A DISCRETE-TIME STATE-SPACE MODEL WITH WAKE INTERFERENCE FOR STABILITY ANALYSIS OF FLEXIBLE AIRCRAFT

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Abstract. This paper investigates the coupled aeroelastic and flight dynamics stability of flexible lightweight aircraft. The aerodynamics are modelled by the discrete-time unsteady vortex lattice method, which can capture the large deformations of the lifting surfaces, and includes 3-D effects and in-plane motions. A geometrically-exact composite beam formulation is used to model the nonlinear flexible-body dynamics, including rigid-body motions, and the equations are accommodated to discrete-time formulation. The governing equations are linearised around an equilibrium configuration, which can be highly deformed, performing a small perturbation analysis and assuming a frozen aeroelastic geometry. The resulting framework is a monolithic discrete-time state-space formulation, which provides a powerful tool for the stability boundary prediction of a flexible vehicle through a direct generalized eigenvalue analysis. It offers increased fidelity as compared to traditional tools, and at very low computational cost. As a suitable test case to illustrate the capabilities of this approach, the flutter of a T-tail is examined. In addition, previous open-loop results are extended in order to assess wake interference effects on flexible aircraft dynamics.

NOMENCLATURE

\begin{align*}
\Delta b & \quad \text{spanwise dimension of aerodynamic vortex ring} \\
C & \quad \text{global tangent damping matrix} \\
C^B_a & \quad \text{coordinate transformation matrix, from } a \text{ to } B \\
\Delta c & \quad \text{chordwise dimension of aerodynamic vortex ring} \\
K & \quad \text{global tangent stiffness matrix} \\
M & \quad \text{global tangent mass matrix} \\
Q & \quad \text{global vector generalized forces in the structural problem} \\
\vec{R} & \quad \text{local position vector along the beam reference line} \\
s & \quad \text{arc length along reference line of the beam elements} \\
t & \quad \text{physical time} \\
\mathbf{u} & \quad \text{input} \\
\vec{v} & \quad \text{inertial translational velocity of the body-fixed frame, } a \\
\vec{V} & \quad \text{inertial translational velocity at a beam location} \\
\mathbf{w} & \quad \text{vector of non-vortical induced velocity at all collocation points} \\
X_{a_m} & \quad \text{coordinates of the aerodynamic lattice in the aerodynamic frame} \\
x & \quad \text{state} \\
y & \quad \text{output}
\end{align*}

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Greek letters
\( \alpha \) angle of incidence
\( \eta \) vector of displacements and rotations at all finite element nodes
\( \Gamma \) circulation strength of vortex ring
\( \xi \) relative position vector within a beam section / aerofoil
\( \Phi \) Euler angles for the orientation of the body-fixed frame, \( a \)
\( \Psi \) Cartesian Rotation Vector (CRV)
\( \bar{\omega} \) inertial angular velocity of the body-fixed frame, \( a \)
\( \bar{\Omega} \) inertial angular velocity at a beam location

Subscript
\( a \) body-fixed (global) reference frame
\( a_m \) aerodynamic frame of surface \( m \)
\( A \) aerodynamic
\( B \) deformed (material) reference frame
\( b \) undeformed reference frame; bound, corresponding to lifting surface
\( FB \) Flexible-Body, including elastic and rigid-body DoF
\( f \) flutter
\( G \) inertial (Earth) reference frame
\( i,j,k \) chordwise, spanwise, and total panel counters
\( R \) rigid-body
\( S \) structure (elastic)
\( w \) wake
\( \infty \) free stream conditions

Superscript
\( \bullet^n \) time step \( n \)
\( \dot{} \) time derivatives, \( \frac{d}{dt} \)
\( \prime \) spatial derivatives, \( \frac{d}{dx} \)
\( \overline{\cdot} \) cross-product operator
\( \bar{\cdot} \) global matrix including relevant transformations for each grid corner point/node

1 INTRODUCTION

New multidisciplinary design tools, able to capture all couplings between aeroelastic and flight dynamic responses, are needed for the analysis of Very Flexible Aircraft (VFA) and for the design of appropriate controllers. A substantial research effort has been carried out towards this goal in recent years in the development of analysis tools for High-Altitude Long-Endurance (HALE) vehicles [1–6] using geometrically-nonlinear beam models for the structural dynamics and 2-D strip theory for the aerodynamics. Three-dimensional aerodynamic effects may be important however in the interference between lifting surfaces and wakes (typically between the wake shed by the main wing and the tail) or in the wing tip effects. The latter is typically considered through corrections on the lift curve, but those are only valid in small ranges of reduced frequencies [7].

A key driving factor in the design of VFA is their response to gusts and atmospheric turbulence, as exposed by the Helios mishap [8]. Those are large vehicles flying at low speeds that are likely to satisfy potential flow assumptions, hence rendering panel methods excellent candidates for the description of the aerodynamics: without incurring in
excessive computational costs, they incorporate 3-D effects, interference and wake modelling. As a drawback, panel methods are not appropriate when the wing enters stalled conditions, to predict drag, or at very low Reynolds number. There are however some methods to correct the method by incorporating ad hoc stall models [9,10] and using the lift-drag polar to predict viscous drag [11]. Panel methods are also not adequate at very high altitudes due to dominance of rarefied flow effects, but the critical load conditions will occur during climb and descend operations in the lower atmosphere. Despite their limitations, panel methods, and in particular the Doublet Lattice Method [12,13], are still the workhorse in aeroelastic analysis and design.

The goal of our ongoing research is to develop a unified model for the aeroelastic and flight dynamics analysis of a highly flexible aircraft, including geometrically-nonlinear deformations and wake interference. For this purpose, the Unsteady Vortex Lattice Method (UVLM) has been coupled with a Geometrically-Exact Composite Beam (GECB) model. In the latter, the primary structural variables are the local displacements and the Cartesian Rotation Vector (CRV), and the rigid-body dynamics of the unsupported structure are captured by the translational and angular velocities of a body-fixed reference frame. This has resulted in a framework for Simulation of High-Aspect Ratio Planes. SHARP provides a modular simulation toolbox for the dynamics of flexible aircraft, allowing static aeroelastic analyses, aircraft trimming, stability studies in frequency domain, and fully nonlinear time-marching simulations. Previously [14], we exercised SHARP in the open-loop response of a flexible HALE aircraft using a loosely-coupled approach to march the equations in time.

The main focus of this paper is to describe a monolithic framework for the linear dynamic stability analysis. A discrete-time state-space formulation is presented, where the coupling between aerodynamic and beam models is analytical. For that purpose, the aerodynamic equations are linearised performing a small perturbation analysis and assuming a frozen geometry (Section 2). This approach is similar to the traditional linearisation performed in the Doublet Lattice Method (DLM). The small perturbation analysis is performed around a deformed configuration (usually a trimmed aircraft, which can be subject to geometrically nonlinear deformations), and hence both lifting surfaces and wakes can be non-planar. Besides, the linearised form of the UVLM accounts for in-plane motions, a cumbersome problem in the DLM [15]. The GECB equations are linearised and discretised in time using a Newmark-β method, in order to accommodate them to discrete-time state-space formulation (Section 3).

Aerodynamics and flexible-body dynamics are considered as independent input-output modules in these sections, and the coupling between them is introduced in Section 4. Finally, the generalized eigenvalue problem that determines the stability boundaries of the system is formulated in Section 5. Numerical results will look first at wake proximity effects on the open-loop response of a flexible aircraft (Section 6.1), extending previous studies [14]. Next, the Goland wing is used as an example to verify the numerical implementation of the newly proposed methodology for stability analysis (Section 6.2), and we finally focus on the effect of in-plane motions on T-tail flutter to exemplify the capabilities of the monolithic formulation (Section 6.3).

The discrete-time state-space formulation of the UVLM presented in this paper was already outlined by Hall [16], who used it for modal aeroelastic analysis and reduced order
modelling. Zhao and Hu [17] also sought a reduced order tool for the study of an aerofoil, but in this case, they converted the discrete-time state-space aerodynamic equations into continuous time. A great deal of insight for the derivation of the linearised state-space equations in this paper has been obtained from the formulation of the UVLM presented by Stanford and Beran [18].

