be described in chapter IV.
2.5.6 Solution for Noisy Data.

If we choose
\[
\langle a_k | a_j \rangle_A = \langle a_k | A a_j \rangle_2
\]
and
\[
\langle b_k | b_j \rangle_B = \langle b_k | B b_j \rangle_2
\]
where the scalar products with subscript 2 denote the
\(L_2\) (continuous) or \(l_2\) (discrete) scalar products and \(A\) and \(B\) are
positive definite operators given by
\[
A = \sigma_s^2 P_2^{-1}, \quad B = \sigma_n^2 R_2^{-1}
\]
with \(\sigma_s^2\) and \(\sigma_n^2\) such that
\[
| A | = | B | = 1
\]
then
\[
P_A = P_2 A = \sigma_s^2 I, \quad R_B = R_2 B = \sigma_n^2 I
\]
The linear operators \(P\) and \(R\) used here are defined in chapter I
after equation (1.7). Thus equation (1.8) can always be reduced to
the Tikhonov form, equation (1.6), and, from equation (2.37),
\((L_{AB}^L + \epsilon)\) will have the tridiagonal matrix representation

\[
\begin{bmatrix}
|\alpha_1|^2 + |\beta_1|^2 + \epsilon & \bar{\beta}_1 \alpha_2 & 0 & 0 & \cdots \\
\bar{\alpha}_2 \beta_1 & |\alpha_2|^2 + |\beta_2|^2 + \epsilon & \bar{\beta}_2 \alpha_3 & 0 & \cdots \\
0 & \bar{\alpha}_3 \beta_2 & |\alpha_3|^2 + |\beta_3|^2 + \epsilon & 0 & \cdots \\
0 & 0 & \bar{\alpha}_4 \beta_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

Inversion of equation (2.51) can then be performed by Gaussian
elimination without pivoting, since the matrix is positive
definite if the $\alpha_n$ and $\beta_n$ are defined real as described earlier.

In this chapter we have provided means of inverting noisy, ill-conditioned linear inverse problems. Will the ability to solve such a problem be of any use in the inversion of experimental data? In order to answer this question we shall, in the next chapter, discuss the origin of the Fourier problem in optics and in chapter IV present the results of inversions using the JL algorithm for such problems.
Chapter III.

Fourier Problems.

3.1 Fourier Optical Image Processing.

The restoration of images which have been degraded is a key problem in image processing and is normally required following acquisition of sensor data and prior to detailed image analysis. In Fourier optics, much work has been devoted to the problem of object restoration. Amongst the many reviews of the field which have been published Frieden [39] and Goodman [40] are recommended.

However, even before discretisation, the well known Fourier transforming property of lenses, Born and Wolf [41], is only an approximation except for an ideal lens. Indeed many optical systems will not be linear and are not directly solvable using the methods above.

Here we shall provide a short review of systems theory with optical problems in mind. We shall show which systems are linear, or approximately so. We shall show how in some cases a non-linear problem can be reduced to a linear problem, or at least a combination of linear problems.
We represent a general two-dimensional system by the operator 
$L$ as before:

$$g(x,y) = L[f(x,y)]$$  \hspace{1cm} (3.1)

and we have the usual definition of the linear system

$$L[af_1(x,y) + bf_2(x,y)] = aL[f_1(x,y)] + bL[f_2(x,y)]$$  \hspace{1cm} (3.2)

for all inputs $f_1$ and $f_2$ and all constants $a$ and $b$. Linearity
implies that an input function can be broken down into elementary
functions, each of which can be separately passed through the
system; the total output is then the sum of the "elementary"
outputs.

Using the so-called sifting property of the delta function,
any input function can be considered to be a linear combination of
weighted and displaced delta functions:

$$f(x,y) = \int \int f(x_1,y_1)\delta(x-x_1)\delta(y-y_1) \, dx_1 \, dy_1$$  \hspace{1cm} (3.3)

giving an output

$$g(x,y) = L[f(x,y)] = \int \int f(x_1,y_1)L[\delta(x-x_1)\delta(y-y_1)] \, dx_1 \, dy_1$$ \hspace{1cm} (3.4)

The system response at $(x,y)$ due to a delta function input at
$(x_1,y_1)$ is called the impulse response function

$$h(x,y;x_1,y_1) = L[\delta(x-x_1)\delta(y-y_1)]$$  \hspace{1cm} (3.5)

In optical imaging systems, the quantity $h$ is called the point
spread function. Thus, for a linear system:

$$g(x,y) = \int \int f(x_1,y_1)h(x,y;x_1,y_1) \, dx_1 \, dy_1$$ \hspace{1cm} (3.6)

If the impulse response function of a linear system depends only
on the coordinate differences $(x-x_1)$ and $(y-y_1)$, and not on each
coordinate separately then:
\[ h(x,y; x_1,y_1) = h(x-x_1,y-y_1) \]  

(3.7)

In this case equation (3.6) becomes a simple convolution relationship

\[ g(x,y) = \int \int f(x_1,y_1) h(x-x_1,y-y_1) dx_1 dy_1 \]  

(3.8)

Such a system is known as stationary. In optical imaging, a stationary optical system is usually called isoplanatic, though some workers refer to the property as shift-invariance or space-invariance. Isoplanaticity requires that the point spread function is the same for all field angles and implies that aberrations are independent of field angles.

Although a large number of physical processes might be satisfactorily modelled by linear systems, the constraint of linearity is too stringent in many situations, as is the constraint of isoplanicity. Here we shall give two examples of linear optical processes, though as we shall see the linearity is not rigorous in a physical sense for the latter example.

3.3 Linearity in Optical Imaging.

We first consider the case where the object plane (figure 3.1) is illuminated by a plane or spherical wave; that is by perfectly spatially coherent light. We denote the complex amplitude immediately after the diffracting obstacle by \( U_{in}(x_1,y_1) \).

Correspondingly the complex amplitude in the image is denoted by \( U_{out}(x,y) \). Usually \( (x_1,y_1) \) and \( (x,y) \) are referred to the image space, Dainty [42], so that for real object coordinates \((a,b)\) we have
Figure (3.1) Linearity of ideal optical imaging: the complex amplitude at \((x,y)\) due to an input impulse at \((x_1,y_1)\) is \(h(x,y;x_1,y_1)\). Integrating to get a total output complex amplitude gives:

\[
U_{\text{out}} = \iint U_{\text{in}}(x_1,y_1) h(x,y;x_1,y_1) \, dx_1 \, dy_1
\]
where $m$ is the magnification of the image.

Let the complex amplitude at $(x,y)$ in the output due to an input impulse at $(x_1,y_1)$ be $h(x,y;x_1,y_1)$. Integrating to get the total output complex amplitude we obtain

$$U_{\text{out}} = \int \int_{-\infty}^{\infty} U_{\text{in}}(x_1,y_1)h(x,y;x_1,y_1)\,dx_1\,dy_1$$

since complex amplitudes are additive. Note that this is exactly the same form as equation (3.6) for an arbitrary linear system and $h(x,y;x_1,y_1)$ is the complex amplitude point spread function (i.e. the complex amplitude of the image of a point). For an isoplanatic system we can make an appropriate modification to the kernel in equation (3.10) but in either case we can see that for this spatially coherent optical system we have a linear transformation in the complex amplitude.

Suppose now that we have light which is not perfectly spatially coherent. The time-varying scalar field expressed in a complex notation is then defined by $V(x,t)$ such that the actual scalar field is given by the real part of $V(x,t)$.

For partially coherent light $V(x,t)$ (also known as the analytic signal due to its analytic properties), can be written as the product of a slowly varying function, the time varying complex amplitude $U(x,t)$, multiplied by the term $e^{-i\omega t}$, that is

$$V(x,t) = U(x,t) e^{-i\omega t}$$

The instantaneous intensity is defined as

$$I(x,t) = |U(x,t)|^2$$

Suppose now that $U(x,t) = U(x)$; this implies perfect coherence.
3.4 Reconciliation of the Bandlimited Inverse Fourier Problem in Amplitude with an Actual Physical Experiment.

There are many examples in the literature where a technique for solving a linear inverse problem is applied to the inversion of bandlimited amplitude Fourier data (e.g. [15]). In an optical system such as that depicted in figure (3.1) it is more likely that we would be measuring optical intensity. Now returning to the discrete notation of chapter I for a moment, consider one component of equation (1.2)

\[ g_i = \sum_j L_{ij} f_j \]  

(3.13)

but the instantaneous optical intensity, for perfectly coherent light is given by

\[ I(x) = |U(x)|^2 \]  

(3.14)

and in this case, having discretised, we see that one pixel in our observation is given by:

\[ I_i = g_i^* g_i = \sum_j L_{ij} f_j \sum_k L_{ik}^* f_k \]

\[ = \sum_{k,j} L_{ij}^* L_{ik} f_j^* f_k \]

\[ I_i = \sum_{k,j} A_{ik,jk} f_j^* f_k \]  

(3.15)

or in continuous form

\[ I(x) = \int \int q(x; x_1, x_2) f(x_1)^* f(x_2) dx_1 dx_2 \]  

(3.16)

Note that the intensity at any point in the observation depends on an integral over two variables. In other words, for a bandlimiting Fourier optical system as depicted in figure (3.1), then, in perfectly spatially coherent light, the system is bilinear in the optical intensity. In fact equation (3.16) is a
special case of the most general representation of bilinear transformations of which more will be said shortly.

Note that we can never have perfectly spatially coherent light though, depending on the circumstances, we may be justified in treating it so. A full treatment of problems involving partial coherence is given by Born and Wolf [41] and in Saleh and Rabbani [43] bilinearly degraded images in partially coherent illumination are reconstructed using finite impulse response digital filters.

Suppose now that we are not justified in treating the radiation as perfectly spatially coherent, that is, \( U(x,t) \neq U(x) \). In that case we must remember that what most detectors actually measure is the time averaged intensity \( \overline{I}(x) \), given by

\[
\overline{I}(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} I(x,t) \, dt
\]  

(3.17)

(A more elegant approach is through the use of ensemble averages; see [41] which also provides a further list of references).

Thus, in the general case the time varying complex averages are given by equation (3.17) where the input and output complex amplitudes are now also dependent on the time \( t \). The time averaged intensity for the non-coherent case is then given by

\[
\overline{I}_{\text{out}}(x,y) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} |U_{\text{out}}(x,y,t)|^2 \, dt
\]  

(3.18)

where

\[
|U_{\text{out}}|^2 = \int_{-\infty}^{+\infty} h(x,y;x_1,y_1)h^*(x,y;x_2,y_2)U_{\text{in}}(x_1,y_1,t)U_{\text{in}}(x_2,y_2,t) \, dx_1 \, dx_2
\]  

(3.19)

and is known as the mutual intensity of the narrowband light which in usual notation is given by

\[
J_{\text{in}}(x_1,x_2) = \langle U_{\text{in}}(x_1,t) | U_{\text{in}}^*(x_2,t) \rangle
\]  

(3.20)
Incoherence is defined to be when
\[ J(x_1, x_2) = \overline{I}(x_1, \delta(x_1 - x_2)) \]  
(3.21)
or, in other words, when two neighbouring points \( x_1 \) and \( x_2 \) have uncorrelated fields for any \( x_1 \neq x_2 \). Clearly this is physically impossible since all optical fields are correlated over distances approximately equal to one wavelength.

Thus substituting for mutual intensity into equation (3.18) we obtain
\[ \overline{I}_{out}(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| h(x, y; x_1, y_1) \right|^2 \overline{I}_{in}(x_1, y_1) \, dx_1 \, dy_1 \]  
(3.22)
and even without the constraint of isoplanaticity (which gives a convolution) we see that in perfectly incoherent illumination the optical system is linear in intensity and so the sort of technique we have applied may be applied directly to such problems.

One method for representing the input / output of non-linear systems comes from the Volterra series. The second order term of the Volterra series expansion is a special case of the transformation connecting the input / output characteristics of a one-dimensional bilinear systems described by the equation
\[ g(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x; x_1, x_2) f_1(x_1) f_2^*(x_2) \, dx_1 \, dx_2 \]  
(3.23)
where \( g(x) \) is the output at \( x \), \( f_i(x_i) \) is the input \( x = x_i \) for \( i = 1, 2 \) and \( q(x; x_1, x_2) \) specifies the double impulse response (DIR) of the bilinear system at the output coordinate \( x \) due to input impulses at input coordinates \( x_1 \) and \( x_2 \).

A special case of the above of interest here is the single input single-output system described for the one-dimensional system by
which is the same as equation (3.16) above.

This expression is frequently referred to in the literature as the bilinear representation, Bose [44] although it is in fact a special case of the more general equation (3.24) as pointed out by Kim and Bose [45] where it is shown that when the degrading phenomenon can be modelled by a shift-variant bilinear system, the data restoration problem can be most conveniently formulated as a special system of linear equations with non-negative coefficients.

Problems represented by this special bilinear equation include coherent imaging through the turbulent atmosphere, Saleh [46], imaging by optical systems with time varying pupils Welford [47], high resolution X-ray imaging systems where the object is illuminated by partially coherent waves, Yamakoshi and Sato [48], and imaging on translucent substrates with thin absorbing patterns on its surface and incoherently illuminated, Goodman and Johnson [49].

In their paper, Kim and Bose [45], describe several different approaches toward restoring bilinearly degraded images. They conclude that their recursive technique performs well on noisy images when the noise and signal are uncorrelated and the noise itself is additive.

Kim and Bose state that when the constraints of causality or weak causality described below are violated, direct recursive schemes for restoration do not exist. In fact most optical systems are non-causal. This suggests that if the bilinear problem we wish
to solve can be reduced to a system of linear equations then we might need to solve them directly by one of the methods described in chapter II. As the dimension of the problem increases this would increase the dimension of the basis vectors required to unmanageable sizes both in terms of the amount of computer space needed to store the vectors in, for example, the JL algorithm, and also the time needed to do the decomposition.

In principle though direct inversion is possible. Hence for higher dimensional systems which need to be inverted many times with different sets of input data, a once and for all inversion, using for example the JL algorithm, may be worthwhile whatever the initial cost. Also, as was mentioned in chapter II some systems of higher dimension may be either so singular or ill-conditioned that the amount of computing required to provide all the available information (according to the prevailing noise conditions) might be less than would be indicated by the dimensions of the full system alone.

For a system which does not lend itself to methods of direct inversion, Kim and Bose state that it is sometimes possible to exploit the advantages of recursion by decomposing a non-causal problem into several causal or weakly causal problems, to each of which the recursive technique can be applied and the results superposed.
3.5 Restoration of One-dimensional Bilinearly Degraded Images via the Two-dimensional Linear Model.

The possibility of deducing certain properties of n-dimensional bilinear systems from the analysis of corresponding two-n-dimensional linear systems is well known. This fact, along with the computational problems mentioned in the previous paragraph, encourages the use of either a recursive scheme such as the Gerchberg-Papoulis algorithm, Gerchberg [3] and Papoulis [4], or another fast recursive restoration scheme such as used by Bose [44].

Let us first see though how we can reduce a one-dimensional bilinearly degraded image to a two-dimensional linear model. The discrete counterpart of equation (3.24) is

\[
g(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2)f(m_1)f(m_2)\]

(3.25)

where \([g(n)]\) and \([f(m)]\) are, respectively, the output and input sequences, assumed real, \(q(n; m_1, m_2)\) is the discrete system DIR at the output coordinate \(n\) due to unit impulses at the input coordinates \(m_1=m_1\) and \(m_2=m_2\), and \(N\) is the finite number of equally spaced points at which the signal is sampled. Note that \(q(n; m_1, m_2)\) is, in general, complex with

\[
q(n; m_1, m_2) = q(n; m_2, m_1)
\]

(3.26)

and also

\[
\text{Re}[q(n; m_1, m_2)] = 0
\]

(3.27)
The output space \([g(n)]\) satisfies the non-negativity constraint
\[ g(n) \geq 0 \] (3.28)
and the objective is to recover \([f(m)]\) in equation (3.25) subject to the additional constraint
\[ f(m) \geq 0 \] (3.29)

In order to use recursive methods such as described in [44] the system must be causal. The causal restriction of equation (3.25) is
\[
g(n) = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} q(n; m_1, m_2) f(m_1) f(m_2) \] (3.30)
which implies that in the causal case
\[ q(n; m_1, m_2) = 0 \quad m_1 > n \text{ and/or } m_2 > n \] (3.31)

Although the restriction imposed by equation (3.31) is invalid in many situations such as the optical one described earlier, it is often possible to decompose a problem described by equation (3.25) into a sum of problems each of which, as described in [45], can be tackled by a recursive method or perhaps by a direct method such as can be used in highly singular or ill-conditioned cases such as was examined in chapter II.

The one-dimensional bilinear system is in fact a special case of a two-dimensional linear system, that is
\[
G(n_1, n_2) = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} Q(n_1, n_2; m_1, m_2) F(m_1, m_2) \] (3.32)
\[ 0 \leq n_1, n_2 \leq N-1 \]
with the proviso
\[ q(n;m_1,m_2) = Q(n_1,n_2;m_1,m_2) \quad (3.33) \]
\[ f(m_1)f(m_2) = F(m_1,m_2) \quad (3.34) \]

and
\[ g(n) = G(n_1,n_2) \quad (3.35) \]

Given the sequences \([g(n)]\) and \([q(n;m_1,m_2)]\), the objective is to recover \([f(m_1)f(m_2)]\) (and consequently the input sequence \([f(m)]\) ) by finding \(\{x(m_{11},m_{12})\}\) from (3.32) after substituting (3.33) and (3.35) into (3.32) and, finally, making use of (3.34).

Note that equation (3.33) can be recast into the standard linear form
\[ G = QF \quad (3.36) \]
where the elements of \(Q\) are now the \(q\) of equation (3.33) and the values of \(G\) are ordered as in the case of the two-dimensional image vector formulation as will be described below, that is, row by row.

Therefore we can express a one-dimensional bilinear transformation in terms of a set of linear equations. It has been shown in Valenzuela and Bose [50], how (3.36) can be solved recursively given that certain conditions hold. Kim [51] describes a restoration algorithm based on an iterative procedure that makes possible the recovery of \(f(m_1,m_2)\) in (3.34) subject to the non-negativity constraint \(f(m) \geq 0\) and also provides a proof of convergence. Kim states that the method can be quite slow to restore bilinearly degraded images because of the large number of iterations that might be necessary to reduce the error in
restoration below an acceptable level.

One method around this problem may be an approach similar to that used in the accelerated Gerchberg algorithm, Jones [52] and also employed in chapter IV to provide accelerated convergence. Such a method is similar in spirit to the method of vector space truncation such as was described in chapter II for the case of singular or highly ill-conditioned problems.

In any case, we have seen how solution of the linear inverse problem ties in with optical problems where measurements are being made in intensity and not amplitude as was the case in the [15].

We can see therefore that, even when making modulus only measurements we can treat certain optical systems as linear and so the techniques of super-resolution as introduced in chapter II may now form a basis for phase-retrieval from modulus measurements. Indeed much work has been put into phase retrieval from measurements of the signal modulus. For a good review of this subject including an extensive bibliography one should refer to Gonsalves [53].

