FINITE ELEMENT DYNAMIC ANALYSIS OF SHELL STRUCTURES

A Thesis submitted to the University of Surrey for the
Degree of Doctor of Philosophy

by

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February 1976
SUMMARY


In this Thesis, Finite Element Models to predict the dynamic behaviour of shell structures, pre-stressed and rotating shell structures, and shells submerged in a fluid medium are developed. Also, a finite element model to predict the instability and stability regions of shell structures subjected to periodic forces is developed. The models are based on a super-parametric shell element representation of the structure. The fluid domain is represented by three dimensional isoparametric elements. The Reduced Integration Technique is used to evaluate the strain energy of the models. The Eigenvalue Economizer is used to reduce the total number of degrees of freedom of the discretized system. The transient response is calculated by the Modal Analysis Method. The Models are applied to several shell structures and the predicted dynamic behaviour is compared with analytical, other finite element models or experimental solutions.

The mathematical foundations of the Finite Element Method, Eigenvalue Economizer, Reduced Integration Technique, Theory of Dynamic Instability and Modal Analysis Method are introduced. The solution of large eigenvalue problems is discussed. The efficiency, accuracy and stability of Direct Integration and Modal Analysis methods are discussed, with particular reference to the transient response of rotating and pre-stressed shell structures.

2.
Several literature surveys of the development and application of the finite element analysis of structures are presented. These surveys include the development of shell elements and their application to the free vibration and transient analysis of shells, the development of the finite element dynamic analysis of rotating structures and structures submerged in a fluid medium, the development of the finite element dynamic instability analysis of structures, and the application of finite elements to blade analysis.

The super-parametric shell element is further developed such that it becomes applicable to the static or dynamic, linear or nonlinear, analysis of shell structures, and to the dynamic analysis of pre-stressed, rotating and submerged shells. The effect of reduced integration on the predicted dynamic characteristics of super-parametric shell element representations is investigated. The accuracy and efficiency of super-parametric shell element representations are compared with other shell element models.

The effect of angular velocity, aspect and thickness ratios, setting and pre-twisted angles, disc radius, pre-stress and surrounding fluid on the dynamic behaviour of shell structures is extensively illustrated.
ACKNOWLEDGEMENTS

The author wishes to express his gratitude and sincere appreciation to Dr. J. Thomas for his guidance, valuable suggestions and critical comments throughout this investigation.

The author is deeply indebted to the Calouste Gulbenkian Foundation, Lisbon, for providing the research scholarship which enabled him to undertake this research.
# CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title Page</td>
<td>1</td>
</tr>
<tr>
<td>Summary</td>
<td>2</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>4</td>
</tr>
<tr>
<td>Contents</td>
<td>5</td>
</tr>
<tr>
<td>List of Tables</td>
<td>8</td>
</tr>
<tr>
<td>List of Figures</td>
<td>9</td>
</tr>
<tr>
<td>List of Symbols</td>
<td>14</td>
</tr>
<tr>
<td>CHAPTER 1 INTRODUCTION</td>
<td>23</td>
</tr>
<tr>
<td>CHAPTER 2 THE FINITE ELEMENT METHOD</td>
<td></td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>30</td>
</tr>
<tr>
<td>2.2 Variational Methods</td>
<td>30</td>
</tr>
<tr>
<td>2.3 Residual Methods</td>
<td>33</td>
</tr>
<tr>
<td>2.4 Finite Element Approximation</td>
<td>33</td>
</tr>
<tr>
<td>2.5 Rayleigh-Ritz Finite Element Method</td>
<td>35</td>
</tr>
<tr>
<td>2.6 Galerkin Finite Element Method</td>
<td>37</td>
</tr>
<tr>
<td>2.7 Conformity, Completeness and Convergence</td>
<td>38</td>
</tr>
<tr>
<td>2.8 Finite Element Dynamic Analysis of Structures</td>
<td>39</td>
</tr>
<tr>
<td>2.8.1 Stress and strain vectors</td>
<td>40</td>
</tr>
<tr>
<td>2.8.2 Potential and kinetic energies</td>
<td>41</td>
</tr>
<tr>
<td>2.8.3 Equations of motion</td>
<td>42</td>
</tr>
<tr>
<td>CHAPTER 3 THE EIGENVALUE ECONOMIZER</td>
<td>44</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>44</td>
</tr>
<tr>
<td>3.2 Mathematical Foundations</td>
<td>47</td>
</tr>
<tr>
<td>3.3 Upper Bound Property</td>
<td>51</td>
</tr>
<tr>
<td>3.4 Further Applications</td>
<td>53</td>
</tr>
<tr>
<td>CHAPTER 4 THE SUPER-PARAMETRIC SHELL ELEMENT</td>
<td>54</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>54</td>
</tr>
<tr>
<td>4.2 Fundamentals</td>
<td>59</td>
</tr>
<tr>
<td>4.3 Stiffness Matrix</td>
<td>65</td>
</tr>
<tr>
<td>4.4 Mass Matrix</td>
<td>67</td>
</tr>
</tbody>
</table>

5.
| CHAPTER 9 | DYNAMIC INSTABILITY OF SHELLS | 228 |
| 9.1 Introduction | 228 |
| 9.2 Dynamic Instability of Multi-degree of Freedom Systems | 230 |
| 9.3 Applications | 238 |
| 9.4 General Discussion | 238 |

| CHAPTER 10 | TRANSIENT ANALYSIS OF SHELLS | 244 |
| 10.1 Introduction | 244 |
| 10.2 Mode Superposition Method | 246 |
| 10.3 Direct Integration Methods | 248 |
| 10.4 Choice of Method | 249 |
| 10.5 Applications | 250 |
| 10.5.1 Analytical solutions | 251 |
| 10.6 General Discussion | 252 |

| CHAPTER 11 | CONCLUSIONS | 271 |

| CHAPTER 12 | FUTURE DEVELOPMENT | 281 |

| APPENDIX A | NUMERICAL INTEGRATION AND CONVERGENCE | 282 |

| REFERENCES | 285 |
### LIST OF TABLES

| TABLE 6.1 | Influence of the primary coordinates on the frequency parameters of a simply supported spherical shell. | 111 |
| TABLE 6.2 | Variation of frequency parameters of pre-twisted blade with degrees of freedom. | 112 |
| TABLE 6.3 | Variation of frequency parameters of simply supported cylindrical shell with degrees of freedom. | 114 |
| TABLE 6.4 | Variation of the natural frequencies of the uniform cylindrical shell blade with degrees of freedom. | 115 |
| TABLE 6.5 | Comparison of the natural frequencies of uniform cylindrical shell blade with other solutions. | 116 |
| TABLE 6.6 | Comparison of the natural frequencies of the uniform cylindrical shell blade calculated by a super-parametric and shallow thin shell element representation. | 117 |
| TABLE 6.7 | Variation of the natural frequencies of the pre-stressed, uniform cylindrical shell blade with stress ratio. | 118 |
| TABLE 6.8 | Variation of the natural frequencies of the tapered cylindrical shell blade with degrees of freedom. | 119 |
| TABLE 6.9 | Comparison of the natural frequencies of the tapered cylindrical shell blade calculated by a super-parametric and a shallow thin shell element representation. | 120 |
| TABLE 6.10 | Variation of the natural frequencies of the pre-stressed, tapered cylindrical shell blade with stress ratio. | 121 |
| TABLE 8.1 | Variation of the fundamental natural frequency of a submerged plate with degrees of freedom of the fluid domain. | 220 |
| TABLE 8.2 | Comparison of experimental and computed fundamental natural frequency of submerged plates. | 221 |
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Three dimensional isoparametric parabolic element.</td>
<td>69</td>
</tr>
<tr>
<td>4.2</td>
<td>Two dimensional isoparametric family of elements.</td>
<td>70</td>
</tr>
<tr>
<td>4.3</td>
<td>Super-parametric parabolic shell element.</td>
<td>71</td>
</tr>
<tr>
<td>4.4</td>
<td>Typical parabolic shape functions.</td>
<td>72</td>
</tr>
<tr>
<td>4.5</td>
<td>Typical cubic shape functions.</td>
<td>73</td>
</tr>
<tr>
<td>5.1</td>
<td>The spurious and correct response of the plane quadrilateral element under pure bending.</td>
<td>80</td>
</tr>
<tr>
<td>5.2</td>
<td>Correct and spurious response of the super-parametric parabolic plate element.</td>
<td>81</td>
</tr>
<tr>
<td>5.3</td>
<td>Correct and spurious response of the super-parametric parabolic plate element.</td>
<td>82</td>
</tr>
<tr>
<td>5.4</td>
<td>Correct and spurious response of the super-parametric parabolic plate element.</td>
<td>83</td>
</tr>
<tr>
<td>6.1</td>
<td>Spherical shell geometry and finite element mesh - simply supported boundary conditions.</td>
<td>122</td>
</tr>
<tr>
<td>6.2</td>
<td>Convergence curves for the finite element models of the spherical shell.</td>
<td>123</td>
</tr>
<tr>
<td>6.3</td>
<td>Vibration modes of a spherical shell.</td>
<td>124</td>
</tr>
<tr>
<td>6.4</td>
<td>Location of primary coordinates of spherical shell.</td>
<td>125</td>
</tr>
<tr>
<td>6.5</td>
<td>Variation of frequency parameters of spherical shell with primary coordinates.</td>
<td>126</td>
</tr>
<tr>
<td>6.6</td>
<td>Variation of the frequency parameters of spherical shells with thickness ratio.</td>
<td>127</td>
</tr>
<tr>
<td>6.7</td>
<td>Relative value of frequency parameters of spherical shells with thickness ratio.</td>
<td>128</td>
</tr>
<tr>
<td>6.8</td>
<td>Variation of frequency parameters of spherical shells with radius/length ratio.</td>
<td>129</td>
</tr>
<tr>
<td>6.9</td>
<td>Side view of pre-twisted blade.</td>
<td>130</td>
</tr>
<tr>
<td>6.10</td>
<td>Convergence curves for the finite element model of a pre-twisted blade - Modes 1 to 7.</td>
<td>131</td>
</tr>
<tr>
<td>6.11</td>
<td>Convergence curves for the finite element model of a pre-twisted blade.</td>
<td>132</td>
</tr>
<tr>
<td>6.12</td>
<td>Vibration modes of a pre-twisted blade - modes 2 to 9.</td>
<td>133</td>
</tr>
<tr>
<td>6.13</td>
<td>Vibration modes of a pre-twisted blade - modes 10 to 15.</td>
<td>134</td>
</tr>
<tr>
<td>6.14</td>
<td>Variation of the frequency parameters of pre-twisted blades with thickness ratio - first and third mode.</td>
<td>135</td>
</tr>
</tbody>
</table>
Figure | Page
--- | ---
6.15 Variation of frequency parameters of pre-twisted blades with thickness ratio - second mode. | 136
6.16 Variation of frequency parameters of pre-twisted blades with thickness ratio - fourth mode. | 137
6.17 Variation of frequency parameters of pre-twisted blades with thickness ratio - fifth mode. | 138
6.18 Variation of frequency parameters of pre-twisted blades with thickness ratio - sixth mode. | 139
6.19 Variation of frequency parameters of pre-twisted blades with thickness ratio - seventh mode. | 140
6.20 Variation of frequency parameters of pre-twisted blades with thickness ratio - eighth mode. | 141
6.21 Variation of frequency parameters of pre-twisted blades with aspect ratio. | 142
6.22 Variation of frequency parameters of cantilever plate. | 143
6.23 Variation of frequency parameters of cantilever plates with thickness ratio. | 144
6.24 Simply supported pre-stressed cylindrical shell. | 145
6.25 Vibration modes of a cylindrical shell - modes 1 to 6. | 146
6.26. Vibration modes of cylindrical shell - modes 7 to 12. | 147
6.27 Accuracy of vibration modes of cylindrical shell - first and second mode. | 148
6.28 Accuracy of vibration modes of cylindrical shell - fourth and fifth mode | 149
6.29 Variation of frequency parameters of pre-stressed cylindrical shell with stress ratio. | 150
6.30 Cylindrical shell blades with uniform and tapered thickness. | 151
6.31 Computed and experimental vibration modes of uniform cylindrical shell blade - modes 1 to 4. | 152
6.32 Computed and experimental vibration modes of uniform cylindrical shell blade - modes 5 to 7. | 153
6.33 Computed and experimental vibration modes of uniform cylindrical shell blade - modes 8 to 10. | 154
6.34 Computed and experimental vibration modes of uniform cylindrical shell blade - modes 11 to 13. | 155
6.35 Variation of natural frequencies of uniform cylindrical shell blades with stress ratio - modes 1 to 3. | 156
6.36 Variation of natural frequencies of uniform cylindrical shell blade with stress ratio - modes 4 to 7. | 157
6.37 Computed and experimental vibration modes of tapered cylindrical shell blade - modes 1 to 4. | 158
Figure | Computed and experimental vibration modes of tapered cylindrical shell blade - modes 5 to 7. | 6.38 | 159
Figure | Computed and experimental vibration modes of tapered cylindrical shell blade - modes 8 to 10. | 6.39 | 160
Figure | Computed and experimental vibration modes of tapered cylindrical shell blade - modes 11 to 13. | 6.40 | 161
Figure | Variation of natural frequencies of tapered cylindrical shell blade with stress ratio - modes 1 to 3. | 6.41 | 162
Figure | Variation of natural frequencies of tapered cylindrical shell blade with stress ratio - modes 4 to 6. | 6.42 | 163
Figure | Effect of the stress ratio on the vibration modes of tapered cylindrical shell blade - fourth mode. | 6.43 | 164

7.1 Rotating structure. | 180
7.2 Rotating blade divided into a 3 x 2 mesh of finite elements. | 181
7.3 Convergence curves for the finite element model of the rotating pre-twisted blade - modes 1 to 5. | 182
7.4 Convergence curves for the finite element model of the pre-twisted blade - modes 6 to 9. | 183
7.5 Vibration modes of rotating pre-twisted blade. | 184
7.6 Variation of frequency parameters of pre-twisted blade with angular velocity. | 185
7.7 Influence of the angular velocity on the vibration modes of pre-twisted blade. | 186
7.8 Variation of frequency parameters of pre-twisted blades with angular velocity ratio. | 187
7.9 Variation of frequency parameters of a pre-twisted blade with angular ratio. | 188
7.10 Variation of the vibration modes of a pre-twisted blade with angular velocity. | 189
7.11 Variation of frequency parameters of pre-twisted blade with radius ratio. | 190
7.12 Variation of frequency parameters of pre-twisted blade with setting angle. | 191
7.13 Variation of frequency parameters of pre-twisted blade with pre-twist angle. | 192
7.14 Variation of frequency parameters of pre-twisted blade with thickness ratio. | 193
7.15 Variation of frequency parameters of pre-twisted blade with aspect ratio. | 194
Figure 7.16 Convergence curves for the finite element model of the rotating, uniform and tapered, cylindrical shell blades. 195
7.17 Vibration modes of uniform cylindrical shell blades. 196
7.18 Variation of the natural frequencies of the uniform cylindrical shell blade with angular velocity. 197
7.19 Variation of the vibration modes of the uniform cylindrical shell blade with angular velocity - first and third mode. 198
7.20 Variation of the vibration modes of the uniform cylindrical shell blade with angular velocity - fourth and sixth mode. 199
7.21 Influence of the steady state deformation on the natural frequencies of the uniform cylindrical shell blade. 200
7.22 Variation of the natural frequencies of the rotating, uniform cylindrical shell blade with radius ratio. 201
7.23 Variation of the natural frequencies of the rotating, uniform cylindrical shell blade with setting angle. 202
7.24 Variation of the natural frequencies of the rotating, tapered cylindrical shell blade with angular velocity. 203
7.25 Variation of the vibration modes of the tapered cylindrical shell blade with angular velocity. 204
7.26 Variation of the vibration modes of the tapered cylindrical shell blade with angular velocity. 205
8.1 Isoparametric parabolic fluid element. 222
8.2 Representation of the submerged plate and fluid domain by finite elements. 223
8.3 Variation of the frequency parameters of submerged plate with aspect ratio. 224
8.4 Variation of the frequency parameters of a submerged plate with thickness ratio. 225
8.5 Variation of the frequency parameters of submerged plate with height ratio. 226
8.6 Variation of the frequency parameters of a submerged plate with depth ratio. 227
9.1 Unit circle in the complex plane. 239
9.2 Principal regions of dynamic instability of simply supported cylindrical shell. 240
9.3 Principal regions of dynamic instability of simply supported cylindrical shell. 241
9.4 Principal regions of dynamic instability of the uniform cylindrical shell blade. 242
Figure 9.5 Principal regions of dynamic instability of the tapered cylindrical shell blade.

10.1 Transient response of the spherical shell subjected to a sinusoidal force.
10.2 Analytical transient response of the spherical shell subjected to a sinusoidal force.
10.3 Transient response of the spherical shell subjected to an impact force.
10.4 Analytical transient response of the spherical shell subjected to an impact force.
10.5 Transient response of the cylindrical shell subjected to a sinusoidal force.
10.6 Analytical transient response of the cylindrical shell subjected to a sinusoidal force.
10.7 Transient response of the cylindrical shell subjected to an impact force.
10.8 Analytical transient response of the cylindrical shell subjected to an impact force.
10.9 Transient response of the pre-stressed cylindrical shell subjected to a sinusoidal force.
10.10 Transient response of the cylindrical shell blade subjected to sinusoidal forces.
10.11 Transient response of the pre-stressed cylindrical shell blade subjected to sinusoidal forces.
10.12 Transient response of the rotating cylindrical shell blade subjected to sinusoidal forces.
10.13 Transient response of the cylindrical shell blade subjected to impact forces.
10.14 Transient response of the pre-stressed cylindrical shell blade subjected to impact forces.
10.15 Transient response of the rotating cylindrical shell blade subjected to impact forces.
LIST OF SYMBOLS

Notations used have been defined as they first appear in the Thesis. Some symbols have in fact been used to represent different parameters in different chapters. A dot over a variable or vector signifies the velocity of this variable or vector. Similarly, two dots signify the acceleration of the variable or vector.

