

On the Capacity of Variable Density Cellular Systems under Multicell Decoding

Symeon Chatzinotas, Muhammad Ali Imran, Costas Tzaras

Abstract—The majority of multicell-decoding cellular models preserve a fundamental assumption which has initially appeared in Wyner’s model, namely the collocation of User Terminals (UTs). Although this assumption produces more tractable mathematical models, it is unrealistic w.r.t. current practical cellular systems. In this paper, we alleviate this assumption by assuming uniformly distributed UTs. The model under investigation is the uplink channel of a planar cellular array in the presence of power-law path loss and flat fading. In this context, we employ a free probability approach to evaluate the effect of UT distribution on the optimal sum-rate capacity of a variable-density cellular system.

Index Terms—Information theory, Information rates, Multiuser channels, Fading channels, Land mobile radio cellular systems, MIMO systems.

I. INTRODUCTION

The first concrete result for the information-theoretic capacity of the Gaussian Cellular Multiple Access Channel (GCMAC) was presented by Wyner in [1]. Using a very simple but tractable model for the cellular uplink channel, Wyner showed the importance of joint decoding at the Base Station (BS) receivers (hyper-receiver) and found the closed forms of the system capacity under the assumption of hyper-receiver (a.k.a. multicell decoding). This model triggered the interest of the research community in the cellular capacity limits and was extended in [2] to include flat fading conditions. One major assumption shared in these models was that the cell density is fixed and only physically adjacent cells interfere. Letzepis in [3], extended the model by assuming multiple-tier interference and incorporated a distance-dependent path loss factor in order to study the effect of cell density in a linear cellular array. However, the assumption of collocation of all users in a single cell was still maintained. In this paper, UTs are spatially distributed within the cell and each channel gain is affected by a distance-dependent path loss factor. The model under investigation is a GCMAC over a planar cellular array in the presence of power-law path loss and flat fading. The presented analysis is based on the principles of free probability.

II. CHANNEL MODEL AND ANALYSIS

Assume that K users per cell are uniformly distributed across a planar cellular system comprising N cells. Assuming

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flat fading, the received signal at BS $n = 1 \dots N$, at time index t , will be given by:

$$y^n[t] = \sum_{m=1}^N \sum_{k=1}^K \varsigma_k^{nm} g_k^{nm}[t] x_k^m[t] + z^n[t], \quad (1)$$

where $x_k^m[t]$ is the t th complex channel symbol transmitted by the k th UT of the m th cell and $\{g_k^{nm}\}$ are independent, strictly stationary and ergodic complex random processes in the time index t , which represent the flat fading processes experienced in the transmission path between the n th BS and the k th UT in the m th cell. The fading coefficients are assumed to have unit power, i.e. $\mathbb{E}[\|g_k^{nm}[t]\|^2] = 1$ for all (n, m, k) and all UTs are subject to an average power constraint, i.e. $\mathbb{E}[\|x_k^m[t]\|^2] \leq P$ for each (m, k) . The path loss coefficients ς_k^{nm} in the transmission path between the m th BS and the k th UT in the n th cell are calculated according to the “modified” power-law path loss model [3], [4]: $\varsigma_k^{nm} = (1 + d_k^{nm})^{-\eta/2}$, where d_k^{nm} denotes the distance along the line of sight of the transmission path and η is the path loss exponent. The model can be more compactly expressed as a vector memoryless channel of the form $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$, where the vector $\mathbf{y} = [y^1 \dots y^N]^T$ represents received signals by the BSs, the vector $\mathbf{x} = [x_1^1 \dots x_K^N]^T$ represents transmit signals by all the UTs of the cellular system and the components of vector $\mathbf{z} = [z^1 \dots z^N]^T$ are independent identically distributed (i.i.d.) complex circularly symmetric (c.c.s.) random variables representing AWGN with $\mathbb{E}[z^n] = 0$, $\mathbb{E}[\|z^n\|^2] = \sigma^2$. The channel matrix \mathbf{H} can be written as $\mathbf{H} = \mathbf{\Sigma} \odot \mathbf{G}$, where $\mathbf{\Sigma}$ is a $N \times KN$ deterministic matrix, $\mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ is a complex Gaussian $N \times KN$ matrix comprising the corresponding Rayleigh fading coefficients and \odot denotes the Hadamard (element-wise) product. The entries of the $\mathbf{\Sigma}$ matrix are defined by the variance profile function

$$\varsigma(u, v) = (1 + d(u, v))^{-\eta/2} \quad (2)$$

where $u \in [0, 1]$ and $v \in [0, K]$ are the normalized indexes for the BSs and the UTs respectively and $d(u, v)$ denotes the distance between BS u and user v . According to [5], the asymptotic sum-rate capacity C_{opt} for this model assuming a very large number of cells, is given by

