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A Tight Closed-Form Approximation of the SISO Energy Efficiency-Spectral Efficiency Trade-Off

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Abstract: Due to the rise of the energy efficiency (EE) as a system performance evaluation criterion, the EE-spectral efficiency (SE) trade-off is becoming a key tool for getting insight on how to efficiently design future communication system. As far as the single-input single-output (SISO) Rayleigh fading channel is concerned, the EE-SE trade-off has been accurately approximated in the past but only at low-SE. In this paper, we propose a novel and more generic closed-form approximation (CFA) of this EE-SE trade-off which is very accurate for any SE values. We compare our CFA with two existing CFAs and show the great accuracy of the former for a wider range of SE in comparison with the latter. As an application, we use our CFA to study the variation of EE-SE trade-off when a realistic power model is assumed and to compare the energy consumption of SISO against a 2x2 multi-input multi-output (MIMO) system over the Rayleigh fading channel.

Keywords: Energy efficiency, spectral efficiency, trade-off, single-input single-output system, closed-form.

1. Introduction

In the current context of carbon footprint reduction and ever growing energy price, energy efficiency (EE) is soon to be the main design criterion, along with the spectral efficiency (SE), for developing the next generation of communication system. The SE, as a metric, indicates how efficiently a limited frequency spectrum is utilized but fails to provide any insight on how efficiently the energy is consumed. In a context of energy saving, the latter will become as important as the former and, thus, it has to be included in the performance evaluation framework by means of an EE metric such as the bit-per-Joule [1] or the traditional energy-per-bit to noise power density [1, 2], which measures the energy consumed per bit.

Maximizing the EE, or equivalently minimizing the consumed energy, while maximizing the SE are conflicting objectives which implies the existence of a trade-off. The concept of power-bandwidth trade-off, or equivalently EE-SE trade-off, has first been introduced in [2], where an approximation of this trade-off has been derived for the white and colored noise, as well as multi-input multi-output (MIMO) fading channels. Recently in [3], we have proposed a closed-form approximation (CFA) of this EE-SE trade-off for MIMO system over a Rayleigh fading channel, which is highly accurate for a wide range of SE values and antenna configurations. Single-input single-output channel is obviously a special case of MIMO, and the approximation method of [2] and our CFA in [3] can also be applied for the SISO case, but both these CFA are only accurate at low SE for SISO system. Thus, there is a need for finding a dedicated and accurate CFA of the SISO EE-SE trade-off, which to the best of our knowledge does not yet exist in the literature.

The EE of a communication system is obviously closely related to its power consumption. In most early studies [1, 2], the EE-SE trade-off has been defined by considering that the total consumed power of the system is solely the transmit power, which is a fair assumption for power-limited applications such as sensor networks but is clearly not realistic for power-unlimited applications such as cellular systems. For instance, in cellular systems, the main power-hungry component is the base station (BS) [4,5]. Here we consider the latest refinements in BS power consumption model (PCM) for assessing the EE of SISO system in a more realistic scenario regarding power consumption.

The rest of the paper is organized as follows, Section 2. introduces the EE-SE trade-off concept and the approximation method of [2]. In Section 3., we recall the classic SISO system model and derive an improved CFA of the ergodic capacity by means of a heuristic curve fitting method [6]. We then use this expression in Section 4. for deriving our tight CFA of the SISO EE-SE trade-off. Numerical results show its great accuracy for a wide range of SE in comparison with the ones in [2] and [3]. As an application, we use our CFA to study the variation of the EE-SE trade-off when a realistic power model is assumed [5] and compare the energy consumption of a SISO system with a 2x2 MIMO system over the Rayleigh fading channel in Section 5.. Conclusions are finally given in Section 6..

