DAMPING CHARACTERISTICS OF MERO-TYPE DOUBLE LAYER GRIDS

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بنام خداوند جان و خرد گذشته کرده،

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تقدیم به همسر عزیزم مرام
و فرزندان دلیل دفاعی، عاطفه و فاطمه،
آنونکه در این راه همراه من بودند و سخاوتنمدادند
مرا و امدادارم حبیب بهریغشا کردن.
ABSTRACT

Damping is a phenomenon in mechanical systems by which the vibrational energy is absorbed and dissipated during oscillation. Much research effort has gone into the investigation of damping since 250 years ago. However, the complexity of damping phenomenon has prevented a complete understanding of the mechanisms by which the vibrational energy is dissipated.

It is important to be able to estimate the amount of damping in structural systems, since this plays a major role in their dynamic behaviour. The most reliable results regarding damping in structures are obtained from dynamic experiments on structures. A number of test methods can be employed for the measurement of damping capacity of structures.

There are three different types of damping. These are, 'viscous damping', 'Coulomb damping' and 'hysteretic (material) damping'. Viscous damping is the resistance offered to a moving body in a fluid. Viscous damping is the most common type of damping. Coulomb damping is caused by friction between surfaces which slide with respect to each other. Material damping is due to friction between the internal planes of a material. The damping capacity of a structure varies due to the variation of structural conditions such as initial strains and stiffness as well as amplitude and frequency of vibration.

The present research experimentally studies the variation of damping in a MERO-type double layer grid due to the variation in the bolt tightness of the connectors. To carry out this study a 10m x 10m MERO-type double layer grid and the necessary equipment including a loading system and a data acquisition system are used. In this study by carrying out more than 120 tests, the damping ratios of the grid for different levels of bolt tightness as well as different support conditions are obtained.

The results show that the bolt tightness has a major effect on the damping characteristics of the grid. The nature of this effect is discussed in Chapters 6 and 7. The results also show that the increasing of the number of supports of the grid will cause an increase in the damping of the grid. In addition, an increase in the amplitude of vibration is found to increase the damping of the grid.
To my parents
and my family
for their support and encouragement.
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TABLE OF CONTENTS

ABSTRACT .................................................................. 3

ACKNOWLEDGMENT ...................................................... 5

Chapter 1: INTRODUCTION
1.1 Organization of Thesis ..................................................... 13

Chapter 2: REVIEW OF PAST WORKS
2.1 Introduction .................................................................. 16
2.1.1 Historical review .......................................................... 16
2.1.2 Motivations for damping studies .............................................. 17
2.2 Vibration Measurement .................................................... 19
2.2.1 Dynamic tests on models .................................................... 20
2.2.2 Test methods ................................................................... 21
2.2.2.1 Initial displacement ......................................................... 21
2.2.2.2 Initial velocity ................................................................. 22
2.2.2.3 Rotating eccentric mass exciter ............................................ 22
2.2.2.4 Electromagnetic exciter .................................................... 23
2.2.2.5 Shake table ................................................................. 23
2.2.2.6 Wind and earthquake ...................................................... 25
2.2.3 Free and forced vibration .................................................... 25
2.3 Damping Variations ........................................................ 26
2.3.1 Amplitude effects on damping ratio ...................................... 27
2.3.2 Frequency effects on damping ratio........................................ 27
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4 Variation Control</td>
<td>28</td>
</tr>
<tr>
<td>2.4.1 Passive damper systems</td>
<td>29</td>
</tr>
<tr>
<td>2.4.1.1 Tuned mass damper (TMD)</td>
<td>29</td>
</tr>
<tr>
<td>2.4.1.2 Tuned liquid damper (TLD)</td>
<td>30</td>
</tr>
<tr>
<td>2.4.2 Active damper system</td>
<td>30</td>
</tr>
<tr>
<td>2.4.3 Hybrid-mass damper system</td>
<td>31</td>
</tr>
<tr>
<td>2.4.4 Examples of auxiliary damping systems</td>
<td>33</td>
</tr>
<tr>
<td>2.4.4.1 Viscous fluid damper</td>
<td>34</td>
</tr>
<tr>
<td>2.4.4.2 Viscoelastic damper</td>
<td>34</td>
</tr>
<tr>
<td>2.4.4.3 Friction damper</td>
<td>34</td>
</tr>
<tr>
<td>2.4.4.4 Electroheological and Magnetorheological dampers</td>
<td>34</td>
</tr>
</tbody>
</table>

Chapter 3: THEORETICAL STUDIES

3.1 Introduction                                                      | 38   |
3.1.1 Basic concepts of vibration                                     | 38   |
3.1.2 Harmonic motion                                                 | 41   |
3.1.3 Degree of freedom                                               | 43   |
3.1.4 Classification of vibration                                     | 44   |
3.1.4.1 Free and forced vibration                                     | 45   |
3.1.4.2 Undamped and damped vibration                                 | 46   |
3.1.4.3 Linear and nonlinear vibration                                | 47   |
3.2 Dynamic Analysis                                                  | 47   |
3.2.1 Single degree of freedom systems                                | 47   |
3.2.1.1 Free vibration of undamped system                             | 49   |
3.2.1.2 Construction of viscous damper                               | 50   |
3.2.1.3 Free vibration of a viscously damped system                  | 53   |
3.2.1.3.1 Oscillatory motion                                         | 55   |
3.2.1.3.2 Non-oscillatory motion                                     | 56   |
3.2.1.3.3 Critically damped motion                                   | 57   |
3.2.1.4 Example of an oscillatory motion                              | 57   |
3.2.1.5 Free vibration with Coulomb damping                          | 62   |
3.2.1.5.1 Equation of motion                                          | 62   |
## Chapter 4: DAMPING: THEORY AND CONCEPTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Introduction</td>
<td>107</td>
</tr>
<tr>
<td>4.1.1 Definition of damping</td>
<td>108</td>
</tr>
<tr>
<td>4.1.2 Damping force</td>
<td>108</td>
</tr>
<tr>
<td>4.2 Nature of damping</td>
<td>108</td>
</tr>
<tr>
<td>4.2.1 Material damping</td>
<td>108</td>
</tr>
<tr>
<td>4.2.2 System damping</td>
<td>109</td>
</tr>
<tr>
<td>4.2.3 Radiation damping</td>
<td>110</td>
</tr>
<tr>
<td>4.2.4 Auxiliary damping</td>
<td>110</td>
</tr>
<tr>
<td>4.2.4.1 Passive systems</td>
<td>110</td>
</tr>
<tr>
<td>4.2.4.2 Active systems</td>
<td>111</td>
</tr>
<tr>
<td>4.3 Representation of Damping</td>
<td>111</td>
</tr>
<tr>
<td>4.3.1 Viscous damping</td>
<td>111</td>
</tr>
<tr>
<td>4.3.2 Coulomb damping</td>
<td>113</td>
</tr>
<tr>
<td>4.3.3. Hysteretic damping</td>
<td>114</td>
</tr>
<tr>
<td>4.4 Measurement of Damping</td>
<td>116</td>
</tr>
<tr>
<td>4.4.1 Logarithmic decrement method</td>
<td>116</td>
</tr>
</tbody>
</table>
Chapter 5: EXPERIMENTAL WORK

5.1 Introduction ................................................................. 154
5.2 The Experimental Double Layer Grid ................................. 155
5.3 The MERO System ........................................................... 157
5.4 Grid Geometry ............................................................... 159
5.4.1 The model ................................................................. 160
5.5 Manufacture of the Components ......................................... 160
5.6 Assembly of the Grid ........................................................ 162
5.6.1 Assembly process ....................................................... 162
5.7 Foundation ................................................................. 165
5.7.1 Formwork for foundation .............................................. 166
5.7.2 Steel reinforcements ..................................................... 167
CHAPTER 6: PRESENTATION AND DISCUSSION OF TEST RESULTS

6.1 Introduction ................................................................ 199
6.2 Calculation of Damping Ratio ................................... 200
  6.2.1 Introduction ............................................................. 200
  6.2.2 Calculation operations .............................................. 202
6.3 Test Results ................................................................. 206
  6.3.1 Introduction ............................................................. 206
  6.3.2 Effects of bolt tightness ............................................ 211
  6.3.3 Effects of support conditions ................................... 218
  6.3.4 Amplitude effects ................................................... 225
## CHAPTER 7: CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

7.1 Introduction ................................................................. 230
7.2 General View ............................................................... 231
7.3 Conclusions ................................................................. 231
  7.3.1 Effects of bolt tightness ............................................. 231
  7.3.2 Effects of supports .................................................. 235
  7.3.4 Effects of amplitude ............................................... 235
7.4 Suggestions for Future Work ........................................... 236

APPENDIX ........................................................................ 237

REFERENCES .................................................................... 244
Damping describes a phenomenon where the amplitude of vibration in a mechanical system steadily diminishes. The effect of damping is to remove energy from the system. Energy in a vibrating system can be dissipated, being converted into other forms of energy such as heat, sound (wave propagation) or radiated away. Dissipation of energy into heat can be experienced simply by bending a piece of metal back and forth a number of times. This will cause the piece of metal to warm up. A lot of research effort has gone into the investigation of damping and understanding it, nevertheless, the complexity of damping phenomenon prevents to present a clear definition of the mechanism by which the vibrational energy is being dissipated. Although, the investigation on damping was started more than two centuries ago, at the present there are a considerable number of institutions and research centres that are involved in damping research around the world.

Damping is present in all oscillatory systems. Every system which possesses mass and elasticity is capable to oscillate. Typical examples of oscillation are swaying of building due to the wind or an earthquake, the motion of an airplane’s wing, the oscillation of a shaver machine and vibration of automobiles on the roads. The elasticity component of an oscillatory system stores potential energy and releases it as kinetic energy in the form of motion. Because
of presence of damping in the system, a portion of the kinetic energy is dissipated in each cycle. Therefore, damping plays an important role in the behaviour of an oscillatory system. Vibration analysis is concerned with the oscillatory motions of masses and forces associated with them. As all engineering structural systems posses mass and elasticity, they are capable to vibrate and their design may require consideration of their oscillatory behaviour.

There are a considerable number of mechanisms of damping by which energy is dissipated in materials. In most systems the vibrational motions are assumed to be undesirable and has to be suppressed. The damping level of a structural system is a measure of the efficiency with which the system dissipates the dynamic energy input. Structural engineers are interested in this energy to be absorbed as much as possible such that the vibration in the structures die down as quickly as possible. Damping of vibration in structural systems may depend on many parameters such as type of material, geometry, chemical composition, temperature, pre-stress, initial strain and amplitude and frequency of vibration. These parameters can be very important in the design of dynamical systems. From the view point of dynamic performance, higher damping can reduce steady state vibration amplitude, and also it can reduce the time needed for a transient vibration to settle.

The present research attempts to find out experimentally the variation of damping in a MERO-type double layer grid due to the variation of structural conditions. In this experimental study the damping ratio of the structure for different levels of tightness of the bolts in the connections are obtained. Also, the damping ratios of the structure for different support conditions are deduced. To find the damping ratios of the structure for different levels of bolt tightnesses as well as different support cases more than 120 tests were carried out. The results of these experiments provide a capability to compare the damping capacity of the structure for different conditions and find out the variation of damping ratio due to the variation of the bolt tightness as well as the variation of the support condition.

1.1 Organization of Thesis

The thesis contains 7 chapters as follows:

- The current chapter that is chapter 1 provides an introduction to the research.
Chapter 2 is connected to research and studies on the damping which have been carried out previously. In this chapter a historical review of damping and motivation of study on damping phenomenon is presented. Also, some researches related to vibration measurement including different methods which are applied for carrying out dynamic tests and some information about test equipment are discussed. In addition, the parameters which affect the damping capacity of the structures, including the amplitude and frequency of vibration are explained.

Chapter 3 is devoted to the theoretical concepts related to this research. In this chapter the basic concepts of vibration are discussed. Also, the dynamic analysis of structures including single and multi-degree of freedom systems in both free and forced vibrations are presented. In addition, the responses of undamped and damped oscillatory systems in the presence of different type of damping are discussed.

Chapter 4 deals with the theory and concept of damping. The definition of damping, representation of the damping by different types of energy dissipation mechanisms are discussed in this chapter. Also, a number of methods for measurement of damping coefficient in mechanical systems are presented. In addition, a number of methods for modelling of damping in oscillatory systems are presented in this chapter.

Chapter 5 is in connection to the experimental work. Construction of the model including, foundation, columns, supports, assembling and placement of the double layer grid is explained in detail in this chapter. In addition, the loading system for initial displacement tests is explained in detail in Chapter 5. A portion of this chapter is devoted to explain the data acquisition system. Finally, the process of carrying out the tests and applying support and bolt tightness conditions is explained.

Chapter 6 is devoted to presenting the results of the tests. In this chapter the method of calculation of the damping ratio from the response of the grid in a test is explained. This method is used to obtain the damping ratios from the responses of all the tests. All the damping ratios are presented in this chapter. Also, the variation of the damping ratio due to the variation of the support conditions as well as the variation of the bolt tightness are shown diagrammatically. In addition, the effect of amplitude on the damping ratio is discussed and the variation of damping ratio due to the variation of amplitude of vibration is shown.
- Chapter 7 is devoted to the discussion of the results of the research and the conclusion together with suggestions for the future work.

- Finally, the thesis contains a number of appendixes dealing with subjects which are referred in the chapters.
CHAPTER TWO

REVIEW OF PAST WORKS

2.1 Introduction

2.1.1 Historical review

Research on the damping properties of solid materials and their engineering significance was started almost 250 years ago [Lazen, B. J., 1968]. In 1784 Coulomb in his ‘Memoir on Torsion’ speculated on the micro-structural mechanisms of damping. He also recognised that mechanisms operating at low strain may be different from those at high strain. He concluded that the damping under torsion is caused by internal losses of energy in the material.

In the nineteenth century, many vibration studies were undertaken on the viscosity of metals and the non-linear nature of their viscosity. Some investigations were carried out on ‘internal friction’ of iron, silver, copper and other metals. Also, the effects of variables such as amplitude and frequency of the vibration as well as the initial strain and the size of the body were studied. Evin and Voigt [Bert, C. W., 1973] performed studies related to the hysteretic
loop, which indicates the dissipation of energy in a cycle. During the nineteenth century about 25 papers on damping were published. John William Strutt, Lord Rayleigh, in the 1890’s and the beginning of the 20th century studied damping in materials and published his conclusions in a number of classical papers. He assumed that damping is proportional to the stiffness of the system. Now it is generally believed that increasing the stiffness will cause an increase in the damping [Kijewski, T., et al, 2000].

Damping measurements on metals have become increasingly important since the middle of the 20th century. Research on damping in polymers in 1960’s was a new interest which increased the field of damping investigations considerably [Lazan, B. J., 1968].

During the last four decades huge efforts were spent by physicists, mechanical and structural engineers to find the damping characteristics in metals, structures and buildings. During these years thousands of papers have been published. However, damping is still not exactly understood and efforts to understand the nature of this phenomenon are continued.

Many of research centres inside and outside of universities were established all over the world to investigate damping in the structures and materials. The ‘Architecture Institute of Japan’ (AIJ) and the ‘Building Research Establishment’ (BRE) in the UK are examples of such research centres. AIJ carried out dynamic tests on 123 steel structures and 66 reinforced concrete structures [Suda, K., et al, 1996]. BRE has been involved with dynamic testing of structures since 1975 and collected data related to the dynamic properties, and in particular, damping characteristics of the buildings in the UK [Ellis, B. R. 1996]. Table 2.1 shows some of the institutes which are involved in research in dynamic properties of buildings and structures.

2.1.2 Motivations for damping studies

Several groups of researchers are involved in research in damping with different aims. Physical metallurgists and solid-state physicists research on damping at the micro-structural level for clarifying the mechanisms that lead to inelastic behaviour and energy dissipation in materials.

Another group which is interested in damping studies uses damping as an inspection tool. For example, the investigators of material science who are concerned with metal purity use damping measurements to determine the purity of metals [Lazan, B. J., 1968].
A third reason for interest is the importance of damping in the field of mechanical engineering. Mechanical engineering investigators are highly interested to identify damping characteristics of materials and design and build auxiliary damping devices.

Table 2.1: A number of research centres which are involved in damping research

<table>
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<tr>
<th></th>
<th>Architectural Institute of Japan (AIJ), Japan</th>
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<tr>
<td>2</td>
<td>Building Research Establishment (BRE), UK</td>
</tr>
<tr>
<td>3</td>
<td>California Universities for Research in Earthquake Engineering (CUREe), USA</td>
</tr>
<tr>
<td>4</td>
<td>Centre for Building Studies (University of Concordia), Canada</td>
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<td>5</td>
<td>Civil Engineering in Research Foundation (CERF), USA</td>
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<tr>
<td>6</td>
<td>Earthquake Engineering Research Institute (EERI), USA</td>
</tr>
<tr>
<td>7</td>
<td>Institute of Sound and Vibration Research (ISVR), UK</td>
</tr>
<tr>
<td>8</td>
<td>International Association of Earthquake Engineering (IAEE), Japan</td>
</tr>
<tr>
<td>9</td>
<td>International Association for Structural safety and Reliability (IASSAR), Denmark</td>
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<tr>
<td>10</td>
<td>International Institute of Acoustics and vibration (IIAV)</td>
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<td>11</td>
<td>International Institute of Earthquake Engineering and seismology (IIEES), Iran</td>
</tr>
<tr>
<td>12</td>
<td>Italian Association for Structural Control, Italy</td>
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<tr>
<td>13</td>
<td>Japan Association for Earthquake Disaster Prevention, Japan</td>
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<td>14</td>
<td>Japan Association for Wind Engineering, Japan</td>
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<td>15</td>
<td>Japan Society for Civil Engineering, Japan</td>
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<td>16</td>
<td>Japanese Building Research Institute, Japan</td>
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<tr>
<td>17</td>
<td>National Center for Earthquake Engineering Research (NCEER), USA</td>
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<tr>
<td>18</td>
<td>Vibration Engineering Research Centre (University of Sheffield), UK</td>
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The mechanical engineers are concerned about many unwanted vibrations in mechanical devices and machines. These vibrations are produced due to the rotating elements in the machines. These vibrations give rise to fatigue in the machine elements. This fact reduces the useful life of the machines and is the main cause of noise production in workshops. Turbines, compressors, power transmission devices and power production engines are examples in which rotating elements with high rotational speed can produce unwanted vibrations. Unwanted vibrations with high amplitude can be the cause of the fatigue failures in machine elements and the reduction of their working life. On the other hand, increasing the damping capacity of the machines by adding dampers can reduce the unwanted responses by absorbing some of the vibrational energy and dissipating it. The above considerations show why the mechanical engineers are interested in damping studies.

Civil engineering investigators are the next group who are involved in research on damping. They are responsible for constructing buildings to resist against powerful winds and earthquakes. The civil engineers are concerned with wind induced vibrations in buildings, although, the level of these vibrations is not normally significant enough to cause structural damage. However, strong winds especially in tall buildings, frequently, cause notable vibrations which can cause anxiety and discomfort to the buildings' occupants [Kijewski, T., et al, 2000]. Absorption and dissipation of the vibrational energy is one of the best methods of reducing the motions induced by the winds and earthquakes in buildings and structures. Also, to predict the behaviour of the structures and analyse them, it is necessary to know the damping characteristics of structures.

2.2 Vibration Measurement

The results obtained from dynamic experiments on structures are the most reliable information for predicting the dynamic behaviour of the structures. Accordingly, investigators have carried out dynamic tests to understand the dynamic characteristics of structures and buildings for over 150 years [Lazan, B. J., 1968]. As the design and construction of instruments for dynamic measurement improved the range of test models as well as the number test types has been increased. Production of new equipment has increased the capability of the investigators to design and set up new tests to find out dynamic characteristics of structures and buildings in more detail. A large number of tests in the field
of engineering and science have been carried out. In general, the test models may be divided into two categories:

- Small-scale models and
- Full-scale (real case) models.

2.2.1 Dynamic tests on models

The process of designing a model for a dynamic test is an important task for an investigator. Depending on the required results, available equipment as well as available space and facilities, the model can be designed and chosen.

The required outcome of the test is the main factor affecting the design of the model. For instance, when the dynamic characteristics of a reinforced concrete beam for specified conditions are required, the construction of the beam as well as creating the specified conditions is unavoidable. The available space is also an important factor affecting for design the model. The scale of the model is dependent on the available space. In the limited space of a laboratory only small-scale models can be employed. In addition, the availability of the equipment and facilities can also play an important role in the design of the model. It is clear that the experimental study on large-scale models such as large buildings and structures depends on the capabilities of the equipment and facilities.

The difficulties and high cost of full-scale tests are the reasons for most experimental investigation in the dynamic properties of structures to be carried out on small-scale models. Although, the experimental study of small-scale models can be useful, the results of these tests cannot be reliably scaled up for application to real structures [Bowkamp, J. G., 1970]. Since the damping of the structures is a complex phenomenon, and is dependent upon a number of structural properties such as the configuration of the structure and the initial strains, the results obtained from small-scale models cannot represent the dynamic characteristics of real structures. In addition, structural damping is affected by stiffness, mass and energy absorption at connections and this can vary from model to model as well as from model to real structures. In fact, the best models are real structures with their own difficulties and cost [Hudson, D. E., 1970].

From the above considerations Suda concluded that accurate results for the dynamic behaviour of a structure can only be found using full-scale tests [Suda, K., et al, 1996].
Inaccuracy of the results from numerical methods and the incapability of small-scale models in predicting the real behaviour of structures, lead the investigators to obtain reliable information related to the dynamic characteristics, and in particular, damping properties, by the employment of full-scale tests. In addition, in carrying out an experiment, the other dynamic properties of a structure like the natural frequencies and mode shapes can be found accurately.


A wide range of data for different levels of response amplitudes, in a variety of different conditions for structural systems and buildings are reported by the investigators. In these reports the variation of damping ratios (see Section 3.2.1.3) of structures due to the variation of the applied loads and frequency of vibration are presented. In the present research the variation of the damping ratio due to the variation of support conditions as well as the variation of the tightness of the bolts in the connections of a full-scale double layer space grid has been experimentally studied.

2.2.2 Test methods

The main types of dynamic tests suitable for full-scale structures are briefly discussed in the sequence.

2.2.2.1 Initial displacement

This type of test is the most common method of carrying out free vibration tests. This method involves deforming a structure by pulling it with a cable that is then suddenly released. This action causes the structure to vibrate freely about its static equilibrium position (see Section 3.1.4.1). By recording the response of the model with respect to time, the natural frequency of the model can be directly determined. Also, due to the energy dissipation of the model, the vibration amplitudes will decay. The ratios of these successive amplitudes can be used in the
‘logarithmic decrement method’ (see Section 4.4.1) allowing the damping ratio of the model to be obtained [Thomson, W. T., 1993].

There is a natural state of vibration with a displacement configuration corresponding to each natural frequency. Each displacement configuration is known as a normal mode (see Section 3.2.2.3). It is possible to excite a model on a natural frequency. This is dependent on the place of loading. The purpose of the present research was to excite the model on its first normal mode. So, the load is applied to the central connector of the top layer of the grid.

Most of the tests in the present research are carried out using this method. For applying initial displacement on the model of this research, a load releasing mechanism constructed. This mechanism is shown schematically in Fig 5.22 (Chapter 5). A view of this mechanism is shown in Fig 5.23. This mechanism is explained in detail in Section 5.10.1.

2.2.2.2 Initial velocity

This is a type of free vibration test which can be applied to a structure by initial velocities rather than initial displacements. This can be achieved by impact forces caused by dropping weights or by a hammer impact [Hudson, D. E., 1970]. Some of tests in the current research are carried out using this method.

2.2.2.3 Rotating eccentric mass exciter

‘Rotating eccentric mass exciter’ is the oldest device used for the generation of sinusoidal dynamic load. This device is also named ‘mechanical exciter’ and has been used since the beginning of the twentieth century [Hudson, D. E., 1970]. The Building Research Establishment (BRE) in the UK has a system of four rotating eccentric mass exciter which was built in 1980. This system works from 0.3 to 20Hz in increments of 0.001Hz and generates a maximum load of 4.1kN at 1Hz [Ellis, B. R., 1996].

A rotating eccentric mass exciter usually consists of two contra-rotating shafts with an eccentric mass on each of them. Fig 2.1 shows a rotating eccentric mass exciter schematically.
The force produced in this device is proportional to the eccentric mass, $m$, the distance from mass centre to the centre of rotation, $r$, and the square of the angular velocity, $\omega^2$ (see Fig 2.1). So, it is possible to vary the output load by varying one or more of the above three parameters.

2.2.2.4 Electromagnetic exciter

Electromagnetic exciters are another type of sinusoidal load generators. These devices are typically used for mode identification (see Section 3.3). These exciters do not involve rotating motion. In these devices, the sinusoidal load is created by the longitudinal stroke of a component of the exciter. These devices are built in a wide range of load and stroke. They can also work in a wide range of frequency from zero to tens of kHz. The configuration of these exciters allows them to be mounted in any position. Figure 2.2 shows a typical medium sized electromagnetic exciter.

2.2.2.5 Shake table

A shake table is another type of exciter for generation of sinusoidal exciting load. Shake tables can be found in different sizes. They work with a force rating from a few Newtons to more than thousands of Newtons. Also, the frequency range of shake table is from 1Hz to several thousands of Hertz. They can be used for large automotive structures, aerospace structures, locomotives and large industrial turbines. Also, they are being used for excitation of large civil engineers structures as well as small models of structures.
One of the first large shake tables was built in Japan in 1950s. This device was used by the 'Japanese Building Research Institute' in order to carry out dynamic tests on a one-third scale model of the core assembly of a graphite-pile of nuclear power reactor [Hudson, D. E., 1970]. Fig. 2.3 shows a picture of a shake table.

Figure 2.2 A typical Electromagnet exciter

(www.labworks.inc.com)
2.2.2.6 Wind and earthquake

Wind and earthquake are natural sources of dynamic excitation. Wind induced motions are useful for evaluating the characteristics of structures and buildings. They are used normally as low amplitude natural sources of excitation, while, earthquakes are natural sources of large amplitude excitation.

Nowadays, many researchers all over the world have undertaken to record the responses of the structures due to wind and earthquake induced motions. Information about these studies can be found in [Suda, K., et al, 1996], [Celebi, M., 1996], [Jeary, A. P., 1997], [Li, Q. S., et al, 2003] and [Li, Q. S., et al, 2004].

Figure 2.3 A typical shake table (http://www.lgarde.com/capabilities/shaker.html)

2.2.3 Free and forced vibration

Understanding the dynamic characteristics of structures, including natural frequencies, mode shapes and damping capacity is the main objective of carrying out experiments on structures. These characteristics can be studied using both free and forced vibration tests.
In some forced vibration tests, the structure has the opportunity to experience a larger size of loads as compared with a free vibration test. In addition, damping ratios have been found to be different for different sizes of applied loads [Marshal, R. D., 1994]. The damping ratio of a structure is an important factor for predicting the dynamic behaviour of the structure. So, the results obtained from free vibration tests as well as the results obtained from forced vibration tests can be used to predict the dynamic behaviour of a structure. In most experimental works, investigators are interested to record the decay of the response of the model. For recording the decay of the response of a model, the employment of free vibration test is unavoidable. During a steady state forced vibration test for recording decay, the exciter needs to be turned off and the oscillation of the model is recorded until it has settled down [Fukuwa, N., et al, 1996].

2.3 Damping Variations

The mechanism of energy absorption (damping) of materials and mechanical systems is a complex phenomenon. This complexity is due to a number of known and unknown factors which affect on damping properties of the materials and mechanical systems [Jeary, A. P., Ellis, B. R., 1981]. Experience shows that different levels of applied loads as well as different support conditions and foundation characteristics give rise to different results for damping ratios. Also, pre-stress condition causes variations in the damping ratios. Another parameter which can effect the damping ratio is the frequency of vibration. Experiments with different frequencies result in different damping ratios [Kareem, A., Gurly, K., 1995], [Suda, K., et al, 1996] and [Chopra, A., K., 2001]. It is now generally accepted that damping capacity of a structure cannot be represented by a constant quantity. Unlike the mass of a structure, the damping capacity of the structure varies due to the variation of parameters which are concerned with the structure, as mentioned above. Two such parameters which have been discussed more frequently among the investigators are the amplitude and frequency of the vibration. The opinions of the investigators in connection with these two parameters are explained in the following sections.
2.3.1 Amplitude effects on damping ratio

The results of all the experiments which were carried out by investigators show that the damping capacity of buildings and structures is affected by the magnitude of amplitude of the vibration. According to these results, it is clear that higher amplitudes of vibration result in higher damping ratios. Some information related to this study can be found in [Marsha, R. D., et al, 1994], [Suda, K., et al, 1996], [Glavine, M. J., 1996],[Celebi, M., 1996], [Fukuwa, N., et al, 1996] and [Chopra, A., K., 2001].

As an example, Celebi recorded low amplitude motions of five buildings induced by wind. He also recorded large amplitude motions of those buildings induced by the Loma Prieta earthquake in October 1989. These records show a larger damping ratio in a larger amplitude vibration. Damping ratios corresponding to that earthquake are 1.6 to 6 times larger than those corresponding to the wind induced vibrations [Celebi, M., 1996]. Another example is the results obtained by Fukuwa in Japan. He carried out free and forced vibration tests on a number of buildings. He concluded that the magnitude of the damping ratio is directly dependent on the amplitude of vibration [Fukuwa, N., et al, 1996]. The results of the experiments on the Milkman Library building in the California Institute of Technology, is another record which confirms the dependence of the damping ratio on the vibration amplitude. Some of these results which are presented by Chopra in his book: 'Dynamics of Structures' are obtained from a forced vibration test using an eccentric mass exciter. Also, the results which correspond to the Lytle Creek earthquake and San Fernando earthquake are presented in this book. These results show that the strong motion of San Fernando earthquake gave rise to high damping ratios, while the damping ratios corresponding to the vibration induced by the eccentric mass exciter are lower [Chopra, A., K., 2001]. From the above discussion one may conclude that the larger the amplitude of vibration the larger will be the damping ratio.

2.3.2 Frequency effects on damping ratio

Another parameter which may effect on the damping ratio is the frequency of vibration. There is no common agreement in connection with the manner of dependence of damping ratio on the frequency of vibration. The results of dynamic tests on the Milkman Library building
presented by Chopra show an inverse dependence of the damping ratio on the frequency of vibration. According to these results, when the vibration frequency increases from 1.9Hz to 8.3Hz, the damping ratio decreases from 2.9% to 1%. Also, another record shows that by increasing the frequency of vibration from 1.6Hz to 7.7Hz, the damping ratio decreases from 6.4% to 4.7% [Chopra, A., K., 2001].

On the other hand, some results which are presented by Suda show a direct dependence of the damping ratio on the frequency of vibration [ Suda, K., et al, 1996]. It turns out that the damping ratio increases when the frequency of vibration has increased. Although, some results presented by Suda show an inverse dependence between the damping ratio and the frequency of vibration.

Kareem believes that in higher frequencies of vibration ‘it is expected a building experiences more flexural and shear deformation which may contribute to higher damping’ [Kareem, A., Gurly, K., 1995].

From the above discussion it can be concluded that there is common agreement among the investigators in relation to the dependence of the damping ratio on the amplitude and frequency of vibration. Also, all the investigators are in agreement with the fact that increasing the amplitude of vibration gives rise to higher damping ratios. However, there is not common agreement about the manner of variation of the damping ratio due to the variation of frequency of vibration.

2.4 Vibration Control

One of the important tasks that a number of investigators have undertaken in the last several decades is to control the vibration of the buildings and structures [Kareem, A., Gurley, 1995]. For the past few decades the technology of energy dissipation has received a great deal of attention from earthquake researchers and engineers. A number of methods have been suggested in order to mitigate the response of structures and a number of auxiliary damping systems were built in recent years [Soon, T. T., Spencer, B. F. 2002]. Auxiliary damping systems are devices which help structural systems to absorb and dissipate vibrational energy. The employment of the auxiliary damping systems and energy dissipation devices results in the reduction of the amplitude of the responses of the structures and buildings effectively.
[Nishimura, S. et al, 1998]. A number of auxiliary damping systems are explained in the following sections.

2.4.1. Passive damper systems

Passive damping systems increase the damping of a vibrating system by modifying the structural characteristics of the system, thereby reducing the structural response. Passive damping systems are oscillatory systems which absorb a portion of vibrational energy of the main system and oscillate out of phase with the main system. The vibrational energy can be dissipated by the dampers in the auxiliary systems during the vibration. The most commonly used passive systems include 'Tuned Mass Dampers' (TMD) and 'Tuned Liquid Dampers' (TLD) [McNamara, R.J., 1977], [Kareem, A., 1983] and [Tamura, Y., 1988].

2.4.1.1 Tuned Mass Damper (TMD)

The most commonly used auxiliary passive damping system is the Tuned Mass Damper (TMD). TMD is a system consisting of a mass, a spring and a viscous or viscoelastic damper that is attached to the structure. A TMD can be planned to move in the horizontal or vertical plane in order to reduce the vibration in horizontal or vertical direction, respectively. Figs 2.4 and 2.5 show schematic views of the TMD in horizontal and vertical positions, respectively. In Fig 2.4, \( m_d, k_d \) and \( c_d \) are the mass, stiffness and damping coefficient of the damping system, respectively. The rollers are between the mass of the main system and the damper system. The rollers are assumed to be frictionless.

The TMD concept was applied firstly by Farham\(^1\) in 1909. Farham used this technique to reduce the rolling motion of ships as well as ship hull vibrations. TMDs have been in service in industry and civil engineering for several decades. Now a large number of TMDs are being used in buildings all over the world.

\(^1\) Reported by [Nishimura, I. et al, 1998]
Chapter 2 Review of Past works

The damping effect of TMD is represented by a dashpot.