2 DISCRETE-TIME STATE-SPACE AERODYNAMIC MODEL

The Unsteady Vortex Lattice Method [19] is an efficient computational technique to solve 3-D potential flow problems about lifting surfaces. Elementary (singularity) solutions are distributed over lifting surfaces and the non-penetration boundary condition is imposed at a number of control (collocation) points, leading to a system of algebraic equations. The UVLM is based on thin-aerofoil approximation and both the elementary solutions and the collocation points are placed over the instantaneous (i.e., deformed) mean surface. The elementary solution is the so-called vortex ring, i.e., a quadrilateral element composed by discrete vortex segments in a closed loop, along which the circulation strength, \( \Gamma_k \), is constant (see Figure 1). As the surface moves along its flight path, a force-free wake is obtained as part of the solution procedure, also represented by vortex rings. The wake is formed, shed, convected and allowed to roll up according to the local flow velocity.

2.1 Non-penetration boundary condition

At discrete time step \( n + 1 \), the non-penetration boundary condition can be formulated as

\[
A_b \Gamma_b^{n+1} + A_w \Gamma_w^{n+1} = w^{n+1},
\]

where \( \Gamma_b \) and \( \Gamma_w \) are the vectors with the circulation strengths in the bound and wake vortex rings, respectively; \( A_b = A_b(X_{am}^{n+\epsilon}) \) and \( A_w = A_w(X_{am}^{n+\epsilon} , X_{wm}^{n+\epsilon}) \) are the wing-wing and wing-wake aerodynamic influence coefficient matrices, and \( X_{am} \) includes the coordinates of the bound vortex rings expressed in the aerodynamic frame of lifting surface \( m \). Elements of these matrices are obtained by projecting the velocity computed using the Biot-Savart law over the vortex ring normal vector, and the time at which they are evaluated within the current time step, determined by \( 0 \leq \epsilon \leq 1 \), depends on the
integration scheme. The right hand side in Eq. (1) is the vector of normal components of the non-vortical induced velocities at the collocation points, and may include gust-induced velocities, wing deformations and rigid-body motions. In the absence of external disturbances, it can be written as

\[ \mathbf{w}^{n+1} = W_b \dot{\mathbf{X}}_{a_m}^{n+1}, \]  

(2)

where \( W_b (\mathbf{X}_{a_m}^{n+1}) \) is a matrix that projects the local velocities along the normal direction to the panels. More details for \( A_b, A_w \) and \( W_b \) can be found in the book by Katz and Plotkin [19]. For a purely aerodynamic problem, the motion of the lifting surfaces will be prescribed and \( \mathbf{X}_{a_m}^{n+1} \) and \( \dot{\mathbf{X}}_{a_m}^{n+1} \) will be part of the inputs to the system. If a coupled aeroelastic and flight dynamics problem is considered, \( \mathbf{X}_{a_m}^{n+1} \) and \( \dot{\mathbf{X}}_{a_m}^{n+1} \) will be a function of the structural and rigid-body states, as described in Section 4.1. Variables \( \mathbf{X}_{a_m} \) and \( \dot{\mathbf{X}}_{a_m} \) may also include other inputs such as wing control surfaces.

### 2.2 Wake propagation

At each time step, a new row of vortex rings will be shed into the wake from the trailing edge of each lifting surface. In addition to this, the existing wake will displace following the local flow velocity (the free wake model). This is written as

\[ \mathbf{X}_w^{n+1} = C_b \mathbf{X}_{a_m}^{n+1} + C_w \mathbf{X}_w^n + \int_{t^n}^{t^{n+1}} \mathbf{V}(t) \, dt, \]  

(3)

where \( \mathbf{X}_w \) is the vector with the wake grid-coordinates, thus defining the wake shape. The vector \( \mathbf{V} \) in this equation includes the local (inertial) flow velocities at the grid points of the wake mesh. \( C_b \) and \( C_w \) in Eq. (3) are very sparse constant matrices that update the position of the prescribed wake: the former closes the newly shed wake panel with the trailing edge of the lifting surface, satisfying the Kutta condition, while the latter preserves the wake of the previous time step unchanged.

For a free wake, the vortex-ring cornerpoints need also to be moved according to the local flow velocity. As shown in Eq. (3), time-integration is necessary to determine the location of the rolled-up wake. This is typically done using an explicit one-step Euler method, but other higher order schemes have also been proposed in the literature [20, 21].

The propagation equations for the wake circulation can be written in discrete time as

\[ \Gamma_w^{n+1} = B_b \Gamma_b^n + B_w \Gamma_w^n, \]  

(4)

where \( B_b \) and \( B_w \) are very sparse constant matrices which account for Kelvin’s circulation theorem (that enforces the condition for wake shedding at the trailing edge) and Helmholtz’s vortex theorem (in the convection of the wake). They map the wake circulation of the previous time step to the current one. As the influence of the wake already decays very rapidly as it is convected away from the lifting surface (due to the Biot-Savart law), no dissipation model [22, 23] has been implemented in this work. Benefiting from this diminishing influence, the wake is truncated several chords downstream to reduce computational expense.
2.3 Aerodynamic loads

Finally, once the distribution of vorticity has been obtained at each time step, the inviscid aerodynamic loads can be computed using the unsteady Bernoulli equation. The induced drag, $D$, is aligned with the local instantaneous velocity, and the lift, $L$, acts along the local vector perpendicular to the local velocity and projected over the normal to the panel. These loads are given by

$$L^n = \rho_\infty G_c \left[ (U_i \Delta_i + U_j \Delta_j) \Gamma^n_b + \dot{\Gamma}_b^n \right], \quad (5)$$

$$D^n = \rho_\infty \left[ -U^* \Delta_i \Gamma^n_b + G_s \dot{\Gamma}_b^n \right], \quad (6)$$

where $\Delta_{i(j)}$ are matrices filled with 1 and $-1$ in the correct positions in order to account for adjacent panels; matrix $G_c = G_c(X_{am}^n)$ and $G_s = G_s(X_{am}^n)$ are diagonal matrices, with element $(k,k)$ given by $(G_c)_{k,k} = (\Delta b \Delta c \cos \alpha^n)_k$, $(G_s)_{k,k} = (\Delta b \Delta c \sin \alpha^n)_k$, and $\Gamma^n_b(X_{am}^n, \dot{X}_{am}^n)$ represents the angle of incidence of vortex ring $k$ at time step $n$; $U_{i(j)} = U_{i(j)}(\Gamma^n_w, X^n_{am}, \dot{X}_w^n)$ and $U^* = U^*(\Gamma^n_b, \Gamma^n_w, X^n_{am}, X^n_w)$ are diagonal matrices that store the induced velocities projected over the relevant vectors and weighted using the corresponding geometric properties. As before, the exact definitions can be found in Ref. [19].

2.4 Linearised Equations for Stability Analysis

In order to perform linear stability analysis on the full aircraft (Section 5), the discrete-time UVLM aerodynamic equations will be coupled with the flexible-body dynamics (Section 3). The linearisation of the aerodynamics is carried out through a small perturbation analysis, under the following assumptions:

- The deformations around the deformed aircraft are small, and as a consequence, the non-penetration boundary condition, Eq. (1), can be enforced at the statically-deformed reference geometry. As a result, the dependencies on $X_{am}$ are neglected, except for matrix $W_b$, Eq. (2), since this is necessary in order to account for local angle of incidence changes as the lifting surface deforms.
- The linearisation is carried out in body-fixed axes. This is the natural description to obtain the aerodynamic forces, but differs from the usual definition of stability axes used in flight dynamics analysis [24].
- The aerodynamic forces will be obtained at the equilibrium configuration. This simplification effectively converts them into dead loads with matrix $G_c$ and $G_s$ assumed not to vary with $X_{am}$.
- Wake rollup around the reference is neglected even though it can be accounted for to accurately trim the aircraft. This assumption reduces the UVLM to a prescribed-wake method, and under this approximation it is not necessary to keep track of the wake shape after trim. The aerodynamic states that fully define the UVLM are only circulation strength distributions and the derivative of the bound circulation\(^1\). Note that the wake is prescribed in this case, but it does not need to be flat, and it will be shed from the deformed lifting surface. As the wake is frozen, Eq. (3) is neglected and $X_w$ will correspond to the shape at the reference (equilibrium) configuration.