\( A_{mn} \) constants

\( AE_{ij} \) coefficients of the element compressibility matrix

\( a \) length

\( a_i \) real part of eigenvalue \( \lambda_i \) (Chapter 9)

\( a_j \) constants

\( a_x, a_y, a_z \) components of \( a \)

\( a \) unit vector defining rotating axis

\( [A] \) compressibility matrix

\( [A] \) differential operator matrix (Chapter 2)

\( [A] \) symmetric band matrix (Chapter 3)

\( [AE] \) element compressibility matrix

\( B \) Bulk modulus

\( BE_j \) coefficients of the element fluid vector

\( b \) width

\( b_i \) imaginary part of \( \lambda_i \)

\( [B] \) symmetric, positive definite, band matrix

\( [B_L] \) linear strain matrix

\( [B_L] \) global linear strain matrix

\( [B_N] \) nonlinear strain matrix
\([B_N]\) global nonlinear strain matrix
\([B_j]\) sub-matrix of the linear strain matrix
\([B]\) fluid vector
\([BE]\) element fluid vector
\([c]\) velocity of sound
\([c_i]\) constants
\([c_{ij}]\) constants
\([C]\) monodromy matrix
\([C]\) boundary operator matrix (Chapter 2)
\([D]\) flexural rigidity of plate
\([d]\) number of independent variables
\([d]\) height of water above structure (Chapter 8)
\([d_i]\) elements of \([d]\)
\([E]\) Elasticity matrix
\([\{d\}]\) vectors of constants
\([E]\) modulus of elasticity
\([F]\) Airy stress function
\([F_i]\) force per unit volume in the \(x_i\) direction
\([f]\) depth of water below structure
\([f_j]\) fluid degrees of freedom
\([\{F\}]\) body force vector
\([\{F_b}\)] body force vector per unit volume
\([\{FE\}]\) Element body force vector
\([\{FE_j}\)] sub-vectors of element body force vector
\([\{f\}]\) degrees of freedom of fluid elements
\([\{f\}]\) vector functions (Chapter 2)
\([g]\) acceleration of gravity
\([\{g\}]\) vector functions

15.
HE_{ij} coefficients of the element fluid stiffness matrix
Hz Hertz. cycles/second
h thickness of shell
[H] fluid stiffness matrix
[H(t)] fundamental matrix
[HB] fluid stiffness matrix after imposing boundary conditions
[HE] element fluid stiffness matrix
\{H_i(t)\} vector of the fundamental matrix
\{h\} vector functions
i, j, k unit vector in the x, y, z direction
[I] unit matrix
j = \sqrt{-1}
[J] Jacobian matrix
|J| Jacobian determinant
Kg kilos
[K] stiffness matrix
[KC] centrifugal stiffness matrix
[KCE] element centrifugal stiffness matrix
[KE] element stiffness matrix
[KE_{ij}] sub-matrices of the element stiffness matrix
[KE_{11ij}], [KE_{12ij}], [KE_{21ij}] and [KE_{22ij}] submatrices of [KE_{ij}]
[KG] geometrical or initial stress matrix
[KGE] element geometrical matrix
[KGE_{ij}] sub-matrices of the element geometrical matrix
[KT] geometric matrix due to dynamic stress
L Lagrangian
L fluid/structure matrix
[LE] element fluid/structure matrix
M_o moment per unit length
m metres
\([M]\) mass matrix
\([MA]\) added mass matrix
\([MAE]\) element added mass matrix
\([MC]\) mass centrifugal matrix
\([MCE]\) element mass centrifugal matrix
\([MCE_{ij}]\) sub-matrices of element mass centrifugal matrix
\([ME]\) element mass matrix
\([ME_{ij}]\) sub-matrices of element mass matrix
\([MG]\) gyroscopic matrix
\([MGE]\) element gyroscopic matrix
\([MGE_{ij}]\) sub-matrices of element gyroscopic matrix

\(N\) Newtons
\(N_i\) displacement shape functions
\(N_i'\) geometrical shape functions
\(N_i^*\) fluid element shape functions
\(N_{ij}\) shape functions
\(N_{x}, N_{y}\) compressible force per unit length in the x and y directions
\(N_{mn}, N_{mn}^x, N_{mn}^y\) dynamic buckling force/unit length in x and y directions

\([N]\) shape function matrix
\([N^*]\) shape function matrix of fluid element
\([N_{ij}]\) sub-matrices of shape function matrix
\([N_{ij}^*]\) sub-matrices of shape function matrix of fluid element
\([N_e]\) local element shape function matrix
\([\bar{N}]\) global shape function matrix
\([\bar{N}_e]\) global element shape function matrix

\(n\) unit vector normal to boundary
\([0]\) null matrix
\({0}\) null vector
P: external forces

$P_i$: boundary force per unit area in the $x_i$ direction

$P_j(\xi)$: Legendre polynomials

$P_0$: magnitude of sinusoidal or impact forces

p: pressure

$p$: highest derivative of function (Chapter 2)

{P}: boundary force vector

{P_s}: boundary force vector per unit area

{PE}: element boundary force vector

{PE_j}: sub-vectors of the element boundary force vector

{p}: degrees of freedom of fluid domain

{p}: Hamiltonian coordinates (Chapter 7)

{p}: principal coordinates (Chapter 10)

{p_a}: sub-vector of {p} corresponding to all nodal pressures not represented by {p_b}

{p_b}: sub-vector representing the nodal pressures subjected to fixed surface boundary conditions

$q_m(t)$: functions of time

{q}: degrees of freedom of structure system

{q_0}: steady state component of vector {q}

{q_1}: oscillatory component of vector {q}

{q_1}: primary coordinate vector (Chapter 3)

{q_2}: secondary coordinates vector (Chapter 3)

{q_r}: eigenvectors of reduced eigenvalue problem

{Q}: force vector

{Q_o}: static component of force vector

{Q_1}: dynamic component of force vector

{Q_C}: centrifugal force vector
\{QCE\} element centrifugal force vector
\{QCE_j\} sub-vector of element centrifugal force vector
\{QE\} element force vector
R radius of spherical or cylindrical shell
\bar{R} radius ratio. Radius of disc/length of blade
R_x, R_y radius of curvature in x and y direction
r_j nodal degree of freedom
r_i instantaneous position vector of a particle
r_o original position vector of a particle
\{r\} degrees of freedom of structural element
\{r_j\} nodal degrees of freedom of element
\{r_o\} steady state displacement vector of an element
S surface of problem
S_\Sigma surface vector
S_e surface of finite element
s seconds
s number of variables (Chapter 2)
T kinetic energy
T period of oscillation (Chapter 9)
t time
t_i nodal thickness
t_o duration of impact force
[T] transformation matrix
[T_e] element transformation matrix
[T_i] nodal transformation matrix
u, v, w displacements in the x, y, z direction
u', v', w' displacements in the x', y', z' direction
u_i displacements in the x_i directions
\( u_{i,j} \) \( = \frac{\partial u_i}{\partial x_j} \)

\( u_n \) fluid velocity in the \( n \) direction

\( u_{x,y,z} \) fluid velocity in the \( x,y,z \) directions

\( u \) displacement vector

\([U]\) modal matrix

\( \{u\} \) displacement vector

\( \{u^r\} \) first approximation of the eigenvectors of original eigenproblem (Chapter 3)

\( V \) potential energy

\( V_{3i} \) nodal thickness vector

\( V_{1i}, V_{2i}, V_{3i} \) vector defining local frame \( x', y', z' \)

\( v_n \) surface velocity normal to boundary

\( v \) velocity of moving boundary

\( v_{1i}, v_{2i}, v_{3i} \) nodal unit vector defining local frame \( x', y', z' \)

\( Xi \) Cartesian frame of reference

\( x_i \) Cartesian frame of reference

\( x,y,z \) Cartesian frame of reference

\( x,y,z \) rotating Cartesian frame of reference (Chapter 7)

\( x_0, y_0, z_0 \) components of \( r_0 \)

\( \{x_i\}, \{y_i\}, \{z_i\} \) coordinates of nodal points

\( w \) total number of degrees of freedom (Chapter 2)

\( \alpha \) angular velocity ratio. Angular velocity/fundamental natural frequency of non-rotating structure

\( \alpha_i \) angular deformation of nodal thickness vector about \( v_{2i} \)

\( \alpha_r \) eigenvalues of the reduced eigenproblem (Chapter 3)
$\beta$ angular velocity

$\vec{\beta}$ angular velocity vector

$\omega$ frequency of dynamic stress (Chapter 9)

$\tau$ time parameter (Chapter 10)

$\beta_i$ angular deformation of nodal thickness vector about $\nu_{ij}$

$\beta_r$ improved eigenvalues (Chapter 3)

$\delta_{ij}$ Kronecker delta

$\varepsilon_{ij}$ strain tensor

$\varepsilon_{xy}$ strain tensor with reference to $x, y, z$ frame

$\varepsilon_{x'y'}$ strain tensor with reference to $x', y', z'$ frame

$\varepsilon_{ij}^L$ linear component of strain tensor

$\varepsilon_{ij}^N$ nonlinear component of strain tensor

$\{\varepsilon\}$ strain vector

$\{\varepsilon^L\}$ linear component of strain vector

$\{\varepsilon^N\}$ nonlinear component of strain vector

$\theta$ setting angle

$\theta_b$ slope due to vertical deflection

$\theta_s$ shear deformation

$\phi_i$ element of vector $\{\phi\}$

$\phi_{mn}$ function satisfying the boundary conditions of the problem

$[\phi_j]$ matrix defining the local axes at nodal points

$\{\phi\}$ vector of unknown functions

$\psi$ pre-twist angle

$\{\psi\}$ vector of functions satisfying the boundary conditions of problem

$\Omega$ domain of problem

$\Omega_e$ domain of finite element

$\omega_i$ natural frequencies of system

$\omega_{mn}$ natural frequencies of system corresponding to mode $(m, n)$
\( \omega_{mn} \) natural frequencies of system without pre-stress

\([\omega^2]\) diagonal matrix containing \( \omega_i^2 \)

\( \lambda \) eigenvalue parameter

\( \lambda_i \) eigenvalues

\( \lambda_n \) frequency parameters corresponding to mode \( n \)

\( \lambda_{mn} \) frequency parameters corresponding to mode \( (m, n) \)

\([\lambda]\) diagonal matrix containing \( \lambda_i \)

\( \mu \) eigenvalue parameter

\( \mu_i \) eigenvalues of secondary eigenproblem

\( \nu \) Poisson ratio

\( \xi, \eta, \zeta \) curvilinear coordinates

\( \rho \) density

\( \tau_{ij} \) stress tensor

\( \tau_{xy} \) stress tensor corresponding to frame \( x, y, z \)

\( \tau_{x'y'} \) stress tensor corresponding to frame \( x', y', z' \)

\( \tau_{xx}^0 \) static stress

\( \tau_{xx}^c \) minimum critical buckling stress

\( (\tau_{xx}^c)_{mn} \) critical stress corresponding to buckling mode \( (m, n) \)

\( \tau_{xx}^d \) dynamic stress

\( \{\tau\} \) stress vector

\( \sigma_{xy} \) initial stress tensor

\( \sigma_{x'y'} \) initial stress tensor corresponding to frame \( x', y', z' \)

\([\sigma]\) initial stress tensor matrix

\([\sigma']\) initial stress tensor matrix corresponding to frame \( x', y', z' \)

\( \{\sigma\} \) initial stress vector

\( \nabla \) gradient operator

\( \nabla^2 \) Laplacian operator

\( \nabla^4 \) biharmonic operator
CHAPTER 1

INTRODUCTION

The objective of this Thesis is the development of Finite Element Models to predict the dynamic behaviour of shell structures, pre-stressed and rotating shell structures, and shells submerged in a fluid medium. Also, the development of a finite element model to predict the instability and stability regions of shell structures subjected to axial periodic forces is a further objective of this investigation. The shell structures have arbitrary geometry, thickness and boundary conditions and are subjected to arbitrary forcing functions.

The Finite Element Models developed in this Thesis are applicable to several important practical problems, such as the dynamic analysis of rotating compressor and turbine blades, propeller blades, shell structures, pre-stressed shell structures and shell structures submerged in a fluid medium. The models are also applicable to the dynamic instability of shells subjected to periodic axial forces.

The prediction of the dynamic behaviour at the design stage of structural systems is essential to improve the design of these structures and, therefore, to extend the working life of the structures. Some structures can be represented by beam or plate models, while others must be represented as thin or thick shells. The practical turbine, compressor and marine propeller blades are in this last category. Any representation of these blades as beams or plates is not an acceptable mathematical model.

The body and boundary forces acting on a structure change significantly the dynamic behaviour of the system. Therefore, the prediction of the dynamic behaviour of pre-stressed shell structures is of practical importance.
The dynamic behaviour of rotating structural systems is dependent on the angular velocity. In order to reduce the fatigue rate of the blades, it is necessary to predict the dynamic behaviour of these blades under rotating conditions.

A structure submerged in a fluid medium has a different dynamic behaviour than the same structure in vacuum. Therefore, it is essential to predict the dynamic behaviour of the submerged structure in order to improve the design of these structural systems.

A survey of the type and form of structural systems developed over this century reveals a continuous trend towards light and thin shell structures. The major disadvantage of the thin shell structure is that it tends to become unstable. Thus, the prediction of the stability and instability regions of thin shell structures subjected to axial periodic forces is of increasing practical importance.

An accurate general shell theory is very complex and virtually impossible to apply. Thus, simplifying assumptions are introduced according to the geometry of the shell. Many simplifications have led to inconsistencies, which lead to further modifications of the theories. Brief details of several shell theories are introduced by Kraus (1, 1967). Full details of the most commonly applied shell theories are presented by Vlasov (2, 1964) and Novoshilov (3, 1970). Leissa (4, 1973) presents an extensive literature survey of the application of shell theories to the dynamic analysis of structures.

Analytical solutions of the dynamic behaviour of shell structures are only possible in a few special cases. In general, a numerical method of solution must be used to analyse shell structures. Greenbaum and Cappeli (5, 1971) present the advantages and disadvantages of several numerical methods when applied to shell structures.
The most important of these methods are the Finite Element, Finite Difference and Direct Integration Methods. These authors concluded that at the present development of the numerical methods of shell analysis, the Finite Element Method is generally superior to the other methods.

The Finite Element Technique is an approximate method of solution of differential equations. The concept of the Finite Element Method was introduced by Argyris (6, 1954) and Turner, Clough, Martin and Topp (7, 1956) with reference to structural analysis. Previously, Levi (8, 1953) introduced the concept of replacing a continuous structural system by several finite elements, generating a stiffness matrix for each element and a stiffness matrix for the global system.

The introduction of the concept of finite elements to the solution of differential equations was presented by Szmelter (9, 1959). This author demonstrated that the Finite Element Technique is a discretized Rayleigh-Ritz method of solution of differential equations. An identical method had already been proposed by Courant (10, 1943). However, the mathematical foundations of the Finite Element Method were developed by Arantes Oliveira (11, 1968).

Since 1960, the Finite Element Method has been a major topic of research in solid mechanics. Recently it has become an important topic of research in fluid mechanics and mathematics. Extensive bibliographies of the development of the theory and its applications are presented by Akins, Fenton and Studdart (12, 1972), Norrie and De Vries (13, 1975) and Whiteman (14, 1975). Details of the mathematical foundations of the Finite Element Method are presented in Chapter 2.

An accurate Finite Element Model of a shell structure requires a large number of degrees of freedom to represent the system. Therefore, a large eigenvalue problem must be solved. Chapter 3 presents a survey of
several methods available to solve large eigenproblems and discusses their efficiency. Also, the mathematical foundations of the Eigenvalue Economizer are presented. This technique reduces the size of the eigenproblem, without loss of accuracy in the lower eigenvalues and eigenvectors.

The Finite Element Method has been successfully developed and applied to the dynamic analysis of shell structures. A literature survey of this development is presented in Chapter 6. This shows that the super-parametric shell element, developed by Zienkiewicz, Irons and Ahmad (15, 1970), and Zienkiewicz, Anderson and Ahmad (16, 1970) is a very efficient thick shell element.

Also, the development of the Reduced Integration Technique, by Zienkiewicz, Taylor and Too (17, 1971), to evaluate the strain energy of structures transforms the super-parametric shell element into a very efficient shell element. The element becomes applicable to thin or thick, shallow or deep shell structures. Also, the element is based on the Theory of Elasticity, therefore the development to nonlinear analysis is simpler than the corresponding development of elements based on shell theories. Consequently, the Super-parametric/Reduced Integration shell element is selected to represent all the Finite Element Models derived in this Thesis. Details of the super-parametric shell element and Reduced Integration Technique are presented in Chapters 4 and 5, respectively.

In Chapter 6, the equations of motion of a Finite Element Model of a pre-stressed, dynamic structural system are derived. This model is further developed for a super-parametric shell element representation and applied to the dynamic analysis of thin or thick, shallow or deep, shell structures with or without pre-stress. The efficiency of the super-parametric shell element with or without the Reduced Integration Technique is investigated and compared with other shell elements. The model is also applied to predict the dynamic characteristics of pretwisted and cylindrical
shell blades and several blade parameters are investigated. The results are compared with experimental, analytical and finite element solutions. A literature survey of the development and applications of shell elements is presented.

In Chapter 7, the equations of motion of a Finite Element Model of a rotating structure are derived. The model is further developed for a super-parametric shell element representation of a rotating shell structure. The Reduced Integration Technique is used to evaluate the strain energy of the structure. The model is applicable to rotating shell structures with arbitrary geometry and thickness and, therefore, to the dynamic analysis of rotating compressor and turbine blades. The dynamic characteristics and blade parameters of pretwisted and cylindrical shell blades are investigated. Also, a literature survey of the development and applications of the finite element dynamic analysis of rotating structures is presented.

At the beginning of this investigation a Finite Element Model of a rotating shell structure had not been developed. However, the finite element dynamic analysis of rotating plates had just been developed by Rawtani and Dokainish (18, 1971).

In Chapter 8, the Finite Element Model of a shell structure submerged in a fluid medium is developed. The model is based on a super-parametric shell element representation of the structure and an assembly of isoparametric fluid elements to represent the fluid domain. This model is capable of predicting the dynamic characteristics of submerged shell structures with arbitrary geometry and thickness, including propeller blades. However, the computer facilities available made it only possible to predict the dynamic characteristics of plates submerged in a compressible fluid. The predicted natural frequencies of submerged plates are compared with experimental values. Also a literature survey of the development and appli-
cations of the finite element dynamic analysis of submerged structures is presented.

At the beginning of this investigation a Finite Element Model of plate or shell structures submerged in a fluid medium had not been developed. However, the theory of the finite element analysis of submerged structures had already been developed by Zienkiewicz and Newton (19, 1969).

In Chapter 9, a Finite Element Model for the dynamic stability analysis of shell structures is developed. Previously, a finite element dynamic stability analysis of shell structures has not been reported. However, an analysis of plate structures has been presented by Hutt and Salama (20, 1971).

Also in this chapter, a literature survey of the development and applications of the finite element dynamic instability analysis of structures is presented. The theory of the dynamic instability analysis of multi-degree of freedom systems is introduced. It is demonstrated that a Finite Element Model of a pre-stressed, dynamic structural system can be used to predict the dynamic stability and instability regions of a structure subjected to axial periodic forces. The dynamic stability and instability regions of cylindrical shell blades subjected to periodic axial forces are calculated.

In Chapter 10, a model for the transient analysis of shell structures due to arbitrary dynamic forces is developed. The model is based on a finite element representation of the shell structure and the Modal Analysis Method to calculate the transient response of the discretized multi-degree of freedom system. The uncoupled equations of motion of the finite element representation are integrated by the Simpson's Rule. The super-parametric shell element is used to represent the shell structure, including pre-stressed and rotating systems. The Reduced Integration
Technique is used to evaluate the strain energy of the structure. A literature survey of the development and applications of the finite element transient analysis of shell structures is presented. The accuracy, efficiency and stability of the Modal Analysis and Direct Integration Methods, with particular reference to pre-stressed and rotating shell structures, are discussed. The Super-parametric Shell Element/Modal Analysis Model is applied to the transient response of shell structures, including pre-stressed and rotating systems. The predicted transient responses are compared with analytical solutions.

Prior to this investigation, finite element transient analysis of rotating and pre-stressed shell structures have not been reported. This is not surprising, since it is only recently that Key and Krieg (21, 1972) used finite elements to calculate the transient response of arbitrary shell structures.

Chapters 11 and 12 present some concluding remarks and possible further developments of this investigation, respectively.
2.1 Introduction

The Finite Element Technique is a numerical method of solution of differential equations. The method is introduced with reference to an equilibrium problem. In section (2.8) and Chapter 8, the method is introduced to structural dynamics and fluid mechanics problems, respectively.

Consider the system of differential equations,

\[ \begin{align*}
[A] \phi &= \{f\} \quad \text{in } \Omega \\
[C] \phi &= \{g\} \quad \text{in } S
\end{align*} \]  

(2.01)

where \([A]\) and \([C]\) are matrices of differential operators, \(\{\phi\}\), \(\{f\}\) and \(\{g\}\) are vector functions of \(d\) independent variables, \(x_1, x_2, \ldots, x_d\), \(\Omega\) and \(S\) are the domain and boundary of the problem.

In general, equation (2.01) cannot be solved analytically. Mikhlin and Smolitzky (22, 1967) present several approximate methods of solution of these equations. The most important of these are the Variational, Residual and Finite Difference Techniques.

In the Variational and Residual methods, a trial function of linearly independent functions matrix \([\beta]\) is assumed, such that

\[ \{\phi\} = [\beta] \{a\}, \]  

(2.02)

where the unknown parameter vector \(\{a\}\) is determined subsequently.

2.2 Variational Methods

In Variational Calculus, the problem of minimizing a functional reduces to the solution of a system of differential equations, subjected to appropriate boundary conditions. Also, in certain cases, the solution of
differential equations with given boundary conditions can be reduced to the problem of minimizing a functional. This is the basis of all Variational Methods of solution of differential equations. The trial functions need only to be Admissible Functions, satisfying the principal boundary conditions of the problem.

When the differential matrix operator \([A]\) in equation (2.01) is linear, self-adjoint and positive definite and the boundary conditions of the problem are homogeneous, the Minimum Functional Theorem states that the functional \(F(\{\psi\})\) to be minimized is given by,

\[
F(\{\psi\}) = \int_{\Omega} \{\psi\}^T [A] \{\psi\} d\Omega - 2 \int_{\Omega} \{\psi\}^T \{f\} d\Omega \tag{2.03}
\]

or, for convenience,

\[
F(\phi) = <A\phi, \phi> - 2<\phi, f>. \tag{2.04}
\]

The definitions of linear, self-adjoint and positive definite differential operators, the proof of the Minimum Functional Theorem and of the minimum value of the functional are presented by Mikhlin (23, 1964).

