

# Closed-form Approximation for the Trade-Off between Energy Efficiency and Spectral Efficiency in the Uplink of Cellular Network

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**Abstract**—In this paper we propose a tight closed-form approximation of the Energy Efficiency vs. Spectral Efficiency (EE-SE) trade-off for the uplink of a cellular communication system. We model the uplink of the cellular system by considering the Wyner model with Raleigh fading. We first demonstrate the accuracy of our expression by comparing it with Monte-Carlo simulation and the EE-SE trade-off expression based on low-power approximation. Results show the great tightness of our expression with Monte-Carlo simulation. We utilize our closed-form for assessing the EE performance of base station (BS) cooperation against non-cooperative system for both a theoretical power model and a realistic power model. The theoretical power model includes only the transmit power, whereas the realistic power model incorporates the backhaul and signal processing powers in addition of the transmit power. Results indicate that BS cooperation is more energy efficient than non cooperative system and the former always outperforms the latter in terms of EE-SE trade-off. This is however no more the case with the realistic power model: the EE performance is then highly dependent on the number of cooperating BSs.

## I. INTRODUCTION

In the past, communication network evolution has mainly been driven by spectral efficiency (SE) improvement. In recent years, the reduction in network energy consumption has become of great importance for network operators. So has the importance of the energy efficiency (EE) as a metric for network performance evaluation.

The SE is the traditional metric for measuring the efficiency of communication systems. It is a measure of how efficiently a limited frequency resource (spectrum) is utilized. It however fails to give any insight on how efficiently energy is utilized. A new metric that provides this insight was introduced in [1], i.e. the bits-per-Joule (bits/J). The bits/J capacity of an energy limited wireless network was defined in [2] as the maximum amount of bits that can be delivered by the network per Joule it consumed to do so. Thus, it is simply the ratio of the capacity to the rate of energy expenditure i.e. the total consumed power.

Research work on EE was initially motivated by limited power applications [1]. Such applications include underwater acoustic telemetry, wireless ad hoc networks such as sensor networks, home networks and data networks. Since most of these systems are operated on batteries, EE is a paramount factor for designing such networks. The global trend towards energy consumption reduction has led to the extension of the

EE concept to unlimited power applications, e.g. devices with constant power supply such as base station (BS) and fixed relay terminal in cellular networks. Moreover, the available spectrum resource needs to be efficiently used for the transmission of information bits and, consequently, the SE also needs to be taken into account in the design of communication networks. However, the two objectives of minimizing the energy consumed in the network and maximizing the bandwidth efficiency, i.e. SE, are not achievable simultaneously and, hence, this creates the need for a trade-off. The EE-SE trade-off in a cellular architecture provides a guide for operators to obtain the best operating point which could either be towards an energy efficient network or a spectral efficient system.

The Shannon's capacity theorem illustrates that there exists a trade-off between bandwidth, transmit power and the coding strategy implemented to achieve a certain rate  $R$ , in other words, the trade-off between EE and SE. The low power approximation technique introduced in [3] has been used to investigate the EE-SE trade-off for single user, multi user [4], single relay networks [5], multiple relay networks [6], [7], BS cooperation [8] and relay assisted BS cooperation [9]. This approach, though easy to implement is only valid for the low power regime, whereas our approach is accurate for all power regimes. As far as the power consumption model for the uplink of cellular system is concerned, three main power components can be distinguished: the users transmit power, the BSs signal processing power and the backhauling power. For instance, a theoretical power model that only takes into account the users transmit power has been utilized in the low power approximation technique. Meanwhile in [10], [11], the authors considered the circuit power (signal processing power) in addition to the transmit power in their model for improving the EE of sensor networks, however, they did not consider the spectrum efficiency. Moreover, in [7], [12], the authors considered the EE-SE trade-off of relay networks based on both the circuit and transmit powers but without including the backhauling power. Whereas, in this work we investigate the EE-SE trade-off of the cellular uplink by considering a more realistic power model than the previous contributions on this topic, since the model we use not only includes the transmit power but the signal processing and backhauling power as well.