2. EE-SE Trade-off

The concept of EE-SE trade-off can simply be described as how to express EE as a function of SE. Let R (bit/s) be the achievable rate of a system and P (W) be its transmit power, then its EE can either be defined as $E_b = P/R$ or $C_J = R/P$, where E_b and C_J are the energy-per-bit and bit-per-Joule capacity , respectively. As far as the channel capacity per unit bandwidth $\mathcal{C}$ (bit/s/Hz) is concerned, it can be expressed in a generic form as

$$\mathcal{C} = f(\gamma) \tag{1}$$

via the Shannon's capacity theorem, where $\gamma = P/(N_0 W)$ is the signal-to-noise ratio, W (Hz) is the bandwidth, N_0 (J) is the noise spectral density. In the general case, $f(\gamma)$ can be described as an increasing function of γ mapping signal-to-noise (SNR) values in $[0, +\infty)$ to capacity per unit bandwidth values in $[0, +\infty)$. As long as $f(\gamma)$ is a bijective function, $f(\gamma)$ would be invertible such that

$$\gamma = f^{-1}(\mathcal{C}), \tag{2}$$

where $f^{-1} : \mathcal{C} \in [0, +\infty) \mapsto \gamma \in [0, +\infty)$ is the inverse function of f. For instance, over the AWGN channel $f(\gamma)$ and $f^{-1}(\mathcal{C})$ are simply given as

$$f(\gamma) = \log_2(1 + \gamma) \text{ and } f^{-1}(\mathcal{C}) = 2^{\mathcal{C}-1}, \tag{3}$$

respectively. As it has been explained in [1], the transmit power P can be expressed as RE_b and hence the SNR, γ, can be re-expressed as a function of the achievable SE, $S = R/W$ (bit/s/Hz) such that

$$\gamma = \frac{P}{N_0 W} = S\frac{E_b}{N_0}. \tag{4}$$

Inserting (4) into (2), the EE-SE trade-off is simply given as

$$\frac{E_b}{N_0} = \frac{f^{-1}(\mathcal{C})}{S}. \tag{5}$$

Equation (5) indicates that a straightforward solution for finding an explicit expression of the EE-SE trade-off boils down to obtaining an explicit expression for $f^{-1}(\mathcal{C})$, as it is shown in (3) for the AWGN channel. However, in cases where $f(\gamma)$ does not have a straightforward formulation, e.g. Rayleigh fading channel, approximating $f^{-1}(\mathcal{C})$ can provide an acceptable solution. In [2], it has been suggested that the EE of a communication system depends mainly on the SE in the low-power/low-SE regime such that the EE-SE trade-off can be approximated as (equation (28) of [2])

$$\frac{E_b}{N_0} \gtrsim \frac{E_b}{N_0}_{\min} 2^{\frac{\mathcal{C}}{S_0}}, \tag{6}$$

where $\frac{E_b}{N_0}_{\min} = \frac{\ln(2)}{\dot{f}(0)}$ and $S_0 = \frac{2[\dot{f}(0)]^2}{-\ddot{f}(0)}$ are the minimum energy-per-bit and the slope of the SE, respectively, and $\dot{f}(0)$ and $\ddot{f}(0)$ are the first and second order derivatives of $f(\gamma)$ when $\gamma = 0$. This method is in effect quite generic and, thus, it can be used to approximate the EE-SE trade-off of any communication channels or systems for which an explicit expression of its SE as a function of γ, i.e. $f(\gamma)$, exists and is twice differentiable. However, the main shortcoming of this universal approach is the rather limited range of SE values for which it is accurate, especially in the SISO channel case, as it is shown later in Fig. 2.

3. Improved CFA of the SISO Ergodic Capacity

Assuming that the number of transmit and receive antennas, t and r, respectively, is equal to one in (2.38) of [7], the ergodic capacity of the SISO Rayleigh fading channel can be formulated as in (1) where $f(\gamma)$ is given in a closed-form as

$$f(\gamma) = e^{\gamma^{-1}} \mathrm{E}_1\left(\gamma^{-1}\right) / \ln(2), \tag{7}$$

with E_1 being the exponential integral function. Due to the nature of $f(\gamma)$, an explicit formulation of its inverse cannot be easily derived. In the MIMO scenario, we have recently proposed in [3] a novel approach for deriving a CFA of the EE-SE trade-off by using the CFA of the ergodic capacity in [8] as a starting point for our derivation. The advantage of the CFA of [8] in comparison with the exact closed-form expression is that its inverse can be explicitly derived [3]. However, this CFA has been derived by assuming a large number of transmit and receive antennas; as a result, this approximation is not very accurate for the SISO case, as it is suggested in [3] and shown in Fig. 1 (a) (left side of Fig. 1), i.e. $h(\gamma) = \widetilde{f}(\gamma)$. Thus, its accuracy must be improved before it can be used for SISO system.