In the case of differential equations with non-homogeneous boundary conditions, as in equation (2.01), the functional to be minimized can be modified. Let \(\{\psi\}\) be a vector function of independent variables, \(x_1, x_2, \ldots, x_d\), which satisfies the boundary conditions of the problem. Defining,

\[
\{\phi\} = \{\phi\} - \{\psi\} \tag{2.05}
\]

then equation (2.01) can be transformed into,

\[
[A] \{\phi\} = \{f\} - [A] \{\psi\} \quad \text{in } \Omega \tag{2.06}
\]

\[
[C] \{\phi\} = \{0\} \quad \text{in } S
\]

where \(\{0\} = \text{Null Vector.}\)
Consequently, equation (2.03) becomes,

\[ F(\phi) = \langle A\phi, \phi \rangle - 2\langle \phi, f \rangle + \langle A\psi, \phi \rangle - \langle A\phi, \psi \rangle + 2\langle \psi, f \rangle - \langle A\psi, \psi \rangle \quad (2.07) \]

It should be noticed that the last two terms of this functional are constants. Therefore they are irrelevant in the minimization process. Using Green's Theorem (23), the second and third terms of the functional can be transformed to a boundary integral. Also, the first term of the functional can be transformed to a domain integral of lower order and a boundary integral.

In general, the functional to be minimized is of lower order than the operator \([A]\) and the boundary integrals have linear and quadratic forms. Thus,

\[ F(\phi) = \int_{\Omega} \{\phi\}^T \left[ G \right]^T [L] [G] \{\phi\} d\Omega - 2\int_{\Omega} \{\phi\}^T \{f\} d\Omega + \int_{S} \{\phi\}^T [H] \{\phi\} dS \]

\[ + \int_{S} \{\phi\}^T \{h\} dS \quad (2.08) \]

where matrices \([G]\), \([L]\) and \([H]\) and vector \(\{h\}\) can be determined for each particular problem.

Structural mechanics and other engineering problems are characterised by Variational Principles. The differential equations defining the problem are the consequence of these Variational Principles. In these problems the Variational Methods can be applied directly to the Variational Principle.

There are several Variational Methods, the most important of these are the Rayleigh-Ritz, Finite Difference, Kantorovitch and Trefftz Techniques. Full details of these variational methods are presented by Mikhlin (23, 1964). Although all the variational methods can be used in the Finite Element Analysis, the Rayleigh-Ritz Finite Element Technique is the most
important and commonly used. This method is presented in this chapter.

2.3 **Residual Methods**

The variational methods are not applicable to all differential equations, since the differential operator must be linear, self-adjoint and positive definite. These methods are less versatile than residual methods, which do not have these restrictions.

In the Residual Methods, the Residual or Error Vector is defined as,

\[ \{ R \} = \{ f \} - [A] \{ a \} \tag{2.09} \]

and it is required to satisfy certain conditions which make this error a minimum. The trial functions are Comparison Functions, satisfying the natural boundary conditions of the problem.

There are several residual methods, the most important of these are the Galerkin, Least Squares, Collocation, Orthogonality, Absolute Error, Sub-Domain and Moments Techniques. Full details of these methods are presented by Finlayson (24, 1972). Although all of the residual methods can be used in the Finite Element Analysis, the Galerkin and Least Square Finite Element Methods are most commonly used. The Galerkin Finite Element Method is briefly presented in this chapter.

2.4 **Finite Element Approximation**

A Finite Element Method is one in which the domain of the differential equations is represented by sub-domains or Finite Elements. Also a functional representation of the solution is adopted within each finite element and the parameters of the assembly become the unknown of the problem. Thus, in the Finite Element Analysis, the original differential equations having an infinite number of degrees of freedom are transformed into approximate equations with a finite number of degrees of freedom.
Let the domain and boundary of the differential equation, defined by equation (2.01), be divided into \( n \) sub-domains \( \Omega_e \) and sub-boundaries \( S_e \). Let \( \{r\} \) be the vector defining the \( m \) nodal values of the \( s \) dependent variables \( \{\psi\} \) and their derivatives within each sub-domain. Also let \( \{q\} \) be the vector defining the \( w \) nodal values of the dependent variables and their derivatives within the domain of the problem. The \( d \) independent variables are \( x_1, x_2, \ldots, x_d \).

Within a finite element, the dependent variables \( \psi_i \) are a function of the independent variables and of the nodal values of the dependent variables and their derivatives, \( r_j \). Thus,

\[
\psi_i = \sum_{j=1}^{m} N_{ij}(x_1, x_2, \ldots, x_d)r_j \quad (i = 1, 2, \ldots, s) \text{ in } \Omega_e
\]  

(2.10)

where \( N_{ij} \) are the shape functions. Clearly, the Lagrange or Hermitian Interpolation Polynomials can be used as shape functions.

In matrix form, this equation becomes,

\[
\{\phi\} = [N]\{r\} \text{ in } \Omega_e
\]  

(2.11)

where \([N]\) is an \( s \times m \) matrix called the Shape Function Matrix.

The relationship between the local degrees of freedom and the global degrees of freedom is the Boolean Matrix \([T_e]\), which is an \( m \times w \) matrix of zeros and ones. Thus, equation (2.11) becomes,

\[
\{\phi\} = [N][T_e]\{q\} = [R_e]\{q\} \text{ in } \Omega_e
\]  

(2.12)

where \([R_e]\) is an \( s \times w \) matrix.

It should be noticed that every element of \([R_e]\) and \( \{\phi\} \) are zero for any point in the domain outside the sub-domain \( \Omega_e \). Thus, the elements of \([R_e]\) are Pyramid and Orthogonal Functions. Consequently, equation (2.12) becomes,
\{q\} = \left[[\mathbf{N}_1]^t \mathbf{N}_2] + \ldots + [\mathbf{N}_e]^t + \ldots + [\mathbf{N}_n]^t\right] \{q\} = \mathbf{N} \{q\} \text{ in } \Omega \quad (2.13)

where \(\mathbf{N}\) is an \(s \times w\) matrix called the Global Shape Function Matrix.

Comparison of this equation with equation (2.02) shows that the Finite Element Technique is a particular case of a more general approximation. In the Finite Element Analysis, the unknown vector \(\{a\}\) of equation (2.02) is the nodal value of the dependent variables and their derivatives, \(\{q\}\). Thus, the Finite Element Technique can be used to evaluate complex trial functions for the Variational or Residual Methods of solution of differential equations.

The major difficulty of the Finite Element Method is to choose the shape functions of any particular element. The problem of Conformity, Completeness and Convergence of finite elements is discussed in section (2.7).

2.5 Rayleigh-Ritz Finite Element Method

The solution of the differential equation (2.01), when the differential matrix operator \([A]\) is linear, self-adjoint and positive definite (23), is identical to the solution of the minimization of the functional defined by equation (2.08).

Let the domain of the differential equation be divided into \(n\) finite elements, whose shape function matrix is defined by equation (2.11). Also let \(\{q\}\) represent the degrees of freedom of the discretized problem.

Substituting equation (2.13) into equation (2.08), the functional to be minimized becomes,

\[
F(\{q\}) = \{q\}^t \int_{\Omega} \left[\mathbf{N}^t \mathbf{G}[L] \mathbf{G} \mathbf{N}\right] d\Omega \{q\} - 2\{q\}^t \int_{\Omega} \left[\mathbf{N}^t \{f\}\right] d\Omega + \{q\}^t \int_{S} \left[\mathbf{N}^t \{h\}\right] dS \quad (2.14)
\]
Minimization of this functional requires that,
\[ \frac{\partial F(q)}{\partial q} = 0 \]  \hspace{1cm} (2.15)

Consequently, the System Matrix Equation becomes,
\[ [K]q = Q, \]  \hspace{1cm} (2.16)

where,
\[ [K] = \int_{\Omega} [N]^t [G]^t [L] [G] [N] d\Omega + \int_{S} [N]^t [H] [N] dS \]  \hspace{1cm} (2.17)

and
\[ q = \int_{\Omega} [N]^t f d\Omega - \frac{1}{2} \int_{S} [N]^t h dS \]  \hspace{1cm} (2.18)

Equation (2.16) is a linear, symmetric matrix equation, which can be solved by standard methods. The solution of this equation is the value of the dependent variables and their derivatives at certain nodal points in the domain of the problem.

It is convenient to derive equations (2.17) and (2.18) with reference to the local shape function matrix. Substituting equations (2.12) and (2.13) into equation (2.17) and integrating this equation, gives

\[ [K] = \sum_{e=1}^{n} [T_e]^t [KE] [T_e] \]  \hspace{1cm} (2.19)

where
\[ [KE] = \int_{\Omega_e} [N]^t [G]^t [L] [G] [N] d\Omega + \int_{S_e} [N]^t [H] [N] dS \]  \hspace{1cm} (2.20)

is the Element Matrix.

Similarly
\[ q = \sum_{e=1}^{n} [T_e]^t \{QE\} \]  \hspace{1cm} (2.21)
where

$$\{qE\} = \int_{\Omega_e} [N]^T \{f\} d\Omega - \int_{S_e} [N]^T \{f\} dS \quad (2.22)$$

is the Element Vector.

It should be noticed that the Element Matrices can be obtained from the System Matrices by replacing global with local shape functions. Also it is not necessary to multiply the matrices in equations (2.19) and (2.21), since the transformation matrix is a Boolean matrix.

2.6 Galerkin Finite Element Method

In the Galerkin method of solving equation (2.01), the Residual, defined by equation (2.09), is required to be orthogonal to the trial functions. Thus,

$$\int [N]^T \{f\} d\Omega - \int [N]^T [A] [N] d\Omega \{q\} = \{0\} \quad (2.23)$$

or

$$\{K\} \{q\} = \{Q\} \Omega_e \quad (2.24)$$

where $[K] = \int [N]^T [A] [N] d\Omega$ and $\{Q\} = \int [N]^T \{f\} d\Omega$.

Consequently, the corresponding Element Matrices are given by,

$$[KE] = \int_{\Omega_e} [N]^T [A] [N] d\Omega \quad \text{and} \quad \{qE\} = \int_{\Omega_e} \{N\}^T \{f\} d\Omega. \quad (2.25)$$

It should be noticed that the Element Matrix is of the same order of operator $[A]$, thus higher than the corresponding matrix operator in the Rayleigh-Ritz Method. However, using Green's Theorem (23), it is possible to reduce the order of the matrix operator if $[A]$ is a linear operator. When the operator $[A]$ is non-linear, the system matrix equation is a non-linear system of equations. When the operator $[A]$ is linear, but not self-adjoint, the System Matrix Equation is an unsymmetric linear system of equations.

37.
2.7 Conformity, Completeness and Convergence

In the Finite Element Technique, it is often required to solve integral equations of the type,

\[ F(\{\phi\}) = \int_{\Omega} f_1(\{\phi\}, \frac{\partial}{\partial x_1}(\{\phi\}), \ldots, \frac{\partial^p}{\partial x_1^p}(\{\phi\}))d\Omega + \]

\[ \int_{\partial\Omega} f_2(\{\phi\}, \frac{\partial}{\partial x_1}(\{\phi\}), \ldots, \frac{\partial^p}{\partial x_1^p}(\{\phi\}))dS \]  

(2.26)

where \( p \) is the highest derivative of the functional.

The exact solution of the dependent variables are continuous functions. This solution has continuous derivatives up to at least order \( p \). In the finite element approximation of the solution, the variables and their derivatives up to order \( p - 1 \) (Principal Derivatives) must be continuous, and the highest derivative has a piecewise continuous representation. As the element size approaches zero, the highest derivative tends to be continuous. The requirement that the representation of the variables and their principal derivative are continuous is known as the Principal Continuity Conditions.

Also, at coincident nodes of adjacent elements, the nodal values of the variables and their derivatives must be identical. This Nodal Compatibility or Reduced Continuity Condition must always be satisfied.

A finite element that satisfies both the Principal Continuity and Compatibility Conditions is called a Conforming element. If the Principal Continuity Conditions are not satisfied, the element is Non-conforming.

Mikhlin (23, 1964) demonstrates that the sufficient condition for convergence of the Rayleigh-Ritz Method is the Completeness criterion of the trial functions. Completeness of a trial function with reference to the differential operator is defined as,
where \( \{ \delta \} \) is a vector of arbitrary positive small numbers.

Thus, in the Rayleigh-Ritz Finite Element Method, Completeness and Conformity are sufficient conditions for monotonic convergence of the solution, as the element sizes decrease to zero. Further, Arantes Oliveira (11, 1968) demonstrated that Completeness is a necessary and sufficient condition for convergence of the Rayleigh-Ritz Finite Element Method.

In general, the functional of equation (2.26) for a variational finite element is of lower order than the corresponding functional for a residual finite element. It is simpler to develop a conforming variational finite element than a conforming residual finite element.

2.8 Finite Element Dynamic Analysis of Structures

Solid mechanics problems are characterized by Variational Principles. Since the Rayleigh-Ritz Finite Element Technique is based on variational principles, this method is almost exclusively used in static and dynamic solid mechanics problems. In this section, the Rayleigh-Ritz Finite Element Method is introduced to the dynamic analysis of structural systems. It is assumed that the displacements and deformations are small.

Consider an elastic solid of volume \( \Omega \), density \( \rho \), modulus of elasticity \( E \), Poisson's ratio \( \nu \) and surface area \( S \), with reference to a Cartesian set of axes \( x_i \) \((i = 1, 2, 3)\). The elastic displacements in the \( x_i \) directions are \( u_i \).

Let the domain of the problem be represented by finite elements of volume \( \Omega_e \) and surface area \( S_e \), whose shape function matrix and degrees of freedom are \( [N] \) and \( \{r\} \), respectively. Also, let \( \{q\} \), \( [R] \), and \( \{Q\} \) be the degrees of freedom, global shape function matrix and force vector of the discretized system. Thus,
\begin{equation}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{pmatrix} = \{u\} = [N]\{r\} \quad \text{and} \quad \{\dot{u}\} = [N]\{\dot{r}\} \quad \text{in} \quad \Omega_e
\end{equation}

\begin{equation}
\{u\} = [N]\{q\} \quad \text{and} \quad \{\dot{u}\} = [N]\{\dot{q}\} \quad \text{in} \quad \Omega
\end{equation}

where \{\dot{q}\}, \{\dot{r}\} and \{\dot{q}\} are the time derivatives of the corresponding displacement vectors.

\subsection{Stress and strain vector}
In the Linear Theory of Elasticity, the linear strain tensor \(\varepsilon_{ij}^L\) is given by,
\begin{equation}
\varepsilon_{ij}^L = \frac{1}{2}(u_{i,j} + u_{j,i})
\end{equation}

where \(u_{i,j} = \frac{\partial u_i}{\partial x_j}\) \(i, j = 1, 2, 3\).

The linear strain vector \(\{\varepsilon\}^L\) and stress vector \(\{\tau\}\) are defined by the components of the strain and stress tensor as,
\begin{equation}
\{\varepsilon\}^L = \begin{pmatrix}
\varepsilon_{11}^L \\
\varepsilon_{22}^L \\
\varepsilon_{33}^L \\
\end{pmatrix} \quad \text{and} \quad \{\tau\} = \begin{pmatrix}
\tau_{11} \\
\tau_{22} \\
\tau_{33} \\
\tau_{12} \\
\tau_{13} \\
\tau_{23} \\
\end{pmatrix}
\end{equation}

The expansion of equation (2.29) gives,
\begin{equation}
\{\varepsilon\}^L = [C]^L\{u\}
\end{equation}
Consequently, the strain vector is given in terms of the local or global degrees of freedom of the discretized system as,

\[ \{c^L\} = [B_L] \{r\} \quad \text{in } \Omega_e \]

\[ \{c^L\} = [B_L] \{q\} \quad \text{in } \Omega \]

(2.32)

where \([B_L] = [C_L][N]\) and \([B_L] = [C_L][N]\).

Matrices \([B_L]\) and \([B_L]\) are known as the Local and Global Linear Strain Matrices, respectively.

The stress and strain tensors are related by the generalised Hooke's Law. For a homogeneous, isotropic, elastic material, this law can be written in the following form,

\[ \tau_{ij} = \frac{E \nu}{(1+\nu)(1-2\nu)} \delta_{ij}c^L_{ss} + \frac{E}{(1+\nu)} c^L_{ij} \]

(2.33)

where \(\delta_{ij} = \text{Kronecker delta.}\)

The expansion of this equation determines the Elasticity Matrix \([D]\),

\[ \{\tau\} = [D]\{c^L\}. \]

(2.34)

2.8.2 Potential and kinetic energies

The Potential Energy \(V\) and the Kinetic Energy \(T\) of the Finite Element Model are given by,

\[ V = \frac{1}{2} \int_{\Omega} \{c^L\}^t \{\tau\} d\Omega \quad \text{and} \quad T = \frac{1}{2} \int_{\Omega} \rho \{q\}^t \{\dot{q}\} d\Omega \]

(2.35)

\[ V = \frac{1}{2} \{q\}^t \int_{\Omega} [B_L]^t [D] [B_L] d\Omega \{q\} \quad \text{and} \quad T = \frac{1}{2} \{q\}^t \int_{\Omega} \rho [\ddot{q}]^t [\ddot{q}] d\Omega \{q\} \]

(2.36)
2.8.3 Equations of motion

The equations of motion of any Conservative, Holonomic, dynamic system can be derived from the Hamilton's Principle,

\[ \int_{t_1}^{t_2} (T - V) dt = \text{Minimum} \quad (2.37) \]

where \( t_1 \) and \( t_2 \) are two instants of time \( t \).

The minimisation of Hamilton's Principle leads to Lagrange's equation,

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{d}{dt} \left( \frac{\partial V}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial \dot{q}_i} = \{Q_i\} \quad (2.38) \]

Applying Lagrange's equations to the kinetic and potential energies of the Finite Element Model, the equation of motion of the discretized system becomes,

\[ [M]\{\ddot{q}\} + [K]\{q\} = \{Q\} \quad (2.39) \]

where

\[ [M] = \int_{\Omega} \rho [\cN]^t [\cN] d\Omega = \text{Mass Matrix} \quad (2.40) \]

\[ [K] = \int_{\Omega} [\cB_L]^t [\cD] [\cB_L] d\Omega = \text{Stiffness Matrix} \quad (2.41) \]

The Mass and Stiffness Matrices were derived with reference to the global degrees of freedom of the system. The Element Matrices can be immediately derived by replacing the global with local shape functions.

Equation (2.39) is a linear, symmetric system of ordinary differential equations. The solution of this equation is presented in Chapter 10. The natural frequencies and mode shapes of structural dynamic systems are given by the solution of the Eigenvalue problem.
\[ [K] \{q\} = \lambda [M] \{q\} \quad \lambda = \text{Eigenvalue parameter} \quad (2.42) \]

A Finite Element Model of structural dynamic systems requires large numbers of degrees of freedom to represent the problem. The solution of these large eigenvalue problems is discussed in Chapter 3.
CHAPTER 3

THE EIGENVALUE ECONOMIZER

3.1 Introduction

In the dynamic and stability analysis of structures using the Finite Element Method, the following equation must be solved,

\[ [A] \{q\} = \lambda [B] \{q\} \]  

(3.01)

where \([A]\) and \([B]\) are symmetric band matrices, \([B]\) is also positive definite, \(\{q\}\) is a vector and \(\lambda\) is an eigenvalue parameter.

When the order of the matrices of this Eigenvalue Problem is large, the computer time required to solve all the eigenvalues and eigenvectors is enormous. In practice, the knowledge of all eigenvalues and eigenvectors is not essential, which can reduce considerably the computer time. However, the computer time required to solve partially the eigenproblem is a major part of the total time required to solve the dynamic or stability problem. Therefore, it is necessary to select an efficient method of solution for the eigenvalue problem.

Wilkinson (25, 1965) presents several methods of solution for the eigenproblem. However, there is not one method which always provides an economical solution. It is necessary to select the most efficient method, dependent on the order and bandwidth of the matrices and the number of eigenvalues and eigenvectors required.

When the matrices are small (less than 50 * 50) and all eigenvalues and eigenvectors are required, the Householder-QR-Inverse Iteration Technique, given by Wilkinson and Reinsch (26, 1971), is regarded as the best available method.
When the matrices are of large order but with a small bandwidth, the method used should take advantage of the banded form of the matrices. If only a few eigenvalues and eigenvectors are required, the Sturm Sequence and Inverse Iteration Technique developed by Gupta (27, 1973) or the Determinant Search Solution developed by Bathe and Wilson (28, 1973) are the most efficient methods. Also the Simultaneous Iteration Technique developed by Jenning (29, 1973) is efficient.

In the Finite Element Analysis of shell structures, the mass and stiffness matrices are of large order and bandwidth. Even after bandwidth reduction, as presented by Collins (30, 1973), the bandwidth of these matrices is very large. Also, it is often not possible to store these matrices in the high speed storage of modern computers, which further complicates the solution of the eigenvalue problem. The Subspace Iteration Technique, developed by Bathe and Wilson (31, 1972), is the most efficient direct method of solution of these large eigenvalue problems. The method is only efficient to calculate the lowest eigenvalues and corresponding eigenvectors of the system.