In the uplink of cellular networks, two main approaches can be followed for decoding the users signal at the BS: 1) in the traditional approach, each BS decodes only the signals from its own cell. 2) in the BS cooperation approach, which is the information theoretic optimal approach, BSs cooperate to jointly decode the signals of all the users in the network and, thus, eliminates the inter-cell interference from the cellular systems. Cooperation is made possible by delayless high speed links such as microwave link or optical fibre connecting the BSs to a central processor. In this work, we consider the scenario in which the backhaul capacity is unlimited such that the network can be modelled as a multiple-input multiple-output (MIMO) multiple access channel (MAC) with spatially distributed antenna.

In this paper, we derive the EE-SE trade-off expression for BS cooperation based on the per-cell sum-rate and the total power consumed in the uplink by using a closed-form approximation. Our closed-form approximation is based on exploiting the random matrix theory for limiting eigenvalue distribution of large random matrices. The closed-form approximation approach presents a considerable advantage over Monte Carlo simulation in terms of computational complexity, as a large number of random values are required for evaluating the capacity with the latter.

In Section II, we introduce the cellular uplink model. In section III, we then derive our tight closed-form approximation of the EE-SE tradeoff for the uplink of cellular system when BS cooperates by considering the Wyner model with Rayleigh fading and a theoretical power model. In the non cooperative case, we extend the closed-form expression of the SE that was derived in [13] for the single user case with  $K$  interferers to the multi-user scenario and use this modified expression for obtaining a closed-form of the EE-SE tradeoff. Section IV then presents the realistic power model of [14] for the uplink of cellular system and we incorporate it into our EE-SE trade-off expression. Section V presents some numerical results based on single user decoding and joint decoding and we show that for the theoretical power model, the EE-SE trade-off for BS cooperation always outperforms that of non cooperative system. This is however not the case with the realistic power model as the EE performance is highly dependent on the number of cooperating BSs. Finally, Section VI concludes the paper.

## II. SYSTEM MODEL

In this paper, we use boldface letters to denote matrices and vectors and refer to the set of complex numbers as  $\mathbb{C}$ . Let  $\mathbf{Z}$  be a matrix, then  $\text{tr}(\mathbf{Z})$  denotes its trace,  $\mathbf{Z}^*$  denotes its complex conjugate,  $\mathbf{Z}^T$  denotes its transpose,  $\mathbf{Z}^\dagger$  denotes its complex conjugate transpose,  $\det(\mathbf{Z})$  denotes its determinant. In addition,  $\log(\cdot)$  denotes the logarithm to base 2,  $\mathbb{E}[\cdot]$  denotes the expectation,  $\otimes$  denotes the Kronecker product,  $\odot$  denotes the Hadamard product and,  $\mathbf{I}_M$  is an identity matrix with size  $M$ .

We consider the uplink of a cellular network where  $K$  user terminals (UTs) and  $M$  BSs in different locations can

communicate with each other. Assuming that each BS is associated with  $l$  UTs, such that  $K = lM$ , where the  $j^{\text{th}}$  BS is equipped with  $r_j$  antennas and the  $k^{\text{th}}$  UT with  $t_k$  antennas, then the signal received at the  $j^{\text{th}}$  BS is given by

$$y_j = \sum_{k=1}^K \alpha_{jk} \mathbf{H}_{jk} \mathbf{x}_k + \mathbf{n}_j, \quad (1)$$

where  $\mathbf{x}_k \in \mathbb{C}^{t_k}$  is the transmitted vector signal by the  $k^{\text{th}}$  user and  $\mathbf{H}_{jk} \in \mathbb{C}^{r_j \times t_k}$  is the channel matrix between the  $k^{\text{th}}$  user and the  $j^{\text{th}}$  BS. The gain elements in  $\mathbf{H}_{jk}$  are independent and identically distributed random variables with zero mean and unit variance. Note that in (1),  $\alpha_{jk}$  is the average channel gain between the  $k^{\text{th}}$  user and the  $j^{\text{th}}$  BS,  $\mathbf{n}_j$  is the additive white Gaussian noise at the  $j^{\text{th}}$  BS with zero mean and  $\sigma^2$  variance. In addition, the signal transmitted by the  $k^{\text{th}}$  user must satisfy the following power constraint :  $\text{tr}(\mathbb{E}(\mathbf{x}_k \mathbf{x}_k^H)) \leq P_k$ . The parameter  $\gamma_k = P_k/\sigma^2$  represents the transmit power of the  $k^{\text{th}}$  UT normalised by the noise at the BS. When the BS cooperates to receive data from UTs, the overall system model can be illustrated by