- $m_d$ is the mass of the damper system.
- $k_d$ is the stiffness of the damper system.
- $c_d$ is the damping coefficient of the damper system.

The rollers are assumed to be frictionless.

Figure 2.4 A tuned mass damper (TMD) system in horizontal position

2.4.1.2 Tuned liquid damper (TLD)

A Tuned Liquid Damper (TLD) is a type of TMD where the mass is replaced by a liquid (usually water). In this technique one or multiple liquid tanks, usually rectangular, are attached to the structures. The TLD absorbs vibration of the structure and dissipates a portion of the vibrational energy of the structure. Accordingly, the damping of the structure is increased [El Damatty, A. A., 2002].

The advantages of TLD systems include low cost and maintenance. Fig 2.6 shows TLD attached to a system.

2.4.2 Active damper systems
Unlike the passive auxiliary damping systems an ‘Active Mass Damper’ (AMD) system consists of mechanisms powered by energy sources. By replacing the damping devise and spring in a tuned mass damper with an actuator an AMD can be obtained as shown in Fig 2.7a. An active tuned mass damper (active TMD) can be obtained by adding an actuator to a TMD, as shown in Fig 2.7b. The operation of an AMD is controlled by a control system. A large number of AMDs are being used in civil engineering structures [Suhrdjo, J., et al, 1992].

2.4.3 Hybrid-mass damper systems

The hybrid mass damper (HMD) is the most common control system employed in full-scale civil engineering applications [Soong, T. T., Spencer, B. F., 2002]. An HMD is a combination of a passive and active control systems. Fig 2.8 shows an HMD. The auxiliary damping systems are gaining in popularity in the world [Kareem, A., et al, 1999].
Chapter 2 Review of Past works

Main TLD
-------------------
MOT-Cd
Liquid

(a): Schematic of a single degree of freedom with a TLD

(b): A multi degree of freedom system with a TLD on it

Figure 2.6 Tuned liquid damper (TLD)

(a): An active mass damper

(b): An active tuned mass damper

Figure 2.7 An active mass damper and active tuned mass damper
2.4.4 Examples of auxiliary damping systems

The following mechanisms are the systems which have been commonly used in recent years in the vibrating systems and buildings in order to mitigate the vibration

- Viscous fluid dampers,
- Viscoelastic dampers,
- Friction dampers and
- ER and MR dampers.

2.4.4.1 Viscous fluid damper
A viscous fluid damper generally consists of a piston within a damper hosing filled with a compound of silicone or similar type of oil. The piston has a number of small orifices on its top surface through which the fluid may pass from one side of the piston to the other [Symans, M. D., Constantinou, M. C., 1999]. Fig 2.9a shows a structure of a viscous damper. A viscous damper is a very attractive option for civil engineering applications [Kareem, A. et al, 1999]. Fig 2.9b shows an application of the viscous damper in a structure.

2.4.4.2 Viscoelastic damper

Viscoelasticity may be defined as material response that presents characteristics of both a viscous fluid and elastic solid. It stores oscillatory energy like a spring and dissipate portion of this energy like a viscous fluid.

To date, viscoelastic dampers have been installed in several buildings in USA and Japan [Kareem, A., et al, 1999].

2.4.4.3 Friction damper

Relative motion between two sliding surfaces produces a force which resists the motion and dissipates the mechanical energy of the moving surfaces. This is named Coulomb damping (see Section 3.2.1.5). This mechanism is used to make 'friction dampers'. Several types of friction dampers have been developed for the purpose of reducing the response of the structures [Soong, T. T., Spencer, B. F., 2002]. Fig 2.10 shows an example of a friction damper which can be used in the bracing system of structures. The damper of this figure is made up from a set of steel plates with slotted holes in them and they are bolted together. At high enough forces, the plates can slide over each other creating friction. The plates are specially treated to increase the friction between them. During cyclic loading, the mechanism conducts slippage in both tensile and compressive directions. Generally, friction devices generate rectangular hysteretic loop (see Section 4.3.2).

2.4.4.4 Electrorheological and Magnetorheological dampers
Electrorheological (ER) and Magnetorheological (MR) dampers are fluid dampers. ER and MR dampers have the capacity to provide large controllable damping forces and several other attractive features such as simplicity, reliability and small power requirement [Ying, Z. G., Zhu, W. Q., 2003]

ER/MR dampers typically consist of a hydraulic cylinder containing micro-sized dielectric particles suspended within a fluid (usually oil). In the presence of a strong electric field (ER dampers) or a strong magnetic field (MR dampers) the particles polarise and become aligned, thus offering an increased resistance to flow. By varying the electric or magnetic field, the dynamic behaviour of an ER/MR damper can be regulated. Fig 2.11 shows an ER/MR schematically.
BMD Structure

Figure 2.9a A viscous damper (http://www.robot.com.tw/bmd.htm)

Figure 2.9b A viscous damper in a structure
Chapter 2 Review of Past works

(a) Friction damper device

(b) Friction damper in a bracing system

Figure 2.10 A friction damper device and its application

Wires to electric device
Bearing and seal
ER/MR fluid
Annular orifice
Coil
Diaphragm
Accumulator

Figure 2.11 Schematic view of ER/MR damper
CHAPTER THREE

THEORETICAL STUDIES

3.1 Introduction

3.1.1 Basic concepts of vibration

Any motion that repeats itself after an interval of time is called ‘vibration’ or ‘oscillation’. The swing of a pendulum, the flutter of airplane wings and the seismic motion of the bridges and buildings under wind induced excitation are typical examples of vibration.

A vibratory system, in general, includes three parts. A part for storing potential energy (elasticity), a part for storing kinetic energy (inertia) and a part by which energy is gradually lost (damper).

Damping is present in all vibratory systems. The vibrational energy is gradually converted to heat. Due to the reduction in the energy, the displacement of the system gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat is known as
‘damping’. In practical systems, it is difficult to determine the causes of damping. So, damping is modelled in various ways, including

- Viscous damping,
- Coulomb damping and
- Material damping.

These will be discussed in detail in Chapter four.

The vibration of a system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a steady state vibration is to be maintained.

As an example, the vibration of the simple pendulum shown in Fig 3.1 can be considered. The pendulum consists of a mass, \( m \), connected to a bar with length, \( l \). The bar is pinned at point \( o \). The mass is released after giving it an angular displacement, \( \theta \). At position 1 the velocity of the mass and, therefore, its kinetic energy is zero. But, it has a potential energy of magnitude \( mgl(1 - \cos \theta) \) with respect to the base line (position 2). This gives the mass a certain angular acceleration in the clockwise direction. The mass starts to swing to the left from position 1. Once it reaches position 2, all of its potential energy will be converted into kinetic energy. So, the mass will not stop at position 2, but will continue to swing to position 3. However, as it passes the position 2, a counterclockwise torque starts acting on the mass due to the gravity and causes the mass to decelerate. The velocity of the mass reduces to zero at the left extreme position (position 3). By this time, all kinetic energy of the mass has been converted to potential energy. Again a moment due to the gravity force, acts on the mass and causes it continues to attain a counterclockwise velocity and passes the mean position (position 2) again. This process repeats and the pendulum will have oscillatory motion. However, in practice, the magnitude of oscillation, \( \theta \), gradually decreases and the pendulum eventually stops due to the resistance (damping) offered by the surrounding medium (air) and friction at the support. This means that some energy is dissipated in each cycle of vibration due to damping.

A vibratory system is a dynamic system for which the variables such as the excitations (inputs) and responses of the system (outputs) are time dependent. The response of a vibratory system generally depends on the initial conditions as well as the external excitations. The
analysis of a vibrating system usually involves mathematical modelling, derivation of the governing equations and solving of the equations.

Mathematical modeling: The purpose of mathematical modelling is to represent all the important features of the system in order to derive mathematical (analytical) equations governing the behaviour of the system.

Equation derivation: Once the mathematical model is available it is possible to use the principals of dynamics and derive the equations that describe the vibration of the system. The equations are usually in the form of ordinary differential equations. The equations may be linear or nonlinear depending on the behaviour of the components of the system. Newton's second law of motion is the most common used approach to derive the governing equations.

Solving of equations: The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem one of the following techniques for finding the solution can be used.

- Standard method of solving differential equations,
- Laplace transformation method,
- Matrix method and
- Numerical method.

The solution of the governing equations gives the displacements, velocities and accelerations of the system.
3.1.2 Harmonic motion

Oscillatory motion may repeat itself regularly, as in the simple pendulum, or may display considerable irregularity, as in an earthquake. When a motion is repeated in equal intervals of time, $T$, it is called 'periodic motion'. And the repetition time, $T$, is called 'period' of oscillation. The inverse of the period, that is,\[
f = \frac{1}{T}
\]is called the 'frequency'. If the motion is designated by the time function, $x(t)$, then any periodic motion must satisfy the relationship\[
x(t) = x(t + T)
\]The simplest form of periodic motion is 'harmonic motion'. The equation of motion for a mass which has a harmonic motion may be written as\[
x(t) = A \sin \omega t \quad [\text{W T. Thomson, 1993}]
\]where
- $x(t)$ is the displacement of the mass,
- $A$ is the amplitude of the motion,
- $\omega$ is the circular frequency of the motion and
- $t$ is the time.

Eqn 3.3 is shown in figure 3.2 graphically.

The circular frequency, $\omega$, is generally measured in radians per second. Because the motion repeats itself in $2\pi$ radians, one may write\[
\omega = \frac{2\pi}{T} = 2\pi f
\]where $T$ is the period and $f$ is the frequency of the harmonic motion. $T$ and $f$ are usually measured in second and cycles per second, respectively.

The velocity and acceleration of a harmonic motion can be simply determined by differentiation of Eqn 3.3. So, the velocity is obtained as
where $dx$ and $dt$ are the displacement differential and time differential, respectively. The equation of velocity (Eqn 3.5) may be alternatively written as

$$\dot{x}(t) = \omega A \sin(\omega t + \pi / 2)$$

(Eqn 3.6)

Eqn 3.6 represents the velocity, $\dot{x}$, and is $\pi/2$ radians (=90°) out of phase with the displacement, $x$. The acceleration of the system can be obtained by differentiation of the velocity which is the second differentiation of the displacement as follows

$$\frac{d^2x(t)}{dt^2} = \ddot{x}(t) = -\omega^2 A \sin \omega t$$

(Eqn 3.7)

The equation of acceleration (Eqn 3.7) may also be written as

$$\ddot{x}(t) = \omega^2 A \sin(\omega t + \pi)$$

(Eqn 3.8)

Eqn 3.8 represents the acceleration, $\ddot{x}$, and is $\pi$ radians (=180°) out of phase with the displacement. The velocity and acceleration equations show that the velocity and acceleration are also harmonic as is the displacement. Figs 3.3 and 3.4 show the velocity and the acceleration of a harmonic motion that are given by Eqns 3.5 and 3.7, respectively.

A question may arise as to why harmonic motion is widely used when dealing with the vibrations of various systems? The answer is partly because of the possibility to represent any
periodic forcing function by a Fourier series which contains harmonic terms, and in dealing with linear systems where one may use the principal of superposition and solve the dynamic equations for each individual harmonic term in the Fourier series and then add the answers for all the harmonic terms in order to find the overall response of the system to a periodic forcing function.

3.1.3 Degree of freedom

The minimum number of independent coordinates required to determine the positions of all parts of a system completely at any instant of time, defines the ‘degree of freedom’ of the system. The simple pendulum shown in Fig 3.1 as well as each of the systems shown in Fig 3.5 represents a single degree of freedom system. For example, in Fig 3.5 the linear coordinate, $x$, can be used to specify the motion.

![Harmonic velocity](image)

Figure 3.3 Harmonic velocity (Eqn 3.5)

Some examples of two and three degree of freedom systems are shown in Figs 3.6 and 3.7, respectively. Fig 3.6a shows a two mass-three spring system that can be described by two linear coordinates, $x_1$ and $x_2$. Fig 3.6b illustrates a two degree of freedom frame whose motion can be specified by two linear coordinates, $x_1$ and $x_2$. 
Chapter 3 Theoretical studies

Figure 3.4 Harmonic acceleration (Eqn 3.7)

The rollers are between the mass and the surface. The rollers are assumed to be frictionless.

Figure 3.5 Single degree of freedom systems.

For the systems shown in Fig 3.7 the coordinates, $x_1$, $x_2$ and $x_3$ can be used to describe the systems.

3.1.4 Classification of vibration

Vibration can be classified in several ways. Some of the main classifications are as follows:
3.1.4.1 Free and forced vibration

**Free vibration.** If a system after an initial disturbance (initial displacement, initial velocity or impulse force) is left to vibrate while there is no external force, the vibration is known as 'free vibration'.

**Forced vibration.** If a system is subjected to an external force (usually a repeating type of force) the resulting vibration is known as 'forced vibration'. The oscillation that arises in rotating machines (because of unbalancing in rotating elements) is an example of forced vibration. If the frequency of the external force coincides with the natural frequency (or one of the natural frequencies) of the system, a condition that is known as 'resonance' occurs and the systems undergoes large oscillations. Sometimes, the failure of a structure or a bridge in weak ground motion or in a wind induced excitation occurs. This is due to coincidence of the excitation frequency with the natural frequency of the structure. The collapse of the first Tacoma narrow suspension bridge due to wind induced excitation can be considered as an example of failure which is associated with the occurrence of resonance.

![Two degree of freedom systems](image_url)
3.1.4.2 Undamped and damped vibration

If no energy is lost or dissipated during the oscillation, the vibration is known as 'undamped vibration'. In contrast, if any energy is lost during the vibration, it is called 'damped vibration'. Damping is present in all vibratory systems. Also consideration of damping in a vibratory system analysis, specially near the state of resonance is extremely important.
3.1.4.3 Linear and nonlinear vibration

If all the components of a vibratory system behave linearly, the resulting vibration is known as 'linear vibration'. On the other hand, if any of the basic components behave nonlinearly, the vibration is called a 'nonlinear vibration'. The differential equations that govern the behaviour of linear or nonlinear vibratory systems are linear or nonlinear, correspondingly.

3.2 Dynamic Analysis

There are two basic ways of analysing a structure. In a case where there is no motion involved, the analysis is referred to as 'static analysis'. In contrast, when the structure involves motion then the analysis is referred to as 'dynamic analysis'. The dynamic analysis of a structure may be carried out without any regard to the damping effect. On the other hand, the analysis may be performed taking into account the dynamic characteristics of the structure. Some details regarding damped or undamped dynamic analysis are discussed in the sequel.

3.2.1 Single degree of freedom systems

The simplest system for vibration analysis is a single degree of freedom system. Fig 3.8 shows a single-story frame as a system with single degree of freedom. The system has horizontal girder and two vertical columns. The horizontal girder is assumed to be rigid and to contain all the mass of the frame. The columns are assumed to be massless and non extendable in the axial direction. As a result of the horizontal displacement of the frame, each column behaves like a spring and provides resistance to girder displacement. The stiffness of each column that represents its capability of resisting the horizontal displacement of the girder is denoted by $k/2$. Thus, the frame has a single degree of freedom, $x(t)$, which is in the direction of the girder. The damping in the frame is assumed as viscous damping. Viscous damping is the most commonly used damping mechanism and is discussed in Section 3.2.1.2 and Chapter 4 in detail. This mechanism of the energy dissipation in the frame is symbolically shown by a dashpot. This mechanism offers a resistance to the motion of the girder proportional to the velocity which is equal to
\[ F_d = c \dot{x} \]

where

- \( F_d \) is the viscous damping force,
- \( c \) is the damping coefficient and
- \( \dot{x} \) is the velocity of the girder

The free body diagram of the frame is shown in Fig 3.9. Using the Newton's second law, one obtains

\[
\sum F_x = m \frac{d^2 x}{dt^2} = m \ddot{x} \tag{3.9}
\]

where

- \( \sum F_x \) indicates the resultant force in the \( x \) direction,
- \( m \) is the mass of the system and
- \( \ddot{x} \) is the acceleration of the system.

![Figure 3.8 A single degree of freedom frame](image)

Considering the forces on the system as shown in Fig 3.9, one may rewrite Eqn 3.9 as

\[-kx - c \dot{x} + F_0 \sin \omega t = m \ddot{x} \]
or
\[ m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \]  \hspace{1cm} (3.10)

where \( F_0 \sin \omega t \) is a harmonic load in which \( F_0 \) and \( \omega \) are the amplitude and the circular frequency of the exciting load, respectively.

Figure 3.9 Free body diagram of frame

The solution of Eqn 3.10 consists of two parts [E. Kreyszig, 1999]. Part one is the solution of the homogenous equation, that is,
\[ m\ddot{x} + c\dot{x} + kx = 0 \]  \hspace{1cm} (3.11)

and part two is the particular solution which will be discussed later.

3.2.1.1 Free vibration of undamped system

Free vibration of a system is such a vibration in which the resultant of external forces is equal to zero. The study of free vibration of single degree of freedom systems is fundamental to understanding of topics in vibration. Of primary interest in vibration analysis is finding the natural frequency (or natural frequencies) of a system.

Ignoring the effects of damping, Eqn 3.11 will be reduced to
\[ m\ddot{x} + kx = 0 \]

or, dividing the above equation by \( m \) (\( m \) is always nonzero and positive) gives
\[ \frac{\ddot{x}}{m} + \frac{k}{m}x = 0 \]  \hspace{1cm} (3.12)

Using the notation
Eqn 3.12 can be written as
\[ \ddot{x} + \omega_n^2 x = 0 \]  
(3.13)

where \( \omega_n \) is the 'natural frequency' of the system. The solution of Eqn 3.13 is given by
\[ x(t) = A \cos \omega_n t + B \sin \omega_n t \]  
[E. Kreyszig, 1999]  
(3.14)

where \( A \) and \( B \) are constants that can be evaluated from the initial conditions \( x(0) \) and \( \dot{x}(0) \). Differentiation of Eqn 3.14 gives
\[ \dot{x}(t) = -A \omega_n \sin \omega_n t + B \omega_n \cos \omega_n t \]  
(3.15)

Assuming,
\[ x(0) = X_0 \]
\[ \dot{x}(0) = 0 \]
from Eqn 3.14
\[ X_0 = A \cos 0 + B \sin 0 \]
or
\[ A = X_0 \]
also, from Eqn 3.15
\[ 0 = -A \omega_n \sin 0 + B \omega_n \cos 0 \]
or
\[ B=0 \]
As a result, Eqn 3.14 can be obtained as
\[ x(t) = X_0 \cos \omega_n t \]  
(3.16)

Eqn 3.16 represents the response of a free vibration undamped single degree of freedom system and is a harmonic vibration. This vibration lasts perpetually mathematically.

3.2.1.2 Construction of a viscous damper
Viscous damping is the most commonly used damping mechanism in vibration analysis. In viscous damping, the dissipated energy is related to the velocity of the vibrating body. A theoretical viscous damping model can be constructed using two parallel plates separated by a distance, \( h \), with a fluid of viscosity, \( \mu \), between the plates, as shown in Fig 3.10a. The lower plate is considered to be fixed and the other plate can move with a velocity, \( V \), in a plane parallel to the fixed plate. The fluid layers in contact with the moving plate move with the velocity, \( V \), while, those in contact with the fixed plate have no motion. The velocities of the intermediate layers of the fluid are assumed to vary linearly between 0 and \( V \), as shown in Fig 3.10a. Due to the different velocities between the fluid layers, shear stresses are developed. This causes a force, which resists the motion of the layers. This force is called 'damping force'. According to Newton's law of viscous flow, the shear stress, \( \tau \), developed in the fluid layer at a distance, \( y \), from the fixed plate is given by

\[
\tau = \mu \frac{dv}{dy} \quad \text{[S. S. Rao, 1990]} \tag{3.17}
\]

where

- \( \tau \) is the shear stress,
- \( \mu \) is the viscosity of the fluid,
- \( dv \) and \( dy \) are the velocity differential and distance differential, respectively.

\( \frac{dv}{dy} \) is the velocity gradient and is equal to \( V/h \). The resisting shear force \( F_d \) developed between the layers of the fluid is

\[
F_d = \tau A = \mu A \frac{dv}{dy} = \mu A \frac{V}{h} \tag{3.18}
\]

where

- \( F_d \) may be interpreted as the damping force developed between plates,
- \( A \) is the surface area of the moving plate,
- \( V \) is the velocity of the moving plate and
- \( h \) is the distance between the plates.

Eqn 3.18 can be alternatively written as

\[
F_d = cV \tag{3.19}
\]

where \( c \) is equal to
and is called 'damping coefficient'.

Fig 3.10b shows the scheme of a viscous damper in practice. Such a damper is used in a structure in order to decay the wind induced or ground motion excitation, as shown in Fig 2.9b. This damper consists of a closed cylinder containing a viscous fluid like oil. A rod is connected to a piston with small holes in its head. The piston can move in and out of the cylinder. As the piston does this, the viscous fluid is forced to flow through the holes, causing friction in the piston head.

When a building in which a viscous damper is installed, starts to vibrates, the friction introduced inside the damper converts some of the building kinetic energy into the heat energy.

Figure 3.10a: Parallel plates with a viscous fluid in between
3.2.1.3 Free vibration of a viscously damped system

The solution of the equation of damped free vibration, that is, Eqn 3.11 has the form

\[ x = e^{\lambda t} \]  

[E.Kreyszig, 1999] (3.20a)

With

\[ \dot{x} = \lambda e^{\lambda t} \]  

(3.20b)

and

\[ \ddot{x} = \lambda^2 e^{\lambda t} \]  

(3.20c)

Substituting Eqns 3.20 into Eqn 3.11 gives

\[ (m\lambda^2 + c\lambda + k)e^{\lambda t} = 0 \]  

(3.21)

Since \( e^{\lambda t} \) is always nonzero, Eqn 3.21 will be satisfied for all values of \( t \) when

\[ m\lambda^2 + c\lambda + k = 0 \]  

(3.22)

Eqn 3.22 is known as the ‘characteristic equation’ of the system and has two roots:

\[ \lambda_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \frac{-c}{2m} \pm \sqrt{\Delta} \]  

(3.23)

where
The general solution of Eqn 3.11 is given by the equation

\[ x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \]  

(3.24)

Where \( A \) and \( B \) are constants that can be evaluated from the initial conditions \( x(0) \) and \( \dot{x}(0) \)

[E. Kreyszig, 1999]

The substitution of Eqn 3.23 into Eqn 3.24 gives

\[ x = e^{-(c/2m)t} (Ae^{\sqrt{\Delta}t} + Be^{-\sqrt{\Delta}t}) \]  

(3.25)

The term

\[ e^{-(c/2m)t} \]

in Eqn 3.25 is simply an exponential function of time. Depending on the numerical value within the radical being positive, zero or negative, there are three cases for the roots \( \lambda_1 \) and \( \lambda_2 \) as follows

**Case 1:** When the term \( (c/2m)^2 \) is larger than \( k/m \), the exponents in Eqn 3.25 are real numbers and no oscillatory motion is possible.

**Case 2:** When the term \( (c/2m)^2 \) is less than \( k/m \), the exponents in Eqn 3.25 become imaginary numbers, \( \pm i\sqrt{\Delta}t \). In this case according to the Euler formula

\[ e^{\pm it} = \cos t \pm i\sin t \]

where,

\[ i = \sqrt{-1} \]

As a result

\[ e^{\pm i\sqrt{\Delta}t} = \cos \sqrt{\Delta}t \pm i\sin \sqrt{\Delta}t \]  

(3.26)

So, the terms of Eqn 3.25 within the parentheses represent an oscillatory motion that corresponds to the equation

\[ x = e^{-(c/2m)t} (A_1 \cos \sqrt{\Delta}t + iB_1 \sin \sqrt{\Delta}t) \]  

(3.27)

**Case 3** In the limiting case between the oscillatory and non-oscillatory motion, the term \( (c/2m)^2 = k/m \) and \( \Delta \) is equal to zero. The damping corresponding to this limiting case is called 'critical damping'. The critical damping is \( c_c \) and denoted by
\[ c_c = 2m \sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{k/m} \]  

(3.28)

Here a non-dimensional number can be defined as:

\[ \zeta = \frac{c}{c_c} \]  

(3.29)

Any damping can then be expressed in terms of the \( \zeta \) which is called 'damping ratio' and \( \lambda_{1,2} \) can be so expressed in terms of \( \zeta \) as follows:

\[ \frac{c}{2m} = \zeta \frac{c_c}{2m} = \zeta \omega_n \]  

(3.29a)

Equation 3.23 then can be written:

\[ \lambda_{1,2} = \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) \omega_n \]  

(3.30)

The three cases of damping discussed above now depend on whether \( \zeta \) is less than, greater than or equal to unity. In the following sections the three cases:

- \( \zeta < 1 \) (oscillatory motion),
- \( \zeta > 1 \) (non-oscillatory motion) and
- \( \zeta = 1 \) (critically damped motion)

will be discussed.

### 3.2.1.3.1 Oscillatory motion

Substitution Eqn 3.30 into Eqn 3.24, the general solution becomes

\[ x = e^{-\zeta \omega_n t} \left( Ae^{i(\omega_n t + \phi_1)} + Be^{-i(\omega_n t + \phi_1)} \right) \]  

(3.31)

Eqn 3.31 can also be written in the following form:

\[ x = X e^{-\zeta \omega_n t} \sin(\sqrt{1 - \zeta^2} \omega_n t + \phi_1) \]  

(3.32)

Where arbitrary constants \( X \) and \( \phi_1 \) are determined from initial conditions \( x(0) \) and \( \dot{x}(0) \).
Similar $\omega_n$ in a undamped system, in Eqn 3.32 the term $\sqrt{1-\zeta^2} \omega_n$ indicates the frequency of response of the damped system and called 'frequency of damped oscillation' which is equal to

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = \frac{2\pi}{T_d}$$

(3.33)

In which, $T_d$ is the period of damped free vibration. [For detail see: W T. Thomson, Theory of Vibration With Application, 1993]

Figure 3.11 shows the general nature of damped oscillatory system

![Diagram of damped free vibration](image)

Figure 3.11 Plot of damped free vibration \[\zeta < 1\]

3.2.1.3.2 Non-oscillatory motion

As $\zeta$ exceeds unity the two roots of Eqn 3.30 become real numbers, one increasing and the other decreasing. The general solution then becomes

$$x = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_s t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_s t}$$

(3.34)

Where $A$ and $B$ can be found from initial conditions. The motion is an exponentially decreasing function of time as shown in Fig 3.12
3.2.1.3.3 Critically damped motion

For $\zeta = 1$ Eqn 3.30 will have double root $\lambda_1 = \lambda_2 = -\omega_n$ and the two terms of Eqn 3.24 combine to form a single term. The correct general solution is

$$x = (A + Bt)e^{-\omega_n t} \quad [\text{E Kreyszig, 1999}]$$

(3.35)

In which $A$ and $B$ can be found by initial conditions. Critically damped motion is shown in Fig 3.13.

3.2.1.4 Example of an oscillatory motion

Consider the system shown in Fig 3.14 with

$m = 20 \text{ kg}$

$c = 40 \text{ kg/s}$
\( k = 1300 \text{ N/m} \)

initial displacement = \( x(0) = 0.2 \text{ m} \)

initial velocity = \( \dot{x}(0) = 0 \text{ m/s} \)

\[ x(t) \]

\[ x(0) \]

\[ \dot{x}(0) > 0 \]
\[ \dot{x}(0) = 0 \]
\[ \dot{x}(0) < 0 \]

Figure 3.13 critically damped motion \( [\zeta = 1] \)

The equation of motion of the system shown in Fig 3.14, is given by Eqn 3.11, That is,

\[ m\ddot{x} + c\dot{x} + kx = 0 \quad (3.11) \text{ repeated} \]

where the damping in the system is assumed to be viscous.

Substituting the above given quantities into Eqn 3.11 gives

\[ 20\ddot{x} + 40\dot{x} + 1300x = 0 \]

or

\[ \ddot{x} + 2\dot{x} + 65x = 0 \quad (3.36) \]

Eqn 3.36 represents the equation of motion of the system of Fig 3.14. It must be noted that the motion of the system is due to the initial displacement. This initial displacement causes potential energy to be stored in the system. The potential energy will then converts to kinetic energy and causes the oscillation of the system.

The general form of the response of the system of Fig 3.14 is given by Eqn 3.32, that is,

\[ x(t) = Xe^{-\zeta \omega_n t} \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \phi \right) \quad (3.32) \text{ repeated} \]
Using the magnitudes of the mass, damping coefficient and stiffness of the system, the natural frequency, $\omega_n$, and the damping ratio, $\zeta$, of the system can be found as follows:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{(see Eqn 3.15)}$$

or

$$\omega_n = \sqrt{\frac{1300}{20}} = \sqrt{65} \quad \text{(3.37)}$$

$$\zeta = \frac{c}{2m\omega_n} \quad \text{(see Eqns 3.28 and 3.29)}$$

or

$$\zeta = \frac{40}{2 \times 20 \times \sqrt{65}} = \frac{1}{\sqrt{65}} \quad \text{(3.38)}$$

Substituting the magnitudes of $\omega_n$ and $\zeta$ into Eqn 3.32 gives

$$x(t) = Xe^{-\frac{1}{\sqrt{65}} \sqrt{65}t} \sin \left( 1 - \frac{1}{65} \sqrt{65}t + \phi \right)$$

or

$$x(t) = Xe^{-\zeta t} \sin (\phi t + \phi) \quad \text{(3.39)}$$
Derivative of Eqn 3.39 with respect to time gives the equation of the velocity as

$$\dot{x}(t) = X \left[ -e^{-t} \sin(8t + \phi_1) + 8e^{-t} \cos(8t + \phi_1) \right]$$  \hspace{1cm} (3.40)

Applying the initial conditions yields

$$x(0) = 0.2$$

or

$$0.2 = X \sin \phi_1$$

or

$$X = \frac{0.2}{\sin \phi_1}$$  \hspace{1cm} (3.41)

Also,

$$\dot{x}(0) = 0$$

or

$$0 = X \left[ -e^0 \sin(0 + \phi_1) + 8e^0 \cos(0 + \phi_1) \right]$$

or

$$X \left[ -\sin \phi_1 + 8 \cos \phi_1 \right] = 0$$  \hspace{1cm} (3.42)

From Eqns 3.41 and 3.42,

$$\frac{0.2}{\sin \phi_1} \left[ -\sin \phi_1 + 8 \cos \phi_1 \right] = 0$$

or

$$-0.2 + 1.6 \frac{1}{\tan \phi_1} = 0$$

or

$$\tan \phi_1 = 8$$

or

$$\phi_1 = 82.7^\circ = 1.45 \text{ radians}$$  \hspace{1cm} (3.43)

Substituting Eqn 3.43 into Eqn 3.41, the amplitude of motion can be found as

$$X = \frac{0.2}{\sin 1.45} = 0.2016 \approx 0.2 \text{ m}$$  \hspace{1cm} (3.44)

Substitution of Eqns 3.43 and 3.44 into Eqn 3.39 gives
\[ x(t) = 0.2e^{-t} \sin(8t + 1.45) \] (3.45)

Fig 3.15 shows the graph of Eqn 3.45. In this figure the response of the system, \( x \), (displacement) is plotted versus the time, \( t \). Eqn 3.45 is the result of multiplying two functions, that are,

\[ f_1(t) = \sin(8t + 1.45) \]
\[ f_2(t) = 0.2e^{-t} \]

\( f_1(t) \) is a sinusoidal harmonic function with unity amplitude and graphically like that of Fig 3.2. \( f_2(t) \) is an exponential function as shown by dashed line in Fig 3.15. Multiplying this function by \( f_1(t) \) causes decay of the displacement, \( x(t) \).

---

**Figure 3.15 Response of the system of Fig 3.14**

- natural frequency = \( \omega_n = \sqrt{65} \) rad/s
- damped frequency = \( \omega_d = 8 \) rad/s
- frequency = \( f = 1.273 \) Hz
- period = 0.785 second
3.2.1.5 Free vibration with Coulomb damping

Coulomb damping is the result of friction force between surfaces sliding to each other. It is assumed that the force resisting the direction of motion is proportional to the normal force between the sliding surfaces and is independent of the magnitude of the velocity. In mechanical systems Coulomb dampers are used because of their mechanical simplicity and convenience [S. S. Rao, 1990]. Also, in vibrating structures, whenever the components slide on each other, Coulomb damping appears internally. Coulomb damping arises when bodies slide on surfaces. Coulomb’s law of friction states that when two bodies are in contact, the force required to producing sliding is proportional to the normal force acting in the plane of contact. Thus, the friction force is given by

\[ F = \mu_s N \]  

(3.46)

Where \( F \) is the friction (damping) force, \( \mu_s \) is the ‘kinetic coefficient of friction’ of two sliding surfaces and \( N \) is the normal force between them. The subscript \( k \) used in \( \mu_s \) indicates the kinetic nature of the friction coefficient. This notation distinguishes the kinetic type of friction coefficient from the static type. (static type of friction coefficient is shown by \( \mu_s \)). In a force analysis the static friction coefficient applies until no relative motion has occurred. Once the kinetic type of friction coefficient is used to determine the friction force. The friction force acts in a direction opposite to the direction of velocity and is a constant force.

3.2.1.5.1 Equation of motion

Fig 3.16 shows a single degree of freedom system with Coulomb damping. Since the direction of the frictional force varies with the direction of velocity, two cases of force classification can be considered as shown in Figs 3.16b and 3.16c

Case 1: When the velocity, \( \dot{x} \), is positive (Fig 3.16b) and the displacement, \( x \), is either positive or negative (i.e., for half cycle during which the mass moves from left to right), the equation of motion can be obtained using Newton’s second law.

\[ -kx - \mu_s N = m\ddot{x} \]

or
This is a second order nonhomogeneous differential equation. The solution to this equation may be written as

\[ x(t) = A_1 \cos \omega_c t + A_2 \sin \omega_c t - \frac{\mu N}{k} \]  

[E.Kreßig, 1999]  

Where \( \omega_c = \sqrt{\frac{k}{m}} \)

is the circular frequency of vibration (natural frequency of the system) and \( A_1 \) and \( A_2 \) are the constants whose values depend on the initial conditions of the first half cycle. The derivative of Eqn 3.48 can be found as

\[ \dot{x}(t) = -A_1 \omega_c \sin \omega_c t + A_2 \omega_c \cos \omega_c t \]  

(3.48a)

The body is in contact with the surface but is not connected to it.

