Information Theoretic Uplink Capacity of the Linear Cellular Array

S. Chatzinotas, M. A. Imran, C. Tzaras
Centre for Communication Systems Research
University of Surrey, United Kingdom, GU2 7XH
Email: {S(Chatzinotas, M.Imran, C.Tzaras}@surrey.ac.uk

Abstract—In the information-theoretic literature, the Wyner’s model has been the starting point for studying the capacity limits of cellular systems. Wyner’s simple infinite cellular model was adopted and extended by researchers in order to incorporate flat fading environments and power-law path loss models. However, the majority of these extensions have preserved a fundamental assumption of Wyner’s model, namely the co-location of User Terminals (UTs). In this paper, we alleviate this assumption and we evaluate the effect of user distribution on the optimal sum-rate capacity. The model under investigation is a Gaussian Cellular Multiple Access Channel over a linear cellular array in the presence of power-law path loss and Rayleigh flat fading. In this context, we show that the effect of user distribution is only considerable in the regime of low cell density and we argue that “collocated” models can be utilized to approximate the “distributed” ones in the high cell density regime. Subsequently, we provide analytical formulas for the calculation of the interference factors of “collocated” models given the cellular system parameters. In addition, the asymptotics of information-theoretic cellular sum-rate capacity are investigated. Finally, the presented results are interpreted in the context of practical cellular systems using appropriate figures of merit.

I. INTRODUCTION

The first concrete result for the information-theoretic capacity of the Gaussian Cellular Multiple Access Channel (GCMAC) was presented by Wyner in [7]. Using a very simple but tractable model for the cellular uplink channel, Wyner showed the importance of joint decoding at the BS receivers (hyper-receiver) and found the analytical formulas of the maximum system capacity under the assumption of hyper-receiver. This model triggered the interest of the research community in the cellular capacity limits and was extended in [5] to include flat fading environments. One major assumption shared in these two models was that the cell density is fixed and only physically adjacent cells interfere. Letzepis in [3], extended the model by assuming multiple-tier interference and incorporating a distance-dependent path loss factor in order to study the effect of cell density. However, the assumption of co-location of all users in a single cell was maintained to keep the model tractable.

The work reported in this paper has formed part of the “Fundamental Limits to Wireless Network Capacity” Elective Research Programme of the Virtual Centre of Excellence in Mobile & Personal Communications, Mobile VCE, www.mobilevce.com. This research has been funded by the following Industrial Companies who are Members of Mobile VCE - BBC, BT, Huawei, Nokia, Nokia Siemens Networks, Nortel, Vodafone. Fully detailed technical reports on this research are available to staff from these Industrial Members of Mobile VCE.

In this paper, we extend these models in order to incorporate the effect of user distribution. Instead of assuming co-located users, we assume that users are spatially distributed within the cell and each channel gain is affected by a distance-dependent path loss factor. To relate the models involving co-located users with our model, we propose an approach to calculate (rather than arbitrarily vary) the interference factors to be used in models assuming co-located users. The calculation of the interference factors is based on the cellular system parameters, such as cell density, path loss exponent and user distribution. Finally, the presented results are interpreted in the context of practical cellular systems using appropriate figures of merit.

The rest of the paper is organised as follows. In the next section, we describe the proposed model and we describe the derivation of the information theoretic capacity of the cellular system. In section III, we show how interference factors can be calculated, followed by the study of the asymptotic capacity of the system. In section IV, the presented results are interpreted in the context of practical cellular systems. The last section concludes the paper.

II. MODEL DESCRIPTION AND ANALYSIS

A. Notation

In the following formulations, $D$ is the coverage range of the linear cellular system, $N$ is the number of BSs, $K$ is the number of UTs per cell and $\eta$ is the power-law path loss exponent. Under these assumptions, $\Pi = N/D$ represents the cell density of the cellular system, $\varnothing = \Pi^{-1} = D/N$ represents the cell diameter of the cellular system and $R = \varnothing/2$ represents the cell radius. Throughout this paper, $E[\cdot]$ denotes the expectation, $(\cdot)^*$ denotes the complex conjugate, $(\cdot)^T$ denotes the Hermitian matrix and $\circ$ denotes the Hadamard product. $\cap (t/T)$ is the rect function, where $T$ is the width of the pulse. The figure of merit studied in this paper is the per-cell sum-rate capacity achieved with optimal decoding and it is denoted by $C_{opt}$.

