On the Existence of Pure-Strategy Equilibria in Large Games

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Abstract

Over the years, several formalizations and existence results for games with a continuum of players have been given. These include those of Schmeidler [18], Rashid [16], Mas-Colell [11], Khan and Sun [10] and Podczeck [15]. The level of generality of each of these existence results is typically regarded as a criterion to evaluate how appropriate is the corresponding formalization of large games.

In contrast, we argue that such evaluation is pointless. In fact, we show that, in a precise sense, all the above existence results are equivalent. Thus, all of them are equally strong and therefore cannot rank the different formalizations of large games.

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1 Introduction

Nash’s [12] celebrated existence theorem asserts that every finite normal-form game has a mixed strategy equilibrium. However, in many contexts mixed strategies are unappealing and hard to interpret, leading naturally to the question of the existence of pure strategy Nash equilibria.

Schmeidler [18] was successful in obtaining an answer to the above question. He showed that in a special class of games — in which each player’s payoff depends only on his choice and on the average choice of the others — a pure strategy Nash equilibrium exists in every such game with a continuum of players.

Schmeidler’s formalization parallels that of Nash in that players have a finite action space, there is a function assigning to each of them a payoff function in a measurable way and the equilibrium notion is formalized in terms of a strategy, i.e., as a measurable function from players into actions. The difference is that, while in [12] there is a finite number of players (which, in particular, makes the measurability conditions trivial), in [18] the set of players is the unit interval endowed with the Lebesgue measure.

Although natural, Schmeidler’s formalization entails serious difficulties. As shown by Khan, Rath, and Sun [7], Schmeidler’s theorem does not extend to general games — in fact, one has to assume that either the action space or the family of payoff functions is denumerable in order to guarantee the existence of a pure strategy equilibrium (see

Motivated by this, several alternative formalizations have been proposed in order to obtain an existence result for the case of a general, not necessarily countable, action space. These include those of Khan and Sun [10] and Podczeck [15], which consider a richer measure space of players, that of Mas-Colell [11], where the equilibrium notion is formalized as a distribution, and that of Rashid [16], which considers approximate equilibria in games with a large but finite set of players.¹

Clearly, an existence theorem that allows for general compact action spaces also allows for finite action spaces. Furthermore, the existence of an equilibrium strategy also implies the existence of an equilibrium distribution, such distribution being the joint distribution of the equilibrium strategy and the function assigning payoff functions to players. Therefore, one might be tempted to use the success of a particular formalization to address the existence problem to argue for it as a more appropriate approach to the modeling of large games. In fact, such an argument has been made in Khan and Sun [10] and in Al-Najjar [1]; Mas-Colell [11] also argues in favor of using equilibrium distributions as opposed to equilibrium strategies. We argue that such an appraisal of the different formalizations is misleading by establishing the equivalence between the existence results they yield.

¹ See also Khan and Sun [10], Kalai [6] and Wooders, Cartwright, and Selten [22] among others.
Indeed, our results roughly show that: (1) the existence of approximate equilibria in large games is equivalent to the existence of an equilibrium distribution in games with a continuum of players; (2) the existence of an equilibrium distribution in games with a continuum of players is equivalent to the existence of an equilibrium strategy in games with a super-atomless space of players; and (3) the existence of an equilibrium strategy in games with a super-atomless space of players is equivalent to the existence of an equilibrium strategy in games with a Lebesgue space of players and a finite action space.

The first equivalence result is important because it confirms the fact that games with a continuum of players are an idealization of games with a large but finite set of players. In fact, as its proof makes clear, equilibria in one class can be constructed using equilibria of the other. The second equivalence result shows formally that, for the solution to the existence problem in games with a super-atomless space of players (which includes those with a Loeb space of players), it makes no difference whether the equilibrium notion is formalized as a strategy or as a distribution. In fact, in such games, a Nash equilibrium exists if and only if an equilibrium distribution exists. Finally, the third equivalence result shows that super-atomless spaces of players provide a space rich enough to solve the (measurability) difficulties that one encounters when working with simpler spaces of players such as Lebesgue spaces. In fact, as shown in [15], we can understand Lebesgue spaces as being the restriction of a super-atomless space to a smaller $\sigma$-algebra, and so a measurability requirement with respect to a Lebesgue space can be viewed as being too demanding.
More broadly, our equivalence results indicate that the assumptions and conclusions present in the several existence theorems for large games compensate each other. Thus, although some of these theorems allow higher levels of generality along some dimensions, this extra generality is exactly compensated by the strengthening of some other condition or the weakening of some other conclusion, rendering all of them as equivalent.

Furthermore, our results provide a non-trivial and unified approach to the existence problem of large games. In fact, they are designed to meet the following two criteria. First, the conditions that we show to be equivalent are stated in such a way that they can be falsified. This goal is obtained by requiring the action space to be merely a separable metric space, rather than compact. Second, by particularizing the action space to be compact, we obtain as a corollary to our results the classical existence theorems of Schmeidler [18], Mas-Colell [11], Khan and Sun [9] and Khan and Sun [10].

We note that an equivalence result similar to ours has been obtained by Balder [2]. There, he shows that the existence of equilibrium in pure strategy is equivalent to the existence of equilibrium in mixed strategies (and, like us, uses this result to obtain several known existence results). Our results are different because: (a) the conditions that we show to be equivalent are different than those considered by [2], (b) his equivalence is between two true propositions, while in ours, the propositions can be true or false, (c) his framework is more general than ours but (d) our arguments are (somewhat) elementary. In contrast with Balder’s work, our goal is not to obtain a
general existence theorem that can generalize or at least encompass most of such results, but rather to show that several standard formalizations of large games yield equivalent existence results.²

The equivalence between the formalizations we consider to yield an existence result is likely to hold more generally. In particular, recent results by Al-Najjar [1] show that this conclusion can be extended to discrete large games, i.e., games with a countable set of players endowed with a finitely additive distribution.

The paper is organized as follows. In Section 2, we introduce our notation and basic definitions. In Section 3, we present our equivalence results. The proof of our main results are presented in Section 4. These proofs rely on three lemmas (stated and proved also in Section 4) that have some interest in their own right. The first provides a characterization of equilibrium distributions in terms of approximate equilibria of games with a large, but finite number of players. The second provides sufficient conditions for the existence of finite-valued approximate equilibria in games with a continuum of players. Finally, the third presents a representation result which implies that, in games with a super-atomless space of players, every equilibrium distribution is the joint distribution of a Nash equilibrium and the function describing the game. Section 5 provides some concluding remarks. Some auxiliary results are in the Appendix.

² Nevertheless, we note that a new existence result follows from our equivalence results, namely, that a Nash equilibrium exists for all games with a super-atomless probability space of players, a compact action space and a measurable payoff-assigning function.
2 Notation and Definitions

In the class of normal-form games we consider, all players have a common pure strategy space and each player’s payoff depends on his choice and on the distribution of actions induced by the choices of all players. Let $X$ denote the common action space; we assume that $X$ is a separable metric space and let $d$ denote the metric on $X$. A distribution of actions is simply a Borel probability measure on $X$. We let $\mathcal{M}(X)$ denote the set of Borel probability measures on $X$ endowed with the Prohorov metric $\rho$, and let $\mathcal{C}$ denote the space of all bounded, continuous, real-valued functions on $X \times \mathcal{M}(X)$ endowed with the sup norm. Thus, each player’s payoff function is an element of $\mathcal{C}$.

The space of players is described by a probability space $(\mathcal{T}, \Sigma, \varphi)$. A game is then specified by the vector of payoff functions, one for each player. To each player $t$, we associate a bounded, continuous function $V(t) : X \times \mathcal{M}(X) \to \mathbb{R}$ with the following interpretation: $V(t)(x, \pi)$ is player $t$’s payoff when he plays action $x$ and faces the distribution $\pi$. Thus, we have defined a function $V : \mathcal{T} \to \mathcal{C}$ and we assume that $V$ is measurable and that it induces a tight probability measure on $\mathcal{C}$. Formally, letting $\mathcal{T}(\mathcal{C})$ denote the set of all tight Borel probability measures on $\mathcal{C}$, we require that $\varphi \circ V^{-1} \in \mathcal{T}(\mathcal{C})$. In such a game, a strategy is a measurable function $f : \mathcal{T} \to X$.

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$^3$ Recall that the Prohorov metric metricizes the weak topology of $\mathcal{M}(X)$.

$^4$ If $(\mathcal{T}, \Sigma, \varphi)$ is a probability space and $g$ a measurable function from $\mathcal{T}$ into a metric space $Z$, $\varphi \circ g^{-1}$ denotes the distribution of $g$, i.e., the measure $\tau$ on $Z$ defined by $\tau(B) = \varphi(g^{-1}(B))$ for every Borel measurable subset $B$ of $Z$. 
Then, for any strategy $f$, player $t$’s payoff function is obtained from $V$ in the following way:

$$U(t)(f) = V(t)(f(t), \varphi \circ f^{-1}).$$

(1)

We denote such a game by $G = ((T, \Sigma, \varphi), V, X)$.

The following particular cases for the space of players play a special role in our results. Our asymptotic results concern games with a large, but finite set of players.

In that case, we denote the space of players by $(T_n, \Sigma_n, \nu_n)$, where $n$ is the number of players, $T_n = \{1, \ldots, n\}$, $\Sigma_n$ equals the family of all subsets of $T_n$ and $\nu_n$ is the uniform measure on $T_n$, i.e., $\nu_n(\{t\}) = 1/n$ for all $t \in T_n$. A game with a finite number of players is then represented by $G_n = ((T_n, \Sigma_n, \nu_n), V_n, X)$. Note that in this case $V_n : T_n \to C$ is measurable and satisfies $\nu_n \circ V_n^{-1} \in \mathcal{T}(C)$ in a trivial way.

Our asymptotic result also concerns tight families of games with a finite number of players, which are defined as follows. Let $\Gamma$ be a family of games with a finite number of players and, for every game $\gamma \in \Gamma$, let $V_\gamma$ be the function describing it and $n_\gamma$ be its number of players. We say that $\Gamma$ is a tight family of games with a finite number of players if the family of Borel probability measures $\{\nu_\gamma \circ V_\gamma^{-1}\}_{\gamma \in \Gamma}$ is tight.

In games with a finite number of players, each player has a small but positive impact on the distributions of actions. This is in contrast with the case of games with a continuum of players. Formally, $G = ((T, \Sigma, \varphi), V, X)$ is a game with a continuum of players if $(T, \Sigma, \varphi)$ is an atomless probability space.