Figure 3.16 Single degree of freedom (spring-mass) system with Coulomb damping

Case 2: When \( \dot{x} \) is negative (Fig 3.16c) and \( x \) is either positive or negative (i.e., for the half cycle during which the mass moves from right to left), the equation of motion, like the previous case, can be written as

\[ -k x + \mu N = m \dot{x} \]
or
\[ m\ddot{x} + kx = \mu_s N \] \hspace{1cm} (3.49)

The general form of the solution of this equation, like the previous case, is
\[ x(t) = A_3 \cos \omega_s t + A_4 \sin \omega_s t + \frac{\mu_s N}{k} \] \hspace{1cm} (3.50)

where \( A_3 \) and \( A_4 \) are constants and can be evaluated by applying the initial conditions. The derivative of the above equation can be obtained as
\[ \dot{x}(t) = -A_3 \omega_s \sin \omega_s t + A_4 \omega_s \cos \omega_s t \] \hspace{1cm} (3.50a)

The initial conditions of the system are
\[ x(t = 0) = X_0 \]
\[ \dot{x}(t = 0) = 0 \] \hspace{1cm} (3.51)

That is, the system starts with zero velocity and displacement \( X_0 \) at \( t=0 \). It must be noted that the condition for starting the motion is
\[ kX_0 > \mu_s N \]

where \( k \), \( \mu_s \) and \( N \) are the stiffness, coefficient of static friction and normal force between the surfaces, respectively.

or
\[ X_0 > \frac{\mu_s N}{k} \]

Clearly, when \( x \) (displacement of motion) becomes less than \( \frac{\mu_s N}{k} \) or
\[ x < \frac{\mu_s N}{k} \]

the movement of the mass stops.

Since \( x = X_0 \) at \( t=0 \), the motion starts from the right to the left. Assuming that \( X_0, X_1, X_2, \ldots \) denote the amplitudes of the motion at successive half cycles and using Eqns 3.50 and 3.51 the constants \( A_3 \) and \( A_4 \) can be evaluated:
\[ X_0 = A_3 + \frac{\mu_s N}{k} \]

or
Chapter 3 Theoretical studies

Also, from Eqns 3.50a and 3.51

\[ 0 = A_4 \omega_n \]

or

\[ A_4 = 0 \]

As a result, Eqns 3.50 and 3.50a become

\[ x(t) = (X_0 - \frac{\mu_s N}{k}) \cos \omega_n t + \frac{\mu_s N}{k} \]

\[ \dot{x}(t) = -\omega_n (X_0 - \frac{\mu_s N}{k}) \sin \omega_n t \] (3.52a)

Eqn 3.52a can also be obtained as the derivative of Eqn 3.52. Eqns 3.52 and 3.52a are valid for the first half the cycle, i.e. for

\[ 0 \leq t \leq \frac{\pi}{\omega_n} \]

When \( t = \pi / \omega_n \), the mass will be at its extreme left position and its displacement from the equilibrium position can be found from Eqn 3.52

\[ X_1 = x(t = \frac{\pi}{\omega_n}) \]

where \( X_1 \) is the displacement of the system at the end of the first half cycle. So, \( X_1 \) can be evaluated from Eqn 3.52, as

\[ X_1 = (X_0 - \frac{\mu_s N}{k}) \cos \pi + \frac{\mu_s N}{k} \]

or

\[ X_1 = -(X_0 - \frac{2\mu_s N}{k}) \] (3.53)

Also, from Eqn 3.52a

\[ \dot{x}(\frac{\pi}{\omega_n}) = 0 \] (3.53a)

Since the motion started with a displacement of \( x(t) = X_0 \), from Eqn 3.53 it can be seen that
the absolute value of reduction of the magnitude of \( x(t) \) in time \( \frac{\pi}{\omega_n} \) is

\[
\frac{2\mu_1 N}{k}
\]

For evaluation of \( A_1 \) and \( A_2 \) of Eqn 3.48, the initial conditions of the system at the beginning of the second half cycle must be applied. Eqns 3.53 and 3.53a that represent the displacement and velocity of the system, respectively, can be considered as the initial conditions of the system in the second half cycle. Equation of motion of the system for the second half cycle is given by Eqn 3.48. Substituting Eqn 3.53 into Eqn 3.48 gives

\[
-(X_0 - \frac{2\mu_1 N}{k}) = A_1 \cos \pi + A_2 \sin \pi - \frac{\mu_1 N}{k}
\]

or

\[
A_1 = X_0 - \frac{3\mu_1 N}{k}
\]

Also, from Eqns 3.48a and 3.53a

\[0 = -A_1 \omega_n \sin \pi + A_2 \omega_n \cos \pi\]

or

\[A_2 = 0\]

Substituting the above values of \( A_1 \) and \( A_2 \) into Eqn 3.48 gives

\[
x(t) = (X_0 - \frac{3\mu_1 N}{k}) \cos \omega_n t - \frac{\mu_1 N}{k} \tag{3.54}
\]

The derivative of Eqn 3.54 can be obtained as

\[
\dot{x}(t) = -\omega_n (X_0 - \frac{3\mu_1 N}{k}) \sin \omega_n t \tag{3.54a}
\]

Eqns 3.54 and 3.54a are valid only for the second half cycle that is for

\[
\frac{\pi}{\omega_n} \leq t \leq \frac{2\pi}{\omega_n}
\]

At the end of this half cycle the value of \( x(t) \) is

\[
X_0 - \frac{4\mu_1 N}{k} \quad \text{(at } t = \frac{2\pi}{\omega_n}) \tag{3.55}
\]

This is equal to \( X_2 \). Furthermore, from Eqn 3.54a the velocity at the end of the first cycle
can be found to be equal to zero ($\dot{x}(t) = 0$). The above values of $x$ and $\dot{x}$ become the initial conditions for the third half cycle and the procedure can be continued until the motion has stopped. Fig 3.17 shows the free vibration of a single degree of freedom system with Coulomb damping.

![Figure 3.17 Free vibration of a system with Coulomb damping](image)

Taking into consideration the above discussion the following characteristics of a free vibration with Coulomb damping can be concluded.

- In each successive cycle the amplitude of motion is reduced by the amount of $\frac{4\mu}{k}$. So, the amplitude of the vibration at the end of any successive cycles are related to

$$X_m = X_{m-1} - \frac{4\mu}{k}$$

(3.56)
• As the amplitude is reduced by an amount, \(4\mu_1 N/k\), in one cycle (i.e., in time \(2\pi/\omega_n\)), the slope of the enveloping straight lines shown in Fig 3.17 is

\[
-\left(\frac{4\mu_1 N}{k}\right) \frac{2\pi}{\omega_n} = -\left(\frac{2\mu_1 N \omega_n}{k\pi}\right)
\]

The final position of the mass is usually displaced from equilibrium \((x=0)\) position and represents a permanent displacement in which the friction force is locked. Slight beating can make the mass come to the equilibrium position.

• The natural frequency of the system has not been changed in Coulomb damping, in contrast to the viscous damping.

3.2.1.6 Forced vibration of viscously damped systems

As discussed in the above sections, in the free vibration of a vibratory system, the amplitude of the vibration diminishes after a number of periods. In contrast, in a forced vibration, the system may experience a steady state vibration without any reduction in the magnitude of the vibration amplitude.

When a system is subjected to harmonic excitation, it is forced to vibrate at the same frequency as the excitation force. For example consider the system shown in Fig 3.19 which is a single degree of freedom system, subjected to the harmonic force \(F_0 \sin \omega t\).

The system shown in Fig 3.19a consists of one mass \(m\), one spring with the stiffness \(k\) and a mechanism for energy dissipation which is designated by a dashpot with coefficient of damping \(c\). Rollers are placed under the mass in order to eliminate the friction.

From the free body diagram of the system which is shown in Fig 3.19b, the equation of motion of the system can be written as

\[m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t\] (3.58)

where \(m\), \(c\) and \(k\) are as usual, the mass, coefficient of damping and stiffness of the system, respectively. Also \(x\), \(\dot{x}\) and \(\ddot{x}\) are the displacement, velocity and acceleration of the mass, respectively, and \(F_0\) is the amplitude of the harmonic applied load.

The solution of the above equation consists of:
1-The solution of the homogeneous equation which has been discussed in Section 3.2.1.3 and
2-the particular solution which is the steady state oscillation with $\omega$ as the frequency of the excitation.

![Figure 3.19](image)

(a): Forced vibration of undamped SDOF system

(b): Free body diagram of mass

The particular solution is of the form

$$x = X \sin(\omega t - \phi_2) \quad [\text{W T. Thomson, 1993}]$$

(3.59)

where $X$ is the amplitude of oscillation and $\phi_2$ is the phase angle of the displacement with respect to the exciting force. Substitution of Eqn 3.59 and its derivatives

$$\dot{x} = X\omega \cos(\omega t - \phi_2) \quad (3.60)$$

$$\ddot{x} = -X\omega^2 \sin(\omega t - \phi_2) \quad (3.61)$$

into Eqn 3.58 leads to the following results:

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \quad (3.62)$$

$$\phi_2 = \tan^{-1} \frac{c\omega}{k - m\omega^2} \quad (3.63)$$

A non-dimensional form of Eqns 3.62 and 3.63 can be found by dividing the numerator and denominator of the equations by $k$ as follows:
\[ X = \frac{F_0 / k}{\sqrt{(1 - \frac{m\omega^2}{k})^2 + \left(\frac{c\omega}{k}\right)^2}} \quad (3.64) \]
\[ \phi_2 = \tan^{-1}\frac{\frac{c\omega}{k}}{1 - \frac{m\omega^2}{k}} \quad (3.65) \]

Using the notation:

\[ \omega_n = \sqrt{\frac{k}{m}} = \text{natural frequency of undamped oscillation} \]
\[ c_c = 2m\omega_n = \text{critical damping coefficient} \]
\[ \zeta = \frac{c}{c_c} = \text{damping ratio} \]

the non-dimensional expressions for the amplitude and phase as given in Eqns 3.64 and 3.65 will then become

\[ \frac{Xk}{F_0} = \frac{1}{\sqrt{[1 - (\omega / \omega_n)^2]^2 + [2\zeta(\omega / \omega_n)]^2}} \quad (3.66) \]
\[ \phi_2 = \tan^{-1}\frac{2\zeta(\omega / \omega_n)}{1 - (\omega / \omega_n)^2} \quad (3.67) \]

These equations indicate that the non-dimensional amplitude \( Xk / F_0 \) and the phase \( \phi \) are functions of the frequency ratio \( \omega / \omega_n \) and the damping ratio \( \zeta \). Eqns 3.66 and 3.67 are plotted in Figs 3.20 and 3.21. These curves show that the damping ratio has a crucial influence on the amplitude and phase angle in the region around \( \zeta = 1 \).

As mentioned before, the complete solution of Eqn 3.58 is the sum of the general solution as given in Eqn 3.32 and the particular solution by Eqn 3.59. The complete solution is then given by

\[ x(t) = \frac{F_0}{k} \sin(\omega t - \phi_2) + X_1 e^{-\zeta \omega_n t} \sin(\sqrt{1 - \zeta^2} \omega_n t + \phi_1) \quad (3.68) \]
Figure 3.20 Plot of equation 3.67

Figure 3.21 Plot of equation 3.67
In this equation the first term in the right-hand side is the particular solution and the second term is the homogeneous (general) solution of Eqn 3.58. The amplitude and phase angle of the homogeneous (general) solution, that is, $X_1$ and $\phi_1$, can be found using the initial conditions.

3.2.2 Two-degree of freedom systems

Consider a system with more than a single degree of freedom (SDOF) as the systems shown in Figs 3.6 and 3.7. When a system requires more than one coordinate to describe its motion, it is called a multi-degree of freedom (MDOF), or an n-DOF system, where n is the required number of coordinates.

In contrast with a SDOF system which has one natural frequency, an n-DOF system has n natural frequencies. Another difference between a SDOF system as compared with a MDOF system is in relation to the normal modes of vibration. To elaborate, there is a natural state of vibration with a displacement configuration, known as the ‘normal mode’.

Normal mode vibrations are free undamped vibrations that depend only on the mass and stiffness of the system and how they are distributed. When the system vibrates at one of these normal modes, all points in the system undergo simple harmonic motions.

Like in the SDOF system, forced harmonic vibration of the n-DOF system takes place at the frequency of excitation. When the excitation frequency coincides with one of the natural frequencies of the system, resonance takes place with large amplitude which is reduced only by damping.

As mentioned before, the damping may be ignored in some problems but in determining the limiting effects on amplitude in resonance and examining the rate of decay of the free vibration, the damping has to be considered.

The simplest n-DOF system is a 2-DOF system which requires two independent coordinates to describe its motion.

3.2.2.1 Equation of motion of undamped free vibration of 2-DOF systems
Fig 3.22 shows an undamped 2-DOF system with its free-body diagram. The system consists of two masses including \( m_1 \) and \( m_2 \) with the coordinates \( x_1 \) and \( x_2 \). Rollers have been applied under masses in order to reduce the friction highly, so the friction force is neglected during analysis.

\[
\begin{align*}
  k_1 &= 38 \text{ N/m} \\
  k_2 &= 3 \text{ N/m} \\
  k_3 &= 2 \text{ N/m}
\end{align*}
\]

\( x_1, x_2 \)

![Free body diagrams of masses](image)

(a)

(b)

Figure 3.22

(a): Free vibration of undamped 2-DOF system

(b): Free body diagrams of masses

The system included three springs, denoted by \( k_1, k_2 \) and \( k_3 \). Each of the spring can apply a force equal to displacement \( x \) multiplied by stiffness coefficient of the spring \( k \)

\[ F = kx \]  \hspace{1cm} (3.69)

in which \( F \) is the force produced in spring.

Thus, \( k_1 x_1 \) and \( k_3 (x_1 - x_2) \) are the applied forces on \( m_1 \) by springs \( k_1 \) and \( k_3 \). Similarly \( k_2 x_2 \) and \( k_3 (x_1 - x_2) \) are the forces applied by springs \( k_2 \) and \( k_3 \) on the mass \( m_2 \).

To analyse the dynamic behaviour of the system the equation of motion must be written for each of masses individually as follows:
\[
\begin{align*}
-k_1x_1 - k_3(x_1 - x_2) &= m_1\ddot{x}_1 \\
-k_2x_2 + k_3(x_1 - x_2) &= m_2\ddot{x}_2
\end{align*}
\]  
(3.70)

Rearranging Eqn 3.70 gives
\[
\begin{align*}
m_1\ddot{x}_1 + (k_1 + k_3)x_1 - k_3x_2 &= 0 \\
m_2\ddot{x}_2 - k_3x_1 + (k_1 + k_3)x_2 &= 0
\end{align*}
\]  
(3.71)

The above equations can be written in the matrix form as follows:
\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
+
\begin{bmatrix}
k_1 + k_3 & -k_3 \\
-k_3 & k_1 + k_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  
(3.72)

or
\[
M\ddot{x} + Kx = 0
\]  
(3.73)

where \(M\) is the mass matrix
\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\]
\(K\) is the stiffness matrix
\[
\begin{bmatrix}
k_1 + k_3 & -k_3 \\
-k_3 & k_1 + k_3
\end{bmatrix}
\]
\(\ddot{x}\) is the acceleration vector
\[
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
\]  
and \(x\) is the displacement vector
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Eqns 3.72 are coupled through the stiffness matrix. That is, one equation cannot be solved independent of the other and the equations have to be solved simultaneously. The general form of the response equation for the 2-DOF system of Fig 3.22 is
\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
sin(\omega t + \phi)
\]  
(3.74)

Substitution of Eqn 3.74 into Eqn 3.71 gives
\[
\begin{align*}
(k_3 + k_1 - m_1\omega^2)X_1 - k_3X_2 &= 0 \\
-k_3X_1 + (k_3 + k_2 - m_2\omega^2)X_2 &= 0
\end{align*}
\]  
(3.75)
In the matrix form the above equation can be written as

\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
(-\omega^2)
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
+
\begin{bmatrix}
k_1 + k_3 & -k_3 \\
-k_3 & k_3 + k_2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(3.76)

or

\[
\begin{bmatrix}
k_1 + k_3 - m_1 \omega^2 & -k_3 \\
-k_3 & k_3 + k_2 - m_2 \omega^2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(3.77)

or

\[
[K - \omega^2 M]
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(3.78)

Clearly the zero vector

\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

is a solution of Eqn 3.78 which implies that \( x(t) = 0 \) that is no motion of the system and it is trivial solution. The Eqn 3.78 will be satisfied if the determinant of the matrix

\[
[K - \omega^2 M]
\]

of equation 3.78 is zero.

\[
\begin{vmatrix}
k_1 + k - m_1 \omega^2 & -k \\
-k & k + k_2 - m_2 \omega^2
\end{vmatrix}
= 0
\]

(3.79)

The solution of the above equation leads to the determination of the eigenvalues \( \omega_1 \) and \( \omega_2 \), that are two natural frequencies of the system.

As an example if

\[
\begin{align*}
k_1 &= k_2 = k, \\
k_3 &= 2k, \\
m_1 &= 2m \quad \text{and} \\
m_2 &= m
\end{align*}
\]

Eqns 3.75 will be as follows


\[
\begin{align*}
(3k - 2m\omega^2)X_1 - 2kX_2 &= 0 \\
-2kX_1 + (3k - m\omega^2)X_2 &= 0
\end{align*}
\]  
\hspace{1cm} (3.80)

and the determinant in Eqn 3.79 will become
\[
\begin{vmatrix}
3k - 2\omega^2 m & -2k \\
-2k & 3k - \omega^2 m
\end{vmatrix} = 0
\]  
\hspace{1cm} (3.81)

Using the notation \(\omega^2 = \lambda\) and expanding the determinant gives
\[
\lambda^2 - \left(\frac{9}{2}\frac{k}{m}\right)\lambda + \frac{5}{2}\left(\frac{k}{m}\right)^2 = 0
\]  
\hspace{1cm} (3.82)

This equation is referred to as the 'characteristic equation' of the system and the two roots \(\lambda_1\) and \(\lambda_2\) of this equation are the eigenvalues of the system. These eigenvalues are found to be
\[
\begin{align*}
\lambda_1 &= \left(\frac{9}{4} - \frac{1}{4}\sqrt{41}\right)\frac{k}{m} = 0.649 \frac{k}{m} \\
\lambda_2 &= \left(\frac{9}{4} + \frac{1}{4}\sqrt{41}\right)\frac{k}{m} = 3.85 \frac{k}{m}
\end{align*}
\]

Therefore, the natural frequencies of the system are
\[
\begin{align*}
\omega_1 &= 0.806\sqrt{\frac{k}{m}} \\
\omega_2 &= 1.96\sqrt{\frac{k}{m}}
\end{align*}
\]  
\hspace{1cm} (3.83) \hspace{1cm} (3.84)

### 3.2.2.2 Normal modes

When a system vibrates at one of its natural frequencies, all points of the system undergo simple harmonic motion passing through their equilibrium positions, simultaneously. This motion which has a specific of displacement configuration of the system is called a 'normal mode'. A normal mode of the system can be expressed by the amplitudes ratio
\[
\frac{X_1}{X_2}
\]
where the $X_1$ and $X_2$ are the amplitudes of $m_1$ and $m_2$ in Fig 3.22. From Eqns 3.80 two expressions for the ratio of amplitudes are found

$$\begin{align*}
\frac{X_1}{X_2} &= \frac{2k}{3k-2\omega^2 m} \\
\frac{X_1}{X_2} &= \frac{3k-\omega^2 m}{2k}
\end{align*}$$

(3.85)

Substitution of the natural frequencies $\omega_1$ and $\omega_2$ from Eqns 3.83 and 3.84 into either of the above equations leads to the ratio of the amplitudes. For

$$\omega_1 = 0.806\sqrt{\frac{k}{m}}$$

Eqns 3.85 give

$$\left(\frac{X_1}{X_2}\right)_1 = \frac{2k}{3k-2m\omega^2} = \frac{2}{3-2\times 0.649} = 1.175$$

(3.86)

which is the amplitude ratio corresponding to the first natural frequency. Similarly, using

$$\omega_2 = 1.96\sqrt{\frac{k}{m}}$$

give

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{2k}{3k-2m\omega^2} = \frac{2}{3-2\times 3.85} = -0.426$$

(3.87)

which is the amplitude ratio corresponding to the second natural frequency. If one of the amplitude is chosen equal to 1 or any other number, it is said that the amplitude is ‘normalised’. The normalised amplitude ratios can then be used to find the normal modes denoted by $\psi_1(x)$ and $\psi_2(x)$.

$$\psi_1(x) = \begin{bmatrix} 1.175 \\ 1 \end{bmatrix}$$

(3.88)

and

$$\psi_2(x) = \begin{bmatrix} -0.426 \\ 1 \end{bmatrix}$$

(3.89)

The mode $\psi_1(x)$ indicates that the two masses $m_1$ and $m_2$ move in phase and $\psi_2(x)$ indicates that the two masses move out of phase with each other. These normal modes are shown graphically in Fig 3.23.
3.2.2.3. Example of free vibration of a 2-DOF system

As a numerical example, the solution of the free vibration of the 2-DOF system shown in Fig 3.22 is carried out. The values of the basic parameters of the system are:

- \( m_1 = 9 \) kg
- \( m_2 = 1 \) kg
- \( k_1 = 38 \) N/m
- \( k_2 = 3 \) N/m
- \( k_3 = 2 \) N/m

The initial conditions of the system are assumed to be:

- \( x_1(0) = 1 \) m
- \( \dot{x}_1(0) = 0 \) m/s
- \( x_2(0) = 0 \) m
- \( \dot{x}_2(0) = 0 \) m/s

**Determination of natural frequencies**

From Eqns 3.72, the mass and stiffness matrices of the system can be found as

\[
M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 40 & -2 \\ -2 & 5 \end{bmatrix}
\]  

The characteristic equation is given by

\[
\text{Det}[K - \lambda M] = 0
\]
where \( \lambda \) is given by
\[
\lambda = \omega^2
\]
From Eqns 3.90 and 3.91
\[
\begin{bmatrix} 40 & -2 \\ -2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} = 0
\]
or
\[
\begin{bmatrix} 40 - 9\lambda & -2 \\ -2 & 5 - \lambda \end{bmatrix} = 0
\]
Expansion the above equation gives
\[
(40 - 9\lambda)(5 - \lambda) - 4 = 0
\]
or
\[
9\lambda^2 - 85\lambda + 196 = 0
\]
or
\[
\lambda_1 = 4 = \omega_1^2
\]
\[
\lambda_2 = \frac{49}{9} = \omega_2^2
\]
or
\[
\omega_1 = 2 \text{ rad/s}
\]
and
\[
\omega_2 = 7/3 \text{ rad/s}
\]

**Determination of mode shapes**

Using Eqns 3.75,
\[
\begin{bmatrix} (k_3 + k_1 - m_1\omega^2)X_1 - k_3X_2 = 0 \\ -k_3X_1 + (k_3 + k_2 - m_2\omega^2)X_2 = 0 \end{bmatrix}
\]
and substituting the values of \( \omega_1, k_1, k_2 \) and \( k_3 \) into one of the above equations (here, the first one) gives
\[
(40 - 9 \times 4)X_1 - 2X_2 = 0
\]
or
\[ \frac{X_1}{X_2} = \frac{2}{4} = \frac{1}{2} \]

By choosing \( X_2 = 1 \) the first normal mode can be found as
\[ \psi_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \]

Also, substituting \( \omega_2 = 7/3 \) into Eqn 3.75 gives
\[ (40 - 9 \times \frac{49}{9}) X_1 - 2X_2 = 0 \]
or
\[ \frac{X_1}{X_2} = \frac{2}{-9} = -\frac{0.222}{1} \]

and the second mode shape is obtained as
\[ \psi_2 = \begin{bmatrix} -0.222 \\ 1 \end{bmatrix} \]

Development of the response equations

The general form of the response of the system is given by Eqns 3.74, that are,
\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega t + \phi) \]

For free vibration to take place in one of the normal modes (say, mode \( i \)) for any initial conditions, the equation of motion for mode \( i \) must be of the form
\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{(i)} = A_i \psi_i \sin(\omega_i t + \phi_i) \quad i=1,2 \]

The constants \( A_i \) and \( \phi_i \) are found to satisfy the initial conditions and \( \psi_i \) ensures that the amplitude ratio for the free vibration is proportional to that of mode \( i \).
For initial conditions, in general, the free vibration contains both modes simultaneously. Depending on the initial conditions, the response may be more close to one mode shape. So, the equations of motion are of the form

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.222 \end{bmatrix} \sin(\omega_1 t + \phi_1) + \begin{bmatrix} 1 \\ -0.222 \end{bmatrix} \sin(\omega_2 t + \phi_2)
\] (3.92)

where \( A_1, A_2, \phi_1 \) and \( \phi_2 \) are four necessary constants for the two differential equations of second order. Constants, \( A_1, A_2 \), establish the amount of each mode and phases, \( \phi_1 \) and \( \phi_2 \), are used to indicate the positions of the masses at time origin. To solve for the four arbitrary constants, two more equation are needed which are available by differentiating Eqn 3.92 for the velocity

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \omega_1 A_1 \begin{bmatrix} 0.5 \\ -0.222 \end{bmatrix} \cos(\omega_1 t + \phi_1) + \omega_2 A_2 \begin{bmatrix} -0.222 \\ 1 \end{bmatrix} \cos(\omega_2 t + \phi_2)
\] (3.93)

Substituting \( t=0 \) into Eqns 3.92 and 3.93 and applying the initial conditions, the four constants can be obtained as follows:

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.222 \end{bmatrix} \sin \phi_1 + \begin{bmatrix} -0.222 \\ 1 \end{bmatrix} \sin \phi_2 \quad t=0
\]

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2A_1 \begin{bmatrix} 0.5 \\ -0.222 \end{bmatrix} \cos \phi_1 + \frac{7}{3} A_2 \begin{bmatrix} -0.222 \\ 1 \end{bmatrix} \cos \phi_2 \quad t=0
\]

or

\[
0.5A_1 \sin \phi_1 - 0.222A_2 \sin \phi_2 = 1
\] (3.94)

\[
A_1 \sin \phi_1 + A_2 \sin \phi_2 = 0
\] (3.95)

\[
A_1 \cos \phi_1 - 0.518A_2 \cos \phi_2 = 0
\] (3.96)

\[
2A_1 \cos \phi_1 + 2.333A_2 \cos \phi_2 = 0
\] (3.97)

Multiplying Eqn 3.96 by \( -2 \) and adding the result to Eqn 3.97 gives

\[
3.369A_2 \cos \phi_2 = 0
\] (3.98)

Also, multiplying Eqn 3.96 by \((2.333/0.518)\) and adding the result to Eqn 3.97 gives

\[
6.504A_1 \cos \phi_1 = 0
\] (3.99)
Eqns 3.98 and 3.99 will be satisfied if

\[
\cos \phi_1 = \cos \phi_2 = 0
\]
or

\[
\phi_1 = \phi_2 = 90^\circ
\]

Also, from Eqns 3.94 and 3.95 it follows that

\[
1.444A_2 = -2
\]

and

\[
A_1 = -A_2
\]
or

\[
A_1 = 1.385
\]
\[
A_2 = -1.385
\]

Finally, the equations of motion (Eqns 3.92) will assume the form

\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = 1.385 \begin{bmatrix}
  0.5 \\
  1
\end{bmatrix} \sin(2t + \frac{\pi}{2}) - 1.385 \begin{bmatrix}
  -0.222 \\
  1
\end{bmatrix} \sin(\frac{7}{3}t + \frac{\pi}{2})
\]

or

\[
x_1 = 0.693 \sin(2t + \frac{\pi}{2}) + 0.307 \sin(\frac{7}{3}t + \frac{\pi}{2}) \tag{3.100}
\]
\[
x_2 = 1.385 \sin(2t + \frac{\pi}{2}) - 1.385 \sin(\frac{7}{3}t + \frac{\pi}{2}) \tag{3.101}
\]

Fig 3.24 shows the variations of \(x_1\) and \(x_2\) with respect to time (Eqns 3.100 and 3.101) graphically.

Since the system of Fig 3.22 is an undamped system, there is no dissipation of energy in this vibrating system. So, the system is conservative and the total amount of the energy of the system remains constant. In a mathematical form this characteristic can be represented by

\[
\frac{dE}{dt} = 0
\]

where \(E\) is the total energy of the system.

3.2.2.4 Forced vibration of undamped 2-DOF systems
Consider a 2-DOF system subjected to the harmonic force

\[ f(t) = F_1 \sin \omega t \]

As shown in Fig 3.25 is a 2-DOF system in forced vibration. The equation of motion for the system can be written as

\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
0
\end{bmatrix} \sin \omega t
\]

(3.108)

The solid line represents the vibration of \( m_1 \)
and the dashed line represents the vibration of \( m_2 \).

Figure 3.24 Free vibration of the system of Fig 3.22
where

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\]

is the mass matrix of the system,

\[
\begin{bmatrix}
  k_{11} & k_{12} \\
  k_{21} & k_{22}
\end{bmatrix}
\]

is the stiffness matrix of the system,

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

are the vectors of displacement and acceleration of the system, respectively, and

\[
\begin{bmatrix}
F_1 \\
0
\end{bmatrix}\sin \omega t
\]

is the vector of applied force. The solution of Eqn 3.108 is of the form

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\sin \omega t \tag{3.109}
\]

in which \(X_1\) and \(X_2\) are the amplitudes of the harmonic displacements of the two masses \(m_1\) and \(m_2\).

Substitution of Eqn 3.109 into Eqn 3.108 gives

\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}(-\omega^2) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\sin \omega t + \begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\sin \omega t = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}\sin \omega t
\]

or

\[
\begin{bmatrix}
k_{11} - m_1 \omega^2 & k_{12} \\
k_{21} & k_{22} - m_2 \omega^2
\end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \tag{3.110}
\]

or

\[
\begin{bmatrix}
K - \omega^2 M
\end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \tag{3.111}
\]

where \(K\) and \(M\) denote the stiffness and mass matrices, respectively. Premultiplying Eqn 3.111 by
\[
\begin{bmatrix}
K - \omega^2 M
\end{bmatrix}^{-1}
gives
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = \begin{bmatrix}
K - \omega^2 M
\end{bmatrix}^{-1} \begin{bmatrix}
F_1 \\
0
\end{bmatrix}
\]  \hspace{1cm} (3.112)

---

Figure 3.25 Forced vibration of undamped 2-DOF system

It is well known fact that
\[
\begin{bmatrix}
K - \omega^2 M
\end{bmatrix}^{-1} = \frac{\text{adj} [K - \omega^2 M]}{|K - \omega^2 M|}.
\]  \hspace{1cm} (3.113)

where

\[
\text{adj} [K - \omega^2 M]
\]
is the adjoint of
\[
[K - \omega^2 M]
\]
and
\[
|K - \omega^2 M|
\]
is the determinant of
\[
\begin{vmatrix}
K - \omega^2 M
\end{vmatrix}
\]
\[
\text{adj}
\begin{vmatrix}
K - \omega^2 M
\end{vmatrix} = \begin{bmatrix}
(k_{22} - m_2 \omega^2) & -k_{12} \\
-k_{21} & (k_{11} - m_1 \omega^2)
\end{bmatrix}
\]

(3.114)

Therefore the Eqn 3.112 becomes
\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = \frac{1}{K - \omega^2 M} \begin{bmatrix}
(k_{22} - m_2 \omega^2) & -k_{12} \\
-k_{21} & (k_{11} - m_1 \omega^2)
\end{bmatrix} \begin{bmatrix}
F_1 \\
0
\end{bmatrix}
\]

(3.115)

From Eqn 3.110, the expansion of the determinant
\[
|K - \omega^2 M|
\]
is found to be
\[
m_1 m_2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)
\]

(3.116)

where \(\omega_1\) and \(\omega_2\) are the natural frequencies of the system.