$$\begin{aligned} C_{\text{opt}} &= \lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{I}(\mathbf{x}; \mathbf{y} | \mathbf{H}) \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \log \left(1 + \frac{\tilde{\gamma}}{K} \lambda_i \left(\frac{1}{N} \mathbf{H}\mathbf{H}^\dagger \right) \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \log\left(1 + \frac{\tilde{\gamma}}{K}x\right) dF_{\frac{1}{N}\mathbf{H}\mathbf{H}^\dagger}(x) \\
&= \mathcal{V}_{\frac{1}{N}\mathbf{H}\mathbf{H}^\dagger}(\tilde{\gamma}/K) = K\mathcal{V}_{\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}}(\tilde{\gamma}/K) \quad (3)
\end{aligned}$$

where $\tilde{\gamma} = KN\gamma$ and $\gamma = P/\sigma^2$ are the system- and UT-transmit power over receiver noise ratio respectively, $\lambda_i(\mathbf{X})$ denotes the eigenvalues of matrix \mathbf{X} and

$$\begin{aligned}
\mathcal{V}_{\mathbf{X}}(y) &\triangleq \mathbb{E}[\log(1 + y\mathbf{X})] \\
&= \int_0^\infty \log(1 + y\lambda) dF_{\mathbf{X}}(\lambda) \quad (4)
\end{aligned}$$

is the Shannon transform [5] of a random square Hermitian matrix \mathbf{X} , whose asymptotic eigenvalue distribution (a.e.d.) has a cumulative function denoted by $F_{\mathbf{X}}(\lambda)$. For a complex Gaussian matrix $\mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$, the a.e.d. of $\frac{1}{N}\mathbf{G}^\dagger\mathbf{G}$ converges almost surely (a.s.) to the nonrandom a.e.d. of the Marčenko-Pastur law [6], whose Shannon transform is given by

$$\mathcal{V}_{\frac{1}{N}\mathbf{G}^\dagger\mathbf{G}}(y) \xrightarrow{\text{a.s.}} \mathcal{V}_{\text{MP}}(y, K), \quad (5)$$

$$\begin{aligned}
\text{where } \mathcal{V}_{\text{MP}}(y, K) &= \log\left(1 + y - \frac{1}{4}\phi(y, K)\right) \\
&+ \frac{1}{K}\log\left(1 + yK - \frac{1}{4}\phi(y, K)\right) - \frac{1}{4Ky}\phi(y, K) \quad (6)
\end{aligned}$$

and $\phi(y, K) =$

$$\left(\sqrt{y(1 + \sqrt{K})^2 + 1} - \sqrt{y(1 - \sqrt{K})^2 + 1}\right)^2. \quad (7)$$

The rest of this section describes the derivation of the a.e.d. of $\frac{1}{N}\mathbf{H}\mathbf{H}^\dagger$ based on the analysis in [3] and using tools from the discipline of Free Probability, which was established by Voiculescu [7]. In this direction, $\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}$ can be written as the sum of $KN \times KN$ unit rank matrices, i.e.

$$\frac{1}{N}\mathbf{H}^\dagger\mathbf{H} = \sum_{i=1}^N \mathbf{h}_i^\dagger \mathbf{h}_i, \quad (8)$$

where $\mathbf{h}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{V}_i)$ denotes the i th $1 \times KN$ row vector of $\frac{1}{\sqrt{N}}\mathbf{H}$. The covariance matrix equals $\mathbf{V}_i = \frac{1}{N}(\text{diag}(\boldsymbol{\sigma}_i))^2$, where $\text{diag}(\boldsymbol{\sigma}_i)$ stands for a diagonal matrix with the elements of vector $\boldsymbol{\sigma}_i$ across the diagonal with $\boldsymbol{\sigma}_i$ being the i th row of $\boldsymbol{\Sigma}$. The unit-rank matrices $\mathbf{W}_i = \mathbf{h}_i^\dagger \mathbf{h}_i$ constitute complex singular Wishart matrices with one degree of freedom and their density according to [8, Theorem 3-4] is

$$\begin{aligned}
f_{\mathbf{V}_i}(\mathbf{W}_i) &= B_{\mathbf{V}_i}^{-1} \det(\mathbf{W}_i)^{1-KN} e^{-\text{tr}(\mathbf{V}_i^{-1}\mathbf{W}_i)} \\
B_{\mathbf{V}_i} &= \pi^{KN-1} \det(\mathbf{V}_i) \quad (9)
\end{aligned}$$