An important special case for the space of players is obtained when it equals the
unit interval $[0, 1]$ endowed with the Lebesgue measure $\lambda$ on its Borel $\sigma$–algebra $B([0, 1])$. Another particular case considered in our results is obtained when the space of players is super-atomless. Formally, $(T, \Sigma, \varphi)$ is super-atomless if for every $E \in \Sigma$ with $\varphi(E) > 0$, the subspace of $L^1(\varphi)$ consisting of the elements of $L^1(\varphi)$ vanishing off $E$ is non-separable. This notion was first introduced by Podczeck [14].

Given a game $G = ((T, \Sigma, \varphi), V, X)$, a strategy $f$, $x \in X$, and $t \in T$ such that $\{t\} \in \Sigma$, let $f \setminus t x$ denote the strategy obtained if player $t$ changes his choice from $f(t)$ to $x$. Formally, $f \setminus t x$ denotes the strategy $g$ defined by $g(t) = x$, and $g(\tilde{t}) = f(\tilde{t})$, for all $\tilde{t} \neq t$. For all measurable subsets $S$ of $X$ and $\varepsilon, \eta \geq 0$, we say that $f$ is an $(\varepsilon, \eta)$-equilibrium of $G$ relative to $S$ if $f(t) \in S$ a.e. $t \in T$ and

$$\varphi(\{t \in T : U(t)(f) \geq U(t)(f \setminus t x) - \varepsilon \text{ for all } x \in S\}) \geq 1 - \eta.$$ \hfill (2)

Thus, in an $(\varepsilon, \eta)$-equilibrium relative to $S$, almost all players play an action in the closure of $S$ and only a small fraction of players can gain more than $\varepsilon$ by deviating from $f$ to an action in $S$. A strategy $f$ is an $\varepsilon$-equilibrium of $G$ relative to $S$ if it

5 Note that $U(t)(f \setminus t x) = V(t)(x, \varphi \circ (f \setminus t x)^{-1})$ if $G$ is a game with a finite number of players, whereas $U(t)(f \setminus t x) = V(t)(x, \varphi \circ f^{-1})$ when $G$ is a game with a continuum of players.

6 Note that the set $\{t \in T : V(t)(f(t), \varphi \circ f^{-1}) \geq V(t)(x, \varphi \circ f^{-1}) - \varepsilon \text{ for all } x \in S\} = (V, f)^{-1}(\{(u, y) \in C \times X : u(y, \varphi \circ f^{-1}) \geq u(x, \varphi \circ f^{-1}) - \varepsilon \text{ for all } x \in S\})$ is measurable. In fact, $\{(u, y) \in C \times X : u(y, \varphi \circ f^{-1}) \geq u(x, \varphi \circ f^{-1}) - \varepsilon \text{ for all } x \in S\}$ is closed and $(V, f)$ is measurable (the latter follows from Fremlin [5, Proposition 418B, p. 111], since $X$ is separable).
is an \((\varepsilon, \eta)\)-equilibrium relative to \(S\) for \(\eta = 0\). Furthermore, a strategy \(f\) is a \textit{Nash equilibrium of} \(G\) \textit{relative to} \(S\) if it is an \(\varepsilon\)-equilibrium of \(G\) relative to \(S\) for \(\varepsilon = 0\). A strategy \(f\) is a \textit{Nash equilibrium of} \(G\) (resp. \((\varepsilon, \eta)\)-equilibrium of \(G\) and \(\varepsilon\)-equilibrium of \(G\)) if \(f\) is a Nash equilibrium of \(G\) (resp. \((\varepsilon, \eta)\)-equilibrium of \(G\) and \(\varepsilon\)-equilibrium of \(G\)) relative to \(X\). We note that in the particular case where \(S\) is finite, we have that \(f\) is a Nash equilibrium of \(G\) relative to \(S\) if and only if \(f\) is a Nash equilibrium of the game \(\tilde{G} = ((T, \Sigma, \varphi), \tilde{V}, S)\) with \(\tilde{V}\) defined by \(\tilde{V}(t) = V(t)|_{S \times \mathcal{M}(S)}\) for all \(t \in T\).

We also describe a game with a continuum of players by a tight Borel probability measure \(\mu\) on \(\mathcal{C}\). This description is, in fact, equivalent to the one provided above: given \(G = ((T, \Sigma, \varphi), V, X)\), we obtain a tight Borel probability measure \(\mu = \varphi \circ V^{-1} \in \mathcal{T}(\mathcal{C})\); conversely, every probability measure \(\mu \in \mathcal{T}(\mathcal{C})\) can be represented by the distribution of a function from the unit interval, endowed with the Lebesgue measure on its Borel \(\sigma\)-algebra, into \(\mathcal{C}\), i.e., there exists a measurable function \(V : [0, 1] \to \mathcal{C}\) such that \(\mu = \lambda \circ V^{-1}\) (see [20, Theorem 3.1.1, p. 281]).

Given a Borel probability measure \(\tau\) on \(\mathcal{C} \times X\), we denote by \(\tau_{\mathcal{C}}\) and \(\tau_{X}\) the marginal distributions of \(\tau\) on \(\mathcal{C}\) and \(X\) respectively. For all subsets \(S\) of \(X\), the expression \(u(x, \tau) \geq u(S, \tau)\) means \(u(x, \tau) \geq u(x', \tau)\) for all \(x' \in S\).

Given a game \(\mu \in \mathcal{T}(\mathcal{C})\), a measurable subset \(S\) of \(X\) and \(\varepsilon \geq 0\), a Borel probability measure \(\tau\) on \(\mathcal{C} \times X\) is an \(\varepsilon\)-equilibrium distribution of \(\mu\) relative to \(S\) if \(\tau_{\mathcal{C}} = \mu\), \(\text{supp}(\tau_{X}) \subseteq \overline{S}\) and

\[
\tau(\{(u, x) \in \mathcal{C} \times X : u(x, \tau_{X}) \geq u(S, \tau_{X}) - \varepsilon\}) = 1. \tag{3}
\]
Roughly, in an \( \varepsilon \)-equilibrium distribution relative to \( S \) almost all players play an action in the closure of \( S \) and cannot gain more than \( \varepsilon \) by deviating to another action in \( S \). An \textit{equilibrium distribution of \( \mu \) relative to \( S \)} is an \( \varepsilon \)-equilibrium distribution of \( \mu \) relative to \( S \) for \( \varepsilon = 0 \). An \textit{equilibrium distribution of \( \mu \)} is an equilibrium distribution of \( \mu \) relative to \( X \). For all \( \varepsilon \geq 0 \), a Borel probability measure \( \xi \) on \( X \) is an \( \varepsilon \)-\textit{equilibrium distribution over actions of \( \mu \)} if there exists an \( \varepsilon \)-equilibrium distribution \( \tau \) of \( \mu \) such that \( \xi = \tau_X \). An \( \varepsilon \)-\textit{equilibrium distribution of} \( G = ((T, \Sigma, \varphi), V, X) \) \textit{relative to} \( S \) is an \( \varepsilon \)-equilibrium distribution of \( \varphi \circ V^{-1} \) relative to \( S \); the notions of an equilibrium distribution of \( G \) relative to \( S \), equilibrium distribution of \( G \) and \( \varepsilon \)-equilibrium distribution over actions of \( G \) are defined analogously.

Let \( K \) be a subset of \( \mathcal{C} \). We say that \( K \) is \textit{equicontinuous} if for all \( \eta > 0 \) there exists a \( \delta > 0 \) such that \( \max\{\rho(\pi, \tau), d(x, y)\} < \delta \) implies \( |V(x, \pi) - V(y, \tau)| < \eta \) for all \( V \in K \) and for all \( x, y \in X \) and \( \pi, \tau \in \mathcal{M}(X) \) (see [17, p. 156]). In our framework, equicontinuity can be interpreted as placing “a bound on the diversity of payoffs” (see [7]).

### 3 Existence of Pure Equilibria in Large Games

In this section we state our equivalence results. Our first result states that the existence of an equilibrium distribution in games with a continuum of players is equivalent to the existence of approximate equilibria in sufficiently large games.

**Theorem 1** Let \( X \) be a separable metric space, \( M \) be a compact subset of \( X \) and
$\mathcal{U} \subseteq \mathcal{C}$. Then, the following conditions are equivalent:

(1) For all games with a continuum of players $\mu \in \mathcal{T}(\mathcal{U})$, there exists an equilibrium distribution $\tau$ of $\mu$ such that $\text{supp}(\tau_X) \subseteq M$.

(2) For all equicontinuous subsets $K$ of $\mathcal{U}$ and $\varepsilon > 0$, there exists $m, N \in \mathbb{N}$ and $\{x_1, \ldots, x_m\} \subseteq M$ such that for all $n \geq N$, all games with a finite number of players $G_n = ((T_n, \Sigma_n, \nu_n), V_n, X)$ with $V_n(T_n) \subseteq K$ have an $\varepsilon$-equilibrium $f_n$ satisfying $f_n(T_n) \subseteq \{x_1, \ldots, x_m\}$.

(3) For all tight families $\Gamma$ of games with a finite number of players satisfying $\{\nu_n \circ V_\gamma^{-1}\}_{\gamma \in \Gamma} \subseteq \mathcal{T}(\mathcal{U})$, and all $\varepsilon, \eta > 0$, there exists $m, N \in \mathbb{N}$ and $\{x_1, \ldots, x_m\} \subseteq M$ such that for all $n \geq N$, all games $G_n \in \Gamma$ have an $(\varepsilon, \eta)$-equilibrium $f_n$ satisfying $f_n(T_n) \subseteq \{x_1, \ldots, x_m\}$.

This result clearly stresses the relationship between equilibrium distributions of games with a continuum of players and approximate equilibria of large finite games. In fact, Theorem 1 shows that the existence problem for large games can be equivalently addressed either in its exact version in games with a continuum of players or in an approximate version in large (equicontinuous or tight) games.

Theorem 1 is established using a characterization of the equilibrium distributions of games with a continuum of players in terms of approximate equilibria of large finite games (see Lemma 5 below). This characterization roughly states that a distribution over actions $\xi$ of a game $G$ with a continuum of players, a finite action space and with finitely many payoff functions belonging to an equicontinuous family is an
equilibrium if and only if for all sequences \( \{G_k\} \) of finite games converging to \( G \) (in the sense that the distributions over \( C \) converge in the Prohorov metric), there exists a corresponding sequence \( \{f_k\} \) of \( \varepsilon_k \)-equilibrium strategies with the property that \( \varepsilon_k \) converges to zero and the sequence of distributions of \( f_k \) converges to \( \xi \). We can then apply this characterization result to games with general action spaces and payoff functions by approximating any such game by games with finite action spaces and with finitely many payoff functions belonging to an equicontinuous family. Although there are characterizations that hold for games with general action spaces and payoff functions (see G. Carmona “Nash Equilibria of Games with a Continuum of Players”, Universidade Nova de Lisboa, 2004), the characterization presented in Lemma 5 is useful due to the bounds on \( \varepsilon_k \) and on the distance between the distributions of \( f_k \) and \( \xi \) that the special case makes possible.