With the above discussions in mind we shall examine, in the next chapter, the problem of retrieving an amplitude object of some known support from its diffraction limited image which has been corrupted with additive noise. This problem has been extensively treated in the literature by both the (iterative) Gerchberg-Papoulis algorithm [3] [4] [52] and by SVD as in Gori and Guattari [54], Sasaki and Yamagami [55] and Abdelmalek and Otsu [56].
For example Howard [57] developed an approach to the restoration of a low pass filtered signal based on the principle of least squares. Rushforth, Crawford and Zhou [15] further discussed the use of least squares techniques for restoration of objects with some high spatial frequency components missing. More recently Abbis and Earwicker [58], in an internal RAE report, have shown how to regularise the SVD for ill-posed synthetic aperture radar systems.

Finally, before considering partitioning of the Fourier transform matrix, we should consider briefly the possibility of linearising more general non-linear inverse problems. In [35] it was suggested that the JL algorithm be applied to the problem of iterative solution of the non-linear inverse problem using Newton's method. We shall say more about this in the final chapter.
3.6 Partial Fourier Transform Data Formulation.

3.6.1 Discrete Fourier Transformation and its Matrix Representation.

In this section we shall give the background for object reconstruction from partial Fourier transform data as is required for the inversion of direct and transform space bandlimited data in chapter IV.

When an optical intensity distribution is processed digitally it must of course be sampled, quantised and spatially truncated. The result of all these operations is a finite-dimensional digital signal \( f^T = (f_0, f_1, \ldots, f_{N-1}) \). The prime denotes a matrix transpose, so \( f \) is an \( N \)-dimensional column vector. Here \( f \) is the true signal and we have ignored any noise present.

The discrete Fourier transform (DFT) of \( f \) is defined as

\[
G_m = \sum_{n=0}^{N-1} f_n \exp\left(-\frac{i2\pi mn}{N}\right) \quad m=0,1,\ldots,N-1
\]  

(3.37)

and we define the transform vector \( G' = (G_0, G_1, \ldots, G_{N-1}) \).

It is convenient to express the DFT in matrix form

\[
G = F f
\]  

(3.38)

where

\[
F_{mn} = \left\{ \exp\left[-\frac{i2\pi mn}{N}\right]\right\} \quad m,n = 0,1,\ldots,N-1
\]  

(3.39)
3.6.2 The Partitioned Discrete Fourier Transformation.

We shall formulate the problem in one-dimension and then show how two-dimensional image data can be treated as a one-dimensional vector and inverted using the one-dimensional formulation. We shall then discuss an iterative method to solve the same problem; the iterative Gerchberg method: [3], [4] and [52], prior to comparison of results obtained in various circumstances using the two methods in the next chapter.

The discrete Fourier transform (equation 2.1) can be written

\[ F = \Omega f \]  

(3.40)

where \( \Omega = [w^r]^t \) and \( w = [-2\pi j/N] \), \( N \) being assumed even so that \( F \) is the Fourier transform of \( f \).

Suppose now that \( f \) is partitioned such that

\[
\begin{bmatrix}
  f_0 \\
  f_1 \\
  f_2
\end{bmatrix}
\]

(3.41)

with \( f_0 \) of dimension \( t+1 \), \( f_1 \) of dimension \( N-2t-1 \), and \( f_2 \) of dimension \( t \); suppose also that \( F \) is similarly partitioned

\[
\begin{bmatrix}
  F_0 \\
  F_1 \\
  F_2
\end{bmatrix}
\]

(3.42)

with \( F_0 \) of dimension \( M+1 \), \( F_2 \) of dimension \( M \) and \( F_1 \) of dimension \( N-2M-1 \). Note that this partitioning differs from that used in [15] where the direct space vector had \( L \) non-zero components. Here we have \( 2t+1=L \) non-zero terms in direct space in order that we can
easily demonstrate that in amplitude terms we will get a real symmetric Fourier transform for a real and symmetric input in the optical problem. The notation used here also differs to that used by Jones [52] whose notation was more concerned with applications to time/frequency domain data where the frequency space data are naturally indexed in ascending order and the time domain data are indexed causally.

The partitioning of the vectors in this way corresponds to the partitioning of the Fourier transform matrix $\Omega$ in the following manner

$$
\Omega = \begin{bmatrix}
\Omega_{00} & \Omega_{01} & \Omega_{02} \\
\Omega_{10} & \Omega_{11} & \Omega_{12} \\
\Omega_{20} & \Omega_{21} & \Omega_{22}
\end{bmatrix}
$$

(3.43)

with the row dimension of $\Omega_{ij}$ that of $F_1$ and the column dimension that of $f_j$. Writing out the partitioned Fourier transform operation explicitly with the known support on $f$ and setting unobserved terms in $F$ to zero we get

$$
\begin{align*}
F_0 & \begin{bmatrix}
F_0 \\
F_M
\end{bmatrix} = \begin{bmatrix}
\Omega_{00} & \Omega_{01} & \Omega_{02} \\
\Omega_{10} & \Omega_{11} & \Omega_{12} \\
\Omega_{20} & \Omega_{21} & \Omega_{22}
\end{bmatrix} \begin{bmatrix}
f_0 \\
f_1 \\
f_{N-1}
\end{bmatrix} = \begin{bmatrix}
f_0 \\
f_1 \\
f_{N-1}
\end{bmatrix}
\end{align*}
$$

(3.44)

and so if we have some a priori knowledge about the support of the object vector $f$ then in the above notation we can identify $f_1$ with the support of the input.
We can express the inverse operation in a similar manner

\[
\begin{bmatrix}
  f_0 \\
  f_1 \\
  \vdots \\
  0 \\
  0 \\
  f_{M-\ell} \\
  f_{N-1}
\end{bmatrix}
= \begin{bmatrix}
  \Omega_{00}^+ & \Omega_{10}^+ & \Omega_{20}^+ \\
  \Omega_{01}^+ & \Omega_{11}^+ & \Omega_{21}^+ \\
  \Omega_{02}^+ & \Omega_{12}^+ & \Omega_{22}^+
\end{bmatrix}
\begin{bmatrix}
  F_0 \\
  F_M \\
  \vdots \\
  0 \\
  0 \\
  F_{N-M} \\
  F_{N-1}
\end{bmatrix}
\]  

(3.45)

where, for example, \( \Omega_{00}^+ \) represents the 00th section of the inverse Fourier transform matrix \( \Omega \). Here we have used the fact that \( [\Omega^\dagger_{ij}] \) is equal to the Hermitian conjugate of the matrix \( [\Omega_{ji}] \) in the forwards transform operator (see appendix II).

In the above we have, in effect, operated with certain parts of the Fourier transform matrix only and the partitioning chosen here reflects this in a way particularly applicable to optical problems. Most optical problems are non-causal and the data is most likely to be captured with the zero spatial frequency component at the centre of the observation vector.

Thus in the optical case we would capture the vector \( F \) as

\[
\begin{bmatrix}
  F_{N/2+1} \\
  \vdots \\
  F_{N-M-1} \\
  F_{N-M} \\
  \vdots \\
  F_{N-1} \\
  F_0 \\
  \vdots \\
  F_M \\
  F_{M+1} \\
  \vdots \\
  F_{N/2}
\end{bmatrix}
\]  

\[ \text{part of } F_1 \]  

or

\[ \text{part of } F_1 \]  

\[ \text{other part of } F_1 \]  

(3.46)
This necessitates blockswapping the vector before either inverse or direct Fourier transformation. This is because the Fourier transform and most algorithms used to perform it, expect the zero spatial frequency component as the first term in the vector to be transformed. The mathematical explanation for this can be found in appendix III.

Hence if we are assuming as additional a priori knowledge the reality of \( f \) then we only need to consider

\[
F_0 = \begin{bmatrix} \Omega_{00} & \Omega_{02} \\ \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ \end{bmatrix}
\] (3.47)

since we have established a symmetry between positive and negative spatial frequency terms using the structure of \( \Omega \). That is we only need to store half of the Fourier components because

\[
\overline{F}_j = F_{(N-j)}
\] (3.48)

Hence equation (3.47) can be expressed with the matrix written explicitly

\[
\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 1 & w & \ldots & w^l & w^{N-l} & \ldots & w^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^m & \ldots & w^{ml} & w^{M(N-l)} & \ldots & w^{M(N-1)} \\ \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ \end{bmatrix}
\] (3.49)

As indicated in appendix III the Fourier transform of an image is essentially a Fourier series representation. For the series representation to be valid the field must be periodic. Thus for the purposes of Fourier transformation, the original image, be it one-dimensional or two-dimensional, must be considered to be periodic as depicted in figure (3.2) for the two-dimensional case. False high spatial frequencies along the coordinate axes of the
Figure (3.2) Periodicity in two-dimensional Fourier transform:

The two-dimensional Fourier transform of an image is essentially a Fourier series representation of a two-dimensional field. For the Fourier series representation to be valid, the field must be periodic. As shown above the two-dimensional image must be considered to be periodic both horizontally and vertically and the left side of the image must abut the right side with the top and bottom also adjacent.
transformation arise from the artificial transition demanded by the Fourier representation. Although these high spatial frequencies do not contribute to the reconstruction of the image within the boundaries they are in fact necessary to form the edges of the image. A useful illustration of the approximate nature of the discrete Fourier transformation is given in appendix IV where we give an example showing how the false spatial frequencies arise.

We shall examine the possibility of inverting a system of this type by the direct Jones-Lanczos method. Although with the conditions already stated the system is fully determined by equation (3.49) when \(2M+1 = L\), it will be more efficient to ignore the symmetry between \(F_0\) and \(F_2\) and solve the system for both \(F_0\) and \(F_2\), that is,

$$
\begin{bmatrix}
F_0 \\
F_2
\end{bmatrix} =
\begin{bmatrix}
\Omega_{00} & \Omega_{02} \\
\Omega_{20} & \Omega_{22}
\end{bmatrix}
\begin{bmatrix}
f_0 \\
f_2
\end{bmatrix}
$$

(3.50)

since this enables us to use the fast Fourier transform (FFT) algorithm.

3.7 Vector Representation of Two-dimensional Image Data.

The first example we shall consider in the next chapter is one-dimensional. Often in optical imaging, however, we will be interested in inverting two-dimensional systems.

Image sampling techniques provide means of obtaining a discrete array \(F(j_1,j_2)\) from continuous image field data sampled
over some rectangular area \(-J_i \leq j_i < J_i\). It is often helpful to regard this sampled image array as an \(N_1 \times N_2\) element matrix \(F = [F(n_1,n_2)]\) for \(1 \leq n_1 \leq N_1\) where the indices of the sampled array are re-indexed for consistency with standard vector space notation.

It is convenient to convert the image matrix to vector form by column or row, scanning the elements together as a long vector. Following Pratt [59] we can express this operation mathematically. We use an \(N_2 \times 1\) operational vector \(v_n\) and an \(N_1 N_2 \times N_1\) matrix \(N_n\) defined as

\[
v_n = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
\vdots \\
0
\end{bmatrix}
\quad \text{and} \quad
N_n = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
\vdots \\
0
\end{bmatrix}
\]

Then the vector representation of the data matrix \(F\) is given by stacking column upon column

\[
f = \sum_{n=1}^{N_2} N_n F v_n
\]

In other words the vector \(v_n\) extracts the \(n\)th column from \(F\) and the matrix \(N_n\) places this column into the \(n\)th segment of \(f\).

Of course an image of any dimension is stored in a computer as a vector so this notation will be of great importance in digital image processing. This one-dimensional vector representation of a two-dimensional image enables us to treat the
problem in essentially the same way as the one-dimensional direct space and this we shall do in section 4.2. for two-dimensional band-limited Fourier data.

3.8 Discrete Two-dimensional Fourier Transform.

The two-dimensional counterpart of the one-Dimensional discrete Fourier transform (equation (3.37) is defined as

\[
G_{mp} = \sum_{n=0}^{N-1} f_{nq} \exp \left[ -i \frac{2\pi mn}{N} \right] \exp \left[ -i \frac{2\pi np}{N} \right] m, p = 0, \ldots, N-1 \tag{3.55}
\]

where \( G_{mp} \) and \( f_{nq} \) are the \( mp \)th and \( nq \)th elements of the matrices \([G_{mp}]\) and \([f_{nq}]\) respectively. Thus equation (3.38) becomes in two dimensions

\[
G = F f \tag{3.56}
\]

where now

\[
F = F_C \circ F_R \tag{3.57}
\]

and \( \circ \) gives the left direct product (defined in appendix V) of the column Fourier transform matrix \( F_C \) and the row Fourier transform matrix \( F_R \). In fact for square images \( F_C = F_R \) and in that case \( F \) is an \( N^2 \times N^2 \) matrix with rank \( \leq N^2 \) as opposed to the one-dimensional case where it is obviously \( \leq N \). With this in mind we can see that when we include the band and domain limiting operators in equation (3.56) then the rank will go over from whichever is smallest out of \( 2M+1 \) and \( L \), in the one-dimensional case, to the smallest out of \( (2M+1)^2 \) and \( L^2 \) in the two-dimensional case.
3.9 The Gerchberg Algorithm: an Iterative Reconstruction Method.

3.9.1 The Gerchberg Algorithm and the Optical Fourier Transform.

We have given a method for reconstructing two-dimensional images from partial Fourier transform data using a direct inversion method. However, except in special cases, this can only be done, for an image containing any detail, if the problem is singular as will be seen in chapter IV.

When dealing with systems of higher rank large amounts of computer resources are necessary for inversion since, for example, when $M=8$ we will have $17^2$ vectors in each space. Thus it seems as though an iterative method may be more appropriate.

Here we shall describe one such method, due to Gerchberg, and show how we can apply it to problems that would be intractable, at least on a minicomputer, using the JL method. We shall also compare how it works on the singular systems inverted by the direct method for comparison, both with and without noise.

It was shown in chapter II that perfectly coherent optical systems are linear in amplitude. Since for the moment we are leaning towards optical problems we shall use the well known Fourier transforming property of a lens to explain how the Gerchberg algorithm is used.

Consider figure (3.3) where the front focal plane $(\xi',\eta')$ is illuminated by a plane monochromatic wave. Ignoring the finite

continued on page 52
Figure (3.3) The Fourier transforming property of an idealised lens: Partitioning an optical observation.

\[ U(x,y) = \int f(\xi',\eta') \exp\left[\frac{2\pi i}{\lambda f}(x\xi' + y\eta')\right] d\xi' d\eta' \]

The front focal plane \((\xi',\eta')\) is illuminated by a plane monochromatic wave. Ignoring the finite size of the lens which produces vignetting effects the lens carries out a Fourier transform. In the diagram above we have a rect function distribution immediately after the front focal plane giving a sinc function in the back focal plane. In Appendix IV we show how the discrete representation output of such an ideal system deviates from this result.

We partition our discrete system such that in the input (front) focal plane \(f\) represents the support of the input which here is zero everywhere. In the exit (back) focal plane \(F_1\) represents the unobserved part of the objects spectrum.
size of the lens which produces vignetting effects we have

\[ U(x,y) = \int \int \mathcal{t}(\xi', \eta') \exp \left[ \frac{2\pi i}{\lambda_f} (x\xi' + y\eta') \right] d\xi' d\eta' \]  

(3.58)

ignoring scaling constants (see for example Born and Wolf [41]).

Thus the complex amplitude in the observation plane, \( U(x,y) \), is related to the complex amplitude in the front focal plane, \( t(\xi', \eta') \), by a two-dimensional Fourier transform. Of course in practice what we would more usually observe is the time averaged intensity (see chapter II) though the above statement is physically correct. \( (U(x,y) \) may possibly be observed in a holographic experiment).

If we divide up the front and back focal planes using the notation given in section (3.6) then we can partition the system as in figure (3.3). Thus in our notation and with this particular system we see that \( f_1 \) represents the masked off sections of the input and these parts of the input are therefore zero.

In transform space we have ordered the observation vector \( F \) with \( F_1 \) giving the support which, in this case, contains the unobserved parts of the image formed by the lens. This partitioning is consistent with that introduced in section (3.6) but differs from that used in [52] where in direct space the support was stored in the vectors \( f_0 \) and \( f_2 \). There is an asymmetry in this partitioning between the front and back focal plane which seems at odds with the fact that it does not matter which way round the lens is orientated. In an optical image there is no reason to define one space as transform space and the other one as direct space. Thus it would seem more natural, in an optical
problem, to partition the two vectors in the same way so that the observed vectors are blockswapped before application of the discrete Fourier transform (or a fast version of it).

In fact if we do not do the block swap in direct space for the pseudo-optical system depicted in figure (3.3) then we do not get the result shown there, that is the sinc function with zero imaginary part. We expect the discrete Fourier transform of a real symmetric function to be real and symmetric and the partitioning we use here ensures that. Note, however, that unless we are in some way observing the complex amplitude, as in a holographic configuration then the notations are the same as far as the Gerchberg algorithm is concerned.

Indeed, as far as the intensity is concerned, ignoring the finite aperture of the lens, it makes no difference to the intensity distribution in the back focal plane where, in the front focal plane we put the object of figure (3.3). In appendix VI we show the difference, in terms of the Gerchberg algorithm, between the notation used here and that in [52].

Mathematically speaking, the Gerchberg algorithm uses the analyticity of the optical transfer function to extrapolate the unknown portions of the image spectrum by analytic continuation. The discrete form of the algorithm thus ensures that if \( f_1 \) is fixed (to zero in this case) and \( F_0 \) and \( F_2 \) are known (the observed central portion of the sinc function in this case) then \( f_0, f_2 \) and \( F_1 \) can be determined by an iterative procedure as described below.

Essentially the algorithm consists of repeated forwards and
inverse Fourier transforms. At the rth iteration and whilst in the appropriate space \( f_1^{(r)} \) is set to zero (or to whatever the known value is) and \( F_0^{(r)} \) and \( F_2^{(r)} \) are set to the observed values \( F_0^{(0)} \) and \( F_2^{(0)} \).

### 3.9.2 The Discrete Gerchberg Algorithm for the Optical Problem

For the optical problem depicted in figure (3.3) we capture \( F_0 \) and \( F_2 \) and define the remainder of the transform space vector, which we denote by \( B^{(0)} \), as

\[
B^{(0)} = \begin{bmatrix}
F_0^{(0)} \\
B_1^{(0)} \\
F_2^{(0)}
\end{bmatrix}
\]  

(3.59)

We then take the inverse Fourier transform of the above to give \( b^{(0)} \) where

\[
b^{(0)} = \frac{1}{N} \Omega^{*} B^{(0)}
\]

(3.60)

and set the values of \( b^{(0)} \) which correspond to the support (\( f_1 \) in this case) to zero and name the resulting vector \( t^{(0)} \) where

\[
t^{(0)} = \begin{bmatrix}
b_0^{(0)} \\
f_1^{(0)} \\
b_2^{(0)}
\end{bmatrix}
\]

(3.61)

We then Fourier transform back to give \( T^{(0)} \) where

\[
T^{(0)} = \Omega t^{(0)}
\]

(3.62)

and substitute in the original observation consisting of \( F_0 \) and \( F_2 \) and rename the resulting vector \( B^{(1)} \) where
which completes the first iteration of the algorithm.