\(^1\)An alternative procedure would have been to discretise $\dot{\Gamma}_b$ in which case this term would not need to be kept explicitly. The current approach was found however more appropriate since it is easier to implement and the penalty in the number of states is not significant.
Under these simplifications, the aerodynamic states and inputs that fully define the system are, respectively

\[ x_A = \begin{pmatrix} \Gamma_b \\ \Gamma_w \\ \dot{\Gamma}_b \end{pmatrix}, \quad \text{and} \quad u_A = \begin{pmatrix} X_{am} \\ \dot{X}_{am} \end{pmatrix}. \tag{7} \]

Using a mid-point integration scheme for the derivatives of the bound circulations, the UVLM linearised propagation equations are obtained as

\[ A_b \Delta \Gamma_b^{n+1} + A_w \Delta \Gamma_w^{n+1} = \left( \frac{\partial W_b}{\partial X_{am}} \dot{X}_{am} \right) \Delta X_{am}^{n+1} + W_b \Delta \dot{X}_{am}^{n+1}, \tag{8} \]

\[ \Delta \Gamma_w^{n+1} = B_b \Delta \Gamma_b^n + B_w \Delta \Gamma_w^n; \tag{9} \]

\[ \Delta \Gamma_b^{n+1} - \frac{1}{2} \Delta t \Delta \dot{\Gamma}_b^n = \Delta \Gamma_b^n + \frac{1}{2} \Delta t \Delta \dot{\Gamma}_b^n, \tag{10} \]

together with the output equations for the aerodynamic loads

\[ \Delta L^n = \rho_\infty G_c [(U_i \Delta_i + U_j \Delta_j) \Delta \Gamma_b^n + \left( \frac{\partial U_i}{\partial \Gamma_w} \Delta_i \Gamma_b + \frac{\partial U_j}{\partial \Gamma_w} \Delta_j \Gamma_b \right) \Delta \dot{X}_{am}^n], \tag{11} \]

\[ \Delta D^n = \rho_\infty \left[ - \left( \frac{\partial U^*}{\partial \Gamma_b} \Delta \Gamma_b + U^* \Delta_i \right) \Delta \Gamma_b^n - \left( \frac{\partial U^*}{\partial \Gamma_w} \Delta \Gamma_b \right) \Delta \Gamma_w^n + G_s \Delta \dot{\Gamma}_b^n \right]. \tag{12} \]

The discrete-time equations that govern the UVLM can then be written in compact form as

\[ E_A \Delta x_A^{n+1} + F_A \Delta u_A^{n+1} = G_A \Delta x_A^n + H_A \Delta u_A^n, \quad \Delta y_A^n = J_A \Delta x_A^n + K_A \Delta u_A^n, \tag{13} \]

where the outputs \( y_A \) are the aerodynamic loads, Eqs. (11-12). This form of the equation, with matrix \( E_A \) premultiplying the updated value on the state variable was preferred to the canonical discrete-state form \( (x^{n+1} = Ax^n) \) because this is the natural expression obtained from the UVLM. This will yield a generalized eigenvalue problem in the determination of the stability characteristics. Also, it will be shown in Section 4.1, that in aeroelastic/flight dynamics problems the inputs to the UVLM will be expressed as a function of the elastic and rigid-body output states, i.e., \( y_S \) and \( y_R \), respectively, as

\[ \Delta u_A^n = p^n_{AS} P_{AS} \Delta y_S^n + p^n_{AR} P_{AR} \Delta y_R^n. \tag{14} \]

where capital \( P \) matrices represent the actual mapping and lower-case \( p^n \) scalar values will depend on the parameters of the integration scheme.
The slender structures in the high-aspect-ratio wing aircraft will be modelled as composite beams, using a finite-element solution methodology based on those of Hodges [25] and Patil et al. [2], but using displacements and the Cartesian Rotation Vector (CRV) as primary degrees of freedom [26]. This solution process was described in Ref. [7] and only a brief summary is presented here (Section 3.1). As the aerodynamic equations obtained from the UVLM (Section 2.4) are formulated in discrete time, the equations of motion of the flexible-body are formulated in a suitable manner and discretised in time for a monolithic coupling (Section 3.2).

Figure 2 sketches the description that will be followed here. The deformation of the structure is described in terms of a moving, body-fixed reference coordinate system \( a \) which moves with respect to an inertial frame \( G \) by the translational velocity of its origin, \( v_a(t) \), and its rotational velocity, \( \omega_a(t) \). Subscripts are used to indicate the coordinate system in which the components of the vectors are expressed. The orientation of the global frame \( a \) with respect to the inertial frame \( G \) is given by the coordinate transformation matrix, \( C^{Ga}(t) \), and will be parametrised using Euler angles, \( \Phi(t) \).

**3.1 Geometrically-Exact Beam Dynamics using the Cartesian Rotation Vector**

The local orientation of the beam cross sections is defined by their local coordinate systems, \( B \), in the deformed (or current) configuration. The orientation of cross-sections at each point in the current configuration is described in terms of finite rotations from the global reference frame \( a \) and the local deformed frame \( B \) using the CRV, \( \Psi(s, t) \). The corresponding coordinate transformation matrix will be \( C^{Ba}(s, t) \). The deformation of the reference line going from the undeformed state \( \{ \vec{R}(s, 0), \vec{B}_i(s, 0) \} \) to the current state \( \{ \vec{R}(s, t), \vec{B}_i(s, t) \} \) will be described by the following force and moment strains [25]

\[
\begin{align*}
\gamma(s, t) &= C^{Ba}(s, t)R_a'(s, t) - C^{Ba}(s, 0)R_a'(s, 0), \\
\kappa(s, t) &= K_B(s, t) - K_B(s, 0),
\end{align*}
\]

where \( \gamma \) and \( \kappa \) are the beam strains and \( (\bullet)' \) is the derivative with respect to the arclength \( s \). The curvature will be computed from the corresponding CRV, \( \Psi \), for the rotation from...
frame $a$ to frame $B$, as $K_B = T(\Psi)\Psi'$, where $T(\Psi)$ is the tangential operator. The inertial properties of the reference line will be determined by its translational and angular inertial velocities at each location defined by the arclength $s$, given, respectively, as

$$V_B = C^{Ba}\left(R_a + \omega_a R_a + v_a\right),$$
$$\Omega_B = T(\Psi) \dot{\Psi} + C^{Ba}\omega_a.$$  \hspace{1cm} (16)

The dynamics of the beam can be then be obtained in the (moving) body-attached reference frame $a$ from standard means (Hamilton’s principle). After a finite-element discretisation of the displacements and rotations, the Geometrically-Exact Composite Beam (GECB) equations can be written in discrete form as \cite{7}

$$M(\eta)\left\{\ddot{\eta} \atop \dot{\nu}\right\} + Q_{ggr}(\eta, \dot{\eta}, \nu) + Q_{stiff}(\eta) = Q_{ext}(\eta, \dot{\eta}, \nu, \Phi, u_{FB}),$$

where $\eta$ is the vector of all nodal displacements and rotations and $\nu = [v_a \omega_a]^T$. Euler angles, $\Phi$, have been included to account for the influence of changes in attitude on the external forces for flight dynamic stability. Matrix $M$ is the discrete mass matrix and $Q_{ggr}, Q_{stiff}$ and $Q_{ext}$ are the discrete gyroscopic, stiffness, and external generalized forces, respectively. The states that fully determine the flexible-body dynamics are therefore

$$x_{FB} = [x_S^T \mid x_R^T]^T = [\eta^T \dot{\eta}^T \mid \nu^T \Phi^T]^T,$$  \hspace{1cm} (18)

The input vector in Eq. (17), $u_{FB} = [u_S^T \mid u_R^T]^T$ includes the dependency of the external loads with any other variable in the most general form. In particular, for the aeroelastic and flight dynamics stability analysis, $u_{FB}$ will depend on the aerodynamic states, Eq. (7), which appear on the output aerodynamic loads of the UVLM, Eq. (13). The linearised (incremental) form of the GECB equations around a dynamic equilibrium is given as

$$M(\eta)\left\{\Delta \ddot{\eta} \atop \Delta \dot{\nu}\right\} + C(\eta, \dot{\eta}, \nu)\left\{\Delta \dot{\eta} \atop \Delta \dot{\nu}\right\} + K(\eta)\left\{\Delta \eta \atop 0\right\} = \Delta Q_{ext},$$

where $M$, $C$, and $K$ are the tangent mass, damping and stiffness matrices.