Alternatively, another method of solution of these large eigenvalue problems is to reduce the size of the matrices using the Eigenvalue Economizer and to solve the much smaller eigenvalue problem. This smaller eigenproblem is efficiently solved by the Householder-QR-Inverse Iteration Technique (26). This method is superior to the previous one.

In this Thesis, the element matrices are assembled and reduced in size by the Frontal Technique and the Eigenvalue Economizer. The reduced eigenvalue problem is solved by the Householder-QR-Inverse Iteration Method (26). Details of the Frontal Technique are presented by Irons (32, 1970).

The Eigenvalue Economizer reduces the size of the eigenvalue problem without significant loss of accuracy in the lowest eigenvalues and corresponding eigenvectors.
In the Eigenvalue Economizer Technique two sets of degrees of freedom are used to describe the eigenvalue problem. The first of these is the set required for the analysis of the problem. The second set is a sub-set of the first set which allows an adequate description of the lowest eigenvectors of the eigenvalue problem. This sub-set is much smaller than the first set, usually 5 to 10% of the original degrees of freedom.

The coordinates of the sub-set are called Primary Coordinates. The coordinates of the full set which do not belong to this sub-set are called Secondary Coordinates. It is these secondary coordinates that are eliminated.

The concept of the Eigenvalue Economizer was introduced by Irons (33, 1963; 34, 1965) and Guyan (35, 1965). Later, Anderson (36, 1968) and Zienkiewicz, Irons and Anderson (37, 1968) used the Eigenvalue Economizer to reduce the number of degrees of freedom of a finite element representation of the dynamic and stability analysis of plates. Ramsden and Stoker (38, 1969) after calculating the eigenvalues and eigenvectors of the reduced eigenproblem used these eigenvectors to estimate the eigenvalues of the original eigenvalue problem.

In all these references, the Eigenvalue Economizer is presented as a natural extension of the concept of Static Condensation, which is an exact method of static analysis. Wright and Miles (39, 1971) and Geradin (40, 1971) developed the mathematical foundations of the Eigenvalue Economizer. Fried (41, 1972) has also contributed to the understanding of the technique by demonstrating that the Eigenvalue Economizer is a combination of the Rayleigh-Ritz and Power Methods.

In this chapter the mathematical foundations of the Eigenvalue Economizer Technique are presented. The Upper Bound property of the approximate eigenvalues is demonstrated. The selection of primary coordinates and its automation is discussed. Further applications of the Eigenvalue Economizers are presented.
3.2 Mathematical Foundations

Consider the standard, symmetric, eigenvalue problem defined by equation (3.01). Let \( n \) be the number of degrees of freedom of the system and \( m \) the number of primary coordinates. Also let the degrees of freedom be defined as follows:

\[
\{q\} = \begin{bmatrix} \{q_1\} \\ \{q_2\} \end{bmatrix}
\]

where \( \{q_1\} \) primary coordinates vector \( \{q_2\} \) secondary coordinates vector (3.02)

It is convenient to define the eigenvalue problem in the sub-matrix form as follows:

\[
\begin{bmatrix}
[A_{11}] & [A_{12}] \\
[A_{21}] & [A_{22}]
\end{bmatrix}
\begin{bmatrix}
\{q_1\} \\
\{q_2\}
\end{bmatrix}
= \lambda
\begin{bmatrix}
[B_{11}] & [B_{12}] \\
[B_{21}] & [B_{22}]
\end{bmatrix}
\begin{bmatrix}
\{q_1\} \\
\{q_2\}
\end{bmatrix}
\]

(3.03)

The solution of this equation is given by,

\[
\{q\} = [T]\{q_1\} = \begin{bmatrix} [I] & [TD] \end{bmatrix}\{q_1\}
\]

(3.04)

where

\[
[TD] = -[[1] - \lambda [A_{22}]^{-1}[B_{22}]]^{-1} [A_{22}]^{-1}[[A_{22}] - \lambda[B_{22}]]
\]

\([I] = Unit\ Matrix.\n\]

After transformation, the eigenvalue problem becomes,

\[
[T]'[A][T]\{q_1\} = \lambda [T]'[B][T]\{q_1\}.
\]

(3.05)

This symmetric eigenvalue problem has \( m \) degrees of freedom. Thus, the original system of \( n \) degrees of freedom is reduced to \( m \) degrees of freedom. Difficulties arise, since the transformation matrix is a function of the eigenvalues. Therefore, some approximation is necessary to evaluate the reduced eigenvalue problem.
The First Approximation of the transformation matrix is given by,

\[ T = \begin{bmatrix} \text{--} & \text{--} \\ -[A_{22}]^{-1} \end{bmatrix} \]  \hspace{1cm} (3.06)

Consequently, the Reduced Eigenvalue Problem becomes,

\[ [C](q_1) = \lambda [D](q_1), \]  \hspace{1cm} (3.07)

where

\[ [C] = [A_{11}] - [A_{12}][A_{22}]^{-1}[A_{21}] \]
\[ [D] = [B_{11}] + [A_{12}][A_{22}]^{-1}[B_{22}][A_{22}]^{-1}[A_{21}] \\
- [A_{12}][A_{22}]^{-1}[B_{21}] - [B_{12}][A_{22}]^{-1}[A_{21}]. \]

Thus, as a First Approximation, the \( n \) degrees of freedom symmetric eigenvalue problem is reduced to an \( m \) degree of freedom, symmetric eigenvalue problem. Rohrle (42, 1972) used the eigenvalues of the Reduced Eigenvalue Problem to improve the transformation matrix, therefore improving the Reduced Eigenvalue Problem.

The Second Approximation of the transformation sub-matrix becomes,

\[ [TD] = -[1] + \lambda [A_{22}]^{-1}[B_{22}]][A_{22}]^{-1}[[A_{21}] - \lambda [B_{21}]] \]  \hspace{1cm} (3.08)

Consequently, the Reduced Eigenvalue Problem, neglecting all second order terms, is given by,

\[ [C](q_1) = \lambda [D](q_1) + \lambda^2 [G](q_1), \]  \hspace{1cm} (3.09)

where

\[ [G] = [B_{12}][A_{22}]^{-1}[B_{21}] + [A_{12}][A_{22}]^{-1}[B_{22}][A_{22}]^{-1}[B_{22}][A_{22}]^{-1}[A_{21}] \\
- [B_{12}][A_{22}]^{-1}[B_{22}][A_{22}]^{-1}[A_{21}] - [A_{12}][A_{22}]^{-1}[B_{22}][A_{22}]^{-1}[B_{21}] \]

This equation is a non-standard, symmetric eigenvalue problem with \( m \) degrees of freedom. This eigenvalue problem can be transformed to
the standard symmetric form as follows:

\[
\begin{bmatrix}
[G] & [0] \\
[0] & [C]
\end{bmatrix}
\begin{bmatrix}
\lambda(q_1) \\
q_1
\end{bmatrix}
= \lambda
\begin{bmatrix}
[0] & [G] \\
[G] & [D]
\end{bmatrix}
\begin{bmatrix}
\lambda(q_1) \\
q_1
\end{bmatrix}
\]

(3.10)

where \([0]\) is a Null Matrix.

This equation is a symmetric eigenvalue problem with 2m degrees of freedom. Although the number of degrees of freedom is twice the number of primary coordinates, this value is much smaller than the original number of degrees of freedom.

The Third and Further Approximations of the original eigenvalue problem are possible. However, each approximation increases the number of degrees of freedom of the reduced eigenvalue problem, which is a disadvantage. Examples presented by Wright and Miles (39, 1971) show that the First Approximation of the eigenvalue problem is, in general, more economical than the Second Approximation.

The validity of any approximation of the eigenvalue problem is limited to cases which the following equation converges,

\[
[I] - \lambda[A_{22}]^{-1}[B_{22}] = [I] + \lambda[A_{22}]^{-1}[B_{22}] + \lambda^2[A_{22}]^{-1}[B_{22}][A_{22}]^{-1}[B_{22}]
\]

+ .... (3.11)

Let the Secondary Eigenvalue Problem be defined as,

\[
[A_{22}]{q_2} = \mu[B_{22}]{q_2}
\]

(3.12)

\(\mu = \) Eigenvalue parameter

and the eigenvalue spectrum be such that,

\[\mu_1 \leq \mu_2 \leq \mu_3 \cdots \leq \mu_{n-m}\] (3.13)

Then, it can be proved that equation (3.11) converges for all values of \(\lambda\) such that,

\[\lambda/\mu_1 < 1.\] (3.14)
Equation (3.11) is convergent for all values of the original eigenvalue problem which are less than $\mu_1$, and, therefore, an approximation of these eigenvalues can be obtained by solving the reduced eigenvalue problem. For all other eigenvalues, equation (3.11) is not convergent and, therefore, the solution of the reduced eigenvalue problem is not accurate.

The accuracy of the approximation of the eigenvalue $\lambda_i$ depends upon the ratio $\lambda_i/\mu_1$ and decreases as the value of $\lambda_i$ increases. Thus, the lower eigenvalues and corresponding eigenvectors are more accurate than the higher eigenvalues and corresponding eigenvectors. Also, the value of the eigenvalues can be improved and the number of approximate eigenvalues increased by selecting the secondary coordinates such that the value of $\mu_1$ is as large as possible. The optimum choice of primary coordinates is the one which gives the maximum value for the lowest eigenvalue of the secondary eigenvalue problem. This secondary system is equivalent to the original system with the primary coordinates constrained to be zero. Therefore, it can be concluded that in regions of high flexibility, a higher proportion of primary coordinates should be selected than in a region of high rigidity.

Consequently, in structural systems an automatic selection of the primary coordinates can be achieved. It is only necessary to assemble the diagonal of the Stiffness and Mass Matrices and compare the magnitude of the ratios $K_i/M_i$, where $K_i$ and $M_i$ are the diagonal elements of the stiffness and mass matrices, respectively. A similar method of automatic selection of primary coordinates has been developed by Henshell and Ong (43, 1975).

In practice, it is not always convenient to number the coordinates of a system such that the primary coordinates are in sequential order. This problem can be avoided by the exchange of certain rows and columns.

50.
of the matrices of the eigenvalue problem. Also, the secondary coordinates can be eliminated individually, as presented by Zienkiewicz, Irons and Anderson (37, 1968). The algorithm is given by,

\[ A^*_{ij} = A_{ij} - A_{is}A_{js}/A_{ss} \]

and

\[ B^*_{ij} = B_{ij} - B_{is}A_{js}/A_{ss} - B_{js}A_{is}/A_{ss} + B_{ss}A_{is}A_{js}/A_{ss}^2 \]

where \( A_{ij} \) and \( B_{ij} \) are elements of the original matrices, \( A^*_{ij} \) and \( B^*_{ij} \) are elements of the reduced matrices and \( s \) is the secondary coordinate to be eliminated.

The main advantage of the Eigenvalue Economizer is not obvious. The technique can be used to reduce the size of the matrices at every stage of the matrix assembly. The assembly of each finite element is followed by a reduction of some secondary coordinates.

3.3 Upper Bound Property

Let the eigenvalues of the Reduced Eigenvalue Problem be

\[ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r \leq \ldots \leq \lambda_m \]  

(3.16)

and the corresponding eigenvectors \( \{q_1^r\} \).

Also, let the eigenvalues of the Original Eigenvalue Problem be

\[ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r \leq \ldots \leq \lambda_n \]  

(3.17)

and the corresponding eigenvector \( \{q^r\} \).

The First Approximation of the eigenvectors of the original equation, \( \{u^r\} \), is given by,

\[ \{u^r\} = \begin{bmatrix} \frac{1}{[A_{22}]^{-1}[A_{21}]} \end{bmatrix} \{q_1^r\} \]  

(3.18)

from which an improved value of the eigenvalue, \( \beta_r \), can be obtained.
Equation (3.01) can be modified such that,

\[ \beta_r = (u^r)^t[A](u^r)/(u^r)^t[B](u^r). \]  \hspace{1cm} (3.19)

This equation is the well-known Rayleigh Quotient. It can be assumed that the vector \( \{u^r\} \) is a linear combination of the eigenvector of the original system,

\[ \{u^r\} = \sum_{i=1}^{n} c_i \{q^i\} \]  \hspace{1cm} (3.20)

where \( c_i \) are constants.

Substituting this equation into the Rayleigh Quotient and using the Orthogonality Properties of the eigenvector,

\[ \beta_r = \sum_{i=1}^{n} c_i^2 \lambda_i / \sum_{i=1}^{n} c_i^2. \]  \hspace{1cm} (3.21)

Since \( \{u^r\} \) is a reasonable approximation of \( \{q^r\} \), the coefficient \( c_r \) is the dominant factor in the expansion. Consequently,

\[ \beta_r = \lambda_r + \sum_{i=1, i \neq r}^{n} (\lambda_i - \lambda_r)(c_i/c_r)^2 \]  \hspace{1cm} (3.22)

When the number of accurate eigenvalues of the reduced equation is \( s \), it can be assumed that the eigenvectors \( \{u^r\} \), for \( r = 1 \) to \( s \), are almost orthogonal to each other. Then, it can be concluded that the improved eigenvalue \( \beta_r \) is higher than the exact eigenvalue \( \lambda_r \), at least for the first \( s \) eigenvalues. Since the eigenvalue \( \beta_r \) is more accurate than the eigenvalue \( \alpha_r \), the solution of the Reduced Eigenvalue Problem provides an Upper Bound to the solution of the Original Eigenvalue Problem.

The Upper Bound property of the eigenvalues of the reduced equation has also been proved by Geradin (40, 1971). This publication presents
a different proof than the one derived in this section.

Also physical arguments can explain the Upper Bound property of the eigenvalues of the reduced equation. The selection of an optimum set of primary coordinates signifies that the secondary coordinates do not give appreciable contribution to the kinetic energy of the system. Also constraints are introduced, thus increasing the potential energy of the system. The neglect of these components of the kinetic and potential energies increases the natural frequencies of the discretized system.

3.4 Further Applications

The development of the Eigenvalue Economiser made possible the elimination of certain degrees of freedom. Thus, it is possible to use complete polynomial expansions to define the displacement field of a finite element and then eliminate some of the internal degrees of freedom. The Eigenvalue Economizer has been used by Lindberg, Olson and Sarazin (44, 1970) to develop a very efficient shell finite element.

The Eigenvalue Economizer can also be used to develop any finite element with a Truncated Mass Matrix, as shown by Vysloukh, Kandidov and Chesnokov (45, 1973).

Another important consequence of the Eigenvalue Economizer is the Super-Element reported by Aralden and Egeland (46, 1974). A Super-Element is a complex finite element developed by the assembly of several finite elements in which some of the internal degrees of freedom have been eliminated.
CHAPTER 4
THE SUPER-PARAMETRIC SHELL ELEMENT

4.1 Introduction

In the early applications of the Finite Element Analysis simple shapes, lower order elements were used almost exclusively. These triangular or rectangular elements had only displacement and slopes as degrees of freedom. Later, it was found that higher order elements with displacement, slopes and higher derivatives as degrees of freedom were more efficient than lower order elements when applied to a system with a given number of degrees of freedom.

In general, the simple shape, higher order elements cannot geometrically represent the structure, unless a large number of elements are used. The number of elements used is determined primarily by the geometry of the system, therefore the need for the development of curved, higher order elements.

The curved element not only represents the geometry of the system more accurately with less elements but also allows the curved beam or shell theories to be used in their development, therefore further improving the performance of the element. One of the most successful curved elements is the Isoparametric Element. A three-dimensional isoparametric element is shown in Figure 4.1.

In this chapter a literature survey of the development and application of the Isoparametric family of elements is presented. The Stiffness and Mass Matrices of the super-parametric shell element are derived.

The concept of the isoparametric element was developed with reference to quadrilateral elements by Ergatondis (47, 1966) and Irons (48, 1966 ; 49, 1966). The concept was later extended to curved elements and applied to two and three-dimensional static analysis of solid...

The three-dimensional isoparametric element was modified by Ahmad (58, 1968; 59, 1969) such that it became applicable to the analysis of thick shells. This super-parametric shell element has been extensively reported and applied to static problems by Zienkiewicz, Irons and Ahmad (60, 1969; 15, 1970), Zienkiewicz, Irons, Ergatoudis, Ahmad and Scott (55, 1970), Zienkiewicz, Irons, Campbell and Scott (56, 1971) and Zienkiewicz (57, 1971). Later, the element has been modified by Too (61, 1971), Razzaque (62, 1972) and Irons and Razzaque (63, 1973).

The performance and versatility of the super-parametric shell element has been improved by the development of the Reduced Integration Technique by Too (61, 1971) and Zienkiewicz, Taylor and Too (17, 1971). Also, the development of the Selective Integration Technique by Pawsey (64, 1970), and Clough and Pawsey (65, 1971) improved the performance and versatility of the super-parametric shell element. With the Reduced or Selective Integration Technique to evaluate the strain energy of the super-parametric shell element, this element becomes applicable to thin or thick, shallow or deep, shell structures.

Anderson (36, 1968) introduced the isoparametric element into the dynamic analysis of solid mechanics problems. Zienkiewicz, Anderson and Ahmad (16, 1970) and Abrahamian and Overaye (66, 1971) applied the super-parametric shell element to the dynamic analysis of thick shells. Pawsey (64, 1970), using the Selective Integration Technique to evaluate the strain energy of the super-parametric element, applied this element to the dynamic analysis of thick and thin shells. Hofmeister and Evensen
(67, 1972) used the Reduced Integration Technique to evaluate the strain energy and applied the super-parametric shell element to the dynamic analysis of thin and thick shells.

Mota Soares and Thomas (68, 1973) further developed the super-parametric shell element such that it became applicable to the nonlinear dynamic analysis of shells and the dynamic analysis of pre-stressed or rotating shell structures. The element was applied to the dynamic analysis of rotating, thin and thick, shell blades. Zienkiewicz and Bossak (69, 1973) further developed the isoparametric element and applied the element to the dynamic analysis of rotating shell structures.

Gupta (70, 1971) and Gupta and Mohraz (71, 1972) have developed the isoparametric element in curvilinear coordinates, therefore eliminating the necessity of introducing intermediate nodes to represent curved boundaries. This new element is very efficient when applied to cylindrical or spherical shell structures.

Bond, Swannell, Henshell and Warburton (72, 1973) have developed the Isoparametric Degraded and Bite Elements and the Contra-parametric Element. These elements are slightly less accurate than the isoparametric elements when the sides are mildly curved but are considerably better when the sides become greatly distorted.

Recently the super-parametric shell element with the Reduced Integration Technique has been applied to the static analysis of marine propellers by Atkinson (73, 1973) and Sontredt (74, 1974). These authors showed the superiority of the element over other shell elements when applied to thick shell problems.

The isoparametric element has also been developed to solve heat transfer problems by Zienkiewicz and Parekh (75, 1970), and plasticity problems by Nayak (76, 1971), Gupta (70, 1971; 77, 1972) and Zienkiewicz and Nayak (78, 1971; 79, 1972).
A literature survey of the development and applications of the Isoparametric family of elements is presented by Zienkiewicz (80, 1973).

The basic concept of the Isoparametric family of elements is simple. The geometry of such elements is described by specific points and by a uniquely defined polynomial which passes through these geometrical nodal points. The element is curved with reference to the global Cartesian coordinates, $x, y, z$. The element is generated from the corresponding simple shape element in the $\xi, \eta, \zeta$ space. The three-dimensional Isoparametric element with parabolic sides is shown in Figure 4.1.

The element has a complex shape in the $x, y, z$ space and a simple shape in the $\xi, \eta, \zeta$ space. To establish the curvilinear coordinates in the $x, y, z$ space, it is required to define the relationship between the Cartesian and curvilinear coordinates. Such a relationship determines the Mapping between both spaces.

The geometry of the element in the $x, y, z$ space is defined by the coordinates $x_i, y_i, z_i$ of all the geometrical nodal points. The mapping between the $x, y, z$ and $\xi, \eta, \zeta$ spaces is defined by,

$$
\begin{align*}
  x &= \sum_{i=1}^{n} N_i'(\xi, \eta, \zeta)x_i; \\
  y &= \sum_{i=1}^{n} N_i'(\xi, \eta, \zeta)y_i; \\
  z &= \sum_{i=1}^{n} N_i'(\xi, \eta, \zeta)z_i
\end{align*}
$$

(4.01)

where $N_i'(\xi, \eta, \zeta)$ are the geometrical shape functions which define the mapping between the two spaces and $n$ is the number of geometrical nodal points.

Such mapping must be uniquely defined. Therefore, each point in the $\xi, \eta, \zeta$ space corresponds only to another point in the $x, y, z$ space. With large distortions between spaces, it is possible to obtain the undesirable situation of non-unique correspondence. The unique mapping requires that within the domain of the element, the Jacobian Determinant defined as,
\[
\begin{vmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}
\end{vmatrix}
\]
\( (4.02) \)

does not change sign.