B. Model

The model under consideration is a linear circular cellular array under power-law path loss and flat fading. The analysis of the planar cellular array can be found in [1]. If $K$ users are uniformly distributed in each cell of a system comprising $N$ base stations distributed in a span of $D$. The received signal
at cell $n$, at time index $i$, is given by:

$$y_i^n = \sum_{k=1}^{K} b_{k,i} x_{k,i}^{n} + \sum_{j=1}^{N/2} \sum_{k=1}^{K} \alpha_{ij}(k) \left( e_{k,i,j}^{n} x_{k,i,j}^{n-j} + d_{k,i} x_{k,i}^{n+j} \right) + z_i^n$$

(1)

where $x_{k,i}^{n}$ is the $i$th complex channel symbol of the $k$th UT in the $n$th cell and $\{b_{k,i}\}, \{e_{k,i,j}\}, \{d_{k,i}\}$ are independent, strictly stationary and ergodic complex random processes in the time index $i$, which represent the flat fading processes experienced by the UTs. The fading coefficients are assumed to have unit power, i.e. $\mathbb{E}[b_{k,i} b_{l,i}^*] = \mathbb{E}[e_{k,i,j} e_{l,i,j}^*] = \mathbb{E}[d_{k,i} d_{l,i}^*] = 1$ and all UTs are subject to an average power constraint, i.e. $\mathbb{E}[|x_{k,i}^{n}|^2] \leq P$ for all $(n,k,i)$. The interference factors $\alpha_{ij}(k)$ of the $k$th user in cell indexed by $n-j$ and $n+j$, are calculated according to the modified power-law path loss model [3], [4]:

$$\alpha_{ij}(k) = (1 + d_{ij}(k))^{-\eta/2}.$$  

(2)

The average distance $d_{ij}(k)$ can be calculated for all users using the fact that the users are assumed to be distributed on a regular grid. Dropping the time index $i$, the aforementioned model can be more compactly expressed as a vector memoryless channel of the form $y = \mathbf{H}x + z$, where $\mathbf{y} = [y^\prime \ldots y^N]^T$ represents received signals by the BSs, the vector $\mathbf{x} = [x^1 \ldots x^K]^T$ represents transmit signals by all the UTs of the cellular system and the components of vector $z = [z^1 \ldots z^N]^T$ are i.i.d. random variables representing AWGN with $\mathbb{E}[z^n] = 0$, $\mathbb{E}[z^n z^m]^* = \sigma^2$. The channel Matrix $\mathbf{H}$ can be written as, $\mathbf{H} = \Sigma \otimes \sqrt{N} \mathbf{G}$, where $\Sigma$ is an $N \times KN$ deterministic matrix and $\mathbf{G}$ is a standard complex Gaussian $N \times KN$ matrix with variance $1/N$, comprising the corresponding fading coefficients. The entries of the $\Sigma$ matrix are defined by the variance profile function

$$\varsigma(r,t) = (1 + d(r,t))^\eta/2$$

(3)

where $r \in [0,1]$ and $t \in [0,K]$ are the normalized indexes for the BSs and the UTs respectively and $d(r,t)$ is the normalized distance between BS $r$ and user $t$.

According to [6], the asymptotic sum-rate capacity $C_{\text{opt}}$ for this model, is given by

$$\lim_{N \to \infty} C_{\text{opt}}(\gamma) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \sum_{i=1}^{N} \log (1 + \lambda_i (\mathbf{H} \mathbf{H}^H)) \right]$$

$$= \int_{0}^{\infty} \log(1 + \gamma x) dF_{\mathbf{HH}^H}(x)$$

$$= \mathcal{V}_{\mathbf{HH}^H}(\gamma) = K \mathcal{V}_{\mathbf{HH}^H}(\gamma)$$

(4)

where $\lambda_i (\mathbf{HH}^H)$ are the eigenvalues of matrix $\mathbf{HH}^H$ and

$$\mathcal{V}_{\mathbf{X}}(\gamma) \triangleq \mathbb{E} \left[ \log(1 + \gamma \lambda_i (\mathbf{X})) \right]$$

$$= \int_{0}^{\infty} \log(1 + \gamma \lambda_i (\mathbf{X})) dF_{\mathbf{X}}(x)$$