$$\mathbf{y} = \tilde{\mathbf{H}} \mathbf{x} + \mathbf{n}, \quad (2)$$

where  $\mathbf{y} = [\mathbf{y}_1^T \cdots \mathbf{y}_M^T]^T$  is the joint received signal vector,  $\mathbf{x} = [\mathbf{x}_1^T \cdots \mathbf{x}_K^T]^T$  is the transmitted signal vector and  $\mathbf{n} = [\mathbf{n}_1^T \cdots \mathbf{n}_M^T]^T$  is the joint received noise vector. The channel matrix can be expressed as:

$$\tilde{\mathbf{H}} = \mathbf{\Omega}_V \odot \mathbf{H}_V, \quad (3)$$

$$\mathbf{H}_V = \begin{bmatrix} \mathbf{H}_{11} & \cdots & \mathbf{H}_{1K} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{M1} & \cdots & \mathbf{H}_{MK} \end{bmatrix}, \mathbf{\Omega}_V = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1K} \\ \vdots & \ddots & \vdots \\ \alpha_{M1} & \cdots & \alpha_{MK} \end{bmatrix}, \quad (4)$$

where  $\mathbf{\Omega}_V$  is a  $Mr \times Kt$  deterministic matrix while  $\mathbf{H}_V$  is a  $Mr \times Kt$  matrix with independent and identically distributed random variables with zero mean and unit variance. As a result of the collocation of the multiple antennas at the UT and the BS,  $\mathbf{\Omega}_V = \mathbf{\Omega} \otimes \mathbf{J}$ , where  $\mathbf{J}$  is a  $r \times t$  matrix with all its elements equal to one and  $\mathbf{\Omega}$  is a  $M \times K$  deterministic matrix.

## III. EE-SE ANALYSIS OF THE SYMMETRICAL CHANNEL MODEL

For simplification purpose, we assume as in [15], an equal transmit power and an equal number of antennas for all UTs such that  $\gamma_k = \gamma$  and  $t_k = t$ ,  $\forall k \in \{1, \dots, K\}$  as well as an equal number of antennas at all BSs such that  $r_j = r$ ,  $\forall j \in \{1, \dots, M\}$ . We consider the generic symmetrical cellular channel model introduced in [15], in which the sum of squared elements of the columns and rows of matrix  $\mathbf{\Omega}_V$  can be expressed as follows:

$$\Upsilon_j = \sum_{k=1}^K \alpha_{jk}^2 = \Upsilon \quad \forall j \in 1, \dots, M \quad (5)$$

$$\Theta_k = \sum_{j=1}^M \alpha_{jk}^2 = \Theta \quad \forall k \in 1, \dots, K. \quad (6)$$

Therefore,  $K\Theta = M\Upsilon$ . Examples of cellular model in which this assumption holds include: the Wyner circular model and the Wyner two dimensional hexagonal array [16]. The per-cell sum-rate of the symmetric scenario is approximated in [15] as

$$C_P = lt \log(1 + \gamma_0 u^*) + r \log(1 + \kappa \gamma_0 w^*) - lt \gamma_0 u^* w^*, \quad (7)$$

where

$$u^* = (1 + \gamma_0 \kappa w^*)^{-1}, \quad (8)$$

$$w^* = (1 + \gamma_0 w^*)^{-1}, \quad (9)$$

and  $\gamma_0 = \gamma_r / \kappa$ . In addition,  $\gamma_r = \gamma \Upsilon$  is the received signal at each antenna and  $\kappa = lt/r$  is the ratio between the horizontal and vertical dimensions of matrix  $\mathbf{H}_V$ . Given that

$$u_0 = \gamma_0 u^* \quad \text{and} \quad w_0 = \kappa \gamma_0 w^*, \quad (10)$$

we prove in the Appendix that the per-cell sum-rate  $C_P$  in bit/s can be re-expressed as

$$C_P = \frac{W}{2 \ln 2} (C_r + C_t), \quad (11)$$

where

$$C_r = r \left( -1 + 2 \ln(1 + w_0) + \frac{1}{(1 + w_0)} \right), \quad (12)$$

$$C_t = \bar{t} \left( -1 + 2 \ln(1 + u_0) + \frac{1}{(1 + u_0)} \right), \quad (13)$$