So in general the rth step is

\[
B^{(r+1)} = \begin{bmatrix}
F_0 \\
T_1^{(r)} \\
F_2
\end{bmatrix}
\]

Using the partitioning scheme introduced in section (3.2), by following [52], in a manner more suited to the optical problem, we can write the Gerchberg algorithm in explicit matrix form

\[
\begin{align*}
b^{(r)} &= \frac{1}{N} \Omega^* B^{(r)} \\
t^{(r)} &= \begin{bmatrix}
b_0^{(r)} \\
f_1 \\
b_2^{(r)}
\end{bmatrix} \\
T^{(r)} &= \Omega t^{(r)} \\
B^{(r+1)} &= \begin{bmatrix}
F_0 \\
T_1^{(r)} \\
F_2
\end{bmatrix}
\end{align*}
\]

where \( B_1^{(r+1)} \) is the \((r+1)\)th estimate of the unobserved part of the spatial frequency spectrum for the optical problem depicted in figure (3.3).
\[
B_1^{(r+1)} = F_1 - \Omega_0 f_0 - \Omega_{12} f_2 + \left[ \begin{array}{cc} \Omega^{\dagger}_{10} & \Omega^{\dagger}_{12} \\ \Omega_{10} & \Omega_{12} \end{array} \right] \left\{ \begin{array}{c} f_0 \\ f_2 \end{array} \right\} - \frac{1}{N} \left[ \begin{array}{c} \Omega^{\dagger}_{10} \\ \Omega^{\dagger}_{12} \end{array} \right] F_1
\]
\[
+ \left[ \begin{array}{cc} \Omega^{\dagger}_{10} & \Omega^{\dagger}_{12} \\ \Omega_{10} & \Omega_{12} \end{array} \right] \frac{1}{N} \left[ \begin{array}{c} \Omega^{\dagger}_{10} \\ \Omega^{\dagger}_{12} \end{array} \right] B_1^{(r)}
\]
and have
\[
B_1^{(r+1)} = \left( (I - P')F_1 \right) + \frac{1}{N} p'B_1^{(r)}
\]
where
\[
P' = \frac{1}{N} \left[ \begin{array}{cc} \Omega^{\dagger}_{10} & \Omega^{\dagger}_{12} \\ \Omega_{10} & \Omega_{12} \end{array} \right]
\]
and
\[
\begin{bmatrix} b_0^{(r+1)} \\ b_2^{(r+1)} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} - \frac{1}{N} \left[ \begin{array}{c} \Omega^{\dagger}_{10} \\ \Omega^{\dagger}_{12} \end{array} \right] F_1 + \frac{1}{N} \left[ \begin{array}{c} \Omega^{\dagger}_{10} \\ \Omega^{\dagger}_{12} \end{array} \right] \left\{ F_1 - \Omega_0 f_0 - \Omega_{12} f_2 \right\}
\]
\[
+ \frac{1}{N} \left[ \begin{array}{cc} \Omega^{\dagger}_{10} & \Omega^{\dagger}_{12} \\ \Omega^{\dagger}_{10} & \Omega^{\dagger}_{12} \end{array} \right] \begin{bmatrix} b_0^{(r)} \\ b_2^{(r)} \end{bmatrix}
\]
simplifies to
\[
\begin{pmatrix}
\mathbf{b}^{(r+1)}_0 \\
\mathbf{b}^{(r+1)}_2
\end{pmatrix} = \left\{ \mathbf{b}^{(r)}_0 \begin{pmatrix}
\mathbf{f}_0 \\
\mathbf{f}_2
\end{pmatrix} \right\} + Q' \begin{pmatrix}
\mathbf{b}^{(r)}_0 \\
\mathbf{b}^{(r)}_2
\end{pmatrix}
\]

where

\[
Q' = \frac{1}{N} \begin{bmatrix}
\Omega^+_{10} \\
\Omega^+_{12}
\end{bmatrix} \begin{bmatrix}
\Omega_{10} & \Omega_{12}
\end{bmatrix}
\]
Chapter IV.

Inversion of Fourier Problems: Model and Results.

Introduction.

In part (4.1) of this section we shall apply the JL method to the problem of inverting one-dimensional direct space band-limited Fourier data. This problem has drawn much attention: e.g. Bertero et al [60] and Sanz and Huang [61] and has been treated using the Gerchberg method by Abbis et al [62] and [2], Walker [63] and Jones [52]. It has also been analysed using the SVD by Rushforth et al [15], Gori and Guattari [54] and Sasaki and Yamagami [55].

In part (4.2) we go on to use the same inversion technique to invert two-dimensional partial Fourier transform data as formulated in chapter III. Finally, we present results of inversions using the iterative Gerchberg algorithm, [3], [4] and [52] and compare the performance of the two methods in various circumstances.

4.1 Reconstruction of an object of known support from its diffraction limited image (One-dimensional case).

Here we consider the problem of retrieving a one-dimensional (amplitude) object of known support from its diffraction-limited (amplitude) image which has been corrupted with additive noise. This problem is closely related to the partial Fourier transformation of chapter III except here both object and image are in direct space and so, in this case, two Fourier transforms
are required to obtain the image from the object, the intermediate stage being equivalent to the partial Fourier transform process formulated in the previous chapter. This is the problem treated by, for example, Rushforth et al [15] and enables a comparison to be made with their results.

The unknown signal \( f \) which we wish to determine is filtered by a perfect low pass digital filter that eliminates all frequencies above its cut-off frequency while leaving components at low frequencies unchanged. The filtered signal is then observed along with any random noise which may be present. The a priori information that we assume at the outset is that the signal is confined to a known and limited region in space. More specifically we assume that the last \( N-L \) components of \( f \) are zero.

To obtain a convenient matrix representation of the problem we first define the ideal band limiting matrix \( B \) to be the matrix

\[
B = \begin{bmatrix}
  [I_{M+1}] \\
  [O_{N-2M-1}] \\
  [I_{M}]
\end{bmatrix}
\]

(4.1)

where \([I_{M}]\) is the unit matrix of dimensions \( M \) by \( M \) and \([O_{N-2M-1}]\) is the null matrix of dimensions \((N-2M-1)\) by \((N-2M-1)\) all other elements of \( B \) being zero. \( B \) can alternatively be written

\[
B = \text{diag}\{ [1, 1, \ldots, 1]_{M+1}, [0, 0, \ldots, 0]_{N-2M-1}, [1, 1, \ldots, 1]_{M} \}
\]

(4.2)

Multiplying the frequency domain vector \( G \) by \( B \) leaves the DC component \( G_0 \), the \( M \) lowest positive frequency components and the corresponding negative frequency components while removing
components at higher frequencies.

We further define the spatial truncation matrix (domain limiting matrix), \( D \) by

\[
D = \text{diag}(1, \ldots, 1, 0, \ldots) \tag{4.3}
\]

Multiplying a vector by \( D \) leaves the first \( L \) components unchanged whilst replacing the remaining \( N-L \) components with zeros. As we discussed in chapter III, as far as optical intensity measurements are concerned, the positioning of the support is irrelevant and we can have the \( L \) non zero components anywhere in equation 4.3. The assumption about the region of support of the unknown signal \( f \) can be expressed as

\[
f = Df \tag{4.4}
\]

Ignoring noise the observed vector would be

\[
g = Af \tag{4.5}
\]

where

\[
A = F^{-1}BDF \tag{4.6}
\]

Thus application of \( A \) entails four operations: truncation (which does not affect \( f \)), discrete Fourier transformation, ideal band-limiting, and inverse discrete Fourier transformation or in other words the same problem as formulated in the previous chapter with the addition of the inverse Fourier transform operation.

We modify the problem slightly and let \( f \) be the \( L \) dimensional vector

\[
f^T = (f_0, \ldots, f_{L-1}) \tag{4.7}
\]

and let \([L_{mn}]\) be the \( N \) by \( L \) matrix consisting of the first \( L \) columns of the matrix \([A_{mn}]\). Then equation (4.5) can be written \( g = Lf \). Furthermore it can be shown (appendix VII) that the \( mn \)th component of the matrix \([L_{mn}]\) is
\[ L_{mn} = \begin{bmatrix} \sin \left( \frac{\pi(n-m)(2M+1)}{N} \right) \\ \sin \left( \frac{\pi(n-m)}{N} \right) \end{bmatrix} \]  

(4.8)

for \( m \neq n \). For \( m = n \) we have

\[ L_{mn} = \frac{2M+1}{N} \]  

(4.9)

### 4.1.1 Relationship between Conditioning and Rank of the Matrix \([L_{mn}]\)

As already stated, there are two problems with the inversion of equation (1.1). Firstly observation noise on the observation vector \( g \) can lead to a meaningless solution and secondly ill-conditioning of the matrix itself as discussed in section (2.3).

The most severe case of ill-conditioning is when one or more of the eigenvalues of the system actually become identically zero; in this case the matrix is said to be singular. In Rushforth et al., [15] it is stated that "for the bandwidth \( M = 4 \), the singular values \( \sigma_{9} \) and \( \sigma_{10} \) are so small that the single precision implementation of SVD cannot accurately compute them or their corresponding singular vectors". It is then discussed how to choose the regularisation parameter in order that the inversion does not explode as described in section (2.3) (the regularisation parameter corresponds to \( \epsilon \) in our procedure).

Now the ill-conditioning gets worse as the bandwidth gets smaller (compare tables (4.1) and (4.2)) and as we have shown in appendix VIII when \( 2M+1 < L \) the rank of the matrix is \( 2M+1 \). So
even with infinite precision $\sigma_9$ and $\sigma_{10}$ are identically equal to zero and to attribute the lack of precision in this case to the limits of the computer word length is incorrect. In fact for the singular case $(2M+1 < L)$ we must truncate the space so that we only solve for as many variables as the rank of the matrix allows us to. Inversion of a truncated singular space using the JL algorithm was described in section (2.4.5).

4.1.2 The New Coordinate Space.

By application of the JL algorithm we obtain the matrix representation of the operator $L$ in bidiagonal form. This produces two (real) sets of vectors along with the corresponding normalisation coefficients which constitute the two non-zero diagonals in the transformed matrix.

In figure 4.1 we illustrate the set of $U$ space JL vectors and in figure 4.2 the $V$ space JL vectors are illustrated for the rank 10 case where $M=5$. (In this case the rank=L since $L<2M+1$).

4.1.3 Pseudo-data.

Having set up the new coordinate space we are in a position to invert the system for some set of pseudo-data. For purposes of comparison we have used the same data as is used in [15], that is,

$$f(n) = \begin{cases} 0.5 \exp[-2(n-3)^2] + \exp[-2(n-7)^2] & n=0,1,...,9 \\ 0 & \text{otherwise} \end{cases}$$

(4.10)

which has also been used by Sasaki and Yamagami [55] and

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Figure 4.1) Jones-Lanczos vectors in V-space for the
one-dimensional band-limited (amplitude) Fourier optical system
(rank 10, M=5 case). The JL algorithm generates these vectors
which are used to reconstruct images in U-space.
Figure (4.2) Jones-Lanczos vectors in $V$-space for the one-dimensional band-limited (amplitude) Fourier optical system (rank 10, $M=3$ case).
Figure (4.3a) The discrete signal $f$ (equation (4.10)) with its band-limited ($M=9$) noiseless image $g$. (b) and (c) are the same as in (a) corrupted with 0.1% and 4% noise respectively.
Abdelmalek and Otsu [56] and we assume that the signal and noise are uncorrelated. For this case \( f \) is the discrete signal shown in figure 4.3(a) along with its band-limited noisy image. The band-limited noise-corrupted signal is sampled over 64 points, so that \( [L_{mn}] \) is the 64 by 10 matrix given by equation (4.8). Figures (4.3b) and (4.3c) show the bandlimited image with 0.1% and 4.0% noise respectively.

**4.1.4 Prior Knowledge.**

We estimate the signal mean \( f_o \) by

\[
f_o = \begin{bmatrix}
0 \\
0 \\
\mu \\
\vdots \\
\mu \\
0 \\
0
\end{bmatrix}, \quad \text{where} \quad \mu = \frac{\sum_{n=0}^{9} f_n}{10}
\]

(4.11)

and the variance, \( \sigma_s^2 \), by

\[
\sigma_s^2 = \frac{\sum (f_n - f_{0n})^2}{10}
\]

(4.12)

In equation (4.11) it is assumed that there is some minimal *a priori* knowledge about \( f \) in that the zero entries in \( f_o \) represent the support of the signal.

The possibility of building prior knowledge into \( f_o \) has not been exploited fully by, for example, Maeda [64] and Darling, Hall and Fiddy [65]. It is clear that for a positive valued signal, \( f_o \) is non-negative. For an object without finite support but subject to known illumination (for example, Gaussian or sinc illumination as in Bertero et al [66]), \( f_o \) can be chosen appropriately. In such a case \( P \) cannot be reasonably estimated as \( \sigma_s^2 \), although it may still be diagonal. Further consideration of
this aspect of reconstruction will be made later in chapter VIII.

4.1.5 Results: Noiseless Singular Case.

For the singular case where \(2M+1 < L\) we can invert equation (4.5) using the technique described in section (2.4.5). In figure (4.4) we display the results of noiseless inversion for each of the singular cases where \(M=0,1,2,3,4\) to illustrate the best possible reconstruction obtainable in each case. Of course for \(M > 4\) i.e. \(2M+1 \geq L\) we can invert the system exactly in the presence of zero noise.

4.1.6 Results: Noisy Singular and Well-Determined (Non-singular) Cases.

We use equation (1.8) (equivalent to equation (1.6)) with \(c = \sigma_n^2/\sigma_s^2\) and also with other values of \(c\). Figures (4.5 a, b and c) show the original signal and its reconstruction with 4% noise for \(c = \sigma_n^2/\sigma_s^2\), \(c = 0.1 \sigma_n^2/\sigma_s^2\) and \(c = 0.01 \sigma_n^2/\sigma_s^2\) for different noise samples. Figure (4.6) follows the same pattern for 0.1% noise. In both of these cases \(M = 9\). Figure (4.7) is the same as figures (4.6) and (4.5) with 0.001% noise for the singular, rank 9, \(M = 4\) case.

These results should be compared with those obtained using the SVD technique as employed by Rushforth et al [15]. The results obtained in [15] are meaningless in the singular case because no information can be incorporated into the reconstruction from vectors associated with singular values which are zero. However

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Figure (4.4) Results of noiseless inversions of the data in figure (4.3a) for each of the singular cases where $M=0,1,2,3,4$ to illustrate the best possible reconstruction obtainable in each case.
Figure (4.5a,b,c) The original signal and its reconstruction from the band-limited signal with 4.0% noise for (a) $\varepsilon=\sigma_n^2/\sigma_s^2$ (b) $\varepsilon=0.1\sigma_n^2/\sigma_s^2$ and (c) $\varepsilon=0.01\sigma_n^2/\sigma_s^2$ with different noise sample and $M=9$

Figure (4.6) The same as figure (4.5) but with 0.1% noise.

Figure (4.7) The same as figures (4.5) and (4.6) with 0.001% noise for the singular, rank 9 $M=4$ case.
the results we present here for the singular case, in figure (4.7), are meaningful. This is because they have been reconstructed from a truncated space and no information is demanded from vectors associated with singular values which are zero.

For some particular noise samples the reconstruction can appear better when we have $\epsilon < \sigma^2_n/\sigma^2_s$. However in this case consistency over several different noise samples, each of the same mean, is lacking. But with $\epsilon = \sigma^2_n/\sigma^2_s$ consistent reconstruction is obtained. In tables (4.1) and (4.2) the values in the

\begin{equation}
\begin{bmatrix}
\alpha_1 \\
\beta_1 \alpha_2 \\
\beta_2 \alpha_3 \\
\vdots \\
\beta_{r-1} \alpha_r \\
\end{bmatrix}
\begin{bmatrix}
\langle a_1 | f \rangle \\
\langle a_2 | f \rangle \\
\langle a_3 | f \rangle \\
\vdots \\
\langle a_r | f \rangle \\
\end{bmatrix}
= 
\begin{bmatrix}
\langle b_1 | g \rangle \\
\langle b_2 | g \rangle \\
\langle b_3 | g \rangle \\
\vdots \\
\langle b_r | g \rangle \\
\end{bmatrix}
+ 
\begin{bmatrix}
\langle b_1 | n \rangle \\
\langle b_2 | n \rangle \\
\langle b_3 | n \rangle \\
\vdots \\
\langle b_r | n \rangle \\
\end{bmatrix}
\end{equation}

where $r = \min (2M+1,10)$, are displayed for $M = 4$ and $M = 9$. In the former case the rank of $[L_{mn}]$ is 9, and so $\alpha_{10}$, $\beta_9$ and $\beta_{10}$ are zero, which in computational practice meant setting to zero $\alpha$'s and $\beta$'s less than $10^{-12}$. In both cases it may be seen that the values of $\alpha$ and $\beta$ for which the noise becomes comparable with the band-limited signal are $\approx \sigma^2_n/\sigma^2_s$, and so choosing $\epsilon < \sigma^2_n/\sigma^2_s$ can result in the amplification of noise.

It is seen therefore that the results presented here are also superior to those presented in, for example [15], since consistency has been demonstrated over several different noise samples whereas in the literature only one plot is shown and no
consistency is demonstrated.

4.1.7 General Comparison of the JL algorithm with SVD.

As for SVD, the JL vectors can be computed once and for all for a given linear operator. The method used here has the additional advantages of including signal and noise correlation and of allowing a degree of flexibility in the initial choice of vector \( a_0 \). Choosing \( a_0 \) judiciously for a particular object can accelerate convergence (For example, \( a_0 = f \) gives \( \langle b_k | g \rangle = 0, k > 1 \), in the noiseless case). In chapter VII we will show how we can avoid unnecessary calculation by using the invariance of the matrix trace.

4.1.8 Speed of the JL algorithm Relative to SVD.

Since the final stage of the SVD uses iterations of the QR algorithm, we cannot obtain a general expression for the relative speeds of the SVD and the JL method. This is because the number of iterations required for each singular value depends on the operator itself. Table (4.3) is a guide to the speed of the SVD algorithm, NAG library [67], relative to our method and gives the number of multiplications involved in each algorithm. For the SVD the value is based on the assumption that two iterations of the QR algorithm are required per singular value, although for a particular problem it may be more.

The expression given for the method we have used allows for multiplications done in the Gram-Schmidt reorthogonalisation, Wilkinson [31], that is performed after the generation of each
vector. As in the standard Lanczos method, when small valued vectors are subtracted, cancellation takes place, and strict orthogonality is lost; use of the Gram-Schmidt process maintains orthogonality at a relatively small extra cost.

Wilkinson [31] points out that in the standard Lanczos method such reorthogonalisation does not produce the same set of vectors as would have been produced in more accurate computation but this has no deleterious effect as far as inversion is concerned. In fact we shall see that, by the time this happens in the two-dimensional problem, the terms of the bidiagonal matrix become small. This allows truncation of the ill-conditioned problem in a similar manner to the purely singular case without seriously degrading the reconstruction, particularly in the presence of noise, since in that case the additional information is inaccessible.