(5)

is the Shannon transform [6] of a random square Hermitian matrix $\mathbf{X}$, whose limiting eigenvalue distribution has a cumulative function denoted by $F_{\mathbf{X}}(x)$. Assuming that $\mathbf{H} = \mathbf{G}$ (all components of matrix $\Sigma$ are unity), the empirical eigenvalue distribution of $\mathbf{HH}^H$ converges almost surely (a.s.) to the nonrandom limiting eigenvalue distribution of the Marčenko-Pastur law, whose Shannon transform is given by

$$\mathcal{V}_{\mathbf{HH}^H}(\gamma) \Rightarrow \mathcal{V}_{\mathbf{HH}^H}(\gamma)$$

(6)

where $\mathcal{V}_{\mathbf{HH}^H}(\gamma) = \log \left( 1 + \frac{\gamma}{\alpha} \right) + \frac{1}{\alpha} \log \left( 1 + \frac{1}{\alpha} \right) - \frac{1}{\alpha} \gamma$ and $\alpha = K/\gamma K$.

(7)

$$\phi(\gamma,K) = \left( \gamma + K \right) + \frac{1}{K} \gamma$$

(8)

C. Marčenko-Pastur Law Approximation

According to the Marčenko-Pastur Law approximation in [3], if $\Sigma$ is a path loss dependent $N \times KN$ deterministic matrix, the limiting eigenvalue distribution of $\mathbf{HH}^H$ and its Shannon transform can be approximated by a scaled version of the Marčenko-Pastur law

$$\mathcal{V}_{\mathbf{HH}^H}(\gamma) \Rightarrow \mathcal{V}_{\mathbf{HH}^H}(q_K(\Sigma)\gamma/K)$$

(9)

where $q_K(\Sigma) \triangleq \|\Sigma\|^2 / KN$ with $\|\Sigma\| \triangleq \sqrt{\text{Tr} \{\Sigma \Sigma^H\}}$ being the Frobenius norm of the $\Sigma$ matrix assuming $K$ UTs per cell and $\gamma = K\gamma K$ is the system power. In the asymptotic case and noticing that $\varsigma(r,t)$ is symmetric about $r = Kt$, $q_K(\Sigma)$ is given by

$$\lim_{N \to \infty} q_K(\Sigma) = \frac{1}{K} \int_{0}^{K/2} \varsigma^2(t)dt, \forall t \in [0,1].$$

(10)

According to [3], this approximation holds for UTs collocated with the BS. In [2] we show that this approximation also holds for the case where the users are distributed within the cells. In this paper, we show how this model can be used to calculate the appropriate values of the path loss factors used in the collocated-users models. We also study the asymptotic behaviour of the cellular capacity using this model and we interpret the presented results in the context of practical cellular systems.

III. RESULTS

In a power-law path loss environment, the variance profile function of linear circular cellular array will be symmetric about the axis $t = K/2$ and hence Equation (10) can be further simplified to

$$\lim_{N \to \infty} q_K(\Sigma) = \frac{2}{K} \int_{0}^{K/2} \varsigma^2 (d(t)) dt$$

(11)
\[
\sum_{j=1}^{M} \alpha_j^2 = \frac{N}{K} \int_0^{K/2} \zeta^2 \left( \frac{1}{2\pi} \int \left( \frac{1}{2\pi} \int \left( N^2 \frac{2}{N} t_i + \frac{1}{N^2} \right) \right) \right) dt - \frac{1}{2}, \text{ where } t_i = t - \frac{K}{N} \tag{A}
\]

User distribution effectively alters \( d(t) \) and therefore it modifies the variance profile function and the resulting sum-rate capacity given by Equation (4) and (9).

**Theorem 1 (from [2] j):** Let us assume that the transmitters of each cell are positioned on a grid generated according to an invertible Cumulative Distribution Function (CDF) \( F_u(r) \), where \( r \in [0, 1] \) corresponds to the normalized single-cell distance from the BS. If \( \zeta(d(t)) \) is the variance profile function w.r.t. the normalized index \( t \), then the variance profile function \( \zeta(t) \) w.r.t. the normalized distance of the distributed users is given by

\[
\zeta(t) = \left( \sum_{i=1}^{K} \left( \frac{1}{2\pi} \int F_u^{-1}(N^2 \frac{2}{N} t_i + \frac{1}{N^2}) \right) \right) \left( \frac{t_i}{K/N} \right) dt - \frac{1}{2}, \text{ where } t_i = t - \frac{K}{N} \tag{12}
\]

where \( t \in [0, K/2] \).

**A. Interference Factors**

On the grounds of Theorem 1, the interference factors \( \alpha_j \) in the high cell density regime can be calculated based on the cellular system parameters, namely the number of BSs \( N \), the number of UTs per cell \( K \), the power-law path loss exponent \( \eta \), the cell density \( \varnothing \) and the user distribution CDF \( F_u(d) \).