Another special property of the mapping is of great practical importance. If a series of adjacent elements in one space is mapped into another space, then the distorted elements remain adjacent and without any inter-element gaps.

The displacement field of the element, \( u, v, w \), with reference to the Cartesian coordinates, can be defined by the displacement shape functions \( N_i(\xi, \eta, \zeta) \) and the displacement at the nodal point, \( u_i, v_i, w_i \) as,
\[
\begin{align*}
  u &= \sum_{i=1}^{m} N_i(\xi, \eta, \zeta)u_i; \\
  v &= \sum_{i=1}^{m} N_i(\xi, \eta, \zeta)v_i; \\
  w &= \sum_{i=1}^{m} N_i(\xi, \eta, \zeta)w_i
\end{align*}
\]
\( (4.03) \)

where \( m \) is the number of displacement nodal points.

The displacement and geometrical continuity of the element are ensured if both the shape functions are compatible. Zienkiewicz (57, 1971) proves that the geometrical shape functions are a linear combination of the displacement shape functions,
\[
N_i' = \sum_{j=1}^{m} c_{ij} N_j \quad \text{and} \quad \sum_{i=1}^{m} N_i = 1
\]
\( (4.04) \)

where \( c_{ij} \) are constants.

When the geometrical shape functions are identical to the displacement shape functions, the element is called an Isoparametric finite element. In this case the number of geometrical nodal points is identical to the number of displacement nodal points, as shown in Figure 4.2a.

When the geometrical shape functions are of higher order than the displacement shape functions, the element is called a Super-Parametric
finite element. In this case there are more geometrical nodal points than displacement nodal points, as shown in Figure 4.2b.

When the geometrical shape functions are of lower order than the displacement shape functions, the element is called a Sub-parametric finite element. In this case, there are more displacement nodal points than geometrical nodal points, as shown in Figure 4.2c.

The sub-parametric elements are advantageous when the structure is slightly curved and with large variation in stress, while the super-parametric elements are advantageous when the structure is deeply curved and with small variation in stress. In other occasions, the isoparametric element is usually superior to either the sub-parametric or super-parametric elements.

4.2 Fundamentals

At first glance it would appear reasonable to expect that the three-dimensional isoparametric element, which is based on the Theory of Elasticity, is applicable to thin plates or shells. However, in practice, the isoparametric element is numerically unstable and uneconomical when applied to thin plates or shells. Numerical instability may occur, because the strain normal to the middle surface will be associated with very large stiffness coefficients as the thickness reduces. The isoparametric element is also uneconomic, since it has too many degrees of freedom across the thickness. Both these difficulties can be eliminated by neglecting the effect of the strain normal to the middle surface.

A super-parametric shell element, with 8 displacement nodes, 16 geometrical nodes and 40 degrees of freedom is shown in Figure 4.3. The geometrical nodal points are at the top and bottom surfaces. The displacement nodal points are the middle surface. The element is referred to a set of curvilinear non-dimensional coordinates $\xi, \eta, \zeta$, and to a set of fixed Cartesian coordinates $x, y, z$. Also a local set of Cartesian
coordinates \( x', y', z' \) are uniquely defined. The coordinates \( x, y, z \) are related to the coordinates \( x', y', z' \) by a variable transformation matrix \([T]\). The element volume and surface area are \( V_e \) and \( S_e \), respectively.

The fixed Cartesian coordinates are related to the curvilinear coordinates by the following expression:

\[
\begin{pmatrix}
  x \\
y \\
z
\end{pmatrix} = \sum_{i=1}^{n} N_i(\xi, n) \left( \frac{1 + \xi}{2} \right) \begin{pmatrix}
x_i \\
y_i \\
z_i
\end{pmatrix} + \sum_{i=1}^{n} N_i(\xi, n) \left( \frac{1 - \xi}{2} \right) \begin{pmatrix}
x_i \\
y_i \\
z_i
\end{pmatrix}
\]

\( \text{top} \quad \text{bottom} \) (4.05)

where \( N_i(\xi, n) \) are the shape functions, \( n \) the number of geometrical nodal points at the top or bottom surfaces and \( x_i, y_i, z_i \) the coordinates of the geometrical nodal points at the top or bottom surfaces.

The shape functions for the parabolic element are given by,

\[
\begin{align*}
N_1 &= \frac{1}{4}(1 - \xi)(1 - n)(-\xi - n - 1) \\
N_2 &= \frac{1}{4}(1 - \xi^2)(1 - n) \\
N_3 &= \frac{1}{4}(1 + \xi)(1 - n)(\xi - n - 1) \\
N_4 &= \frac{1}{4}(1 + \xi)(1 - n^2) \\
N_5 &= \frac{1}{4}(1 + \xi)(1 + n)(\xi + n - 1) \\
N_6 &= \frac{1}{4}(1 - \xi^2)(1 + n) \\
N_7 &= \frac{1}{4}(1 - \xi)(1 + n)(-\xi + n - 1) \\
N_8 &= \frac{1}{4}(1 - \xi)(1 - n^2)
\end{align*}
\]

(4.06)

Some typical shape functions are shown diagrammatically in Figure 4.4.

The shape functions for the cubic element are given by,

\[
\begin{align*}
N_1 &= \frac{1}{32}(1 - \xi)(1 - n)(-10 + 9(\xi^2 + n^2)) \\
N_2 &= \frac{9}{32}(1 - \xi^2)(1 - 3\xi)(1 - n) \\
N_3 &= \frac{9}{32}(1 - \xi^2)(1 + 3\xi)(1 - n)
\end{align*}
\]
\[
\begin{align*}
N_4 &= \frac{1}{32} (1 + \xi)(1 - \eta)(-10 + 9(\xi^2 + \eta^2)) \\
N_5 &= \frac{9}{32}(1 + \xi)(1 - \eta^2)(1 - 3\eta) \\
N_6 &= \frac{9}{32}(1 + \xi)(1 - \eta^2)(1 + 3\eta) \\
N_7 &= \frac{1}{32}(1 + \xi)(1 + \eta)(-10 + 9(\xi^2 + \eta^2)) \\
N_8 &= \frac{9}{32}(1 - \xi^2)(1 + 3\xi)(1 + \eta) \\
N_9 &= \frac{9}{32}(1 - \xi^2)(1 - 3\xi)(1 + \eta) \\
N_{10} &= \frac{1}{32}(1 - \xi)(1 + \eta)(-10 + 9(\xi^2 + \eta^2)) \\
N_{11} &= \frac{9}{32}(1 - \xi)(1 - \eta^2)(1 + 3\eta) \\
N_{12} &= \frac{9}{32}(1 - \xi)(1 - \eta^2)(1 - 3\eta)
\end{align*}
\]  

Some typical shape functions are shown diagrammatically in Figure 4.5.

It is convenient to rewrite equation (4.05) as,

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \sum_{i=1}^{n} N_i \begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix} + \sum_{i=1}^{n} \frac{1}{2} N_i \xi \begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix}
\]

(4.07)

where,

\[
V_{3i} = \begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix} - \begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix} = \begin{pmatrix}
  \delta x_i \\
  \delta y_i \\
  \delta z_i
\end{pmatrix} \quad ; \quad V_i = \frac{1}{2} \left( \begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix} + \begin{pmatrix}
  x_i \\
  y_i \\
  z_i
\end{pmatrix} \right)
\]

(4.08)

In thin or moderately thick shells, the strain in the direction normal to the middle surface is negligible, therefore, the displacement throughout the element can be uniquely defined by the Cartesian coordinates of the middle surface node and two rotations of the nodal vector \(V_{3i}\) about orthogonal directions normal to it. Let such orthogonal directions, which will be uniquely defined later, be given by unit vector \(v_{2i}\) and \(v_{1i}\) with corresponding rotations \(\alpha_i\) and \(\beta_i\).

The nodal vector \(V_{3i}\) is already uniquely defined. Two orthogonal axes normal to vector \(V_{3i}\) can be uniquely defined as:
\[
\mathbf{v}_{1i} = \mathbf{i} - \mathbf{v}_{3i} \quad \text{and} \quad \mathbf{v}_{2i} = \mathbf{v}_{3i} - \mathbf{v}_{1i} \quad (4.09)
\]

where \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are the unit vector in the \( x, y, z \) directions, respectively.

The vectors \( \mathbf{v}_{1i}, \mathbf{v}_{2i}, \mathbf{v}_{3i} \) represent a set of Cartesian frames of reference, \( x', y', z' \) at node \( i \). The orthogonal matrix \( [T_i] \), relating the fixed Cartesian coordinates to the local Cartesian coordinates at node \( i \), is obtained by normalising the vectors \( \mathbf{v}_{1i}, \mathbf{v}_{2i} \) and \( \mathbf{v}_{3i} \). Let \( \mathbf{v}_{1i}', \mathbf{v}_{2i}' \) and \( \mathbf{v}_{3i}' \) be such normalised vectors. The orthogonal matrix is given by the second order transformation tensor at node \( i \), as

\[
[T_i] = [\mathbf{v}_{1i}', \mathbf{v}_{2i}', \mathbf{v}_{3i}'] = \begin{bmatrix}
\frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\
\frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\
\frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'}
\end{bmatrix} \quad (4.10)
\]

In general, \( \mathbf{v}_{1i} \) and \( \mathbf{v}_{2i} \) will lie only approximately in the middle surface plane and \( \mathbf{v}_{3i} \) is only approximately normal to the middle surface.

The displacement field can be described by the nodal displacements as,

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{n} N_i \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} + \sum_{i=1}^{n} \{ \mathbf{v}_{1i}, \mathbf{v}_{2i}, \mathbf{v}_{3i} \} \begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix} \quad (4.11)
\]

where \( u_i, v_i, w_i \) are the displacements at node \( i \), with reference to the global coordinates \( x, y, z \). Also, \( \alpha_i \) and \( \beta_i \) are the rotations of vector \( \mathbf{v}_{3i} \) about the local axes \( x' \) and \( y' \). Equation (4.11) can be transformed into,

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = [N] \{ \mathbf{r} \} = [\begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix}] \ldots [\begin{bmatrix} N_j \\ \vdots \\ N_n \end{bmatrix}] \begin{bmatrix}
\{ \mathbf{r}_1 \} \\
\{ \mathbf{r}_j \} \\
\{ \mathbf{r}_n \}
\end{bmatrix} \quad (4.12)
\]
\[ [N_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times t_j[\phi_j]; \quad [\phi_j] = [\nu_{1j}, \nu_{2j}] \]

\[
\{r_j\} = \begin{bmatrix} u_j \\ v_j \\ w_j \\ \alpha_j \\ \beta_j \end{bmatrix}
\]

where matrix \([N] \) is the Shape Function Matrix, \([N_j] \) is a sub-matrix of this matrix and vector \(\{r\} \) defines the degrees of freedom of the element.

Neglecting the strain normal to the surface, \(\varepsilon_{zz} = \text{constant} \), the strain and stress vector with reference to the local axes are given by,

\[
\{\varepsilon\} = \begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ 2\varepsilon_{x'y'} \\ 2\varepsilon_{x'z'} \\ 2\varepsilon_{y'z'} \end{bmatrix} = [C_L] \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \quad \text{and} \quad \{\tau\} = \begin{bmatrix} \tau_{x'x'} \\ \tau_{x'y'} \\ \tau_{x'z'} \\ \tau_{y'x'} \\ \tau_{y'y'} \\ \tau_{y'z'} \end{bmatrix}
\]

\[
[C_L] = \begin{bmatrix} \frac{a}{a x'} & 0 & 0 \\ 0 & \frac{a}{a y'} & 0 \\ \frac{a}{a y'} & \frac{a}{a x'} & 0 \\ \frac{a}{a z'} & 0 & \frac{a}{a x'} \\ 0 & \frac{a}{a z'} & \frac{a}{a y'} \end{bmatrix}
\]

where \(\varepsilon_{x'y'} \) and \(\tau_{x'y'} \) are the components of the strain and stress tensor.

In shells, the normal stress perpendicular to the middle surface is negligible. This constraint is imposed in the Elasticity Matrix \([\mathcal{D}]\),

63.
which is defined by equation (2.33) for an isotropic, elastic material. Consequently,

\[
[D] = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & (1-\nu)/2 & 0 & 0 \\
0 & 0 & 0 & (1-\nu)/2k & 0 \\
0 & 0 & 0 & 0 & (1-\nu)/2k
\end{bmatrix}
\] (4.15)

where the factor \( k \) is introduced to account more accurately for shear strain energy. As the displacement varies linearly across the thickness of the shell, the beam type shear stress can only be approximated by an average value. However, these shear stresses are known to be parabolic. Therefore, the shear strain energy would be reduced to \( 1/1.2 \) of the true value.

The derivatives of a function with reference to the fixed Cartesian coordinates are related to the derivatives of the same function with reference to the curvilinear coordinates by,

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix} = [J]^{-1} \begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial \zeta}
\end{bmatrix}
\] (4.16)

where

\[
[J] = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
\]

is the Jacobian Matrix.

Also, the derivatives of a function in the local Cartesian coordinates are related to the derivatives of the same function in the fixed Cartesian coordinates by,
\[
\begin{bmatrix}
\frac{a}{\partial x'} \\
\frac{a}{\partial y'} \\
\frac{a}{\partial z'}
\end{bmatrix}
= \begin{bmatrix}
\frac{a}{\partial x} & \frac{a}{\partial y} & \frac{a}{\partial z'} \\
\frac{a}{\partial y} & \frac{a}{\partial y} & \frac{a}{\partial z'} \\
\frac{a}{\partial z'} & \frac{a}{\partial y} & \frac{a}{\partial z'}
\end{bmatrix}^t
\begin{bmatrix}
\frac{a}{\partial x} \\
\frac{a}{\partial y} \\
\frac{a}{\partial z}
\end{bmatrix}
\]
or
\[
\begin{bmatrix}
\frac{a}{\partial x'} \\
\frac{a}{\partial y'} \\
\frac{a}{\partial z'}
\end{bmatrix}
= \begin{bmatrix}
\frac{a}{\partial x'} \\
\frac{a}{\partial y'} \\
\frac{a}{\partial z'}
\end{bmatrix}
\begin{bmatrix}
\frac{a}{\partial x} \\
\frac{a}{\partial y} \\
\frac{a}{\partial z}
\end{bmatrix}
\]

Zienkiewicz, Taylor and Too (17, 1971) have demonstrated that matrix \([A]\) has the following form:

\[
[A] = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} & 0 \\
0 & 0 & A_{33}
\end{bmatrix}
\]

In the derivation of the Element Matrices, it is necessary to transform the integration from the curvilinear to the Cartesian space. This transformation is achieved by the following equation,

\[
d\Omega = |J|d\xi d\eta d\zeta
\]

where \(|J|\) is the determinant of the Jacobian Matrix.

It should be noticed that,

\[
\begin{align*}
\int_{-1}^{1} d\zeta &= 2; & \int_{-1}^{1} \xi d\zeta &= 0; & \int_{-1}^{1} \xi^2 d\zeta &= 2/3.
\end{align*}
\]

4.3 Stiffness Matrix

The Stiffness Matrix of any Displacement Finite Element is developed in Chapter 2 to be,
For the super-parametric shell element, the Strain Matrix is given by,

\[ \left[ B_L \right] = \left[ C_L \right] \left[ T \right] \left[ N \right] \] (4.22)

It is convenient to define a sub-matrix of the strain matrix as follows:

\[ \left[ B_j \right] = \left[ C \right] \left[ T \right] \left[ N_j \right] \] (4.23)

Consequently,

\[ \left[ B_j \right] = \left[ \left[ B_{1j} \right] \left[ T \right] \left[ \xi \right] \left[ C_{1j} \right] \left[ T \right] \left[ \phi_j \right] \right] \] (4.24)

where,

\[ \left[ B_{1j} \right] = \begin{bmatrix} B_{1j} & 0 & 0 \\ 0 & B_{2j} & 0 \\ B_{2j} & B_{1j} & 0 \\ 0 & 0 & B_{1j} \\ 0 & 0 & B_{2j} \end{bmatrix} \]

\[ \left[ C_{1j} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{1j} & 0 & 0 \\ 0 & C_{1j} & 0 \end{bmatrix} \]

\[ B_{1j} = A_{11} \frac{\partial N_j}{\partial \xi} + A_{12} \frac{\partial N_j}{\partial n} \]

\[ B_{2j} = A_{21} \frac{\partial N_j}{\partial \xi} + A_{22} \frac{\partial N_j}{\partial n} \]

\[ C_{1j} = A_{33} N_j \]

It is advantageous to divide the stiffness matrix into sub-matrices \([KE_{ij}]\) and to divide further these sub-matrices, such that

\[ \left[ KE_{ij} \right] = \int_{\Omega_e} \left[ B_L \right]^t \left[ D \right] \left[ B_L \right] d\Omega \quad \text{and} \quad \left[ KE_{ij} \right] = \begin{bmatrix} KE_{11ij} & KE_{12ij} \\ KE_{21ij} & KE_{22ij} \end{bmatrix} \] (4.25)
After substitutions and integration in the $\zeta$ direction, these sub-matrices become,

$$[KE_{11,ij}] = \frac{1}{2} \int \left[ T \right] [B_{1,i}]^t [D] [B_{1,j}] [T]^t |J| d\xi d\eta$$

$$[KE_{12,ij}] = t_j \int \left[ T \right] [B_{1,i}]^t [D] [C_{1,j}] [T]^t [\phi_j] |J| d\xi d\eta$$

$$[KE_{21,ij}] = t_i \int \left[ T \right] [C_{1,i}]^t [D] [B_{1,j}] [T]^t |J| d\xi d\eta$$

$$[KE_{22,ij}] = t_i t_j \int \left[ C_{1,i} \right]^t [T] \left[ t \left[ B_{1,j} \right]^t [D] [B_{1,j}] + \left[ C_{1,j} \right]^t [D] [C_{1,j}] \right]$$

$$\cdot [T]^t [\phi_j] |J| d\xi d\eta$$

(4.26)

In Appendix A, the numerical integration of matrices by the Gauss Method is presented. Also the minimum and correct order of integration are defined.

The Minimum order of integration for the Stiffness Matrix of the super-parametric parabolic shell element is a $2 \times 2$ Gaussian Mesh. The Correct order of integration is a $3 \times 3$ Gaussian Mesh.

4.4 Mass Matrix

The Mass Matrix of any Displacement Finite Element is developed in Chapter 2 to be,

$$[ME] = \int_{\Omega_e} \rho [N]^t [N] d\Omega$$

(4.27)

The sub-matrix of this equation is given by,

$$[ME_{1,j}] = \int_{\Omega_e} \rho [N_i]^t [N_j] d\Omega.$$
After substitutions and integration in the $\zeta$ direction, this equation becomes,

$$[M E_{i j}] = 2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2^2 t_i t_j [\phi_i] ^t [\phi_j] \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \int \frac{p N_i N_j J d \xi d \eta}{\lambda}$$

(4.29)

For the parabolic element, this equation is numerically integrated by a 3 * 3 Gaussian Mesh.
o Displacement and geometrical nodal points.

b) X, Y, Z space.

FIG. 4.1. THREE DIMENSIONAL ISOPARAMETRIC PARABOLIC ELEMENT.
a) Isoparametric element.

b) Super-parametric element.

c) Sub-parametric element.

FIG. 4.2. TWO DIMENSIONAL ISOPARAMETRIC FAMILY OF ELEMENTS.
FIG. 4.3. SUPER-PARAMETRIC PARABOLIC SHELL ELEMENT.
FIG. 4.4. TYPICAL PARABOLIC SHAPE FUNCTIONS.
FIG. 4.5. TYPICAL CUBIC SHAPE FUNCTIONS.
CHAPTER 5

THE REDUCED AND SELECTIVE INTEGRATION TECHNIQUES

5.1 Introduction

In the Finite Element Analysis, the behaviour of an element is fully dictated by the shape functions, therefore, all elements have some constraints. In some cases, especially with a coarse mesh of elements, these constraints can cause an undesirable behaviour of the element. When this behaviour is caused by unnecessary high order terms in the strain energy of the element and the stiffness matrix is evaluated using numerical integration, the element behaviour can be improved by evaluating the strain energy using a Gaussian mesh of lower order than the correct order.

In the numerical integration of finite elements, a Gaussian mesh is called Correct if it is capable of the exact integration of all the terms in the Element Matrices. A Gaussian mesh of lower order than the correct one, but of the same or higher order than the minimum order of integration which ensures convergence, is called Reduced.