Assuming that $r = 1$, $t = 1$ and $\beta \triangleq r/t = 1$ in (E.41) and (4.51) of [8], the ergodic capacity of the SISO Rayleigh fading channel can then be approximated as $\mathcal{C} \approx \widetilde{f}(\gamma)$, where

$$\widetilde{f}(\gamma) = \frac{2}{\ln(2)}\left[-c + \frac{1}{1+\sqrt{1+4\gamma}} + \ln\left(1+\sqrt{1+4\gamma}\right)\right] \tag{8}$$

and $c = 1/2 + \ln(2)$. In order to illustrate the inaccuracy of this approximation for the SISO case, we plot in Fig. 1 (a) the relative approximation error in percentage

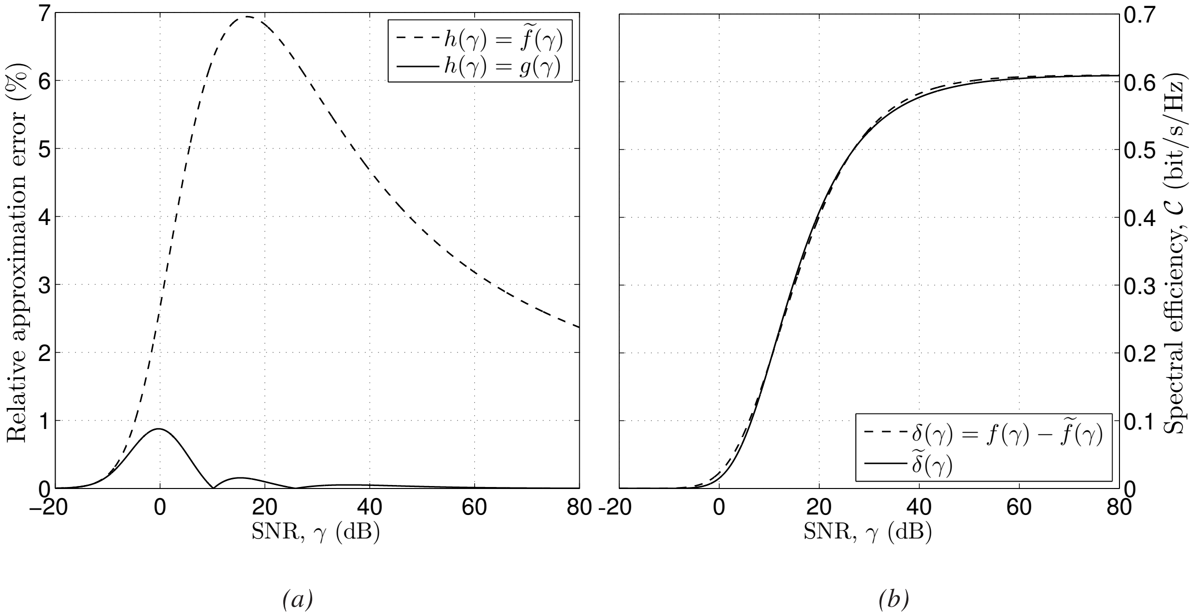

Figure 1: (a) Relative approximation error between $f(\gamma)$ and $\widetilde{f}(\gamma)$, as well as between $f(\gamma)$ and $g(\gamma)$ as a function of γ / (b) Approximation error function $\delta(\gamma)$ and its approximation $\widetilde{\delta}(\gamma)$ vs. γ.