We remark that the conditions in Theorem 1 are neither always true nor always false. For instance, when \( U = C \), they hold if and only if \( X \) is compact.\(^7\) Furthermore, they hold if, for example, \( U \) is the subspace of \( C \) consisting of the constant functions.

We note that the compact support assumption used in conditions 1 – 3 plays an important role in Theorem 1 since it allows us to obtain equilibrium distributions using a limit argument and to establish the existence of a finite-valued Nash equilibrium in games with a continuum of players. Its existence cannot be dispensed with. In fact, if conditions 1 and 2 were to be changed to

\(^7\) See the working paper version of this paper for details.
(a) For all games with a continuum of players \( \mu \in T(\mathcal{U}) \), there exists an equilibrium distribution \( \tau \) of \( \mu \)

and

(b) For all equicontinuous subsets \( K \) of \( \mathcal{U} \) and \( \varepsilon > 0 \), there exists \( m, N \in \mathbb{N} \) and \( \{x_1, \ldots, x_m\} \subseteq X \) such that for all \( n \geq N \), all games with a finite number of players \( G_n = ((T_n, \Sigma_n, \nu_n), V_n, X) \) with \( V_n(T_n) \subseteq K \) have an \( \varepsilon \)-equilibrium \( f_n \) satisfying \( f_n(T_n) \subseteq \{x_1, \ldots, x_m\} \),

respectively, then neither would condition (a) imply condition (b) nor would condition (b) imply condition (a) (a similar conclusion holds regarding an analogous variation of condition 3). Thus, Theorem 1 would be false without the compact support requirement.

The above claim is established by the following examples. The first shows that condition (b) does not imply condition (a). Let \( X = (0, 1) \), \( v \in \mathcal{C} \) defined by \( v(x, \pi) = x \) for all \( x \in X \) and \( \pi \in \mathcal{M}(X) \) and let \( \mathcal{U} = \{v\} \). Then, for all \( \varepsilon > 0 \), let \( N = 2 \), \( m = 1 \) and \( x_1 = 1 - \varepsilon \). Thus, \( f_n \equiv x_1 \) is an \( \varepsilon \)-equilibrium of every game with a finite number of players \( G_n \) satisfying \( V_n(T_n) \subseteq \mathcal{U} \). However, it is clear that no \( \mu \in T(\mathcal{U}) \) has an equilibrium distribution.

The second example shows that condition (a) does not imply condition (b). Let \( X = \mathbb{R} \) with metric \( d(x, y) = |x - y|/(1 + |x - y|) \), \( v_x \in \mathcal{C} \) be defined by \( v_x(x', \pi) = -d(x, x') \) for all \( x, x' \in X \) and \( \mathcal{U} = \{v_x\}_{x \in X} \). Let \( \mu \in T(\mathcal{U}) \) and \( V : [0, 1] \to \mathcal{U} \) be such that \( \lambda \circ V^{-1} = \mu \). Note that \( h : \mathbb{R} \to \mathcal{U} \) defined by \( h(x) = v_x \) is a homeomorphism between
\( \mathbb{R} \) and \( \mathcal{U} \). Then, \( f : [0, 1] \to \mathbb{R} \) defined by \( f(t) = h^{-1} \circ V \) is a Nash equilibrium of \( G = \left( ([0,1], \mathcal{B}([0,1]), \lambda), V, X \right) \) and so \( \tau = \lambda \circ (V, f)^{-1} \) is an equilibrium distribution of \( \mu \). Let \( K = \mathcal{U} \) and so \( K \) is equicontinuous. Since \( X \) is not totally bounded, there exists \( \varepsilon > 0 \) such that for all finite subsets \( F \) of \( X \), there exists \( x \in X \) such that \( d(x, x') > \varepsilon \) for all \( x' \in F \). Let \( m, n \in \mathbb{N} \) and \( \{x_1, \ldots, x_m\} \subseteq X \) be given and let \( x \in X \) be such that \( d(x, x') > \varepsilon \) for all \( x' \in \{x_1, \ldots, x_m\} \). Then, letting \( n = N \) and \( G_n \) be such that \( V_n(t) = v_x \) for all \( t \in T_n \), it follows that if \( f_n \) is an \( \varepsilon \)-equilibrium of \( G_n \), then \( f_n(t) \not\in \{x_1, \ldots, x_m\} \).

Our second equivalence result states that the existence of an equilibrium distribution in games with a continuum of players is equivalent to the existence of a Nash equilibrium in games with a super-atomless space of players.

**Theorem 2** Let \( X \) be a separable metric space and \( \mathcal{U} \subseteq \mathcal{C} \). Then, the following conditions are equivalent:

1. An equilibrium distribution exists for all games with a continuum of players \( \mu \in T(\mathcal{U}) \).
2. A Nash equilibrium exists for all games \( G = ((T, \Sigma, \varphi), V, X) \) with \( V(T) \subseteq \mathcal{U} \) and a super-atomless probability space of players.

Theorem 2 implies that the existence of pure strategy Nash equilibria in games with a super-atomless space of players can be addressed either in terms of strategies or in terms of distributions. Furthermore, the proof of Theorem 2 (namely, Corollary 8) establishes a close relationship between equilibrium distributions and Nash equilibria.
of games with a super-atomless space of players. Indeed, Corollary 8 shows that for any game \( G = ((T, \Sigma, \varphi), V, X) \) with a super-atomless space of players, a probability measure \( \tau \) on \( C \times X \) is an equilibrium distribution for \( \varphi \circ V^{-1} \) if and only if there is a Nash equilibrium \( f \) of \( G \) such that \( \tau = \varphi \circ (V, f)^{-1} \).

We emphasize that Theorem 2 and the above representation result (Corollary 8) would be false if the space of players were merely atomless. In fact, the example in [7, Section 2] consists of a game with Lebesgue space of players, a compact action space and a continuous payoff-assigning function that has an equilibrium distribution (by [11, Theorem 1]) but fails to have a Nash equilibrium. The reason for this failure is that a Lebesgue space of players may not offer enough measurable functions. This, in turn, can be viewed as a consequence of the fact that, by Lusin’s Theorem, a measurable function on a Lebesgue space to a Polish space must be “almost continuous”. In contrast, if the space of players is extended to a super-atomless one by enlarging the original \( \sigma \)-algebra (which implies that there are more measurable functions), it is possible not only to obtain a Nash equilibrium of the game with the extended, super-atomless space of players, but also to represent each equilibrium distribution of the original game as the joint distribution of a Nash equilibrium of the extended game and the payoff-assigning function of the original game. Thus, in particular, Theorem 2 implies that super-atomless spaces are rich enough to solve the measurability problems that one encounters when working with simpler spaces, and which prevent, in general, the existence of an equilibrium strategy.
Our third equivalence result states that the existence of an equilibrium distribution in games with a continuum of players is equivalent to the existence of a Nash equilibrium in games with a finite action space and a Lebesgue space of players. Such an equivalence is obtained through the use of relative equilibrium since it allows for the common action space $X$ to be used in both statements even though $X$ may not be finite. Ideally, the statement would assert the equivalence of the following two conditions:

For all games with a continuum of players $\mu \in \mathcal{T}(\mathcal{U})$ and all non-empty closed subsets $S$ of $X$, there exists an equilibrium distribution $\tau$ of $\mu$ relative to $S$,

and

For all games with a continuum of players $G = (([0, 1], \mathcal{B}([0, 1]), \lambda), V, X)$ with $V([0, 1]) \subseteq \mathcal{U}$ and all non-empty finite subsets $F$ of $X$, there exists a Nash equilibrium $f$ of $G$ relative to $F$.

However, as in Theorem 1, a common compact support assumption is needed for such result. But assuming the existence of a compact subset $M$ of $X$ such that $\text{supp}(\tau_X) \subseteq M$ and $\text{supp}(\lambda \circ f^{-1}) \subseteq M$ is not enough now. In fact, when $X$ is not compact, there given any compact subset $M$ of $X$, there is a finite subset $F$ of $X$ such that $F \cap M = \emptyset$, and so neither $\mu$ can have an equilibrium distribution $\tau$ relative to $F$ such that $\text{supp}(\tau_X) \subseteq M$, nor $G$ can have an equilibrium $f$ relative to $F$ such that $\text{supp}(\lambda \circ f^{-1}) \subseteq M$. Thus, without assuming that both $M \cap S$ and $M \cap F$ are nonempty, the above conditions would be false even when it is assumed that the supports are
Theorem 3 Let $X$ be a separable metric space, $M$ be a compact subset of $X$ and $U \subseteq C$. Then, the following conditions are equivalent:

$(1)$ For all games with a continuum of players $\mu \in T(U)$ and all closed subsets $S$ of $X$ such that $M \cap S$ is nonempty, there exists an equilibrium distribution $\tau$ of $\mu$ relative to $S$ such that $\text{supp}(\tau_X) \subseteq M$.

$(2)$ For all games with a continuum of players $G = (([0, 1], \mathcal{B}([0, 1]), \lambda), V, X)$ with $V([0, 1]) \subseteq U$ and all countable, closed subsets $C$ of $X$ such that $M \cap C$ is nonempty, there exists a Nash equilibrium $f$ of $G$ relative to $C$ such that $\text{supp}(\lambda \circ f^{-1}) \subseteq M$.

$(3)$ For all games with a continuum of players $G = (([0, 1], \mathcal{B}([0, 1]), \lambda), V, X)$ with $V([0, 1]) \subseteq U$ and all finite subsets $F$ of $X$ such that $M \cap F$ is nonempty, there exists a Nash equilibrium $f$ of $G$ relative to $F$ such that $\text{supp}(\lambda \circ f^{-1}) \subseteq M$.

Theorem 3 shows, in particular, that although the hypothesis of a finite action space is restrictive, a Nash equilibrium existence result for such action spaces and a Lebesgue space of players is strong enough to imply, for a general compact metric action space, the existence of an equilibrium distribution and, due to Theorem 2, of a Nash equilibrium in games with a richer space of players.