In the calculation which gave the results in section (4.7) bidiagonalisation is achieved in less than one third of the time it took using the NAG library SVD on the same computer. Table (4.4) shows how greater savings can be obtained in other problems.
### Table 4.1
Values of Parameters in the JL Equations (M=4)

| i  | α      | β      | $<b_{i}|g>$ | $<b_{i}|n>$ |
|----|--------|--------|------------|------------|
| 1  | 7.285x10^{-1} | 3.768x10^{-1} | 1.830x10^{-1} | -9.053x10^{-4} |
| 2  | 6.589x10^{-1} | 4.554x10^{-1} | 5.253x10^{-1} | -1.891x10^{-4} |
| 3  | 2.967x10^{-1} | 4.548x10^{-2} | 3.110x10^{-1} | -7.671x10^{-5} |
| 4  | 4.255x10^{-2} | 3.757x10^{-3} | -6.384x10^{-3} | -9.384x10^{-4} |
| 5  | 4.930x10^{-3} | 2.559x10^{-4} | -2.572x10^{-3} | 3.035x10^{-4}  |
| 6  | 3.992x10^{-4} | 1.206x10^{-5} | -1.818x10^{-4} | 4.897x10^{-4}  |
| 7  | 2.220x10^{-5} | 3.570x10^{-7} | 7.825x10^{-6}  | -4.565x10^{-4} |
| 8  | 7.939x10^{-7} | 5.353x10^{-9} | 5.952x10^{-7}  | -5.984x10^{-4} |
| 9  | 1.517x10^{-8} | 0.0       | 3.992x10^{-9}  | 1.872x10^{-4}  |
| 10 | 0.0     | 0.0     | 0.0         | 0.0         |

### Table 4.2
Values of Parameters in the JL Equations (M=9)

| i  | α      | β      | $<b_{i}|g>$ | $<b_{i}|n>$ |
|----|--------|--------|------------|------------|
| 1  | 7.596x10^{-1} | 3.477x10^{-1} | 6.795x10^{-2} | -1.458x10^{-3} |
| 2  | 5.900x10^{-1} | 3.907x10^{-1} | -3.908x10^{-3} | -1.576x10^{-3} |
| 3  | 7.319x10^{-1} | 3.788x10^{-1} | -1.766x10^{-1} | -6.077x10^{-3} |
| 4  | 1.970x10^{-1} | 1.925x10^{-1} | 2.392x10^{-2}  | -2.295x10^{-3} |
| 5  | 9.690x10^{-1} | 1.373x10^{-1} | 6.870x10^{-1}  | -3.213x10^{-3} |
| 6  | 3.523x10^{-2} | 3.022x10^{-3} | 7.191x10^{-2}  | -2.979x10^{-3} |
| 7  | 5.346x10^{-3} | 2.787x10^{-4} | 1.194x10^{-3}  | -3.361x10^{-3} |
| 8  | 6.095x10^{-4} | 1.766x10^{-5} | 4.201x10^{-4}  | -1.614x10^{-3} |
| 9  | 4.956x10^{-5} | 6.100x10^{-7} | 1.404x10^{-5}  | -1.025x10^{-3} |
| 10 | 2.526x10^{-6} | 0.0       | 7.394x10^{-7}  | -3.488x10^{-4} |
Table 4.3 Number of Multiplications in SVD and JL Algorithm

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Condition</th>
<th>Number of Multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVD(^a)</td>
<td>( m \geq n )</td>
<td>( 3n^2( n + 4n) )</td>
</tr>
<tr>
<td></td>
<td>( n &gt; m )</td>
<td>( 5m^2( n + 2m) )</td>
</tr>
<tr>
<td>JL</td>
<td>( m \geq n )</td>
<td>( 3n(n + m) + 2n^2m + m + \frac{(m - 1)}{2} + \frac{n(n - 1)}{2} )</td>
</tr>
<tr>
<td></td>
<td>( n &gt; m )</td>
<td>( 3m(n + m) + 2nm^2 + \frac{m(m - 1)}{2} + \frac{n(n - 1)}{2} )</td>
</tr>
</tbody>
</table>

\(^a\)The SVD algorithm is described in [34]

Table 4.4 Relative Speeds of the SVD and JL Algorithms: a Sample case

<table>
<thead>
<tr>
<th>Algorithm used</th>
<th>m</th>
<th>n</th>
<th>Number of Multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVD</td>
<td>64</td>
<td>10</td>
<td>-31,000</td>
</tr>
<tr>
<td>JL</td>
<td>64</td>
<td>10</td>
<td>-17,000</td>
</tr>
<tr>
<td>SVD</td>
<td>10</td>
<td>10</td>
<td>-15,000</td>
</tr>
<tr>
<td>JL</td>
<td>10</td>
<td>10</td>
<td>-2,700</td>
</tr>
</tbody>
</table>

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4.2 Reconstruction of an Object of Known Support from its *Diffraction-limited Image (Two-dimensional Case)*.

4.2.1 Generation of the New Coordinate Space.

Although image data is stored in the computer in the form of a one-dimensional vector described in section (3.7), it is more convenient (and faster) to use the two-dimensional FFT instead of 2N one-dimensional FFTs. Optimisation procedures in the NAG two-dimensional FFT make it faster than using a series of one-dimensional transforms.

Thus we generate our image basis vectors using the same procedure as used in section (4.1) except now the operator $L$ is given by $B'F'D'$ where $B'$ and $D'$ are the two-dimensional versions of the band-limiting and domain-limiting operators of section (4.1.2) and $F'$ is now the two-dimensional FFT operator. Thus for the $n=32$ random image, figure (4.8a), application of the domain-limiting operator for $L=24$ gives $u_0$, figure (4.8b). Following the two-dimensional FFT we obtain figure (4.8d) which is then band-limited with $M=6$ and this gives figure (4.8e) (real part) and figure (4.8f) (imaginary part). Following inverse Fourier transformation of figures (4.8e) and (4.8f) the process continues in the same manner.

Inspection of equation (3.49) and (3.50) shows that the complex 'v's should be even in the real part and odd in the imaginary part at each stage. This is born out, for the $n=32, M=6$, continued on page 81
Figure (4.8) (Overleaf)

(a) (top left) An n=32 image whose pixels have values of either 0 or 1 according to a random Gaussian distribution about zero.

(b) (top right) The n=32 image \( u_0 \) obtained by domain-limiting figure (4.8a) with \( L=24 \).

(c) (middle left) The n=32 image \( u'_0 \) used to restart the generation process when it ends prematurely in the \( N=32, M=1, L=24 \) case (section 4.2.2). The breakdown occurs due to the partitioned nature of the matrix representation of the operator in the JL reference frame. This type of breakdown should not be confused with the case where \( \beta_n = 0 \) due to the singular nature of the matrix. Nor should it be confused with breakdown when, using the precision available, the system is too large to enable orthogonality to be maintained even when using reorthogonalisation.

(d) (middle right) The real part of the image obtained after application of the two-dimensional FFT algorithm to figure (4.8b).

(e) (bottom left) The image in figure (4.8d) after application of the band-limited operator.

(f) (bottom right) The imaginary band-limited part of the image obtained after application of the two-dimensional FFT algorithm to figure (4.8b).
Figure (4.9) The non-zero parts of the real parts of the first few $V$-space JL vectors generated in the JL procedure for the inverse band-limited problem with $N=32, M=6, L=24$. Note the evenness.
Figure (4.10) The non-zero parts of the imaginary parts of the first few $V$-space JL vectors generated in the JL procedure for the inverse band-limited problem with $N=32, M=6, L=24$. Note the oddness.
Figure (4.11) The non-zero parts of the first few of the purely real $U$-space JL vectors. A linear combination of the full set of 169 of which can be used to reconstruct band-limited images in the $N=32,M=6,L=24$ case. That is the case of a $32\times32$ image with a support such that there are $24\times24$ non-zero pixels and composed of $13\times13$ spatial frequencies.
L=24 case, in figure (4.9) where we have plotted a few of the real parts of the \( V \) space vectors and in figure (4.10) which shows the corresponding imaginary parts. The (real) \( U \)-space vectors are plotted in figure (4.11).

### 4.2.2 Premature Breakdown of Vector Generation

(Restartable Case).

For the highly singular case: \( n=32, \ M=1, \ L=24 \), we find that when \( u_0 \) was chosen as indicated in figure (4.8b) then, before normalisation, \( v_6 \) is generated as zero within machine accuracy, in other words \( \beta_6 = 0 \) (Infact this happens whichever \( u_0 \) is chosen). This indicates a premature breakdown of the generation process. However as discussed in section (2.5.5) such an occurrence does not necessarily indicate that the system's rank is 5 but that we have accidentally arrived at a vector which is linearly dependent on the vectors already generated. If we ignore this breakdown we find that the orthogonality of the subsequently generated vectors is lost and furthermore, before normalisation, \( v_6 \neq 0 \) to machine accuracy.

However we can restart the process at \( v_6 \). To do this we generate a new \( v_6 \) in exactly the same way as we generated \( v_0 \) from \( u_0 \) but with a different random vector \( u_0' \) (figure 4.8c). After reorthogonalisation of the new \( v_6 \), with respect to the vectors already generated the process can be continued as before. In this case orthogonality is maintained and the unnormalised \( v_9 = 0 \) within
Table (4.5) (Overleaf) Several successive stages in the vector generation process are illustrated here. We show the output of the computer program implementation of the JL algorithm to show how the magnitude of the unnormalised $V$-space JL vectors varies at each stage. Also shown are the inner products of the vectors before and after reorthogonalisation to illustrate how $\beta_6 = 0$, that is the unnormalised $v_6 \neq 0$ within machine accuracy. The process is seen to continue successfully after restarting the process using $u'_0$. We only illustrate the output from $v_5$ to $v_6$. 

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Scalar product of new v with previous v s
\[ -1.346 \times 10^{-14}, -1.076 \times 10^{-15}, 2.123 \times 10^{-15}, 6.884 \times 10^{-14}, 5.733 \times 10^{-14} \]
\[ 1.008 \times 10^{-14}, 9.806 \times 10^{-15} \]

Scalar product of new u with previous u s
\[ 2.327 \times 10^{-14}, 2.251 \times 10^{-14}, 2.097 \times 10^{-14}, 1.833 \times 10^{-14}, 3.411 \times 10^{-14}, 6.995 \times 10^{-15} \]
\[ 0.000 \times 0.000 \]

Scalar product of new u with previous u s
\[ 9.298 \times 10^{-16}, 0.000 \times 0.000, 2.651 \times 10^{-15}, 0.000 \times 0.000, 2.496 \times 10^{-15}, 0.000 \times 0.000 \]
\[ 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000 \]

Scalar product of new u with previous u s
\[ 1.345 \times 10^{-13}, 3.081 \times 10^{-13}, 1.345 \times 10^{-13}, 3.081 \times 10^{-13}, 1.345 \times 10^{-13}, 3.081 \times 10^{-13} \]
\[ 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000 \]

Scalar product of new u with previous u s
\[ 1.342 \times 10^{-15}, 0.000 \times 0.000, -1.355 \times 10^{-15}, 0.000 \times 0.000, 9.888 \times 10^{-16}, 0.000 \times 0.000, 1.297 \times 10^{-15} \]
\[ 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000, 0.000 \times 0.000 \]

Scalar product of new u with previous u s
\[ -1.557 \times 12, 1.042 \times 10^{-15}, 1.171 \times 13, 3.686 \times 16, 1.932 \times 10^{-13}, 1.273 \times 10^{-14}, 1.343 \times 10^{-13}, 1.273 \times 10^{-14}, 4.917 \times 10^{-15}, 4.943 \times 10^{-15}, 3.065 \times 10^{-15} \]
\[ 3.932 \times 10^{-12}, 3.932 \times 10^{-12}, 3.932 \times 10^{-12}, 3.932 \times 10^{-12}, 3.932 \times 10^{-12}, 3.932 \times 10^{-12} \]
machine accuracy. The same process is necessary for the $M=2$ case when we need to restart at $v_{15}$ in order that $v_{25}=0$.

In table (4.5) we show the successive stages in the vector generation process to illustrate how the magnitude of the unnormalised vectors $v_n$ varies at each stage. Also shown are the values of the inner products of the vectors generated before and after reorthogonalisation to illustrate how orthogonality is maintained until $v_6$ where the process is restarted. Note how $v_9=0$ which shows that we have a closed space for the case when the rank=9 and $M=1$.

Of course for this noiseless but highly singular case we can only reconstruct a highly blurred version of the original image as plotted in figure (4.12a) with the real part of its spectrum in figure (4.12b) and its reconstruction using the JL algorithm in figure (4.12c). This is because with $M=1$ we only measure 9 spatial frequencies out of the complete field of 1024 spectral components in the $n=32$ case. Figure (4.12e) shows the result obtained with $M=2$ and not surprisingly slightly more detail is then obtained. In figures (4.12d) and (4.12f) we have the results for exactly the same problem but obtained using the iterative Gerchberg algorithm and as expected for the noiseless case they give identical results.
Figure (4.12) (Overleaf) Comparison of Solutions to the Hyper-singular band-limited problem using the Gerchberg and JL algorithms.

(a) (top left) Original image 'THE/END' before band-limiting.
(b) (top right) Real part of the spectrum of 'THE/END'.
(c) (middle left) Reconstruction of 'THE/END' after severe (M=1) band-limiting. This image has been reconstructed from only 3*3 spatial frequencies using the JL algorithm on noiseless data.
(d) (middle right) Reconstruction under the same circumstances as (c) but using the iterative Gerchberg algorithm.
(e) (bottom left) Reconstruction of 'THE/END' after severe (M=2) band-limiting. This image has been reconstructed from only 5*5 spatial frequencies using the JL algorithm on noiseless data.
(f) (bottom right) Reconstruction under the same circumstances as (e) using the iterative Gerchberg algorithm.
THE END
4.2.3 Premature Breakdown of Vector Generation
(Non-restartable Case).

In obtaining the results in figures (4.12c) and (4.12e) we have employed three techniques to maintain stability in the generation process and these are:

1. Reorthogonalisation of each new vector with respect to those already generated.

2. Enforcing symmetry in the real part and antisymmetry in the imaginary part of the $V$-space vectors.

3. Enforcing reality in the $U$-space vectors.

Enforcing symmetry was achieved merely by comparing each half of the vector. However, even when using these techniques we find that for the two-dimensional band-limiting problem the process of vector generation eventually breaks down. For the $n=32$ case we find that when $M>2$ and we are using double precision (as used to obtain all the results herein) we simply can not obtain a complete set of vectors. This is due to accumulation of errors during the generation process, even when using the techniques 1-3 above and is not due to any singular or ill-conditioned property of the matrix.

In figure (4.13a) we plot the results obtained before the breakdown occurs for the $M=3$ case. In figure (4.13b) we have the result obtained by the Gerchberg technique, which as figures (4.12d) and (4.12f) show us what we would obtain using the JL method if sufficient precision were available. In obtaining figure

continued on page 89
(a) Reconstruction of figure (4.12a) after band-limiting with $M=3$. That is the image has been reconstructed using $7 \times 7$ spatial frequencies. With the precision available on our computer using the inversion algorithm in its present form the basis generation process breaks down since orthogonality can not be maintained. This reconstruction has been achieved using only 28 of the full complement of $(2M+1)^2=49$ vectors.

(b) The same as figure (4.13a) as reconstructed using the Gerchberg algorithm. This is the result which would be obtained using a direct method if perfect precision were available.
(4.13a) we used 28 of the full complement of \((2M+1)^2 = 49\) vectors. The point where generation breaks down is indicated by a simple check using our knowledge (in this case) of the pseudo-data. This would not be useful in practice except in the hyper-singular case (see section 4.2.5) but merely shows us how close to the full solution we can get under these conditions.

The technique is as follows. Having solved the problem we obtain the components of the original object \(f\) in \(U\)-space, that is we have obtained \(\langle u_i | f \rangle, i=0,..81\). Then knowing our original pseudo-object we evaluate the actual \(\langle u_i | f \rangle_{\text{actual}}\) and compare the two. Once the two inner products differ by more than \(10^{-3}\) we ignore subsequent components and this is how we obtained figure (4.13a).

4.2.4 Storage Problems with the JL Method.

Although double precision is not sufficient to obtain the complete solution in the above case it must be remembered that for a fixed focus optical system one need only solve the problem once and store the new coordinate frames in order for many fast inversions later. With many inversion then possible at little cost any expense occurred in the first case would be made worthwhile.

However as \(M\) increases we find that the storage space required becomes increasingly restrictive. One way around this problem would be to invert and generate vectors simultaneously so that only a few vectors need be stored at any one time. But because we are using Gram-Schmidt reorthogonalisation this process in not
possible.

It looks increasingly likely that an iterative technique, such as that due to Gerchberg, will be the only viable method for observations with larger values of M. There is one more case, however, where the direct inversion technique may still be worth considering.

4.2.5 Direct Inversion in a Hyper-singular case.

Whenever \(2M+1 < L\) we have a singular problem of dimension \((2M+1)^2\). This problem considered alone is also likely to be more or less ill-conditioned. That is after application of the JL algorithm the terms \(\alpha_i\) and \(\beta_i\) may become small when \(i<2M+1\). This hyper-singular case may be such that most of the available information in a given problem is obtained before the vector generation process breaks down itself, particularly in a noisy problem.

The problem considered here becomes increasingly hyper-singular as the support on the original object becomes more and more restricted as \(L\) becomes smaller. This is illustrated in figure (4.14a) and (4.14b) where we plot the spectrums of \(\alpha\) and \(\beta\) for the case \(n=32,M=3\) and \(L=24\) and similarly in figure (4.15a) and (4.15b) for \(L=10\). It is clear how as \(L\) restricts the support more the problem becomes more hyper-singular.

In figure (4.16a) we plot the reconstruction of the letter E for the hyper-singular case \(M=6, L=10\), using the JL method. In
Figure (4.14)

(a) The spectrum of the JL normalisation coefficients $\alpha_n$ produced in the JL decomposition when $N=32$, $M=3$ and $L=24$, as used in obtaining the reconstruction in figure (4.13a).

(b) The spectrum of the JL normalisation coefficients $\beta_n$ produced simultaneously with the $\alpha_n$ of figure (4.14a).
Figure (4.15)

(a) The spectrum of the JL normalisation coefficients $\alpha_n$ produced in the JL decomposition for the $N=32, M=3, L=10$ system.

(b) The spectrum of the JL normalisation coefficients $\beta_n$ for the same system as in figure (4.15a). Note how the restriction of the support with $L$ made smaller makes the singular problem ill-conditioned in itself, that is it becomes hyper-singular.
Figure (4.16)

(a) Reconstruction of the letter E using the JL algorithm for the noiseless hyper-singular case, N=32, M=6, L=10. Although the full complement of vectors is \((2M+1)^2=169\) a successful reconstruction can be obtained because of the hyper-singular nature of the system. Unlike in the case whose reconstructed image is illustrated in figure (4.13a) all the useful information is contained in the vectors generated before the breakdown process occurs. (Refer to figure (4.17a) and (4.17b).

(b) Reconstruction of the letter E under the same circumstances as in figure (4.16a) but using the iterative Gerchberg algorithm.
Figure (4.17)

(a) Distribution of the JL normalisation coefficients $\alpha_n$ for the system inverted in figure (4.16a).

(b) Distribution of the JL normalisation coefficients $\beta_n$ for the system inverted in figure (4.16a).
figure (4.16b) we give the result for the same problem using the Gerchberg method. In this case the direct method actually outperforms the Gerchberg method. The direct method works well here because of the hyper-singular nature of the problem with most of the information being extracted before the vector generation breaks down, coupled with the restriction on the support to L=10. In figures (4.17a) and (4.17b) we plot the spectrum of alphas and betas and again the hypersingularity is clear to see.

4.2.6 Techniques for Extending the Basis Generation Process

If we are determined to provide a direct inversion system for repeated use then we can use increased precision using, for example using the C like language BC. Alternatively we can look at more sophisticated methods for extending the process by maintaining orthogonality and symmetry.

One possible technique might be to use an alternative form of the FFT algorithm which guarantees symmetry in its output. For example, Jones [68] suggests the use of a centred Fourier transform which enables one to have the origin of both the image and its Fourier transform at the geometrical centre of the picture.