**Corollary 1:** In the high cell density regime, the “distributed” cellular model can be represented by the “collocated” one, using the interference factors \( \alpha_j \) given by Equation (A) at the top of the page, where \( M \in [1, N/2] \) denotes the number of interfering neighboring cells taken into account.

**Proof:** Based on the Marčenko-Pastur approximation in [3] of Somekh-Shamai’s model [5], the sum rate capacity is given by

\[
\lim_{N \to \infty} C_{opt}(\gamma) \approx \text{K}/V_FK \left( \frac{1 + 2 \alpha^2}{\gamma} \right). \tag{13}
\]

By considering interference from \( M \) tiers and by following the same derivation as in [3], it can be easily proved that the sum rate capacity is given by

\[
\lim_{N \to \infty} C_{opt}(\gamma) \approx \text{K}/V_FK \left( \frac{1 + \sum_{j=1}^{M} \alpha_j^2}{\gamma} \right). \tag{14}
\]

By combining Equations (13) and (14), the interference factors can be calculated by using Equation (A) recursively.

**Corollary 2:** As shown in [2], the sum-rate capacities for the “collocated” and the “distributed” models converge in the high cell density regime. Therefore, the “collocated” model can be used to approximate the “distributed” one, by approximating the interference factors \( \alpha_j \) using Equation (2)

\[
\alpha_j \approx (1 + j/\Pi)^{-\eta/2}. \tag{15}
\]

**B. Asymptotics of Sum-rate Capacity**

In order to study the asymptotics of the per-cell sum-rate capacity, the cell density \( \Pi = N/D \) and the cell diameter \( \varnothing = \Pi^{-1} = D/N \) of the cellular system are kept constant, while both \( N \) and \( D \) grow large. Considering the uniform user distribution, the sum-rate capacity is given by

\[
\lim_{N,D \to \infty} C_{opt}(\gamma) = K/VI_FK \left( \frac{\gamma}{(\eta - 1/R)} \right). \tag{17}
\]

According to [6], the asymptotic of the Shannon transform for \( K > 1 \) is given by

\[
\lim_{x \to \infty} K/VI_FK(x) = \log(Kx) - (K - 1)\log(K - 1/K) - 1. \tag{18}
\]

Therefore, for large values of \( \gamma \) the asymptotic sum-rate capacity is given by combining Equations (16) and (18),

\[
\lim_{N,D,\gamma \to \infty} C_{opt}(\gamma) = \log \left( \frac{\gamma K}{(\eta - 1/R)} \right) - (K - 1)\log(K - 1/K) - 1. \tag{19}
\]

Furthermore, the asymptotic sum-rate capacity for a very large number of users per cell converges to

\[
\lim_{N,D,\gamma,K \to \infty} C_{opt}(\gamma) = \log \left( \frac{\gamma K}{(\eta - 1/R)} \right). \tag{20}
\]

since \( \lim_{K \to \infty} \left( 1 + \frac{1}{K} \right)^K = e. \)
IV. PRACTICAL CONSIDERATIONS

A. Path loss denormalization

The employed power-law path loss model of Equation (3) provides a variance profile coefficient as a function of the normalized distance $d(t)$. Similar path-loss models have been already utilized in the information-theoretic literature [3], [4]. However, in order to apply the aforementioned results to real-world cellular systems, a reference distance $d_0$ is required to interconnect the normalized distance $d(t)$ and the actual distance $d(t)$. Assuming that the power loss at the reference distance $d_0$ is $L_0$, the scaled variance profile function is given by

$$\varsigma(d(t)) = \sqrt{L_0 \left(1 + \frac{d(t)}{d_0}\right)^{-\eta}}.$$  \hfill (21)

In the context of a macro-cellular scenario, the typical parameters of Table I will be considered. Figures 1 and 2 depict the per-cell capacity of the linear cellular system versus the cell radius $R$ and the UT transmit power $P_T$ respectively.