between $f(\gamma)$ and $h(\gamma)$, i.e. $100(f(\gamma)-h(\gamma))/f(\gamma)$ vs. γ (dB) for $h(\gamma)=\widetilde{f}(\gamma)$ in (8) and $h(\gamma)=g(\gamma)$ in (9). The results show that $f(\gamma)$ and $\widetilde{f}(\gamma)$ differs by up to 7% and by more than 2% for γ between 0 to 80 dB, which cannot be considered as satisfactory in terms of accuracy. In order to improve the accuracy of $\widetilde{f}(\gamma)$, we propose the following function

$$g(\gamma)=\widetilde{\delta}(\gamma)+\widetilde{f}(\gamma), \tag{9}$$

where $\widetilde{\delta}(\gamma)$ is defined by means of a heuristic curve fitting method [6] such that it accurately approximates the difference between $f(\gamma)$ and $\widetilde{f}(\gamma)$, i.e. $\widetilde{\delta}(\gamma)\approx\delta(\gamma)=f(\gamma)-\widetilde{f}(\gamma)$. In [6], a parametric function is designed in terms of elementary functions and three independent parameters for solving a curve fitting problem. Following this method, we first numerically evaluated $\delta(\gamma)$ as a function of γ (dB) in Fig. 1 (b) (right side of Fig. 1). It can be noticed that $\delta(\gamma)$ clearly presents the feature of a hyperbolic tangent function "tanh" such that $\delta(\gamma)\overset{0}{\sim}0$ and $\delta(\gamma)\overset{\infty}{\sim}\eta_0$. The parameter η_0 can easily be obtained by deriving the limits of $f(\gamma)$ and $\widetilde{f}(\gamma)$ for $\gamma\rightarrow\infty$ such that $\eta_0=(1-\phi)/\ln(2)\simeq 0.61$, where $\phi=0.57721...$ is the Euler-Mascheroni constant [9]. In the effort of obtaining the function that best fits $\delta(\gamma)$, the curve fitting method leads to

$$\widetilde{\delta}(\gamma)=\eta_0(1-\tanh(\eta_1\gamma^{-\eta_2})), \tag{10}$$

where $\eta_1=2.193$ and $\eta_2=0.402$, which provides a satisfying approximation since $\delta(\gamma)$ and $\widetilde{\delta}(\gamma)$ well-fits each other, as it is illustrated in Fig. 1 (b). Inserting (10) into (8), we obtain our improved CFA of the SISO ergodic capacity, $g(\gamma)$, which differs from the exact formulation, $f(\gamma)$, by at most 0.9% and on average by less than 0.1%, as it is depicted in Fig. 1 (a).

4. CFA of the SISO EE-SE Trade-off

As we previously mentioned, the main advantage of $\widetilde{f}$ over f is the fact that its inverse function $\widetilde{f}^{-1}$ can be expressed into a closed-form. Knowing that the real branch of the Lambert W function, W_0, satisfies $W_0(x)e^{W_0(x)} = x$, where $x \in \mathcal{D}_{W_0} = [-e^{-1}, +\infty)$ [10], that W_0 is monotonically increasing over its domain $\mathcal{D}_{W_0}$ and $-e^{-(\widetilde{f}(\gamma)\ln(2)/2+c)} \in \left[-\frac{1}{2}e^{-\frac{1}{2}}, 0\right]$ belongs to $\mathcal{D}_{W_0}$, it implies that (8) is equivalent to

$$-\left(1+\sqrt{1+4\gamma}\right)^{-1} = W_0\left(-2^{-\left(\frac{\widetilde{f}(\gamma)}{2}+1\right)}e^{-\frac{1}{2}}\right), \tag{11}$$

which further simplifies as $\gamma =$

$$\widetilde{f}^{-1}(\widetilde{\mathcal{C}}) = 0.25\left\{-1+\left[1+W_0\left(-2^{-\left(\frac{\widetilde{\mathcal{C}}}{2}+1\right)}e^{-\frac{1}{2}}\right)^{-1}\right]^2\right\}, \tag{12}$$

where $\widetilde{\mathcal{C}} = \widetilde{f}(\gamma)$ and, hence, $\gamma = \widetilde{f}^{-1}(\widetilde{\mathcal{C}})$. Note that (12) is equivalent to (13) of [3] for $t = r = 1$. Moreover, we know that $\mathcal{C} = f(\gamma) \approx g(\gamma) = \widetilde{f}(\gamma) + \widetilde{\delta}(\gamma)$, which is equivalent to $\widetilde{f}(\gamma) \approx \mathcal{C} - \widetilde{\delta}(\widetilde{f}^{-1}(\widetilde{\mathcal{C}}))$ and in turn implies $\gamma \approx \widetilde{f}^{-1}(\mathcal{C} - \widetilde{\delta}(\widetilde{f}^{-1}(\widetilde{\mathcal{C}})))$. Consequently,

$$\gamma = f^{-1}(\mathcal{C}) = \widetilde{f}^{-1}(\widetilde{\mathcal{C}}) \approx \widetilde{f}^{-1}(\mathcal{C} - \widetilde{\delta}(\widetilde{f}^{-1}(\widetilde{\mathcal{C}}))). \tag{13}$$