When only reduced integrations are used to evaluate the strain energy of an element, the stiffness matrix is said to be calculated by a Reduced Integration Technique. If the strain energy is evaluated partly by reduced and partly by correct integrations, then the stiffness matrix is said to be calculated by a Selective Integration Technique.

Doherty, Wilson and Taylor (81, 1968) used a reduced integration to evaluate the stiffness matrix of a quadrilateral, plane stress element. If correct integration is applied to this element, the flexure response of beam structures are accompanied by spurious shear distortion, as shown in Figure 5.1a. The correct response of the beam is shown in Figure 5.1b. The reduced integration neglects the extraneous shear strain imposed by
the displacement functions and improves considerably the performance of this element.

A theoretical analysis of the reduced integration has been developed by Pawsey (64, 1970) and Too (61, 1971). This analysis has also been reported by Zienkiewicz, Taylor and Too (17, 1971) and Clough and Pawsey (65, 1971). Further theoretical arguments have been presented by Pawsey (82, 1972), Key and Kavanagh (83, 1972) and Cook (84, 1972; 85, 1975).

Too (61, 1971) and Zienkiewicz, Taylor and Too (17, 1971) used, with astonishing success, the Reduced Integration Technique to improve the super-parametric shell element. The improved element was applied to the static analysis of thin and thick shells. Also, the improved element has been applied to the dynamic analysis of thin and thick shells by Hofmister and Evensen (67, 1972).

The Selective Integration Technique has been successfully used to improve the super-parametric shell element by Pawsey (64, 1970) and Clough and Pawsey (65, 1971). The element was applied to the static and dynamic analysis of thin and thick shells.

The Reduced Integration Technique has been applied to the dynamic analysis of rotating shell structures. Mota Soares and Thomas (68, 1973) use the technique to evaluate the strain energy of a super-parametric shell element representation of rotating shells. Zienkiewicz and Bossak (69, 1973) used the Reduced Integration Technique to evaluate the strain energy of an isoparametric element representation of a rotating shell.

The Reduced Integration Technique has also been applied to large deformation static problems by Nayak (76, 1971) and Zienkiewicz and Nayak (78, 1971).

In this chapter, the mathematical foundations of the Reduced Integration Technique are presented. This technique improves the efficiency
and applicability of the super-parametric shell element. It reduces the computer time required to integrate the strain energy of the element by 60%. Also the number of elements required to represent the shell structure is decreased, therefore, further improving the efficiency of the element. The technique transforms the super-parametric element into an efficient thin shell element.

The Reduced Integration Technique is continuously used to evaluate the strain energy of all the Finite Element models developed in this Thesis.

5.2 Reduced Integration in Gauss Quadrature

This section presents the effect of the reduced integration on a function. For simplicity, one dimensional functions are investigated. Since the strain energy density has a quadratic form, the square of a function is considered.

Let the function \( \phi(\xi) \) be expressed in terms of the first \( m + 1 \) Legendre polynomials, \( P_j(\xi) \), as

\[
\phi(\xi) = \sum_{j=0}^{m} a_j P_j(\xi)
\]  

(5.01)

\( a_j \) = constants

Integrating the square of equation (5.01) and using the orthogonality property of the Legendre polynomials,

\[
\int_{-1}^{1} \phi^2(\xi) d\xi = \sum_{j=0}^{m} a_j^2 \int_{-1}^{1} P_j^2(\xi) d\xi
\]  

(5.02)

The exact integration of this equation, using \( n \) Gaussian points, is given by,
Where \( A_i \) and \( \xi_i \) are the Gaussian Weights and the coordinates of the Gaussian points, respectively.

Reducing the order of integration to \( n - 1 \) Gaussian points, the integration is exact up to the terms \( m - 1 \). Since the Legendre polynomial \( P_r(\xi) \) has zero values at the \( r \) Gaussian points, the last term of equation (5.03) is zero. It can be concluded that the highest order Legendre polynomial term can be deleted. Also the order of the function is reduced and the value of the integral equation (5.03) is reduced.

The strain energy is always positive. The terms deleted by the reduced integration are positive, therefore, the Reduced Integration Technique reduces the stiffness of an element. As a consequence of this decrease in stiffness, a compatible finite element loses its bound properties. Therefore, the solution does not have monotonic convergence as the number of elements is increased. This is a major disadvantage of the Reduced Integration Technique.

5.3 Reduced Integration and Spurious Response

The effects of applying the Reduced Integration Technique to evaluate the strain energy of the super-parametric plate element are presented. For simplicity, the plate element is considered, instead of the shell element.

Consider one element with two edges simply supported and subjected to clockwise bending moments, as shown in Figure 5.2a. The vertical deflection and slope are given by,
\[ w = M_0 a^2 \left(-x/12 + x^3/3a^2\right)/D \]

\[ \frac{dw}{dx} = M_0 a \left(-1/12 + x^2/a^2\right)/D \]  

(5.04)

where \( M_0 \) is the moment/unit length, \( a \) is the length of plate in the \( x \) direction and \( D \) is the flexural rigidity of the plate. The shear strain \( \varepsilon_{xz} \) is constant. The vertical deflection and shear deformation of the plate are shown in Figure 5.2b. However, these deformations are not possible, since the element cannot be adapted to the required geometry.

The boundary conditions of the problem are given by,

\[ w_i = \beta_i = 0 \quad i = 1, 2, \ldots, 8 \]  

(5.05)

\[ \alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 \quad \alpha_2 = \alpha_6 \]

Consequently, the vertical deflection is zero anywhere in the element domain. Also the shear strain is not constant, since

\[ \varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} = \frac{\partial w}{\partial z} = \sum_{i=1}^{8} N_i \alpha_i = \xi^2 \alpha_1 + (1 - \xi^2) \alpha_2 \]  

(5.06)

Thus, the shear strain is forced to vary parabolically.

The total slope of the plate is determined by the slope due to the vertical deflection and the slope due to the constant shear deformation \( \theta_s \). Then,

\[ \alpha_1 = \theta_b + \theta_s; \quad \alpha_2 = -\theta_b + \theta_s; \quad \theta_b = M_0 a/6D \]  

(5.07)

Consequently, the shear strain becomes,

\[ \varepsilon_{xz} = \frac{3}{2} \theta_b \xi^2 - \theta_b + \theta_s \]  

(5.08)

The actual shear strain is constant, therefore, equation (5.08) is only accurate at the points \( \xi = \pm 1/\sqrt{3} \), which are precisely the points of the two point Gauss Quadrature. Thus, the integration of the shear
energy of the element by a $2 \times 2$ Gaussian mesh is accurate, while the integration by a $3 \times 3$ Gaussian mesh is inaccurate and too large. Figure 5.2c shows the expected and predicted shear strain and the corresponding integration points.

It has been demonstrated that for the linearly bending moment mode, the Reduced Integration Technique improves the performance of the super-parametric element. However, all the possible modes of deformation must be investigated. A finite element with $n$ nodes and $m$ degrees of freedom at each node has $n \times m$ independent deformation modes. Since these modes are independent, they can be combined to give another set of independent modes. This new set of independent modes can be selected to satisfy some chosen boundary conditions. This set of modes is called the Physical Modes of an element. The boundary conditions of the modes are satisfied by the shape functions, but the deformation pattern of the element is not necessarily correct. Thus, for a set of boundary conditions, there may be a deformation pattern which is determined by the shape functions and another deformation pattern which the element should follow. The former is called the Predicted Mode and the latter the Expected Mode. When the two modes coincide, the element gives Correct Response, otherwise the element gives Spurious Response. Therefore, spurious response is an intrinsic error, inherited from the inability of the element to respond correctly. Too (61, 1971) has analysed all the independent physical modes of the super-parametric parabolic plate element and demonstrated that the spurious response is eliminated, when using a $2 \times 2$ Gaussian mesh to evaluate the strain energy of the element. Some of these physical modes and the corresponding spurious response, where applicable, are shown in Figures 5.3 and 5.4.
FIG. 5.1. THE SPURIOUS AND CORRECT RESPONSE OF THE PLANE QUADRILATERAL ELEMENT UNDER PURE BENDING.

(a) Spurious shear response

(b) Correct bending response.
(a) Superparametric parabolic plate element.

\( w \) - Displacement due to bending.

(b) Correct response

1 2 3

\[ w = 0 \]

Constant shear deformation.

1 2 3

\[ E_{xz} = \text{constant} \]

E_{xz} E_{xz}

(c) Spurious response.

3 - Point gauss quadrature
2 - Point gauss quadrature
Constant shear response

Spurious shear strain

\(-\frac{1}{2} \theta_b + \theta_s\)

\( \theta_b + \theta_s \)

FIG. 5.2. CORRECT AND SPURIOUS RESPONSE OF THE SUPER-PARAMETRIC PARABOLIC PLATE ELEMENT.
FIG. 5.3. CORRECT AND SPURIOUS RESPONSE OF THE SUPER-PARAMETRIC PARABOLIC PLATE ELEMENT.
FIG. 5.4. CORRECT AND SPURIOUS RESPONSE OF THE SUPER-PARAMETRIC PARABOLIC PLATE ELEMENT.
6.1 Introduction

Since the earliest phases of its development, the Finite Element Method has appeared to be ideally suited to the analysis of general shell structures, because of its flexibility in accounting for arbitrary geometries, loading and material properties variations. Indeed, it was evident that a general shell analysis program could be developed as soon as effective plate bending and plane stress elements were available. Thus, early efforts presented by Adini (86, 1961) with the pioneering plate and plane stress elements clearly demonstrated the feasibility of the application of finite elements to shell analysis.

Attempts to develop finite elements to general shell structures has followed three different courses. In the first approach, the shell is replaced by an assembly of plate elements, therefore introducing a geometrical approximation. The elements are based on the theories of plates and plane stress, therefore neglecting the coupling between the bending and stretching energies. Also, complex finite elements cannot be successfully used, since a reduction in the number of elements introduces further geometrical approximation.

The second approach is to develop curved shell elements based on shell theories, therefore allowing better geometrical and elastic representation.

In the third approach a three-dimensional isoparametric element is developed into a shell element by imposing geometrical and elastic constraints. The concept of the Isoparametric family of elements is introduced in Chapter 4.

The development and applications of flat, shell elements in the analysis of shell structures was initiated by Zienkiewicz and Cheung (87, 1965),
Zienkiewicz (88, 1965) and Clough and Tocher (89, 1965) using a non-compatible, rectangular element with 4 nodes and 20 degrees of freedom. The transverse and axial displacements are given by incomplete quartic and quadratic polynomials, respectively. This element was used to analyse arch dams and cylindrical shell structures.

When an arbitrary doubly curved shell is to be idealized by flat elements, only triangular elements are acceptable. A general description of triangular and quadrilateral shell elements was presented by Argyris (90, 1966). The first successful attempt to analyse doubly curved shells by finite elements was presented by Carr (91, 1967; 92, 1967). This author developed a triangular shell element based on a compatible plate and membrane elements. This element, which has 27 degrees of freedom, has the same interpolation polynomial to define the transverse and axial displacements.

At the same time, Dungar, Severn and Taylor (93, 1967) developed a hybrid, triangular shell element. The stress and axial displacement fields are linear, while the transverse displacement is given by an incomplete cubic function.

Johnson (94, 1967) and Clough and Johnson (95, 1968) developed a compatible triangular shell element based on the standard constant strain element and on a compatible triangular plate element. Parekh, Zienkiewicz and King (96, 1968) developed a non-compatible triangular element based on the constant strain element and a non-compatible plate bending element. Parekh (97, 1969) demonstrated the general superiority of this element over some of the previously mentioned shell elements.

Rawtani and Dokainish (98, 1969) also developed a triangular shell element based on standard plate and plane stress elements. Although the element has only 9 bending degrees of freedom and 6 tangential degrees of freedom, it was necessary to introduce another 3 pseudo degrees of freedom. The element was applied to the dynamic analysis of pre-twisted cantilever plates.
The development of an efficient, compatible, flat, triangular element was impeded by the late development of a good triangular plate element. However, by 1969 the age of the curved shell element had arrived and flat shell elements had only historical value. An excellent and extensive survey of plate elements is presented by Gallagher (99, 1969). Also, a survey of the early developments of conical shell elements is presented by Jones and Strone (100, 1966).

The overall development of the curved shell element follows the expected path. Firstly, elements for shells of revolution followed by cylindrical, swallow and deep shell elements. The first successful curved shell element was developed by Jones and Strone (101, 1966) and Strickland, Navaratha and Pian (102, 1966), with reference to shells of revolution. Shortly after, Bogner, Fox and Schmid (103, 1967) developed a rectangular cylindrical shell element. This element has 48 degrees of freedom, the transverse displacement is a first order Hermite polynomial, while the membrane displacements are represented by an incomplete quintic interpolation polynomial. The element is based on cylindrical shell theory. Although some rigid body modes are not included, the results converge monotonically but slowly. This element is superior to any of the previously developed shell elements. However, its application is limited to cylindrical shell structures.

Cantin and Clough (104, 1968) developed a rectangular cylindrical shell element with 24 degrees of freedom, having an incomplete quadratic polynomial to represent the membrane displacements and an incomplete quintic polynomial to represent the transverse displacement. The element is based on the Novoshilov theory (3) of thin cylindrical shells. The displacement functions are modified to include all the rigid body modes. This element has been modified and improved by Sabir and Lock (105, 1972).

Lindberg and Olson (106, 1968; 107, 1969) developed a rectangular cylindrical shell element based on the Love theory (1) of thin cylindrical shells. The transverse displacement is represented by an incomplete
quartic polynomial, while the tangential displacement is defined by another incomplete quartic polynomial. The element has 28 degrees of freedom, being 12 transverse and 16 membrane degrees of freedom. The element is applied to the dynamic analysis of a cylindrical fan blade.

The first attempt to develop a general curved shell element was presented by Utku and Melosh (108, 1967) and was only partially successful. The triangular shell element is based on the Marguerre Theory (4) of thin, shallow shells and the displacements are represented by linear functions. The element does not incorporate all the required rigid body modes and it is too stiff and inefficient.

Connor and Brebbia (109, 1967) developed a rectangular curved shell element based on the Marguerre Theory (4) of thin, shallow shells. The element has 20 degrees of freedom and does not include all the required rigid body modes. The element is also too stiff and inefficient.

Strickland and Loden (110, 1969) developed a curved triangular shell element based on the Novoshilov Theory (3) of thin shallow shells. The transverse displacement is represented by a cubic variation and the tangential displacement by a linear function. Although this element is an improvement over the previous shallow shell elements, it is not more efficient than some of the flat shell elements.

Bonnes, Shatt, Giroux and Robichaud (111, 1969) have introduced a triangular shell element based on the Reissner Theory (1) of thin, shallow shells. All the displacements are represented by a cubic polynomial. The element is superior to any previous curved element but its superiority over flat elements is marginal.

Dhatt (112, 1969) developed several triangular shell elements based on the theory of thin shallow shells presented by Washizu (113, 1975). All displacements are represented by a cubic polynomial and the transverse slopes are quadratic. The element is divided into 3 sub-elements, being the
interior degrees of freedom eliminated by static condensation. The elements are superior to any previously developed flat or curved shell element. It also showed the need to replace the shallow shell element formulation by a more accurate curved element based on curved coordinates.

The first attempt to develop a quadrilateral shell element is presented by Key and Beisinger (114, 1969). This element is based on the Washizu Theory (113) of thin shells. Hermitian polynomial representation of the displacement field, coupled with isoparametric element concepts are used. Extensive applications to shell problems demonstrate that the rigid body modes are not well represented and that the convergence is too slow.

By 1969 some curved shell elements were superior to the flat shell elements. However, an efficient, compatible, triangular shell element had not been developed. This is not surprising and is a consequence of the late development of an efficient, compatible, triangular plate element. Lindberg, Olson, Cowper and Kosko (115, 1969) developed such an element.

A very successful triangular shell element was developed by Lindberg, Olson and Cowper (116, 1969; 117, 1970), Lindberg and Olson and Saragin (44, 1970) and Lindberg and Olson (118, 1971) based on the Novoshilov Theory (3) of thin shallow shells. The transverse displacement is represented by a quintic polynomial and the tangential displacements are represented by a cubic polynomial. Static condensation is used to eliminate the internal degrees of freedom and reducing the element to 38 degrees of freedom. The static and dynamic applications of this element demonstrate its superiority to any other previously developed shell element. This element has been further developed to analyse deep thin shells. This successful deep shell element is presented by Lindberg, Olson and Cowper (119, 1971), Lindberg and Cowper (120, 1971) and Olson (121, 1973). Also, substituting cylindrical shell theory for shallow shell theory, the original element was
transformed into a very efficient triangular, cylindrical shell element by Lindberg and Olson (122, 1970; 123, 1971). The development of these elements is a major achievement in the progress of the Finite Element Technique.

A quadrilateral shell element has been developed by Key and Beisinger (124, 1971) and Key (125, 1972; 126, 1972), based on the Washizu Theory (113) of non-shallow shells. Extensive applications documented in these references demonstrate its successful behaviour in the static analysis of shells.

Another quadrilateral shell element has been developed by Petyt and Fleischer (127, 1973) based on the Novoshilov Theory (3) of thin shallow shells. The element has 48 degrees of freedom and the displacements are represented by cubic polynomials defined over each triangle formed by the diagonals of the quadrilateral. The element has been applied to the dynamic analysis of shells with success.

Hybrid and mixed formulation shell elements have also been developed. The hybrid element of Dungar, Severn and Taylor (93, 1967) has already been described. A hybrid, rectangular, cylindrical shell element has been developed by Henshell, Neale and Warburton (128, 1971), based on the Novoshilov Theory (3) of cylindrical shells and successfully applied to dynamic problems. An extensive survey on the development and applications of mixed formulation shell elements is presented by Hermann and Mason (128, 1971).

The Finite Element Method has been developed and applied to the non-linear dynamic analysis of shell structures. An extensive survey of this development and application is presented by Marcal, Dupuis, Hibbit and McNamara (130, 1971).

The development of the Super-parametric shell element by Zienkiewicz, Irons and Ahmed (15, 1970) is a major achievement in the progress of the Finite Element Technique. This element is a very efficient thick shell element. Also, the development of the Reduced Integration Technique by Zienkiewicz, Taylor and Too (17, 1971) improved the efficiency and applica-
bility of the element, such that it became a very efficient thin or thick, shallow or deep, shell element. An extensive literature survey of the development and applications of the super-parametric shell element is presented in Chapter 4.

Although the Finite Element Method has been often applied to the dynamic analysis of shells and pre-stressed plates, its application to the dynamic analysis of pre-stressed shells has seldom been reported. Mota Soares and Thomas (68, 1973) and Zienkiewicz and Bossack (69, 1973) have developed and applied the super-parametric and isoparametric elements, respectively, to the dynamic analysis of pre-stressed, shell structures.

Hartung (131, 1971) presents an extensive literature survey of the computer programs developed to analyse shell problems. These programs are based on Finite Element and Finite Difference Methods.

The Finite Element Method was introduced to the dynamic analysis of non-rotating blades by Dokumacy, Thomas and Carnegie (132, 1967). These authors represented the blades as an assembly of beam finite elements. Later, Dokainish and Rawtani (98, 1969), Rawtani (133, 1970) and Barten, Scheurenbrand and Scheer (134, 1970) represented a blade as an assembly of plate finite elements.

The first representation of a blade by curved shell finite elements was presented by Lindberg and Olson (106, 1968; 107, 1969). The blade is represented as an assembly of rectangular, cylindrical shell elements. Later Lindberg, Olson and Saragin (44, 1970) and Lindberg and Olson (118, 1971) represented the same blades by triangular, shallow shell elements.

The super-parametric shell element has been applied to the dynamic analysis of actual turbine blades by Zienkiewicz, Anderson and Ahmad (16, 1970), Aprahamian and Overage (66, 1971) and Hofmeiter and Evensen (67, 1972). In this last report, the Reduced Integration Technique is used to evaluate the strain energy of the structure.

90.
Recently, the super-parametric shell element with Reduced Integration Technique has been applied to the static analysis of marine propellers by Atkinson (73, 1973) and Sontredt (74, 1974). These authors showed the superiority of the element over other shell elements.

An important contribution to the development of the Finite Element Method to the dynamic analysis of blades has been reported by Kirhope and Wilson (135, 1971). These authors represented a bladed disc by an assembly of annular plate elements to idealise the disc and an assembly of beam elements to model the blades. The effect of the disc elasticity on the dynamic characteristics of the blades is presented by Ewins (136, 1973).

From this literature survey, it can be concluded that the Super-parametric/Reduced Integration shell element is a very efficient element and applicable to thin or thick, shallow or deep shells. Also, the element is based on the Theory of Elasticity, therefore, the development to nonlinear analysis is simpler than the corresponding development of elements based on shell theories. Consequently, the Super-parametric/Reduced Integration shell element is selected to develop all the Finite Element Models derived in this Thesis.