It is worthwhile to note that the argument used in the proof of Theorem 3 shows that any limit point of a sequence of equilibrium distributions, each relative to a finite subset of the action space, is an equilibrium distribution of the original game.
provided that the sequence of those finite sets is increasing and its union is dense. This result means that, in order to establish the existence of an equilibrium distribution, it suffices to approximate the action space with increasingly large finite sets and to obtain a Nash equilibrium relative to each of such finite action spaces.

As already noted, as in Theorem 1, the compact support assumption cannot be dispensed with in Theorem 3. However, dropping this requirement from conditions 1 and 2 produces two conditions that are still equivalent and that imply the one resulting from dropping the same requirement from condition 3. On the other hand, the example used in the discussion of Theorem 1 to show that condition (b) does not imply condition (a) can still be used to show that the converse is not true.

Our equivalence results provide a unifying approach to the existence problem in large games. In fact, by considering the particular case when $X$ is compact and $\mathcal{U} = \mathcal{C}$, we obtain the classical existence results of Schmeidler [18], Mas-Colell [11], Khan and Sun [9] and Khan and Sun [10]. It is interesting to note that each of these existence theorems can be coupled with our main results to derive the others. Thus, for instance, Schmeidler’s theorem, together with Theorem 3, implies the result in [9] (simply by taking $X$ to be countable and $M = C = X$) and the one in [11] (simply by taking $X$ to be an arbitrarily compact space and $M = S = X$). This conclusion, together with Theorem 2, then implies the existence of a Nash equilibrium in games with superatomless space of players, and so, in particular, in games with an atomless Loeb space of players. Thus, we obtain the existence result for games with a continuum of players in [10]. Furthermore, by Theorem 1, it also implies the existence result in [10] for tight
games with a finite but sufficiently large number of players. It is interesting to note that the existence of a Nash equilibrium in games with a super-atomless space of players is a new existence result, not covered by previous existence theorems.

**Corollary 4** Suppose that $X$ is non-empty and compact. Then,

(1) For all equicontinuous subsets $K$ of $C$ and $\varepsilon > 0$, there exists $m, N \in \mathbb{N}$ and \{x$_1$, …, x$_m$\} $\subseteq$ X such that for all $n \geq N$, all games with a finite number of players $G_n = ((T_n, \Sigma_n, \nu_n), V_n, X)$ with $V_n(T_n) \subseteq K$ have an $\varepsilon$-equilibrium $f_n$ satisfying $f_n(T_n) \subseteq \{x_1, \ldots, x_m\}$.

(2) (Khan and Sun) For all tight families $\Gamma$ of games with a finite number of players, and all $\varepsilon, \eta > 0$, there exists $m, N \in \mathbb{N}$ and \{x$_1$, …, x$_m$\} $\subseteq$ X such that for all $n \geq N$, all games $G_n \in \Gamma$ have an ($\varepsilon, \eta$)-equilibrium $f_n$ satisfying $f_n(T_n) \subseteq \{x_1, \ldots, x_m\}$.

(3) (Mas-Colell) An equilibrium distribution exists for all games with a continuum of players $\mu \in \mathcal{M}(C)$.

(4) A Nash equilibrium exists for all games $G = ((T, \Sigma, \varphi), V, X)$ with a super-atomless probability space of players.

(5) (Khan and Sun) A Nash equilibrium exists for all games $G = ((T, \Sigma, \varphi), V, X)$ with an atomless Loeb probability space of players.

(6) (Khan and Sun) A Nash equilibrium exists for all games with a continuum of players $G = (([0, 1], \mathcal{B}([0, 1]), \lambda), V, X)$ with $X$ countable.

(7) (Schmeidler) A Nash equilibrium exists for all games with a continuum of players $G = (([0, 1], \mathcal{B}([0, 1]), \lambda), V, X)$ with $X$ finite.
4 Proofs and Further Results

The proof of our main results, Theorems 1 — 3, relies on three lemmas which have some interest in their own right. The first provides a characterization of equilibrium distributions in terms of approximate equilibria of games with a large, but finite number of players. Lemma 6 provides sufficient conditions for the existence of finite-valued approximate equilibria in games with a continuum of players. Finally, Lemma 7 presents a representation result which implies that, in games with a super-atomless space of players, every equilibrium distribution is the joint distribution of a Nash equilibrium and the function describing the game. These lemmas are stated and proved in Subsections 4.1, 4.2 and 4.3, respectively, while the proofs of Theorems 1, 2 and 3 are presented in Subsections 4.4, 4.5 and 4.6, respectively.

In the rest of the paper, for a metric space $Z$, $\mathcal{M}(Z)$ denotes the space of all Borel probability measures on $Z$, and $\mathcal{T}(Z)$ the space of all tight Borel probability measures on $Z$. Convergence of Borel probability measures on a metric space is always understood with respect to the Prohorov metric.

4.1 A Characterization of Equilibrium Distributions

In this section we characterize the equilibrium distributions, supported on a given finite set, of some simple games with a continuum of players. These are games with a finite number of characteristics and with payoff functions selected from an equicontin-
uous family. Despite all these restrictive assumptions, this result is enough to deduce the existence of pure strategy approximate equilibria in large finite games from the existence of an equilibrium distribution with compact support in games with a continuum of players.

The following notation is used in Lemma 5 and its proof. When $F$ is a finite set and $\pi$ a probability measure on $F$, we write $\pi_l$ instead of $\pi(\{l\})$, whenever $l \in F$, and also $\pi = (\pi_1, \ldots, \pi_L)$, with $L = |F|$. This notation also suggests that a measure with a finite support can be thought of as a vector in some Euclidean space. We will also write $||\pi|| = \max_{l \in F} |\pi_l|$, i.e., $||\pi||$ is the sup norm of the vector $(\pi_1, \ldots, \pi_L)$. Note that a sequence of measures $\{\pi_n\}_{n=1}^{\infty}$ on $F$ converges to $\pi$ in the Prohorov metric if and only if $\lim_{n \to \infty} ||\pi_n - \pi|| = 0$. Furthermore, for all equicontinuous subsets $K$ of $C$ and $V \in K$, let $\omega_V : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\omega_V(\delta) = \sup\{|V(x, \pi) - V(y, \tau)| : \max\{d(x, y), \rho(\pi, \tau)\} \leq \delta\}$ for all $\delta > 0$, denote the modulus of continuity of $V$ and $\omega_K(\delta) = \sup_{V \in K} \omega_V(\delta)$. Of course, since $K$ is equicontinuous, then $\lim_{\delta \to 0} \omega_K(\delta) = 0$.

**Lemma 5** Let $S$ be a finite subset of $X$, $m = |S|$ and $K$ be an equicontinuous subset of $C$. Then, the following holds for all games $G = ((T, \Sigma, \varphi), V, X)$ with a continuum of players such that $V(T)$ is a finite subset of $K$ and for all $\varepsilon \geq 0$:

A Borel probability measure $\xi$ on $X$ is an $\varepsilon$-equilibrium distribution over actions of $G$ with $\text{supp}(\xi) \subseteq S$ if and only if for all games $G_n = ((T_n, \Sigma_n, \nu_n), V_n, X)$ with a finite number of players in which $V_n(T_n)$ is a subset of $V(T)$ there exists a strategy $f_n : T_n \to S$ such that
\((1)\) \(f_n\) is an \(\varepsilon + 2\omega_K (m||\varphi \circ V^{-1} - \nu_n \circ V_n^{-1}|| + (m^2 + 1)/n)\)-equilibrium of \(G_n\) and 
\[(2)\] \[||\nu_n \circ f_n^{-1} - \xi|| \leq ||\varphi \circ V^{-1} - \nu_n \circ V_n^{-1}|| + \frac{m}{n}.\]

In order to illustrate the idea of Lemma 5, consider the particular case of a sequence of games \(\{G_n\}\) with a finite number of players with \(V_n(T_n) \subseteq V(T)\) and with \(||\varphi \circ V^{-1} - \nu_n \circ V_n^{-1}||\) converging to zero. In this case, we can, intuitively, say that the sequence \(\{G_n\}\) converges to \(G\). If \(\xi\) is an equilibrium distribution over actions of \(G\), then Lemma 5 guarantees the existence of an \(\varepsilon_n\)-equilibrium \(f_n\) of \(G_n\) such that \(\varepsilon_n \to 0\) and \(||\nu_n \circ f_n^{-1} - \xi|| \to 0\). That is, finite games that are close to \(G\) have approximate equilibria, with a degree of approximation close to zero, whose distributions are close to \(\xi\).

Conversely, the existence of approximate equilibria of games converging to \(G\), with a vanishing degree of approximation and with distributions converging to \(\xi\), is enough to show that \(\xi\) is an equilibrium distribution over actions of \(G\).

The strength of Lemma 5, which is crucial to the asymptotic result, is that the degree of approximation involved depends only on \(\varepsilon\), on the equicontinuous family \(K\), on the number of pure strategies \(m\) of the set \(S\), on the distance between the distributions of characteristics \(||\varphi \circ V^{-1} - \nu_n \circ V_n^{-1}||\) and on the number of players \(n\). In particular, it is independent of the particular games \(G\) and \(G_n\) that we are considering. So, if \(\varepsilon\) and the set of actions is fixed, and we are considering games \(G\) and \(G_n\) with the same distribution of characteristics, then the degree of approximation depends only on \(n\). This fact is at the core of our asymptotic result: once \(n\) is sufficiently large,
equilibrium distributions of $G$ induce approximate equilibria of $G_n$.

**Proof of Lemma 5.**  
Let $S = \{x_1, \ldots, x_m\}$ be a finite subset of $X$ and $K$ be an equicontinuous subset of $C$. Let $\epsilon \geq 0$ and let $G = ((T, \Sigma, \varphi), V, X)$ be a game with a continuum of players such that $V(T)$ is a finite subset of $K$. Let $\beta = \varphi \circ V^{-1}$ and let $\text{supp}(\beta) = \{V_1, \ldots, V_L\}$.

(Necessity) Let $\xi$ be an $\epsilon$-equilibrium distribution over actions of $G$ with $\text{supp}(\xi) \subseteq S$ and let $\psi$ be an $\epsilon$-equilibrium distribution of $G$ such that $\xi = \psi_X$. For all $1 \leq l \leq L$ and $1 \leq i \leq m$, let $\psi_{l,i} = \psi(\{(V_l, x_i)\})$ and note that $\sum_{i=1}^m \psi_{l,i} = \beta_l$ and $\sum_{l=1}^L \psi_{l,i} = \xi_i$. Since $\psi$ is an $\epsilon$-equilibrium distribution, it follows that if $\psi_{l,i} > 0$ then, for all $x \in X$,

$$V_l(x_i, \xi) \geq V_l(x, \xi) - \epsilon. \quad (4)$$

Let $G_n$ be a game with a finite number of players such that $V_n(T_n)$ is a subset of $V(T)$. For all $1 \leq l \leq L$, let $T_{n,l} = \{t \in T_n : V_n(t) = V_l\}$ and $\gamma_{n,l} = |T_{n,l}|$. Then, $\gamma_n = (\gamma_{n,1}, \ldots, \gamma_{n,L})$ is such that $\gamma_n/n = \nu_n \circ V_n^{-1}$.

