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Structure of the string R-matrix

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Abstract

By requiring invariance directly under the Yangian symmetry, we rederive Beisert's quantum R-matrix, in a form that carries explicit dependence on the representation labels, the braiding factors, and the spectral parameters u_i. In this way, we demonstrate that there exist a rewriting of its entries, such that the dependence on the spectral parameters is purely *of difference form*. Namely, the latter enter only in the combination $u_1 - u_2$, as indicated by the shift automorphism of the Yangian. When recasted in this fashion, the entries exhibit a cleaner structure, which allows to spot new interesting relations among them. This permits to package them into a practical tensorial expression, where the non-diagonal entries are taken care by explicit combinations of symmetry algebra generators.

1 Introduction

Integrability in AdS/CFT [1, 2, 3, 4] (see [5] for reviews) is strictly related to the existence of an R-matrix [6], which is a solution of the quantum Yang-Baxter equation [7, 8], and satisfies the Hopf-algebraic analog of crossing symmetry [9, 10]. This R-matrix exhibits a certain infinite-dimensional Yangian-type symmetry [11], based on the central extension of the Lie superalgebra $\mathfrak{psu}(2|2)$, plus an additional Yangian generator of $\mathfrak{u}(2|2)$ signature [12, 13]. A universal form of the R-matrix would be desirable, particularly in view of its potential use in studying finite-size effects [14, 15, 16]. Such universal R-matrix is still unknown, mainly due to the fact that the dual Coxeter number of the algebra vanishes, and therefore traditional mathematical techniques do not straightforwardly apply [17, 18]. Nevertheless, huge progress has come in several aspects [19, 20, 21, 22, 23], like in the study of the correspondent classical r-matrix [24, 25, 13, 23, 26], in deriving higher representations [27], and in giving the Yangian an almost canonical form [28].

The way the R-matrix was originally derived [6] was by using the Lie superalgebra symmetry in the fundamental representation. The combination of the peculiar features of the tensor products of two fundamentals [7], together with the non-trivial braiding of the coproduct [10, 4, 8] allowed to fix all entries up to an overall scalar factor [29, 30]. Yangian symmetry was explicitly discovered only later [11], initially hardly visible due to the consistency relations that mix together all the parameters in the game. In higher representations the Yangian may impose useful conditions [26], but the Yang-Baxter equation could probably be sufficient to fix the R-matrix [27].

Were it not for the algebraic peculiarity mentioned above, then probably, as it happens for more traditional Yangians, the Lie superalgebra symmetry would not be enough, and the invariance under the Yangian would determine much of the form of the R-matrix. Since the Yangian carries the spectral parameter u, this normally gives the familiar rational R-matrices, depending only on the difference $u_1 - u_2$ due to a constant shift automorphism $u \rightarrow u + c$. From there, investigation of the universal R-matrix is easier.

In this paper, we want to exploit this privileged viewpoint that the Yangian gives on the R-matrix, and imagine we were to fix the latter by purely using the (first level) Yangian generators. Our strategy is to try to avoid explicit use of the constraints, and to carry on the various parameters almost until the end as if they were unrelated. In particular, the relevant Yangian still possesses the shift automorphism [11, 28]. Our calculation produces explicit expressions for the entries, where the dependence on the spectral parameters u_i is indeed purely *of difference form*, namely $u_1 - u_2$, as expected for the above reasons. Moreover, the other parameters (representation labels, braiding factors) are also recognizable, which helps in finding new relations among the entries, and tells us much about the combination of generators that is going to produce them. Upon imposing the constraints, we exactly recover the original quantum R-matrix found by Beisert. Tempted by this new perspective, we also provide a tensorial repackaging of the entries which further clarifies their hidden structure, and might serve as a device to investigate the properties of the universal R-matrix.