If $\mathbf{h}_i^\dagger = \mathbf{Q}_i \mathbf{S}_i$ is a singular value decomposition, then the density can be written as

$$f_{\mathbf{V}_i}(\mathbf{W}_i) = B_{\mathbf{V}_i}^{-1} \det(\mathbf{S}_i \mathbf{S}_i^\dagger)^{1-KN} e^{-\text{tr}(\mathbf{V}_i^{-1} \mathbf{Q}_i \mathbf{S}_i \mathbf{S}_i^\dagger \mathbf{Q}_i^\dagger)} \quad (10)$$

It can be easily seen that if $\mathbf{V}_i = \mathbf{I}$, the matrices would be unitarily invariant [9, Definition 17.7] and therefore asymptotically free [10]. Although in our case $\mathbf{V}_i = \frac{1}{N}\text{diag}(\boldsymbol{\sigma}_i^2)$,

we assume that the asymptotic freeness still holds. Similar approximations have been already investigated in an information-theoretic context, providing useful analytical insights and accurate numerical results [11], [12]. In this context, the R-transform of each unit rank matrix [5, Example 2.28] is given by

$$\mathcal{R}_{\mathbf{h}_i^\dagger \mathbf{h}_i}(w) = \frac{1}{KN} \frac{\|\mathbf{h}_i\|^2}{1 - w \|\mathbf{h}_i\|^2} \quad (11)$$

and the asymptotic R-transform of $\mathbf{H}^\dagger\mathbf{H}$ is equal to the sum of the R-transforms of all the unit rank matrices [5, Theorem 2.64]

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathcal{R}_{\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}}(w) &\simeq \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathcal{R}_{\mathbf{h}_i^\dagger \mathbf{h}_i}(w) \\
&= \lim_{N \rightarrow \infty} \frac{1}{KN} \sum_{i=1}^N \frac{\|\mathbf{h}_i\|^2}{1 - w \|\mathbf{h}_i\|^2}. \quad (12)
\end{aligned}$$

Since the variance profile function of Equation (2) defines a rectangular block-circulant matrix with $1 \times K$ blocks which is symmetric about $u = Kv$, the channel matrix \mathbf{H} is asymptotically row-regular [5, Definition 2.10] and thus the asymptotic norm of \mathbf{h}_i converges to a deterministic constant for every BS, i.e. $\forall i$

$$\lim_{N \rightarrow \infty} \|\mathbf{h}_i\|^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^{KN} \varsigma_{ij}^2 = \int_0^K \varsigma^2(u, v) dv, \quad (13)$$

where ς_{ij} is the (i, j) th element of the $\boldsymbol{\Sigma}$ matrix. In addition, based on the row-regularity it can be seen that

$$\int_0^K \varsigma^2(u, v) dv = \int_0^1 \int_0^K \varsigma^2(u, v) dudv. \quad (14)$$

Therefore, while $\lim_{N \rightarrow \infty}$, Equation (12) can be simplified to [5, Theorem 2.31, Example 2.26]

$$\begin{aligned}
\mathcal{R}_{\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}}(w) &\simeq \frac{1}{K} \int_0^1 \frac{\int_0^K \varsigma^2(u, v) dv}{1 - w \int_0^K \varsigma^2(u, v) dv} du \\
&= \frac{1}{K} \frac{\int_0^1 \int_0^K \varsigma^2(u, v) dudv}{1 - w \int_0^1 \int_0^K \varsigma^2(u, v) dudv} \\
&= q(\boldsymbol{\Sigma}) \frac{1}{1 - Kwq(\boldsymbol{\Sigma})} \\
&= R_{q(\boldsymbol{\Sigma}) \frac{1}{N}\mathbf{G}^\dagger\mathbf{G}}(w), \quad (15)
\end{aligned}$$

$$\text{where } q(\boldsymbol{\Sigma}) \triangleq \|\boldsymbol{\Sigma}\|^2 / KN^2 \quad (16)$$

with $\|\boldsymbol{\Sigma}\| \triangleq \sqrt{\text{tr}\{\boldsymbol{\Sigma}^\dagger \boldsymbol{\Sigma}\}}$ being the Frobenius norm of the $\boldsymbol{\Sigma}$ matrix. In the asymptotic case, $q(\boldsymbol{\Sigma})$ is given by

$$\lim_{N \rightarrow \infty} q(\boldsymbol{\Sigma}) = \frac{1}{K} \int_0^K \varsigma^2(u, v) dv. \quad (17)$$