Let $1 \leq l \leq L$ be given. Define $E_l = \{e_i \in E : \psi_{l,i} > 0\}$, where $E = \{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{R}^m$. Define $E_t = E_l$ if $t \in T_{n,l}$. If $\gamma_{n,l} > 0$, it follows that $E_l \subseteq \frac{1}{\gamma_{n,l}} \sum_{t \in T_{n,l}} E_t$. Also, we have that $\psi_l/\beta_l = (\psi_{l,1}/\beta_l, \ldots, \psi_{l,m}/\beta_l) \in \text{co}(E_l)$ and so $\psi_l/\beta_l \in \text{co} \left( \frac{1}{\gamma_{n,l}} \sum_{t \in T_{n,l}} E_t \right) = \frac{1}{\gamma_{n,l}} \sum_{t \in T_{n,l}} \text{co}(E_t)$.

---

8 Lemmas 9-12 which are appealed to in this proof may be found in the Appendix.
Define
\[
\tau = \sum_{i=1}^{L} \frac{\gamma_{n,i}}{n} \psi_{i} = \sum_{t: \gamma_{n,t} > 0} \frac{\gamma_{n,t}}{n} \beta_{t}.
\] (5)

Then, for all \(1 \leq i \leq m\), it follows that \(|\xi_{i} - \tau_{i} \leq \sum_{t=1}^{L} \frac{\psi_{i}}{n} \left| \frac{\gamma_{n,t}}{n} - \beta_{t} \right| \leq \left\| \beta - \frac{2n}{n} \right\|\), and so \(\|\xi - \tau\| \leq \left\| \beta - \frac{2n}{n} \right\|\). Hence, by Lemma 9,
\[
\rho(\xi, \tau) \leq m \left\| \beta - \frac{2n}{n} \right\|.
\] (6)

Furthermore, \(\tau \in \sum_{t: \gamma_{n,t} > 0} \frac{\gamma_{n,t}}{n} \sum_{t \in T_{n}, \alpha} \text{co}(E_{t}) = \frac{1}{n} \sum_{t \in T_{n}} \text{co}(E_{t})\). Thus, by the Shapley-Folkman Theorem (see [21, Corollary, p. 35]), it follows that there are \(n\) points \((\alpha_{t})_{t \in T_{n}}\) such that \(\alpha_{t} \in \text{co}(E_{t})\) for all \(t \in T_{n}\), \(|\{t \in T_{n} : \alpha_{t} \not\in E_{t}\}| \leq m\) and
\[
\tau = \frac{1}{n} \sum_{t \in T_{n}} \alpha_{t}.
\] (7)

Let \(1 \leq l \leq L\) and define \(P_{n} = \{t \in T_{n} : \alpha_{t} \in E\}\). Define a strategy \(f_{n}\) as follows: if \(t \in P_{n}\), then let \(e_{i}\) be such that \(\alpha_{t} = e_{i}\) and define \(f_{n}(t) = x_{i}\); if \(t \in P_{n}^{c} := T_{n} \setminus P_{n}\) and \(V_{t} = V_{l}\), choose \(1 \leq i \leq m\) such that \(\psi_{t,i} > 0\) and define \(f_{n}(t) = x_{i}\). By (4), it follows that \(V(t)(f_{n}(t), \xi) \geq V(t)(x, \xi) - \varepsilon\) for all \(t \in T_{n}\) and \(x \in X\).

Let \(\sigma = \nu_{n} \circ f_{n}^{-1}\). We claim that \(\|\tau - \sigma\| \leq \frac{m}{n}\). In fact, for all \(1 \leq i \leq m\) we have that
\[
\sigma_{i} = \sum_{t \in P_{n}} \frac{\alpha_{t,i}}{n} + \frac{|P_{n}^{e} \cap f_{n}^{-1}(x_{i})|}{n} = \sum_{t \in P_{n}} \frac{\alpha_{t,i}}{n} + \sum_{t \in P_{n}^{e} \cap f_{n}^{-1}(x_{i})} \frac{1}{n}
\] (8)

and \(\tau_{i} = \sum_{t=1}^{n} \alpha_{t,i}/n\). Therefore, letting \(\chi_{f^{-1}(x_{i})}\) denote the characteristic function of \(f^{-1}(x_{i})\), we obtain that \(\|\tau_{i} - \sigma_{i}\| = \frac{1}{n} \sum_{t \in P_{n}} \left| \alpha_{t,i} - \chi_{f^{-1}(x_{i})}(t) \right| \leq \frac{|P_{n}^{e}|}{n} \leq \frac{m}{n}\) and so \(\|\tau - \sigma\| \leq m/n\).

Since \(\nu_{n} \circ f^{-1} = \sigma\) and \(\|\xi - \tau\| \leq \|\beta - \gamma_{n}/n\|\), then \(\|\nu_{n} \circ f^{-1} - \xi\| \leq \|\beta - \gamma_{n}/n\| + m/n\).
This establishes assertion 2 in the statement of the Lemma.

By Lemma 9, \( \rho(\tau, \nu_n \circ f_n^{-1}) \leq m^2/n \) since \( \nu_n \circ f_n^{-1} = \sigma \). Also, by Lemma 10, it follows that \( \rho(\nu_n \circ f_n^{-1}, \nu_n \circ (f_n \setminus t x)^{-1}) \leq 1/n \) for all \( t \in T_n \) and \( x \in X \). Hence, using (6), it follows that \( \rho(\nu_n \circ f_n^{-1}, \xi) \leq m \| \beta - \gamma_n/n \| + m^2/n \) and

\[
\rho(\nu_n \circ (f_n \setminus t x)^{-1}, \xi) \leq m \left\| \beta - \frac{\gamma_n}{n} \right\| + \frac{m^2 + 1}{n}.
\] (9)

For convenience, let \( \theta = m \left\| \beta - \frac{\gamma_n}{n} \right\| + \frac{m^2 + 1}{n} \). Hence, for all \( t \in T_n \) and \( x \in X \), we obtain

\[
V(t)(f_n(t), \nu_n \circ f_n^{-1}) \geq V(t)(f_n(t), \xi) - \omega_K(\theta)
\geq V(t)(x, \nu_n \circ (f_n \setminus t x)^{-1}) - \varepsilon - 2\omega_K(\theta).
\] (10)

Therefore, \( f_n \) is an \( \varepsilon + 2\omega_K(m \| \beta - \gamma_n/n \| + (m^2 + 1)/n) \)-equilibrium of \( G_n \).

(Sufficiency) Let \( \xi \) be a distribution over \( X \) satisfying the condition. Let \( \{q_n\} \subseteq \mathbb{Q}_+^L \) be such that \( q_n \rightarrow \beta \). For all \( n \in \mathbb{N} \), there exist \( \gamma_n = (\gamma_{n,1}, \ldots, \gamma_{n,L}) \in \mathbb{N}^L \) and \( k_n \in \mathbb{N} \) such that \( q_n = \gamma_n/k_n \). By multiplying both \( k_n \) and \( \gamma_n \) by \( n \) if necessary, we may assume that \( k_n \geq n \). Define, for all \( n \), a game \( G_{k_n} = ((T_{k_n}, \Sigma_{k_n}, \nu_{k_n}), V_{k_n}, X) \) where \( V_{k_n} \) satisfies \( |\{t \in T_{k_n} : V_{k_n}(t) = V_l\}| = \gamma_{n,l} \) for all \( 1 \leq l \leq L \).

For all \( n \), let \( f_{k_n} \) satisfy 1 and 2. Consider the sequence \( \{\nu_{k_n} \circ (V_{k_n}, f_{k_n})^{-1}\} \subseteq \mathcal{M}(\{V_1, \ldots, V_L\} \times S) \). Since \( \mathcal{M}(\{V_1, \ldots, V_L\} \times S) \) is compact (being a closed and bounded subset of a finite dimensional space), taking a subsequence if necessary, we may assume that it converges. Let \( \tau = \lim_n \nu_{k_n} \circ (V_{k_n}, f_{k_n})^{-1} \). Then, \( \tau_C = \beta = \lambda \circ V^{-1}, \tau_X = \xi \) and \( \text{supp}(\xi) \subseteq S \) since, respectively, \( \nu_{k_n} \circ V_{k_n}^{-1} = \gamma_n/k_n \rightarrow \beta \),
\[ |\nu_{k_n} \circ f_{k_n}^{-1} - \xi| \leq |\beta - \gamma_n/k_n| + m/k_n \to 0 \] and \( f_{k_n}(T_{k_n}) \subseteq S \) for all \( n \). Since \( \nu_{k_n} \circ (V_{k_n}, f_{k_n})^{-1} \) converges to \( \tau \), \( f_{k_n} \) is an \( \varepsilon + 2\omega_K(m \|\beta - \gamma_n\| + \frac{m^2+1}{k_n}) \) - equilibrium of \( G_{k_n} \) and \( \lim_n (m \|\beta - \gamma_n\| + \frac{m^2+1}{k_n}) = 0 \), it follows, by Lemma 11, that \( \tau \) is an \( \varepsilon \)-equilibrium distribution of \( G \). Thus, \( \xi \) is an \( \varepsilon \)-equilibrium distribution over action of \( G \) with \( \text{supp}(\xi) \subseteq S \). ■

4.2 Finite-valued Equilibria

In this subsection we address the existence of finite-valued approximate equilibria. Lemma 6 considers games with a continuum of players where the set of players’ characteristics is a countable subset of an equicontinuous family. It guarantees the existence of a finite set of actions with the property that all such games have an approximate equilibrium strategy taking values in this finite set. The strength of this result is that the finite set works uniformly for all such games, i.e., it depends only on the equicontinuous set and on the degree of approximation desired. Since Lemma 5 only applies to games with finite action space, these properties are useful in order to demonstrate part of Theorem 1 using that lemma.