The centred Fourier transform is achieved by dispensing with the "d.c" term and having the origin of each space not at a sample point but half way between. This leads to $F_q^c = F_{-q}^c$ for all L.

Fast methods are available for the CFT as for the DFT based on the identity
\[ F_{q_0+lm} = \sum_{p_0=-(L-1)/2}^{(L-1)/2} e^{-2\pi i (q_0+lm)p_0/N} + \sum_{m=-(M-1)/2}^{(M-1)/2} e^{-2\pi i (q_0+lm)m/N} f_{p_0+ml} \]  

(4.14)

\[ (q_0=-(M-1)/2,\ldots,(M-1)/2, \quad l=-(L-1)/2,\ldots,(L-1)/2 \text{ where } L,M \text{ and } N=LM \text{ are all even.} \]

The particular form of the fast centred Fourier transform should be carefully chosen with reference to appropriate butterfly diagrams as this can also affect the symmetry of the vectors produced. Many different variations on the basic FFT algorithm are available and their respective butterfly diagrams can be found in such places as Oppenheim and Schafer [69], and Brigham [70].

4.2.7 Comparison of problems caused by eigenvalue distributions in the JL algorithm and the Gerchberg algorithm. Application of the accelerated Gerchberg technique to Image Data.

In [52] it is shown that if \( 2M+1>L \) then \( P \) is positive semi-definite, that is, \( 0\leq \lambda \leq 1 \) and so for \( r \to \infty, \ P^r \to 0 \). So for an infinite number of iterations we see that \( b_1^{r \to \infty} \to f_1 \) in exact precision.

Note that in our notation
\[
I-P= I- \frac{1}{N} [\Omega_{10}^{\dagger} \Omega_{10} + \Omega_{12}^{\dagger} \Omega_{12}] = \frac{1}{N} \Omega_{11}^{\dagger} \Omega_{11} \]

(4.15)

and convergence depends on \( \Omega_{11} \) as opposed to the direct method of solution where it depends on \( \Omega_{00} \Omega_{02} \).

The technique described in [52] depends on an expansion of \( F_1 \)
in eigenvectors of $P$. We have, from section (3.9.2).

$$B_1^{(r)} = (1-P^r)F_1 \tag{4.16}$$

assuming $B_1^{(0)} = 0$ (for many optical problems it would at least be small). The eigenvector expansion then gives

$$F_1 = \sum_j \alpha_j v_j \tag{4.17}$$

Therefore

$$B_1^{(r)} = \sum_j (1-\lambda_j^r) \alpha_j v_j \tag{4.18}$$

We can see that for eigenvalues close to unity we may need many iteration before $\lambda_j^r$ becomes negligible and the contribution to $B_1^{(r)}$ from the corresponding eigenvector becomes constant. This compares with the JL method where the problem of ill-conditioning occurred when $[\Omega_0 \Omega_{02}]$ had small eigenvalues as opposed to here where slow convergence is caused by $[\Omega_{11}]$ having eigenvalues close to unity.

The results plotted here-in for reconstruction were produced using the acceleration technique, a full description of which can be found in [52] where it was applied to one-dimensional signal data. The acceleration technique depends on the condition

$$\lambda_j^r \to 0 \text{ rapidly for } r \to \infty \text{ when } \lambda_j \neq 1 \tag{4.19}$$

With this condition in mind, after a few iterations we can simplify the summation in equation (4.18) above. This is because after a few iterations of the Gerchberg algorithm the contribution to $B_1^{(r)}$ from the eigenvectors corresponding to the smaller eigenvalues of all the $\lambda_j^r$ will remain constant. We can then iterate solely for the components corresponding to the larger eigenvalues which require more iterations before convergence is
achieved.

In figure (4.18a) we plot the spectrum of the reconstruction of

after 32 iterations of the Gerchberg algorithm for the case

M=6, L=24. In figure (4.18b) we have the same plot but having employed the acceleration process. Although the improvement obtained is not spectacular one can see how more high spatial frequency components have been brought into the reconstruction using the acceleration process. The fact that in this case the improvement in reconstruction is not great can be attributed to the relevant part of the band-limiting matrix, \([R_{11}]\), not having any eigenvalues near unity.

4.2.8 Another Choice of Algorithm for the Band-limiting Problem.

Bearing in mind the various problems encountered above and how they may vary in importance from one system to another it is worth considering the statement made by Wilkinson in [31], "for a positive definite matrix Cholesky decomposition has all the virtues". With this in mind note that the general matrix representation arrived at in section (2.5.6), equation (2.51) is rendered positive definite by the terms \(\epsilon\) on the diagonal and is therefore ideally suited for Cholesky decomposition.

Now in the technique of Cholesky there is no need to store
Figure (4.18)

(a) Spectrum of the reconstruction of figure (4.12a) after 32 iterations of the Gerchberg algorithm for the case N=32,M=6,L=24.

(b) The same as in figure (4.18a) but having used the accelerated Gerchberg algorithm. Note the presence of high spatial frequencies in the reconstruction not present in the unaccelerated reconstruction indicating improved super-resolution. (The DC components of the images have been clipped to show up the high spatial frequencies.)
the vectors which is part of the problem faced when using the JL algorithm in two-dimensional problems where $2M+1 \leq L$. However once the JL decomposition has been accomplished it is easy to invert quickly with different values of the regularisation parameter $c$ whereas in the Cholesky decomposition one would have to do the full decomposition for each new value of $c$. In any case further work may demonstrate the Cholesky decomposition as an alternative to iterative techniques such as that of Gerchberg.
Chapter V.

Electromagnetic Theory of Eddy Currents.

Introduction to Eddy Current Imaging.

So far we have considered the inverse problem and its solution for problems of the type encountered in optical imaging. These applications are very important due to the large range of optical inverse problems over the whole electromagnetic spectrum.

Inverse imaging problems do not only occur in optical situations. Various acoustic problems demand the solution of an inverse problem: medical acoustic imaging, underwater acoustic imaging and quality control in the manufacture of musical instruments being just some examples.

In chapter I we introduced the Fredholm equation of the first kind and chapters III and IV gave the results of inverse problems where the kernel of the Fredholm equation being inverted was something closely related to the Fourier transform operator. In this section we shall outline the electromagnetic theory which leads to an inverse problem in the form of a Fredholm equation of the first kind whose kernel is a modified complex Laplace transform. This is the eddy current imaging problem and it is both complex and highly ill-conditioned.

Suppose we know the magnetic field above the plane surface of a conductor resulting from the scattering of eddy currents by a subsurface crack. If the crack has a negligible opening and acts
as a surface barrier to eddy currents, its effect on the field is the equivalent of a surface layer of dipoles Bowler [71]. Because the dipole moment is zero elsewhere in an otherwise homogeneous conductor, if we can image the dipole distribution, then within the limits of the ill-posedness of the problem we can deduce the size, shape and orientation of the flaw.

In general the dipole distribution associated with a given magnetic field outside the conductor is not unique (the problem is ill-posed) and therefore one cannot in general determine the crack geometry from the external magnetic field at a particular frequency, without additional a priori information. However by assuming the crack lies in a known plane, the extent of the dipole distribution in the plane may be found and hence the flaw geometry inferred.

5.1 Formulation of Forward Problem: Calculating the Induced Dipole Distribution on a Flaw for a Given Incident Field.

Bowler [72] has used Maxwell's equations along with the usual boundary conditions for conductor/ non-conductor interfaces in order to formulate the first forward problem of general validity for eddy current imaging. The first step is to calculate the dipole distribution induced on an arbitrary crack by some arbitrary incident field. This can be done in four steps as in [72]:

1. The incident field due to a probe is calculated in the case of
an unflawed conductor.

2. The crack is then represented by its equivalent source; a current dipole distribution \( \mathbf{p} \) on a surface.

3. The field due to this distribution is expressed as an integral with a Green's function kernel.

4. An integral equation for the dipole distribution is then obtained by using the boundary condition that at the crack surface the field due to the crack is such that it exactly cancels out the normal component of the incident electric field.

This method has been used by Travis, Bowler and Jones [73] to calculate pseudo-data for the dipole distribution around a predefined crack in a known plane.

5.2 Formulation of the Inverse Problem: Calculating the Magnetic Field at the Surface of a Conductor caused by a Given Dipole Distribution on the Flaw.

In a similar spirit to the use of pseudo-data in the optical imaging problems described above we aim to reconstruct such dipole distributions as calculated in the previous section from pseudo-measurements of the corresponding magnetic field at the surface near the flaw. As before consistency of reconstruction for pseudo-measurements corrupted with different noise samples gives us a measure of the efficacy of the inversion algorithm.

In formulating the inverse problem we neglect displacement current. The magnetic field in air \((z>0)\) scattered by a flaw in the conductor \((z<0)\) is expressed in terms of the transverse
electric (TE) Hertz potential \( W(r) \) as, Bowler [71]

\[
H(r) = \nabla \times \nabla_z W(r)
\]  
(5.1)

\( a_z \) being the unit vector normal to the surface of the conductor.

In source free regions the Hertz potential satisfies the Laplace equation. Thus

\[
\nabla^2 W(r) = 0
\]

(5.2)

Assuming the \( z \)-component of the magnetic field is known in a plane \( z = z_0 \) in air \( (z_0 > 0) \), then \( W(r) \) may be related to the field sample in this plane through the two-dimensional transform of the \( z \)-component of equation (5.1). Writing the two dimensional Fourier transform of the potential as

\[
\tilde{W}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x,y,z) \exp[iux +ivy] dxdy
\]  
(5.3)

where the tilde implies a dependence on the Fourier-space variables \( u \) and \( v \) and we find

\[
s^2 \tilde{W}(z) = H(u,v,z_0) \exp[-s(z-z_0)]
\]

(5.4)

With \( s \) as the positive root of \( s = \sqrt{u^2 + v^2} \), the potential vanishes as \( z \to \infty \) and (5.2) is satisfied.

Using the Green's function method one can get an integral relationship between the source distribution on the crack and \( W(r) \). For a flaw giving rise to an effective volume source \( P(r) \) in a half space conductor, the TE Hertz potential representing the scattered field is given by

\[
W(r) = \int_{\text{flaw}} [\nabla_x U(r,r')] \cdot P(r) dr'
\]

(5.5)

where \( P(r) \) is the current dipole density of the flaw and \( U(r,r') \) is the half-space Green's function transforming an electrical source distribution in the conductor \( (z'<0) \) into the TE Hertz

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potential in air \((z>0)\). The derivation of the Green's function for a source inside a conductor is similar to the analysis leading to scalar Green's functions for external sources \([71]\) and gives

\[
\nabla x a_z U(r,r') = \frac{1}{4\pi} \int \int \frac{i(v a - u a)}{s^2(s + \gamma)} \exp[-sz+\gamma z'+iu(x-x')+iv(y-y')]dudv
\]

\((5.6)\)

where \(\gamma = \sqrt{u^2+v^2-i\omega \mu_0 \sigma}\) taking the root with a positive real part.

For the case of a crack of negligible opening in the plane \(x=0\), the dipole distribution induced on the flaw has the form \([71]\)

\[
P(r) = p(y,z)\delta(x)a_x
\]

\((5.7)\)

where \(p(y,z)\) is the surface current dipole density excited on the crack by the induced current in the conductor. With this form for the dipole distribution, the two dimensional Fourier transform of \((5.5)\) may be combined with \((5.4)\) to give

\[
H_z(u,v,z) = \int_{-\infty}^{0} iv/(s+\gamma)p(v,z')dz'
\]

\((5.8)\)

which relates the normal component of the magnetic field in the plane \(z\) above the flaw with the dipole distribution \(p(v,z')\) in the plane in which we have assumed the crack to be present.

If we can invert the modified complex Laplace transform of equation \((5.8)\) well, then given the magnetic field normal to the surface of the conductor in some sample plan \(z=z_0\), we can hope to image the dipole distribution \(p(v,z')\).

Before we attempt to invert equation \((5.8)\) we should point out that solution of the inverse problem is not only necessary for what we might call imaging problems but is also needed for various other situations. For example certain problems in the fields of sedimentation equilibrium, photon spectroscopy and fluorescent
decay require the inversion of the real Laplace transform of some unknown distribution and various *ad hoc* methods exist for the solution of such problems. However no single method has yet been given for general use and indeed some very pessimistic views have been expressed about the possibility of inverting the Laplace transform, Varah [74].

The ability to invert the real Laplace transform using a method such as the JL algorithm is a prerequisite for solution of the eddy current imaging problem. Since the demand for solution of the real Laplace transform is considerable it makes sense therefore to solve the real problem first before going on to solve the complex eddy current problem.
Chapter VI.

Inversion of the Laplace Transform.

Introduction.

In chapter I we introduced the Fredholm equation of the first kind and chapters III and IV gave solutions to problems where the kernel of the equation being inverted was something closely related to the Fourier transform operator. In this section we have a different Fredholm equation whose kernel is the Laplace transform

\[ L(p,t) = e^{-pt} \]  \hspace{1cm} (6.1)

and instead of imaging oscillatory components the experiment is concerned with resolving exponential relaxation rates.

As in the case of the Fourier problem, finer detail than is accessible within the resolution limit is not obtainable due to ill-conditioning. In fact the inverse Laplace transform is always highly ill-conditioned compared to the systems encountered in the Fourier problem which, except in the explicitly singular cases discussed in chapter III and IV, were not ill-conditioned.

6.1 Demonstration of the Ill-conditioned Nature of the Inverse Laplace Transform.

To demonstrate ill-conditioning of the inverse Laplace transform first consider the forwards Laplace transform (which is well conditioned and continuous). Let \( f(t) \) be a function for which
\[ \int_0^\infty |f(t)| e^{-\lambda t} \, dt \leq \varepsilon \]  \hspace{1cm} (6.2)

then if \( \text{Re}(s) \geq k \)

\[ |L(f+g)-L(g)| = \int_0^\infty f(t)e^{-st}dt \leq \int_0^\infty g(t)e^{-\text{Re}(s)t}dt \leq \varepsilon \]  \hspace{1cm} (6.3)

and so a small change in \( g \) produces an equally small change in \( Lg \).

That is \( L(g) \) is stable under perturbations of this type.

The inverse of the Laplace transform is however not stable under small perturbations. To demonstrate this we consider two examples.

The Laplace transform of \( \sin(at) \) is given by

\[ L(\sin(at)) = \frac{a}{s^2 + a^2} \]  \hspace{1cm} (6.4)

As \( a \) increases, the function \( \sin(at) \) oscillates more and more rapidly, but remains a constant amplitude. The Laplace transform is uniformly bounded by \( 1/a \) for \( s \geq 0 \) and thus approaches 0 uniformly as \( a \to \infty \). Thus a small change on the right hand side of equation (6.4) will produce a large change in its inverse. Consequently any small error will result in a large discrepancy in the inverse.

Also consider the formula

\[ L\left\{ \frac{a}{(2\sqrt{\pi})}e^{-\frac{a^2}{4t^2}}t^{3/2} \right\} = L(u(t)) = e^{-\sqrt{a}s} \]  \hspace{1cm} (6.5)

where, as \( a \to 0 \), the function \( e^{-\sqrt{a}s} \) remains uniformly bounded by 1 for \( s \geq 0 \). Observe however how \( u(t) \) behaves as a function of \( t \). At \( t = a^2/4 \) we see that it has the value \( c_1/a^2 \) where \( c_1 \) is a positive constant. Nonetheless, at \( t=0 \), for all \( a>0 \), \( u(t) \) assumes the value 0. Hence \( u(t) \) tends towards a delta function as \( a \to 0 \). Again, as in the first example, a small error \( \varepsilon \) on the right hand side will correspond to a greatly different inverse or put conversely any particular given value for the right hand side expressed to a
certain degree of accuracy will correspond to many possible inverses within that accuracy.

6.2 Reconstruction and Inversion of the Laplace Transform.

In the case of bandlimiting imaging and communication systems there is a well developed theory of resolution as associated with names like Nyquist [75], Shannon [76] etc. and stretching back to Abbe [77], and Lord Rayleigh [78]. For the usual linear transformation.

\[ g(x) = \int_{a}^{b} L(x,y) f(y) \, dy \quad (6.6) \]

we find that in the presence of noise the object \( f(y) \) can be recovered from the image up to a limit of resolution governed by the Shannon or Nyquist number or Rayleigh criterion. This limit of resolution is determined by the properties of the eigenvalue spectrum of the transformation \( f \circ g \). In the case where the kernel is \( e^{-pt} \), finer details are irrecoverable owing to the highly ill-conditioned nature of the inversion.

The classical theory of resolution is concerned with the problem of recovering and resolving natural spatial or temporal oscillatory components for the bandlimited Fourier problem considered in chapters III and IV.

McWhirter and Pike [79] have calculated analytically the eigenfunctions and eigenvalues of the Laplace transform and identified an analogue of the Shannon number of Fourier optics. In contrast to the example of Fourier optics this number is strongly dependent on noise. The Shannon number in this case represents the number of exponential relaxation rates (as opposed to the number
of oscillatory components) which can be successfully determined. In their paper, McWhirter and Pike show that these rates must be placed in a geometrical sequence in the independent variable.

In all the examples we will consider here the inversions will be carried out with linear sample point spacing and if we were to follow the techniques of McWhirter and Pike, as used by Bertero et al [80] then improved resolution may be expected. We will then demonstrate how significantly improved resolution can be obtained by restriction of the object support. However here we will incorporate various amounts of a priori knowledge into the linear minimum variance estimation to demonstrate how significantly improved reconstruction can be obtained.

The explicit Laplace transform is

$$g(p) = \int_{0}^{\infty} e^{-pt} f(t) dt \quad 0 < p < \infty$$  \hspace{1cm} (6.7)

In the problem of photon correlation spectroscopy, Bertero et al, [81], [80], [82] and [83], p is real whilst in the problem of electrical eddy current imaging as described in the previous chapter p is complex. As demonstrated above, although the forwards Laplace transform operation is continuous, the inverse is not. In other words knowing only an approximation to the true Laplace transform g(p) means that the solution of the problem is indeterminate. Thus the set of functions whose Laplace transforms approximate g(p) within a given error is not bounded with respect to the norm $L^2(0,\infty)$ and consequently the inversion of the Laplace transform is highly ill-posed.
6.3 Previous Work on the Inversion of the Laplace Transform.

The inverse of the Laplace transform, equation (6.7), is defined by the Bromwich or Fourier-Mellin integral (see e.g. Boas [84] or Arfken [85])

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{pt} g(p) dp \quad t > 0$$

(6.8)

The derivation of this expression demands that $f(t)$ be piecewise continuous and of exponential order $\alpha$, that is,

$$|f(t)| \leq e^{\alpha t}$$

(6.9)

where $\alpha$ is the minimum exponent. The integration is performed along the line $p=a$ in the complex plane where $a > \alpha$, this is equivalent to saying that the line $p=a$ lies to the right of all singularities in $g(p)$. For this reason the value of $\alpha$ is crucial to the correct evaluation of the inverse. It is not essential to know $\alpha$ exactly, but an upper bound must be known (this is another example where a priori knowledge is required for a useful solution of the problem).

In the case where we wish to invert the real Laplace transform then, in addition to the requirement of a priori knowledge, we must know how $g(p)$ varies in the complex plane and so in terms of the Bromwich integral we can see the problem is poorly determined.

Even in a problem where we measure complex $p$ lack of a priori information about the exponential order of $f(t)$ and the position of the poles in the plane of $p$ prevent us from obtaining a useful
inversion. Krylov and Skoblya [86] have investigated inversion using the Bromwich integral, whilst Bohn and Flynn [87], have derived the inverse Laplace transform using only real variables.