In this chapter, the equation of motion of a pre-stressed Finite Element Model of a structural system is derived. This model is further developed for a super-parametric shell element representation of the structure. The computer programs based on this model are applied to the dynamic analysis of thin or thick, shallow and deep, shell structures, with or without pre-stress. The efficiency of the super-parametric shell element and of the Reduced Integration Technique are investigated. The computer programs are also applied to pretwisted and cylindrical shell blades and several blade parameters are investigated. The results are compared with experimental, analytical and other finite element solutions.
6.2 Theoretical Analysis

Consider an elastic solid of volume $\Omega$, surface area $S$, density $\rho$, modulus of elasticity $E$, and Poisson ratio $v$, with reference to a Cartesian set of coordinates $x_i$ ($i = 1, 2, 3$). The solid is subjected to body forces $F_i$ per unit volume and boundary forces $P_i$ per unit area, in the $x_i$ direction. The displacements and velocities in the $x_i$ directions are $u_i$ and $\dot{u}_i$. The corresponding displacement and velocity vectors are $\{u\}$ and $\{\dot{u}\}$.

It is assumed that the oscillatory displacements are small, although the steady state displacements are large. However, the deformations of the structural system are small, such that a Lagrangian Formulation of the problem is possible.

Let the solid be represented by finite elements of volume $\Omega_e$ and surface area $S_e$, whose shape function matrix and degrees of freedom are $[N]$ and $\{r\}$ respectively. Also let $\{q\}$ and $[\bar{N}]$ be the global degrees of freedom and shape function matrix of the discretized system. Thus,

\[
\begin{align*}
\{u\} &= [N]\{r\} & \{\dot{u}\} &= [N]\{\dot{r}\} \quad \text{in} \quad \Omega_e \\
\{u\} &= [\bar{N}]\{q\} & \{\dot{u}\} &= [\bar{N}]\{\dot{q}\} \quad \text{in} \quad \Omega 
\end{align*}
\]

where $\{r\}$ and $\{q\}$ are the velocity of the corresponding displacements.

In the large elastic displacement, small deformation Theory of Elasticity, the strain tensor $\varepsilon_{ij}$ is given by,

\[
\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{s,i}u_{s,j})
\]

where $u_{i,j} = \partial u_i / \partial x_j \quad i,j,s = 1, 2, 3$.

It is convenient to separate the linear and nonlinear components of the strain tensor, $\varepsilon_{ij}^L$ and $\varepsilon_{ij}^N$, respectively, thus
\[ \varepsilon_{ij}^L = \frac{1}{2}(u_{i,j} + u_{j,i}) \]

and

\[ \varepsilon_{ij}^N = \frac{1}{2}(u_{s,i} \times u_{s,j}) \] (6.03)

The expansion of this equation gives the linear and nonlinear components of the strain vector. Let \( \{\varepsilon\}, \{\varepsilon_L\} \) and \( \{\varepsilon_N\} \) be the strain vector and the linear and nonlinear components of the strain vector, respectively. Then,

\[ \{\varepsilon\} = \{\varepsilon_L\} + \{\varepsilon_N\} = [[C_L] + [C_N]]\{u\} \] (6.04)

where \([C_L]\) and \([C_N]\) are the matrix expansion of equation (6.03).

The strain vector is given in terms of the local or global degrees of freedom of the discretized system as,

\[ \{\varepsilon\} = [[B_L] + [B_N]]\{r\} \text{ in } \Omega_e \]

or

\[ \{\varepsilon\} = [[\bar{B}_L] + [\bar{B}_N]]\{q\} \text{ in } \Omega \] (6.05)

where \([B_L] = [C_L][N]\) \( [B_N] = [C_N][N] \)

\([\bar{B}_L] = [C_L][N]\) \( [\bar{B}_N] = [C_N][N] \)

The stress and strain tensors are related by the Generalized Hooke's Law. For a homogeneous, isotropic material this law can be written in the following form,

\[ \tau_{ij} = \frac{E}{(1+\nu)(1-2\nu)} \delta_{ij}\varepsilon_{ss} + \frac{E}{(1+\nu)} \varepsilon_{ij} \] (6.06)

where \( \delta_{ij} = \) Kronecker delta

or

\[ \{\tau\} = [D]\{\varepsilon\} \] (6.07)
where \([D]\) is the Elasticity Matrix corresponding to the expansion of equation (6.06).

The Potential Energy \(V\) and the Kinetic Energy \(T\) of the system are,

\[
V = \frac{1}{2} \int_{\Omega} \{e\}^t \{\sigma\} d\Omega - \int_{\Omega} \{u\}^t \{F_b\} d\Omega - \int_{\Omega} \{u\}^t \{F_s\} d\Omega
\]

and

\[
T = \frac{1}{2} \int_{\Omega} \rho \{\dot{u}\}^t \{\dot{u}\} d\Omega
\]

where

\[
\{F_b\} = \begin{cases} F_1 \\ F_2 \\ F_3 \end{cases} \quad \text{and} \quad \{F_s\} = \begin{cases} F_1 \\ F_2 \\ F_3 \end{cases}
\]

In terms of the global degrees of freedom of the discretized system, neglecting second order terms, this equation becomes,

\[
V = \frac{1}{2} \{q\}^t \left[ \begin{bmatrix} B_L \end{bmatrix}^t [D] \left[ \begin{bmatrix} B_L \end{bmatrix} \right] + \{q\}^t \left[ \begin{bmatrix} B_N \end{bmatrix}^t [D] \left[ \begin{bmatrix} B_L \end{bmatrix} \right] \right] d\Omega(q) \right. \\
- \{q\}^t \left[ \begin{bmatrix} B_L \end{bmatrix}^t \{F_b\} d\Omega - \{q\}^t \left[ \begin{bmatrix} B_L \end{bmatrix}^t \{F_s\} dS \right. \right]
\]

and

\[
T = \frac{1}{2} \{q\}^t \int_{\Omega} \rho \left[ \begin{bmatrix} B_L \end{bmatrix}^t \left[ \begin{bmatrix} B_L \end{bmatrix} \right] \right] \{q\} d\Omega(q)
\]

Let \(\{q_0\}\) and \(\{q_1\}\) represent the steady state and oscillatory components of the displacement vector. Also let \(\{Q_1\}\) be the force vector acting on the coordinate \(\{q_1\}\). Then,

\[
\{q\} = \{q_0\} + \{q_1\} \quad \{q_0\} \gg \{q_1\}
\]

\[
\{q\} = \{q_1\}
\]

It should be noticed that \([B_N]\) and \([B_N]\) are only functions of \(\{q_0\}\).
Also, the Lagrangian \( L = T - V \) is a function of \( \{q_0\} \), \( \{q_1\} \) and \( \{q_1\} \).

Applying the Lagrange Equations to the kinetic and potential energy of the structure and neglecting second order terms, the equations of the discretized system become,

\[
[K]\{q_0\} + [KG]\{q_0\} = \{F\} + \{P\} = \{Q_0\}
\]

and

\[
[M]\{\ddot{q}_1\} + [K]\{q_1\} + [KG]\{q_1\} = \{Q_1\}
\]

where

\[
[M] = \int_\Omega [\bar{N}]^t [\bar{N}] d\Omega \quad \text{Mass Matrix} \tag{6.12}
\]

\[
[K] = \int_\Omega [\bar{B}_L]^t [D] [\bar{B}_L] d\Omega \quad \text{Stiffness Matrix} \tag{6.13}
\]

\[
[KG] = 2\int_\Omega [\bar{B}_N]^t [D] [\bar{B}_L] d\Omega \quad \text{Geometric or Initial Stress Matrix} \tag{6.14}
\]

\[
\{F\} = \int_\Omega [\bar{N}]^t \{F_b\} d\Omega \quad \text{Body Force Vector} \tag{6.15}
\]

\[
\{P\} = \int_S [\bar{N}]^t \{P_s\} dS \quad \text{Boundary Force Vector} \tag{6.16}
\]

Therefore, the pre-stressed system can be replaced by a Nonlinear Static and a Linear Dynamic System. The nonlinear static system can be solved by an Iterative Method (57).

The First Approximation of the Nonlinear Static System is to neglect the influence of the Geometric Matrix on the displacement but considering the influence of the displacement on the Geometric Matrix. With this approximation the system becomes linear and is given by,

\[
[K]\{q_0\} = \{Q_0\} \tag{6.17}
\]

The steady state stress \( \{\sigma\} \) due to the displacements, neglecting secondary terms, is given by,
\[ \{\sigma\} = [D][B_L]\{q_0\} \quad \text{in } \Omega \]

or
\[ \{\sigma\} = [D][B_L]\{r_0\} \quad \text{in } \Omega_e \]

where \( \{r_0\} \) is the local degrees of freedom, corresponding to the steady state displacement of the element.

When the static, body and boundary forces are zero, equation (6.11) becomes,
\[ [M]\{\ddot{q}_1\} + [K]\{q_1\} = \{Q_1\} \quad (6.19) \]

which is the equation of motion of a discretized system without pre-stress.

Equation (6.11) defines a Finite Element Model of any pre-stressed structure. In the following section, this model is further developed for a super-parametric shell element representation.

6.3 Super-parametric Shell Element

The Mass and Stiffness matrices for the super-parametric shell element are developed in Chapter 4. In this section, the geometric matrix, body and boundary force vectors for the super-parametric shell element are developed. The nomenclature of Chapter 4 is used.

6.3.1 Geometric matrix

The geometric matrix is given by equation (6.14) as a function of the steady state displacement vector. Alternatively, this matrix can be formulated in terms of the stress vector. Comparing equations (6.14) and (6.18), it can be concluded that,
\[ [KG]\{q_0\} = \int_{\Omega} [\tilde{B}_N]^t \{\sigma\} d\Omega \quad (6.20) \]

Also, it can be proved that, in general,
\[ 2 \{B_N\}^t \{\sigma\} = [\widehat{G}]^t [H] \{\sigma_o\} \quad (6.21) \]

where

\[
[\widehat{G}] = \begin{bmatrix}
\frac{\partial}{\partial x} [N] \\
\frac{\partial}{\partial y} [N] \\
\frac{\partial}{\partial z} [N]
\end{bmatrix} \quad [H] = \begin{bmatrix}
\sigma_{xx}[1] & \sigma_{xy}[1] & \sigma_{xz}[1] \\
\sigma_{yx}[1] & \sigma_{yy}[1] & \sigma_{yz}[1] \\
\sigma_{zx}[1] & \sigma_{zy}[1] & \sigma_{zz}[1]
\end{bmatrix}
\]

\(\sigma_{xy}\) = Initial stress tensor

\([I]\) = 3 * 3 Unit Matrix

Let \([KGE_{ij}]\) be a sub-matrix of the Element Geometric Matrix \([KGE]\).

For the super-parametric shell element this sub-matrix becomes,

\[
[KGE_{ij}] = \int_{\Omega_e} [G_i]^t [H] [G_j] d\alpha \quad (6.22)
\]

where

\[
[G_j] = \begin{bmatrix}
\frac{\partial}{\partial x} [N_j] \\
\frac{\partial}{\partial y} [N_j] \\
\frac{\partial}{\partial z} [N_j]
\end{bmatrix} \quad [N_j] = N_j \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

or

\[
[KGE_{ij}] = \int_{\Omega_e} \begin{bmatrix}
\{aN_i/ax\}^t \{\sigma\} \{aN_j/ax\} [I] \{t_j\} \\
\{aN_i/ay\}^t \{\sigma\} \{aN_j/ay\} \\
\{aN_i/az\}^t \{\sigma\} \{aN_j/az\}
\end{bmatrix} \begin{bmatrix}
\{aN_i/ax\} \{C_j\} \{\phi_j\} \\
\{aN_i/ay\} \\
\{aN_i/az\}
\end{bmatrix} d\alpha \quad (6.23)
\]
where

\[
\{C_j\} = \xi \left\{ \frac{\partial N_j}{\partial x} \right\} + N_j \left\{ \frac{\partial \xi}{\partial x} \right\} \quad \text{and} \quad \{\sigma\} = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\]

This equation can be given only in terms of the curvilinear coordinates using equations (4.16) and (4.19), which are as follows:

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\
\frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\
\frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z}
\end{bmatrix}
\]

Consequently, transforming equation (6.23) to the curvilinear coordinates gives,

\[
[KGE_{ij}] = \begin{bmatrix}
1 & 1 & 1 \\
-1 & -1 & -1
\end{bmatrix} \times
\]

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial \xi}^t & \frac{\partial N_j}{\partial \xi} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_j}{\partial \zeta} \\
0 & \frac{\partial N_i}{\partial \eta} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_j}{\partial \eta} \\
0 & 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_j}{\partial \zeta} \\
N_i & 0 & 0 & N_j
\end{bmatrix} \begin{bmatrix}
\frac{\partial N_i}{\partial \xi} & \frac{\partial N_j}{\partial \xi} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_j}{\partial \zeta} \\
0 & \frac{\partial N_i}{\partial \eta} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_j}{\partial \eta} \\
0 & 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_j}{\partial \zeta} \\
N_i & 0 & 0 & N_j
\end{bmatrix} \begin{bmatrix}
\frac{\partial N_i}{\partial \xi} & \frac{\partial N_j}{\partial \xi} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_j}{\partial \zeta} \\
0 & \frac{\partial N_i}{\partial \eta} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_j}{\partial \eta} \\
0 & 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_j}{\partial \zeta} \\
N_i & 0 & 0 & N_j
\end{bmatrix}
\]

\[
\times |J|d\xi d\eta d\zeta
\]

(6.25)

where \([F] = [J]^{-t}[\sigma][J]\).

Although this equation can be integrated, it is advantageous to transform the stress tensor matrix \([\sigma]\) into another stress tensor matrix \([\sigma']\), which is referred to the local coordinate system \(x', y', z'\). Thus,
$$\mathbf{[F]} = [\mathbf{J}]^t [\mathbf{T}] [\sigma'] [\mathbf{T}]^t [\mathbf{J}]^{-1} = [\mathbf{A}]^t [\sigma'] [\mathbf{A}] \quad (4.26)$$

Consequently,

$$[\mathbf{KGE}_{ij}] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \times$$

$$\begin{bmatrix} \{B_{1i}\}^t \{\sigma'\} \{B_{1j}\} \{1\} & \{t_j\} \{B_{1i}\}^t \{\sigma'\} \{\zeta B_{1j}\} \{\phi_j\} \\
\{B_{2i}\} \{B_{2j}\} & \{B_{2i}\} \{B_{2j}\} \{C_{1j}\} \end{bmatrix}$$

$$\begin{bmatrix} \{t_i\} \{\zeta B_{1i}\}^t \{\sigma'\} \{B_{1j}\} \{\phi_i\}^t & \{t_i\} \{t_j\} \{\zeta B_{2i}\}^t \{\sigma'\} \{\zeta B_{1j}\} \{\phi_i\}^t \{\phi_j\} \\
\{C_{1i}\} \{B_{2j}\} \{0\} & \{B_{2i}\} \{N_1\} \{N_j\} \end{bmatrix}$$

$$\times |J|d\xi d\eta d\zeta \quad (6.27)$$

where $B_{1j}$, $B_{2j}$ and $C_{1j}$ are defined by equation (4.24).

The Correct and Minimum order of integration for this equation are a $3 \times 3 \times 2$ and a $2 \times 2 \times 1$ Gaussian mesh, respectively.

### 6.3.2 Body force vector

The Element Body Force Vector is given by,

$$\{\mathbf{F}_{E} \} = \int_{\Omega_e} [\mathbf{N}]^t \{\mathbf{F}_{b}\} d\Omega \quad (6.28)$$

The sub-vector $\{\mathbf{F}_{E,j}\}$ of this force becomes,

$$\{ \mathbf{F}_{E,j} \} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \{t_j\}^t [\phi_j]^t \{\mathbf{F}_{b}\} |J|d\xi d\eta d\zeta \quad (6.29) \end{bmatrix}$$

99.
For the parabolic element, this equation can be integrated using a $3 \times 3 \times 2$ Gaussian mesh.

### 6.3.3 Boundary Force Vector

The Element Boundary Force Vector is given by,

$$
\{PE\} = \int_{S_e} [N]^t \{P_s\} dS = \int_{S_e} p [N]^t dS \quad (6.30)
$$

where $dS = n \, dS$,

$n$ is the unit vector normal to the surface and $p$ the pressure. When the pressure is acting on the surface $\xi = \pm 1$,

$$
dS = d\xi \cdot dn = \left\{ \begin{array}{c}
\frac{\partial y}{\partial n} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial n} \\
\frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial n} - \frac{\partial x}{\partial n} \frac{\partial z}{\partial \zeta} \\
\frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial n} - \frac{\partial x}{\partial n} \frac{\partial y}{\partial \zeta}
\end{array} \right\} d\eta d\zeta \quad (6.31)
$$

Consequently, the sub-vector of the Boundary Force Vector becomes,

$$
\{PE_j\} = \int_{-1}^{1} \left\{ pN_j \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} \frac{\partial y}{\partial n} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial n} \\
\frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial n} - \frac{\partial x}{\partial n} \frac{\partial z}{\partial \zeta} \\
\frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial n} - \frac{\partial x}{\partial n} \frac{\partial y}{\partial \zeta}
\end{bmatrix} \right\} d\eta d\zeta
$$

$$
\{PE_j\} = \int_{-1}^{1} \left\{ pN_j \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} \frac{\partial y}{\partial n} \frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial n} \\
\frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial n} - \frac{\partial x}{\partial n} \frac{\partial z}{\partial \zeta} \\
\frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial n} - \frac{\partial x}{\partial n} \frac{\partial y}{\partial \zeta}
\end{bmatrix} \right\} d\eta d\zeta
$$

(6.32)

For the parabolic element, this equation can be integrated numerically using a $3 \times 2$ Gaussian mesh.

### 6.4 Applications

The computer programs developed are applied to the dynamic analysis of spherical, pre-twisted and cylindrical shells and to pre-stressed cylindrical shells. The results are compared with experimental, analytical and
other finite element solutions. The efficiency of the Reduced Integration Technique and of the Eigenvalue Economizer are investigated.

Unless otherwise stated, the Reduced Integration Technique and the Eigenvalue Economizer are used. The primary coordinates are all the transverse displacements. In all the applications the Poisson ratio is 0.3.

A spherical shell, with a square base and simply supported boundary conditions, is shown in Figure 6.1. This problem is selected since there is an analytical solution and, therefore, the computed and analytical natural frequencies and mode shapes can be compared. Also the efficiency of the Reduced Integration Technique and of the Eigenvalue Economizer can be studied. The influence of the thickness of the shell on the frequency parameter is also studied. The analytical solution of the dynamic analysis of this shell is presented in this section.

A pre-twisted blade is shown in Figure 6.9. The influence on the natural frequencies of the aspect ratio \((a/b = 1.0 \text{ to } 2.0)\), thickness ratio \((b/h = 2.5 \text{ to } 15.0)\) and pre-twisted angle \((\psi = 0.0, 30.0, 45.0 \text{ degrees})\) are investigated. The results are compared with the solution of Rawtani and Dokainish (98, 1969) and Leissa (137, 1969).

A simply supported, pre-stressed, cylindrical shell is shown in Figure 6.24. The computed natural frequencies and mode shapes of the shell with or without pre-stress are compared with the analytical solution of the problem, which is presented in this section.

The dynamic characteristics of the uniform or tapered, cylindrical shell blades, which are shown in Figure 6.30, are calculated. The natural frequencies and mode shapes are compared with the experimental and finite element solutions of Lindberg and Olson (107, 1969; 118, 1971) and Lindberg Olson and Saragin (44, 1970). The effect of pre-stress on the dynamic characteristics of the blades is investigated.
6.4.1 Analytical solutions

The equation of motion of a thin, shallow shell subjected to tangential stresses is presented by Vlasov (2, 1964) as,

$$\nabla^8 F + 12(1 - v^2)/h\nabla_1^4 F + \frac{1}{D}(N_x \frac{\partial^2}{\partial x^2} + N_y \frac{\partial^2}{\partial y^2} + h \frac{\partial^2}{\partial t^2} )\nabla^4 F = P \quad (6.33)$$

where

- $F =$ Airy stress function
- $w = \nabla^4 F =$ vertical displacement
- $x, y =$ Cartesian coordinates of the projection of the shell
- $h =$ thickness of shell
- $D = \frac{E h^3}{12(1 - v^2)}$
- $E =$ Modulus of Elasticity
- $v =$ Poisson ratio
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} =$ Laplacian
- $\nabla_1^2 = \frac{1}{R_y} \frac{\partial^2}{\partial x^2} + \frac{1}{R_x} \frac{\partial^2}{\partial y^2}$
- $R_x, R_y =$ radius of curvature in $x, y$ direction
- $N_x, N_y =$ compressive force/unit lengths in the $x, y$ direction
- $P(x, y, t) =$ external force
- $t =$ time

In solving this equation, it is necessary to separate the variables, then it can be assumed that,

$$F(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(x, y) q_{mn}(t) \quad (6.34)$$

where $\phi_{mn}$ is only a function of the spatial coordinates and $q_{mn}(t)$ is only a function of time. Also, the function $\phi_{mn}$ must satisfy the boundary conditions of the problem.