**Lemma 6** Let \( M \) be a compact subset of \( X \) and \( K \) be an equicontinuous subset of \( C \). Then, for all \( \varepsilon > 0 \), there exists a finite subset \( \{x_1, \ldots, x_m\} \) of \( M \) such that if \( G = ((T, \Sigma, \varphi), V, X) \) is a game with a continuum of players such that \( V(T) \subseteq K \) is countable, and \( \tau \) is an equilibrium distribution of \( G \) with \( \text{supp}(\tau_X) \subseteq M \), then there exists an \( \varepsilon \)-equilibrium strategy \( g \) such that \( g(T) \subseteq \{x_1, \ldots, x_m\} \) and \( \hat{\rho}(\varphi \circ \)
\[(V, g)^{-1}, \tau) < \varepsilon, \text{ where } \hat{\rho} \text{ is the Prohorov metric on } M(C \times X).\]

**Proof.** Let \(\varepsilon > 0\). Since \(K\) is equicontinuous, there exists \(\delta > 0\) such that \(\max\{d(x, y), \rho(\pi, \psi)\} < \delta\) implies that \(|u(x, \pi) - u(y, \psi)| < \varepsilon/2\) for all \(x, y \in X, \pi, \psi \in M(X)\) and \(u \in K\).

We can choose \(\delta < \varepsilon\).

Let \(\{x_1, \ldots, x_m\} \subseteq M\) be such that \(M \subseteq \bigcup_{j=1}^{m} B_{\delta/2}(x_j)\). Define \(B_1 = B_{\delta/2}(x_1)\) and \(B_j = B_{\delta/2}(x_j) \setminus \left(\bigcup_{l=1}^{j-1} B_{\delta/2}(x_l)\right)\) for all \(2 \leq j \leq m\).

Let \(G = ((T, \Sigma, \varphi), V, X)\) be a game with a continuum of players such that \(V(T)\) is a countable subset of \(K\) and let \(\tau\) be an equilibrium distribution of \(G\). It follows from [4, Theorem 1] that there exists a Nash equilibrium \(f\) of \(G\) such that \(\tau = \varphi \circ (V, f)^{-1}\) and \(f(T) \subseteq M\).

Define \(g : T \to \{x_1, \ldots, x_m\}\) by \(g(t) = x_j\) if \(f(t) \in B_j\). Then, \(g\) is measurable and \(d(f(t), g(t)) < \delta/2\) for all \(t \in T\). This implies that \(\{t \in T : (V(t), g(t)) \in D\} \subseteq \{t \in T : (V(t), f(t)) \in \overline{B}_{\delta/2}(D)\}\) for all Borel measurable subsets \(D\) of \(C \times X\) and so \(\varphi \circ (V, g)^{-1}(D) \leq \varphi \circ (V, f)^{-1}(\overline{B}_{\delta/2}(D)) + \delta/2\). Similarly, one can show that \(\varphi \circ (V, f)^{-1}(D) \leq \varphi \circ (V, g)^{-1}(\overline{B}_{\delta/2}(D)) + \delta/2\). Thus, \(\hat{\rho}(\varphi \circ (V, g)^{-1}, \tau) \leq \delta/2 < \varepsilon\). Analogously, we can show that \(\hat{\rho}(\varphi \circ g^{-1}, \varphi \circ f^{-1}) \leq \delta/2 < \delta\).

This implies that for almost all \(t \in T\) and all \(x \in X, V(t)(g(t), \varphi \circ g^{-1}) > V(t)(f(t), \varphi \circ f^{-1}) - \varepsilon/2 \geq V(t)(x, \varphi \circ f^{-1}) - \varepsilon/2 > V(t)(x, \varphi \circ g^{-1}) - \varepsilon\). Hence, \(g\) is an \(\varepsilon\)-equilibrium of \(G\). \(\blacksquare\)
4.3 A Representation Result for Distributions

In this subsection, we characterize equilibrium distributions of games with a super-atomless space of players in terms of its Nash equilibria. Such a characterization (Corollary 8) is a direct consequence of Lemma 7, which is a representation result for tight measures. In general, every tight measure \( \tau \) on a metric space \( Y \) can be represented as the distribution of a measurable function \( h \), mapping the unit interval with Lebesgue measure into \( Y \). However, if \( Y = A \times B \) and \( \tau_A = \lambda \circ g^{-1} \), where \( g : [0, 1] \to A \), in general there is no \( f : [0, 1] \to B \) such that \( \tau = \lambda \circ (g, f)^{-1} \).

Lemma 7 shows the existence of such a function \( f \) provided that the probability space \(( [0,1], B([0,1]) , \lambda ) \) is replaced by a super-atomless one.

**Lemma 7** Let \( A \) and \( B \) be metric spaces, \(( T, \Sigma, \varphi) \) a super-atomless probability space, \( \tau \) a tight Borel probability measure on \( A \times B \) and \( g : T \to A \) a measurable function such that \( \tau_A = \varphi \circ g^{-1} \). Then, there is a function \( f : T \to B \) such that \( (f, g) \) is measurable and \( \tau = \varphi \circ (g, f)^{-1} \).

**Proof.** We claim that we may assume, without loss of generality, that \( A \) and \( B \) are compact metric spaces. In order to establish this claim, we first show that if the conclusion of the lemma holds when \( A \) and \( B \) are Polish spaces, it also holds when they are arbitrary metric spaces.

In order to establish this latter claim, let \( A \) and \( B \) be arbitrary metric spaces. Since \( \tau \) is tight, so are \( \tau_A \) and \( \tau_B \). Thus, we can find an increasing sequence \( \{ A_n \}_{n=1}^{\infty} \) of compact
subsets of $A$ such that $\tau_A(\cup_n A_n) = 1$, and an increasing sequence $\{B_n\}_{n=1}^{\infty}$ of compact subsets of $B$ such that $\tau_B(\cup_n B_n) = 1$. Note that we must have $\tau((\cup_n A_n) \times (\cup_n B_n)) = 1$ and $(\cup_n A_n) \times (\cup_n B_n) = \cup_n (A_n \times B_n)$, so we may view $\tau$ as a tight Borel probability measure on $(\cup_n A_n) \times (\cup_n B_n)$. Furthermore, changing $g$ on a null set if needed, we may assume that it takes all of its values in $\cup_n A_n$, and we may then assume as well that $A = \cup_n A_n$ and $B = \cup_n B_n$.

It then follows by [19, Corollary 2, p. 102, Corollary 3, p. 103, and Definition 2, p. 94] that there is a Polish topology $\eta_A$ on $A$ which is stronger than the original topology of $A$, but which generates the same Borel $\sigma$-algebra on $A$ as the original one. Similarly, there is a Polish topology $\eta_B$ on $B$, stronger than the original topology of $B$, but generating the same Borel $\sigma$-algebra on $B$ as the original topology of $B$. In particular, then, the product topology $\eta_A \times \eta_B$ is stronger than the original product topology of $A \times B$. Note also that since $\eta_A$ and $\eta_B$ are Polish topologies, the product of the Borel $\sigma$-algebras generated by $\eta_A$ and $\eta_B$ coincides with the Borel $\sigma$-algebra generated by the product topology $\eta_A \times \eta_B$. Consequently, the Borel $\sigma$-algebra generated by the product topology $\eta_A \times \eta_B$ coincides with the original Borel $\sigma$-algebra of $A \times B$. In view of these facts, we may assume that $A$ and $B$ are Polish spaces.

Finally, we show that if the conclusion of the lemma holds when $A$ and $B$ are compact metric spaces, it also holds when they are just Polish spaces. Recall that if $C$ and $D$ are any two Polish spaces of the same cardinality, then they are Borel isomorphic, i.e., there is a bijection $\xi: C \to D$ such that both $\xi$ and its inverse $\xi^{-1}$ are Borel measurable, and recall that the cardinality of a Polish space is finite, countable infinite,
or that of the continuum (see [5, Corollary 424D, p. 166]). Thus, since any compact metric space is a Polish space, we may assume, in fact, that both $A$ and $B$ are compact metric spaces. This establishes the above claim.

Let $x \mapsto \tau_x$ be a disintegration of $\tau$, $x \in A$ (see [5, Corollary 452N, p. 436]). Thus, for each $x \in A$, $\tau_x$ is a Borel probability measure on $B$, and for each Borel set $C \subseteq A \times B$,

$$\tau(C) = \int_A \tau_x(C_x) d\tau_A(x), \quad (11)$$

where $C_x \subseteq B$ is the $x$-section of $C$. Note that for each $x \in A$ and each Borel set $C \subseteq A \times B$, writing $\delta_x$ for the Dirac measure at $x \in A$,

$$\delta_x \otimes \tau_x(C) = \int_A \tau_x(C_{x'}) d\delta_x(x') = \tau_x(C_x), \quad (12)$$

where the first equality follows by Fubini’s theorem. Let $\phi: T \to \mathcal{M}(A \times B)$ be the mapping defined by $\phi(t) = \delta_{g(t)} \otimes \tau_{g(t)}$. Then $\phi$ is measurable in the sense that $t \mapsto \phi(t)(C)$ is measurable for each Borel set $C \subseteq A \times B$ (because $t \mapsto \phi(t)(C)$ is the composition of the measurable mapping $g$ with the measurable mapping $x \mapsto \tau_x(C_x)$), and for any Borel set $C \subseteq A \times B$,

$$\int_T \phi(t)(C) d\varphi(t) = \int_T \delta_{g(t)} \otimes \tau_{g(t)}(C) d\varphi(t)$$

$$= \int_A \delta_x \otimes \tau_x(C) d(\varphi \circ g^{-1})(x) = \int_A \delta_x \otimes \tau_x(C) d\tau_A(x) \quad (13)$$

$$= \int_A \tau_x(C_x) d\tau_A(x) = \tau(C),$$

where the two last equalities follow from (12) and (11), respectively. Since $(T, \Sigma, \varphi)$ is super-atomless, Corollary 1 in [15] provides a measurable function $h: T \to A \times B$ such that $h(t) \in \text{supp}(\phi(t))$ for almost all $t \in T$, and for all Borel sets $C \subseteq A \times B$,
\[ f_T \phi(t)(C) \, d\varphi(t) = \varphi(h^{-1}(C)). \] 

Thus, it follows from above that \( \tau = \varphi \circ h^{-1} \). For the function \( h \) we can write \( h = (e, f) \) with \( e = \text{proj}_A \circ h \) and \( f = \text{proj}_B \circ h \); in particular, both \( e \) and \( f \) are measurable and \( \tau = \varphi \circ (e, f)^{-1} \). Also, for almost all \( t \in T \), \( (e(t), f(t)) = h(t) \in \text{supp}(\phi(t)) = \text{supp}(\delta_{g(t)} \otimes \tau_{g(t)}) \subseteq \{ g(t) \} \times B \). Consequently \( e(t) = g(t) \) for almost all \( t \in T \), and hence \( \tau = \varphi \circ (g, f)^{-1} \). \( \blacksquare \)

Lemma 7 immediately implies the following characterization of equilibrium distributions in games with a super-atomless space of players.