Finally, we obtain our CFA of the SISO EE-SE trade-off as

$$f^{-1}(\mathcal{C}) \approx \widetilde{f}^{-1}(\mathcal{C} - \widetilde{\delta}(\widetilde{f}^{-1}(\mathcal{C}))) \tag{14}$$

by assuming that $\widetilde{\delta}(\widetilde{f}^{-1}(\widetilde{\mathcal{C}})) \approx \widetilde{\delta}(\widetilde{f}^{-1}(\mathcal{C}))$. In effect, we assume that $\mathcal{C} \approx \widetilde{\mathcal{C}}$, or equivalently that $f(\gamma) \approx \widetilde{f}(\gamma)$. We know from Fig. 1 (a) that $f(\gamma)$ and $\widetilde{f}(\gamma)$ can differ by up to 7%, which implies that $\widetilde{\delta}(\widetilde{f}^{-1}(\widetilde{\mathcal{C}}))$ and $\widetilde{\delta}(\widetilde{f}^{-1}(\mathcal{C}))$ will also differ by the same margin. However, since $\widetilde{\delta}(\gamma)$ is bounded, i.e. $\widetilde{\delta}(\gamma) \leq 0.61$ (bit/s/Hz), the maximum absolute approximation error between $\widetilde{\delta}(\widetilde{f}^{-1}(\widetilde{\mathcal{C}}))$ and $\widetilde{\delta}(\widetilde{f}^{-1}(\mathcal{C}))$ will be 7% of 0.61, i.e. 0.043 bit/s/Hz, which is negligible, in comparison with the maximum absolute approximation error between $f(\gamma)$ and $\widetilde{f}(\gamma)$.

In order to show the accuracy of our novel CFA of the SISO EE-SE trade-off in (14), we compare it in Fig. 2 with our EE-SE trade-off CFA for MIMO in (13) of [3], the approximation method of [2] and the nearly-exact E_b/N_0 as a function of $\mathcal{C}$, which has been obtained via (7). Indeed, (7) provides us with the SE $\mathcal{C}$ for a given SNR γ. However, one can obtain the SNR $\gamma = f^{-1}(\mathcal{C})$ for a given SE $\mathcal{C}$ by using this expression in conjunction with a simple line search algorithm where we set the target $\mathcal{C}$ to differ by less than 10^{-8} from the actual $\mathcal{C}$. Using this approach, we have obtained $f^{-1}(\mathcal{C})$ for $\mathcal{C} = 10^{-2}$ to 16 bit/s/Hz with an increment step of 0.1 bit/s/Hz and then plugged it into (5) for plotting the nearly-exact E_b/N_0 as a function of $\mathcal{C}$ when $S = \mathcal{C}$. For the method of [2], the values of $\dot{f}(0)$ and S_0 can be found in (213) and (215) of [2], respectively, such that $\dot{f}(0) = 1$ and $S_0 = 1$ when $t = r = 1$.

The results in Fig. 2 clearly demonstrate the tight fit between $f^{-1}(\mathcal{C})$ and our CFA in (14), hence, they graphically show the accuracy of the latter. These two expressions differ on average by less than 0.02 dB for $\mathcal{C}$ spanning from 0 to 16 bit/s/Hz. Furthermore, they also indicate that both the approximations of [2] and [3] are only accurate