### 3.2 Discrete-Time Formulation of the Linearised Equations

For a discrete-time state-space monolithic description of the aeroelastic and flight dynamics coupled problem, the linearised flexible-body states are discretised in time using the Newmark-$\beta$ method. The elastic and rigid-body states are time-marched as

$$\Delta \eta^{n+1} = \Delta \eta^n + \Delta t \Delta \dot{\eta}^n + \left(\frac{1}{2} - \beta_2\right) \Delta t^2 \Delta \ddot{\eta}^n + \beta_2 \Delta t^2 \Delta \dot{\eta}^{n+1},$$
$$\Delta \dot{\eta}^{n+1} = \Delta \dot{\eta}^n + (1 - \gamma_2) \Delta t \Delta \ddot{\eta}^n + \gamma_2 \Delta t \Delta \dot{\eta}^{n+1},$$
$$\Delta \nu^{n+1} = \Delta \nu^n + (1 - \gamma_2) \Delta t \Delta \dot{\nu}^n + \gamma_2 \Delta t \Delta \ddot{\nu}^{n+1},$$
$$\Delta \Phi^{n+1} = \Delta \Phi^n + \frac{1}{2} \Delta t \Delta \dot{\Phi}^n + \frac{1}{2} \Delta t \Delta \ddot{\Phi}^{n+1},$$

9
\[ \Delta t \] represents the time step, and the tuning parameters \( \gamma_1, \gamma_2, \) and \( \beta_2 \) are chosen for the desired accuracy and stability properties of the Newmark-\( \beta \) integration scheme. Note that Euler angles are marched following a mid-point integration scheme, analogous to \( \dot{\varPhi} \) in Eq. (10), and that for small perturbations it is \( \Delta \dot{\varPhi} = \Delta \omega_a \). The values of \( \dot{j} \) and \( \dot{\nu} \) at time steps \( n \) and \( n + 1 \) are obtained as a function of the states, \( x_{FB} \), and inputs, \( u_{FB} \), from Eq. (19), which can be alternatively written as

\[
\begin{bmatrix}
\Delta \eta^{n+\epsilon} \\
\Delta \nu^{n+\epsilon}
\end{bmatrix} = -M^{-1} \left[ (C + C_{ext}) \begin{bmatrix}
\Delta \eta^{n+\epsilon} \\
\Delta \nu^{n+\epsilon}
\end{bmatrix} + (K + K_{ext}) \begin{bmatrix}
\Delta \eta^{n+\epsilon} \\
0
\end{bmatrix} \\
+ \frac{\partial Q_{ext}}{\partial \Phi} \Delta \Phi^{n+\epsilon} + \frac{\partial Q_{ext}}{\partial u_{FB}} \Delta u_{FB}^{n+\epsilon}
\right],
\]

\[
M \begin{bmatrix}
\Delta \eta \\
\Delta \nu
\end{bmatrix} + (C - C_{ext}) \begin{bmatrix}
\Delta \eta \\
\Delta \nu
\end{bmatrix} + (K - K_{ext}) \begin{bmatrix}
\Delta \eta \\
0
\end{bmatrix} = \frac{\partial Q_{ext}}{\partial \Phi} \Delta \Phi + \frac{\partial Q_{ext}}{\partial u_{FB}} \Delta u_{FB},
\]

where the following definitions have been used

\[
K_{ext} = - \begin{bmatrix}
\frac{\partial Q_{ext}}{\partial \eta} & 0
\end{bmatrix},
\]

\[
C_{ext} = - \begin{bmatrix}
\frac{\partial Q_{ext}}{\partial \eta} & \frac{\partial Q_{ext}}{\partial \nu}
\end{bmatrix}.
\]

The flexible-body model equations presented in this section can be wrapped up as

\[
E_{FB} \begin{bmatrix}
\Delta x_s \\
\Delta x_R
\end{bmatrix}^{n+1} + F_{FB} \begin{bmatrix}
\Delta u_s \\
\Delta u_R
\end{bmatrix}^{n+1} = G_{FB} \begin{bmatrix}
\Delta x_s \\
\Delta x_R
\end{bmatrix}^n + H_{FB} \begin{bmatrix}
\Delta u_s \\
\Delta u_R
\end{bmatrix}^n,
\]

\[
\begin{bmatrix}
\Delta y_s \\
\Delta y_R
\end{bmatrix}^n = \begin{bmatrix}
\Delta x_s \\
\Delta x_R
\end{bmatrix}^n.
\]

Note that the output equation, Eq. (25), in the coupling with the UVLM is the full deformed state. It is then transformed to suitable inputs for the UVLM through Eq. (14). Finally, the elastic and rigid-body inputs will be expressed as a function of the aerodynamic outputs as

\[
\Delta u_s^n = p_{SA}^n P_A \Delta y_A^n, \quad \text{and} \quad \Delta u_R^n = p_{RA}^n P_R \Delta y_A^n,
\]

where, as in the UVLM, capital \( P \) matrices represent the actual mapping and lowercase \( p^n \) scalar values will depend on the tuning parameters of the Newmark-\( \beta \) and the time-step, Eq. (20).

4 FLUID/STRUCTURE COUPLING

The previous flexible-body and unsteady aerodynamic models will be used to represent the complete dynamics of a flexible air vehicle. The aerodynamic equations (Section 2.4) and the flexible-body equations (Section 3.2) have been outlined as independent modules, and their interdependency has been formulated as given by certain inputs, \( u \), and outputs, \( y \). In this section, the necessary relationships are obtained, both for a loosely-coupled
nonlinear solution methodology and for a linearised monolithic discrete-time state-space description of the problem. As the structural model is based on beams (curves in space) and the aerodynamic lattice is distributed over a lifting surface (see Figure 3), a mapping procedure is required between both meshes.

First of all, the structural displacements and velocities are transformed to the aerodynamic model in Section 4.1. The linearised version of the position and velocity mappings are necessary for the non-penetration boundary condition in the UVLM, Eq. (8), since the angle of attack and velocities of the aerodynamic grid are determined from the elastic and rigid-body states. This dependency on velocities will also affect the aerodynamic loads (lift, to be precise), Eq. (11), which is mapped to the flexible-body model under the assumptions of frozen geometry, as elucidated in Section 4.2.

Figure 3: Full representation of flexible aircraft: beam-like structure, vortex-ring lattice and rigid-body motions.

4.1 Mapping Structural Displacements and Velocities to the Aerodynamic Model

Firstly, displacements and rotations of the beam nodes, $R_a$ and $\Psi$, and the corresponding velocities, $\dot{R}_a$ and $\dot{\Psi}$, have to be transformed to deformations and velocities of the grid points of the aerodynamic lattice, $X_{am}$ and $\dot{X}_{am}$. Vortex-ring cornerpoints and collocation points are expressed in the aerodynamic coordinate system, $a_m$, defined independently but rigidly linked to the body-fixed global one, $a$, and can be mapped to and from the other reference frames through transformation matrices. It would be possible to include camber deformations on this approach [27], but it will be assumed here that aerofoils remain rigid under wing deformations.

In the initial configuration a mapping between the structural nodes and the aerodynamic grid can be defined, as illustrated in Figure 4(a). For the sake of simplicity, the finite-element discretisation of the beam coincides with the spanwise aerodynamic grid, and hence vortex ring cornerpoints and beam nodes lie along the same rigid aerofoil at each spanwise station – note, however, that cambered aerofoils are allowed and that non-coinciding meshes have also been implemented. The variable $\xi$ measures the distance between a vortex ring cornerpoint and the relevant node, and it will be expressed in the nodal material frame, i.e., $\xi_B$. This quantity will remain constant under the assumption
of rigid aerofoils, and as a consequence, it is possible to determine the aerodynamic grid in the lifting surface aerodynamic frame of reference, $a_m$, at any deformed configuration of the member.