The probability density function (p.d.f.) of the a.e.d. of $\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}$ follows a scaled version of the Marčenko-Pastur law and hence the Shannon transform of the a.e.d. of $\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}$ can be approximated by

$$\mathcal{V}_{\frac{1}{N}\mathbf{H}^\dagger\mathbf{H}}\left(\frac{\tilde{\gamma}}{K}\right) \simeq \mathcal{V}_{\text{MP}}\left(q(\boldsymbol{\Sigma}) \frac{\tilde{\gamma}}{K}, K\right). \quad (18)$$

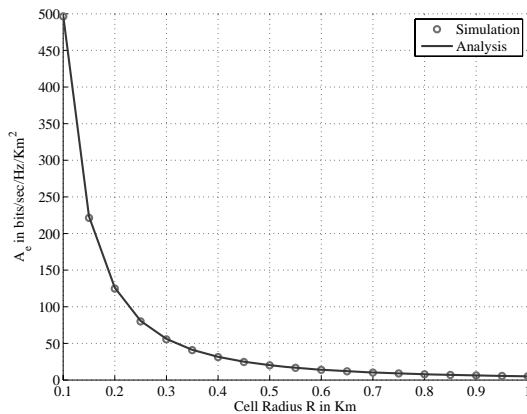


Fig. 1. Mean Area Spectral Efficiency A_e (bits/sec/Hz/Km²) vs. the cell radius R . Note that there are 127 UTs in each Km².

It should be noted that the limiting eigenvalue distribution converges to the Marčenko-Pastur law, as long as Σ is asymptotically doubly-regular [5, Definition 2.10, Theorem 2.49]. In this paper, it is shown that on the grounds of free probability, a scaled version of the Marčenko-Pastur law can be effectively utilized in cases where Σ is just asymptotically row-regular. Furthermore, according to [3], this analysis holds for UTs collocated with the BS in a linear cellular array. Herein, we show that the convergence holds for the case of distributed UTs. Closed-form expressions for $q(\Sigma)$ in the case of a planar cellular array are studied in [13].

III. NUMERICAL RESULTS

In this section, we present some simulation results in order to verify the accuracy of the asymptotic analysis for the per-cell sum-rate capacity C_{opt} of a planar cellular system with uniformly distributed UTs. In the context of the asymptotic analysis, Equations (3), (17) and (18) are utilized, while the distances $d(u, v)$ are calculated assuming that the UTs are positioned on a uniform planar grid. In the context of Monte Carlo finite-system simulations, the UT positions are randomly generated according to the considered distribution and the capacity is calculated by evaluating the capacity formula [14]:

$$C_{\text{opt}} = (1/N) \mathbb{E} [\log \det (\mathbf{I}_N + \gamma \mathbf{H} \mathbf{H}^\dagger)]. \quad (19)$$

The presented results refer to the mean Area Spectral Efficiency (ASE)

$$A_e = C_{\text{opt}} / \pi R^2 \quad (\text{bits/sec/Hz/Km}^2) \quad (20)$$

averaged over a large number of fading realizations g_k^{nm} and UT positions d_k^{nm} and they are plotted vs. a variable cell radius R . In this point, it should be noted that the ASE of conventional interference-limited cellular systems has been studied in [15]. In the current setting, while scaling the cell radius, the UT density (users/Km²) K_0 is kept fixed and hence the per-cell number of UTs $K = K_0 \pi R^2$ and the system power scales accordingly. The path loss coefficients are calculated by considering a nominal power loss L_0 at a reference distance d_0 (see Table I for parameter values) $\zeta(d) = \sqrt{L_0(1 + d/d_0)^{-\eta}}$.

TABLE I
SIMULATION PARAMETERS

Parameter	Symbol	Value/Range (units)
Cell Radius	R	0.1 – 1 km
Reference Distance	d_0	1 m
Path Loss at ref. distance	L_0	−38 dB
Path Loss Exponent	η	3.5
UTs per Km ²	K_0	127
UT Transmit Power	P_T	200 mW
Thermal Noise Density	N_0	−169 dBm/Hz
Channel Bandwidth	B	5 MHz

IV. CONCLUSION

In conclusion, this paper has considered the capacity performance of a multicell joint decoding system under two realistic assumptions: planar coverage area and distributed UTs. Using a free probability approach, the a.e.d. of the matrix $\frac{1}{N} \mathbf{H} \mathbf{H}^\dagger$ has been derived and the proposed analysis was validated with Monte Carlo simulations for variable cell-density systems. The importance of the presented methodology lies in the fact that it allows a more realistic representation of the UTs' spatial arrangement. Therefore, this approach can be employed in order to investigate various practical UT distributions and their effect on the sum-rate capacity.

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