2 Structural relations

In this section, we will follow the approach of rederiving the quantum R-matrix, by using the fundamental representation not of the Lie superalgebra, as it is traditionally done, but rather using from the very beginning the first Yangian generators. The correspondent coproducts are given in [11, 28]. In doing this, we will leave explicit the dependence on the representation labels, the braiding factors, and the spectral parameters, as if they were not related to each other. This will allow us to trace them back through the entries of the R-matrix. Of course, only when they satisfy the familiar relations one can find a solution, therefore it is impossible to really terminate the process without imposing those relations. Nevertheless, it is possible to proceed further enough to reach a satisfactory form for all the entries, which we report below. At that point, there is no need to continue, and we can just read the entries we obtain. As a check, we can then verify that, on the constraints, we recover Beisert's quantum R-matrix.

Let us first remind the basic definition of the labels of the fundamental representation:

$$
\begin{aligned}
&\mathfrak{R}^a{}_b|\phi^c\rangle = \delta^c_b|\phi^a\rangle - \frac{1}{2}\delta^a_b|\phi^c\rangle \ , \quad \mathfrak{L}^\alpha{}_\beta|\phi^\gamma\rangle = \delta^\gamma_\beta|\phi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta|\phi^\gamma\rangle \ ,\\
&\mathfrak{Q}^\alpha{}_a|\phi^b\rangle = a\,\delta^b_a|\psi^\alpha\rangle \ , \quad \mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle = b\,\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b\rangle \ ,\\
&\mathfrak{S}^a{}_\alpha|\phi^b\rangle = c\,\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\beta\rangle \ , \quad \mathfrak{S}^a{}_\alpha|\psi^\beta\rangle = d\,\delta^\beta_\alpha|\phi^a\rangle \ .
\end{aligned} \tag{1}
$$

The labels satisfy $ad - bc = 1$. The central (braiding) operator $\mathfrak{U}$ has eigenvalue U, and u will be the spectral parameter.

Let us write the R-matrix in the familiar $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant fashion:

$$
\begin{aligned}
\mathcal{R}_{12}|\phi^a_1\phi^b_2\rangle &= R^{12}_{12}|\phi^a_1\phi^b_2\rangle + R^{12}_{21}|\phi^b_1\phi^a_2\rangle + R^{12}_{34}\,\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\alpha_1\psi^\beta_2\rangle \ ,\\
\mathcal{R}_{12}|\psi^\alpha_1\psi^\beta_2\rangle &= R^{34}_{34}|\psi^\alpha_1\psi^\beta_2\rangle + R^{34}_{43}|\psi^\beta_1\psi^\alpha_2\rangle + R^{34}_{12}\,\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^a_1\phi^b_2\rangle \ ,\\
\mathcal{R}_{12}|\phi^a_1\psi^\beta_2\rangle &= R^{13}_{13}|\phi^a_1\psi^\beta_2\rangle + R^{13}_{31}\psi^\beta_1\phi^a_2\rangle \ ,\\
\mathcal{R}_{12}|\psi^\alpha_1\phi^b_2\rangle &= R^{31}_{31}|\psi^\alpha_1\phi^b_2\rangle + R^{31}_{13}|\phi^b_1\psi^\alpha_2\rangle \ .
\end{aligned} \tag{2}
$$

Choosing the overall normalization to be such that $R^{33}_{33} = R^{34}_{34} + R^{34}_{43} = U_2/U_1$, the expressions for the entries which we obtain by imposing invariance under the Yangian are given by the following formulas:

$$
\begin{aligned}
R^{31}_{31} &= U_2\,\frac{u_1 - u_2}{u_1 - u_2 - a_1 d_1 + a_1 b_1 (c_2/a_2)\,U_2^2} \ ,\\
R^{31}_{13} &= R^{31}_{31}\,\frac{1}{U_2}\,\frac{(a_2 d_1 - b_1 c_2\,U_2^2)}{u_1 - u_2} \ ,\\
R^{13}_{13} &= \frac{U_2^2}{U_1}\,\frac{u_1 - u_2}{u_1 - u_2 + b_2 c_2 - a_2 b_2 (d_1/b_1)\,U_1^2} \ ,\\
R^{13}_{31} &= R^{13}_{13}\,\frac{1}{U_2}\,\frac{(-b_2 c_1 + a_1 d_2\,U_2^2)}{u_1 - u_2} \ ,\\
R^{12}_{34} &= R^{12}_{12}\,\frac{1}{U_2}\,\frac{(-c_1 a_2 + a_1 c_2\,U_2^2)}{u_1 - u_2 - 1} \ ,\\
R^{12}_{21} &= R^{12}_{12}\,\frac{1}{u_1 - u_2}\left(1 + \frac{1}{U_2^2}\,\frac{(c_1 a_2 - a_1 c_2\,U_2^2)(d_1 b_2 - b_1 d_2\,U_2^2)}{u_1 - u_2 - 1}\right) \ ,\\
R^{34}_{12} &= R^{34}_{34}\,\frac{1}{U_2}\,\frac{(d_1 b_2 - b_1 d_2\,U_2^2)}{u_1 - u_2 + 1} \ ,\\
R^{34}_{43} &= R^{34}_{34}\,\frac{1}{u_1 - u_2}\left(-1 + \frac{1}{U_2^2}\,\frac{(c_1 a_2 - a_1 c_2\,U_2^2)(d_1 b_2 - b_1 d_2\,U_2^2)}{u_1 - u_2 + 1}\right) \ ,
\end{aligned} \tag{3}
$$

$$R_{11}^{11} = R_{12}^{12} + R_{21}^{12} = \frac{U_2^2}{U_1^2}\,\frac{u_1 - u_2 - b_1 c_1 + a_1 b_1 (d_2/b_2)\,U_2^2}{u_1 - u_2 + b_2 c_2 - a_2 b_2 (d_1/b_1)\,U_1^2}\ , \tag{4}$$

As one can immediately see, the dependence of the entries on the spectral parameters u_i is purely *of difference form*, as indicated by the fact that the Yangian is invariant under an automorphism that shifts the spectral parameter by a constant.

One can check that these expressions indeed reproduce the entries of Beisert's R-matrix [6]. It is sufficient to recall the parametrization

$$a = \sqrt{g}\gamma\ , \quad b = \sqrt{g}\frac{\alpha}{\gamma}\left(1 - \frac{x^+}{x^-}\right)\ , \quad c = \sqrt{g}\frac{i\gamma}{\alpha x^+}\ , \quad d = \sqrt{g}\frac{x^+}{i\gamma}\left(1 - \frac{x^-}{x^+}\right)\ , \tag{5}$$

together with $U = \sqrt{x^+/x^-}$ and the relation $x^+ + 1/x^+ - x^- - 1/x^- = i/g$, and that $u = (ig/2)\,(x^+ + x^-)(1 + 1/x^+ x^-)$. Then, we compare with the familiar formulas

$$\begin{aligned}
\mathcal{R}_{12}|\phi_1^a \phi_2^b\rangle &= \frac{1}{2}(A_{12} - B_{12})|\phi_1^a \phi_2^b\rangle + \frac{1}{2}(A_{12} + B_{12})|\phi_1^b \phi_2^a\rangle + \frac{1}{2}C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_1^\alpha \psi_2^\beta\rangle\ ,\\
\mathcal{R}_{12}|\psi_1^\alpha \psi_2^\beta\rangle &= -\frac{1}{2}(D_{12} - E_{12})|\psi_1^\alpha \psi_2^\beta\rangle - \frac{1}{2}(D_{12} + E_{12})|\psi_1^\beta \psi_2^\alpha\rangle - \frac{1}{2}F_{12}\epsilon^{\alpha\beta}\epsilon_{ab}|\phi_1^a \phi_2^b\rangle\ ,\\
\mathcal{R}_{12}|\phi_1^a \psi_2^\beta\rangle &= G_{12}|\phi_1^a \psi_2^\beta\rangle + H_{12}|\psi_1^\beta \phi_2^a\rangle\ ,\\
\mathcal{R}_{12}|\psi_1^\alpha \phi_2^b\rangle &= L_{12}|\psi_1^\alpha \phi_2^b\rangle + K_{12}|\phi_1^b \psi_2^\alpha\rangle\ ,
\end{aligned} \tag{6}$$