**Corollary 8** Let \((T, \Sigma, \varphi)\) be a super-atomless probability space. Then, \( \tau \) is an equilibrium distribution of \( G = ((T, \Sigma, \varphi), V, X) \) if and only if there exists a Nash equilibrium \( f \) of \( G \) such that \( \tau = \varphi \circ (V, f)^{-1} \).

### 4.4 Proof of Theorem 1

We start by establishing that condition 1 implies condition 2. Let \( K \) be an equicontinuous subset of \( \mathcal{U} \) and \( \varepsilon > 0 \). Let \( \{x_1, \ldots, x_m\} \subseteq X \) be given according to Lemma 6, with \( \varepsilon \) there replaced by \( \varepsilon/2 \). Finally, let \( N \in \mathbb{N} \) be such that \( 2\omega_K(\frac{m^2+1}{n}) < \varepsilon/2 \) for all \( n \geq N \).

Let \( G_n \) be a game with a finite number of players with \( V_n(T_n) \subseteq K \) and \( n \geq N \). Consider the following game with a continuum of players: \( G = ([0, 1], \mathcal{B}([0, 1]), \lambda), V, X) \) where \( V(t) = V_n(i) \) if \( t \in [\frac{i-1}{n}, \frac{i}{n}) \) for all \( 1 \leq i \leq n-1 \) and \( V(t) = V_n(n) \) if \( t \in [\frac{n-1}{n}, 1] \).

\(^9\) Recall that, for all \( \delta > 0 \), \( \omega_K(\delta) = \sup_{V \in K} \sup \{|V(x, \pi) - V(y, \tau)| : \max\{d(x, y), \rho(\pi, \tau)\} \leq \delta\} \).
Note that $\lambda \circ V^{-1} = \nu_n \circ V_n^{-1} \in \mathcal{T}(\mathcal{U})$. By condition 1, $G$ has an equilibrium distribution $\tau$ such that $\text{supp}(\tau_X) \subseteq M$. Since $V([0,1])$ is a finite subset of $K$, it follows by Lemma 6 that $G$ has $\varepsilon/2$-equilibrium $f$ with $f([0,1]) \subseteq \{x_1, \ldots, x_m\}$.

By Lemma 5, there exists a $\varepsilon/2 + 2\omega_K((m^2 + 1)/n)$-equilibrium $f_n$ of $G_n$. Since $2\omega_K((m^2 + 1)/n) < \varepsilon/2$, then $f_n$ is an $\varepsilon$-equilibrium of $G_n$. This concludes the proof that condition 1 implies condition 2.

The same scheme can be used to prove that condition 1 implies condition 3. Let $\Gamma$ be a tight family of games with a finite set of players, $\varepsilon > 0$ and $\eta > 0$. Let $K$ be a compact subset of $\mathcal{U}$ satisfying $\nu_n \circ V_n^{-1}(K) > 1 - \eta$ for all $G_n \in \Gamma$. Then, let $\{x_1, \ldots, x_m\}$ be given according to Lemma 6, with $\varepsilon$ there replaced by $\varepsilon/2$. Following the same arguments used above, we can show that $V_n(t)(f(t), \nu_n \circ f^{-1}) \geq V_n(t)(x, \nu_n \circ (f \setminus t x)^{-1}) - \varepsilon$ for all $x \in X$ and all $t \in V_n^{-1}(K)$. Since, $\nu_n \circ V_n^{-1}(K) > 1 - \eta$ for all $G_n \in \Gamma$, the result follows.

We turn to the proof that condition 2 implies condition 1. Let $\mu \in \mathcal{T}(\mathcal{U})$ be a game with a continuum of players. Since $\mu$ is tight, it has a separable support. Hence, it follows by [13, Theorem II.6.3, p.44] that there exists a sequence $\{\mu_k\}_k \in \mathcal{T}(\mathcal{U})$ converging to $\mu$ such that $\text{supp}(\mu_k)$ is a finite subset of $\text{supp}(\mu)$ with $\mu_k(\{v\}) \in \mathbb{Q}$ for all $v \in \text{supp}(\mu_k)$ and $k \in \mathbb{N}$. For all $k$, let $\text{supp}(\mu_k) = \{V_k^1, \ldots, V_k^{L_k}\}$; also, let $t_k \in \mathbb{N}$ and, for all $1 \leq l \leq L_k$, $\beta_k^l \in \mathbb{N}$ be such that $\beta_k^l/t_k = \mu_k(\{V_k^l\})$.

Let $k \in \mathbb{N}$ be fixed. Then $\{V_k^l\}_{1 \leq l \leq L_k}$ is an equicontinuous subset of $\mathcal{U}$. Define, for all $\gamma \in \mathbb{N}$, a game $G_{\gamma t_k} = ((T_{\gamma t_k}, \nu_{\gamma t_k}), V_{\gamma t_k}, X)$ as follows: $G_{\gamma t_k}$ has $\gamma t_k$ players, each
has $X$ as his choice set and their payoff functions are defined in the following way: $V_{\gamma t_k} : T_{\gamma t_k} \to \mathcal{U}$ is such that it associates $V_k^l$ to $\gamma/\beta_k$ players, for all $1 \leq l \leq L_k$.

By condition 2, $G_{\gamma t_k}$ has a $1/k$-equilibrium $f_{\gamma t_k} : T_{\gamma t_k} \to M$ if we choose $\gamma_k$ sufficiently large. We may also choose $\gamma_k$ so that $\gamma_k t_k > k$, which implies that the sequence $\{\gamma_k t_k\}_{k=1}^\infty$ converges to infinity. Let $\tau_k = \nu_{\gamma t_k} \circ \left(V_{\gamma t_k}, f_{\gamma t_k}\right)^{-1} \in \mathcal{M}(\mathcal{U} \times X)$.

Since $\{\tau_{U,k}\}_k$ converges to $\mu$ and both $\mu$ and $\tau_{U,k}$ are tight for all $k$, it follows that $\{\mu, \tau_{U,1}, \tau_{U,2}, \ldots\}$, and so $\{\tau_{U,k}\}_k$ is tight by [3, Theorem 8, p. 241]. Also, since $M$ is compact and $\text{supp}(\tau_{k,X}) \subseteq M$ for all $k$, then $\{\tau_{X,1}, \tau_{X,2}, \ldots\}$ is tight. Thus, $\{\tau_k\}_k$ is tight ([3, Exercise 6, p. 41]) and, taking a subsequence if necessary, we may assume that $\{\tau_k\}$ converges ([3, Theorem 6.1, p. 37]). Let $\tau = \lim_k \tau_k$. Then, by Lemma 11, it follows that $\tau$ is an equilibrium distribution of $\tau_U = \mu$. Furthermore, $\tau_X(M) \geq \limsup_k \tau_{k,X}(M) = 1$ and so $\text{supp}(\tau_X) \subseteq M$.

Similarly, we show that condition 3 implies condition 1. We can use the same argument used in the proof that condition 2 implies condition 1, except that $f_{\gamma t_k}$ is only a $(1/k, 1/k)$-equilibrium of $G_{\gamma t_k}$. However, Lemma 11 still applies and the conclusion follows.

4.5 Proof of Theorem 2

It follows from Corollary 8 that condition 1 implies condition 2. So, it suffices to show that condition 2 implies condition 1. Let $\mu \in \mathcal{T}(\mathcal{U})$ be a game with a continuum of players. By [20, Theorem 3.1.1, p. 281], there exists a Borel measurable function
$V : [0, 1] \to \mathcal{U}$ such that $\mu = \lambda \circ V^{-1}$ (recall that $\lambda$ denotes the Lebesgue measure). By [15, Appendix], there exists a super-atomless measure $\varphi$ on $[0, 1]$ such that, denoting by $\Sigma$ the domain of $\varphi$, $\mathcal{B}([0, 1]) \subseteq \Sigma$ and $\varphi$ agrees with $\lambda$ on $\mathcal{B}([0, 1])$. Clearly, $V$ is still measurable when $\mathcal{B}([0, 1])$ is replaced by $\Sigma$ and $\mu = \varphi \circ V^{-1}$. Indeed, for all measurable $C \subseteq \mathcal{U}$, $V^{-1}(C) \in \mathcal{B}([0, 1])$, and so $\mu(C) = \lambda(V^{-1}(C)) = \varphi(V^{-1}(C)) = \varphi \circ V^{-1}(C)$. By condition 2, $G = (([0, 1], \Sigma, \varphi), V, X)$ has a Nash equilibrium $f$. Hence, $\tau = \varphi \circ (V, f)^{-1}$ is an equilibrium distribution of $\mu$.

4.6 Proof of Theorem 3

Note that condition 2 trivially implies condition 3. Thus, it suffices to prove that condition 1 implies condition 2 and that condition 3 implies condition 1.

Assume that condition 1 holds. Let $G = ([0, 1], \mathcal{B}([0, 1]), \lambda), V, X)$ be a game with a continuum of players with $V([0, 1]) \subseteq \mathcal{U}$ and $C$ be a countable, closed subset of $X$ such that $M \cap C \neq \emptyset$. By condition 1, there exists an equilibrium distribution $\tau$ of $\mu = \lambda \circ V^{-1}$ relative to $C$. Note that $\text{supp}(\tau_X) \subseteq C$, since $C$ is closed. It then follows by [8, Theorem 2] that there exists a Nash equilibrium $f$ of $G$ relative to $C$ such that $\tau = \lambda \circ (V, f)^{-1}$. Thus, condition 2 holds.

Assume that condition 3 holds. Let $\mu \in \mathcal{T}(\mathcal{U})$ be a game with a continuum of players and let $S$ be a closed subset of $X$ such that $M \cap S \neq \emptyset$. Let $C = \{x_m\}_{m=1}^{\infty}$ be a countable dense subset of $S$ such that $x_1 \in M \cap S$. We claim that it suffices to establish that there exists an equilibrium distribution $\tau$ of $\mu$ relative to $C$ with
supp(τ_X) ⊆ M. In fact, given such a distribution τ, since C is dense in S, it follows that supp(τ_X) ⊆ C = S and that the set \{ (u, x) \in U \times X : u(x, \tau_X) \geq u(S, \tau_X) \} is equal to \{ (u, x) \in U \times X : u(x, \tau_X) \geq u(C, \tau_X) \}. Hence, τ(\{ (u, x) \in U \times X : u(x, \tau_X) \geq u(S, \tau_X) \}) = 1 and so τ is an equilibrium distribution of µ relative to S.