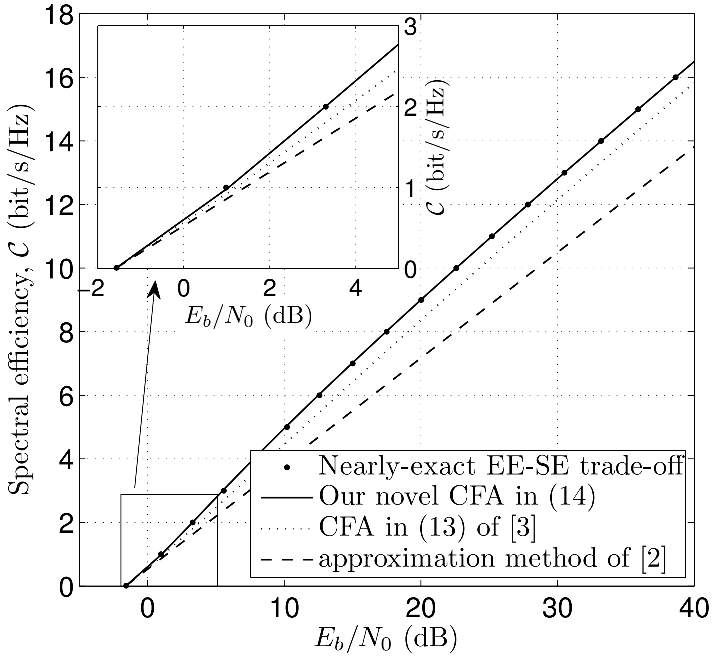

Figure 2: Comparison of our novel CFA of the SISO EE-SE trade-off in (14) with the approximations of [2] and [3], as well as the nearly-exact EE-SE trade-off.

at low SE, i.e. from 0 to 1 bit/s/Hz, and then the former loosens linearly with E_b/N_0 (dB) (up to 7 dB at $\mathcal{C} = 14$ bit/s/Hz), whereas the latter differs from $f^{-1}(\mathcal{C})$ on average by 2 dB at high SE.

5. Applications for our CFA

The EE of a communication system is obviously closely related to its power consumption. Assuming the realistic PCM for different types of BS in [5] where the total consumed power, P_Σ, is given by

$$P_\Sigma = t(\Delta_P P + P_0), \tag{15}$$

the EE-SE trade-off expression in (5) can be re-expressed as

$$\frac{E_b}{N_0} = \frac{1}{S}\left[t\Delta_P f^{-1}(\mathcal{C}) + \frac{tP_0}{N}\right]. \tag{16}$$

In (16), Δ_P and P_0 are the slope and overhead power of the PCM, $N = N_0 W$ is the noise power, and $P \in [0, P_{\max}]$ with $P_{\max}$ being the maximum radio frequency output power.

As an application for our CFA, we first plot in Fig. 3 (a) the SISO EE-SE trade-off in the linear PCM of [5] by relying on the values of Δ_P, P_0 and $P_{\max}$ for various types of BS in [5], different noise power values, when inserting $t = 1$ as well as $f^{-1}(\mathcal{C})$ in (14) into (16). Comparing the results of Macro BS for different N values, it obviously shows that a lower noise power implies a lower energy consumption per bit, i.e. lower E_b/N_0 values. We also depicts by means of a circle the point that corresponds to the maximum power $P_{\max}$ and the results show that this point is not always the most energy