with the functions $A_{12}, B_{12}, \ldots$ given by

$$\begin{aligned}
A_{12} &= \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\ , \quad B_{12} = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1 - 2\frac{1 - 1/x_1^+ x_2^-}{1 - 1/x_1^+ x_2^+}\frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\right)\ ,\\
C_{12} &= \frac{2\gamma_1\gamma_2 U_2}{\alpha x_1^+ x_2^+}\frac{1}{1 - 1/x_1^+ x_2^+}\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\ ,\\
D_{12} &= -\frac{U_2}{U_1}\ , \quad E_{12} = -\frac{U_2}{U_1}\left(1 - 2\frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^- x_2^-}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\right)\ ,\\
F_{12} &= -\frac{2\alpha(x_1^+ - x_1^-)(x_2^+ - x_2^-)}{\gamma_1\gamma_2 U_1 x_1^- x_2^-}\frac{1}{1 - 1/x_1^- x_2^-}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\ ,\\
G_{12} &= \frac{1}{U_1}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\ , \quad H_{12} = \frac{\gamma_1 U_2}{\gamma_2 U_1}\frac{x_2^+ - x_2^-}{x_2^- - x_1^+}\ ,\\
L_{12} &= U_2\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\ , \quad K_{12} = \frac{\gamma_2}{\gamma_1}\frac{x_1^+ - x_1^-}{x_2^- - x_1^+}\ .
\end{aligned} \tag{7}$$

By making use of (3), (4), (5), (7), it is possible to verify that the two R-matrices indeed exactly match.

As an interesting remark, we notice that in formulas (3) and (4) it is possible to *formally* switch off the braiding ($U_i \to 1$), and also send everywhere b_i, c_i to zero[1], and a_i, d_i to 1. This

[1] Undetermined expressions like b_1/b_2 are sent to 1.

should correspond to scatter two representations of $\mathfrak{gl}(2|2)$. Indeed, if one does that, one finds out that the R-matrix becomes formally equal to

$$\mathcal{R}_{12}^{lim} = \frac{u_1 - u_2}{u_1 - u_2 - 1} \left(1 \otimes 1 + \frac{1}{u_1 - u_2} \sum_{i,j=1}^{4} (-)^j \, E_{ij} \otimes E_{ji} \right) , \tag{8}$$

where E_{ij} are the unit matrices with all zeroes but 1 in position (i, j), and bosonic and fermionic indices are altogether numbered from 1 to 4. The combination $\sum_{i,j=1}^{4} (-)^j \, E_{ij} \otimes E_{ji}$ is the quadratic Casimir of $\mathfrak{gl}(2|2) \otimes \mathfrak{gl}(2|2)$, which already emerged from one-loop gauge theory [2, 14], and from the classical analysis of [24]. The R-matrix which we obtain by this formal procedure is recognized this time as the quantum (Yang-type) R-matrix of $\mathfrak{gl}(2|2)$ in the fundamental representation, and it is well known to solve the Yang-Baxter equation. We have thus find a consistent practical way of tuning on and off the central extensions (which are proportional to b and c respectively) in the formula for the R-matrix.

3 Tensorial repackaging

We would like to exploit here the rewriting achieved in the previous section, in order to express the whole R-matrix in a more compact tensorial form. This will be far from enough to be able to provide the universal R-matrix, but it will be a useful exercise which may teach us where some of the terms are likely to come from. We (re)write here below some of the important identities that one obtains from the expressions given in the previous section:

$$\begin{aligned}
R_{13}^{31} &= R_{31}^{31} \, \frac{1}{U_2} \, \frac{(a_2 d_1 - b_1 c_2 \, U_2^2)}{u_1 - u_2} \; , \\
R_{31}^{13} &= R_{13}^{13} \, \frac{1}{U_2} \, \frac{(-b_2 c_1 + a_1 d_2 \, U_2^2)}{u_1 - u_2} \; , \\
R_{34}^{12} &= R_{12}^{12} \, \frac{1}{U_2} \, \frac{(-c_1 a_2 + a_1 c_2 \, U_2^2)}{u_1 - u_2} \, \frac{u_1 - u_2}{u_1 - u_2 - 1} \; , \\
R_{12}^{34} &= R_{34}^{34} \, \frac{1}{U_2} \, \frac{(d_1 b_2 - b_1 d_2 \, U_2^2)}{u_1 - u_2} \, \frac{u_1 - u_2}{u_1 - u_2 + 1} \; , \\
R_{21}^{12} &= R_{12}^{12} \, \frac{1}{U_2^2} \, \frac{(c_1 a_2 - a_1 c_2 \, U_2^2)}{u_1 - u_2} \, \frac{(d_1 b_2 - b_1 d_2 \, U_2^2)}{u_1 - u_2} \, \frac{u_1 - u_2}{u_1 - u_2 - 1} + R_{12}^{12} \, \frac{1}{u_1 - u_2} \; , \\
R_{43}^{34} &= R_{34}^{34} \, \frac{1}{U_2^2} \, \frac{(c_1 a_2 - a_1 c_2 \, U_2^2)}{u_1 - u_2} \, \frac{(d_1 b_2 - b_1 d_2 \, U_2^2)}{u_1 - u_2} \, \frac{u_1 - u_2}{u_1 - u_2 + 1} - R_{34}^{34} \, \frac{1}{u_1 - u_2} \; .
\end{aligned} \tag{9}$$

This seems to suggest a not too unfamiliar matrix pattern. After trying different possibilities, one can see that a relatively simple tensorial repackaging of the R-matrix that is able to produce these relations can be found:

$$\mathcal{R}_{12} = \mathcal{R}_1^F \, \mathcal{R}_2^F \, \mathcal{R}_{12}^H + \mathcal{R}_1^B \, \mathcal{R}_1^H + \mathcal{R}_2^B \, \mathcal{R}_2^H \; , \tag{10}$$

where

$$\mathcal{R}_1^F = 1\otimes 1 + \frac{1}{u_1-u_2}\left(\mathfrak{Q}^1{}_1\otimes\mathfrak{S}^1{}_1\mathfrak{U} + \mathfrak{Q}^2{}_2\otimes\mathfrak{S}^2{}_2\mathfrak{U} - \mathfrak{S}^1{}_1\otimes\mathfrak{Q}^1{}_1\mathfrak{U}^{-1} - \mathfrak{S}^2{}_2\otimes\mathfrak{Q}^2{}_2\mathfrak{U}^{-1}\right) ,$$
$$\mathcal{R}_2^F = 1\otimes 1 + \frac{1}{u_1-u_2}\left(\mathfrak{Q}^1{}_2\otimes\mathfrak{S}^2{}_1\mathfrak{U} + \mathfrak{Q}^2{}_1\otimes\mathfrak{S}^1{}_2\mathfrak{U} - \mathfrak{S}^1{}_2\otimes\mathfrak{Q}^2{}_1\mathfrak{U}^{-1} - \mathfrak{S}^2{}_1\otimes\mathfrak{Q}^1{}_2\mathfrak{U}^{-1}\right) ,$$

$$\mathcal{R}_1^B = \frac{\Pi_B\otimes\Pi_B}{1-u_1+u_2} + \frac{1}{u_1-u_2}\left(\mathfrak{R}^1{}_2\otimes\mathfrak{R}^2{}_1 + \mathfrak{R}^2{}_1\otimes\mathfrak{R}^1{}_2\right) , \qquad \mathcal{R}_1^H = R_{12}^{12}\; 1\otimes 1 ,$$
$$\mathcal{R}_2^B = \frac{\Pi_F\otimes\Pi_F}{1+u_1-u_2} - \frac{1}{u_1-u_2}\left(\mathfrak{L}^1{}_2\otimes\mathfrak{L}^2{}_1 + \mathfrak{L}^2{}_1\otimes\mathfrak{L}^1{}_2\right) , \qquad \mathcal{R}_2^H = R_{34}^{34}\; 1\otimes 1 ,$$