and  $\bar{t} = lt$ . Notice that expressions (11), (12) and (13) are somehow equivalent to expressions (5) and (6) that we derived in [14]. Thus, we can obtain the EE-SE trade-off closed-form approximation by using the same approach that we proposed in [14]. Given that  $\bar{u}_0 = 2u_0 + 1$  and  $\bar{w}_0 = 2w_0 + 1$ , it can be shown that

$$\bar{u}_0 \bar{w}_0 = 1 + 2\gamma_0(1 + \kappa), \quad (14)$$

$$\bar{u}_0 = - \left[ 1 + \frac{1}{W_0(f(\bar{t}, C_t))} \right], \quad (15)$$

where  $f(a, b) = -\exp(-([\frac{b}{2a} + 1] \ln(2) + \frac{1}{2}))$  and  $W_0$  is the Lambert function. It represents the inverse function of  $f(w) = w \exp(w)$  and is such that  $W(z)e^{W(z)} = z$ , where  $w, z \in \mathbb{C}$  [17]. Similarly,

$$\bar{w}_0 = - \left[ 1 + \frac{1}{W_0(f(r, C_r))} \right]. \quad (16)$$

Therefore,

$$\gamma_0 = \frac{[1 + \frac{1}{W_0(f(\bar{t}, C_t))}] [1 + \frac{1}{W_0(f(r, C_r))}] - 1}{2(1 + \kappa)}. \quad (17)$$

Moreover,  $\gamma$  can be expressed as  $\gamma = \frac{RE_b}{N_0 W}$ , where  $R \leq C_P$  is the achievable rate,  $E_b$  is the energy-per-bit,  $N_0$  is the noise spectral density and  $W$  is the bandwidth. Thus, (17) can be re-expressed as

$$\frac{E_b}{N_0} \geq \frac{\kappa W ([1 + \frac{1}{W_0(f(\bar{t}, C_t))}] [1 + \frac{1}{W_0(f(r, C_r))}] - 1)}{2\Upsilon R(1 + \kappa)}. \quad (18)$$

We obtain  $C_t$  and  $C_r$  as a function of  $C_P$  by using the parametric solutions provided in (19) and (20) of [14]. The energy efficiency  $C_j$  which is equivalent to  $\frac{1}{E_b}$  can then be expressed as

$$C_j \leq \frac{2R\Upsilon(1 + \kappa)}{\kappa N_0 W ([1 + \frac{1}{W_0(f(\bar{t}, C_t))}] [1 + \frac{1}{W_0(f(r, C_r))}] - 1)}, \quad (19)$$

which simplifies as

$$C_j \leq \frac{4\Upsilon R}{N_0 W ([1 + \frac{1}{W_0(f(\bar{t}, C))}]^2 - 1)}, \quad (20)$$

for the special case where  $\kappa = 1$  i.e.  $Mr = LMt$ . Notice that for the Wyner circular model  $\Upsilon = 1 + 2\alpha^2$ , where  $\alpha$  is the attenuation scaling factor of the adjacent (next neighbouring) cells. While for the Wyner two dimensional hexagonal array (Planar model)  $\Upsilon = 1 + 6\alpha^2$ .

In the case of the symmetrical Wyner model with no cooperation at the BS. We assume intra cell time-division multiple access (TDMA) and, consequently, each user performs single user decoding. Based on (56) and (57) in [13], the average per-cell sum-rate is given as

$$C_{Psud} = t \log \left( 1 + \frac{l\gamma}{\kappa} \eta_1 \right) + Zt \log \left( \frac{1 + \frac{\alpha^2 l\gamma}{\kappa} \eta_1}{1 + \frac{\alpha^2 l\gamma}{\kappa} \eta_2} \right) + r \log \left( \frac{\eta_2}{\eta_1} \right) + r(\eta_1 - \eta_2) \log(e), \quad (21)$$

when the single user decoding approach is applied and where  $\eta_1$  and  $\eta_2$  satisfy

$$\begin{aligned} \eta_1 + \frac{l\gamma\eta_1}{1 + \frac{lP\eta_1}{\kappa}} + Z \frac{l\gamma\alpha^2\eta_1}{1 + \frac{lP\alpha^2\eta_1}{\kappa}} &= 1, \\ \eta_2 + Z \frac{l\gamma\alpha^2\eta_2}{1 + \frac{l\gamma\alpha^2\eta_2}{\kappa}} &= 1. \end{aligned} \quad (22)$$