Substitution of Eqn 3.116 into Eqn 3.115 gives
\[
\begin{align*}
X_1 &= \frac{(k_{22} - m_2 \omega^2)F_1}{m_1 m_2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} \\
X_2 &= \frac{-k_{12}F_1}{m_1 m_2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}
\end{align*}
\]

(3.117)

As an example, consider the system shown in Fig 3.25 with the following assumptions:

\(k_1 = k_2 = k,\)

\(k_3 = 2k,\)

\(m_1 = 2m\) and

\(m_2 = m\)

Eqns 3.115 become
\[
X_1 = \frac{(3k - m\omega^2)F_1}{2m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} \\
X_2 = \frac{kF_1}{2m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}
\] (3.118)

In a non-dimensional form, Eqns 3.118 can be written as

\[
\begin{align*}
X_1 k &= \frac{k(3k - m\omega^2)}{F_1 2m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} \\
X_2 k &= \frac{k^2}{F_1 2m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}
\end{align*}
\] (3.119)

The natural frequencies of the system as obtained in Section 3.2.2.1 are

\[
\omega_1 = 0.806 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.96 \sqrt{\frac{k}{m}}
\]

Substitution of these values in Eqns 3.119 gives

\[
\begin{align*}
X_1 k &= \frac{3 - 1.95(\frac{\omega}{\omega_1})^2}{0.845[1 - (\frac{\omega}{\omega_1})^2][5.9 - (\frac{\omega}{\omega_1})^2]} \\
X_2 k &= \frac{1}{0.845[1 - (\frac{\omega}{\omega_1})^2][5.9 - (\frac{\omega}{\omega_1})^2]}
\end{align*}
\] (3.120)

Eqns 3.120 are non-dimensional equations that are shown in Figs 3.26a and 3.26b

### 3.2.2.5 Example of viscously damped 2-DOF system

Fig 3.27 shows a 2-DOF system. The system is viscously damped. Taking into consideration the free body diagram of the masses, the equation of motion for each mass can be written individually as follows:

\[
-k_1x_1 - k_3(x_1 - x_2) - c_1\ddot{x}_1 - c_3(\ddot{x}_1 - \ddot{x}_2) = m_1\ddot{x}_1
\]

\[
-k_2x_2 + k_3(x_1 - x_2) - c_2\ddot{x}_2 + c_3(\ddot{x}_1 - \ddot{x}_2) = m_2\ddot{x}_2
\] (3.121)
Rearranging Eqns 3.121 gives

\[ m_1 \ddot{x}_1 + (k_1 + k_3)x_1 - k_3x_2 + (c_1 + c_3)\dot{x}_1 - c_3\dot{x}_2 = 0 \]  

(3.122)

\[ m_2 \ddot{x}_2 - k_3x_1 + (k_2 + k_3)x_2 - c_3\dot{x}_1 + (c_2 + c_3)\dot{x}_2 = 0 \]

Eqns 3.122 can be rewritten in the matrix form as

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
\end{bmatrix}
+ 
\begin{bmatrix}
    c_1 + c_3 & -c_3 \\
    -c_3 & c_2 + c_3 \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
\end{bmatrix}
+ 
\begin{bmatrix}
    k_1 + k_3 & -k_3 \\
    -k_3 & k_2 + k_3 \\
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
    0 \\
    0 \\
\end{bmatrix}
\]  

(3.123)

or

\[ M\ddot{x} + C\dot{x} + Kx = 0 \]  

(3.124)

where

- \( M \) is the mass matrix of the system,
- \( C \) is the damping matrix of the system and
- \( K \) is the stiffness matrix of the system.
As an example consider the damped two degree of freedom system of Fig 3.27 with the following numerical values:

\[
\frac{X_{x,k}}{F_1},
\]

Figure 3.26b  Plot of equation 3.120 (second equation)

Figure 3.27  Free vibration of damped 2-DOF system
$m_1 = 20$ kg
$m_2 = 14$ kg
$k_1 = 380$ N/m
$k_2 = 105$ N/m
$k_3 = 160$ N/m
$c_1 = 20$ kg/s
$c_2 = 5$ kg/s
$c_3 = 7.5$ kg/s

Substituting the above values into Eqn 3.123 gives

$$\begin{bmatrix} 20 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 20 + 7.5 & -7.5 \\ -7.5 & 5 + 7.5 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 380 + 160 & -160 \\ -160 & 105 + 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$20\ddot{x}_1 + 27.5\dot{x}_1 - 7.5\ddot{x}_2 + 540x_1 - 160x_2 = 0$$
$$14\ddot{x}_2 - 7.5\dot{x}_1 + 12.5\ddot{x}_2 - 160x_1 + 265x_2 = 0 \tag{3.125}$$

Before the solution of Eqns 3.125, ignore the damping in the system and determine the natural frequencies and normal modes. Ignoring the damping in the system reduces Eqns 3.125 to

$$20\ddot{x}_1 + 540x_1 - 160x_2 = 0 \tag{3.126}$$
$$14\ddot{x}_2 - 160x_1 + 265x_2 = 0$$

In a similar manner as used to determine the natural frequencies and normal modes in Section 3.2.2.3, the natural frequencies and normal modes can be determined as

$$\omega_1 = 3.55 \text{ rad/s}$$
$$\omega_2 = 5.78 \text{ rad/s}$$

and

$$\Psi_1 = \begin{bmatrix} 0.555 \\ 1 \end{bmatrix} \quad \text{(first normal mode)}$$
$$\Psi_1 = \begin{bmatrix} -1.26 \\ 1 \end{bmatrix} \quad \text{(second normal mode)}$$

Solving of Eqns 3.125 analytically is difficult. Using MATLAB software to find out the variations of $x_1$ and $x_2$ versus time for the different set of initial conditions results in two graphs as shown in Figs 3.28 and 3.29.
The first and second degree of freedom in Figs 3.28 as well as Fig 3.29 are plotted by dotted line and solid line, respectively. In other words, the motion of the mass, $m_1$, is shown by dotted line and the motion of the mass, $m_2$, is shown by solid line. The masses, $m_1$ and $m_2$, correspond to $x_1$ and $x_2$, respectively.

The applied initial conditions for the plots shown in Fig 3.28a are assumed as

$$
\begin{align*}
    x_1(0) &= 0.555 \text{ m} \\
    \dot{x}_1(0) &= 0 \text{ m/s} \\
    x_2(0) &= 1.0 \text{ m} \\
    \dot{x}_2(0) &= 0 \text{ m/s}
\end{align*}
$$

The ratio of the initial displacement is

$$
\frac{x_1(0)}{x_2(0)} = \frac{0.555}{1}
$$

which is equal to the first normal mode. Also, the applied initial conditions for the plots shown in Fig 3.28b are assumed as

$$
\begin{align*}
    x_1(0) &= -1.26 \text{ m} \\
    \dot{x}_1(0) &= 0 \text{ m/s} \\
    x_2(0) &= 1.0 \text{ m} \\
    \dot{x}_2(0) &= 0 \text{ m/s}
\end{align*}
$$

In this case the ratio of the initial conditions is

$$
\frac{x_1(0)}{x_2(0)} = \frac{-1.26}{1}
$$

which is equal to the second normal mode.

Figs 3.29a and 3.29b show the plots of motions that subjected arbitrary initial conditions. The ratio of these initial displacements are not proportional to the first or second normal modes.

The applied initial conditions for the plots shown in Fig 3.29a are assumed as

$$
\begin{align*}
    x_1(0) &= 0 \text{ m} \\
    \dot{x}_1(0) &= 0 \text{ m/s} \\
    x_2(0) &= 0.50 \text{ m} \\
    \dot{x}_2(0) &= 0 \text{ m/s}
\end{align*}
$$
The solid line represents the vibration of $m_1$ and the dashed line represents the vibration of $m_2$.

The ratio of the initial displacements is proportional to the first normal mode of the undamped system.

\[
\begin{align*}
x_1(0) &= 1.110 \\
x_2(0) &= 2 = 1
\end{align*}
\]

The ratio of initial displacement for Fig 3.29a, that is,

\[
\frac{x_1(0)}{x_2(0)} = \frac{0}{0.50}
\]

is closer to the first normal mode than the second one. On the other hand, the ratio of initial displacement for Fig 3.29b, that is,

\[
\frac{x_1'(0)}{x_2'(0)} = \frac{-0.80}{0} = 0
\]

Also, the applied initial conditions for the plots shown in Fig 3.29b are assumed as

\[
\begin{align*}
x_1(0) &= -0.80 \text{ m} \\
x_1'(0) &= 0 \text{ m/s} \\
x_2(0) &= 1.0 \text{ m} \\
x_2'(0) &= 0 \text{ m/s}
\end{align*}
\]
\[
\frac{x_1(0)}{x_2(0)} = -0.8 \quad \frac{1}{1}
\]

is closer to the second normal mode than the first one.

The solid line represents the vibration of \(m_1\)
and the dashed line represents the vibration of \(m_2\).

The ratio of the initial displacements is proportional
to the second normal mode of the undamped system.

\[
\begin{align*}
x_1(0) &= -1.26 \\
x_2(0) &= 1
\end{align*}
\]

Figure 3.28b:

Regarding to the ratio of the initial displacement in Fig 3.28a, which is equal to the first normal mode, it can be seen from the Fig 3.28a that the motions of \(m_1\) and \(m_2\) are in phase. This motion of the system is similar to the first mode shape exactly. Also, from Fig 3.28b, it can be seen that the motions of \(m_1\) and \(m_2\) are 180° out of phase which is similar to the second mode shape, exactly. In this case, the ratio of initial displacements is equal to the second normal mode.
In Fig 3.29a the ratio of the initial displacements are more close to the first normal mode than the other one. In this figure it can be seen that the system motion is more close to the first mode shape than the other one. Unlike that of Fig 3.29a, the ratio of the initial displacements in Fig 3.29b is more close to second normal mode than the other one. Here it can be seen that the system motion is more close to the second mode shape than the first one.

The solid line represents the vibration of \( m_1 \)
and the dashed line represents the vibration of \( m_2 \).

The initial displacements are arbitrary but the ratio of them is more close to the first normal mode.

\[
\begin{align*}
x_1(0) &= 0 \\
x_2(0) &= 0.50
\end{align*}
\]

From the above discussion, it can be concluded that the nature of motion is related to the initial displacements. In other words, if the ratio of initial displacements is proportional to one of the normal modes, the system motion is exactly similar to the modal shape, corresponding
to that normal mode. In other cases, the similarity of the system motion is related to the nearness or the distance of the ratio of the initial displacements to that of normal modes.

\[ x_1(0) = -0.8 \]
\[ x_2(0) = 1 \]

Figure 3.29b

3.3 Multi-Degree of Freedom Systems

The general form of a multi-degree of freedom system (MDOF), like that of a 2-DOF system, can be given by Eqn 3.106, that is,

\[ M \ddot{x} + C \dot{x} + Kx = 0 \quad \text{(null square matrix)} \quad (3.106) \text{ repeated} \]
where $M$, $C$ and $K$ are the mass, damping and stiffness matrices, respectively, and where, $x$, $\dot{x}$ and $\ddot{x}$ are the displacement, velocity and acceleration vectors, respectively. For a system with $n$-degree of freedom, the mass matrix as well as the damping and stiffness matrices are square and have $n \times n$ elements. The displacement vector as well as the velocity and acceleration vectors are column vectors and have $n$ components.

The solution of the system of equations given by Eqn 3.106 is obtained first for the undamped vibration. The undamped free vibration of the system is given by Eqn 3.73, that is,

$$ M\ddot{x} + Kx = 0 $$

(3.73) repeated

Like the system discussed in Section 3.2.2.1, an $n$-degree of freedom system has $n$ natural frequencies. The general form of the solution of Eqn 3.73 may be given as

$$ \{x\} = \{X\} e^{i\omega t} $$

(W.T. Thomson, 1993) (3.109)

where

$$ i = \sqrt{-1} $$

$$ \{x\} = [x_1, x_2, \ldots, x_n]^T = x $$

is the response vector and

$$ \{X\} = [X_1, X_2, \ldots, X_n]^T = X $$

is the amplitude vector. The components of the amplitude vector, $X_1, X_2, \ldots, X_n$, correspond to components of the response vector $x_1, x_2, \ldots, x_n$, respectively. The first and second derivatives of Eqn 3.109 can be found as

$$ \{\dot{x}\} = i\omega \{X\} e^{i\omega t} = i\omega x $$

(3.110)

$$ \{\ddot{x}\} = -\omega^2 \{X\} e^{i\omega t} = -\omega^2 x $$

(3.111)

Substituting Eqns 3.109 and 3.111 into Eqn 3.73 gives

$$ -\omega^2 MXe^{i\omega t} + KXe^{i\omega t} = 0 $$

(null vector)

(3.112)

or

$$ (-\omega^2 M + K)X = 0 $$

(3.113)

where 0 is a vector with zero components. As discussed in Section 3.2.2.1, Eqn 3.113 will be satisfied if the determinant of matrix
\(- \omega^2 M + K\)

is equal to zero, that is,

\[\left| K - \omega^2 M \right| = 0 \quad (3.114)\]

This is the characteristic equation of the system. Assuming that

\[\omega^2 = \lambda \quad (3.115)\]

Eqn 3.114 may be rewritten as

\[\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n = 0 \quad (3.116)\]

Eqn 3.116 is the expanded characteristic equation of an n-degree of freedom system. The roots of Eqn 3.116 are the eigenvalues of matrix

\[K - \omega^2 M\]

The natural frequencies of the system are the square roots of the eigenvalues.

The solution vector, \(X\) of Eqn 3.113 corresponding to a particular eigenvalue is an eigenvector. The vector, \(X\), represents a deformation pattern of the structure for a particular frequency of vibration. Since Eqn 3.113 is homogeneous, there is not a unique solution for \(X\) [E. Kreyszig, 1999]. Only a ratio among the components of \(X\) can be obtained. In other words, vector \(X\) can be solved in terms of one of the components of \(X\). Therefore, the deflected deformation of the structure, which describes a mode shape of the vibration, is defined by the known ratios of the amplitudes of the motion. These amplitudes correspond to various points of the structure. Thus, the actual amplitudes of the vibration of various points of the structure are not direct property of a natural mode of vibration. Furthermore, these amplitudes depend on the locations and characteristics of the excitation forces. Also, these amplitudes are affected by the initial conditions of the system.

As discussed in Section 3.2.2.1, the amplitude ratios can be normalised by choosing one of the amplitudes to be equal to 1 (or any other number). The resulting amplitude ratio vectors are referred to as normal modes as represented by

\[\psi_i(x) = \left[1, \frac{X_2}{X_1}, \ldots, \frac{X_n}{X_1}\right]^T \quad \omega = \omega_i \quad (3.117)\]
where \( \psi_i \) denotes the \( i^{th} \) normal mode of the system and where the first component of the vector is chosen as 1.

An n-DOF system has n normal mode vectors (mode shapes). The mode \( \psi_i \) gives the mode shape of the free vibration of the system at the frequency \( \omega_i \) which is the \( i^{th} \) natural frequency of the system. Sometimes an unnormalised amplitude vector

\[
X_i = [X_1, X_2, \ldots, X_n]^T
\]

is used as the modal shape for \( \omega = \omega_i \).

### 3.3.1 Orthogonality of modal vectors

The normal modes of a system can be shown to be orthogonal with respect to the mass and stiffness matrices. Eqn 3.113 for the \( i^{th} \) natural frequency can be written as

\[
(-\omega_i^2 M + K)X_i = 0 \quad \text{(null vector)}
\]

or since, \( X_i \) is not a null vector

\[
-\omega_i^2 M + K = 0 \quad \text{(null square matrix)} \quad (3.118)
\]

Substituting Eqns 3.115 into Eqn 3.118 gives

\[
\lambda_i M = K \quad (3.119)
\]

Post-multiplying by the \( i^{th} \) normal mode and pre-multiplying by a different normal mode, say \( \psi_j^T \) gives

\[
\lambda_i \psi_j^T M \psi_i = \psi_j^T K \psi_i \quad (3.120)
\]

In the same manner, one can deduce that

\[
\lambda_j \psi_i^T M \psi_j = \psi_i^T K \psi_j \quad (3.121)
\]

Transposing both sides of the above equation

\[
(\lambda_i \psi_i^T M \psi_j)^T = (\psi_i^T K \psi_j)^T \quad (3.122)
\]

or
\[
\lambda_j \psi_j^T M \psi_i = \psi_i^T K \psi_i \tag{3.123}
\]
or since, \(M\) and \(K\) are symmetric \((M^T = M\) and \(K^T = K)\)
\[
\lambda_i \psi_i^T M \psi_i = \psi_i^T K \psi_i \tag{3.124}
\]

Subtracting Eqn 3.124 from Eqn 3.120 gives
\[
(\lambda_i - \lambda_j) \psi_j^T M \psi_i = \psi_i^T K \psi_i - \psi_j^T K \psi_i = 0 \tag{3.125}
\]
and since, \(\lambda_i \neq \lambda_j\)
\[
\psi_j^T M \psi_i = 0 \tag{3.126}
\]

Also, from Eqns 3.120 and 3.126 it may be concluded that
\[
\psi_j^T K \psi_i = 0 \tag{3.127}
\]

Eqns 3.126 and 3.127 are statements of orthogonality properties of the modal vectors with respect to the mass and stiffness matrices of the system. The concept of the orthogonality can be looked at from a vector analysis viewpoint. In vector analysis, vectors are orthogonal if their dot product equals zero. This means that the ‘projection’ of one vector on the other is zero and, therefore, the two vectors are ‘perpendicular’ to each other. As an example the unit vectors, \(i\), \(j\) and \(k\) for the 3-dimensional Cartesian coordinate system can be considered as orthogonal vectors. Eigenvectors of an n-DOF system can be viewed as \(n\) orthogonal vectors in an \(n\)-dimensional space.

### 3.3.2 Modal mass and modal stiffness

In Eqn 3.125, if \(i=j\), the equation can be rewritten as
\[
(\lambda_i - \lambda_i) \psi_i^T M \psi_i = 0
\]
Clearly, the term \(\psi_i^T M \psi_i\) is a scalar equal to a quantity which is denoted by \(M_{ii}\). To elaborate, consider an \(n\)-DOF system with mass matrix
The stiffness matrix

\[
M = \begin{bmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & & \vdots \\
m_{n1} & m_{n2} & \cdots & m_{nn}
\end{bmatrix}
\]

and the \(i\)th unnormalised mode

\[
\psi_i = \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}
\]  
(3.128)

Premultiplying \(M\) by \(\psi_i^T\) gives

\[
\psi_i^T M = B_i = [b_{i1}, b_{i2}, \ldots, b_{in}]
\]  
(3.129)

where

\[
b_{i1} = X_1 m_{11} + X_2 m_{21} + \cdots + X_n m_{n1}
\]

\[
b_{i2} = X_1 m_{12} + X_2 m_{22} + \cdots + X_n m_{n2}
\]

\[
\vdots
\]

\[
b_{in} = X_1 m_{1n} + X_2 m_{2n} + \cdots + X_n m_{nn}
\]

Postmultiplying Eqn 3.129 by \(\psi_i\) gives

\[
\psi_i^T M \psi_i = B_i \times \psi_i = b_{i1} X_1 + b_{i2} X_2 + \cdots + b_{in} X_n
\]
or

\[
\psi_i^T M \psi_i = M_{ii}
\]  
(3.131)

\(M_{ii}\) is the generalised mass and is referred to as the ‘modal mass’ for the \(i\)th mode of vibration, \((\omega = \omega_i)\). Also, in a similar manner the above operation can be applied to the stiffness matrix which results in the following equation

\[
\psi_i^T K \psi_i = K_{ii}
\]  
(3.132)
K_n is the generalised stiffness and is referred to as the 'modal stiffness' for the i\textsuperscript{th} mode of the vibration (\( \omega = \omega_i \)).

### 3.3.3 Modal matrix

An n-DOF system has n normal modes (eigenvectors). These normal modes can be assembled into a square matrix in which each column represents a normal mode. This matrix is called the 'modal matrix' and is denoted by \( P \). Thus, the modal matrix for an n-DOF system can be written as

\[
P = \begin{bmatrix}
X_1^{(1)} & X_2^{(1)} & \cdots & X_i^{(n)} \\
X_1^{(2)} & X_2^{(2)} & \cdots & X_i^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
X_1^{(n)} & X_2^{(n)} & \cdots & X_i^{(n)} 
\end{bmatrix} = [\psi_1 \psi_2 \cdots \psi_n] \quad (3.133)
\]

The result of the product \( P^T M P \) or \( P^T K P \) will be a diagonal matrix, because the off-diagonal terms are simply equal to zero due to the orthogonality properties of the normal modes.

For example consider a 3-DOF system. Performing the operation discussed above, gives

\[
P^T M P = [\psi_1 \psi_2 \psi_3]^T M [\psi_1 \psi_2 \psi_3] \quad (3.134)
\]

\[
= \begin{bmatrix}
\psi_1^T M \psi_1 & \psi_1^T M \psi_2 & \psi_1^T M \psi_3 \\
\psi_2^T M \psi_1 & \psi_2^T M \psi_2 & \psi_2^T M \psi_3 \\
\psi_3^T M \psi_1 & \psi_3^T M \psi_2 & \psi_3^T M \psi_3 
\end{bmatrix} = \begin{bmatrix}
M_{11} & 0 & 0 \\
0 & M_{22} & 0 \\
0 & 0 & M_{33} 
\end{bmatrix} \quad (3.135)
\]

In Eqn 3.135, the off-diagonal terms are equal to zero and the diagonal terms are the generalised masses, \( M_a \).

In a similar manner, the above operation can be applied to the stiffness matrix resulting in the following equation

\[
P^T K P = \begin{bmatrix}
K_{11} & 0 & 0 \\
0 & K_{22} & 0 \\
0 & 0 & K_{33} 
\end{bmatrix} \quad (3.136)
\]
The diagonal terms in Eqn 3.136 are referred to as the generalised stiffnesses of the system, $K_{ii}$.

### 3.3.4 Modal coordinates

Consider the equations of motion for an undamped 2-DOF system given by Eqn 3.72,

$$
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
x_1'' \\
x_2''
\end{bmatrix}
+
\begin{bmatrix}
k_1 + k_3 & -k_3 \\
-k_3 & k_1 + k_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

The major drawback in solving the equations to find the responses is the coupling between the equations. Coupling is represented in terms of nonzero off diagonal components of the mass, stiffness or both matrices of the system. Generally, two types of coupling exist for an undamped system as follows:

- **Static coupling** which is caused due to the nondiagonal stiffness matrix.
- **Dynamic coupling** which is caused due to the nondiagonal mass matrix.

Eqns 3.72 represent a system which is only statically coupled. If Eqns 3.72 could be uncoupled that is, diagonal mass and diagonal stiffness matrices, then each equation of 3.72 can be sorted independent of the other equation. In other words, each uncoupled equation would look like the equation of motion for SDOF system whose solution can very easily be obtained. Therefore, if a set of coupled equations can be reduced to an uncoupled set, the solution will become straightforward.

The procedure used to uncouple a set of coupled system of equations is basically a coordinate transformation. In other words, the aim is to find a coordinate transformation that transforms the original coordinate, $x$ into another set of coordinates, say, $q$ which causes the system statically and dynamically to be uncoupled. This new set of coordinates is typically referred to as ‘modal coordinates’.

The problem of finding a coordinate transformation that makes the original equations uncoupled, is simple. It can be done due to the orthogonality properties of the normal modes. From Eqns 3.126 and 3.127, it can be seen that, if either the mass and stiffness matrices are pre- and post-multiplied by different normal modes, the result will be zero. However, if the
same normal mode is used to pre- and post-multiply the mass and stiffness matrices (Eqns 3.131 and 3.132), the results will be constants. Therefore, the new coordinate system can be defined by the following transformation

\[ x = Pq \]  
(3.137a)

\[ \dot{x} = P\dot{q} \]  
(3.137b)

\[ \ddot{x} = P\ddot{q} \]  
(3.137c)

where \( P \) is the modal matrix of the system, \( x \) is the original coordinate system and \( q \) is the new coordinate system.

Consider the general form of the equations of motion of an undamped MDOF system, that is,

\[ M\ddot{x} + Kx = F \]  
(3.138)

where \( M \) and \( K \) are, as usual, the mass and stiffness matrices, \( x \) and \( \ddot{x} \) are the displacement and acceleration vectors, respectively, and \( F \) is the vector of the applied forces on the system. Substituting Eqns 3.137 into Eqn 3.138 gives

\[ MP\ddot{q} + KPq = F \]  
(3.139)

Premultiplying Eqn 3.139 by \( P^T \) gives

\[ P^T MP\ddot{q} + P^T KPq = P^T F \]  
(3.140)

Eqn 3.140 is the equivalent of Eqn 3.138 but in a different coordinate system. Regarding the orthogonality properties of the normal modes

\[ P^T MP = M^d \]  
(3.141)

where \( M^d \) is a diagonal matrix. Also,

\[ P^T KP = K^d \]  
(3.142)

where \( K^d \) is a diagonal matrix. The diagonal property of the mass and stiffness matrices is denoted by the letter \( d \) which appears as the superscript for \( M \) and \( K \). Eqn 3.140 can be rewritten as

\[ M^d \ddot{q} + K^d q = P^T F \]  
(3.143)
Since, both the new mass and new stiffness matrices are diagonal, Eqns 3.143 are completely uncoupled. So, each equation of the set of Eqns 3.143 can be considered as an equation of motion for a single degree of freedom system. The $i^{th}$ equation of Eqns 3.143 is

$$M_i \ddot{q}_i + K_i q_i = \psi_i^T F$$  \hspace{1cm} (3.144)

where $M_i$ and $K_i$ are the modal mass and the modal stiffness, respectively, for the $i^{th}$ mode and $\psi_i$ is the $i^{th}$ normal mode.

### 3.3.5 Modal damping

The equation of motion of an n-DOF system with viscous damping and arbitrary excitation force, $F(t)$ can be presented in the matrix form as

$$M\ddot{x} + C\dot{x} + Kx = F$$  \hspace{1cm} (3.145)

where $M$, $C$, and $K$ are, as usual, the mass, damping and stiffness matrices, respectively, and where $x$ is the displacement vector, $\dot{x}$ and $\ddot{x}$ are the velocity and acceleration vectors of the motion, respectively. Eqn 3.145, generally, is a set of n coupled equations. It was found that the solution of the homogeneous undamped equation

$$M\ddot{x} + Kx = 0$$

leads to the eigenvalues and eigenvectors that describe the modal matrix, $P$ and the normal modes of the system. Substituting Eqn 3.137 into Eqn 3.145 transforms the original coordinate system, $x$ to $q$, as

$$MP\ddot{q} + CP\dot{q} + KPq = F$$  \hspace{1cm} (3.146)

Premultiplying Eqn 3.146 by $P^T$, gives

$$P^T MP\ddot{q} + P^T CP\dot{q} + P^T KPq = P^T F$$  \hspace{1cm} (3.147)

In the previous section it has been shown that $P^T MP$ and $P^T KP$ are diagonal matrices. On the other hand, in general, $P^T CP$ is not diagonal. Therefore, Eqn 3.146 is coupled by the damping matrix. If $C$ is proportional to $M$ or $K$, it is evident that $P^T CP$ will become diagonal. In the diagonal case of damping matrix, it is said that the system has proportional damping. Eqn 3.147 is then completely uncoupled. So, Eqn 3.147 can be rewritten as
\[ M^d \ddot{q} + C^d \dot{q} + K^d q = Q \]  

(3.148)

where \( C^d \) is the diagonal damping matrix and \( Q \) is the load vector equal to \( P^T F \). Eqn 3.148 can be rewritten as

\[
\begin{bmatrix}
M_{11} & 0 & \cdots & 0 \\
0 & M_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & M_{nn}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\vdots \\
\dot{q}_n
\end{bmatrix}
+
\begin{bmatrix}
C_{11} & 0 & \cdots & 0 \\
0 & C_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & C_{nn}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\vdots \\
\dot{q}_n
\end{bmatrix}
+
\begin{bmatrix}
K_{11} & 0 & \cdots & 0 \\
0 & K_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & K_{nn}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_n
\end{bmatrix}
=
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_n
\end{bmatrix}
\tag{3.149}
\]

The \( i \)th equation of the Eqns 3.149 can be obtained as

\[ M_{ii} \ddot{q}_i + C_{ii} \dot{q}_i + K_{ii} q_i = Q_i \]  

(3.150)

or

\[ \ddot{q}_i + \frac{C_{ii}}{M_{ii}} \dot{q}_i + \frac{K_{ii}}{M_{ii}} q_i = \frac{Q_i}{M_{ii}} \]  

(3.151)

The construction \( C_{ii}/M_{ii} \) in Eqn 3.151 is equal to \( 2\zeta_i \omega_i \) (see Eqn 3.29a). Also, the ratio \( K_{ii}/M_{ii} \) equals to \( \omega_i^2 \) (see Eqn 3.12a). So, Eqn 3.151 can be written as

\[ \ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = Q_i \]  

(3.152)

where \( q_i \) is the \( i \)th degree of freedom of the transformed coordinate system, \( \dot{q}_i \) and \( \ddot{q}_i \) are the velocity and acceleration of the \( i \)th degree of freedom in the transformed coordinate system, \( \zeta_i \) is the damping ratio corresponding to the \( i \)th mode of vibration, \( \omega_i \) is the \( i \)th natural frequency of the system and \( Q_i \) is the generalised force associated with the \( i \)th degree of freedom of the system. So, for an n-DOF system n damping ratio can be obtained. The damping ratio for \( i \)th normal mode is referred to as the \( i \)th 'modal damping ratio' of the system.

### 3.3.6 Rayleigh damping in modal analysis

As discussed in Chapter 2, Rayleigh introduced proportional damping in the form

\[ C = \alpha M + \beta K \]  

(3.153)

where \( C \), \( M \) and \( K \) are, as usual, the damping, mass and stiffness matrices, respectively, and \( \alpha \) and \( \beta \) are constants. Substituting Eqn 3.153 into Eqn 3.147 gives
\[ P^T M \ddot{q} + P^T (\alpha M + \beta K) \dot{q} + P^T K P q = P^T F \]  
(3.154)

or

\[ P^T M \ddot{q} + (\alpha P^T M P + \beta P^T K P) \dot{q} + P^T K P q = Q \]  
(3.155)

Substituting Eqns 3.141 and 3.142 into Eqn. 3.155 gives

\[ M^d \ddot{q} + (\alpha M^d + \beta K^d) \dot{q} + K^d q = Q \]  
(3.156)

The \(i^{th}\) equation of the above set of equations may be written as

\[ M_{ii} \ddot{q}_i + (\alpha M_{ii} + \beta K_{ii}) \dot{q}_i + K_{ii} q_i = Q_i \]  
(3.157)

or

\[ \ddot{q}_i + (\alpha + \beta \frac{K_{ii}}{M_{ii}}) \dot{q}_i + \frac{K_{ii}}{M_{ii}} q_i = \frac{Q_i}{M_{ii}} \]  
(3.158)

In a similar manner used for Eqn 3.154, the above equation can be written as

\[ \ddot{q}_i + (\alpha + \omega^2 \beta) \dot{q}_i + \omega^2 q_i = Q_i \]  
(3.159)

Using the equivalent viscous damping, the construct

\[ \alpha + \omega^2 \beta \]

may be written as

\[ \alpha + \omega^2 \beta = 2(\zeta_i)\omega_i \]

where \((\zeta_i)\) is the equivalent modal damping ratio for the \(i^{th}\) normal mode.
4.1 Introduction

Damping is present in all oscillatory systems. Its effect is to remove energy from the system. Energy in a vibrating system is either dissipated into heat or radiated away. In vibration analysis, the analysts are generally concerned with damping in terms of system response. The loss of energy from the oscillatory system results in the decay of the amplitude of free vibration. In steady-state forced vibration, the loss of energy is balanced by the energy that is supplied by the excitation.

This chapter attempts to define damping and discuss the basic theories and concepts of damping related to the present research.
4.1.1 Definition of damping

Damping is associated with the energy dissipation properties of a material or system under cyclic stress (Lazen, 1968). Damping in a vibrating structure is associated with the dissipation of mechanical energy, usually by converting into thermal energy.

4.1.2 Damping force

Energy dissipation is usually determined under conditions of cyclic oscillation. Depending on the type of damping present, the force-displacement relationship when plotted can differ greatly. However, in all cases the force-displacement curve will enclose an area that is equal to the energy loss per cycle. The force involved is referred to as 'damping force' and is indicated by \( F_d \). Fig. 4.1 shows an enclosed area which is equal to the work done by the damping force. The vertical and horizontal axes of the graph show the damping force and the displacement, respectively. The energy dissipation equals to the work done by the damping force. Damping force only exists in vibrating systems or materials. So, damping is basically the dissipation of energy which occurs only in vibrating systems or materials. The enclosed area of Fig. 4.1 is a measure of energy dissipation through damping in a cycle. The larger the area the greater the damping will be.

4.2 Nature of Damping

Damping sources are normally classified into the following different types:

- Material Damping
- System Damping
- Radiation damping
- Auxiliary Damping

These different types of damping are described in the sequel.

4.2.1 Material damping
Material damping is the result of a complex molecular interaction within the material. Material damping is the energy dissipated within the material of a vibrating body due to internal hysteresis in the material arising from nonlinear stress-strain behaviour, internal surface friction and thermoelasticity (Smith, 1988). Therefore, material damping is dependent on the type of material, the methods used in manufacturing the material and the final finishing processes. The complexity is further increased by the fact that material properties often differ from sample to sample, which could result in significant differences in energy losses among similar members of a structural system (T. Kijewski & A. Kareem, 2000).

![Diagram](image_url)

The enclosed area shows the work done by damping force.

**Figure 4.1**

### 4.2.2 System damping

System damping is the energy dissipated at structural discontinuities, e.g. bolted and riveted connections and construction joints. This type of damping results from friction in sliding of joints, supports, cladding or various other parts of the structure during relative motion (Smith, 1988). This type of damping can occur in prefabricated space structures more than other type
of structures because of the large number of bolted connections. In contrast to the bolted connections, welded connections do not give rise to system damping.

Material damping can be measured and predicted by testing, but it is much more difficult to predict the level of the system damping. The degree of system damping may vary greatly even for the structures which are nominally identical.

4.2.3 Radiation damping

Radiation damping is the energy dissipated in the environment of the structures, e.g. air and water resistance and foundations. Again it is difficult to predict the degree of dissipation of energy due to radiation damping which occurs in the foundation even when the foundation material is linearly elastic because of the propagation of stress waves through the foundation. Soil structure interaction also contributes towards the overall damping, depending on the soil characteristics (Wolf, 1988).

4.2.4 Auxiliary damping

Auxiliary damping systems are devices which help structural systems to absorb and dissipate the vibrational energy. The employment of an auxiliary damping device in a structure, results in the increase in the damping level of the structure and a reduction in the motion of the structure.

Auxiliary damping sources can be categorised into passive and active systems. Both passive and active systems may be further categorised based on their mechanisms of energy dissipation.

4.2.4.1 Passive systems

Passive systems offer indirect damping to a structure through modification of the structural characteristics. In other words, passive devices increase the damping of a structure by modifying the frequency response of the structure. Such systems include ‘Tuned Mass Dampers’ (TMD) and ‘Tuned Liquid Dampers’ (TLD) (Tumara, 1988).
A TMD typically consists of a mass attached near the top of the building through a spring and damping mechanism. Viscous and viscoelastic dampers are typically used to provide the damping in a TMD system. In these systems the mass can move in either the horizontal or vertical plane.