For a given vortex ring cornerpoint\(^2\), the following transformation is defined at time step $n$:

$$X_{an} = C_{an} \left[ R_a + C^{anB} (\Psi^n) \xi_B \right], \quad (27)$$

where the coordinate transformation between the body-fixed global coordinate system, $a$, and the aerodynamic frame, $a_m$, is given by the constant $C^{anm}$ matrix. Each cornerpoint of the vortex rings is updated analogously. In turn, the positions of the collocation points are obtained through interpolation of the corresponding four vortex ring cornerpoints. The linearised mapping between grids is required to account for the change in angle of attack in the non-penetration boundary condition, Eq. (8), and it is given by

$$\Delta X_{am} = C^{anm} \left[ \Delta R_a - C^{anB} \xi_B T \Delta \Psi \right]. \quad (28)$$

The transformation for the velocities is

$$\dot{X}_{an} = C^{anm} \left[ v_a + \tilde{\omega}_a R_a + \dot{R}_a + C^{anB} (\Psi^n) \tilde{\Omega}_{B} \xi_B \right], \quad (29)$$

where the local inertial angular velocity, $\Omega_B^n$, was given in Eq. (16). As in the case of positions, the velocities of the aerodynamic vortex ring cornerpoints are obtained using Eq. (29), and the velocities of the collocation points are obtained through interpolation.

The velocity mapping is necessary for the stability analysis of the full aircraft, since it provides the dependency of the non-penetration boundary condition, Eq. (1), and of the

---

\(^2\)Note that variables are not bold since the transformation corresponds to a single vortex ring cornerpoint. The full matrix will be filled with the relevant transformation for each cornerpoint and node.
 aerodynamic loads, Eqs. (5-6), on the elastic and rigid-body states. The linearised velocity mapping can be computed as

\[
\Delta \dot{X}^n_{am} = C^{a_{m}a} \left\{ \omega_a \Delta R^u_a + C^a R^b \left[ (\tilde{\xi}_B \Omega_B) - \tilde{\xi}_B A_1 (\Psi, \dot{\Psi}) - \tilde{\xi}_B C^{Ba} \omega_a T \right] T \Delta \Psi^n + \Delta \dot{R}^u_a - C^a B \tilde{\xi}_B T \Delta \dot{\Psi}^n \right. \\
+ \Delta v^n_a - \left. \left( \tilde{R}_a + C^a B \tilde{\xi}_B C^{Ba} \right) \Delta \omega_a^n + \frac{\partial C^{aG}}{\partial \Phi} v_G \Delta \Phi^n \right\},
\]

(30)

where \( A_1 (\Psi, \dot{\Psi}) \) is given in Ref. [26]. The last term of the equation, \( (\partial C^{aG} / \partial \Phi) v_G \Delta \Phi^n \), does not appear explicitly on Eq. (29), and it vanishes in the linearisation process. However, it is necessary to incorporate it, since it represents the influence of the free stream speed with perturbations on the aircraft attitude.

Eqs. (28) and (30) relate the inputs to the aerodynamic module \( \Delta u_A \) and the outputs of flexible-body equations, \( \Delta y_S \) and \( \Delta y_R \), Eq. (14).

4.2 Mapping External Loads to the Structural Model

Next, the mapping of external forces and moments to the flexible-body equations is tackled. For stability analysis, these will include aerodynamics and weight.

First of all, it is necessary to transform the inviscid aerodynamic loads computed in Eqs. (5-6) to forces and moments acting upon the beam nodes – any estimation of the viscous drag will be included in the inputs to the system. For that purpose, it is assumed that they can be approximated by isolated aerodynamic loads applied in the centre of the leading segment of each vortex-ring.

The aerodynamics forces act on the plane defined by the instantaneous inertial velocity of the vortex ring (computed at the collocation point), and the normal vector of the vortex ring. The pressure differential acts along the normal vector, but due to the inability of the UVLM to account for the leading edge suction, only the component normal to the inertial velocity is considered. In turn, the induced drag acts along the vector defined by the local instantaneous velocity. As a result, the inviscid aerodynamic forces at vortex ring \( k \), expressed in the three axis defined by the inertial (ground) frame of reference, \( G \), are given by

\[
(F^n_{aero})_k = C^{GA} \begin{pmatrix} D \\ 0 \\ L \end{pmatrix}^n,
\]

(31)

with \( L^n \) and \( D^n \) the lift and the induced drag of the panel, as given by Eqs. (5-6); \( C^{GA} = C^{GA} (X^n_{am}, \Phi^n) \) is the coordinate transformation matrix between the ground frame
and the local aerodynamic frame linked to the wing aerofoils (determined by the local instantaneous inertial velocity and the normal vector the wing).

These forces are then lumped into the nodes of the deformed beam, splitting them between adjacent nodes as illustrated in Figure 4(b) – note that this mapping will give rise to moments acting upon the corresponding nodes. Once the resulting nodal forces and moments have been computed, \((F_G, M_G)\), they are transformed to the body-fixed \(a\) frame, in order to be consistent with the flexible-body equations, Eq. (17). These operations can be summarized as

\[
\begin{bmatrix}
F_a \\
M_a
\end{bmatrix}^n = \bar{C}^{aG} \begin{bmatrix}
F_G \\
M_G
\end{bmatrix}^n = \bar{C}^{aG} \chi_{\text{vr} \rightarrow \text{no}} F_{\text{aero}}^n, \tag{32}
\]

where \(\bar{C}^{aG}\) is a block diagonal matrix, being each block given by the corresponding coordinate transformation matrix from the inertial to the body-fixed frame, \(C^{aG} = C^{aG}(\Phi^n)\); \(\chi_{\text{vr} \rightarrow \text{no}} = \chi_{\text{vr} \rightarrow \text{no}} (R^n_a, \Psi^n, X^n_{am})\) is a very sparse matrix that lumps the forces acting on the aerodynamic lattice vortex rings, \(F_{\text{aero}}\), into forces and moments applied on the beam nodes expressed in the inertial frame, \((F^n_G, M^n_G)\).

The aerodynamic loads will also affect the rigid-body motions of the aircraft. The forces and moments acting at the origin of the body-fixed frame of reference, \((f_a, m_a)\), are obtained by integrating the nodal values and can be expressed as

\[
\begin{bmatrix}
f_a \\
m_a
\end{bmatrix}^n = \chi_{\text{no} \rightarrow \text{bf}} \begin{bmatrix}
F_a \\
M_a
\end{bmatrix}^n = \chi_{\text{no} \rightarrow \text{bf}} \bar{C}^{aG} \chi_{\text{vr} \rightarrow \text{no}} F_{\text{aero}}^n, \tag{33}
\]

where \(\chi_{\text{no} \rightarrow \text{bf}} = \chi_{\text{no} \rightarrow \text{bf}} (R^n_a, \Psi^n)\) is the matrix that computes the resultant forces and moments integrating contributions of all nodes of the discretisation. Hence, the generalized aerodynamic forces can be written as

\[
Q_a^n = \begin{bmatrix}
Q_{\text{aero}}^S \\
Q_{\text{aero}}^R
\end{bmatrix}^n, \tag{34}
\]

with

\[
Q_{\text{aero}}^S = \bar{C}^{aG} \chi_{\text{vr} \rightarrow \text{no}} F_{\text{aero}}^n \quad \text{and} \quad Q_{\text{aero}}^R = \chi_{\text{no} \rightarrow \text{bf}} \bar{C}^{aG} \chi_{\text{vr} \rightarrow \text{no}} F_{\text{aero}}^n. \tag{35}
\]

These generalized aerodynamic forces will be part of the generalized external forces presented in Eq. (17), which will also encompass any other applied loads. For stability analysis, a frozen geometry will be assumed (see Section 2.4), and thus aerodynamic loads will only depend on velocities. As a result, lumping matrices \(\chi_{\text{vr} \rightarrow \text{no}}\) and \(\chi_{\text{no} \rightarrow \text{bf}}\) are constant. In addition, lift and drag are considered to act along the same directions as in the equilibrium configuration, and hence \(C^{GA}\) remains unchanged, as well. The change in attitude of the full vehicle is however included through matrix \(\bar{C}^{aG} = C^{aG}(\Phi)\).