Given that  $Z$  is the number of interfering cells, which is 2 for the Wyner circular model and 6 for the planar model, the EE-SE trade-off expression for intra cell TDMA with no cooperation based on the per-cell sum-rate (equal rate in all the cells) and the per-cell transmit power is thus given by

$$C_{jnc} = \frac{C_{Psud}}{lP}. \quad (23)$$

#### IV. POWER MODEL

We have derived our EE-SE trade-off expression in Section III by considering only the transmit power. Whereas, in this section, we incorporate the realistic power model of [18] for BS cooperation in our closed-form, which defines the average consumed power  $P_M$  as

$$P_M = \frac{aP_{sp}}{M} + \frac{(M-1)}{M} P_{bh} + lP_{ms}, \quad (24)$$

instead of  $P_M = lP_{ms}$  in the theoretical model. This realistic model incorporates the signal processing power  $P_{sp}$ , the backhaul power  $P_{bh}$  in addition of the total power consumed by the UT  $P_{ms}$ . We underestimate  $P_{ms}$  by assuming that

$P_{ms} \approx P$ . The effects of cooling and battery backup are taken into account via the factors  $c_c$  and  $c_{bu}$  respectively, such that  $a = r(1 + c_c)(1 + c_{bu})$ . The backhaul Power  $P_{bh}$  is given as  $P_{bh} = \frac{C_{sum}}{C_{bh}} \cdot p_b W$ , where  $C_{sum}$  is the system sum rate,  $C_{bh}$  is the capacity limit of the backhaul link with dissipation power  $p_b$ . Note that  $P_{sp}$  is given as

$$P_{sp} = p_{sp}(0.8 + 0.1N_c + 0.1N_c^2), \quad (25)$$

where  $p_{sp}$  is the base line signal processing power per-BS and  $N_c$  is the number of cooperating BS. The effective energy efficiency  $C_{jef}$  of the uplink cellular system with BS cooperation and joint user detection is given by

$$C_{jef} = \frac{C_j l P}{P_M}, \quad (26)$$

where  $C_j$  is given in (19). For the non cooperative case, we assume that the backhaul power  $P_{bh}$  is zero. Thus, the total average power consumed per-cell in the uplink is given by  $P_{sud} = a p_{sp} + l P_{ms}$  and the effective EE-SE trade-off is obtained as

$$C_{jef-nc} = \frac{C_{P_{sud}}}{P_{sud}}. \quad (27)$$

## V. NUMERICAL RESULTS

In this section, we evaluate the EE and SE performances of the linear cellular architecture based on the well-known Wyner model. We evaluate performance by using the joint user decoding (JUD) for the case with full BS cooperation and single user decoding (SUD) for the case with no cooperation. We fix the number of BSs to 10, the attenuation scaling factor for the Wyner model to  $\alpha = 0.4$  and  $\sigma^2 = 1$  unless otherwise stated. We assume an orthogonal multiple access scheme, e.g. TDMA, within each cell such that only one user is active per-cell at each instance of time. The other parameters used in our evaluation are listed in Table I.

Figure 1 depicts the trade-off between EE and SE of the circular Wyner model for various antenna combinations when using the theoretical power model. It could be observed that our closed-form approximation results closely match those obtained through Monte-Carlo simulation, whereas the low power approximation approach of [3] is mainly accurate in the low-SE regime. Increasing the number of antennas at the UT or BS node results in an increase in the EE and SE of the system since the slope of the trade-off curve becomes steeper in this case. Figure 2 depicts the EE performance for a range of attenuation scaling factor of  $\alpha = 0 - 1$  for both the full cooperation and the non cooperative scheme when  $P = 27\text{dBm}$ . As it can be observed, on the one hand in the theoretical power model, increasing the attenuation scaling factor leads to an increase in EE for the full BS cooperation scheme as a result of the increase in diversity gain. On the other hand, increasing  $\alpha$  leads to a reduction in EE for the non cooperative scheme due to the increase in the interference. Figure 3 compares the EE-SE trade-off performance of BS cooperation with the traditional non-cooperative cellular system. As it is depicted, the full cooperation scheme always outperforms the traditional scheme in terms of EE and SE for a