B. Figures of Merit

In the practical engineering design of cellular systems, the main figure of merit that determines the capacity rate of a UT is the $SINR = \frac{P_T}{P_T + N_R}$, where $P_T$ is the received power at the BS of interest, $N_R$ is the thermal AWGN at the receiving BS and $I$ is the inter-cell and intra-cell interference received from other UTs of the system. However, in the information-theoretic analysis of hyper-receiver cellular systems, the main figure of merit that determines the per-cell capacity is $\gamma = \frac{P_T}{N_{HR}}$, where $P_T$ is the transmit power of the UT and $N_{HR}$ is the AWGN thermal noise at the hyper-receiver. The main reason that $SINR$ does not constitute an appropriate figure of merit for multi-cell joint processing analysis is that inter-cell interference is not harmful and thus the term $I$ can be added to the nominator. Since there is no harmful interference, there is no need for power control and thus the UTs constantly transmit with the maximum available power $P_T$. In this context, the transmit power $P_T$ remains fixed for all the UTs, whereas the received power differs for each UT. In addition, since the objective function is the per-cell capacity, the power variable affecting the value of this function should have a constant value throughout the whole cell. Taking this into account, it is reasonable to calculate the per-cell capacity as a function of $P_T$ or $\gamma$, which is a fixed system parameter, common for all the UTs of a cell.

In order to compare the capacity performance of hyper-receiver and practical cellular systems, a common figure of merit is required, which can be used effectively for both cases. In this context, three approaches which are described in the following paragraphs can be employed. For each approach, the per-cell capacity will be evaluated based on Equation (20) for the aforementioned macro-cellular scenario.

1) Cell-edge SNR: In practical engineering design of cellular systems, the objective is to provide network coverage to all the subscribers. Therefore, the cellular system has to be designed in a way that it even allows cell-edge UTs to communicate effectively with the receiving BS. Thus, it would be reasonable to consider the cell-edge $SINR$ as the figure of merit that determines the per-cell capacity. Assuming that $N_{HR} = N_R = N_c$, the cell-edge $SINR$ can be defined as:

$$SINR_{CE} = \gamma_0 (1 + R)^{-\eta}.$$  \hfill (22)

Figure 3 depicts the per cell capacity vs. cell-edge $SINR$.

2) Average cell SNR: A second figure of merit which could be used to determine the per-cell capacity is the average cell $SINR$, which is defined as the average of the received SNRs of all the UTs in a cell. Assuming that $N_{HR} = N_R = N_c$ and uniformly distributed UTs, the cell-edge $SINR$ can be defined as

$$SINR_{AC} = 2\gamma_0 \int_0^R (1 + r)^{-\eta} dr.$$  \hfill (23)

Figure 4 depicts the per cell capacity vs. average cell $SINR$.

3) Rise over Thermal: In hyper-receiver cellular networks, the Rise over Thermal (RoT) of the system is defined as the ratio of the total signal power received from all the UTs of the system at the hyper-receiver to the thermal AWGN. More
specifically, assuming uniformly distributed UTs, RoT is given by:

$$RoT = 2\gamma L_0 \int_0^{D/2} (1 + r)^{-\eta} \, dr.$$  \hfill (24)

For an infinite cellular array, the coverage span $D$ grows to infinity and therefore

$$RoT = \gamma \cdot 2 \int_0^{\infty} L_0 (1 + r)^{-\eta} \, dr = \frac{2\gamma L_0}{n-1}.$$  \hfill (25)

Figure 5 depicts the per cell capacity vs. $RoT$. The $RoT$ curves (thick lines) have been drawn on top of the log(1 + $x$) curve (thin line).

V. Conclusion

The already existing information-theoretic models for cellular systems are based on the assumption that the users of each cell are collocated. In this paper, we have investigated the optimal information-theoretic capacity under the assumption of distributed users. Based on the presented results, we can conclude that a cellular model assuming distributed users can be approximated by a model assuming collocated users only in the high cell density regime. In this case, we have proposed an approach for calculating the interference factors of the “equivalent” collocated model based on the system’s parameters. Furthermore, the asymptotic cellular capacity has been studied and plotted by varying the path loss exponent and the user transmit power. Finally, the presented results were interpreted in the context of practical cellular systems using appropriate figures of merit, such as Rise over Thermal.

ACKNOWLEDGMENT

The work reported in this paper has formed part of the “Fundamental Limits to Wireless Network Capacity” Elective Research Programme of the Virtual Centre of Excellence in Mobile & Personal Communications, Mobile VCE, www.mobilevce.com. This research has been funded by the following Industrial Companies who are Members of Mobile VCE - BBC, BT, Huawei, Nokia, Nokia Siemens Networks, Nortel, Vodafone. Fully detailed technical reports on this research are available to staff from these Industrial Members of Mobile VCE. The authors would like to thank Prof. G. Caire and Prof. D. Tse for the useful discussions.

REFERENCES