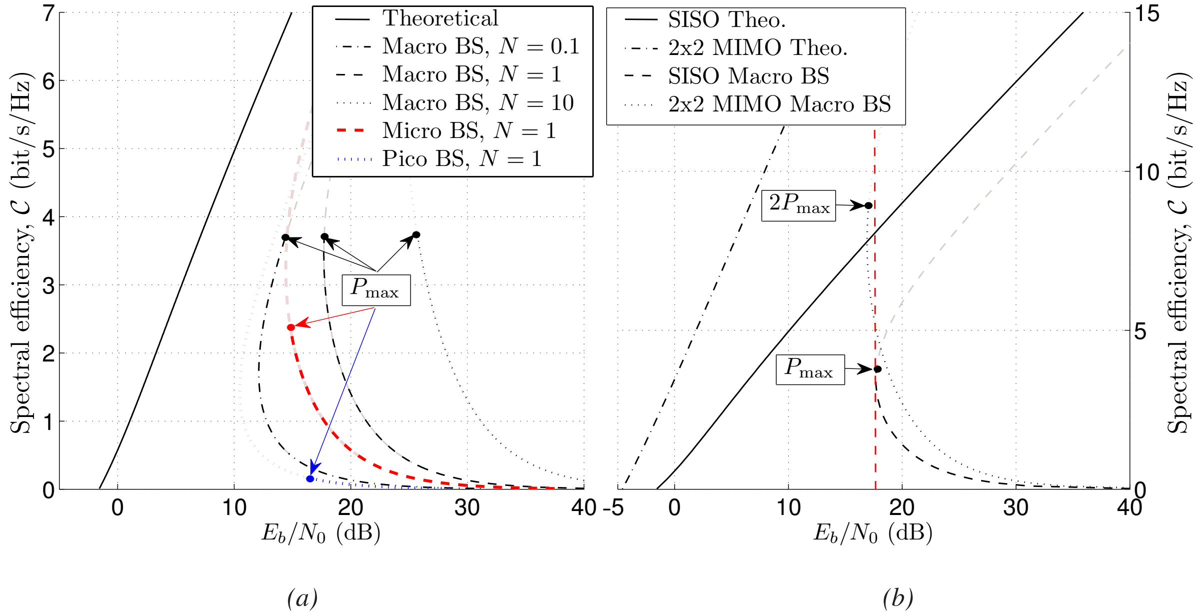

Figure 3: (a) SISO EE-SE trade-off for various types of BS and noise values when considering the linear PCM of [5] / (b) SISO vs. 2x2 MIMO EE-SE trade-off when considering the linear PCM of [5].

efficient operation point. Comparing, the results for the 3 different types of BS, our results indicate that Micro BS has the lowest energy consumption per bit, even though it consumes more power than a pico BS.

In Fig. 3 (b), we compare the EE-SE trade-off of SISO with a 2x2 MIMO system in the linear PCM of [5]. In order to plot the EE-SE trade-off for the 2x2 MIMO case, we have used (13) of [3] for $t = r = 2$. As far as the theoretical EE-SE trade-off is concerned, i.e. equation (5), 2x2 MIMO has always a lower energy consumption per bit than SISO and the difference between the two systems increases as the SE increases due to the MIMO diversity gain. However, in a realistic PCM, it appears that SISO is more energy efficient than 2x2 MIMO for SE up to 4.5 bit/s/Hz. This is mainly due to the fact that in this PCM, 2x2 MIMO consumes twice as much overhead power than SISO and that the extra SE provided by MIMO over SISO does not compensate for this handicap. When the SE is between 5 to around 10 bit/s/Hz, then 2x2 MIMO has a lower energy consumption per bit than the minimum E_b/N_0 of SISO, which occurs at $E_b/N_0 = 17.69$ dB, $\mathcal{C} = 3.5$ bit/s/Hz and for $P = 14.8$ W, $P_\Sigma \simeq 200$ W. Meanwhile, the minimum E_b/N_0 of 2x2 MIMO occurs at $E_b/N_0 = 16.91$ dB, $\mathcal{C} = 7.9$ bit/s/Hz and for $P = 26.3$ W, $P_\Sigma \simeq 380$ W. Thus, the 2x2 MIMO system can be up to 0.8 dB better than SISO in terms of energy-per-bit but it will consumes around 180 W of extra power for achieving this maximum EE gain.

6. Conclusions

In this paper, a tight CFA of the EE-SE trade-off over the SISO Rayleigh fading channel has been derived. First we have proposed an improved approximation of the SISO ergodic capacity and then utilized this approximation to derive our CFA. The great accuracy of our novel CFA has been experimentally shown for a wide range of SE values, and the results have indicated the extend of its accuracy in comparison with the existing approximations. As an application for our CFA, we have shown the variations of the SISO EE-SE trade-off in a realistic power model and have compared the energy consumption of SISO against 2x2 MIMO system over the Rayleigh fading channel. Results have indicated that Micro BS is more energy efficient than Macro or Pico BS and that a SISO system is more energy efficient than a 2x2 MIMO for low SE values (up to 4.5 bit/s/Hz) when a realistic power model is considered.

7. Acknowledgment

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