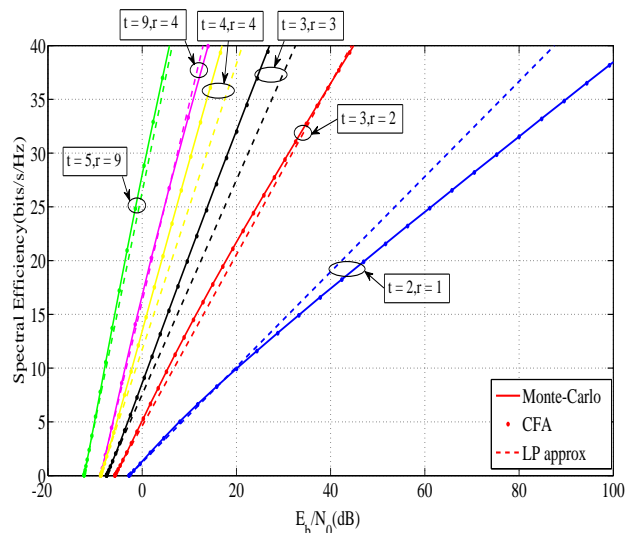


Fig. 1. Comparison of our closed-form approximation (CFA), Monte-Carlo simulation and Low power approximation (LP approx) based on theoretical power model.

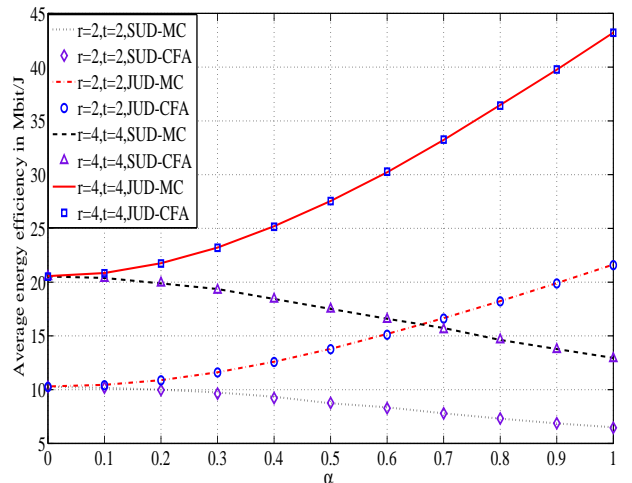


Fig. 2. Comparison of the EE performance of non cooperative BS with BS cooperation based on the theoretical power model with the attenuation factor  $\alpha$  varied from 0 – 1 and  $P = 27\text{dBm}$ .

given number of antennas at the nodes and a fixed attenuation factor ( $\alpha = 0.4$ ) when only the transmit power is considered. In Figure 4, we introduce the realistic power model which incorporates the signal processing and backhaul powers into our evaluation. The results show that increasing the number of cooperating BSs results in a loss in EE as no gain in per-cell sum-rate is achieved by increasing  $N_c$  beyond three. When the  $N_c$  increases then the backhaul power increases at the same time, thus, leading to a loss in EE. In addition, for very large  $N_c$ , non cooperating scheme with single user decoding can outperform the cooperative scheme with joint decoding over a significant range of attenuation scaling factor.

TABLE I  
POWER MODEL PARAMETERS [18]

Parameter	Value
$p_{sp}$	128W
$c_c$	0.29
$c_{bu}$	0.14
$C_{bh}$	100Mbit/s
$p_b$	50W
$W$	5MHz

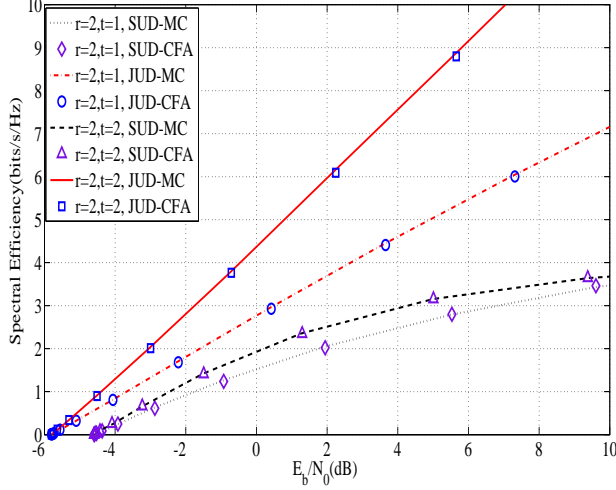


Fig. 3. Comparison of the EE-SE performance of non cooperative BS with BS cooperation based on the theoretical power model.