4.2.4.2 Active systems

Active systems reduce the structural response by means of an external energy source (Soong, 1990). Structural motion can be controlled by an active mass damper (AMD). In an AMD the main components consist of an actuator which supplies the external energy to the structure, a controller which controls the actuator, a mass and a spring.

A classification of important types of damping is shown diagrammatically in Fig. 4.2

4.3 Representation of Damping

Damping from all sources is represented in three ways (not necessarily specifically related to a particular method of energy dissipation).

- Viscous Damping
- Coulomb Damping
- Hysteretic Damping

It must be stressed that these are only ways of representing damping. They do not imply a mechanism for damping.

4.3.1 Viscous damping

Viscous damping is the most common type of damping in structural vibration. In considering the damping forces in the dynamic analysis of a structure, it is usually assumed that these forces are proportional to the magnitude of the velocity of the vibrating body and opposite to the direction of motion. This type of damping is known as viscous damping and is analogous to the damping produced by the motion in a fluid.
As the viscous type of damping can be expressed in a simple mathematical way, other more complex types of damping are often expressed as an equivalent viscous damping in the analysis. The assumption of the viscous damping is often made regardless of the actual dissipative characteristics of the system. In fact there is a widespread belief that if the gross nature of the dissipation is accounted for, the actual details are irrelevant for engineering calculations (Irwin, 1986).

The mechanism of the energy dissipation by the viscous damping is usually symbolised by a 'dashpot'. For instance, a single degree of freedom frame with viscous damping may be represented as shown in Fig 4.3.
The mathematical representation of this type of damping is given by

\[ F_d = c\dot{x} \]  \hspace{1cm} (4.1)

where \( F_d \) is the damping force, \( c \) is the damping coefficient and \( \dot{x} \) is the velocity of the vibrating body.

### 4.3.2 Coulomb damping

The relative motion between two surfaces, sliding with respect to each other, causes Coulomb damping force. The damping is caused by friction between the two surfaces.

Damping in structures is not strictly due to viscosity but is mostly caused by friction at interfaces such as in bolted connections, in joints of cladding and in the cracks. The frictional forces are independent of the amplitude and frequency of the vibration [Beard, 1983]

The Coulomb damping force is constant and opposite in direction to the motion of the vibrating body. The term 'friction damping' or 'dry friction damping' is sometimes used to refer to Coulomb damping. The Coulomb damping force may be represented by

\[ F_d = \mu_k N \]  \hspace{1cm} (4.2)
Where $F_d$ is the damping force, $\mu_k$ is the ‘kinetic friction coefficient’ of two sliding surfaces and $N$ is the normal force between them. The subscript $k$ used in $\mu_k$ indicates the kinetic nature of the friction coefficient. This notation distinguishes the kinetic type of friction coefficient from the static type. (static type of friction coefficient is shown by $\mu_s$). In a force analysis the static friction coefficient applies until no relative motion has occurred. Once the relative motion begins a kinetic type of friction coefficient is used to determine the friction force.

Fig 4.4a shows a vibrating system with Coulomb damping. This system consists of a mass and a spring. The mass slides against a surface. The damping force results from friction between the mass and the surface. This force is in a direction opposite to the motion of the mass. So, the direction of the damping force changes when the direction of motion changes. Two free body diagrams for two cases are shown in Figs 4.4b and 4.4c. One of these diagrams applies for one direction and the other applies when the direction is reversed.

In free body diagrams:
- $\mu_kN$ is the damping force resulting from friction,
- $kx$ is the spring force applied on the mass due to displacement of the mass,
- $N$ is the normal reaction of the surface and
- $W$ is the weight of the mass.

When a system with Coulomb (frictional) damping is subjected to vibration, the friction force-displacement diagram shows a closed loop as shown in Fig 4.4d. The area closed by the loop in Fig 4.4d represents the dissipated energy in a cycle.

4.3.3 Hysteretic damping

Hysteretic damping is another form of energy dissipation. When a piece of material is deformed, a part of the deformation energy is absorbed and dissipated by the material. This is due to the friction between the internal planes of the material, which slip or slide as the deformation takes place. Hysteretic damping is the result of complex molecular interaction within the material. Therefore, it is dependent on the type of material, the method used in manufacturing and the final finishing process. Experiments on damping that occurs in solid materials and structures which have been subjected to cycling stress, have shown that the
The body is in contact with the surface but is not connected to it.

\[ \text{Direction of motion} \]

\[ \text{Friction force} \]

\[ \text{Displacement} \]

Figure 4.4
(a) Single degree of freedom (spring-mass) system with Coulomb damping
(b) and © Free body diagram of the system
(d) Friction force-displacement diagram for the system in a cycle

Damping force is independent of the frequency of the exciting force. In order to indicate this type of damping, the term hysteretic damping is used [Bishop and Johnson, 1960]. This definition of hysteretic damping happens to coincide with the definition of 'structural damping' as stated by Clough and Penzin (Clough and Penzin, 1993). For steady state excitation, Clough and Penzin defined structural damping for a single degree of freedom system being such that the damping force is proportional to displacement and opposes the motion.

When a body with a hysteretic damping, is subjected to vibration, the stress-strain diagram shows a hysteretic loop, as shown in Fig 4.5. The area enclosed by the loop in Fig 4.5
represents the dissipated energy in a cycle. The term ‘solid damping’ and ‘material damping’ are also used to refer to hysteretic damping.

4.4 Measurement of Damping

There are several ways of defining the damping capacity of materials and/or structures. The procedure adopted will depend mainly on the method of measurement employed. There are a variety of techniques commonly employed to measure the damping capacity. The main techniques are as follows:

- Logarithmic decrement method
- Half power method
- Energy method

Figure 4.5 Hysteresis loop for elastic material

4.4.1 Logarithmic decrement method
A convenient way of finding the viscous damping ratio $\zeta$ through experimental measurements is by determining of the decay of free vibration. The approach is based on the fact that the higher damping effect the greater will be the rate of decay. This is the simplest and most frequently used method for the determination of viscous damping coefficient. This method is referred to as the 'logarithmic decrement' method. To elaborate, reconsider Eqn 3.32 in Section 3.2.1. This equation which has been renumbered here, expresses the damped free oscillatory motion.

$$x = Xe^{-\zeta \omega_n t} \sin\left(\sqrt{1-\zeta^2} \omega_n t + \phi\right)$$  \hspace{1cm} (4.3)

where
- $x$ is the response (displacement) of the system,
- $X$ is the amplitude of the response,
- $\zeta$ is the damping ratio of the system,
- $\omega_n$ is the natural frequency of the system,
- $t$ is the time and
- $\phi$ is the phase angle of the displacement with respect to the velocity.

Eqn 4.3 is shown in Fig 4.6 graphically.

![Figure 4.6 Plot of damped free vibration with $\zeta < 1$](image)
For any two successive amplitudes, for example $x_1$ and $x_2$ in Fig 4.6, the natural logarithm of
\[
\frac{x_1}{x_2},
\]
is referred to as the 'logarithmic decrement'. The logarithmic decrement is denoted by $\delta$.

\[
\delta = \ln \frac{x_1}{x_2} \quad (4.4)
\]

From Eqn 4.3
\[
x_1 = e^{-\zeta \omega_n t} \sin \left[ \sqrt{1 - \zeta^2} \omega_n t + \phi \right] \quad (4.5)
\]
where $x_1$ is the response of the system at time $t = t_1$ and
\[
x_2 = e^{-\zeta \omega_n (t_1 + T_d)} \sin \left[ \sqrt{1 - \zeta^2} \omega_n (t_1 + T_d) + \phi \right] \quad (4.6)
\]
is the response of the system at time
\[
t_2 = (t_1 + T_d) \quad (4.7)
\]
$T_d$ is the period of damped free oscillation

Substituting Eqns 4.5 and 4.6 into Eqn 4.4 gives
\[
\delta = \ln \frac{x_1}{x_2} = \ln \frac{e^{-\zeta \omega_n t} \sin \left[ \sqrt{1 - \zeta^2} \omega_n t_1 + \phi \right]}{e^{-\zeta \omega_n (t_1 + T_d)} \sin \left[ \sqrt{1 - \zeta^2} \omega_n (t_1 + T_d) + \phi \right]} \quad (4.8)
\]
The values of sine functions in the numerator and denominator of Eqn 4.8 are equal, that is
\[
\sin \left[ \sqrt{1 - \zeta^2} \omega_n t_1 + \phi \right] = \sin \left[ \sqrt{1 - \zeta^2} \omega_n (t_1 + T_d) + \phi \right] \quad (4.9)
\]
therefore, Eqn 4.8 reduces to
\[
\delta = \ln \frac{x_1}{x_2} = \ln \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + T_d)}} \quad (4.10)
\]
The right-hand side of Eqn 4.10 can be written
\[
\ln \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + T_d)}} = \ln e^{-\zeta \omega_n (t_1 + T_d)} - \ln e^{-\zeta \omega_n t_1} = \zeta \omega_n T_d \quad (4.11)
\]
As the result Eqn. 4.10 reduces to:

\[ \delta = \ln \frac{x_1}{x_2} = \zeta \omega_n T_d \]  

(4.12)

In general, the ratio of any successive amplitudes such as \( x_n \) and \( x_{n+1} \) can be written

\[ \ln \frac{x_n}{x_{n+1}} = \ln \frac{e^{-\zeta \omega_n t_n} \sin[\sqrt{1 - \zeta^2} \omega_n t_n + \phi]}{e^{-\zeta \omega_n (t_n + T_d)} \sin[\sqrt{1 - \zeta^2} \omega_n (t_n + T_d) + \phi]} \]  

(4.13)

Applying Eqn. 4.9 on Eqn. 4.13 leads to:

\[ \ln \frac{x_n}{x_{n+1}} = \ln \frac{e^{-\zeta \omega_n t_n}}{e^{-\zeta \omega_n (t_n + T_d)}} \]  

(4.14)

Eqn. 4.14 is similar to Eqn. 4.10. So can be reduced to

\[ \ln \frac{x_n}{x_{n+1}} = \zeta \omega_n T_d \]  

(4.15)

Combination of the above Eqn. and Eqn. 4.12 leads to

\[ \delta = \ln \frac{x_1}{x_2} = \ln \frac{x_n}{x_{n+1}} = \zeta \omega_n T_d \]  

(4.15)

This is the general form of natural logarithm of the ratio of any two successive amplitudes and can be made use to determine the damping coefficient of the system as follows:

Every oscillatory system with viscous damping in free vibration has a response curve as shown in Fig. 4.6. Such a response can be obtained experimentally and then \( x_n \) and \( x_{n+1} \) can be determined from the recorded curve. Substituting the experimentally obtained values of \( x_n \) and \( x_{n+1} \) into Eqn. 4.15 gives the logarithmic decrement \( \delta \). In Eqn. 4.15, \( \delta \) is a function of

- the damping ratio \( \zeta \),
- the natural frequency, \( \omega_n \) and
- period of damped free vibration system, \( T_d \) that may be written as

\[ T_d = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \]

as discussed before (see Eqn. 3.33)
Substitution of this equation into Eqn. 4.15 gives

\[
\delta = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}
\]  

(4.16)

or

\[
\delta^2 = \frac{4\pi^2 \zeta^2}{1 - \zeta^2}
\]

or

\[
\delta^2 - \delta^2 \zeta^2 = 4\pi^2 \zeta^2
\]

that is,

\[
\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}
\]  

(4.17)

Since \( \delta \) has been found experimentally, the value of \( \zeta \) then can be determined from Eqn. 4.17. When \( \zeta \) is very small say \( \zeta < 0.2 \), then

\[
\sqrt{1 - \zeta^2} \approx 1
\]

and therefore Eqn. 4.16 can be written as

\[
\delta \approx 2\pi \zeta
\]  

(4.18)

that is,

\[
\zeta = \frac{\delta}{2\pi}
\]  

(4.19)

Eqns. 4.17 and 4.19 gives \( \zeta \) as function of logarithmic decrement \( \delta \). These Eqns are plotted in Fig. 4.7. Eqn. 4.17 determines the exact value of \( \zeta \) whereas, Eqn. 4.19 gives an approximate value for \( \zeta \). From Fig. 4.7, it can be seen that for a small value of \( \delta \) (say \( \delta < 1.5 \)) the results of the exact and approximate values of

- \( \zeta \) (exact) = 0.157
- \( \zeta \) (approximate) = 0.159

In contrast, for the value of \( \delta = 15 \)

- \( \zeta \) (exact) = 0.922
- \( \zeta \) (approximate) = 2.39
Comparing the two values of $\zeta$ when $\delta = 1$ indicates that there is 0.2% difference between the exact and approximate values of the damping ratio $\zeta$. However for $\delta = 15$ the difference between the exact and approximate values of $\zeta$ is 61%.

It is interesting to have also the logarithmic decrement as a function of $\zeta$. Eqns 4.16 and 4.18 give $\delta$ as a function of $\zeta$. Eqn 4.16 gives the exact value of $\delta$ whereas Eqn 4.18 gives its approximate value. The plots of these equations are shown in Fig 4.8. It may be seen that for $\zeta < 0.25$ the two curves are close to each other. So, for values of $\zeta$ less than 0.25 the results of the approximate equation is acceptable.

![Figure 4.7 Plot of equations 4.17 and 4.19](image)

Eqn. 4.15 gives the logarithmic decrement $\delta$ using any two successive amplitudes. It is possible to give $\delta$ using any two non-successive amplitudes as follows:

If $x_1$ and $x_{n+1}$ are the first and $(n+1)^{th}$ amplitudes in a vibration, respectively,

$$\ln \frac{x_1}{x_{n+1}} = \ln \left(\frac{x_1}{x_2} \frac{x_2}{x_3} \cdots \frac{x_n}{x_{n+1}}\right)$$
or
\[
\ln \frac{x_1}{x_{n+1}} = \ln \frac{x_1}{x_2} + \ln \frac{x_2}{x_3} + \ldots + \ln \frac{x_n}{x_{n+1}}
\]
or
\[
\ln \frac{x_1}{x_{n+1}} = \delta + \delta + \ldots + \delta = n\delta
\]
or
\[
\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}} \tag{4.20}
\]

Eqn 4.20 gives the logarithmic decrement $\delta$ as the form of natural logarithm of the ratio of any two amplitudes.

### 4.4.2 Half power method

Another way of finding the viscous damping ratio $\zeta$ is named the ‘half power method’. Before discussing of this method, it is useful to explain what is implied by the term ‘half power’.
The amount of dissipation of energy in an oscillatory system is a measure of damping of the system. Therefore, the greater the dissipation of energy, the higher will be the damping.

In dynamics, the relation between energy, force and displacement can be written as

$$W = \int_{x_1}^{x_2} F dx$$  \[4.21\]

where
- $W$ is the spent or released energy in the $x_1$ to $x_2$
- $F$ is the force acting in the interval $x_1$ and $x_2$
- $dx$ is the differential of displacement $x$
- $x_1$ and $x_2$ are the lower and upper limit of the integral, in other words $x_1$ and $x_2$ are the start and end points of the movement of the force $F$, respectively, between which the energy is calculated.

Eqn 4.21 implies that when a force, say $F$, has a differential movement $dx$, the energy is equal to $Fdx$. By integrating $Fdx$ from $x_1$ to $x_2$, the total energy spent or released can be obtained.

For a forced oscillatory system with viscous damping the dissipated energy per cycle is

$$W_d = \oint F_d dx$$  \[4.22\]

where
- $W_d$ is the energy dissipated by damping in a cycle,
- $F_d$ is the viscous damping force,
- $dx$ is the differential movement $x$ and
- $\oint$ indicates that the integration carried out for one cycle.

The viscous damping force $F_d$ is proportional to the velocity, that is

$$F_d = c\dot{x}$$  \[4.23\]

in which $c$ is the damping coefficient and $\dot{x}$ is the velocity.

The steady state response equation of the system, is given as Eqn. 3.59 in Section 3.2.1. This equation is of the form

$$x = X \sin(\omega t - \phi)$$  \[4.24\]

The differential and first derivative with respect to time of $x$ from the above equation may be written as
\[ dx = X \omega \cos(\omega t - \phi) \, dt \]

\[ \dot{x} = \frac{dx}{dt} = X \omega \cos(\omega t - \phi) \quad (4.25) \]

Substituting Eqn 4.25 into Eqn 4.23 gives

\[ F_d = cX \omega \cos(\omega t - \phi) \quad (4.26) \]

Therefore, from Eqn 4.22, \( W_d \) can be written as

\[ W_d = \int_{\phi}^{\phi_0} c \omega^2 X^2 \cos^2 (\omega t - \phi) \, dt \quad (4.27) \]

The lower and upper limits of the integral in the above equation are the times at the start and end points of the cycle.

In dynamics, power is defined as the rate of energy with respect to the time. This, in a mathematical form may be written as

\[ P = \frac{dW}{dt} \quad (4.28) \]

where

- \( P \) is the power,
- \( dW \) is the differential of the energy and
- \( dt \) is the differential of time.

So, the power corresponding to the energy dissipation in an oscillatory system can be written as

\[ P_d = \frac{dW_d}{dt} \quad (4.29) \]

in which \( P_d \) is the power corresponding to the energy dissipation, \( dW_d \) is the differential of dissipated energy and \( dt \) is the differential of time.

Substituting Eqn 4.27 into Eqn 4.29 gives

\[ P = c \omega^2 X^2 \cos^2 (\omega t - \phi) \quad (4.30) \]

This equation indicates that the power corresponding to the energy dissipation in the system (that is, \( P \)) is proportional to the square of the amplitude \( X \).

The power corresponding to the energy dissipation in the system at resonance can then be written as
\[ P_{\text{res}} = cX_{\text{res}}^2 \omega^2 \cos^2(\omega t - \phi) \]  \hspace{1cm} (4.31)

where \( P_{\text{res}} \) is the power at resonance and \( X_{\text{res}} \) is the amplitude at resonance. This amplitude is the peak amplitude in the response curve of a forced excited damped system. The frequency of response corresponding to the resonance amplitude is referred to as the 'resonance frequency'.

Now, looking for a point, say \( p_1 \), at which the power in the system is half of the power at the resonance. If the power corresponding to the \( p_1 \) is \( P_1 \) it can be written as

\[ P_1 = \frac{P_{\text{res}}}{2} \]  \hspace{1cm} (4.32)

where \( P_1 \) and \( P_{\text{res}} \) are the powers at \( p_1 \) and resonance, respectively. If the amplitude corresponding to the \( p_1 \) is the \( X_1 \) the power can be written as

\[ P_1 = c\omega^2 X_1^2 \cos^2(\omega t - \phi) \]  \hspace{1cm} (4.33)

substituting Eqns 4.31 and 4.33 into Eqn 4.32 gives

\[ c\omega^2 X_1^2 \cos^2(\omega t - \phi) = \frac{1}{2} c\omega^2 X_{\text{res}}^2 \cos^2(\omega t - \phi) \]  \hspace{1cm} (4.34)

or

\[ X_1^2 = \frac{1}{2} X_{\text{res}}^2 \]

or

\[ X_1 = \frac{\sqrt{2}}{2} X_{\text{res}} \]  \hspace{1cm} (4.35)

The amount of power corresponding to the amplitude that is obtained from the above equation is half of the power at resonance

The point corresponding to such amplitude (amplitude given in Eqn 4.35) has been focused in a method, which determines the damping ratio \( \zeta \). That is why this method is named as 'half power' method.

In order to introduce the half power method, consider the resonance curve of the damped forced vibration of a single degree of freedom system shown in Fig 4.9.
Fig 4.9 shows a resonance curve with \( \zeta = 0.1 \). In this curve, the ratio of response amplitude \( X \) to the static amplitude \( F_0 / k \) that is \( \frac{Xk}{F_0} \) with respect to the frequency ratio \( \frac{\omega}{\omega_n} \) is plotted.  

This is a plot of Eqn 3.66, that is,

\[
\frac{Xk}{F_0} = \frac{1}{\sqrt{1 - (\frac{\omega}{\omega_n})^2}} \text{ for } 2\zeta(\frac{\omega}{\omega_n})^2
\]

(3.66) repeated

![Frequency response curve for a forced damped system \([\zeta = 0.1]\)](image)

When \( \frac{\omega}{\omega_n} = 1 \) the system encounters resonance. The resonance amplitude can be obtained by substitution of \( \frac{\omega}{\omega_n} = 1 \) in Eqn 3.66 that is

\[
\frac{X_{res}k}{F_0} = \frac{1}{2\zeta}
\]

(4.36)
or

\[ X_{\text{res}} = \frac{F_0 / k}{2\zeta} \]  

(4.37)

in which \( X_{\text{res}} \) is the amplitude of the response at \( \frac{\omega}{\omega_n} = 1 \). \( F_0 \) is the amplitude of the applied load, \( k \) is the stiffness of the system and \( \zeta \) is the damping ratio. In either side of the curve around the resonance \( \left( \frac{\omega}{\omega_n} = 1 \right) \), two points \( p_1 \) and \( p_2 \) can be found with the same amplitude

\[ X = \frac{\sqrt{2}}{2} X_{\text{res}} \]  

(4.38)

The frequencies corresponding to these points are \( \omega_1 \) and \( \omega_2 \). These points are referred to as the 'half power' points as shown in Fig 4.9.

Squaring Eqn 3.66 gives

\[ \left( \frac{Xk}{F_0} \right)^2 = \frac{1}{1 - (\omega / \omega_n)^2} \]  

(4.39)

Using the following notation

\[ \left( \frac{\omega}{\omega_n} \right)^2 = \alpha \]

gives

\[ \left( \frac{Xk}{F_0} \right)^2 = \frac{1}{(1 - \alpha)^2 + (2\zeta)^2 \alpha} \]  

(4.40)

Substituting

\[ X = \frac{\sqrt{2}}{2} X_{\text{res}} \]

into Eqn 4.40 gives

\[ \frac{1}{2} \left( \frac{1}{2\zeta} \right)^2 = \frac{1}{(1 - \alpha)^2 + (2\zeta)^2 \alpha} \]  

(4.41)

or

\[ \alpha^2 - 2(1 - 2\zeta^2)\alpha + (1 - 8\zeta^2) = 0 \]  

(4.42)

or
\[ \alpha = (1 - 2\zeta^2) \pm 2\zeta^2 \sqrt{1 + \zeta^2} \]  
(4.43)

Assuming \( \zeta \) is very small and neglecting the higher order terms of \( \zeta \), give

\[ \alpha = \left( \frac{\omega}{\omega_n} \right)^2 = 1 \pm 2\zeta \]  
(4.44)

If \( \omega_1 \) and \( \omega_2 \) are the roots of Eqn 4.18, this equation can be written as

\[ 4\zeta = \frac{\omega_1^2 - \omega_2^2}{\omega_n^2} = 2 \left( \frac{\omega_2 - \omega_1}{\omega_n} \right) \]

or

\[ \frac{1}{2\zeta} = \frac{\omega_n}{\omega_2 - \omega_1} = \frac{f_n}{f_2 - f_1} \]  
(4.45)

Form Eqn 4.45 can be inferred that low damping systems have sharp resonance curve. It means the sharper the resonance curve, the smaller will be the damping.

Here according to Eqn 4.45 a quantity related to the damping can be defined, that is a measure of the sharpness of the resonance denoted by \( Q \)

\[ Q = \frac{1}{2\zeta} \]  
(4.46)

The above two methods deduce the damping capacity of any vibrating system in the form of the damping ratio \( \zeta \). The damping capacity of any material or vibrating system can also be represented in other quantities depending on the technical areas to which they are applied.

One of these is the 'specific damping capacity' defined as the energy loss per cycle \( W_d \) (see Section 4.4.3) divided by the peak potential or strain energy \( U \), that is

\[ \varphi = \frac{W_d}{U} \]

The second quantity is the 'loss coefficient' defined as the ratio of damping energy loss per radian \( \frac{W_d}{2\pi} \) divided by the peak potential or strain energy \( U \), that is

\[ \eta = \frac{W_d}{2\pi U} \]

In the current research the quantity of the damping ratio is used to represent the damping capacity of the test model.
4.4.3 Energy method

As discussed in the previous section, the energy lost per cycle of vibration is indicated by $W_d$.

The simplest case of energy dissipation is that of a spring-mass in a single degree of freedom system with viscous damping in a steady state forced vibration. This energy is equal to the work done by the damping force which can be determined by Eqn 4.27, that is,

$$ W_d = \int_0^{2\pi/\omega} c\omega^3 X^2 \cos^2(\omega t - \phi) dt $$

where

- $W_d$ is the work done by damping force in a cycle,
- $c$ is the damping coefficient,
- $\omega$ is the frequency of exciting force,
- $\phi$ is the phase of displacement with respect to the exciting force and
- $X$ is the amplitude of vibration

$c$ and $\omega$ are constant and $X$ which is the amplitude of vibration at a constant $\omega$ is constant as well. So, $c\omega^3 X^2$ as a constant quantity and the above equation can be written as

$$ W_d = c\omega^3 X^2 \int_0^{2\pi/\omega} \cos^2(\omega t - \phi) dt $$

or

$$ W_d = c\omega^3 X^2 \left[ \frac{\pi}{\omega} \right] $$

or

$$ W_d = \pi c\omega X^2 $$

The above equation is the work done by the damping force in a cycle which is equal to the energy lost in the cycle. Thus

$$ E = \pi c\omega X^2 \quad (4.47) $$

The energy dissipated at resonance can be found by replacing $\omega$ by

$$ \omega = \omega_n = \sqrt{\frac{k}{m}} \quad (\text{see Eqn 3.12a, Section 3.2.1}) $$

where $k$ is the stiffness of the system and $m$ is the mass of the system.
From Eqn 3.28 and 3.29, Section 3.2.1, the damping coefficient can be written as

\[ c = 2\zeta \sqrt{km} \]

Where \( \zeta \) is the damping ratio.

By substituting for \( \omega \) and \( c \) from the above equations into Eqn 4.47, the energy dissipated per cycle becomes

\[ E = 2\zeta \pi kX^2 \tag{4.48} \]

The energy dissipated per cycle due to the viscous damping force can be represented graphically as discussed below. Fig 4.10 shows a single degree of freedom with a viscous damper. For this system, the force, \( F \) needed to cause a displacement, \( x(t) \), is equal by

\[ F = kx + c\dot{x} \tag{4.49} \]

The equation for harmonic motion of frequency \( \omega \) and amplitude \( X \) is given by Eqn 4.24 that is

\[ x = X \sin(\omega t - \phi) \]

The velocity of the vibration of the system is given by Eqn 4.25 that is

\[ \dot{x} = \omega X \cos(\omega t - \phi) \]

Substituting Eqn 4.25 into Eqn 4.49 gives

\[ F(t) = kx + cX\omega \cos(\omega t - \phi) \]

or

\[ F(t) = kx \pm cX\omega \sqrt{1 - \sin^2(\omega t - \phi)} \]

or

\[ F(t) = kx \pm c\omega \sqrt{X^2 - X^2 \sin^2(\omega t - \phi)} \]

\[ F(t) = kx \pm c\omega \sqrt{X^2 - x^2} \]

or

\[ (F - kx)^2 = c^2 \omega^2 (X^2 - x^2) \]

or

\[ \frac{(F - kx)^2}{c^2 \omega^2 X^2} + \frac{x^2}{X^2} = 1 \tag{4.50} \]
The rollers are inserted between the mass and surface. The rollers cause the friction to be negligible.

This equation can be recognised as the equation of an ellipse with $F$ and $x$ being the vertical and horizontal axes, respectively, as shown in Fig 4.11. The energy dissipated per cycle is then given by the area of the ellipse.

It is possible to replace $F_d$ instead of $F$ in Eqn 4.49 and obtain the dissipated energy graphically

$$F_d = c\dot{x}$$

By using the same manner as used previously

$$F_d = \pm c\omega \sqrt{X^2 - x^2}$$

or

$$F_d^2 = c^2 \omega^2 (X^2 - x^2)$$

or

$$F_d^2 = c^2 \omega^2 X^2 (1 - \frac{x^2}{X^2})$$

or

$$\left( \frac{F_d}{c\omega X} \right)^2 + \left( \frac{x}{X} \right)^2 = 1 \quad (4.51)$$
This equation can be recognised as the equation of an ellipse as well with $F_0$ and $x$ being the vertical and horizontal axes, respectively. Eqn 4.51 is shown in Fig 4.1. The area enclosed in Fig 4.1 is the same as that in Fig 4.11. However, the existence of $kx$ in the first term of Eqn 4.50 makes different position for ellipse comparing with the ellipse in Fig 4.1 although, it does not affect the magnitude of the enclosed area.

4.4.3.1 Equivalent viscous damping

The primary influence of damping on an oscillatory system is that of limiting the amplitude of response at the resonance. As seen from the response curves of Fig 3.20, Section 3.2.1, damping has little influence on the response in the frequency regions away from resonance. In the case of viscously damped, harmonically forced vibration, the amplitude of the vibration is given by

![Figure 4.11 Plot of equation 4.50](image-url)
where $X$ is the amplitude of the vibration, $F_0$ and $\omega$ are the amplitude and frequency of the exciting force, respectively, $k$ is the stiffness of the system and $m$ is the mass of the system. At resonance the frequency of exciting force is equal to the natural frequency of the vibrating system. By replacing

$$\omega_n = \sqrt{\frac{k}{m}} \quad (\omega_n \text{ is the natural frequency of the system})$$

for $\omega$ into the above equation the amplitude of the vibration at resonance can be found as

$$X = \frac{F_0}{c \omega_n} \quad (4.52)$$

For other types of damping no such simple expression exists. However, it is possible to approximate the amplitude of the vibration at resonance by substituting an equivalent viscous damping coefficient say, $c_{eq}$, into Eqn 4.52. The equivalent viscous damping coefficient $c_{eq}$, can be obtained by equating the dissipated energy caused by a particular damping with the energy dissipated by the viscous damping.

4.5 Damping Models

4.5.1 Introduction

There are several different mechanisms of energy dissipation in vibrating systems. Some of the different causes of damping are discussed in Section 4.3. The mechanisms have been modelled in the time and frequency domain in various forms. The developed models are either linear or non-linear. The advantage of a linear model is carrying out its simplicity in analysis and estimating model parameters. Actually, it turns out linear damping is applicable in structures with low amplitude motion [A.P.Jury, 1997].

4.5.1.1 Time and frequency domain
The equation of motion for a vibrating system can be described either in the time domain or in the frequency domain. A differential equation in the time domain is a function of time. In other words, an equation given in the time domain is dependent on the time, while the time is an independent variable. For example, the following differential equation is the equation of motion of a system in a damped forced vibration in time domain

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \]  
(see Section 3.2.1) \hspace{1cm} (4.53)

Normally, the equations of motion that appear as differential equations are given in the time domain. Solving a differential equation in the time domain, requires the integration of the equation with respect to the time under given initial conditions. The steady state solution of Eqn 4.52 is given by

\[ x = X \sin(\omega t - \phi) \]  
(see Eqn 3.59, Section 3.2.1)

This equation is in the time domain.

In the frequency domain an equation of motion appears as an algebraic equation, which is usually much simpler to solve. The Laplace transform of Eqn 4.53 with initial conditions \( x(0) \) and \( \dot{x}(0) \) is found as

\[ ms^2X(s) + csx(0) + \dot{x}(0) + ksX(s) + x(0) + kX(s) = F(s) \]  
[E. Keryszig, 1999]

where

- \( m \), \( k \) and \( c \) are, as usual, the mass, the stiffness and the damping coefficient of the system,
- \( X(s) \) is the Laplace transform of \( x(t) \) and
- \( F(s) \) is the Laplace transform of \( F(t) \)

By applying the initial conditions

\[ x(0) = \dot{x}(0) = 0 \]

into the above equation gives

\[ ms^2X(s) + csX(s) + kX(s) = F(s) \]
or
\[(ms^2 + cs + k)X(s) = F(s)\]  
(4.54)

which is an algebraic equation. As discussed in Section 3.2.1, the response of viscously damped system is given by

\[X = \frac{F_0/k}{\sqrt{(1 - \frac{m\omega^2}{k})^2 + (\frac{c\omega}{k})^2}}\]  
(see Eqn 3.64)  
(4.55)

This equation is given in the frequency domain. It means that the variation of response is a function of variation of \(\omega\).

4.5.2 Viscous damping model

Viscous damping is the simplest damping model available from the theoretical viewpoint. It is a linear damping model. An equation of motion incorporating this type of damping can be solved for any type of exciting force. As a result, this type of damping is simple to deal with mathematically. A viscous damper is a device, which opposes the velocity and is proportional to it. The damping force in a viscous model is given by

\[F_d = c\dot{x}\]  
(see Eqn 4.1, Section 4.3.1)

The energy lost per cycle in a steady state viscously damped system is formulated in section 4.4.3 as follows

\[E = \pi\omega X^2\]  
(4.56)

4.5.3 Coulomb damping model

In Coulomb damping, it is assumed that the damping force is proportional to the normal force between the sliding surfaces. This damping force is independent of the velocity (see Sections 3.2.1 and 4.3). A single degree of freedom vibrating system with Coulomb damping is illustrated in Fig 4.4. As discussed in Section 3.2.1, the Coulomb damping force always opposes the movement of the mass in a vibrating system. A general form of a steady state response of a vibrating system with Coulomb damping is given as

\[x(t) = X \cos(\omega t - \phi)\]  
(see Section 3.2.1)  
(4.57)
where
- \( x(t) \) is the response of the system,
- \( X \) is the amplitude of the vibration,
- \( \omega \) is the frequency of the vibration and
- \( \phi \) is the phase angle of the displacement with respect to the exciting force.