$$\mathcal{R}_{12}^H = R_{13}^{13}\;\Pi_B\otimes\Pi_F\; + R_{31}^{31}\;\Pi_F\otimes\Pi_B\; + \mathcal{R}_B^H , \tag{11}$$

Π_B and Π_F being the projectors onto the bosonic and fermionic subspaces respectively, $\Pi_B = diag\{1,1,0,0\}$ and $\Pi_F = diag\{0,0,1,1\}$. The only non-zero entries of $\mathcal{R}_B^H$ are

$$\mathcal{R}_B^H|\phi_1^a\phi_2^a\rangle = \left(R_{11}^{11} + \frac{R_{12}^{12}}{u_1-u_2-1}\right)|\phi_1^a\phi_2^a\rangle , \qquad \mathcal{R}_B^H\,\epsilon_{ab}|\phi_1^a\phi_2^b\rangle = R_{12}^{12}\,\frac{u_1-u_2}{u_1-u_2-1}\,\epsilon_{ab}|\phi_1^a\phi_2^b\rangle ,$$
$$R_B^H|\psi_1^\alpha\psi_2^\alpha\rangle = \left(R_{33}^{33} - \frac{R_{34}^{34}}{u_1-u_2+1}\right)|\psi_1^\alpha\psi_2^\alpha\rangle , \qquad \mathcal{R}_B^H\,\epsilon_{\alpha\beta}|\psi_1^\alpha\psi_2^\beta\rangle = R_{34}^{34}\,\frac{u_1-u_2}{u_1-u_2+1}\,\epsilon_{\alpha\beta}|\psi_1^\alpha\psi_2^\beta\rangle . \tag{12}$$

This has to be seen as a practical tool to organize the entries, useful for a subsequent attempt to find the universal R-matrix. The Cartan part is traditionally the most complicated to reproduce, as shown by formulas (3), (4). But also the root part seems to be a little different from standard cases, even when taking into account that in the fundamental representation the roots are nilpotent, and therefore what appears to be linear could well be the first term of a series expansion. Nevertheless, in the classical limit, the formula seems to reproduce the non-diagonal part of the classical r-matrix. In any case, no conclusions are to be drawn at the moment on the persistence of this kind of pattern at the universal level, and the device is purely practical.

4 Conclusions

In this paper we have shown that there exist a way of rewriting Beisert's quantum R-matrix, such that the dependence on the spectral parameters is purely of difference form. This was hidden before by the complicated relations connecting the spectral parameter with the representation labels and the braiding factors, and can be achieved by using the Yangian symmetry. In particular, this dependence is expected from the shift automorphism that the Yangian possesses. The remaining dependence on the two representations, which is responsible for the ultimately observed non-difference pattern, is encoded in a precise sequence of representation labels and braiding factors. This, in turn, is to be expected, if the R-matrix has to come from some universal combination of symmetry generators. We provide some hints at this structure, suggested by novel relations among the entries, which in this new rewriting are easier to spot.

When comparing with the classical r-matrix analysis of [13], one notices that a suitable readjusting of the classical variables can be used to abandon a pure $u_1 - u_2$ dependence on the classical spectral parameters, in order to reach a convenient normalization for the extra generator to insert in the classical Yangian. The quantum version of this generator [12, 13] has not yet been canonically embedded in the quantum Yangian, and it is hard to judge on its role at the moment. In other words, it is possible that one may have to eventually introduced a few uncoupled u_i's in order to get a truly universal expression. What we have shown here is that, nevertheless, *there exist* at least one rewriting where the spectral parameters come in differences, all the rest being taken care of by recognizable algebraic quantities.

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