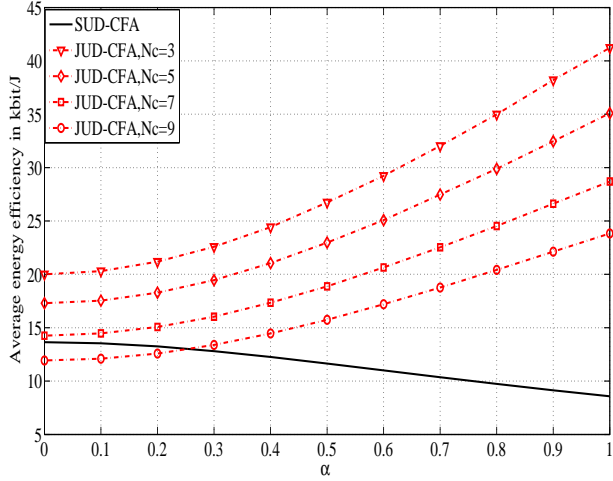


Fig. 4. Comparison of the EE-SE performance of non cooperative BS with BS cooperation based on the realistic power model. Parameters:  $r = 2, t = 2, P = 27\text{dBm}$ .

## VI. CONCLUSION

In this paper, we have derived a tight closed-form approximation of the EE-SE trade-off for the uplink of cellular system both in the BS cooperation and the non-cooperative scenarios by considering the Wyner model with Raleigh fading and a theoretical power model. We then presented the realistic power model of [18] for the uplink of cellular system and incorporated it into our EE-SE trade-off expression. Our closed-form approximation can be used for network simulations as it provides tightly matched results with the Monte-Carlo approach but in a much faster way than the latter.

The findings in this paper can be summarized as follows: When only the transmit power is considered in the EE-SE

trade-off analysis, increasing the number of antennas at the UT or BS nodes results in an increase in both the EE and SE. BS cooperation with joint user decoding approach outperforms the non cooperative approach for the same antenna configuration settings. When the signal processing power and the backhaul powers are introduced, it was revealed that increasing the number of cooperating BS can results in a reduction in EE.

## APPENDIX

From [15], for the symmetric scenario, the spectral efficiency per-receive antenna is given by:

$$C_P = \kappa \log(1 + \gamma_0 u^*) + \log(1 + \kappa \gamma_0 w^*) - \kappa \gamma_0 u^* w^* \quad (28)$$

, where  $u^*$  and  $w^*$  are expressed in (8) and (9). However, closed form solution in (28) does not meet our purpose of expressing EE in terms of SE. We prove in this appendix that (28) can be expressed as in (11), which is the formulation that allowed us to derive our closed-approximation of the EE-SE trade-off. By solving (8) and (9), the positive roots are obtained as

$$u^* = \frac{\gamma_0 - \mathcal{F}(\gamma_0, \kappa)}{\gamma_0}, \quad (29)$$

$$w^* = \frac{\kappa \gamma_0 - \mathcal{F}(\gamma_0, \kappa)}{\kappa \gamma_0}, \quad (30)$$

where

$$\mathcal{F}(x, z) = \frac{1}{4} \left( \sqrt{1 + x(1 + \sqrt{z})^2} - \sqrt{1 + x(1 - \sqrt{z})^2} \right)^2 \quad (31)$$

Expanding (8) and (9) we obtain

$$u^* = \frac{\gamma_0(1 - \kappa) - 1 + \sqrt{1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2}}{2\gamma_0} \quad (32)$$

$$w^* = \frac{\gamma_0(\kappa - 1) - 1 + \sqrt{1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2}}{2\kappa\gamma_0} \quad (33)$$

let

$$u_0 = \gamma_0 u^* \quad \text{and} \quad w_0 = \kappa \gamma_0 w^* \quad (34)$$

such that  $u_0$  and  $w_0$  can be expressed as

$$u_0 = \frac{\gamma_0(1 - \kappa) - 1 + \sqrt{1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2}}{2}, \quad (35)$$

$$w_0 = \frac{\gamma_0(\kappa - 1) - 1 + \sqrt{1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2}}{2}. \quad (36)$$