Like the viscous damping, the energy dissipated in a cycle with Coulomb damping is given as

\[
E = \int F_d dx
\]  

(4.58)

where
- \( E \) is the dissipated energy in a cycle,
- \( F_d \) is the damping force,
- \( dx \) is the displacement differential and
- \( \int \) indicates that the integration region is limited to a cycle.

Differential of the displacement, \( dx \), can be obtained from Eqn 4.57 as

\[
dx = -X\omega \sin(\omega t - \phi) dt
\]  

(4.59)

Substituting Eqn 4.59 into Eqn 4.58 gives

\[
E = \int_{-\frac{2\pi}{\omega}}^{\frac{2\pi}{\omega}} F_d X\omega \sin(\omega t - \phi) dt
\]  

(4.60)

where the upper limit of the integral, that is, \( 2\pi / \omega \), indicates the end of the cycle.

Since the direction of the Coulomb damping force changes with the direction of the movement in each half cycle, therefore, it is necessary to split up the integral (Eqn 4.60) into segments corresponding to the change of the direction of motion. So, Eqn 4.60 can be rearranged as follows

\[
E = -\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} F_d X\omega \sin(\omega t - \phi) dt + \int_{-\frac{2\pi}{\omega}}^{\frac{2\pi}{\omega}} F_d X\omega \sin(\omega t - \phi) dt
\]  

(4.61)

Arrangement of the limits of the above integral depends on the position of the velocity change points as shown in Fig 4.12. Since \( F_d \), \( \omega \) and \( X \) are constant, therefore, Eqn 4.61 can be written as
Chapter 4 Damping: Theory and Concepts

\[ E = -F_d X \omega \left( \int_0^{\pi/\omega} \sin(\omega t - \phi) \, dt + \int_{\pi/\omega}^{2\pi/\omega} \sin(\omega t - \phi) \, dt \right) \]  
(4.62)

or

\[ E = -F_d X \omega \left( \left[ \frac{1}{\omega} \cos(\omega t - \phi) \right]_0^{\pi/\omega} - \left[ \frac{1}{\omega} \cos(\omega t - \phi) \right]_{\pi/\omega}^{2\pi/\omega} \right) \]

or

\[ E = -F_d X \left( \cos \pi - \cos 0 \right) - \left( \cos 2\pi - \cos \pi \right) \]  
(4.63)

or

\[ E = -F_d X \left( -1 - 1 \right) - \left( 1 - (-1) \right) \]

or

\[ E = -F_d X (-4) \]

or

\[ E = 4F_d X \]  
(4.64)

Figure 4.12 Velocity of a steady state harmonic vibration
It must be remembered that this result is dependent on assuming that the nonlinearities in a system with Coulomb damping are small. Therefore, the system can be approximated by a linear vibrating system.

4.5.3.1 Equivalent viscous damping coefficient for Coulomb damping

It is possible to use the concept of equivalent viscous damping coefficient in a system with Coulomb damping. Equivalent viscous damping is determined by equating the energy dissipated by two models, one with Coulomb damping and the other with viscous damping [Inman, 1994]. By assuming a single degree of freedom system and equating the energies dissipated by the models for one cycle gives

\[ \pi c_{eq} \omega X^2 = 4 F_d X \]  

(see Eqns 4.56 and 4.64) \hspace{1cm} (4.65)

or

\[ c_{eq} = \frac{4 F_d X}{\pi \omega X^2} \]

or

\[ c_{eq} = \frac{4 F_d}{\pi \omega X} \] \hspace{1cm} (4.66)

where

- \( c_{eq} \) is the equivalent viscous damping coefficient,
- \( \omega \) is the frequency of exciting force,
- \( X \) is the amplitude of vibration and
- \( F_d \) is the Coulomb damping force.

4.5.4 Velocity squared damping model

This type of damping model is directly proportional to the square of the velocity and opposes the direction of the motion. 'Velocity squared damping' models the behaviour observed when systems vibrate in a fluid or when a fluid is rapidly forced through an orifice. In mathematical form this model may be written as
\[ F_d = a\dot{x}^2 \]  

(4.67)

Where \( a \) is the 'velocity squared damping constant' and is dependent on the characteristics of the fluid in which damping has occurred. This is a nonlinear model. An equation of motion in which this model appears can be solved numerically.

The equation of motion for a single degree of freedom system with velocity squared damping is similar to that of a single degree of freedom with Coulomb damping except for the magnitude of the damping. Fig 4.13 shows a single degree of freedom system, which experiences the velocity squared damping. In this system the mass is in contact with a fluid. Damping force resists the movement of the mass. This force is proportional to the square of the velocity and opposes to the direction of the velocity. Figs 4.13b and 4.13c show the free body diagrams of the system for two variations. The equations of motion of the free vibration of the system can be found as follows:

For the first case (Fig 4.13b), the condition of dynamic equilibrium may be written as

\[ -kx - a\dot{x}^2 = m\ddot{x} \]

or

\[ m\ddot{x} + a\dot{x}^2 + kx = 0 \]  

(4.68)

For the second case (Fig 4.13c), one may write

\[ -kx + a\dot{x}^2 = m\ddot{x} \]

or

\[ m\ddot{x} - a\dot{x}^2 + kx = 0 \]  

(4.69)

Assuming a steady state harmonic motion for a system with velocity squared damping results

\[ x(t) = X \cos(\omega t - \phi) \]  

(see Section 3.2.1)  

(4.70)

The energy dissipated per cycle becomes

\[ E = \frac{1}{2} F_d \Delta x \]  

(4.71)

This has a similar form to the equation used for Coulomb damping (Eqn 4.65) except for the \( F_d \) which has a magnitude of

\[ F_d = a\dot{x}^2 \]
The bottom surface of the mass is in contact with a fluid. The fluid applies the force $ax^2$ to the mass. The force opposes to the motion of the mass.

As a result, Eqn 4.71 can be written as

$$E = \int ax^2 dx$$

(4.72)

where $a$ is the velocity squared damping constant and $x$ is the velocity of the system. Displacement differential, $dx$, and the velocity, $\dot{x}$, can be obtained from Eqn 4.70 as follows

$$dx = -X\omega \sin(\omega t - \phi)dt$$

(4.73)

and

$$\dot{x} = \frac{dx}{dt} = -X\omega \sin(\omega t - \phi)$$

(4.74)

Substituting the above equation in Eqn 4.72 gives

$$E = \int a\left[X^2 \omega^2 \sin^2(\omega t - \phi) - X\omega \sin(\omega t - \phi)\right] dt$$

or

$$E = \int -aX^3 \omega^3 \sin^3(\omega t - \phi) dt$$

(4.75)

In a manner similar to the case of Coulomb damping Eqn 4.75 should be split into two integrals corresponding to two half cycles. That is,
\[
E = \int_{0}^{\pi/\omega} -aX^3\omega^3 \sin^3(\omega t - \phi)\,dt + \int_{\pi/\omega}^{2\pi/\omega} -aX^3\omega^3 \sin^3(\omega t - \phi)\,dt
\]  
(4.76)

where the limits of the integrals are as follows:

- \(\pi/\omega\) is the end point of the first half cycle (start point of second half cycle) and
- \(2\pi/\omega\) is the end point of the cycle.

The velocity of steady state harmonic vibration of a system is shown in Fig 4.12. The velocity is plotted versus the time. The horizontal and vertical axes in Fig 4.12 relate to time and velocity axes, respectively.

Since the absolute value of the energy in the two half cycles are to be added together, Eqn 4.76 can be written as

\[
E = \left| -\int_{0}^{\pi/\omega} aX^3\omega^3 \sin^3(\omega t - \phi)\,dt \right| + \left| -\int_{\pi/\omega}^{2\pi/\omega} aX^3\omega^3 \sin^3(\omega t - \phi)\,dt \right|
\]  
(4.77)

Since \(a, X,\) and \(\omega\) are independent of time, \(t,\) therefore, Eqn 4.77 can be written as

\[
E = aX^3\omega^3 \left( -\int_{0}^{\pi/\omega} \sin^3(\omega t - \phi)\,dt \right) + \left| -\int_{\pi/\omega}^{2\pi/\omega} \sin^3(\omega t - \phi)\,dt \right|
\]  
(4.78)

The result of evaluation of the integral

\[
\int \sin^3(\omega t - \phi)\,dt
\]

is

\[
\frac{1}{\omega} \left( -\cos(\omega t - \phi) + \frac{\cos^3(\omega t - \phi)}{3} \right)
\]

or

\[
\int_{0}^{\pi/\omega} \sin^3(\omega t - \phi)\,dt = \frac{1}{\omega} \left[ -\cos(\omega t - \phi) + \frac{\cos^3(\omega t - \phi)}{3} \right]_{0}^{\pi/\omega}
\]

or

\[
\int_{0}^{\pi/\omega} \sin^3(\omega t - \phi)\,dt = \frac{1}{\omega} \left[ \left( -(-1) + \frac{1}{3} \right) - \left( -1 + \frac{1}{3} \right) \right]
\]

or

\[
\int_{0}^{\pi/\omega} \sin^3(\omega t - \phi)\,dt = \frac{1}{\omega} \left[ \left( \frac{-2}{3} \right) - \left( -\frac{4}{3} \right) \right]
\]

or

\[
\int_{0}^{\pi/\omega} \sin^3(\omega t - \phi)\,dt = \frac{1}{\omega} \left[ \left( \frac{2}{3} \right) - \left( -\frac{4}{3} \right) \right]
\]
\[ \int_{0}^{\pi/\omega} \sin^3(\omega t - \phi) \, dt = \frac{4}{3\omega} \]  
\hspace{2cm} (4.79)

and in a similar manner

\[ \int_{\pi/\omega}^{2\pi/\omega} \sin^3(\omega t - \phi) \, dt = -\frac{4}{3\omega} \]  
\hspace{2cm} (4.80)

Substituting Eqns 4.79 and 4.80 into Eqn 4.78 gives

\[ E = aX^3 \omega^3 \left( \frac{4}{3\omega} + \frac{4}{3\omega} \right) \]

or

\[ E = \frac{8}{3} a \omega^2 X^3 \]  
\hspace{2cm} (4.81)

where \( a \) is the velocity squared damping constant and \( \omega \) and \( X \) are the frequency and amplitude of the vibration, respectively.

4.5.4.1 Equivalent viscous damping coefficient

for velocity squared damping

The equivalent viscous damping coefficient for the velocity squared damping can be derived using the same method as used in finding the equivalent coefficient for Coulomb damping. The energy dissipated in the viscously damped model (Eqn 4.56) and the energy dissipated in the velocity squared damped model (Eqn 4.81), can be equated as follows

\[ \pi c_v \omega X^2 = \frac{8}{3} a \omega^2 X^3 \]

the equivalent viscous damping coefficient then can be obtained as

\[ c_v = \frac{8a \omega^4 X^3}{3\pi \omega X^2} \]

or

\[ c_v = \frac{8}{3\pi} a \omega X \]  
\hspace{2cm} (4.82)
4.5.5 Displacement squared damping

One of the other types of damping models is ‘displacement squared damping’. In this model the damping force opposes to the movement of the system like Coulomb damping model. In this type of damping model the damping force is proportional to the square of the displacement of the system, \( x^2 \). So, the damping force can be written as

\[ F_d = e x^2 \]  

(4.83)

where

- \( e \) is the displacement squared damping constant and
- \( x \) is the displacement of the mass of the vibrating system

By using similar assumptions of linearity to previous damping models the general form of the equation of motion for steady state vibration of a system with displacement squared damping can be written as

\[ x(t) = X \cos(\omega t - \phi) \]  

(4.84)

Substituting the above equation in Eqn 4.83 gives

\[ F_d = eX^2 \cos^2(\omega t - \phi) \]  

(4.85)

The energy dissipated per cycle can be found, as usual by the following equation

\[ E = \int F_d \, dx \quad \text{(see Eqn 4.58)} \]  

(4.86)

This has a similar form to the equation used for the velocity squared damping model (Eqn 4.71), except for \( F_d \) which can be obtained from Eqn 4.83. Displacement differential, \( dx \), is given by Eqn 4.73 and repeated here that is

\[ dx = -X\omega \sin(\omega t - \phi) \]  

(4.87)

Substituting Eqns 4.85 and 4.87 into Eqn 4.86 gives

\[ E = \int eX^2 \cos^2(\omega t - \phi)[-X\omega \sin(\omega t - \phi)\, dt] \]  

(4.88)

or

\[ E = \int -eX^3 \omega \cos^2(\omega t - \phi) \sin(\omega t - \phi)\, dt \]  

(4.89)
Since $e$, $X$ and $\omega$ are constant, Eqn 4.89 may be written as

$$E = eX^3 \int - \cos^2(\omega t - \phi) \sin(\omega t) dt \quad (4.90)$$

The motion of a steady state harmonic vibration of single degree of freedom system is shown in Fig 4.14. The direction of damping force changes with the sign of the displacement. The points $\pi / 2\omega$ and $3\pi / 2\omega$ that are indicated on the graph correspond to the direction change points. As can be seen from Fig 4.14, Eqn 4.90 may be written as

$$E = eX^3 \int_{0}^{\pi / 2\omega} - \cos^2(\omega t - \phi) \sin(\omega t) dt \quad (4.91)$$

The evaluation of integral

$$\int \cos^2(\omega t - \phi) dt$$

is

$$\frac{\cos^3(\omega t - \phi)}{3\omega}$$

So, Eqn 4.91 can be written as

$$E = eX^3 \omega \left[ \frac{\cos^3(\omega t - \phi)}{3\omega} \right]_{0}^{\pi / 2\omega}$$

or

$$E = eX^3 \omega \left( \frac{4}{\omega} \right) \left[ 0 - \frac{1}{3} \right]$$

or

$$E = -\frac{4}{3} eX^3 \quad (4.92)$$

In damping analysis the magnitude of dissipated energy is of interest regardless of its sign. Therefore, one may write Eqn 4.92 as follows

$$E = \frac{4}{3} eX^3 \quad (4.93)$$
4.5.5.1 Equivalent viscous damping coefficient
for displacement squared damping

The equivalent viscous damping coefficient for the displacement squared damping model can be derived like that of the previous damping models. The energy dissipated in the viscously damped model (Eqn 4.56) and the energy dissipated in the displacement squared damped model (Eqn 4.93), can be equated as follows

\[ \pi c_e \omega X^2 = \frac{4}{3} eX^3 \]

the equivalent viscous damping coefficient can be obtained as

\[ c_e = \frac{4eX}{3\pi \omega} \]  

(4.94)

4.5.6 Solid damping model

As discussed previously, when a material is cyclically strained, energy is dissipated internally within the material itself. This give rise 'solid damping' which is a type of damping that
Chapter 4 Damping: Theory and Concepts

absorbs the energy of a vibrating system and dissipates it. In this type of damping, the damping force is considered to be proportional to the displacement and opposes the movement of the system. Experiments indicate that for most structural materials such as steel and aluminium, the energy dissipated per cycle is proportional to the square of the amplitude of the vibration [Lazen, 1968]. The energy dissipated in a system with solid damping, as will be evaluated later, is in agreement with the results of these experiments. Solid damping is sometimes referred to as the 'structural damping'. The proportionality of the solid damping force with the displacement, $x$, as stated above can be presented mathematically as follows

$$F_d = bx$$

(4.95)

where $b$ is the 'solid damping constant' and $x$ is the displacement of the system.

The energy dissipated per cycle can be found, as usual, by the integral

$$E = \int F_d dx$$

(see Eqn 4.58)

(4.96)

where $F_d$ is the damping force in a system with solid damping as given by Eqn 4.95 and $dx$ is the displacement differential. By using similar assumptions of linearity to previous the damping models, $x$ can be given by Eqn 4.84, that is,

$$x(t) = X \cos(\omega t - \phi)$$

(4.97)

Also, $dx$ can be obtained from the above equation as

$$dx = -X\omega \sin(\omega t - \phi)dt$$

(4.98)

The damping force may now be obtained by substituting Eqn 4.97 into Eqn 4.95, that is,

$$F_d = bX \cos(\omega t - \phi)$$

(4.99)

Substitution of Eqns 4.98 and 4.99 into Eqn 4.96 gives

$$E = \int bX \cos(\omega t - \phi)[-X\omega \sin(\omega t - \phi)dt]$$

or

$$E = \int -bX^2 \omega \cos(\omega t - \phi)\sin(\omega t - \phi)dt$$

(4.100)

Since $b$, $X$ and $\omega$ are constants, Eqn 4.100 can be written as
The above integral for a cycle of vibration of a steady state harmonic response of a system is given by

\[ E = -bX^2 \omega \int_0^{\pi/2\omega} \cos(\omega t - \phi) \sin(\omega t - \phi) \, dt \]  

(4.102)

This is obtained in the same manner as used in Eqn 4.91 in Section 4.5.5. The presence of the factor 4 in Eqn 4.102 is due to the fact that the integral covers only a quarter of a cycle.

Evaluation of the integral

\[ \int_0^{\pi/2\omega} \cos(\omega t - \phi) \sin(\omega t - \phi) \, dt \]

from Eqn 4.102 gives

\[ \left[ \frac{\sin^2(\omega t - \phi)}{2\omega} \right]_{0}^{\pi/2\omega} = \frac{1}{2\omega} \]

Applying the limits of the integral yields

\[ \left[ \frac{\sin^2(\omega t - \phi)}{2\omega} \right]_{0}^{\pi/2\omega} = \frac{1}{2\omega} \]

So, Eqn 4.102 can be written as

\[ E = -4bX^2 \omega \frac{1}{2\omega} \]

or

\[ E = -2bX^2 \]

(4.103)

The magnitude of the dissipated energy is of interest regardless of its sign. Therefore, one may write Eqn 4.103 as

\[ E = 2bX^2 \]

(4.104)

4.5.6.1 Equivalent viscous damping coefficient

for solid damping
The equivalent viscous damping coefficient for the solid damping model can be derived in a manner similar to that of the previous damping models. The energy dissipated in a viscously damped model (Eqn 4.56) and the energy dissipated in a model with solid damping (Eqn 4.104) can be equated as follows

$$\pi c_{eq}\omega X^2 = 2bX^2$$

the equivalent viscous damping coefficient can then be obtained as

$$c_{eq} = \frac{2b}{\pi \omega} \quad (4.105)$$

4.5.7 Complex stiffness

In the calculation of the flutter speeds of airplane wings, the concept of 'complex stiffness' is used [W. T. Thompson, 1993]. Complex stiffness is related to a material, which is called viscoelastic. A material is called viscoelastic because exhibits both elastic and viscous behaviour. In other words, material behaviour is termed viscoelastic if the material stores part of the deformation energy elastically as potential energy and dissipate the rest, simultaneously. Complex stiffness is a concept that involves both the stiffness and damping characteristics of a system. In other words, the complex stiffness represents not only the stiffness but also the damping characteristics of a system. Complex stiffness in a vibrating system involves the particulars of the elasticity and energy dissipation of the system. Extensive investigations have shown that the complex stiffness concept provides a simpler approach in the modelling of damped vibrating system [Maly, J. R. 2000]. The complex stiffness can be written as

$$k(1 + i\eta) = k + ik\eta$$

where

- $k$ is the stiffness of the system,
- $\eta$ is a representation of the damping characteristics of the vibrating system and $i = \sqrt{-1}$

The force induced by the complex stiffness due to the displacement, $x$, is equal to

$$k(1 + i\eta)x = kx + ik\eta x \quad (4.106)$$
The first term in the above equation is related to the elasticity of the system and the second term is related to the energy dissipation in the system. The analysis of a system with complex stiffness is simpler than that with viscous damping and it (complex stiffness model) is a linear model [Barkanov, 1999].

The equation of motion of an undamped single degree of freedom system in free vibration is given by

\[ m\ddot{x} + kx = 0 \]  (see Section 3.2.1)  (4.107)

where \( m \) and \( k \) are, as usual, the mass and stiffness of the system.

For a system with complex stiffness, Eqn 4.107 can be written as

\[ m\ddot{x} + k(1 + i\eta)x = 0 \]  (4.108)

or

\[ m\ddot{x} + kx + ik\eta x = 0 \]  (4.109)

In contrast with Eqn 4.107, which is the equation of free vibration of an undamped system, Eqn 4.109 is the equation of free vibration of a damped system. The construct

\[ k\eta x \]

in the third term on the left-hand side of Eqn 4.109 is the damping force in the system.

The auxiliary equation to solve Eqn 4.108 is given by

\[ mA\dot{x} + k(1 + \eta) = 0 \]  \[ \text{[E. Kreyszig, 1999]} \]  (4.110)

Assuming

\[ \lambda = a + ib \]  (4.111)

and substituting Eqn 4.111 into Eqn 4.110 gives

\[ m(a^2 + 2iab - b^2) + k(1 + i\eta) = 0 \]

Separating the real and imaginary parts gives

\[ [m(a^2 - b^2) + k] + i[2mab + k\eta] = 0 \]  (4.112)

Eqn 4.112 implies that the real and imaginary parts must be equal to zero separately. So,

\[ m(a^2 - b^2) + k = 0 \]  (4.113)
2mab + kη = 0 \quad (4.114)

Solving Eqns 4.113 and 4.114 yields

\[ a = \pm \sqrt{\frac{k}{2m}} \left[ -1 \pm \sqrt{1 + \eta^2} \right]^{1/2} \quad (4.115) \]

\[ b = \pm \sqrt{\frac{k}{2m}} \left[ 1 \pm \sqrt{1 + \eta^2} \right]^{1/2} \quad (4.116) \]

As can be seen from Eqns 4.115 and 4.116, the ± signs are present at two places in each of the equations. Possible combinations of these signs provide four solutions for $a$ and four solutions for $b$. The process of solving Eqns 4.113 and 4.114 involves polynomials that contain the terms $a^4$ and $b^4$. This is why there are four separate solutions for $a$ and $b$. However, some of these solutions are not acceptable as discussed below.

The complex stiffness force as given in Eqn 4.106 represents a force relating to the displacement, $x$. The force consists of a real component and an imaginary component. The real component of the complex stiffness force, that is, $kx$, is the force associated with the stiffness of the system and it is in phase with the displacement. The imaginary component, that is, $k\eta x$, is a force that is associated with damping. This force is $90^\circ$ out of phase with the displacement and is associated with damping. Damping force causes dissipation of energy and causes decay of the amplitude of the vibration of the system.

The general form of the solution of Eqn 4.108 is given by

\[ x(t) = e^a (A_1 \cos bt + A_2 \sin bt) \quad \text{[E. Kreyszig, 1999]} \quad (4.117) \]

where $a$ and $b$ are as given by Eqns 4.115 and 4.116 and $A_1$ and $A_2$ are constants that can be obtained from the initial conditions of the system. The terms in the parentheses in Eqn 4.117 describe a harmonic motion and the term $e^a$ is an exponential function. This exponential function relates to the amplitude of the vibration. Eqn 4.108 is homogeneous representing a transient response of the system. Accordingly a positive value of $a$, is not acceptable physically since a positive value of $a$ will implies an increasing amplitude of the vibration which is in contrast with the transient state of vibration. Eqns 4.115 and 4.116 represent, respectively, the real and imaginary roots of the complex differential equation given by Eqn 4.108. For a transient response, the real part of the root (Eqn 4.115) must be negative. So, the value of $a$ is chosen as
\[ a = -\sqrt{\frac{k}{2m} \left[ -1 + \sqrt{1 + \eta^2} \right]} \]

or

\[ a = -\sqrt{\frac{k}{m} \left[ \frac{1}{2} \left( -1 + \sqrt{1 + \eta^2} \right) \right]} \]  

(4.118)

Using the notation

\[ \sqrt{\frac{k}{m}} = \omega_* \]

one obtains

\[ a = -\omega_* \sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}} \]  

(4.119)

where \( \omega_* \) is the natural frequency of the system. Using the same notation, the imaginary part of the root (Eqn 4.116) can be found as

\[ b = \pm \omega_* \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} \]  

(4.120)

Substitution of Eqns 4.119 and 4.120 into Eqn 4.111 yields

\[ \lambda_1 = -\omega_* \sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}} + i \omega_* \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} \]  

(4.121)

\[ \lambda_2 = -\omega_* \sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}} - i \omega_* \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} \]  

(4.122)

As a result, the solution of Eqn 4.108 as given in Eqn 4.117 can be written as

\[ x(t) = e^{-\omega_* \sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}}} \left( A_1 \cos \omega_* \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} t + A_2 \sin \omega_* \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} t \right) \]  

(4.123)

[E. Kreyszig, 1999]

where \( A_1 \) and \( A_2 \) are constants and can be obtained from the initial conditions of the system.

Eqn 4.123 can be simplified to
\begin{equation}
    x(t) = Ae^{-\sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}}} \cos \left( \omega_n \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} t - \phi \right)
\end{equation}

where

\[ A = \sqrt{A_1^2 + A_2^2} \]

and

\[ \phi = \tan^{-1} \frac{A_2}{A_1} \]

The term

\[ e^{-\sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}}} \]

in Eqn 4.124 causes the displacement, \( x \), to decay and represents the energy dissipation in the system. The rate of the energy dissipation depends on the absolute value of

\[ -\omega_n \sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}} \]

which has appeared as the power of \( e \). The above term is a function of \( \eta \), where \( \eta \) represents the damping characteristics of the system. The greater the value of \( \eta \), the greater will be the damping. Therefore, the rate of damping decay increases as damping increases.

The construct

\[ \omega_n \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} \]

in Eqn 4.124 can be considered as damped natural frequency of the system. The damped natural frequency increases with increased damping. The limits of the damped natural frequency as \( \eta \) approaches zero and infinity are

\[ \lim_{\eta \to 0} \omega_n \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} = \omega_n \]

\[ \omega_n \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} = \omega_n \]

(4.125)

and
Eqn 4.125 shows that the natural frequency of the system with no damping ($\eta = 0$) is simply $\omega_n$ and Eqn 4.126 shows that the natural frequency of the system becomes infinite as the damping becomes infinite.
CHAPTER FIVE

EXPERIMENTAL WORK

5.1 Introduction

As the spans covered by space structures are increasing continuously and these structures are becoming more popular, the importance of the understanding of their dynamic behaviour, in particular, damping characteristic is also increasing. To investigate the dynamic behaviour of a space structure, it is necessary to have complete information about the stiffness of the component parts of the structure as well as the mass distribution and damping characteristics. Without sufficient and accurate information about the dynamic characteristics of a structure, the assessment of the behaviour of structures under dynamic loads, like earthquakes is not possible.
The difficulties and high cost of full-scale tests are the reasons for most experimental investigations into the damping properties of structures to be carried out on small-scale models. Although, the experimental study of small-scale models can be useful, the results of these tests cannot be reliably scaled up for application to real structures. Since the damping characteristic of a real structure is a complex phenomenon, and is dependent upon a number of structural properties such as the configuration of the structure, the initial strains, support and connection conditions, the results obtained from small-scale models cannot represent the dynamic characteristics of the structure. In addition, structural damping is affected by stiffness, mass and energy absorption at connections in a structure and this can vary from model to model as well as from model to a real structure. In fact, useful models are real structures which are not normally available for experiments. One may say that accurate results for damping capacity of a structure can only be found using full-scale tests.

5.2 The Experimental Double Layer Grid

Double layer grids constitute an important family of space structures used to cover large areas. For ease of handling, flexibility in fabrication and speedy on-site erection, these structures are usually prefabricated. Therefore, the components of double layer grids are normally mass produced in factory with a high standard of quality control. Several industrialised systems have been developed and applied in construction of double layer grids and one of the most popular of these is the MERO system.

The objective of the present work is to experimentally investigate the effects of the degree of bolt tightness in damping properties of a double layer grid that is constructed using the MERO system. The experimental work is carried out on a 10m by 10m double layer grid. The double layer grid is constructed at the University of Mazandaran in Iran. Two general views of the constructed double layer grid are shown in Figs 5.1 and 5.2. The basic model which is used for this research is the same as that used by M Davoodi. However, the dynamic loading system and the data acquisition system used for the experiments are specific to this research. Correspondingly, the description of the basic model, as appear in pages 135 to 155 and 175 to 179, is taken from M Davoodi's PhD Thesis and after applying required changes on them, they are used in this Thesis. This opportunity is taken to acknowledge the help of M Dvoodi in this regard.
Chapter 5 Experimental work

Figure 5.1 A general view of the experimental double layer grid

Figure 5.2 Another view of the experimental grid
5.3 The MERO System

The MERO jointing system is one of the oldest proprietary space frame jointing systems available, having been developed in Germany in 1940 [Allen, D. 1982]. This system is a multidirectional system allowing up to fourteen members to be connected together at various angles to a spherical node. The jointing system is suitable for both single and multilayer space structures of various geometry including those with curved and irregular shapes. Consider Fig 5.3a where four tubular elements are connected together by means of a MERO jointing system. The details of the connection are shown in Figs 5.3b and 5.3c. The constituent parts of the system are:

Figure 5.3 The MERO jointing system

(a) General view of four connected elements

(b) Section AA

(c) Outside view

Steel ball
High tensile bolt
Dowel pin
Window
Sleeve
Conical end piece

MH Pashaei 157
• A forged steel ball,
• A forged steel conical end piece,
• A high tensile bolt,
• A sleeve with a window and
• A dowel pin.

The ball is located at the intersection of the longitudinal axes of tubular elements. The conical end pieces are welded to the end of the tube element. A high tensile bolt is passing through the conical end piece and is screwed into the ball by means of a sleeve. A dowel pin is used to constrain the bolt to the sleeve in order to allow the turning of the bolt. There is a window on the sleeve that allows the movement of the dowel pin and the penetration of the bolt.

The interaction between the tubular elements and the connectors in a structure usually involves tensile, compressive and bending effects. The tension in a member is transferred to the ball through the conical end piece and the ball, as shown in Fig 5.4a. In this case the bolt is under tension and the sleeve is inactive.

A compressive force in a tubular member is transferred to the ball by direct bearing through the conical end piece, sleeve and the ball, as shown in Fig 5.4b. In this case, the sleeve is under compression and the bolt is inactive.

Under the action of a sagging bending moment the connection assumes the deformed shape shown in Fig 5.4c. Because of the bending deformation, the lower part of the sleeve is separated from the ball and the conical end piece. Thus, the lower part of the sleeve does not contribute to the transmission of the bending moment due to the gaps. Therefore, the bending moment is transmitted by the bolt and the upper part of the sleeve both of which will be under bending stresses.

The three cases of load transfer as shown in Fig 5.4 are idealised forms. In practice, the loads will be either a combination of a tensile force together with some bending or a combination of a compressive force together with some bending.
5.4 Grid Geometry

The grid on which the experimental work was carried out is shown in Fig 5.5. Some of the factors that have influenced the choice of the grid geometry are as follows:

- The square-on-square offset pattern has been chosen due to the fact that this type of pattern is simple as well as being frequently used in practice.
- The geometry of the grid has been chosen such that the elements and the connectors are identical.
- The overall size of the grid had to be limited to 10m due to the limitation on the available space
Chapter 5 Experimental work

- The geometry of the grid was chosen to be symmetric. Also, the grid was tested under symmetric support conditions.

5.4.1 The model

The dimensions of the grid are shown in Fig 5.5. In this figure the top layer elements are shown by thick lines and the bottom layer elements as well as the web elements are shown by thin lines.

The centre-to-centre distance between the top and bottom layers is 1000mm. Thus, the angles of the web members with the horizontal plane are 45° and all the members had the same length. The grid has twelve supports that are shown by little squares at twelve bottom layer nodes in Fig 5.5. The grid consists of 360 elements of which 84 top layer elements, 180 web elements and 96 bottom layer elements. These elements are connected together using 109 connectors.

5.5 Manufacture of the Components

The connector which is used in this grid is type of the MERO jointing system. A MERO-type jointing system consists of a number of pieces including bolts. These pieces are from different materials involving high tensile steel (that is, bolt) and different grades of mild steel. In this system, discontinuities will be inevitable between the connecting parts due to the presence of bolts.

The jointing system was manufactured by the Sepah Industrial Factory in Tehran, Iran. Due to the simplicity of the pattern and the fact that all the grid elements and the nodes were identical, all the components were produced in factory with a reasonable standard of quality control.

The dimensions of the components that are used in the experimental grid are shown in Fig 5.6. The diameter of the ball is 90mm and the diameter of the holes on the ball is 20mm. Most connectors in the structure required eight member locating holes. However, the same balls were used where fewer members were required. The dowel pin is a cylindrical piece with 5mm diameter and 35mm length. The tubular elements for the experimental grid were
prepared from steel tubes of nominal external diameter 76.2mm and wall thickness 3.3mm. The tubular elements were prepared by first cutting 1200mm lengths of tube and then welding conical pieces to the ends of the tubes. The dimensions of a tubular element are shown in Fig 5.7.

![Plan and Elevation Diagram](image)

**Figure 5.5** The plan and elevation of the grid with 12 supports.
5.6 Assembly of the Grid

The components of the grid were transported to the experimental site near the Engineering Faculty of the University of Mazandaran, Iran. All of the components were unloaded carefully and under continuous supervision. It was decided to assemble the grid on a flat concrete surface. Attention was paid to ensuring that the entire surface was level. The assembly started from one side and carried on until the entire grid was completed. A general view of the assembled grid is shown in Fig 5.8.

5.6.1 Assembly process

The assembly process was started by a small portion of the grid at one of the corners. This portion was a pyramid consisting of four bottom layer elements and four web elements that were connected together by means of five connectors. This portion is shown in Fig 5.9a with thick lines. The assembly process was continued by connecting three bottom layer elements, four web elements and one top layer element. This step is shown in Fig 5.9a with thin lines.

Further continuation of the assembly is shown in Fig 5.9b, where the assembly of the first row is complete and the assembly of the second row has started. The assembly of the second row was continued in six steps. In each step two bottom layer elements, four web elements and two top layer elements were connected to the assembled part of the structure by means of two connectors as shown in Fig 5.9b with thin lines. The rest of the structure was assembled using the same procedure.