We then establish the existence of an equilibrium distribution τ of µ relative to C. Since supp(µ) is a closed (hence, complete) and separable subset of U, it follows by [20, Theorem 3.1.1, p. 281] that there exists a Borel measurable function V : [0, 1] → U such that µ = λ ◦ V^{-1} and so we can represent the game µ by G = (\([0, 1], \mathcal{B}([0, 1]), \lambda\), V, X). For all k ∈ N, define F_k = \{ x_1, \ldots, x_k \} and note that M \cap F_k ≠ ∅ since x_1 ∈ M \cap F_k. By condition 3, for all k, there exists a Nash equilibrium f_k of G relative to F_k with supp(λ ◦ f_k^{-1}) ⊆ M. Let τ_k = λ ◦ (V, f_k)^{-1} for all k. Since τ_{k,l} = λ ◦ V^{-1} and supp(τ_{k,X}) ⊆ C \cap M for all k, it follows that the sequence \{ τ_k \} is tight and so, taking a subsequence if necessary, we may assume that it converges. Let τ = \lim_k τ_k. Clearly, supp(τ_X) ⊆ C \cap M. We next establish that τ is an equilibrium distribution of µ relative to C.

Let (u, x) ∈ supp(τ) and m ∈ N. For all k ∈ N, define A_k = \{ (u, x) \in U \times X : u(x, \tau_{k,X}) < u(y, \tau_{k,X}) \} for some y ∈ F_k. Since τ_k is an equilibrium distribution of G relative to F_k, it follows that τ_k(A_k) = 0 for all k ∈ N. By Lemma 12 in the Appendix, there exists a subsequence \{ τ_{k_j} \}_{j=1}^\infty of \{ τ_k \}_{k=1}^\infty and, for each j ∈ N, \( (u_j, x_j) \in \text{supp}(τ_{k_j}) \setminus A_{k_j} \) such that \lim_j (u_j, x_j) = (u, x). Then, \( u_j(x_j, \tau_{k_j,X}) \geq u_j(x_m, \tau_{k_j,X}) \) for all j ∈ N such that k_j ≥ m. Since \( \lim_j u_j = u, \lim_j \tau_{k_j} = \tau, \text{ and } \lim_j x_j = x \), it follows that u(x, \tau_X) ≥ u(x_m, \tau_X). Since (u, x) ∈ supp(τ) and m ∈ N were chosen arbitrarily,
it follows that \( \text{supp}(\tau) \subseteq \{(u, x) \in U \times X : u(x, \tau_X) \geq u(x_m, \tau_X) \text{ for all } m \in \mathbb{N}\} \) and so \( \tau \) is an equilibrium distribution of \( \mu \) relative to \( C \).

5 Concluding Remarks

In this paper, we have considered several existence results for large games with the purpose of establishing their equivalence. In our view, such equivalence is important since it expresses the close relationship between the different formalizations of large games and their corresponding equilibrium notions. In particular, all the existence results are equally strong and so none of the formalizations we consider should be regarded as better suited to address the existence problem of large games.

Furthermore, our equivalence results also imply that the relative strengths and weaknesses of the different equilibrium concepts and formalizations are more apparent than substantial. In fact, as their proofs make clear, it is possible to obtain an equilibrium in one model using an equilibrium (or a sequence of equilibria) in another one. Thus, a critique (resp. praise) to a particular equilibrium concept in some given formalization, implies a critique (resp. praise) to all the other equilibrium concepts (from which the original one can be obtained).

To illustrate this point consider a game with an uncountable and compact action space. Then, an equilibrium strategy will exist if the space of players is super-atomless. However, an equilibrium distribution for this game exists under exactly the same measurability and compactness assumptions. Furthermore, as noted in the remarks
following Theorem 3, all games with the same distribution of players’ characteristics as the original game but with a Lebesgue space of players have an equilibrium relative to a (finitely) discretised action space, and, by choosing the discrete action space close to the original one, the distribution of these strategies can be made arbitrarily close to an equilibrium distribution of the original game. Thus, as long as one is only interested in the strategic behavior displayed in such equilibrium, all three methods of reaching such equilibrium should deserve the same appraisal.

In conclusion, as far as the existence problem is concerned, all the formalizations of large games that we have considered should be regarded as equivalent and the choice of which one to use in practice regarded as a matter of taste and convenience. In particular, anyone interested in using large games in applications does not have to worry about which formalization is the most appropriate, but rather choose the one that he or she feels more comfortable with. Furthermore, our results provide a mean to compare the results of theoretical and applied models that have been formalized in a different way.

A Appendix

In this appendix, we prove several results needed for our main results. Lemma 9 deals with measures with a finite support, which can be thought of as a vector in some Euclidean space. Roughly, Lemma 9 says that the Prohorov distance between two measures whose support is contained in some finite set is proportional to their sup
norm as vector in such an Euclidean space.¹⁰

**Lemma 9** Let \( \tau, \mu \in \mathcal{M}(X) \) be such that \( \text{supp}(\tau) \cup \text{supp}(\mu) \subseteq \Psi \), where \( \Psi \) is a finite set. If there exists \( \varepsilon > 0 \) such that \( |\tau_l - \mu_l| \leq \varepsilon \) for all \( 1 \leq l \leq |\Psi| \), then \( \rho(\tau, \mu) \leq |\Psi|\varepsilon \).

**Proof.** Let \( \varepsilon > 0 \) and \( B \subseteq X \) be Borel measurable. Then,

\[
\tau(B) = \sum_{l \in \Psi \cap B} \tau_l \leq \sum_{l \in \Psi \cap B} (\mu_l + \varepsilon) \leq \sum_{l \in \Psi \cap B} \mu_l + |\Psi|\varepsilon \leq \mu(\overline{B}_{|\Psi|\varepsilon}(B)) + |\Psi|\varepsilon. \tag{A.1}
\]

Similarly, we can show that \( \mu(B) \leq \tau(\overline{B}_{|\Psi|\varepsilon}(B)) + |\Psi|\varepsilon \). This implies that \( \rho(\tau, \mu) \leq |\Psi|\varepsilon \). □

Lemma 10 shows that in large games, deviations by a small fraction of players have a small impact on the distribution of actions.

**Lemma 10** Let \( G_n \) be a game with a finite number of players and let \( f \) and \( g \) be strategies. If \( |\{ t \in T_n : f(t) \neq g(t)\}|/n \leq \gamma \), then \( \rho(\nu_n \circ f^{-1}, \nu_n \circ g^{-1}) \leq \gamma \).

**Proof.** Let \( \mu = \nu_n \circ f^{-1} \) and \( \tau = \nu_n \circ g^{-1} \). Let \( B \subseteq X \) be Borel measurable. Then,

\[
\tau(B) = \frac{|\{ t : g(t) \in B \}|}{n} \leq \frac{|\{ t : f(t) \in B \}|}{n} + \frac{|\{ t : f(t) \neq g(t)\}|}{n} = \mu(B) + \frac{|\{ t : f(t) \neq g(t)\}|}{n} \leq \mu(\overline{B}_\gamma(B)) + \gamma. \tag{A.2}
\]

Similarly, we can show that \( \mu(B) \leq \tau(\overline{B}_\gamma(B)) + \gamma \). This implies that \( \rho(\tau, \mu) \leq \gamma \). □

In particular, we have that \( \rho(\nu_n \circ f^{-1}, \nu_n \circ (f \setminus t x)^{-1}) \leq 1/n \) for all strategies \( f \), players \( t \in T_n \) and actions \( x \in X \).

¹⁰ In the statement and the proof of this lemma we use the notational conventions introduced prior to the statement of Lemma 5.
Lemma 11 draws conclusions for games with a continuum of players from properties of large finite games and was used in the proof of Lemma 5. It considers a more general case in which a game $G_n = ((T_n, \Sigma_n, \nu_n), V_n, X)$ with finitely many players has $|T_n|$ players (not necessarily equal to $n$), and $\nu_n$ is the uniform measure on $T_n$.

**Lemma 11** Let $G = ((T, \Sigma, \phi), V, X)$ be a game with a continuum of players, $\tau$ be a distribution on $C \times X$ satisfying $\tau_C = \lambda \circ V^{-1} \in \mathcal{T}(C)$ and $\varepsilon \geq 0$. Suppose that $\{G_n\}_{n=1}^\infty$ is a sequence of games with a finite number of players and $\{f_n\}_{n=1}^\infty$ is a sequence of strategies such that $|T_n| \to \infty$, $f_n$ is an $(\varepsilon_n, \eta_n)$-equilibrium of $G_n$ for all $n$, $\lim_n \varepsilon_n = \varepsilon$, $\lim_n \eta_n = 0$ and $\lim_n \nu_n \circ (V_n, f_n)^{-1} = \tau$, then $\tau$ is an $\varepsilon$-equilibrium distribution of $G$.

This lemma can be established using an argument analogous to the one employed to prove that condition 3 implies 2 in Theorem 3. Both results rely on the following lemma.

**Lemma 12** Let $Z$ be a metric space, $\{\tau_k\}_{k=1}^\infty$ be a sequence in $\mathcal{T}(Z)$ converging to $\tau \in \mathcal{T}(Z)$, and $\{A_k\}_{k=1}^\infty$ be a sequence of Borel subsets of $Z$ with $\lim_k \tau_k(A_k) = 0$. Then, for all $z \in \text{supp}(\tau)$, there exists a subsequence $\{\tau_{k_j}\}_{j=1}^\infty$ of $\{\tau_k\}_{k=1}^\infty$ and an element $z_j \in \text{supp}(\tau_{k_j}) \setminus A_{k_j}$ for all $j \in \mathbb{N}$ such that $\lim_j z_j = z$.

**Proof.** Let $z \in \text{supp}(\tau)$ and suppose the assertion were false. Then there is an open neighborhood $U$ of $z$ such that $U \cap (\text{supp}(\tau_k) \setminus A_k) = \emptyset$ for all sufficiently large $k$. Thus, $U \cap \text{supp}(\tau_k) \subseteq A_k$ for all sufficiently large $k$ and hence $\lim_k \tau_k(U \cap \text{supp}(\tau_k)) = 0$. Since $\tau_k(U) = \tau_k(U \cap \text{supp}(\tau_k)) + \tau_k(U \setminus \text{supp}(\tau_k)) = \tau_k(U \cap \text{supp}(\tau_k))$, it follows that
0 \leq \tau(U) \leq \liminf_k \tau_k(U) = \lim_k \tau_k(U) = 0. \text{ Hence, } \tau(U) = 0, \text{ contradicting the hypothesis that } z \in \text{supp}(\tau). \blacksquare

References