Let  $u_0 = (q + \sqrt{s})/2$  and  $w_0 = (r + \sqrt{s})/2$ , where  $q = \gamma_0(1 - \kappa) - 1$ ,  $r = \gamma_0(\kappa - 1) - 1$  and  $s = 1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2$ . By multiplying  $u_0$  and  $w_0$  with  $(q - \sqrt{s})/2$  and  $(r - \sqrt{s})/2$  respectively, we obtain

$$w_0 = \frac{2\kappa\gamma_0}{(1 + (1 - \kappa)\gamma_0 + \sqrt{1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2})}, \quad (37)$$

$$u_0 = \frac{2\gamma_0}{1 - \gamma_0(1 - \kappa) + \sqrt{1 + 2\gamma_0(1 + \kappa) + \gamma_0^2(\kappa - 1)^2}}, \quad (38)$$

Equations (35, 37) and (36, 38) indicates that  $w_0$  and  $u_0$  are related as follows

$$w_0 = \frac{\kappa\gamma_0}{1 + u_0}, \quad u_0 = \frac{\gamma_0}{1 + w_0} \quad (39)$$

Let  $v_0 = \frac{u^*\gamma_0}{\gamma_0} = \frac{u_0}{\gamma_0} = \frac{1}{1 + w_0}$  and  $c_0 = \frac{\kappa\gamma_0 w^*}{\gamma_0} = \frac{w_0}{\gamma_0} = \frac{\kappa}{1 + u_0}$ , we can express the first two terms of (28) as

$$\begin{aligned} \kappa \log(1 + \gamma_0 u^*) + \log(1 + \kappa\gamma_0 w^*) \\ = \kappa \log(1 + u_0) + \log(1 + w_0) \end{aligned} \quad (40)$$

The third term of (28) can be expressed as

$$\kappa\gamma_0 u^* w^* = \kappa\gamma_0 w^* \cdot \frac{u^*\gamma_0}{\gamma_0} = w_0 v_0 = \frac{w_0}{1 + w_0} \quad (41)$$

$$\kappa\gamma_0 u^* w^* = \gamma_0 u^* \cdot \frac{\kappa w^*\gamma_0}{\gamma_0} = u_0 c_0 = \frac{u_0 \kappa}{1 + u_0} \quad (42)$$

such that

$$\begin{aligned} \kappa\gamma_0 u^* w^* &= \frac{w_0}{2(1 + w_0)} + \frac{u_0 \kappa}{2(1 + u_0)} \\ &= \frac{1}{2} - \frac{1}{2(1 + w_0)} + \frac{\kappa}{2} - \frac{\kappa}{2(1 + u_0)}. \end{aligned} \quad (43)$$

Therefore, the SE per-receive antenna in (28) can be express as;

$$\begin{aligned} C_c &= \kappa \log(1 + u_0) + \log(1 + w_0) \\ &\quad - \left( \frac{1}{2} - \frac{1}{2(1 + w_0)} + \frac{\kappa}{2} - \frac{\kappa}{2(1 + u_0)} \right), \\ &= -\frac{1}{2} + \log(1 + w_0) + \frac{1}{2(1 + w_0)} \\ &\quad + \kappa \left( -\frac{1}{2} + \log(1 + u_0) + \frac{1}{2(1 + u_0)} \right). \end{aligned} \quad (44)$$

The system sum capacity  $C = Mr.W.C_c$

$$\begin{aligned} C &= WMr \left( -\frac{1}{2} + \log(1 + w_0) + \frac{1}{2(1 + w_0)} \right) + \\ &\quad Wkt \left( -\frac{1}{2} + \log(1 + u_0) + \frac{1}{2(1 + u_0)} \right). \end{aligned} \quad (45)$$

The per-cell sum-rate of the symmetric model is then given by  $C_P = C/M$

$$\begin{aligned} C_P &= \frac{W}{2 \ln 2} \left[ r \left( -1 + 2 \ln(1 + w_0) + \frac{1}{(1 + w_0)} \right) \right. \\ &\quad \left. + \bar{t} \left( -1 + 2 \ln(1 + u_0) + \frac{1}{(1 + u_0)} \right) \right], \\ &= \frac{W}{2 \ln 2} (C_r + C_t), \end{aligned} \quad (46)$$

which is the same equation as (11) and, thus, our proof is concluded.

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