Apart from aerodynamic loads only the weight will be considered relevant for stability analysis, and in particular, its dependency with the orientation of the aircraft, introduced
through Euler angles, $\Phi$. Under these approximations, the external loads in the flexible-body equations, Eq. (19) are written as

$$\Delta Q_{ext}^n = \frac{\partial Q_{aero}}{\partial x_S} \Delta x_S^n + \frac{\partial Q_{aero}}{\partial x_R} \Delta x_R^n + \frac{\partial Q_{aero}}{\partial x_A} \Delta x_A^n + \frac{\partial g}{\partial \Phi} \Delta \Phi^n. \quad (36)$$

Furthermore, through the velocity mapping given in Eq. (30), the derivatives of the aerodynamic loads with respect to any of the flexible-body states can be written as

$$\frac{\partial Q_{aero}}{\partial x_S} = \frac{\partial Q_{aero}}{\partial \dot{X}_a} \frac{\partial \dot{X}_a}{\partial x_S}, \quad \text{and} \quad \frac{\partial Q_{aero}}{\partial x_R} = \frac{\partial Q_{aero}}{\partial \dot{X}_a} \frac{\partial \dot{X}_a}{\partial x_R}. \quad (37)$$

The equations presented in this subsection provide the interface between the output states of the UVLM, $\Delta y_A$, and the inputs to the flexible-body model, $\Delta u_S$ and $\Delta u_R$, which can be cast into a compact form as in Eq. (26).

### 5 MONOLITHIC DISCRETE-TIME STATE-SPACE FORMULATION FOR LINEAR STABILITY ANALYSIS

The discrete-time state-space formulation that couples the aerodynamics and flexible-body dynamics can be used to obtain directly the linear stability characteristics of the vehicle. For that purpose, the linearised dynamic equations are used to define a generalised eigenvalue problem. These equations have been obtained by performing a small perturbation analysis around an equilibrium configuration, and hence all relevant derivatives are evaluated at these conditions. For stability, only the homogeneous part of the equations is required.

All the dependencies are cast into a simple expression by merging Eqs. (13-14) and (24-26). The system propagation is written as

$$E_A \Delta x_A^{n+1} + F_A \Delta u_A^{n+1} = G_A \Delta x_A^n + H_A \Delta u_A^n, \quad (38)$$

$$E_{FB} \begin{bmatrix} \Delta x_S \\ \Delta x_R \end{bmatrix}^{n+1} + F_{FB} \begin{bmatrix} \Delta u_S \\ \Delta u_R \end{bmatrix}^{n+1} = G_{FB} \begin{bmatrix} \Delta x_S \\ \Delta x_R \end{bmatrix}^n + H_{FB} \begin{bmatrix} \Delta u_S \\ \Delta u_R \end{bmatrix}^n, \quad (39)$$

with the following mapping relationships

$$\Delta u_A^n = p_{AS}^n P_{AS} \Delta x_S^n + p_{AR}^n P_{AR} \Delta x_R^n, \quad (40)$$

$$\Delta u_S^n = p_{SA}^n P_{SA} J_A \Delta x_A^n + p_{SA}^n P_{SA} K_A \Delta u_A^n, \quad (41)$$

$$\Delta u_R^n = p_{RA}^n P_{RA} J_A \Delta x_A^n + p_{RA}^n P_{RA} K_A \Delta u_A^n. \quad (42)$$

where it has been taken into account that the GECB output and state vector are identical. Note again that the interfaces given in Eqs. (40-42) correspond to time step $n$. The mapping relations for time step $n + 1$ are identical, but the factor that multiplies them is different according to the integration scheme. Figure 5 displays the flow chart of the coupled homogeneous model.

Equations (38-42) define a generalised eigenvalue problem of the form
Figure 5: Flow chart of the aeroelastic and flight dynamics coupled stability solver.

\[ Y_{sys} \Delta \mathbf{x}^{n+1} = Z_{sys} \Delta \mathbf{x}^n, \]  

where the state vector that completely determines the system is

\[
\mathbf{x} = \begin{bmatrix} \mathbf{x}_A^T & \mathbf{x}_S^T & \mathbf{x}_R^T \end{bmatrix}^T = \begin{bmatrix} \Gamma_b^T & \Gamma_w^T & \dot{\Gamma}_b^T \mid \dot{\eta}^T & \eta^T \mid \dot{v}_a^T & \omega_a^T & \Phi^T \end{bmatrix}^T.
\]

The entries to matrices \( Y_{sys} \) and \( Z_{sys} \) depend on the equilibrium conditions and are given in the Appendix. The stability of the system is determined via a direct eigenvalue analysis on Eq. (43). For the system to be stable, \( |z_i| \leq 1, \forall i \), where \( |z_i| \) represents the magnitude of the \( i^{th} \) discrete time eigenvalue, and equality corresponds to the neutral stability boundary. Alternatively, the discrete time eigenvalues can be transformed to the more familiar continuous time counterparts \( \lambda_i \), given by \( z_i = e^{\lambda_i \Delta t} \). In this case, a positive real part of any of the \( \lambda_i \)'s will imply instability.

This formulation provides a very powerful tool for the stability boundary prediction. As all derivatives have been obtained analytically, finite differences are not necessary. The typical size of the problem for a model flexible aircraft is of the order of 2000 states, which can be solved in a few seconds on a single-processor computer. Note that the GECB model was used to introduce nonlinear static equilibrium conditions, but the stability analysis could be also based on the linear normal modes of the structure.

6 NUMERICAL EXAMPLES

The equations described in previous chapters have been implemented in a new simulation framework, codenamed SHARP (Simulation of High Aspect-Ratio Planes). SHARP was previously exercised in the open-loop response of a flexible aircraft [14]. In this section, earlier time-domain results are first extended investigating wake interference effects (Section 6.1). However, the main focus of this paper is the stability analysis tool, so its numerical implementation is tested next. The Goland wing is used for code verification purposes (Section 6.2), and finally, T-tail flutter is investigated to evince the applicability of the outlined methodology (Section 6.3).
6.1 Wake interference on open-loop response

First of all, the loosely-coupled equations are solved for the open-loop response of a model flexible aircraft. A representative HALE aircraft was described in Ref. [14], and its open-loop dynamics were explored for sinusoidal elevator inputs. In this section, the interference effects of the wake shed by the main wing and the tail are briefly discussed.

The aircraft consists of a large aspect ratio flexible wing (1 m chord and 16 m semi-span), a rigid fuselage and a rigid T-tail. The influence of wake proximity is examined for longitudinal motions, so the fin of the T-tail is modelled as a beam, hence neglecting its aerodynamics. The detailed characteristics of the aircraft are given in Ref. [14].

In order to study the influence between the wake shed by the main wing and the tail, a sinusoidal elevator deflection is commanded around the trim configuration, and the time-domain response of aircraft is monitored. The vehicle is free to follow the trajectory that will result from the elevator input, and no particular path has been sought for.

First of all, the aircraft is trimmed for steady level longitudinal flight at a given velocity. The trimming of the aircraft is performed using the nonlinear solver in SHARP, but then the time-marching solution is obtained by linearising the equations with respect to this equilibrium condition, assuming a frozen wake. The trim configuration is found using a Newton’s method for three inputs, namely angle of attack, $\alpha$, thrust per propeller, $T$, and elevator deflection, $\delta$, balancing lift/weight, thrust/induced drag and cancelling out pitching moments. In this case $V_\infty = 25$ m/s has been chosen, for which the corresponding trim values are $\alpha_{\text{trim}} = 4.56$ deg, $T_{\text{trim}} = 2.42$ N, and $\delta_{\text{trim}} = 9.85$ deg. At these conditions, and including gravity effects only on the payload, the tip deflection of the main wing is $z_{\text{tip}} = 1.06$ m.

Around this trim configuration, the elevator perturbation will be given by $\delta = \delta_{\text{trim}} + \delta^* \sin(\omega t)$, where the oscillation frequency is $\omega = 5$ rad/s, close enough to the first bending mode of the main wing, $\omega_1 = 5.1$ rad/s. In Ref. [14] different values of $\delta^*$ were studied, making the wake pass close to the tail but avoiding direct collisions. Here, in order to expose the interference due to wake proximity, the case of $\delta^* = \pm \delta_{\text{trim}}$ is investigated in more detail.