Other published work on the inversion of the Laplace transform includes Bellman et al [88] and more recently Fioravanti [89] and Essah and Delves [90]). One means of usefully inverting ill-conditioned systems is by using an SVD (Wilkinson [34] as used in [80],[81],[82] and [83]). Again we use here the faster alternative JL algorithm previously applied to reconstructing bandlimited noisy data in the Fourier space problems of chapter III and IV.

To invert (6.7) we first discretise, writing \( g_n \) for \( g(p_n) \). For constant sample point spacing along \( z \) (for now) the discretised form of (6.7) is

\[
g_n = \Delta t \sum_{k=0}^{K-1} f(t_k) e^{-p_n t_k}
\]

or

\[
g_n = \Delta t \sum_{k=0}^{K-1} L_{(n,k)} f_k
\]

where

\[
L_{nk} = (e^{-p_n \Delta z})^k = \lambda_n^k
\]

Explicitly

\[
\begin{bmatrix}
  g_0 \\
g_1 \\
\vdots \\
g_{K-1}
\end{bmatrix}
= \begin{bmatrix}
  1 & \lambda_0 & \lambda_0^2 & \ldots & \lambda_0^{K-1} \\
  1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{K-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & \ldots & \ldots & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
p_0 \Delta z \\
p_1 \Delta z \\
\vdots \\
p_{K-1} \Delta z
\end{bmatrix}
\]

(6.13)

The matrix \([L_{nk}]\) is a van der Monde matrix \((L_{nk} = L_n^k)\) for which the inverse can be computed for example using the expression

\[\text{112}\]
\[
L_{ij}^{\text{inverse}} = (-1)^{j+1} \sum (L_{k1} L_{k2} \ldots L_{k(n-j)}) / L\prod (L_k - L_1)
\] (6.14)

where \(L_{ij}^{\text{inverse}}\) is the \(ij\)th element of the inverse matrix \([L_{ij}]^{-1}\) and where the sum in the numerator is the coefficient of \(L^{j-1}\) in the polynomial \((L - L)(L - L)(\ldots)(L - L)/(L - L)\).

In the sum
\[
1 \leq k_1 \leq \ldots \leq k_{n-j} \leq n
\] (6.15)

and
\[
k_1, \ldots, k_{n-j} = i
\] (6.16)

whilst in the product \(1 \leq k \leq n\) and \(k = i\). However difficulty will be found in inverting the system due to the highly ill-conditioned nature of the matrix \([L_{jk}]\) and also because of attendant observation noise in the elements \(g_n\).

6.4 Setting up for the Bidiagonalisation.

For the standard linear measurement model of equation (1.2), that is
\[
g = Lf + n
\] (6.17)

where \(L\) is now the Laplace transform operator we need to evaluate the adjoint operator \(L_{AB}^\dagger\). For the simplest case \(L_{AB}^\dagger\): \(W^\dagger W\) will be the Hermitian conjugate of \(L\). More generally though an inner product \(\langle g|h \rangle_v\) will be defined in \(V\) space such that:
\[
\langle g|h \rangle_v = \sum_{n=0}^{K-1} W_n^* g(p_n) h(p_n)
\] (6.18)

where for the simplest case \(W\) is the identity matrix. If \(W\) is diagonal but not the identity then this corresponds to unequal sample point spacing as prescribed in [80] and [82]. In addition a non-identity diagonal \(W\) may reflect some additional weighting as prescribed in [83]. More will be said about this in chapter VIII.
With the definition of the inner product we can obtain

\[
L^* = \begin{bmatrix}
  w_0 & w_1 & w_2 & \cdots & w_{N-1} \\
  w_0 \tilde{\lambda}_0 & w_1 \tilde{\lambda}_1 & w_2 \tilde{\lambda}_2 & \cdots & w_{N-1} \tilde{\lambda}_{N-1} \\
  & \ddots & & \ddots & \vdots \\
  & & w_0 \tilde{\lambda}_{N-1} & w_1 \tilde{\lambda}_{N-1} & w_2 \tilde{\lambda}_{N-1} & \cdots & w_{K-1} \tilde{\lambda}_{N-1}
\end{bmatrix}
\begin{bmatrix}
p_0 \Delta z \\
p_1 \Delta z \\
\vdots \\
p_K \Delta z
\end{bmatrix}
\]  

(6.19)

and so using this along with \( L \) we can invert the system using the JL algorithm along with linear minimum variance estimation.

6.5 Regularisation in the Laplace Transform case and a Criterion for the Termination of Basis Vector Generation using the Matrix Trace.

In fig. (6.1) we plot the real and imaginary parts of the first few vectors in the two spaces produced by bidiagonalisation of the matrix using the JL algorithm. In table (6.1) are plotted the corresponding elements of the bidiagonalised matrix, the alphas and betas, and it is clear they fall off very rapidly. Since there is obviously no point in fully decomposing such an ill-conditioned system we need some criterion for terminating vector generation.

Referring to equation (2.49) we see that the diagonal term consists of \( \alpha^2 + \beta^2 + \epsilon \) and obviously once \( \alpha \) and \( \beta \) become small enough then \( \epsilon \) will dominate. Hence all we need to do is terminate generation of more vectors once \( \epsilon \) dominates by an appropriate amount. As long as the subsequent \( \alpha \) and \( \beta \) are known to be smaller than the previous ones then we know that we are not ignoring any meaningful information by truncating the space.
By using the theorem that the trace of a matrix is unchanged under an orthogonal transformation (see appendix IX) we can simply terminate generation once the trace of the elements already calculated exceeds a certain percentage of the known trace of the original representation of the operator.

If an operator $D$ has a matrix representation $[D_{jk}]$ referred to an $A$-orthonormal basis, then

$$\text{trace}(D) = \sum_{k} D_{kk} \quad (6.20)$$

For $D = L^\dagger L$, as for the Tikhonov regularisation problem, then

$$\text{trace}(L^\dagger L) = \sum_{jk} L^H_{jk} L_{jk} = \sum_{jk} |L_{jk}|^2 \quad (6.21)$$

In the singular value decomposition

$$\sum_{j} |L_{jk}|^2 = \gamma_k^2 \quad (6.22)$$

so

$$\text{trace}(L^\dagger L) = \sum_{k} \gamma_k^2 \quad (6.23)$$

For the bidiagonal representation

$$\text{trace}(L^\dagger L) = \sum_{k} (|\alpha|^2 + |\beta|^2) \quad (6.24)$$

Alternatively, following the work of McWhirter and Pike we can use Laplace transform inversion resolution theory to predict beforehand, given a certain noise level and sampling scheme, the number of exponential terms accessible (see for example [81]).

For the Laplace transform we have (Appendix X)

$$\text{Trace}(LL^\dagger) = \frac{1}{2} \ln \left[ \frac{b}{a} \right] \quad (6.25)$$

enabling us to evaluate the trace immediately.
6.6 Regularisation in the Complex Case and Choice of $\varepsilon$.

In the case of a complex system we can formulate the problem in two ways. We can either solve it as one of complex vectors in complex spaces of dimension $N$ as in equation (6.27) or as one of real vectors in real spaces of dimension $2N$ as in equation (6.28).

In the former case and one data point

$$[g_r + ig_i] = [L_r + iL_i][f_r + if_i]$$  \hspace{1cm} (6.26)

and for the latter case

$$\begin{bmatrix} g_r \\ g_i \end{bmatrix} = \begin{bmatrix} L_r & -L_i \\ L_i & L_r \end{bmatrix} \begin{bmatrix} f_r \\ f_i \end{bmatrix}$$  \hspace{1cm} (6.27)

If we choose the complex vector space formulation then having decomposed the operator $L$ to its bidiagonal form then the final stage of the inversion consists of the inversion of the two systems (equations (6.29) and (6.30)). (Here we assume $P_A$ and $R_B$ to be equal to the identity).

$$g_r = (L^*L + \varepsilon)f_r$$  \hspace{1cm} (6.28)

and

$$g_i = (L^*L + \varepsilon)f_i$$  \hspace{1cm} (6.29)

The separability of the real and imaginary systems is a consequence of the real nature of the decomposed $L$. This separability represented in equations (6.29) and (6.30) suggests that we define $\varepsilon$ as a complex regularisation parameter such that we might have two values of the regularisation parameter $\varepsilon$, $\varepsilon_r$ for the real part of the decomposed system and $\varepsilon_i$ for the imaginary system. However the definition of the variance, a real number, is

$$\sigma^2 = E(f - f_0)(f - f_0)$$  \hspace{1cm} (6.30)
which gives
\[ \sigma^2 = \sigma^2_r + \sigma^2_i \]  
(6.31)

and so we must take
\[ \varepsilon = \frac{\sigma^2_{nr} + \sigma^2_{ni}}{\sigma^2_{sr} + \sigma^2_{si}} \]  
(6.32)

If we choose the real vector space representation this allows us to write the vectors with the real parts stored in the first \( N \) points with the imaginary parts in the second \( N \) points. Hence any \textit{a priori} knowledge about the covariance of the real and imaginary parts of the source considered separately can be made use of. The information is simply built in to the top left and bottom right parts of the source covariance matrix. However the definition of the linear minimum variance estimator is such that \( \varepsilon \) must be the same in both parts. Hence in the real representation we may have equations (6.29) and (6.30) look like

\[
\begin{bmatrix}
\text{Re}(g) \\
\text{Im}(g)
\end{bmatrix}
= \begin{bmatrix}
\text{Re}(P_A) + \varepsilon & 0 \\
0 & \text{Im}(P_A) + \varepsilon
\end{bmatrix}
\begin{bmatrix}
\text{Re}(f) \\
\text{Im}(f)
\end{bmatrix}
\]

(6.33)

Here though we shall tackle the problem as an \( N \)-dimensional complex one.
Chapter VII.

Results of Laplace Transform Inversions.

7.1 Real Laplace Transform Inversions.

We present here the results of our inversion of equation (6.7) for the one-dimensional case

\[ g(p) = \int_{0}^{\infty} e^{-pt} f(t) dt \]  \hfill (7.1)

where figure (7.1a) shows the real and imaginary parts of a test function \( f(t) \) where

\[ \text{Re}( f(t) ) = 16^{\frac{1}{16}} t^{15} e^{-16t} / 15! \]  \hfill (7.2)

(as used by Bertero et al [83]), and

\[ \text{Im}( f(t) ) = \text{Re}( f(t) ) / 2 \]  \hfill (7.3)

Figure (7.1b) shows \( g(p) \) along with the same corrupted with noise whose variance as a percentage of the noiseless observation variance is 1%.

Figure (7.2) shows the function \( f(t) \) along with the real part of the reconstruction obtained in the noiseless case when the \( n^{th} \) vector generated \( \nu_n \) is set to zero and subsequent vectors are left out of the reconstruction. The graph shows the reconstructions for the cases \( n = 1, 3, 5, 7 \) and 9 along with the actual function \( f(t) \), the reconstruction getting nearer the more terms are included. The Laplace transform operator is highly ill-conditioned and so it is found in practice that elements of the bidiagonalised matrix fall off rapidly down the two diagonals.

continued on page 122
Figure 7.1.(a) The real and imaginary parts of the test function $f(t)$ where

\[ \text{Re}(f(t)) = 16^{16}t^{15}e^{-16t}/15! \quad \text{and} \]

\[ \text{Im}(f(t)) = \text{Re}(f(t))/2 \]

Figure 7.1.(b). The Laplace transform of $f(t)$ as defined by equation (7.1) for complex $p$, $g(p)$ along with the same corrupted with noise whose variance is 1% of the variance of $f(t)$.(The percentage is not of the function to which the noise is added but of the function which is being reconstructed ).
Figure 7.2. The function $\text{Re}(f(t))$ (figure 7.1a) along with the real part of the reconstruction as obtained in the noiseless case when the $n^{th}$ vector generated, $v_n$, is set to zero and subsequent vectors are left out of the reconstruction.
Figure 7.3. Reconstruction from noiseless Laplace transform data of two delta functions. In a similar spirit to figure 7.2, vector generation was terminated after $v_9$, $v_{13}$, and $v_{17}$ respectively to demonstrate the impossibility of perfect reconstruction of this function without unlimited machine accuracy. No better reconstruction could be obtained under double precision since the ill-conditioning of the Laplace transform operator imposes, in effect, a small noise level on the system (see section 2.3).
We thus obtain a graphically perfect reconstruction in the noiseless case (for this particular function) using far less than the full complement of $n$ vectors in each of the object and image spaces for the case of an $n \times n$ operator $M$.

For the function in figure (7.1a) it was found that a graphically perfect reconstruction could be obtained with basis vector generation terminated after $v_{11}$ or $v_{12}$ though it should be said that by this stage the coefficients of the bidiagonalised matrix are very small, much smaller than we could hope to image in practice. This is a consequence of the highly ill-conditioned nature of the Laplace transform operator and should not be attributed to the algorithm itself.

For a test function with discontinuities we attempted a noiseless reconstruction of two delta functions (imaginary part zero) as illustrated in figure (7.3). This graph shows the two delta functions being reconstructed along with the real part of the noiseless reconstruction with vector generation terminating after $v_9$, $v_{13}$ and $v_{17}$, the latter being the best reconstruction possible since after that machine noise became greater than the elements of the bidiagonal matrix being generated even in double precision.

To test the algorithm in the presence of additive noise on the observation vector we reconstructed the function $p(z)$ where

\[
\text{Re}(f(t)_2) = f(t)_1 + 32^{32} t^{31} e^{-32t} / (31!) 3 \quad (7.4)
\]

\[
\text{Im}(f(t)_2) = \text{Re}(f(t)_1) \quad (7.5)
\]

Figure (7.4a) shows the function $f(t)_2$ along with the continued on page 124.
Figure 7.4(a) and (b) The functions $\text{Re}(f(t))$ and $\text{Im}(f(t))$ along with the Laplace transform of $f(t)$ when the observation was corrupted with noise whose variance was 1% of the variance of the original image $f(t)$. The two peaks in the real part are not resolved due to the high noise level as depicted in figure 7.1(b).

Figure 7.5(a) and (b) is the same as figure 7.4, but with an observation noise level as depicted in figure 7.6. In this case the two peaks are resolved consistently with the different noise samples at that level.

Figure 7.6. The Laplace transform of $f(t)$ with noise whose variance is .001% of the original signal $f(t)$.
reconstructions obtained when Gaussian noise whose variance was 1% of the variance of the function \( f(t) \) was added to the observation before reconstruction. Each curve shows a reconstruction corresponding to a noise sample generated using a different seed in the NAG noise generating routine we used. Figure (7.4b) shows the same for the imaginary part of the reconstruction. Though there is consistency in the reconstruction with respect to different noise samples it seems there is too much noise present in the observation to make resolution of the two peaks in the real part of the test function possible in this case. Bearing in mind we have used noise of the same magnitude as depicted in figure (7.1b) this is not surprising.

Figure (7.5a) and (b) follow the same pattern as figure (7.4) but use noise whose variance is .01% of the variance of the observation function as shown in figure (7.6) and we see in this case that the noise is small enough to allow resolution of the two peaks in the real part of the test function.

Note how in figure (7.5), for example, the deviation from the original of the reconstruction is always worst near the edge of the function. Reducing such edge effects was the topic of recent work, [83], where weighted spaces were used to produce a more accurate solution. The techniques used in [83] was to redefine the inner product and this option is also open to us in our technique. However we also have the ability to build in a priori knowledge to the object mean \( f_0 \).
7.2 Eddy Current Inversion Results.

Here we give results of eddy current inversion as explained in chapter VI. We have for the end result of the forwards eddy current problem the expression

\[ H_z(u,v,z) = \int_{-\infty}^{0} \frac{iv/(s+\gamma)e^{-szz'}}{s+r} p(v,z')dz' \quad (7.6) \]

which relates the magnetic Bowler potential, Nakagawa [91], in the plane \( z \) above the flaw with the dipole distribution \( p(v,z') \) in the plane of the crack in which we have assumed the crack to be present.

If we can invert equation (7.6) well, then given the magnetic field normal to the surface of the conductor in some sample plane \( z = z_0 \), we can hope to image the dipole distribution \( p(v,z') \). In an experiment we would be measuring \( H_z(x,y,z) \), in fact, and hoping to image \( p(y,z') \). Here though we shall invert and regularise equation (7.6) and so the two-dimensional Fourier transform of the observation data will not be included under the regularisation and we shall image the dipole distribution with its first coordinate in transform space.

Discretising equation (7.6) for \( K \) sample points of constant spacing \( \Delta z' \) along \( z' \) we obtain

\[ H_n^{(m)} = \Delta z \sum_{k=0}^{K-1} \frac{iv^{(m)}}{(s_n^{(m)} + \gamma_n^{(m)})} \exp\left\{-s_n z_0 + \gamma_n^{(m)} z_k \right\} p(v^{(m)},z_k) \quad (7.7) \]

where \( H_n^{(m)} \) has been written for \( H_z(u,v^{(m)},z) \), \( \gamma_n^{(m)} \) for \( \gamma(u,v^{(m)}) \) and similarly for \( s_n \). Note that (7.7) is a modified complex Laplace transform, and so retrieving the dipole distribution \( p(v^{(m)},z_k) \) is essentially a problem of inverting \( M \) complex Laplace...
transforms \((m=1,M)\). Each inversion consists of finding \(p(v^{(m)}z_k)\) from the values \(H_z(u,v^{(m)}z)\) at constant \(v\).

The system represented by equation (7.7) can be written as a matrix operation and in figure (7.6a) we plot the real and imaginary parts of the first few vectors in the two spaces produced in the bidiagonalisation of the matrix. The even nature of the \(V\)-space vectors is a consequence of the assumption that the dipole distribution on the crack is even in the \(y\) direction. In general this may not be true and such \textit{a priori} knowledge is not necessary for good reconstruction but was simply a device used to halve the computing.

In this case the bidiagonalisation is achieved using the complex version of the JL algorithm and the system to which it is applied is pictured in schematically in figure (7.7). In table (7.1) are plotted the corresponding elements of the bidiagonalised matrix, the alphas and betas, and it is clear they fall off very rapidly. Since there is obviously no point in fully decomposing such an ill-conditioned system we need some criterion for terminating vector generation. We use the trace method as described in chapter VI.

Figure (7.8a-c) shows the real and imaginary parts at each stage during the generation of the pseudo-data. Figure (7.8a) shows the dipole distribution on an idealised rectangular crack calculated by using boundary integral techniques, [72]. Figure (7.8b) shows the same after it has been Fourier transformed with respect to the \(y\)-coordinate and figure (7.8c) shows the

\[\text{continued on page 134}\]
Figure (7.6)

The real parts (a) and the imaginary parts (b) of the first six $U$-space JL vectors produced for the system of equations for the eddy-current problem equation (7.6).
The real parts (c) and the imaginary parts (d) of the first six \( V \)-space JL vectors produced for the system of equations for the eddy-current problem equation (7.6).
Table (7.1)

The normalisation coefficients, $a_n$ and $b_n$, the first six each of which correspond to the $U$-space vectors of figures (7.6a-b) and to the $V$-space vectors of figures (7.6c-d) respectively. Note how they fall off to zero rapidly (by the 11th component in the case of the $V$-space vectors) even though the space contains 32 vectors; the problem is highly ill-conditioned.
Figure (7.7). Schematic Relationship between the Dipole Density around a flaw in a known plane and the normal component of the magnetic field at the surface of the conductor (The Bowler potential).
Figure (7.8.) Generation of Pseudo-data and Reconstruction with low noise.