In the case of prefabricated structures the accumulation of the imperfections of the components usually shows itself in the assembly process. These imperfections are usually inevitable and will change the theoretical configuration of the structure to a certain degree. In the initial steps of the assembly process the discrepancy is negligible and the components can be located in their positions properly. In the following steps especially in the case of large structures, the discrepancies increase gradually and significant distortions will appear. The main problem is when there are some constraints for the nodes or elements of the structure such as supports. In the case of the experimental grid, due to the nature of the method chosen
Figure 5.6 The details of jointing system
and the freedom of movement at the supports, all of the nodes and elements fitted together easily. Actually, in this case the accumulation of the imperfections only affected the overall dimensions of the grid. The overall dimensions of the grid, when the assembly was completed, were measured to be 9908±2mm by 9908±2mm. (the theoretical values were 9898mm by 9898mm).

Figure 5.7 Tubular element

Figure 5.8 A general view of the assembled grid
In the assembly process, the connectors were tightened with an open-end spanner or a monkey wrench. At this stage the main concern was the assembly of all the elements and the bolts of the connectors were not fully tightened. Subsequently, all the bolts were tightened after the completion of the whole assembly of the grid. The retightening process was started from one corner and continued in the order of the assembly of the components.

5.7 Foundation

The foundation for the test set-up of the experimental grid was constructed at a site at the University of Mazandaran, Iran. A general view of this foundation is shown in Fig 5.10. The foundation consists of four pad footings that are connected by a rectangular grid of strip footings, Fig 5.11. Twelve reinforced concrete columns were built on the foundation as possible support positions for the grid. A plate was placed on the top of each column by means of four anchor bolts. Furthermore, thirteen plates were placed on the foundation as possible support positions for loading frame and measurement equipment. The strip and pad footings as well as columns are identical.
Fig 5.11 shows the plan of the foundation and the cross-sections of pad and strip footings. The building of the foundation involved the construction of formwork, preparation of steel reinforcements and the preparation and casting of concrete.

### 5.7.1 Formwork of foundation

The experimental site was a flat concrete surface. The formwork for the foundation was a short wall with 600mm height that was built using concrete blocks (Fig 5.11). The internal surface of the formwork wall was covered with sand cement plaster (Fig 5.11). This formwork was not removed when the foundation was completed.

In order to provide a clean and level surface for the steel reinforcement bars a layer of lean concrete was laid on the bottom surface (Fig 5.11). The thickness of the lean concrete layer was about 100mm.
5.7.2 Steel reinforcement

The strip and pad footings are reinforced in both directions at both the upper and lower sides, Fig 5.11. The concrete cover to the outermost reinforcement bars was 50mm. The steel reinforcement of the foundation comprised 20mm diameter (type III) mild steel ribbed bars. The bars were delivered in lengths of about 12m. They were cut and bent and then were fixed with mild steel tie wires.

Figure 5.12 shows the layout of a typical column that is constructed as a possible support position. The longitudinal reinforcements of columns were 20mm diameter (type III) mild steel ribbed bars. Each stirrup was made of a sufficient length of 10mm diameter ribbed bar. The reinforcements of the columns were carefully formed to the dimensions indicated in Fig 5.12. The vertical reinforcing bars and stirrups were tied together with tie wires to form the column cage. The column cages were accurately placed and secured against displacement by using tie wires.

The base plates that are placed on the strip footings have eight anchor bolts while the base plates that are located on the columns have four anchor bolts. The entire anchor bolts are from 20mm diameter ribbed bars.

5.7.3 Concreting of foundation

The concrete for the foundation was manufactured at the experimental site. The aggregate including sand, gravel and crushed gravel were derived from natural deposits that were available. Ordinary Portland type I cement was used in the concrete. The mix was designed to achieve 28 days cylinder compressive strength (fc) of 20MPa. The concrete mix had 350kg/m³ cement content and 42% water/cement ratio. The following steps were carried out under continuous supervision:

- The accurate weights of cement, gravel and sand as well as the amount of water were used for each batch according to the mix design.
- The mixing time was 60 seconds or more after all the material was in the mixer.
- The concrete was sampled from mixer discharge point that was near to the placement point on a random basis.
Figure 5.11
• A slump test was taken from most of the batches. The measured slumps were between 40mm to 60mm.

• At the time of concrete placement, all the steel reinforcements were free from flaky rust, mud, grease or any other coating that might have reduce the bond with the concrete.

• Every endeavour was made to avoid steel reinforcement bars from being displaced or depressed.

• During concreting, adequate compaction was achieved using a vibrator to ensure good quality concrete.

• The surface of freshly placed concrete, specially between anchor bolts, was levelled.

• The base plates with slotted holes were bolted to the pre-installed anchor bolts on the strip footings.

• The entire surface of the concrete was kept damp for a number of days.

Minimum compressive strength obtained from cub tests after 7 days and 28 days moist-cured concrete were 18MPa and 28MPa, respectively. The cubes dimensions were 150×150×150mm.

5.7.4 Formwork for columns

Since all of the columns were identical, it was decided to build them in three stages. In each stage four columns were constructed. The timber formworks were prepared on site for the construction of columns. Timber is generally used in Iran for in-situ concrete work. The formwork parts were substantial and sufficiently tight to prevent leakage of mortar. The corners of the columns were bevelled by moulding placed in the formworks. The formworks finished surface was coated with oil before the assembling process.

The column reinforcements as mentioned before were put in place together with the footing reinforcements. Each one of the constructed column formworks was placed in the correct position and supported by four screw jacks. The screw jacks helped to adjust the alignment and the vertical position of the formwork. The height of the formworks was about 2.20m.
In order to level the top surfaces of the columns a wooden guide strip was provided at the proper location on each formwork so that the top surface of the columns can be levelled.
5.7.5 Concreting of columns

The concrete aggregate and cement type for the columns' concrete were the same as those for the foundation. The concrete mix had 400kg/m³ cement content and 41% water/cement ratio. In order to have a better workability for concrete, the slump range was changed to 60mm-100mm. Particular attention was paid to ensure that, at the time of concrete placement, all the reinforcement and anchor bolts were clean and in correct positions.

The concrete was sampled and slump test was taken from each batch. Since the slump amount in columns’ concreting was important, the batching operation was slowed down to allow completion of each slump test before the next batch was made, so that the test results could be communicated to the operator and the necessary corrections could be made. During the concreting process the main concern was that the reinforcement bars were not displaced.

The concrete was placed in uniform layers not more than 500mm thick. Each layer of concrete was fully compacted before placing the next one, and each new layer was placed while the underlying layer was still responsive to vibration. The top surfaces of columns were finished so that the base plates can be installed at the same level and in firm contact with columns. Finally, exposed parts of all anchor bolts were cleaned using a cloth.

Since the formworks for columns were not supporting the weight of the concrete, they could be removed when the concrete has attained sufficient strength to resist damage from the removal operation. Since the formworks were to be reused, they were thoroughly cleaned, repaired, and coated with oil again.

The column surfaces were cured after removal of formworks. In order to obtain good curing, wet hessian was used to cover the top and the sides of the columns. The hessian was kept wet by frequent watering for four weeks.

The remaining eight columns were constructed in two further stages. At each stage formworking, concreting, striping formworks, and curing process were carried out as mentioned for the first stage. The base plates with slotted holes on the top of the columns were bolted to the pre-installed anchor bolts. All concrete cube samples were placed in a pool and compressive strength test carried out after seven and twenty eight days. Minimum compressive strength for seven days moist-curved concrete was 20MPa and for twenty eight days samples was 30MPa.
5.8 Placement of the Grid

The process of creating a flat and level surface for the experimental work started with filling the voids between the strip footings. These voids were filled with two 25cm layers, using the available sand and gravel at site. Each layer was compacted using a compactor. A 10cm thick layer of concrete was poured on the top of the fill. The concrete had 250kg/m$^3$ cement content and was produced at site using the available material. The poured concrete was levelled using a screed board.

For the placement of the double layer grid a telescopic crane was used. In spite of the fact that the total weight of the double layer grid was about 3800kg, the large dimensions of the grid and the presence of the columns required quite a big crane. A 40 ton truck crane with about 30m boom was used for the purpose.

It was decided to hoist the grid from four top layer nodes. Accordingly, the grid was analysed for this loading and boundary condition to ensure that the grid can be safely hoisted. Four cables were connected to the four top layer nodes in the central part of the grid. Before moving the grid the tightness of the nodes was checked with inspection of the dowel pins on the sleeves that show the amount of penetration of the bolts into the balls. The grid was picked up carefully and placed on the top of columns. The placement of the grid on the top of the columns is shown in Figs 5.13 to 5.15. Fig 5.16 shows the grid in the correct position after the placement.

5.9 Supports

The main objective of the present work is to investigate the effects of the degree of bolt tightness of connectors and number of supports on the damping characteristics of MERO-type double layer grids. Roller supports in two directions in the horizontal plane were chosen for all of the supports of the test grid.

A support assembly consists of two hardened steel plates, a rigid PVC plate with seven holes and seven ball-bearings. A cross-section of a support assembly is shown in Fig 5.17. The lower hardened steel plate was placed on the base plate of the reinforced concrete column. The PVC plate had seven cylindrical holes for the ball-bearings as shown in Fig 5.18. This
Figure 5.13 The grid about to be placed on the columns

Figure 5.14 Ensuring the accurate placement of the grid
Chapter 5 Experimental work

Fig 5.15 The final inspection of the grid after placement

Figure 5.16 The grid in its correct position on the top of the columns
plate together with the seven ball-bearings were placed on the lower hardened plate. The upper hardened plate was placed on the ball-bearings.

![Diagram](image)

**Figure 5.17 The cross-section of a support assembly**

A simple clamping mechanism fixed the supports in the vertical direction as shown in Fig 5.19. The clamping mechanism consisted of a bent bar with a clamping plate and a pair of nuts. A 17mm diameter steel rod with a length of 40cm was bent as shown in Fig 5.19. One end of the bent rod was placed into a hole of the connector and the other end of the rod was held by the clamping plate. Anchor bolts of the supports were used for clamping down the rod.

### 5.10 Loading System

Most of the tests which were carried out in this work are of type ‘initial displacement test’. An initial displacement test is a type of free vibration test in which the model is displaced initially by applying a load. The load is then released rapidly and the model is left to vibrate. To provide such a loading condition for the grid, a loading system was designed, built and installed. The loading system consisted of two main parts as follows:

- Load releasing mechanism and
- Hung load of 1kN to 2.5kN.
Figure 5.18 Rigid PVC plate with holes together with seven ball-bearings

Figure 5.19 Clamping mechanism
5.10.1 Load releasing mechanism

Fig 5.20 shows the elevation and side view of a load releasing mechanism. The frame of this mechanism is built using two 40x80mm steel rectangular hollow sections (RHS) at the top and two 30x60mm steel RHS's on the sides. The hook holder piece is pivoted on the 30x60mm RHS on the left side of the frame, as shown in Fig 5.20. Also, an L shaped bar is pivoted at its end on the right side of the frame, as shown in Fig 5.20. The hook holder is supported by the L shaped bar at its right hand end. The contact surfaces between the hook holder and L shaped bar are polished. A helical spring keeps the L shaped bar in its correct position. A regulatory bolt which is inside the spring (as shown in Fig 5.20) is used to regulate the length of the spring. The spring is between the frame and a thin round plate which is connected to the end of the bolt. The adjustment is done by a nut which is screwed on the bolt. This nut may be seen at the back of the 30x60mm RHS on the right side of the frame. As it is shown in Fig 5.20 a string is connected to the L shaped bar. When the L shaped bar is pulled to the right by the string, the hook holder will be free to rotate about its end pivot and the load will be released.

The load releasing mechanism was connected firmly to the central node of the top layer of the grid by a 16mm diameter bolt. A view of the load releasing mechanism is shown in Fig 5.21. This figure shows the load releasing mechanism which is connected to the central connecter of the grid.

5.10.2 Loading arrangement

A cylindrical barrel which is filled with water is the main part of the loading arrangement. The full capacity of the barrel is 220 litres. A sketch of the loading arrangement is shown in Fig 5.22.

A view of the loading arrangement is shown in Fig 5.23. A cylindrical frame is used to hold the barrel, as shown in the figure. The cylindrical frame is constructed from steel strips of cross-section 5x50mm. The barrel is held by three 1.1m chains that are hooked to the frame. The other ends of the chains are held by the hook of the load releasing mechanism. The barrel is lifted by a chain hoist crane. The lifting of the barrel by the crane was repeated for each test. The chain hoist crane was attached to an independent steel frame.
Figure 5.20 Elevation and side view of the load releasing mechanism
In order to prevent any damage to the loading arrangement when it is released and hits the ground, two layers of 100mm sponge are used on the ground.
Figure 5.22 A sketch of loading arrangement.
Figure 5.23 A view of the loading arrangement.

The loading for the full capacity of the barrel weighs 2.5kN. During the tests a number of load levels were applied. This was done by adding water to the barrel or by removing water from the barrel. Different levels of the loading result in different amplitudes in the response curves. Figs 5.24 and 5.25 show two views of the loading arrangement.
5.11 Data Acquisition System

In order to record the response of the grid in each test a set of data acquisition equipments was used. This set comprises of:

- Front-end module 6-1 channel,
- Transducer,
- Connecting cable and
- Personal computer.

5.11.1 Front-end module 6-1 channel

The 6-1 channel front-end module is the main part of the data acquisition system. The term 6-1 means that this module has 6 input channels and one output channel. Input for this module is the output of a transducer such as accelerometer, tachometer, load cell, thermometer, microphone and so on. A transducer can be linked to the module by a cable. An input to the module as well as an output of a transducer is the form of an electrical signal. The signals
which are input to the module will be digitised by analogue to digital converters. These converters are installed in the module. There is an analogue to digital converter in the module for each input channel. Figs 5.26 and 5.27 show the 6-1 channel module in horizontal and vertical positions, respectively.

The 6-1 channel module consists of a number of electronic boards on which some of the necessary software is installed. The software analyses the data which are being input to the module during the experiments. All of the operations of this module is controlled by a main program which is named 'Pulse'. This module and the Pulse software were designed and developed by a Denmark based company named 'Brüel & Kjær'.

Pulse software controls analysers which are installed in the module. These analysers can analyse the input signals of the module which come from different types of transducers. Also, the Pulse software provides different graphs and output results. These graphs include Time-Acceleration curves, Time-Speed curves and Time-Displacement curves in the time domain. Also, Fourier Spectrum and Auto Spectrum graphs in frequency domain can be displayed by Pulse. In addition, Pulse software can provide data in digital form which can be exported to
any other software that is capable of using digital form of data. For example, Microsoft Excel software is such a program to which the Pulse produced digital form data files can be imported.

Each of the experiments had a corresponding file which created in the Pulse software before carrying out the test. In other words, Pulse controls the process of carrying out the test through this file. All of the devices that are involved in the test have to be defined in this file. Also, the form of output as well as the response curves need to be defined in this file. For instance, the type and specifications of a transducer which is being used in the test, the number of channel or channels through which the data are being transmitted, the types of processes which are expected to be done on data and the output response curves which are to be displayed need to be defined clearly in this file. This file is called in Pulse and will be activated before carrying out the test.

Figure 5.26 The 6-1 channel in horizontal position
As mentioned previously, the 6-1 channel module has only one output channel. The output channel of this module may be used in a test in order to control an electrical equipment which is being used in the test. Such a control is operated by Pulse. In this work, the output channel of the module was not used. There are two types of connections to the ports of input and output channels. So, there are two ports for each channel. As a result, for 6 input channels there are 12 ports. Also, for one output channel there are two ports.

The 6-1 channel module is able to work using both direct electrical power or batteries which are placed in the module. The batteries are charged when the electrical power is connected.

5.11.2 Transducers

During a test the 6-1 channel module receives the electrical signals and analyses them. The electrical signals are being produced by transducers and transmitted to the analogue to digital converters of the module. A transducer is an electric/electronic device that can sense energy
like mechanical energy or a physical quantity like pressure. Also, a transducer converts the energy from one form to another. A common example of transducers includes load cell or force transducer, accelerometer, microphone and thermometer. An accelerometer is a transducer that converts the mechanical energy to electrical energy. Fig 5.28 shows a view of the accelerometer used in this present research. This transducer is a piezoelectric sensor made by D.J. BRICHALL in UK.

As it may be seen from Fig 5.28, a cable is connected to the transducer and transmits the signals which are produced by the transducer to the 6-1 channel module. The accelerometer was pasted on the top horizontal surface of the central connector of the test model by using a type of wax. Fig 5.29 shows the accessories which were used for sticking the transducer on the model. The accessories include a chemical washing liquid to dissolve the wax after the experiment. Also, there is an electrical device in which a cylindrical of wax can be placed. The wax will melt when heated by this electrical device. The melted wax is poured on the place where the transducer is to be installed.
5.11.3 Connecting cables

Cables are the connective device between the 6-1 channel module and the other equipment. A cable which is plugged into an input channel of the module transmits the signals from a transducer to the module. On the other hand, a cable which is connected to the output channel of the module can transmit Pulse program commands to other equipment. A local network cable creates the connection between the module and computer. The data acquisition system is designed for inside and outside experimental work.

5.11.4 Personal computer (PC)

An important part of the data acquisition system is a computer in which the main program of the system, that is, Pulse, is installed. Pulse controls all the operations of the 6-1 channel module during the test. Also, all the graphs and response curves are displayed on the monitor.
of the computer. In the present work a personal computer (PC) was connected to the 6-1 channel module through a local connection port.

5.12 Data production and processing

Fig 5.30 shows the flowchart of the production, recording and processing of data in a test. The first step in carrying out a dynamic test is to make the model vibrate. In this work two methods were used in order to vibrate the model. In the first method the grid was loaded at its central connector. After releasing the load the model was left to vibrate. This method is known as initial displacement test. In the second method a hammer load was applied at various points near around the central connector.

In the second step in carrying out a test the vibration of the model needs to be detected. A transducer functions as a detector. Also, the transducer converts the detected mechanical energy to electrical energy. In the third step the electrical signals were delivered to the 6-1 channel module through a connecting cable. In the module an analogue to digital converter converts the analogue input signals to digital form. The digitised signals were delivered to the analysers inside the module. The analysers measure the time records, Fourier spectrum for the signals. In this work the measurements of the time records were used. So, the ‘Fast Fourier Transform’ (FFT) analyser was the default analyser. An FFT analyser is an algorithm for converting data from the time domain to frequency domain.

In the next step in carrying out a test the results of the measurements may be seen as a response curves on the monitor of the PC or can be stored as a data file. Subsequently, the digital form of data file was exported to the Excel software and the damping ratio of the test model was deduced, finally, as shown in the flowchart of Fig 5.30.

5.13 The Test

The material in the previous sections explains the equipment involved in carrying out the tests. The different support cases of the grid will be explained in the following section. Also, the process of applying different levels of bolt tightness for the connections will be discussed in the sequel.
5.13.1 Support arrangement

The tests were carried out using three support conditions. The first support case corresponds to the tests which were carried out for four supports. In these tests there are four support assemblies on the top of the reinforced concrete columns. These supports are at four corners of the grid. The supports of the grid are indicated by small squares shown in Fig 5.31. In this figure the top layer elements are shown by thick lines and the bottom layer elements as well as the web elements are shown by thin lines.

The second support condition corresponds to the tests which were carried out for eight supports. This support case was created by adding four supports to the first support case. In this case eight support assemblies (see Section 5.9) are placed on the top of the reinforced concrete columns. The supports in these tests are arranged in two parallel lines, as shown in Fig 5.32.

Finally, the third support condition corresponds to the tests which were carried out for twelve supports. The arrangement of the supports in this case is shown in Fig 5.33
Figure 5.30 Flowchart of production, recording and processing of data
Figure 5.31 The grid on four supports
Figure 5.32 The grid on eight supports
Each of the above three support conditions was used for four different levels of bolt tightness as follows:

- Very low tightness (loose),
- 60Nm tightness,
- 120Nm tightness and
- 180Nm tightness.

Figure 5.33 The grid on twelve supports
5.13.2 Setting up the grid with a specified bolt tightness

The tightening of the connector bolts of the grid to a specified degree was carried out as follows:

1- All the nodes of the bottom layer of the grid were levelled using the adjustable steel props as shown in Figs 5.34 and 5.35. An adjustable steel prop, Fig 5.36, was manufactured from an outer tube that constituted the bottom part of prop and an inner tube with some holes that constituted the top part of the prop. Also a threaded portion with a slot was welded to the outer tube. A high tensile steel pin that was located through the slot in the outer tube and two holes of the inner tube was used for coarse adjustment. A threaded collar located below the pin was used for fine adjustment and accurate levelling of nodes (Figs 5.34, 5.35 and 5.36). Particular attention was paid to ensure that all props were in firm contact with the concrete base and the balls of the connectors.

2- When all bottom layer nodes were levelled, all the bolts of the connectors of the grid were loosened using a spanner. The loosening operation was started from bottom layer and continued to top layer and completed with web layer.

3- After loosening of all the bolts of the grid, the bottom layer nodes were levelled again using the adjustable collars of the props.

4- All the bolts were tightened using an adjustable torque wrench (with a spanner) as shown in Fig 5.37. The torque wrench was model EVT2000A and made by Britool Limited Co, UK. Its adjusting torque range was 50 to 225Nm and its length was 597mm. The amount of the torque of the torque wrench for tightening the bolts in the first stage was set at 50Nm. The torque wrench indicated with touch, sound and sight when the required torque was reached. The square peg at the end of the torque wrench was not suitable for gripping and fastening of the sleeves of the connectors. So, as shown in Fig 5.37, a spanner was attached to the wrench torque for tightening of the bolts. In order to connect the spanner to the torque wrench a square hole was cut on the handle of the spanner. The position of the hole was determined so that the overall length of the torque wrench and the attached spanner was 730mm. So, the applied torque to the bolts was 20% more than the torque that was set on the torque wrench and was 60Nm in the first stage. The tightening process by adjustable torque wrench-spanner started from one side of the bottom layer and carried on.
Figure 5.34 Levelling of bottom layer nodes using adjustable steel props

Figure 5.35 Closer view of levelling of bottom layer nodes using adjustable steel props
until all the connectors of this layer were tightened. Then the top layer and finally, web layer
connectors were tightened in a manner similar to that of the bottom layer.

5- All adjustable props were removed after the completion of the tightening process.

6- The barrel (see Section 5.10.2) was filled with water.

7- The accelerometer (see Section 5.11.2) was mounted on the top of the central connector of
the grid.

8- The transducer (accelerometer) was connected to an input channel of the 6-1 channel
module (this input channel was specified in the Pulse software in advance), see Section
5.11.1.

9- The computer and 6-1 channel module were switched on and the test program was run.
10- The barrel was lifted by the chain hoist crane and was hooked to the load releasing mechanism (see Section 5.10.2).

11- The load was released by the load releasing mechanism and the response of the grid was recorded by the data acquisition system (see Section 5.11).

12- The output of the test was displayed on the monitor and saved as both a graph and a data file in the computer.

13- Steps 10 to 12 were repeated again.

In this work, each test was carried out twice. This was done to ensure the reliability of the results. The results of all the second tests were the same as the results of the first ones, however, if there were any differences between the results of the tests, the test would have been carried out again.

For each degree of tightness with a support condition, five levels of loading were applied. The load was in the range 1kN to 2.5kN. Different levels of loading were achieved by adding to or removing water from the barrel. The above steps 10 to 13 were repeated for each load case. In addition, the responses of the grid under different cases of hammer loading were recorded. In the hammer loading, the grid was hit by a 2kg hammer. Hammer loading was applied by a person on different points around the central connector of the top layer of the grid. Testing of the grid with 120Nm and 180Nm degrees of tightness were carried out in a manner similar to that mentioned in the above steps. To achieve to the degree of looseness in the connections, the steel props were made removed after step 3. In this condition, five tests were carried out totally.
Figure 5.37 Tightening of bolts using adjustable torque wrench (with a spanner)
CHAPTER SIX

PRESENTATION AND DISCUSSION OF TEST RESULTS

6.1 Introduction

Depending on the type of the test carried out and the resulting curves, a suitable method for deducing the damping ratio of the experimental model can be employed. There are a variety of techniques commonly used to measure the damping capacity, as mentioned in Section 4.4. In particular, the following techniques are frequently employed:

- Logarithmic decrement method
- Half power method
- Energy method

The 'logarithmic decrement' technique has been employed in this work to obtain the damping ratio values. The logarithmic decrement technique is explained in Section 4.4.1 in detail. Eqn. 4.17 gives the damping ratio of an oscillatory system as follows.
where $\zeta$ is the damping ratio and $\delta$ is the logarithmic decrement which is given by Eqn. 4.20

$$\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}$$

where $x_1$ and $x_{n+1}$ are the amounts of the first and $(n+1)^{th}$ amplitudes of the curve of the structural response.

### 6.2 Calculation of Damping Ratio

#### 6.2.1 Introduction

The response of each experiment is presented as a time-displacement curve. These results, which have been recorded by the data acquisition system during the test, were plotted by the Pulse software (see Section 5.11.1). The results are stored as a picture as well as a data file. Each data file contains a test result and could be plotted again by using such software as Excel. Fig 6.1 shows a typical time-displacement response which is produced in Pulse. The horizontal axis of the coordinate system shows the time in seconds and the vertical axis shows the displacements. The vertical scale is not given since it is only the ratios of the amplitudes that are of interest. The response shown in Fig 6.1 is a cyclic curve and is plotted in the time interval of $t=0$ to $t=1.6$ [second].

It may be seen from Fig 6.1 that initially the amplitudes of the cycles reduce successively up to around $t=0.6$. Thereafter the response experiences a steady low amplitude oscillation. For clarity, a part of the response of Fig 6.1 is plotted for the interval $t=0$ to $t=0.8$ in Fig 6.2. This figure is plotted by Excel using the digital information of the response of Fig 6.1. Now, the cycles in Fig 6.2 are shown more clearly than those of Fig 6.1. Also, the value of the amplitudes could be read in Excel easily. These values are required for the calculation of the damping ratio.
Figure 6.1 The resulting curve of the experiment of the grid under the condition of four supports and low bolt tightness

Figure 6.2 Fig 6.1 replotted by Excel
Eqn 6.2 gives the logarithmic decrement $\delta$ using the first and $(n+1)^{th}$ amplitudes of the response. The parameter $n$ in this equation can be chosen by 1 or 2 or any other integer. Usually the second amplitude of the response experiences a high reduction compared with the first one. Such a rate of reduction does not happen for the subsequent amplitudes. If the first and second amplitudes are used for the determination of the damping ratio (that is, $n=1$), the result will be inaccurate due to the initial high reduction rate of the amplitude. On the other hand, after a large number of cycles (say $n>10$) the response of the structure is likely to have entered a steady state region. Calculation of the damping ratio in this region leads to an inaccurate result. In view of the above considerations, the first and fourth (that is, $n=3$) amplitudes were used to calculate the damping ratio for all the responses in this work.

In some cases, the response of a test shows a very low reduction in the successive amplitudes (very low damping). In such a case, the damping ratio can not be obtained using the first four amplitudes. Some of the responses that correspond to 180Nm bolt tightness experience such a very low rate reduction of the amplitudes. In these cases the parameter $n$ requires to be increased to 10 or more.

### 6.2.2 Calculation operations

Fig 6.3 shows the resulting response from an experiment carried out on the model. This response corresponds to an initial displacement test (see Section 5.10) which is carried out for the condition of eight supports and low bolt tightness. The digital form of this result was exported to Excel and is plotted again by Excel as shown in Fig 6.4.

The response which is shown in Fig 6.3 or 6.4 has a periodic form like a sinusoidal graph. Also, as it may be seen from Fig 6.3 or 6.4, the response has an envelope which is sinusoidal as well. Since the response has a sinusoidal envelope, the value of any amplitude is determined as the mean value of the upper peak and lower peak of the cycle. This is illustrated for a regularised version of a part of the response of Fig 6.4 in Fig 6.5.
Figure 6.3 The response of the model under the condition of eight supports and low bolt tightness

As may be seen from Fig 6.5, a cycle has two amplitudes, an upper one and a lower one. The mean amplitude of a cycle is defined as the mean of the upper and lower amplitudes of the cycle, as shown in Fig 6.5. For the mean amplitudes the first and fourth cycles of Fig 6.4 the following data are read in Excel.

For the first cycle

- Lower peak value = -2.1 at $t = 0.133$ sec.
- Upper peak value = 2.15
- Mean value of the first amplitude = $(2.1 + 2.15)/2 = 2.125$

or

$x_1 = 2.125$ ($x_1$ in Eqn 6.2)
For the fourth cycle

Upper peak value = -0.551 at t = 0.188

Lower peak value = 1.1

Mean value of the fourth amplitude = (0.551+1.1)/2 = 0.8255

or

\[ x_4 = 0.8255 \] (\( x_4 \) in Eqn 6.2)

Replacing \( n \) by 3 in Eqn 6.2 gives

\[ \delta = \frac{1}{3} \ln \frac{x_1}{x_4} \] (6.3)

Substituting the value of \( x_1 \) and \( x_4 \) into the above equation gives

\[ \delta = \frac{1}{3} \ln \frac{2.125}{0.8255} \]

or

\[ \delta = 0.31518 \] (6.4)
Substituting this value into Eqn 6.1 gives

\[ \zeta = \frac{0.31518}{\sqrt{0.31518^2 + 4\pi^2}} \]

or

\[ \zeta = 0.050 \quad (6.5) \]

Eqn 6.5 shows a value of the damping ratio. This value depends on the logarithmic decrement \( \delta \). Also, the logarithmic decrement depends on the amplitudes ratio as well. Low rate of reduction of the amplitudes in a response leads to a low value of the amplitude ratio. Such a low rate of amplitude ratio leads to a low value of \( \delta \). Finally, a low value of \( \delta \) results in a low value of damping ratio for the model. In contrast, a high reduction of the amplitudes in a response curve results in a high value of the damping ratio.

![Figure 6.5 A regularised version of a part of the response of Fig 6.4](image)
The data processing technique which is presented above is used to determine the damping ratios of all the test cases in this work. The results and the related information are presented in Table 6.1

This table contains seven columns. The first column lists the file names in which the results are stored. The results are stored in a digital form as well as picture. It should be mentioned that all the data are produced in the Pulse software initially during the tests and then exported to Word and Excel soft wares. The digital form of the data is stored in Excel while the picture is stored in Word programs.

The second column of Table 6.1 shows the value of the tightness of the bolts of the connections. All the experiments are carried out for four levels of the bolt tightness. The number of the supports for the experiments may be found in the third column. All the tests are carried out for three support conditions. Column four of Table 6.1 shows the loading conditions. Most of the tests were carried out using hung loads and some of the experiments were carried out using hammer impact.

Columns five and six present the resulting damping ratios. Column five shows the damping ratio deduced within four cycles. However, column six shows the damping ratio deduced within a larger number of cycles. Finally, the seventh column of Table 6.1 shows the corresponding filenames in Pulse software.

The damping ratio which is given by Eqn 6.5 can be seen in fourth row of Table 6.1.

6.3 Test Results

6.3.1 Introduction

As mentioned in the previous chapter, all the experiments on the test grid involved free vibration only (see Section 3.1.4.1). These tests were carried out for different support and bolt tightness conditions. The tests were carried out for four different levels of bolt tightness. The experiment for each level of the bolt tightness was repeated for three different support conditions.