Figure 6 depicts the pitch rate for these two cases during the first two periods of elevator perturbation, with and without wake interference. For the case without wake interference, the influence of the wake shed by the main wing over the tail is switched off – note that the influence over the shedding surface itself, i.e., the main wing, is accounted for nonetheless. Above the pitch rate, snapshots of the flight trajectory every half-period are presented for both values of $\delta^*$, including the wake of the main wing, which as historian of the flow, represents the path followed by the root of the main wing. The top case corresponds to $\delta^* = \delta_{\text{trim}}$, which leads to a negative pitching at the beginning of the motion due to an increased force on the tail. The snapshots below represent $\delta^* = -\delta_{\text{trim}}$, where the aircraft pitches up first.

It can be observed that the interference has an effect on the pitch rate of the aircraft when the wake gets close to the tail. When the wake and the tail are at a distance roughly beyond a chord length, the results with and without interference agree very well, except for the first quarter period, which is caused by the difference in trim conditions.
However, when the wake approaches the tail a significant discrepancy can be seen due to the downwash created by the wake. Not including interference effects leads to an overestimation of the pitching rate, and the error reaches values of up to 20%.

Note finally that after these two oscillatory cycles a residual pitch rate persists, and hence, the perturbation on the elevator yields a resultant motion that departs from the trimmed state.

\[ \delta^* = \delta_{\text{trim}} \]

\[ \delta^* = - \delta_{\text{trim}} \]

\( \delta_{\text{trim}} \) (no interference)

\( \delta_{\text{trim}} \) (no interference)

Figure 6: Impact of wake proximity on pitch rate of a model HALE aircraft. Snapshots of the cross-section of the vehicle, including the wake-trajectory of the main wing for \( \delta^* = \delta_{\text{trim}} \) (top), and \( \delta^* = - \delta_{\text{trim}} \) (centre), and pitch rate evolution for different values of \( \delta^* \) with and without interference (bottom), as a function of time during two periods of elevator sinusoidal oscillation.

6.2 Flutter of the Goland wing

The method for evaluation of dynamic stability is first verified estimating the flutter onset point of the Goland wing, a stiff cantilever wing for which the relevant properties can be obtained in Ref. [28].

Figure 7 presents the linear stability plot for the this wing, using symmetry conditions and neglecting gravity. In this case, the flutter speed is computed around the undeformed configuration, at zero angle of attack. Air density is assumed to be \( \rho_\infty = 1.020 \text{ kg/m}^3 \), which corresponds to an altitude of 1500 m. The stability diagram indicates that the first torsion mode becomes unstable at a velocity \( V_f = 166 \text{ m/s} \), which is due to a torsion-bending coupling. This is in very good agreement with other estimations (see Ref. [14] for a complete comparison).

It should be noted that very low free stream velocities were not included in Figure 7, since they lead to a prohibitively large number of wake circulation states in the state-space model. This is because the chordwise size of the wake panels in the UVLM is determined by \( \Delta c_w = V_\infty \Delta t \), and hence the number of wake panel rows needed to retain a representative wake length is inversely proportional to the free stream velocity. This poses no real problem for highly flexible structures, since the dominant modes will have relatively small frequencies, allowing larger time steps and less wake states. However, for more rigid structures with high vibration frequencies, the analysis must resolve in the small time steps required to capture them. This challenge can be overcome by taking into account that flutter is actually defined by a dynamic pressure, \( q_f = \frac{1}{2} \rho_\infty V_f^2 \). Hence, by
Figure 7: Root locus for the undeformed Goland wing, computed with velocity increments of $\Delta V_\infty = 1$ m/s, starting at $V_\infty = 35$ m/s. Flutter occurs at 166 m/s. $[\rho_\infty = 1.020 \text{ kg/m}^3 \text{ and } \alpha = 0 \text{ deg}]$

reducing the air density at which simulations are run, the values of the free-stream speed can be significantly increased keeping a small enough time step.

As remarked above, results in Figure 7 have been obtained using a symmetry condition for the wing, and therefore antisymmetric modes are not captured. However, it has been found that if the symmetry condition is not applied, it is actually the antisymmetric counterpart of the torsion-bending coupling which becomes unstable first. Both symmetric and antisymmetric modes have the same frequency in vacuo, but the aerodynamic loading is not exactly the same. This leads to a different aeroelastic coupling, and thus to slight variations in the mode behaviour. The antisymmetric mode flutters at $(V_{fas} = 162 \text{ m/s, } \omega_{fas} = 73 \text{ rad/s})$, whereas the symmetric one becomes unstable at $(V_f = 166 \text{ m/s, } \omega_f = 72 \text{ rad/s})$. Figure 8 depicts the symmetric mode and its antisymmetric counterpart. To the authors’ knowledge, this antisymmetric flutter mode had not been previously reported.

Figure 8: Unstable torsion-bending coupling of the Goland wing: (a) symmetric mode and (b) antisymmetric counterpart.

6.3 Aeroelastic Stability Boundaries of a T-Tail

In order to exercise the methodology for stability evaluation, the flutter of a typical T-empennage has been explored. Two original test cases have been defined, for which the relevant properties are highlighted in Table 1. As it can be seen, the structural properties for both cases coincide. T-Tail 1 corresponds to an assembly of unswept rectangular vertical fin and horizontal stabilizer, whereas T-Tail 2 has tapered lifting surfaces and a
30 deg sweep angle.

<table>
<thead>
<tr>
<th>Property</th>
<th>T-Tail 1</th>
<th>T-Tail 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root chord of vertical fin, c</td>
<td>2 m</td>
<td>2 m</td>
</tr>
<tr>
<td>Tip chord of vertical fin</td>
<td>2 m</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Span of vertical fin</td>
<td>6 m</td>
<td>6 m</td>
</tr>
<tr>
<td>Sweep angle of vertical fin</td>
<td>0 deg</td>
<td>30 deg</td>
</tr>
<tr>
<td>Root chord of horizontal stabilizer</td>
<td>2 m</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Tip chord of horizontal stabilizer</td>
<td>2 m</td>
<td>1 m</td>
</tr>
<tr>
<td>Semi-span of horizontal stabilizer</td>
<td>4 m</td>
<td>4 m</td>
</tr>
<tr>
<td>Sweep angle of horizontal stabilizer</td>
<td>0 deg</td>
<td>30 deg</td>
</tr>
<tr>
<td>Elastic axis (from l.e.)</td>
<td>25% chord</td>
<td></td>
</tr>
<tr>
<td>Centre of gravity (from l.e.)</td>
<td>35% chord</td>
<td></td>
</tr>
<tr>
<td>Mass per unit length</td>
<td>35 kg/m</td>
<td></td>
</tr>
<tr>
<td>Sectional moment of inertia (around e.a.)</td>
<td>8 kg·m</td>
<td></td>
</tr>
<tr>
<td>Torsional stiffness</td>
<td>$1\times10^6$ N·m²</td>
<td></td>
</tr>
<tr>
<td>Bending stiffness (spanwise)</td>
<td>$1\times10^7$ N·m²</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Properties of the T-tail test cases

The flutter speed of these two empennages has been computed using the described stability analysis tool. As opposed to methods based on the Doublet Lattice [15], no modification is needed on the standard procedure based on the UVLM, which is able to capture all relevant kinematics. The vertical fin is discretised using 12 elements, and 8 are used for each of the horizontal members. The aerodynamic lattice consists of 6 panels chordwise for all surfaces, and the number of spanwise panels coincides with the finite-element mesh. The time step guarantees the capture of modes with frequencies up to 200 rad/s. 20 chord lengths of wake have been retained for the simulations. The following values have been used for the Newmark-β method: $\gamma_1 = 0.50$, $\gamma_2 = 0.55$, and $\beta_2 = 0.28^3$. In this set of results, gravity has been included in the computation of the equilibrium configuration, even though its effect is minor due to the relatively high stiffness of the members.

Figure 9 presents results of the flutter dynamic pressure for varying angles of attack of the empennage, normalized with the flutter dynamic pressure for $\alpha = 5$ deg, i.e., $q_{f,\alpha}^* = q_{f,\alpha}/q_{f,\alpha=5\text{deg}}$. It is $q_{f,\alpha=5\text{deg}} = 200$ Pa for T-tail 1 and $q_{f,\alpha=5\text{deg}} = 512$ Pa for T-tail 2. The trends shown by the stability boundary are in good agreement with the findings reported in Ref. [29] for a different T-tail, exhibiting a decrease in flutter speed as incidence increases. For both test cases and in the range of incidence angles examined, the mode that becomes unstable is the torsion of the fin coupled with a lead-lag motion of the horizontal stabilizer.