Figure (7.8a) shows the real and imaginary parts of the dipole distribution around the flaw calculated as in [67]. After one-dimensional Fourier transformations in the y direction and operation of the modified complex Laplace transform we obtain our pseudo-observation as in figure (7.8c). After application of the JL algorithm and one-dimensional Fourier transformation we obtain the reconstruction as in figure (7.8e).
Figure (7.9a) Reconstruction of $p(v_{16}, z)$ using four different noise samples each of 0.5%. (b) The original $p(v_{16}, z)$. This graph demonstrates how consistent results are obtained in the eddy current problem by showing one row only of the reconstruction.
Figure (7.10) Pseudo-observation along with (a) 0.05% noise and (b) 5.0% noise along with the corresponding reconstructions (b) and (c).
two-dimensional Fourier transform of the normal component of the magnetic field found using equation (7.6).

Various amounts of noise were added to $H_z$ and then the dipole density reconstructed using the JL algorithm. Noise levels are given by the noise variance as a percentage of the variance of the imaginary part of the observation $H_z$ (as opposed to in chapter IV where it was given with respect to the original function. Figure (7.8d) shows the reconstruction obtained at low noise levels ($10^{-9}\%$). Figure (7.8e) shows the same reconstruction after inverse Fourier transforming $p(v,z)$ in the v direction and is to be compared with the original dipole distribution shown in figure (7.8a).

A comparison of the original dipole density and the reconstruction is shown in figure (7.9) for a number of different noise samples each of 0.5%. The results show that a consistent estimate of $p(v^{(m)},z_k)$ was found for different noise samples having the same variance. Reconstructions in two dimensions with experimental noise levels were carried to obtain $p_{\text{rec}}$. These are built up from the results of thirty-two one-dimensional inversions. Figure (7.10a) shows the real and imaginary parts of the pseudo-observation $H_z(u,v,z)$ with 0.05% noise. Figure (7.10b) shows the corresponding reconstruction $p(y,z)_{\text{rec}}$. Figures (7.10c) and (7.10d) show the same for 5% noise.

The images plotted in figures (7.8) - (7.10) show that within the fundamental limits imposed by the problem of skin depth we can deduce stable, consistent images of dipole distributions around sub-surface cracks.
In addition the techniques of super-resolution employed via the linear minimum variance estimator allow us the best possible chance to incorporate any a priori information available. This allows us to improve the resolution of such images beyond the limit of resolution which is governed by the equivalent of the Rayleigh criterion of Fourier optics as described in [79]. The results also illustrate the great premium available when observation noise is kept low, as expected.

The primary motivation for this thesis was to demonstrate, for the first time, the possibility of imaging the structure of sub-surface flaws in conductors. Given that the magnetic dipole distributions which have been reconstructed tell us a great deal about the actual crack structure, the basic aim of this thesis has been completed. Also, as discussed in Chapter VI, the technique used to carry out the inversion has been shown to be ideal for the particular problem of eddy current imaging. Finally it should be pointed out that since the problem is so ill-conditioned, the technique we have used may well be just as good in higher dimensions as was shown for the hyper-singular two-dimensional optical problem of Chapter IV.
Chapter VIII.

Introduction of \textit{A Priori} Knowledge to the Linear Minimum Variance Estimator and Future Work.

8.1 Determining the Position of Edges.

We have shown that the methods described above do indeed give consistent and meaningful results for a number of different noisy and ill-conditioned inverse problems. So far we have only used \textit{a priori} knowledge of the image support for the one-dimensional and two-dimensional bandlimited Fourier inverse problem in chapter IV. Figures (8.1a) and (8.1b) illustrate that, when reconstructing the letter E, relaxing the support constraint degrades the solution thus illustrating the importance of \textit{a priori} knowledge.

It is clear that we still have not fully exploited the use of \textit{a priori} knowledge in the LMVE. In many cases of practical interest the signal \( f \) giving rise to the measurement is a function in \( \mathbb{R}^n \), and may have discontinuities. Indeed, the localisation of these for \( n=1,2,3.. \) may be the primary purpose of the measurement. Since the discontinuities are associated with high spatial frequencies, and these are associated with the smaller singular values of the matrix representation of \( L \), regularisation techniques such as that of Tikhonov, which we have used, tend to smooth the solution across edges. Indeed in [14] an extra term is introduced into equation (1.8) such that a constrained least

\[ \text{continued on page 138} \]
Here we illustrate the importance of *a priori* knowledge for effective reconstruction. In (a) we show the reconstruction of the $M=3$ band-limited image where *a priori* knowledge gave a support with $L=10$. In (b), where we were only able to define the support with $L=24$, the reconstruction is not so effective.
squares solution is obtained by minimising the functional

\[ \langle g - Lf | g - Lf \rangle_B + \varepsilon \langle f | Df \rangle_A \]  

(8.1)

where \( D \) is a positive operator in \( U \) such that \( D \) is an explicit smoothness constraint on \( f \) giving

\[ \hat{f} = (L^\dagger AB L + \varepsilon D)^{-1} L^\dagger AB g \]  

(8.2)

where \( D=I \) corresponds to the simplest case of the LMVE as used above.

Attempts to overcome this problem have been made by Geman and Geman [92], Blake and Zisserman [93], Terzopoulos [94], Kashyap and Eon [95] and Jeng and Woods [96]. The last two references use autoregressive random field models and it is not clear how these relate to the methods of the first three which can all be considered as maximum a posteriori estimates though Blake and Zisserman use a constrained least squares method.

In Blanchard, Travis and Jones [14] it is shown that edges (d) can be determined by minimising

\[ -\ln(N_d) + \frac{1}{2} \langle g - L\mu_d | R + LP_d L^\dagger \rangle^{-1} (g - L\mu_d) \]  

(8.3)

where \( N_d \) is a normalising factor, \( \mu_d \) denotes the mean in the presence of edges, \( d, \) and \( P_d \) is the corresponding variance. In [14] it is premised that \( \mu_d \) is piecewise constant with discontinuities at edges and \( P_d \) was likewise uniform between edges. To determine the position of edges equation (8.3) must be minimised with respect to the position of edges and the discontinuities of \( \mu_d \) across them. The full linear minimum variance estimate is then used and \( \hat{f} \) will reflect the discontinuities in \( \mu_d. \)
In [14] it is discussed in detail how equation (8.3) can be minimised and it is clear that there is much scope for future work to apply these techniques. Here we shall use *ad hoc* estimates of the object mean with edges to demonstrate how when using real experimental data (as opposed to the pseudo-data here) significant improvement may be obtained. The results we shall give here are similar to those produced in Blanchard, Travis and Jones [97]. More rigorous application of the techniques described in [14] will be made in Blanchard, Travis and Jones [98].

In figure (8.2) we plot the function we are reconstructing with prominent edges d and d'. This function is operated on using the real Laplace transform operator (equation (6.7)) to give a pseudo measurement to which 5% noise is added as indicated by figure (8.3). Reconstructions were then obtained using different estimates of the mean, $\mu_d$, as indicated by the dashed lines, figure (8.4).

The examples looked at here are all concerned with edges in the mean and no *a priori* knowledge about the object covariance has been included. Future work should include the reconstruction of images of known covariance from incomplete data. One application of such a reconstruction might be the imaging of a submarine which had surfaced at sea in a satellite image. The submarine itself may not reflect any more or less light than the surrounding water. However the sea surface is likely to have a certain texture whereas the submarine is likely to be smooth textured.

continued on page 142
Figure (8.2)

(a) A test function with prominent edges $d$ and $d'$. The Laplace transform of this function is taken to give a pseudo-observation to which 5% noise is added as shown in (b).
Figure (8.3)

Using the noisy observation of figure (8.2b) reconstructions are plotted of the function in figure (8.2a) using three different estimates of the mean $m$ as indicated by the thick line in (a), (b) and (c). In each case the inversion is done three times with the observation having three different noise samples.
8.2 The JL Algorithm and Iterative Solution of Non-linear Problems using Newton's Method.

Finally, we have only given results here for linear systems, although possible techniques for bilinear optical problems were looked at in chapter II. In [35] it was suggested that the JL algorithm be applied to the iterative solution of non-linear inverse problems using Newton's method, Lanczos [30].

Suppose we have a function \( f(y) \) which is non-linear in \( y \) and is the solution which minimises the \( L_2 \) norm

\[
D(y) = \| g(x) - Lf(y) \| \quad (8.4)
\]

Suppose also that \( f(y) \) is differentiable with respect to all its variables, \( y \). Therefore the least squares estimate is also differentiable with respect to \( y \) and

\[
\frac{\partial D(y)}{\partial y_1} = \frac{\partial D(y)}{\partial y_2} = \ldots \frac{\partial D(y)}{\partial y_M} \quad (8.5)
\]

and this gives us a set of equations

\[
F_1(y) = \sum_{k=1}^{n} (g-Lf(y)) \frac{\partial f(y)}{\partial y_1}
\]

\[
\vdots
\]

\[
F_M(y) = \sum_{k=1}^{n} (g-Lf(y)) \frac{\partial f(y)}{\partial y_M}
\]

or

\[
F(y) = \begin{bmatrix}
F_1(y) \\
F_2(y) \\
\vdots \\
F_M(y)
\end{bmatrix} = 0 \quad (8.6)
\]

If \( F(y) \) is non-linear in \( y \) then \( F(y) \) is a system of non-linear equations and in that case one can use Newton's method to find a \( y \) that minimises \( D(y) \). The iterative Newton method is defined by

\[
F'[y^{(k)}][y^{(k+1)} - y^{(k)}] = -F[y^{(k)}] \quad (8.7)
\]

where \( F' \) is the Jacobian
To use Newton's method we begin with \( k=0 \) and the initial estimate of \( y, y^{(0)} \). At each iteration we evaluate the vector function

\[
F'(y) = \begin{bmatrix}
\frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} & \cdots & \frac{\partial F}{\partial y_M} \\
\frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} & \cdots & \frac{\partial F}{\partial y_M} \\
\vdots & \vdots & & \vdots \\
\frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} & \cdots & \frac{\partial F}{\partial y_M}
\end{bmatrix}
\]

(8.8)

for the increment vector and set

\[
y^{(k+1)} = y^{(k)} + \Delta^{(k)}
\]

(8.10)

This technique is used by Hall [100] to determine parametrically specified curves describing boundary regions in a non-linear least squares image processing problem. Bellman and Kalaba [101] apply this method to a Laplace transform problem which arises in time dependent and time independent neutron transport processes. This leads to the use of a more powerful variant of the method based on the theory of quasilinearisation as described in [88] using the Newton-Raphson-Kantorovich technique.

In a similar manner to that above quasilinearisation of a non-linear Laplace transform problem in [101] leads to a linear equation with variable coefficients and the transform is highly ill-conditioned. The results obtained in this thesis leave us well prepared to tackle such problems.
Chapter IX.

Conclusion and Future Work.

With the aim of solving the eddy current imaging problem the linear inverse problem was introduced in Chapter I. The connected concepts of ill-conditioning, super-resolution and instability due to noise in the inverse problem were then defined. The regularisation method of Tikhonov and Miller was introduced as a method of constructing reliable, consistent approximate solutions for systems which are ill-conditioned or otherwise unstable. The regularisation parameter $\epsilon$ of Tikhonov's method was then shown to be simply connected to the noise in the system so that a knowledge of the noise level in the system enabled the best possible and most consistent solutions to be obtained. Hence it was shown how a probabilistic approach to the problem using the linear minimum variance estimator could be unified with the deterministic approach of Tikhonov.

It was also pointed out that the ill-conditioned problem is an example of a chaotic system and that the unification of the deterministic and probabilistic approaches was closely connected with the latest ideas of deterministic chaos.

In Chapter II inversion methods for the solution of sets of linear equations

$$g = Lf$$  \hspace{1cm} (9.1)

were discussed where $f$ is a vector in a Hilbert space $U$, $L$ is a bounded linear transformation of $U$ with image space $L(U) = V$ so
that $g$ is a vector in a Hilbert space $\mathcal{H}$ that contains $V$ as a subspace.

It was discussed how the best algorithm for a particular problem depends on the nature of the matrix representation of the problem being looked at.

After pointing out that further insight into a system can be gained by choosing a new frame of reference, the method of singular value decomposition (SVD) was introduced as one such coordinate transform. The mathematical origin of ill-conditioning was then described.

The various algorithms which are used to perform SVD were discussed before introducing a new method of matrix decomposition, the method of Jones-Lanczos direct bidiagonalisation or the JL algorithm, Jones and Travis [35]. The solution of the inverse problem using the JL algorithm and incorporating linear minimum variance estimation was then given.

In Chapter III a short review of systems theory was provided with a bias towards the optical problem. The non-physical nature of the problem treated in much published work on super-resolution, e.g. Rushforth et al [15], was pointed out and the band-limited inverse Fourier problem in amplitude of [15] was contrasted with an actual physical experiment.

With a view to treating such an experiment, it was shown which systems are linear or approximately so. It was discussed how in some cases a non-linear problem can be reduced to one which is either linear or a combination of linear problems, thereby justifying the large amount of work put into a general solution...
for the linear inverse problem. (The eddy current problem itself is one in which a non-linear problem is reduced to a linear problem, in this case via the Born approximation, Bowler [71], and not in the inversion algorithm stage.

The background for object restoration from partial Fourier transform data was reviewed and the Gerchberg-Papoulis algorithm as an alternative iterative solution for the inverse problem of optics discussed.

Before looking at the two-dimensional problem a detailed comparison of the SVD and JL algorithms was made and it was shown how considerable savings in computer time could be made using the technique of Jones and Travis.

The JL algorithm was then applied to the two-dimensional band-limited problem and methods of maintaining the correct orthogonality discussed. The technique was applied to the reconstruction of a simple image and the result was shown to be the same as that obtained when using the Gerchberg algorithm. It was demonstrated how when the vector generation process breaks down in the JL method the Gerchberg gives the better result. However it was also shown how the JL algorithm was the ideal algorithm to use in the hyper-singular problem when it out-performed the Gerchberg algorithm in every way. Further techniques were suggested to extend the usefulness of the JL algorithm in two-dimensional problems which are not hyper-singular, that is where a larger set of basis vectors is required for the full solution.

Finally the Cholesky decomposition was suggested as a further alternative to both the JL and Gerchberg algorithms in the case of
a positive definite matrix for problems where the basis need not be stored.

In Chapter IV the new technique of JL bidiagonalisation was applied to the reconstruction of an object of known support from its diffraction limited image in the one-dimensional case as treated by Rushforth et al [15]. Although, as already mentioned, this is a non-physical problem, in the sense that it is amplitude only, it was necessary to compare the effectiveness of the technique with other workers. It was pointed out that the results were at least as good as those of Rushforth et al. Moreover, the results using the JL technique along with Tikhonov regularisation and linear minimum variance estimation gave consistent results with different noise samples, an attribute not demonstrated elsewhere in the literature. In fact meaningful results were obtained even in the singular case in comparison with [15] where wholly meaningless results were published. A discussion of the relationship between the conditioning of the system and the rank of the matrix was made and the possibility of including further a priori knowledge mentioned.

In Chapter V the problem which was the original motivation for this thesis, the eddy current problem, was arrived at. After reviewing the theory of eddy current imaging the inverse eddy current problem was arrived at and it was shown to be in the form of a modified complex Laplace transform. It was shown how this transform connected the normal component of the magnetic field in a plane z above the flaw with the magnetic dipole distribution in the plane in which the crack had been assumed to be.
Since the inversion of the Laplace transform is a prerequisite for successful inversion of the eddy current problem, the one-dimensional Laplace transform was then tackled. This was done using the same technique as was used for the Fourier optical problem, the JL algorithm. This technique was described in Chapter VI along with a review of all the previous methods with which attempts to invert the Laplace transform had been made.

In Chapter VII we gave the results of inversions of the Laplace transform using data as previously used in the literature. It was seen that the JL method was the ideal technique because the Laplace transform is so ill-conditioned. Hence in a similar way to the hypersingular two-dimensional Fourier optical problem the JL technique is seen to be superior to any so far used. This is true in terms of both computer time as already shown (and more so in this case because the system need not be fully decomposed in such an ill-conditioned case). Also, since linear minimum variance estimation had been employed there was far more scope to make maximum use of any a priori knowledge available and this had not been done in any of the published work.

Again the technique employed was shown to cope very well with this highly ill-conditioned system even when there was noise added to the observation. This was demonstrated with sets of reconstructions showing consistency over different noise samples. The fact that the technique worked so well in both the Fourier optical case and in the Laplace transform problem shows it is of general value rather than being useful in only one area of problems.
In the second part of Chapter VIII we went on to invert the two-dimensional complex eddy current problem with pseudo data. In fact the problem was to solve a set of one-dimensional modified complex Laplace transforms.

For the first time it was demonstrated that stable and consistent images of magnetic dipole distributions on sub-surface cracks could be imaged even in the presence of observation noise, although it was quite clear that the skin depth problem represented a fundamental limit beyond which only the techniques of super-resolution developed and described in this thesis could penetrate.

In Chapter VIII we briefly introduced the problem of determination of the position of edges since there is still much scope to employ further a priori knowledge into our results by way of the linear minimum variance estimator. It was suggested that the JL algorithm might be used to solve a non-linear problem using Newton's method.

Finally one should point out that implicit in the techniques introduced to solve the linear inverse problem was the assumption by many workers in image processing of the following:

1. Linearity of the image distortion system.
2. Space invariance of the image distortion system.
3. The deterministic nature of the image distortion system.
4. That any noise was additive, stationary and signal independent.

In addition to any departure from the conditions 1-4 above, future work should also bear in mind these other departures from the ideal imaging systems:
1. Distortion that is described by a linear system followed by a point non-linearity, such as that of a photographic film or of the human visual system, Hunt [102] and [24], Hunt and Trussell [103], Sawchuk [104] and Oppenheim et al [105].

2. Special cases of space variant imaging systems, Sawchuk [106], Robbins and Huang [107] and Sawchuk' [108] (This offers no additional difficulties using the JL algorithm).

3. Random imaging systems, such as imaging through a turbulent medium, Knox and Thompson [109] and Goodman and Belsher [110] (Here the stochastic optimisation we have employed will be of special value).

4. Signal-dependent noise (which is non-stationary or multiplicative), such as Poisson photon noise (speckle) and film-grain noise, [104], Walkup and Choens [111], Naderi and Sawchuk [112], Froelich et al [25], Frieden and Wells [21] and Prucnal and Saleh [113].

The prime motivation for the work was to provide stable and consistent results for the eddy current inverse imaging problem. At the time of writing the techniques used above for pseudo eddy current data are about to be applied to experimentally obtained measurements in The University of Surrey eddy current laboratory. The ultimate aim is to develop an eddy current "camera" and future developments are likely to see transputer based parallel processing architectures applied to the inversion of three-dimensional eddy current systems. In addition future models are likely to invert eddy current probe impedance responses rather than, as above, measurements of the surface field itself. Bowler
et al [99] have produced a model for the forward three dimensional eddy current model, connecting electrical dipole distributions with probe impedance responses.

This thesis has shown that the ill-conditioned linear inverse problem can be solved consistently and meaningfully provided one accepts and determines the fundamental limits of the system we are looking at. The techniques have been shown to work equally well in one-dimensional Fourier optical problems and in the Laplace type skin effect problem of eddy current imaging in which field this thesis represents a major advance.
Appendix I.