More than 120 experiments are carried out altogether, of which the results of 69 experiments are listed in Table 6.1
Table 6.1: The damping ratios

<table>
<thead>
<tr>
<th>File name</th>
<th>Tightness [N.m]</th>
<th>Number of Supports</th>
<th>Loading</th>
<th>Damping Ratio1</th>
<th>Damping Ratio2</th>
<th>File name in Pulse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test1</td>
<td>Loose</td>
<td>4</td>
<td>Hung load</td>
<td>0.047</td>
<td></td>
<td>Realtest1 measurement2</td>
</tr>
<tr>
<td>Test2</td>
<td>Loose</td>
<td>4</td>
<td>Hung load</td>
<td>0.031</td>
<td></td>
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<tr>
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<td></td>
<td>Realtest1 measurement2</td>
</tr>
<tr>
<td>Test5</td>
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<td>Hung load</td>
<td>0.062</td>
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<td>Realtest1 measurement</td>
</tr>
<tr>
<td>Test11</td>
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<td></td>
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</tr>
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<td>Test14</td>
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<td>Hung load</td>
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<td>0.025</td>
<td></td>
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<td>Hung load</td>
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<tr>
<td>Test26</td>
<td>60</td>
<td>8</td>
<td>Hung load</td>
<td>0.016</td>
<td></td>
<td>Sealtest3 Measurement6</td>
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<td>Test27</td>
<td>60</td>
<td>8</td>
<td>Hammer</td>
<td>0.024</td>
<td></td>
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</tr>
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<td>8</td>
<td>Hung load</td>
<td>0.020</td>
<td></td>
<td>Sealtest3 measurement11</td>
</tr>
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<td>Test30</td>
<td>60</td>
<td>8</td>
<td>Hung load</td>
<td>0.040</td>
<td></td>
<td>Sealtest4 measurement5</td>
</tr>
<tr>
<td>Test</td>
<td>Duration</td>
<td>Frequency</td>
<td>Load Type</td>
<td>Load</td>
<td>Real Test</td>
<td>Measurement</td>
</tr>
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<td>Test31</td>
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<td>8</td>
<td>Hung load</td>
<td>0.025</td>
<td>Srealtest4</td>
<td>measurement3</td>
</tr>
<tr>
<td>Test32</td>
<td>60</td>
<td>8</td>
<td>Hung load</td>
<td>0.034</td>
<td>Srealtest4</td>
<td>measurement2</td>
</tr>
<tr>
<td>Test34</td>
<td>60</td>
<td>8</td>
<td>Hung load</td>
<td>0.034</td>
<td>Srealtest5</td>
<td>measurement2</td>
</tr>
<tr>
<td>Test40</td>
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<td>12</td>
<td>Hung load</td>
<td>0.017</td>
<td>Realtest1</td>
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<td>Test41</td>
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<td>Test43</td>
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<td>12</td>
<td>Hung load</td>
<td>0.012</td>
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<td>measurement8</td>
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<tr>
<td>Test44</td>
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<td>12</td>
<td>Hung load</td>
<td>0.025</td>
<td>Realtest1</td>
<td>measurement10</td>
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<td>Test45</td>
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<td>12</td>
<td>Hung load</td>
<td>0.031</td>
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<td>Hung load</td>
<td>0.039</td>
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</tr>
<tr>
<td>Test47</td>
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<td>12</td>
<td>Hung load</td>
<td>0.033</td>
<td>Realtest1</td>
<td>measurement14</td>
</tr>
<tr>
<td>Test50</td>
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<td>12</td>
<td>Hammer</td>
<td>0.015</td>
<td>Realtest1</td>
<td>measurement16</td>
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<tr>
<td>Test51</td>
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<td>12</td>
<td>Hammer</td>
<td>0.011</td>
<td>Realtest1</td>
<td>measurement12</td>
</tr>
<tr>
<td>Test63</td>
<td>120</td>
<td>12</td>
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<td>0.023</td>
<td>Frealtest4</td>
<td>measurement6</td>
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<td>Test64</td>
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<tr>
<td>Test66</td>
<td>120</td>
<td>12</td>
<td>Hammer</td>
<td>0.036</td>
<td>Frealtest4</td>
<td>measurement17</td>
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<tr>
<td>Test67</td>
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<td>Hammer</td>
<td>0.043</td>
<td>Frealtest4</td>
<td>measurement12</td>
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<tr>
<td>Test70</td>
<td>120</td>
<td>12</td>
<td>Hammer</td>
<td>0.026</td>
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<td>120</td>
<td>12</td>
<td>Hammer</td>
<td>0.060</td>
<td>Frealtest4</td>
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<tr>
<td>Test160</td>
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<td>8</td>
<td>Hung load</td>
<td>0.019</td>
<td>Frealtest2</td>
<td>measurement</td>
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<tr>
<td>Test161</td>
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<td>8</td>
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<td>0.029</td>
<td>Frealtest2</td>
<td>Measurement4</td>
</tr>
<tr>
<td>Test162</td>
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<td>8</td>
<td>Hung load</td>
<td>0.035</td>
<td>Frealtest2</td>
<td>measurement2</td>
</tr>
<tr>
<td>Test163</td>
<td>120</td>
<td>8</td>
<td>Hung load</td>
<td>0.038</td>
<td>Frealtest2</td>
<td>measurement5</td>
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<td>Force</td>
<td>Size</td>
<td>Load Type</td>
<td>Force Value</td>
<td>Measurement</td>
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<td></td>
</tr>
<tr>
<td>Test164</td>
<td>120</td>
<td>8</td>
<td>Hung load</td>
<td>0.028</td>
<td>Frealtest2</td>
<td></td>
</tr>
<tr>
<td>Test165</td>
<td>120</td>
<td>8</td>
<td>Hammer</td>
<td>0.034</td>
<td>measurement6</td>
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<tr>
<td>Test167</td>
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<td>8</td>
<td>Hammer</td>
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<td>Frealtest2</td>
<td></td>
</tr>
<tr>
<td>Test168</td>
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<td>8</td>
<td>Hammer</td>
<td>0.028</td>
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<tr>
<td>Test150</td>
<td>120</td>
<td>4</td>
<td>Hung load</td>
<td>0.020</td>
<td>Frealtest1</td>
<td></td>
</tr>
<tr>
<td>Test151</td>
<td>120</td>
<td>4</td>
<td>Hung load</td>
<td>0.031</td>
<td>measurement5</td>
<td></td>
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<tr>
<td>Test152</td>
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<td>4</td>
<td>Hung load</td>
<td>0.025</td>
<td>Frealtest1</td>
<td></td>
</tr>
<tr>
<td>Test154</td>
<td>120</td>
<td>4</td>
<td>Hung load</td>
<td>0.026</td>
<td>measurement3</td>
<td></td>
</tr>
<tr>
<td>Test155</td>
<td>120</td>
<td>4</td>
<td>Hung load</td>
<td>0.026</td>
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<tr>
<td>Test83</td>
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<td>8</td>
<td>Hung load</td>
<td>0.008</td>
<td>Frealtest7</td>
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<td>8</td>
<td>Hung load</td>
<td>0.008</td>
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<td></td>
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<tr>
<td>Test88</td>
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<td>8</td>
<td>Hung load</td>
<td>0.009</td>
<td>Frealtest7</td>
<td></td>
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<tr>
<td>Test90</td>
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<td>8</td>
<td>Hammer</td>
<td>0.002</td>
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<td>Test91</td>
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<td>8</td>
<td>Hammer</td>
<td>0.002</td>
<td>measurement14</td>
<td></td>
</tr>
<tr>
<td>Test98</td>
<td>180</td>
<td>8</td>
<td>Hammer</td>
<td>0.007</td>
<td>Frealtest7</td>
<td></td>
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<td>Test103</td>
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<td>12</td>
<td>Hung load</td>
<td>0.010</td>
<td>Frealtest7</td>
<td></td>
</tr>
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<td>Test104</td>
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<td>12</td>
<td>Hung load</td>
<td>0.005</td>
<td>measurement27</td>
<td></td>
</tr>
<tr>
<td>Test105</td>
<td>180</td>
<td>12</td>
<td>Hung load</td>
<td>0.009</td>
<td>Frealtest7</td>
<td></td>
</tr>
<tr>
<td>Test111</td>
<td>180</td>
<td>12</td>
<td>Hammer</td>
<td>0.004</td>
<td>Frealtest7</td>
<td></td>
</tr>
<tr>
<td>Test112</td>
<td>180</td>
<td>12</td>
<td>Hammer</td>
<td>0.003</td>
<td>measurement31</td>
<td></td>
</tr>
<tr>
<td>Test113</td>
<td>180</td>
<td>12</td>
<td>Hammer</td>
<td>0.008</td>
<td>Frealtest7</td>
<td></td>
</tr>
<tr>
<td>Test114</td>
<td>180</td>
<td>12</td>
<td>Hammer</td>
<td>0.005</td>
<td>measurement34</td>
<td></td>
</tr>
</tbody>
</table>
The remaining curves of all the tests that are not listed in Table 6.1 have unsuitable forms for analysis. As mentioned in Section 3.2.2.3, in general, a response of a test case involves the effects of several mode shapes. The combination of the mode shapes sometimes makes the resulting curves unsuitable for analysis. If the response in a test case oscillates in one of its modes or one mode shape is dominant during the oscillation, an analysis on the resulting curve could be done easily. The dominant mode or a combination of modes that may be present in the response depends on the place as well as the amount of the applied load. In this work, a variety of hung loads were applied. Also, a variety of places of hammer loading was chosen. These varieties gave rise to different load conditions. Some of these conditions led to responses that are combinations of several mode shapes. The results of the tests that are listed in Table 6.1 are those in which one mode shape is dominant and can be analysed easily.

As discussed in Chapter 3, the nature of the motion of a system in a free vibration is related to the initial conditions. In particular, a free vibration may be induced by an initial displacement. In such a motion, if the shape of the initial displacement is proportional to the shape of one of the normal modes, the motion of the system will be similar to the modal shape corresponding to that normal mode (Section 3.2.2.5). In other cases, the similarity of motion of the system to a mode shape depends on the closeness of the initial displacement shape to that of the normal mode. Figs 3.28a, 3.28b, 3.29a and 3.29b (Chapter 3) show the responses of a two degree of freedom system for different shapes of initial displacement. For detail see Sections 3.2.2.3 and 3.2.2.5.
Making the grid to excite in its first normal mode is targeted in this work. This is due to the fact that excitation of the grid in its first normal mode is more conveniently achieved than the others.

In order to demonstrate the mode shapes of the grid, a modal analysis of the grid was carried out using the ‘ANSYS’ software. In this numerical analysis, the first ten mode shapes were extracted. The extracted mode shapes of the grid are shown in the Appendix in detail.

According to the first mode shape of the grid extracted from the numerical analysis, the load should be applied at the centre of the test grid. Most of the tests corresponding to the central loading led to response curves that are convenient for analysis and determination of the damping ratios. Hammer loading was usually applied off centre. So, most of the responses corresponding to hammer loading do not follow a clear pattern and cannot be analysed for damping ratio determination easily.

As an example, Fig 6.3 or 6.4 is a response curve which was analysed and the corresponding damping ratio was obtained as listed in the fourth row of Table 6.1. Also, Fig 6.6 shows a response curve which is recorded during a test on the grid for the 12 support case and 60Nm bolt tightness. The curve of Fig 6.6 is plotted again by Excel as shown in Fig 6.7. In contrast with the curve of the Fig 6.4, which involves a single dominant mode shape, the curve of Fig 6.7 involves a combination of several mode shapes. Although, Fig 6.6 shows a general trend of decreasing amplitudes of the successive cycles, but it does not follow a clear pattern. This fact is more apparent from Fig 6.7. So, the determination of the damping ratio using such a curve can not lead to a reliable outcome.

The above discussion explains why a number of the tests were not listed in Table 6.1.

6.3.2 Effects of bolt tightness

The results of the experiments are categorised from different view points. The first series of categories are established based on a specific number of supports. From this view point, the results are divided into three categorises, as presented in Tables 6.2, 6.3 and 6.4. Table 6.2 contains the mean damping ratios obtained from test result corresponding to four supports. The mean damping ratios obtained for the condition of eight supports are shown in Table 6.3.
Also, Table 6.4 relates to the condition of twelve supports. Each of these tables list four different levels of bolt tightness as follows:

- Very low tightness (loose),
- 60Nm tightness,
- 120Nm tightness and
- 180Nm tightness.

Tables 6.2 to 6.4 show the results for three different support conditions.

From Table 6.2 it may be seen that under the condition of looseness, the grid experiences a high damping ratio. The mean damping ratio for this condition is given in the first row of this table, that is,
This is the highest value in this table. For 60Nm bolt tightness the mean damping ratio is reduced to:

\[ \zeta = 0.025 \]

Increasing the tightness from 60Nm to 120Nm give rise to increasing damping ratio to

\[ \zeta = 0.026 \]

In the next step the tightness is increased to 180Nm and the mean damping ratio experiences a high rate of reduction. The mean damping ratio in this condition is obtained as

\[ \zeta = 0.003 \]

which is the lowest in this table. The lowest mean damping ratio is given for the highest tightness while the highest mean damping ratio is given for lowest bolt tightness.
Table 6.2: Mean damping ratios corresponding to four supports

<table>
<thead>
<tr>
<th>Tightness [N.m]</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loose</td>
<td>2</td>
<td>0.039</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>0.025</td>
</tr>
<tr>
<td>120</td>
<td>5</td>
<td>0.026</td>
</tr>
<tr>
<td>180</td>
<td>4</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Table 6.3 shows the mean damping ratio for grid with eight supports and different bolt tightnesses. The pattern of variation of the mean damping ratio versus the bolt tightness for eight supports is similar to those presented in Table 6.2. Namely, for the eight support case, the highest mean damping ratio is obtained for the condition of the lowest bolt tightness, that is,

\[ \zeta = 0.052 \]

Table 6.3: Mean damping ratios corresponding to eight supports

<table>
<thead>
<tr>
<th>Tightness [N.m]</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loose</td>
<td>2</td>
<td>0.052</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>0.025</td>
</tr>
<tr>
<td>120</td>
<td>8</td>
<td>0.030</td>
</tr>
<tr>
<td>180</td>
<td>6</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Also, for 60Nm bolt tightness the mean damping ratio reduces to

\[ \zeta = 0.025 \]

Increasing the bolt tightness from 60Nm to 120Nm makes the damping ratio to increase to
Finally, for 180Nm bolt tightness the mean damping ratio experiences a high rate of reduction. The mean damping ratio in this case is obtained as:

\[ \zeta = 0.006 \]

So, it may be seen that the lowest damping ratio is given for the highest bolt tightness. Also, the highest damping ratio is given for the lowest bolt tightness. This is similar to what happens for the four support case previously.

Table 6.4 shows the mean damping ratios for the twelve support case with different bolt tightnesses. The pattern of the variation of the mean damping ratios versus the tightness for twelve support case is similar to those presented in Table 6.3 as well as Table 6.2. The highest mean damping ratio for the twelve support case is obtained for the loose condition as:

\[ \zeta = 0.062 \]

<table>
<thead>
<tr>
<th>Tightness [N.m]</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loose</td>
<td>1</td>
<td>0.062</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>0.029</td>
</tr>
<tr>
<td>120</td>
<td>6</td>
<td>0.030</td>
</tr>
<tr>
<td>180</td>
<td>4</td>
<td>0.007</td>
</tr>
</tbody>
</table>

Also, the lowest damping ratio is obtained for the highest bolt tightness namely, 180Nm as:

\[ \zeta = 0.007 \]

Figs 6.8 to 6.10 show diagrammatically the variations of the damping ratio versus the bolt tightness for each support arrangement. The horizontal axis of these diagrams shows the tightness and the vertical axis shows the damping ratio.
The variation of the damping ratio versus the tightness corresponding to the four support case is illustrated in Fig 6.8. Figs 6.9 and 6.10 illustrate the variations of the damping ratio versus the tightness for the eight and twelve support cases, respectively.

For the comparison, the diagrams of the variations of the damping ratio versus the bolt tightness for all three different support conditions are illustrated in Fig 6.11. In this figure, the damping ratio for every bolt tightness is shown by three columns. Every column shows the damping ratio for a specific support case.
Figure 6.9 Damping ratios for the eight support case with different bolt tightnesses

Figure 6.10 Damping ratios for the twelve support case with different bolt tightnesses
6.3.3 Effects of support conditions

Based on bolt tightness, the results are divided into four categories, as shown in Tables 6.5 to 6.8. Table 6.5 contains the mean damping ratios corresponding to the low bolt tightness (loose). The mean damping ratios corresponding to the condition of 60Nm bolt tightness are shown in Table 6.6. Also, Table 6.7 relates to the condition of 120Nm bolt tightness. Finally, Table 6.8 shows the mean damping ratio obtained for the 180Nm bolt tightness. Each of these four tables lists three different support conditions as follows:

- Four supports,
- Eight supports and
- Twelve supports.

From Table 6.5 it may be seen that the lowest damping ratio is obtained under the condition of four supports. The mean damping ratio for this condition is given in the first row of the table as:

$$\zeta = 0.039$$
Table 6.5: Mean damping ratios corresponding to low bolt tightness

<table>
<thead>
<tr>
<th>Number of supports</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>0.039</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>0.052</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>0.062</td>
</tr>
</tbody>
</table>

Increasing the number of supports from four to eight, makes the mean damping ratio to increase to:

\[ \zeta = 0.052 \]

Also, increasing the number of supports from eight to twelve will increase the damping ratio to:

\[ \zeta = 0.062 \]

So, it may be seen that the lowest damping ratio is obtained for the smallest number of supports. Also, the highest damping ratio is attained for the largest number of supports. In other words, increasing the number of supports makes the mean damping ratio to increase.

Table 6.6 shows the mean damping ratio of the grid for 60Nm bolt tightness.

Table 6.6: Mean damping ratios corresponding to 60Nm tightness

<table>
<thead>
<tr>
<th>Number of supports</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>0.025</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>0.025</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>0.029</td>
</tr>
</tbody>
</table>
The lowest and highest mean damping ratios in this case are obtained for the smallest and largest number of supports, respectively. This fact is similar to that of Table 6.5. The lowest mean damping ratio in this table is given as:

\[ \zeta = 0.025 \]

Also, the highest mean damping ratio in this table is obtained as:

\[ \zeta = 0.029 \]

For the eight supports condition, the mean damping ratio in Table 6.6 is given as:

\[ \zeta = 0.025 \]

Table 6.7 shows the mean damping ratio of the grid for 120Nm bolt tightness with different support conditions.

Table 6.7: Mean damping ratios corresponding to 120Nm tightness

<table>
<thead>
<tr>
<th>Number of supports</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>0.026</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>0.030</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>0.039</td>
</tr>
</tbody>
</table>

In this table, the pattern of variation of the mean damping ratio due to the variation of the number of supports is similar to that presented in Table 6.5. In other words, increasing the number of supports give rise to increasing the mean damping ratio. So, the lowest mean damping ratio is obtained for the four support case which has the lowest number of supports. The mean damping ratio in this case is obtained as:

\[ \zeta = 0.026 \]

Also, the highest mean damping ratio is obtained for the twelve support case which has the highest number of supports. The mean damping ratio in this case is obtained as:


\[ \zeta = 0.039 \]

For the eight support case the mean damping ratio is between the lowest and highest values of the mean damping ratio:

\[ \zeta = 0.030 \]

The damping ratios of the grid for the 180Nm bolt tightness and different supports conditions are given in Table 6.8. From this table, it is clear that the pattern of the variations of the mean damping ratios for 180Nm bolt tightness is similar to those of 120Nm bolt tightness. Namely, the lowest mean damping ratio is obtained for the four support case as:

\[ \zeta = 0.003 \]

Also, increasing the number of supports from four to eight makes the mean damping ratio to increase to:

\[ \zeta = 0.006 \]

Finally, increasing the number of supports to larger one, namely, 12 will increase the mean damping ratio to:

\[ \zeta = 0.007 \]

Table 6.8: Mean damping ratio corresponding to 180Nm tightness

<table>
<thead>
<tr>
<th>Number of supports</th>
<th>Number of results</th>
<th>Mean damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>0.003</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>0.006</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>0.007</td>
</tr>
</tbody>
</table>

Figs 6.12 to 6.15 show diagrammatically the variations of the damping ratio versus the number of supports for each level of bolt tightness. The horizontal axis of these diagrams shows the number of supports and the vertical axis shows the damping ratio.
The variation of the damping ratio versus the number of supports corresponding to the loose case is illustrated in Fig 6.12. Fig 6.13 illustrates the variations of the damping ratio versus the number of supports for 60Nm bolt tightness. Also, Figs 6.14 and 6.15 illustrate the variations of the damping ratio with respect to the variation of the number of supports for 120Nm and 180Nm bolt tightnesses, respectively.

For comparison, the diagrams of the variations of the damping ratios with respect to the number of supports for four different levels of bolt tightnesses are illustrated in Fig 6.16. In this figure, the damping ratio for every support condition is shown by four columns. Where, every column shows the damping ratio for a specific bolt tightness.
Figure 6.13 Damping ratio for 60Nm bolt tightness with different support conditions

Figure 6.14 Damping ratio for 120Nm bolt tightness with different support conditions
Figure 6.15 Damping ratio for 180Nm bolt tightness with different support conditions

Figure 7.16: Damping ratio for different levels of bolt tightnesses with different support cases
6.3.4 Amplitude effects

One of the aspects in which investigators are highly interested is the effect of the amplitude of vibration on the damping capacity of an oscillatory system. The opinions of a number of investigators in relation to the effect of the amplitude of vibration on damping capacity of the oscillatory systems are presented in Chapter 2. In addition, some results of experiments obtained by investigators are presented. These results show the direct dependence of damping on the amplitude of vibration. The results of the experiments of the current research also confirm that the damping coefficient of the test model is effected by the amplitude of the vibration.

To show the effect of the amplitude of the vibration on the damping of the test model, the results of 12 pairs of tests are presented in the sequel. Both tests of each pair are carried out for the same bolt tightness and the same support conditions, but they correspond to different load conditions. They also are carried out successively. Application of different loads on the grid results in different amplitudes of the responses. Comparison of the damping ratios obtained from the responses of the tests for each pair reveals the effects of the amplitude of vibration on the damping ratio. The notation:

\[ A = \text{initial amplitude of vibration and} \]
\[ \zeta = \text{damping ratio} \]

is used for the following examples. All of the following tests are selected from Table 6.1.

Example 1:

<table>
<thead>
<tr>
<th>Tightness: Loose</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Supports: 4</td>
<td>A = 2.1</td>
<td>A = 1.2</td>
</tr>
<tr>
<td></td>
<td>( \zeta = 0.047 )</td>
<td>( \zeta = 0.031 )</td>
</tr>
<tr>
<td>Change in amplitude= 75%</td>
<td>Change in damping ratio= 52%</td>
<td></td>
</tr>
</tbody>
</table>
Example 2:

<table>
<thead>
<tr>
<th>Tightness: 60Nm</th>
<th>Number of Supports: 4</th>
<th>Test 13</th>
<th>$A = 3.4$</th>
<th>$\zeta = 0.019$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Test 14</td>
<td>$A = 14.3$</td>
<td>$\zeta = 0.024$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 320%</td>
<td>Change in damping ratio = 26%</td>
<td></td>
</tr>
</tbody>
</table>

Example 3:

<table>
<thead>
<tr>
<th>Tightness: 60Nm</th>
<th>Number of Supports: 8</th>
<th>Test 25</th>
<th>$A = 2.3$</th>
<th>$\zeta = 0.024$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Test 26</td>
<td>$A = 1.4$</td>
<td>$\zeta = 0.012$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 64%</td>
<td>Change in damping ratio = 100%</td>
<td></td>
</tr>
</tbody>
</table>

Example 4:

<table>
<thead>
<tr>
<th>Tightness: 60Nm</th>
<th>Number of Supports: 8</th>
<th>Test 29</th>
<th>$A = 1.3$</th>
<th>$\zeta = 0.020$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Test 30</td>
<td>$A = 8$</td>
<td>$\zeta = 0.040$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 515%</td>
<td>Change in damping ratio = 100%</td>
<td></td>
</tr>
</tbody>
</table>
### Example 5:

<table>
<thead>
<tr>
<th>Tightness: 60Nm</th>
<th>Test 31</th>
<th>$\Lambda = 5.0$</th>
<th>$\zeta = 0.025$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Supports: 8</td>
<td>Test 32</td>
<td>$\Lambda = 8.1$</td>
<td>$\zeta = 0.034$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 62%</td>
<td>Change in damping ratio = 36%</td>
</tr>
</tbody>
</table>

### Example 6:

<table>
<thead>
<tr>
<th>Tightness: 60Nm</th>
<th>Test 44</th>
<th>$\Lambda = 3.7$</th>
<th>$\zeta = 0.025$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Supports: 12</td>
<td>Test 45</td>
<td>$\Lambda = 7.7$</td>
<td>$\zeta = 0.031$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 108%</td>
<td>Change in damping ratio = 24%</td>
</tr>
</tbody>
</table>

### Example 7:

<table>
<thead>
<tr>
<th>Tightness: 120Nm</th>
<th>Test 63</th>
<th>$\Lambda = 4.9$</th>
<th>$\zeta = 0.023$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Supports: 12</td>
<td>Test 64</td>
<td>$\Lambda = 7.7$</td>
<td>$\zeta = 0.043$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 57%</td>
<td>Change in damping ratio = 87%</td>
</tr>
</tbody>
</table>
## Example 8:

<table>
<thead>
<tr>
<th>Tightness: 120Nm</th>
<th>Number of Supports: 12</th>
<th>Test 64</th>
<th>$A = 7.7$</th>
<th>$\zeta = 0.043$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Test 66</td>
<td>$A = 6.6$</td>
<td>$\zeta = 0.036$</td>
</tr>
</tbody>
</table>

- Change in amplitude = 17%
- Change in damping ratio = 19%

## Example 9:

<table>
<thead>
<tr>
<th>Tightness: 120Nm</th>
<th>Number of Supports: 12</th>
<th>Test 70</th>
<th>$A = 6.3$</th>
<th>$\zeta = 0.026$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Test 71</td>
<td>$A = 10.4$</td>
<td>$\zeta = 0.060$</td>
</tr>
</tbody>
</table>

- Change in amplitude = 65%
- Change in damping ratio = 150%

## Example 10:

<table>
<thead>
<tr>
<th>Tightness: 120Nm</th>
<th>Number of Supports: 8</th>
<th>Test 160</th>
<th>$A = 4.1$</th>
<th>$\zeta = 0.019$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Test 161</td>
<td>$A = 5.0$</td>
<td>$\zeta = 0.029$</td>
</tr>
</tbody>
</table>

- Change in amplitude = 22%
- Change in damping ratio = 53%
Example 11:

<table>
<thead>
<tr>
<th>Tightness: 120Nm</th>
<th>Test 164</th>
<th>$A = 3.9$</th>
<th>$\zeta = 0.028$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Supports: 8</td>
<td>Test 165</td>
<td>$A = 8.7$</td>
<td>$\zeta = 0.034$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 123%</td>
<td>Change in damping ratio = 21%</td>
</tr>
</tbody>
</table>

Example 12:

<table>
<thead>
<tr>
<th>Tightness: 120Nm</th>
<th>Test 150</th>
<th>$A = 2.5$</th>
<th>$\zeta = 0.020$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Supports: 4</td>
<td>Test 151</td>
<td>$A = 4.2$</td>
<td>$\zeta = 0.031$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Change in amplitude = 68%</td>
<td>Change in damping ratio = 55%</td>
</tr>
</tbody>
</table>

The above results show that the damping ratio is dependent on the amplitude of vibration. Also, the results show that the dependency is direct. That is, an increase in the amplitude corresponds to an increase in the damping ratio and vice versa. From the above results it is clear that although, it is not possible to obtain a specific relation between the amplitude of the vibration and the damping ratio, but, it has been shown that any increment in the amplitude of vibration leads to an increment in the damping ratio of the grid.
CHAPTER SEVEN

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

7.1 Introduction

In the present work the damping capacity of a MERO-type double layer grid is studied experimentally. The grid is tested for different conditions including four different levels of bolt tightness and three different support conditions for each level of bolt tightness. Also, different levels of loading are applied and the responses of the grid for all the cases are recorded. Finally, the damping ratios are obtained from the responses of the grid.
7.2. General View

The test model for this research has been tested for four different levels of bolt tightness as follows:

- Very low tightness (loose),
- 60Nm tightness,
- 120Nm tightness and
- 180Nm tightness.

Each of the above levels of bolt tightness was used with three different support conditions as follows:

- Four supports,
- Eight supports and
- Twelve supports.

In the following discussion, the effects of some parameters on the damping ratios are explained.

7.3 Conclusions

7.3.1 Effects of bolt tightness

The results obtained from the tests show that the highest damping ratio is found for the loose condition. This fact is true for all the three different support cases. Moving from the loose condition to the 60Nm bolt tightness, the grid will show a lower damping ratio. In the loose condition, there is a clearance between the engaged threads of the bolts and the threads of the connector balls. This clearance allows the surfaces of the threads to slide on each other during the oscillation. Sliding of the surfaces of the threads produces damping as a result of friction. This frictional damping (Coulomb damping, see Section 4.3.2)) together with the material damping (see Section 4.3.3) absorb the vibrational energy and dissipate it. So, it may be said that the main reason for the higher damping ratio for the loose condition is the strong presence of Coulomb damping.
When the bolt tightness is increased to 60Nm the damping ratio is reduced. Any increase in the bolt tightness will cause:

a) A reduction in the clearance between the threads. This reduction will reduce the Coulomb damping.

b) An increase in the lack-of-fit strains which will increase the stiffness of the grid.

The reduction of damping ratio for the 60Nm bolt tightness is due to the bolt tightening operation which reduces the interaction between the thread surfaces. Although, in this condition the friction force between the thread surfaces is increased, but, the reduction of the sliding of the surfaces reduces the Coulomb damping. So, in this case the presence of Coulomb damping is not as strong as it was for the loose condition. This leads to a lower damping ratio for the grid. On the other hand, increasing the bolt tightness gives rise to an increase in the lack-of-fit strains in the grid. This will increase the stiffness of the grid. However, in the case under consideration, the reduction in the Coulomb damping is more effective than the increase in the material damping. As a result, damping ratio for the 60Nm bolt tightness is lower than that for the loose condition.

For the condition of 120Nm bolt tightness the damping ratio of the grid tends to increase. This fact can be seen more clearly in the eight and twelve support cases as compared with the four support case. Increasing the bolt tightness gives rise to an increase in the lack-of-fit strains which will cause an increase in the stiffness of the grid. This, in turn, increases the damping ratio of the grid. On the other hand, increasing the bolt tightness reduces the clearance between the threads. This reduction reduces the Coulomb damping, but, there is no major change in the Coulomb damping as compared with the 60Nm bolt tightness. As a result, the damping ratio for 120Nm bolt tightness is higher than that for the 60Nm bolt tightness. It can be said that unlike the loose condition, the material damping has a strong effect in the 120Nm bolt tightness.

For the condition of 180Nm bolt tightness, the damping ratio of the grid experiences a reduction. Actually, the damping ratio corresponding to 180Nm bolt tightness is the lowest amongst the four levels of bolt tightness. Due to the high bolt tightness in this condition, the Coulomb damping can be negligible. On the other hand, an increase in the tightness causes an increase in the stiffness of the grid. Due to the increase in the stiffness, it should be expected that the material damping has a strong presence. However, the actual low damping ratio suggests a weak presence of the material damping. So, it is difficult to make a judgment on
what is happening in this case. Actually, this phenomenon repeats for three series of results which show independently the lowest damping ratio corresponding to the 180Nm bolt tightness.

One may wonder what is happening to the grid during the change of the bolt tightness from 120Nm to 180Nm which is causing a large reduction in the damping ratio. Could this be due to the increase in the lack-of-fit strains? Is the grid with 180Nm bolt tightness stiffer than that with 120Nm bolt tightness? Does this phenomenon show a limitation of direct dependency of the damping ratio on the stiffness or the strain? In Chapter 4, it is explained that the material damping is due to the friction between the internal planes of the material (see Section 4.3.3). So, it may be said that changes in material damping can occur due to microscopic effects in the material. Do the high strains put the internal planes in positions that reduce the friction between them? Although, most investigators believe in the influence of the initial strains on the damping, but, a clear picture regarding this effect cannot be found in the past researches.

The actual results of the present research show two different effects of bolt tightness on the damping ratio. These are the increase in damping due to the increase in the stiffness of the grid and the reduction in damping due to the increase in the lack-of-fit strains.

Regarding the above considerations, one may explain the behaviour of the grid as follows: It is clear that the variation of the bolt tightness has an influence on both the stiffness of the grid and the presence of the lack-of-fit strains. So, the material damping may be affected by the bolt tightness through stiffness and lack-of-fit strains differently. An estimate of the effects on the material damping due to the stiffness of the grid is shown in Fig 7.1. This shows that, in the region of higher bolt tightness, the bolt tightness has no major influence on the stiffness. On the other hand, an estimate of the effects on the material damping due to the presence of lack-of-fit strains is shown in Fig 7.2. Also, the variation of Coulomb damping due to the variation of bolt tightness is estimated to be of the form shown in Fig 7.3.

Figs 7.1 to 7.3 show estimations of three different effects on the damping due to the variation of the bolt tightness of the connections. The damping capacity of the grid in the form of damping ratio is a function of these three effects. The combination of the effects shown in Figs 7.1 to 7.3 can explain the experimental results obtained in the present research.
Chapter 7 Conclusions and suggestions for future work

Effects on material damping due to the increase in the stiffness

Figure 7.1 The influence of tightness on material damping caused by the stiffness

Effects on material damping due to lack-of-fit strains

Figure 7.2 The influence of tightness on material damping caused by the lack-of-fit strain
7.3.2 Effects of supports

The test model has been tested for three support cases: with 4 supports, with 8 supports and with 12 supports. The results obtained from the tests show that the lowest damping ratio is for the four support case. Also, the damping ratio corresponding to the eight support case is larger than that for the four support case and is smaller than that for the twelve support case.

Increasing the number of supports will increase the stiffness of the grid. So, the grid with eight supports is stiffer than the grid with four supports. Also, the grid with twelve supports is stiffer than the grid with eight supports. Since the damping ratio is dependent on the stiffness, it should be expected that the lowest damping ratio corresponding to the smallest number of supports. Also, the highest damping ratio is expected to correspond to the largest number of supports. These expectations are found to be in agreement with the actual test results.

7.3.4 Effects of amplitude
Another interesting aspect is the effects of the amplitude of vibration on the damping ratio. The results, which are presented in Chapter 6, show that an increase in the amplitude of vibration will increase the damping ratio. So, one may conclude that the damping ratio is directly dependent on the amplitude of vibration. However, it is not possible to obtain a specific relation between the amplitude of vibration and the damping ratio, but, it can be said that the larger the amplitude of vibration the larger the damping ratio will be.

7.4 Suggestions for Future Work

Because of the complexity of damping phenomenon, the employment of real scale model tests is unavoidable. The model for the current research has been tested for four levels of bolt tightness. In order to investigate more clearly the changes in damping due to the changes of bolt tightness, it is recommended to carry out tests for more levels of tightness. For instance, one can carry out tests for 12 levels of bolt tightness from the loose condition to the 120Nm case. Such an investigation will produce a clearer picture corresponding to the changes of damping ratio with respect to the changes of bolt tightness.
APPENDIX

This appendix contains the first 10 mode shapes of the experimental grid with 4 supports. These mode shapes are obtained using the ANSYS program. The numbers on the grid for the last 4 mode shapes are the node numbers.
Appendix

Fig a.1.1 First mode shape of the grid

Fig a.1.2 First mode shape of the grid
Appendix

![Modal analysis of the grid](image1)

**Fig a.2 Second mode shape of the grid**

![Modal analysis of the grid](image2)

**Fig a.3 Third mode shape of the grid**
Fig a.4 Fourth mode shape of the grid

Fig a.5 Fifth mode shape of the grid
Fig a.6 Sixth mode shape of the grid

Fig a.7 Seventh mode shape of the grid
Fig a.8 Eighth mode shape of the test grid

Fig a.9 Ninth mode shape of the test grid
Fig a.10 Tenth mode shape of the grid
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