Table 2 displays damping and frequency of the first 3 dominant oscillatory eigenvectors for both T-tails. In turn, Figure 10 presents the root loci for T-tail 1 and Figure 11 illustrates the dominant eigenmodes for T-Tail 1 – the evolution of poles and the eigenmode shapes are analogous for T-Tail 2. Results in Table 2 and Figures 10 and 11 correspond to an angle of attack $\alpha = 5$ deg, and they have been obtained from the eigenvectors at the corresponding flutter dynamic pressure.

---

3This introduces a small amount of numerical damping into the system, which can be seen as the margin to establish the onset of flutter.

20
Figure 9: Normalized flutter dynamic pressure of T-Tail 1 (left) and T-Tail 2 (right).

<table>
<thead>
<tr>
<th>Eigenmode</th>
<th>T-Tail 1</th>
<th>T-Tail 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fin torsion + stabilizer lead-lag</td>
<td>$0 \pm 10.47j$</td>
<td>$0 \pm 7.55j$</td>
</tr>
<tr>
<td>Fin bending</td>
<td>$-0.0205 \pm 18.05j$</td>
<td>$-0.0157 \pm 15.20j$</td>
</tr>
<tr>
<td>Fin bending + stabilizer antisymmetric torsion</td>
<td>$-0.2130 \pm 62.91j$</td>
<td>$-0.0952 \pm 48.99j$</td>
</tr>
</tbody>
</table>

Table 2: Dominant eigenmodes for T-tail 1 and T-tail 2. [$\alpha = 5$ deg, $q_\infty = q_f$]

All poles obtained from the eigenvalue analysis are included in Figure 10(a), where most of them are aerodynamic and have very large negative damping. The structural modes of the members also appear, such as bending and torsion of the horizontal stabilizer, and they are heavily damped in the range of interest. Figure 10(b) zooms in the region in which the dominant poles are visible. As mentioned above, Eigenmode 1 corresponds to coupling between the fin torsion and a lead-lag motion of the horizontal stabilizer. Eigenmode 2, in turn, is a pure fin bending motion. It can be seen that the damping of Eigenmode 2 is very small when Eigenmode 1 becomes unstable. The damping on both eigenmodes approaches zero roughly at the same rate, but it is the torsion-lead-lag combination which becomes unstable first. This behaviour has been observed for both test cases and across the range of angles of incidence, and it was also reported in Ref. [29].

Finally, the dominant eigenmodes for T-Tail 1 are depicted in Figure 11. The state-space stability analysis also provides non-oscillatory (zero frequency) eigenvectors, and an example has been included in Figure 11(d).
Finding a reproducible good test case of T-tail flutter for model validation purposes has proved difficult. The qualitative agreement with the conclusions reported in Ref. [29] provides a reasonable degree of confidence in the numerical implementation. However, an exhaustive verification of this tool is still part of ongoing efforts, and the results presented in this section should be be taken with caution.

7 CONCLUDING REMARKS

A monolithic discrete-time state-space formulation has been presented for the linear stability analysis of flexible aircraft. The Unsteady Vortex Lattice has been formulated in a suitable way following other contributions in the literature, such as Ref. [16]. The flexible-body dynamics are modelled using a displacement-based Geometrically-Exact Composite-Beam, with the position and orientation given by the Cartesian Rotation Vector as independent degrees of freedom. The equations have been discretised in time and accommodated to a discrete-time formulation. The linearisation of the governing equations has been carried out under the assumption of frozen geometry, which is fully consistent with standard unsteady aerodynamic methods, i.e., Doublet Lattice.

The different modules have been presented independently as input-output systems, and the relevant mappings have been described analytically. The resulting integration can be cast into a very compact and simple form, and it leads to a generalized eigenvalue analysis that can be solved directly, without pre-computing aerodynamic forces in the frequency domain or projecting structural modes. The implementation of the methodology has been found to be very efficient, performing stability analyses in a few seconds on a single-processor computer.
Even though the approach is most adequate to solve very flexible structures, it has been shown that it can also be used for traditional more rigid cases. An example has been included in order to highlight the capabilities of the tool, looking at the flutter of a T-empennage. This test case has been chosen due to challenges it exhibits for benchmark aeroelastic methods, such as the DLM. The stability boundaries of such a tail can be computed requiring no corrections in the present approach, by just performing a direct eigenvalue analysis.

In addition to stability results, previous open-loop results have been extended by evaluating the influence of wake proximity on the dynamics of a flexible aircraft. Future research will target the stability prediction of High-Altitude Long-Endurance flexible platforms, where the coupling between elastic and rigid body modes is a crucial factor. The present model is expected to provide an appropriate framework for the analysis of those situations. Furthermore, the way in which the stability equations have been formulated should enable straightforward reduced order modelling, and this will be particularly useful in the design of linear controllers to alleviate the dynamic response of the aircraft to gusts and atmospheric turbulence.

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APPENDIX

The system matrices for the full aeroelastic and flight dynamics stability analysis are given by

\[
Y_{sys}(:,1) = \begin{cases} 
E_A \\
F_{SPSA}P_{SA}J_A + F_{SRPRA}P_{RA}J_A \\
F_{RS}P_{SA}J_A + F_{RPRA}P_{RA}J_A 
\end{cases}, \quad (45)
\]

\[
Y_{sys}(:,2) = \begin{cases} 
F_{APAS}P_{AS} \\
E_S + F_{SPSA}P_{SA}K_{APAS}P_{AS} + F_{SRPRA}P_{RA}K_{APAS}P_{AS} \\
E_{RS} + F_{RS}P_{SA}K_{APAS}P_{AS} + F_{RPRA}P_{RA}K_{APAS}P_{AS} 
\end{cases}, \quad (46)
\]

\[
Y_{sys}(:,3) = \begin{cases} 
F_{APAR}P_{AR} \\
E_S + F_{SPSA}P_{SA}K_{APAR}P_{AR} + F_{SRPRA}P_{RA}K_{APAR}P_{AR} \\
E_{R} + F_{RS}P_{SA}K_{APAR}P_{AR} + F_{RPRA}P_{RA}K_{APAR}P_{AR} 
\end{cases}, \quad (47)
\]

\[
Z_{sys}(:,1) = \begin{cases} 
G_A \\
H_{SPSA}P_{SA}J_A + H_{SRPRA}P_{RA}J_A \\
H_{RS}P_{SA}J_A + H_{RPRA}P_{RA}J_A 
\end{cases}, \quad (48)
\]

\[
Z_{sys}(:,2) = \begin{cases} 
H_{APAS}P_{AS} \\
G_S + H_{SPSA}P_{SA}K_{APAS}P_{AS} + H_{SRPRA}P_{RA}K_{APAS}P_{AS} \\
G_{RS} + H_{RS}P_{SA}K_{APAS}P_{AS} + H_{RPRA}P_{RA}K_{APAS}P_{AS} 
\end{cases}, \quad (49)
\]

\[
Z_{sys}(:,3) = \begin{cases} 
H_{APAR}P_{AR} \\
G_S + H_{SPSA}P_{SA}K_{APAR}P_{AR} + H_{SRPRA}P_{RA}K_{APAR}P_{AR} \\
G_{R} + H_{RS}P_{SA}K_{APAR}P_{AR} + H_{RPRA}P_{RA}K_{APAR}P_{AR} 
\end{cases}, \quad (50)
\]
where the following definitions have been used

$$E_{FB} = \begin{bmatrix} E_S & E_{SR} \\ E_R & E_R \end{bmatrix}, \quad F_{FB} = \begin{bmatrix} F_S & F_{SR} \\ F_R & F_R \end{bmatrix},$$

(51)

$$G_{FB} = \begin{bmatrix} G_S & G_{SR} \\ G_R & G_R \end{bmatrix}, \quad H_{FB} = \begin{bmatrix} H_S & H_{SR} \\ H_R & H_R \end{bmatrix}.$$  

(52)

8 REFERENCES