Details of Tridiagonalisation using the standard
Lanczos Method.

Note that $[H^a_{j,k}]$ and $[H^{T a}_{j,k}]$ are upper triangular since each operation of $M$ produces a vector, part of which is a linear combination of the previously generated vectors. Also since

$$\langle b_r | a_1 \rangle_A = 0 \quad \text{for } r \geq 1 \quad (A1)$$

and

$$\langle a_r | b_1 \rangle_A = 0 \quad \text{for } r \geq 1 \quad (A2)$$

therefore

$$\langle b_1 | a_k \rangle_A = \delta_{1,k} \quad (A3)$$

Now from (2.23)

$$[H^{T a}_{j,k}] = \langle a_j | M^{T a} a_k \rangle_A \quad (A4)$$

$$[H^{T a}_{j,k}]^T = \langle a_j | M a_k \rangle_A \quad (A5)$$

and so

$$\langle b_k | a_q \rangle_A [H^{T a}_{j,k}]^T \langle a_1 | b_m \rangle_A = \langle b_p | a_q \rangle_A \langle a_j | M a_k \rangle_A \langle a_1 | b_m \rangle_A$$

$$= \langle b_p | M b_m \rangle_A \quad (A6)$$

since

$$\langle a_k | a_1 \rangle_A = \delta_{1,k} \quad (A7)$$

but

$$\langle b_p | M b_m \rangle_A = \langle H^a_{p,m} \rangle = \langle b_k | a_q \rangle_A [H^{T a}_{j,k}]^T \langle a_1 | b_m \rangle_A \quad (A8)$$

Now $[H^a_{j,k}]$ is upper Hessenberg and $[H^{T a}_{j,k}]$ is lower Hessenberg and so both sides of equation (A8) must be tridiagonal. Also $\alpha_{i,r} = \alpha_{i,r} = 0$ for $i=0,...,r-2$. This means we only need to orthogonalise $M b_r$ with respect to $a_r$ and $a_{r-1}$ since all other $b_i$ are automatically orthogonal. Since the matrices $[H^a_{j,k}]$ and $[H^{T a}_{j,k}]$ are tridiagonal we simplify (1.20) and (1.21) and write
\[ H_{r} - \alpha_{r} b \alpha_{r} - \beta_{r} b_{-1} = \beta_{r+1} b_{r+1} \quad \text{(A9)} \]

and

\[ H^\top a_{r} - \alpha_{r} a_{r} - \beta_{r} a_{r-1} = \beta_{r+1} a_{r+1} \quad \text{(A10)} \]

Appendix II.

Symmetry property in the Discrete Fourier Transform Matrix.

Neglecting the factor \( \frac{1}{N} \) the inverse Fourier Transform matrix is \( \Omega^\dagger \) where the dagger denotes complex conjugate.

Plotting the top left of the forwards DFT we have

\[
\begin{bmatrix}
  w_{00} & w_{01} \\
  w_{10} & w_{11}
\end{bmatrix}
\begin{bmatrix}
  \Omega_{00} & w_{02} & w_{03} \\
  w_{12} & w_{13}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  w_{00} & w_{01} \\
  w_{10} & w_{11}
\end{bmatrix}
\begin{bmatrix}
  \Omega_{00} & w_{02} & w_{03} \\
  w_{12} & w_{13}
\end{bmatrix}
\]

and similarly for the inverse DFT

\[
\begin{bmatrix}
  w_{00} & w_{01} \\
  w_{10} & w_{11}
\end{bmatrix}
\begin{bmatrix}
  \Omega_{00} & w_{02} & w_{03} \\
  w_{12} & w_{13}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  w_{00} & w_{01} \\
  w_{10} & w_{11}
\end{bmatrix}
\begin{bmatrix}
  \Omega_{00} & w_{02} & w_{03} \\
  w_{12} & w_{13}
\end{bmatrix}
\]

and we can see that \( \Omega_A = \Omega_{01}^\dagger \) and \( \Omega_B = \Omega_{10}^\dagger \) for example.
Appendix III.

Relationship between DFT and Continuous Fourier Transform.

In order that a real symmetric function give a real symmetric Fourier transform we have to blockswap the observation vector in the single lens optical system of figure (3.3). The reason for this derives from the formulation of the discrete Fourier transform. The Fourier transform representation of a discrete image is essentially a Fourier series representation of the field. For the Fourier series representation to be valid the field must be periodic so the discrete Fourier transform can only be exact within the limits of the sampling theorem, Dainty [112].

If one takes the continuous Fourier transform, that is

\[ F(u) = \int_{-\infty}^{\infty} f(x) \exp[-2\pi iux] \, dx \]

and discretise symmetrically about the origin by sampling at \( N \) points \(-N/2 \ldots +N/2-1\) then it is easy to show that, within the limits of the sampling theorem, the equations thus obtained are the same as the defined discrete Fourier transform with the exception of a factor \( \exp[\pi i m] \) where \( m \) is the index on each vector component. We need to multiply by such a factor in both direct and transform space in order to get the discrete Fourier transform. But multiplying by such a factor in one space is the equivalent of a shift in the opposite space. As we can write \( \exp[\pi i m] = \exp\left[\frac{2\pi i m}{N}\right] \) then using the shift theorem we can see how the need
for a blockswap in each space arises.

Appendix IV.

Demonstration of approximate nature of DFT for a Non-bandlimited Function.

From the definition of the DFT

\[ G_k = \sum_{n=0}^{N-1} f_n \exp\left[-\frac{2\pi i kn}{N}\right] \]  \hspace{1cm} (A12)

then for the rect function of figure (3.3) we see

\[ G_k = \sum_{n=N/2-L/2}^{N/2+L/2} f_n \exp\left[-\frac{2\pi i kn}{N}\right] \]  \hspace{1cm} (A13)

and using the sum for \( L \) terms we obtain

\[ S_L = \exp\left[-\frac{2\pi i k}{N} (N/2-L/2)\right] \frac{\left\{1 - \exp\left[-\frac{2\pi i L}{N}\right]\right\}}{\left\{1 - \exp\left[-\frac{2\pi i k}{N}\right]\right\}} \]  \hspace{1cm} (A14)

\[ = \exp[-\pi i k] \left\{ \frac{\sin\left\{\frac{\pi k L}{N}\right\}}{\sin\left\{\frac{\pi k}{N}\right\}} \right\} \]

The factor outside the brackets reflects the fact that we did not block swap as discussed in appendix (III) and we see that with the approximation \( \sin \theta \approx \theta \) then we have the sinc function which is the continuous Fourier transformation of the rect function. Note that the approximation is worst at high frequencies \( (|k| \rightarrow N/2) \) in transform space where the approximation breaks down.
Appendix V.

Direct Product Notation.

The left direct product of a PxQ matrix A and an MxN matrix B is a PM x QN matrix defined by

\[
C = A \otimes B = \begin{bmatrix}
B(1,1)A & B(1,2)A & \cdots & B(1,N)A \\
B(2,1)A & B(2,2)A & \cdots & B(2,N)A \\
\vdots & \vdots & \ddots & \vdots \\
B(M,1)A & \cdots & \cdots & B(M,N)A
\end{bmatrix}
\]  

(A15)

Appendix VI.

Comparison of the optical Gerchberg notation of chapter III with that used in [42].

In [42] we have instead of equations (2.47) and (2.49)

\[
B_1^{(r+1)} = (I - Q)F_1 + QB_1^{(r)}
\]  

(A16)

and

\[
b_1^{(r+1)} = (I - P)f_1 + Pb_1^{(r)}
\]  

(A17)

where

\[
Q = \frac{1}{N_{11}}n_1^{\dagger}n_1
\]  

(A18)

and

\[
P = \frac{1}{N_{11}}n_1^{\dagger}n_1
\]  

(A19)

(The prime indicates partitioning under the scheme of [42].

We can see therefore that equations A16,A17 and (2.47),(2.49)
are identical if

\[
\frac{1}{N} \begin{bmatrix} \Omega_{10} & \Omega_{12} \end{bmatrix} \begin{bmatrix} \Omega^{\dagger}_{10} \\ \Omega^{\dagger}_{12} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \Omega'_{11} & \Omega^{\dagger}_{11} \end{bmatrix}
\] (A20)

and

\[
\frac{1}{N} \begin{bmatrix} \Omega^{\dagger}_{10} \\ \Omega^{\dagger}_{12} \end{bmatrix} \begin{bmatrix} \Omega_{10} & \Omega_{12} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \Omega'_{11} & \Omega^{\dagger}_{11} \end{bmatrix}
\] (A21)

where \( \Omega'_{11} \) and \( \begin{bmatrix} \Omega_{10} & \Omega_{12} \end{bmatrix} \) both have 2l+1 columns and N-2M-1 rows.

Now

\[
\begin{bmatrix}
1 & w^{M+1} & \ldots & w^{(M+1)L} \\
\vdots & \ddots & \ddots & \vdots \\
1 & w^{N-M-1} & \ldots & w^{(N-M-1)L}
\end{bmatrix}
\]

\[
= \Omega_{10}
\]

\[
\begin{bmatrix}
w^{(M+1)(N-L)} & \ldots & w^{(M+1)(N-1)} \\
\vdots & \ddots & \vdots \\
w^{(N-M-1)(N-L)} & \ldots & w^{(N-M-1)(N-1)}
\end{bmatrix}
\]

\[
= \Omega_{12}
\] (A22-23)

and since \( w^{(M+1)(N-1)} = w^{-(M+1)} \) we can see that if we swap the ordering of \( \Omega_{10} \), \( \Omega_{12} \) to give \( \Omega_{12} \Omega_{10} \) then the rows of the combined matrix form geometric progressions. Block-swapping means that the notations for direct space of [42] and chapter III are connected by the statement \( [f_1] = [f_2 f_0] \) and not \( [f_0 f_2] \) and this accounts for the need to swap \( \Omega_{10} \) and \( \Omega_{12} \). Therefore we can write

\[
\begin{bmatrix} \Omega_{12} & \Omega_{10} \end{bmatrix} =
\begin{bmatrix}
w^{-(M+1)L} & \ldots & 1 & \ldots & w^{(M+1)L} \\
w^{-(M+2)L} & \ldots & 1 & \ldots & w^{(M+2)L} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
w^{-(N-M-1)L} & \ldots & 1 & \ldots & w^{(N-M-1)L}
\end{bmatrix}
\] (A24)

whereas
These two matrices can be made identical if we extract a diagonal matrix $D$ from equation (A25) whose diagonal elements are $-(M+1)(L-s)$, $-(M+2)(L-s)$, etc., $-(N-M-1)(L-s)$.

Therefore we can write

$$N.P = [D][Q_{11}][D] = [Q_{11}][D][Q_{11}] = [Q_{11}][Q_{11}]$$

or

$$P = P'$$

and so equations (A17) and (2.49) are identical. However

$$Q = D Q' D^\dagger$$

and so equations A25 and 2.49 differ because of the phase change in $D$ and $D^\dagger$.

Appendix VII.

Derivation of the Matrix $[L_{mn}]$.

$$X_m = \frac{1}{N} \sum_{n=0}^{N-1} x_n \exp \left( -i \frac{2\pi mn}{N} \right)$$

or in operator notation

$$X = F x$$

where
\[ F_{mn} = \exp \left( \frac{-i 2\pi mn}{N} \right) \quad n,m = 0,1,\ldots,N-1 \]  
(A31)

\( C \) is defined as
\[ C_{mn} = F^{-1}B \ F \]  
(A32)

which explicitly is
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & w_{11} & w_{12} & w_{13} & 1 \\
1 & w_{21} & w_{22} & w_{23} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & w_{N-1,1} & w_{N-1,2} & w_{N-1,3} & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_{M+1} \\
O_{N-2M-1} \\
I_M \\
\vdots \\
O_{N-2M-1} \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & w_{11} & w_{12} & w_{13} & 1 \\
1 & w_{21} & w_{22} & w_{23} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & w_{N-1,1} & w_{N-1,2} & w_{N-1,3} & 1 \\
\end{bmatrix}
\]  
(A33)

where \( I_M \) is the \( M \) by \( M \) unit matrix and \( O_{N-2M-1} \) is a null matrix of dimension \( N-2M-1 \) by \( N-2M-1 \), all other terms in \( B \) being zero. The effect of multiplying together the first two of these matrices is to set to zero the \( (M+2) \)th to the \( (N-M) \)th columns of the inverse Fourier transform matrix.

Hence one can write the combination of the three matrices as
\[ C_{mn} = \sum_{j=-M}^{M} w_{mj} w_{jn} + \sum_{j=N-M}^{N} w_{mj} w_{jn} \]  
(A34)

where the second term can be written explicitly as
\[ \sum_{j=N-M}^{N-1} \exp \left( \frac{-i 2\pi mj}{N} \right) \exp \left( \frac{2\pi in}{N} \right) = \sum_{j=-M}^{-1} \exp \left( \frac{-i 2\pi (j+N)(m-n)}{N} \right) \exp \left( \frac{-i 2\pi jN}{N} \right) \exp \left( \frac{-i 2\pi j(m-n)}{N} \right) \]  
(A35)

and combining this with the first part gives
\[ C_{mn} = \sum_{j=-M}^{M} \exp \left[ \frac{-i 2\pi j(m-n)}{N} \right] \quad \text{where } \exp \left[ \frac{-i 2\pi jN}{N} \right] = 1 \]  
(A36)
Now using the well known formular for the sum of a geometric progression (here there are \((2M+1)\) terms)

\[
S_{2M+1} = a_0 \frac{1-r^{2M+1}}{1-r} \quad \text{where } a_0 = w^{(n-m)^M} \quad \text{and } r = w^{(n-m)} \quad (A37)
\]

hence

\[
S_{2M+1} = \frac{w^{(n-m)M}(1-w^{(n-m)(2M+1)})}{(1-w^{(n-m)})} \quad (A38)
\]

Extracting a factor \(w^{[(n-m)(2M+1)/2]}\) from the bracket in the numerator and a factor \(w^{[(n-m)/2]}\) from the denominator gives

\[
S_{2M+1} = \frac{w^{(n-m)M}w^{[(n-m)(2M+1)/2]} - w^{[(n-m)(2M+1)/2]}}{w^{[(n-m)/2]}} - w^{[(n-m)/2]}
\]

\[
= \frac{w^{(n-m)M}w^{[(n-m)/2]} - w^{[(n-m)/2]}}{w^{[(n-m)/2]}} \quad (A39)
\]

Now \(\sin \theta = \frac{1}{2} \left\{ \exp[i\theta] - \exp[-i\theta] \right\} \)

And so finally

\[
S_{2M+1} = \frac{\sin \left( \frac{\pi (n-m)2M+1}{N} \right)}{\sin \left( \frac{\pi (n-m)}{N} \right)} \quad (A40)
\]

Appendix VIII.

Relationship between Rank of the Matrix \([L_{mn}]\) and the Bandwidth \(M\).

As already shown above

\[
L_{mn} = \sum_{j=-M}^{+M} w^n j w^{-m j} \quad (A41)
\]

and the matrix \([L_{mn}]\) can thus be written as the product of two matrices
Now the rank of a matrix is the dimension of the largest non-zero determinant in the matrix or the maximum number of linearly independent rows or columns in the matrix. Since \( L < N \) and \( 2M+1 < N \) then the rank obviously cannot be greater than the smallest out of \( L \) and \( 2M+1 \) since these are the actual numbers of columns/rows in the two component matrices.

For a system with a large bandwidth then only a few components will be missing and since in that case \( 2M+1 \) will be greater than \( L \) then the rank of the bandlimiting operator \([L]\) will be \( L \). For a severely bandlimited case when for example \( M = 3 \) then the rank of \([L]\) will be 7.

For an alternative derivation of the rank of this system see [51].
Appendix IX.

Invariance of the Matrix Trace under Coordinate Transformation.

The trace of a linear operator $D$ in a Hilbert space $U$ is

$$\text{trace}(D) = \sum_k \langle a_k | Da_k \rangle_A$$

(A43)

where $(a_k)$ is an $A$-orthonormal basis in $U$.

The trace is invariant under an $A$-unitary transformation

$$a'_k = \sum_j u_{jk} a_j$$

(A44)

$$\text{trace}'(D) = \sum_k \langle a'_k | Da'_k \rangle = \sum_k \langle a'_k | a_j \rangle_A \langle a_j | Da_j \rangle_A \langle a_j | a'_k \rangle_A$$

$$= \sum U_{jk} \langle a_j | Da_j \rangle_A U_{1k} = \sum_j \langle a_j | Da_j \rangle_A \sum_k u_{jk} u_{k1} = \sum_j \langle a_j | Da_j \rangle_A \delta_{j1}$$

$$= \sum_j \langle a_j | Da_j \rangle_A = \text{trace}(D)$$

(A45)

Appendix X.

The Trace of the Laplace Transform Matrix.

For the Laplace transform we have (chapter VI)

$$g(p) = \int_a^b e^{-px} f(x) dx$$

(A46)

and for the $L_2$ norm we have

$$\text{trace}(LL^*) = \int_a^b e^{-2px} dx \ p = j\Delta \ (j=0,1,..)$$

(A47)

which gives
In the limit $\Delta \to 0$ (A47) becomes

$$\text{Trace}(LL\dagger) = \frac{1}{2}\ln\left[\frac{b}{a}\right]$$

Note that in order for the trace to be finite we must have $a > 0$ which in the eddy current problem means taking the first reconstruction point below the surface of the conductor.
Bibliography.

[1]: Numerical Algorithms Group library section F04, Simultaneous Linear Equations.
[13]: V. I. Smirnov, "Course of Higher Mathematics, Volume V", 164


[32]: NAG library, F01 page 6.
[37]: [31] section 6.26, Reduction of a General Matrix to Condensed
Form; Generalised Hessenberg process, pp. 377-378.


[59]: W.K. Pratt, "Digital Image Processing," Section 5.3, Vector


[67]: NAG FORTRAN library routine F02WCF


[78]: Lord Rayleigh, Phil. Mag. 8, no.5, pp.261, 1879.


[81]: M. Bertero, P. Boccacci and E.R. Pike, "On the recovery and resolution of exponential relaxation rates from experimental data:
a singular value analysis of the Laplace Transform in the presence

[82]: M. Bertero, P. Brianzi and E.R. Pike, "On the recovery and
resolution of exponential relaxation rates form experimental data.
III. The effect of sampling and truncation of data on the Laplace

[83]: M. Bertero, P. Brianzi and E.R. Pike, "On the recovery and
resolution of exponential relaxation rates from experimental
data. Laplace transform inversions in weighted spaces", Inverse

[84]: M.L. Boas, "Mathematical methods in the physical sciences," Wiley, Section 13.6, Integral transforms: Inverse Laplace

[85]: G. Arfken, "Mathematical Methods for Physicists", Section
15.11, Integral Transforms: Inverse Laplace transformation, pp.
715-720.

[86]: V.I. Krylov and N.S. Skoblya, "Handbook of Numerical Inversion
of Laplace transforms," translated by D. Lauvish, Jerusalem: Israel

[87]: C.L. Bohn and R.W. Flynn, "Real Variable Inversion of Laplace

[88]: R.E. Bellman, R.E. Kalaba and J. Lockett, "Numerical Inversion of

[89]: R. Fioravanti, "Numerical inversion of the Laplace transform


Reconstruction of 'THE END' using the accelerated Gerchberg algorithm for the n=32, M=6, L=24 case.