Consider a rectangular base, shallow shell with simply supported
boundary conditions. Let \( u \) and \( v \) be the displacement in the \( x \) and \( y \) directions, \( a \) and \( b \) the length of the shell in the \( x \) and \( y \) directions.

The boundary conditions are,

at \( x = 0.0 \) or \( x = a \) \( w = v = \frac{\partial w}{\partial y} = 0.0 \) \hspace{1cm} (6.35)

at \( y = 0.0 \) or \( y = b \) \( w = u = \frac{\partial w}{\partial x} = 0.0 \)

For these boundary conditions, the functions \( \phi_{mn} \) are,

\[
\phi_{mn} = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \hspace{1cm} (6.36)
\]

where \( A_{mn} \) are constants.

Consequently, the equations of motion of the shell, without external forces, become,

\[
\frac{d^2 q_{mn}(t)}{dt^2} + \omega_{mn}^2 q_{mn} = 0 \hspace{1cm} m, n = 1, 2, 3, \ldots \hspace{1cm} (6.37)
\]

where

\[
\omega_{mn}^2 = \omega_{0mn}^2 \left(1 - \frac{N_x}{N_{mn}} \right) \left(1 - \frac{N_y}{N_{mn}} \right)
\]

\[
\omega_{0mn}^2 = D B_{mn}/\rho h; \hspace{1cm} N_{mn}^x = D a^2 B_{mn}/m^2 \pi^2; \hspace{1cm} N_{mn}^y = D b^2 B_{mn}/n^2 \pi^2
\]

\[
B_{mn} = \pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \left(1 + \frac{\nu^2}{R_y a^2} + \frac{n^2 R_y b^2}{b^2}\right)^2 /
\]

\[
h^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2
\]

It is clear that \( \omega_{mn} \) and \( \omega_{0mn} \) are the natural frequencies of shell, with or without pre-stress, respectively. Also, \( N_{mn}^x \) and \( N_{mn}^y \) are the Dynamic Buckling Loads of the shell in the \( x \) and \( y \) directions, respectively.

The natural frequencies and dynamic buckling loads can further be expanded in the particular cases of spherical or cylindrical shells.
6.5 General Discussion

The super-parametric parabolic shell element, with and without the Reduced Integration Technique to evaluate the stiffness and geometric matrices, has been applied to the dynamic analysis of thin or thick, shallow or deep, shells. The convergence and accuracy of the element is investigated and compared with other shell elements. The effect on the natural frequencies of the aspect ratio, thickness ratio and pre-twist angle of pre-twisted blades is investigated.

In this Thesis, the Eigenvalue Economizer is continuously used to reduce the number of degrees of freedom of shells. The magnitude of the errors introduced by this reduction and consequent analysis of the reduced eigen problem is demonstrated in Table 6.1 and Figure 6.5. In the spherical shell of Table 6.1, the average error of the first six natural frequencies in analysing a 16 degrees of freedom eigen problem instead of the original 120 degrees of freedom system is only 1.2%. The example of Figure 6.5 shows that about 80% of the original coordinates can be eliminated, without any major error in the lower natural frequencies. However, major errors are introduced in the higher natural frequencies.

The effect of the reduced integration on the accuracy and convergence of the super-parametric parabolic shell element applied to the dynamic analysis of spherical shells is presented in Figures 6.2, 6.6 and 6.7. It can be seen that the super-parametric element with reduced integration converges quicker. Therefore, more accurate and efficient. For example, the representation of the spherical shell of Figure 6.2 by 36 elements with corrected integration is as accurate as the representation of the shell by 16 elements with reduced integration. Thus, in this particular example, the reduced integration makes the super-parametric parabolic shell element approximately 500% more efficient. From Figures 6.6 and 6.7, it can be concluded that the effect of the reduced integration is greater with thin
shells and at higher modes of vibration. The element with correct integration
is too stiff and inefficient when applied to thin shell problems. Thus the
reduced integration makes the super-parametric element applicable to thin
and thick shell problems.

It can be seen in Figure 6.3 that the predicted vibration modes of the
spherical shell are very accurate. Only mode (3,3) and higher modes begin
to be distorted. However, the distortion of the higher modes are caused by
the small number of primary coordinates.

The applicability of super-parametric elements to deep shell problems is
demonstrated in Figure 6.8. It is clear that as long as the shallow shell
theory is applicable, R/a < 1, the computed and theoretical frequency para-
meters are identical. Also for deep shells, the shallow shell theory pre-
dicts higher natural frequencies.

The applicability of the super-parametric element to thick shells is
demonstrated in Figure 6.6. The computed and theoretical value of the
frequency parameter is almost identical in thin shell structures (h/a <
0.04). For thick shell structures, the thin shell theory predicts higher
natural frequencies.

It can be concluded that the thin shallow shell theory is only applicable
to shells of thickness ratio (h/a) less than 0.04 and shallowness ratio
(R/a) less than 1.0. For deep and thick shells, the thin shallow shell
theory predicts higher natural frequencies.

The Reduced Integration Technique transforms the super-parametric para-
bolic shell element into a much more efficient, accurate and versatile
element. It makes it applicable to the static and dynamic analysis of thin
or thick, shallow or deep, shells. The accuracy and efficiency of the
element as a thick shell element has never been in question but it is neces-
sary to demonstrate its efficiency and accuracy as a thin, shallow or
deep shell finite element.
The super-parametric element is developed with reference to curvilinear coordinates and not to the projected Cartesian coordinates, as in the majority of thin shallow elements. Therefore, the shallowness or deepness of the shell is almost irrelevant to the efficiency and accuracy of the element.

The dynamic characteristics of the uniform or tapered cylindrical shell blades of Figure 6.30 are ideal to demonstrate the efficiency and accuracy of the super-parametric element as a thin shallow shell element. This is the worst representation of the element. Besides, there are several experimental and finite element solutions of the natural frequencies and mode shapes of these blades. The computed natural frequencies are compared with the other solutions in Tables 6.4-6.9.

It is clear, from Table 6.5, that only the Lindberg shallow shell element (118), which is one of the most efficient and accurate thin shallow shell elements ever developed, is as accurate as the super-parametric element. However, from Tables 6.6 and 6.9, it can be concluded that for representations with equivalent numbers of degrees of freedom and primary coordinates, the super-parametric element is superior to the Lindberg element (118). For example, the first five natural frequencies of the uniform cylindrical shell can be predicted with an average error of 7.4% by a representation of super-parametric elements with 80 degrees of freedom and 16 primary coordinates. The corresponding average error for a representation of thin shallow elements with 72 degrees of freedom and 39 primary coordinates is 11.3%.

The first five and ten natural frequencies of the tapered cylindrical shell are predicted with an average error of 3.5 and 4.0% by a representation of thin shallow elements with 240 degrees of freedom and predicted with an average error of 3.1% and 5.9% by a representation of super-parametric elements with 280 degrees of freedom. However, it should be noticed that the number of primary coordinates are 125 and 56 for the shallow shell and
super-parametric representations, respectively. Also, the average error of
the first ten natural frequencies of a thin shallow shell representation
with 76 primary coordinates is 7.3%.

It can also be concluded from Tables 6.4 and 6.8 that the non monotonic
convergence of the super-parametric element due to the reduced integration is
not of practical importance.

The predicted vibration modes of the uniform and tapered cylindrical
shells are compared with the experimental nodal lines in Figures 6.31 to
6.34 and Figures 6.37 to 6.40. It is clear that the predicted modes of vib-
ration are also very accurate.

Thus, it can be concluded that the reduced integration makes the super-
parametric parabolic shell element one of the most accurate and efficient
thin shallow shell finite elements.

The simply supported, pre-stressed, cylindrical shell of Figure 6.24
is ideal to test the validity of the pre-stressed, super-parametric Finite
Element Model developed in this chapter, since there is an analytical solution
of this problem. This application also proves the validity of the Reduced
Integration Technique in non-linear dynamic problems. The super-parametric
shell element cannot be compared with other shell elements, since the thin
shallow shell elements have not been developed to analyse pre-stressed
shells. Such development has been retarded by the complexity of the non-
linear thin shallow shell theories.

The accuracy of the predicted frequency parameters of a simply supported,
cylindrical shell is presented in Table 6.3. The corresponding modes of
vibration are shown in Figures 6.25 to 6.28. These modes of vibration are
very accurate, almost identical to the theoretical modes of vibration. It
can be concluded, from Table 6.3, that a 5 * 5 finite element mesh can pre-
dict the frequency parameter of five and ten modes with an error of 1.3 and
3.4%, respectively. The corresponding error of a 4 * 4 mesh is 1.4% and
7.2%, respectively.
Figure 6.29 presents the variation of the computed and analytical frequency parameters with stress ratio, for a pre-stressed, simply supported cylindrical shell. All the predicted frequency parameters, dynamic buckling stresses and mode shapes are very accurate, except the frequency parameter and buckling stress corresponding to mode (2, 2). This can be explained, since an analysis of the shell, Table 6.3, shows that a 4 × 4 finite element mesh predicts mode (2, 2) with an error of 11.3%. (It was not possible to analyse a pre-stressed cylindrical shell with a 5 × 5 mesh because of large computer storage requirements.)

Also, the Reduced Integration Technique can be used to evaluate accurately the non-linear strain energy, therefore, applicable to non-linear dynamic analysis of the super-parametric shell elements.

The variation of the natural frequencies of the cantilever, uniform and tapered cylindrical shells with stress ratio is presented in Tables 6.7 and 6.10, and Figures 6.35, 6.36, 6.41 and 6.42. With a 3 × 3 mesh, the critical buckling stress is $\sigma_{xx} = -8.83 \times 10^7 \text{N/m}^2$ and $\tau_{xx} = -6.28 \times 10^7 \text{N/m}^2$ for the cantilever, uniform and tapered, cylindrical shell blades, respectively. These Tables and Figures demonstrate clearly the sensitivity of the natural frequencies with axial stress.

The variation of the modes of vibration with stress ratio is presented in Figure 6.43 with reference to the tapered cylindrical shell. Thus, it is not absolutely correct to analyse pre-stressed structures by assuming identical eigenfunctions as the structure without pre-stress. Also, it is not absolutely correct to assume the dynamic and buckling modes of a structure to be identical. However, these assumptions are usually assumed in the Theory of Stability of structures and in the Theory of Pre-stressed Structures.

The frequency parameters and mode shapes of a pre-twisted blade ($a/b = 2.0$, $b/h = 10.0$, $\psi = 30.0$), shown in Figure 6.9, are presented in Table 6.2 and Figures 6.10 to 6.13. The table shows that the representation of

108.
the blade by only one super-parametric element, a 25 degrees of freedom system with 5 primary coordinates, is capable of predicting the first two natural frequencies with only 5.0% error. A representation of the blade by a 2*1 finite element mesh, with 50 degrees of freedom and 10 primary coordinates, predicts the first four natural frequencies with an average error of 4.0%. Also a representation of the blade by a 4*2 mesh, with 160 degrees of freedom and 32 primary coordinates, predicts the first ten natural frequencies with an average error of 3.2%.

The effect of the aspect ratio (a/b = 1.0, 2.0), thickness ratio (b/h = 2.5 to 15.0) and pre-twist angle (ψ = 0.0, 30.0, 45.0) on the frequency parameters of several modes of vibration is shown in Figures 6.14 to 6.21. Also some results of Rawtani and Dokainish (98, 1969) and the Upper and Lower Bound predicted by the theory of thin plates (137) are shown.

The frequency parameters of bending modes of thin, pre-twisted blades are not affected by the thickness ratio. However, for thick blades the frequency parameter of the higher modes shows a large variation with thickness ratio. For example, in the particular case of a blade (a/b = 1.0) with thickness ratio varying from 5.0 to 15.0, the effect on the frequency parameter of the first, third, fifth and eighth modes is 3.2, 10.8, 20.6, and 30.0, respectively. In these modes a decrease of the aspect ratio reduces the effect of the thickness ratio. Also a variation of the pre-twist angle does not change the relative effect of the thickness ratio.

The frequency parameters of torsion and bending-torsion modes of thin blades are significantly affected by a small variation in the thickness ratio. This effect is also significantly increased with pre-twist angle. The same phenomenon does not occur in thick blades.

A small variation in the aspect ratio changes significantly the frequency parameter of torsion and bending-torsion modes of blades, without any drastic change in the frequency parameter of bending modes, as shown in Figure 6.21.
Variation of the pre-twist angle changes significantly the frequency parameters of the torsion and bending-torsion modes of thin blades. The corresponding change in thick blades is small. Also, an increase of the pre-twist angle reduces the frequency parameters of the bending modes of thin and thick blades. These reductions are smaller with thick blades and at higher modes of vibration.

The limitations of the theory of thin plates is illustrated in Figures 6.22 and 6.23. The Theory is not applicable to plates with thickness ratios \((b/h)\) less than 15. It predicts higher natural frequencies when applied to thick plates. These Figures also illustrate the inaccuracy of the Rawtani and Dokainish (98, 1969) element in predicting the frequency parameters of plates and, consequently, of pre-twisted blades.
TABLE 6.1. Influence of the primary coordinates on the frequency parameters of a simply supported spherical shell. (R/a = 1.0, Rh/a² = 0.02). (3 x 3 finite element representation with 120 degrees of freedom).

<table>
<thead>
<tr>
<th>Frequency Parameter</th>
<th>Exact Value</th>
<th>120 Primary Coordinates</th>
<th>16 Primary Coordinates (All transverse displacements)</th>
<th>Relative % Errors (100(C_2 - C_1)/C_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Computed value</td>
<td>% Error</td>
<td>Computed value</td>
</tr>
<tr>
<td>(\lambda_{11})</td>
<td>1.014</td>
<td>1.004</td>
<td>-0.99</td>
<td>1.005</td>
</tr>
<tr>
<td>(\lambda_{12})</td>
<td>1.089</td>
<td>1.097</td>
<td>0.73</td>
<td>1.123</td>
</tr>
<tr>
<td>(\lambda_{21})</td>
<td></td>
<td>1.228</td>
<td>13.7</td>
<td>1.410</td>
</tr>
<tr>
<td>(\lambda_{22})</td>
<td></td>
<td>1.357</td>
<td>38.0</td>
<td>1.886</td>
</tr>
</tbody>
</table>

Average relative error 1.01%

\[ \lambda_{nm} = \text{frequency parameter} = \rho R^2 \omega_{nm}^2 / E \]

\[ \omega_{nm} = \text{natural frequency corresponding to mode (n, m)} \]
TABLE 6.2. Variation of frequency parameters of pre-twisted blade with degrees of freedom.
(a/b = 2.0, b/h = 10.0, ψ = 30.0).

<table>
<thead>
<tr>
<th>Freq. No.</th>
<th>(1x1, 25,5)</th>
<th>(2x1, 50,10)</th>
<th>(2x2, 80,16)</th>
<th>(3x2, 120,24)</th>
<th>(4x2, 160,32)</th>
<th>(3x3, 165, 33)</th>
<th>(4x3, 220,44)</th>
<th>(6x3, 330,66)</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>15.00</td>
<td>14.61</td>
<td>16.00</td>
<td>15.89</td>
<td>15.86</td>
<td>15.83</td>
<td>15.80</td>
<td>15.79</td>
</tr>
<tr>
<td>4</td>
<td>58.52</td>
<td>49.31</td>
<td>51.53</td>
<td>49.66</td>
<td>49.13</td>
<td>49.32</td>
<td>48.79</td>
<td>48.56</td>
</tr>
<tr>
<td>5</td>
<td>90.89</td>
<td>67.44</td>
<td>68.63</td>
<td>60.04</td>
<td>56.92</td>
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<td>56.93</td>
<td>55.75</td>
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<tr>
<td>6</td>
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<td>91.54</td>
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<td>88.46</td>
<td>88.31</td>
<td>88.31</td>
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<td>91.21</td>
<td>94.18</td>
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<td>113.7</td>
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<td>125.0</td>
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<td>124.0</td>
<td>118.6</td>
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<tr>
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<td>152.2</td>
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<td>207.3</td>
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<td>228.8</td>
<td>228.8</td>
<td>213.6</td>
<td></td>
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<tr>
<td>14</td>
<td>308.1</td>
<td>239.4</td>
<td>284.2</td>
<td>239.0</td>
<td>239.0</td>
<td>239.0</td>
<td>223.5</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>283.3</td>
<td>260.4</td>
<td>264.8</td>
<td>244.6</td>
<td>244.6</td>
<td>244.6</td>
<td>241.8</td>
<td></td>
</tr>
</tbody>
</table>

\[ \lambda_r = \text{frequency parameter} = \omega_r \sqrt{\frac{Eh^2}{12(1-\nu^2)}} a^4 \]

\[ \omega_r = \text{natural frequency} \]

\[ \nu = 0.3 \]
<table>
<thead>
<tr>
<th>Freq. No.</th>
<th>Finite element mesh, degrees of freedom and primary coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(7 x 3, 385, 77) (8 x 3, 440, 88) (10 x 3, 550, 110) (8 x 4, 560, 112)</td>
</tr>
<tr>
<td></td>
<td>3.344</td>
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<tr>
<td>1</td>
<td>15.79</td>
</tr>
<tr>
<td>2</td>
<td>19.17</td>
</tr>
<tr>
<td>3</td>
<td>48.53</td>
</tr>
<tr>
<td>4</td>
<td>55.61</td>
</tr>
<tr>
<td>5</td>
<td>88.28</td>
</tr>
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<td>88.65</td>
</tr>
<tr>
<td>7</td>
<td>118.6</td>
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<td>9</td>
<td>162.2</td>
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<td>14</td>
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</tr>
<tr>
<td>15</td>
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</tr>
</tbody>
</table>

113.
TABLE 6.3. Variation of frequency parameter of simply supported cylindrical shell with degrees of freedom \((a/b = 1.0, \text{Rh}/a^2 = 0.02)\).

<table>
<thead>
<tr>
<th>Frequency Number</th>
<th>Mode</th>
<th>Finite element mesh, degrees of freedom and primary coordinates</th>
<th>Exact Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>((3 \times 3, 120, 16))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>((4 \times 4, 221, 33))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>((5 \times 5, 352, 56))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Result % Error</td>
<td>Result % Error</td>
</tr>
<tr>
<td>1</td>
<td>(1, 2)</td>
<td>0.153 18.6</td>
<td>0.128 0.0</td>
</tr>
<tr>
<td>2</td>
<td>(1, 1)</td>
<td>0.270 2.2</td>
<td>0.265 0.3</td>
</tr>
<tr>
<td>3</td>
<td>(1, 3)</td>
<td>0.649 76.8</td>
<td>0.359 -2.2</td>
</tr>
<tr>
<td>4</td>
<td>(2, 2)</td>
<td>0.745 55.9</td>
<td>0.554 11.3</td>
</tr>
<tr>
<td>5</td>
<td>(2, 3)</td>
<td>1.021 46.3</td>
<td>0.714 2.3</td>
</tr>
<tr>
<td>6</td>
<td>(2, 1)</td>
<td>0.774 6.2</td>
<td>0.735 0.8</td>
</tr>
<tr>
<td>7</td>
<td>(1, 4)</td>
<td>1.149 10.8</td>
<td>0.956 -8.2</td>
</tr>
<tr>
<td>8</td>
<td>(3, 2)</td>
<td>1.507 39.3</td>
<td>1.174 8.5</td>
</tr>
<tr>
<td>9</td>
<td>(3, 1)</td>
<td>1.678 43.8</td>
<td>1.212 3.8</td>
</tr>
<tr>
<td>10</td>
<td>(3, 3)</td>
<td>2.549 81.3</td>
<td>1.882 33.8</td>
</tr>
<tr>
<td>11</td>
<td>(2, 4)</td>
<td>2.264 18.1</td>
<td>2.007 4.7</td>
</tr>
<tr>
<td>12</td>
<td>(4, 1)</td>
<td>2.532 22.2</td>
<td>2.072 4.7</td>
</tr>
<tr>
<td>13</td>
<td>(4, 2)</td>
<td>3.751 58.9</td>
<td>2.360 4.7</td>
</tr>
<tr>
<td>14</td>
<td>(3, 4)</td>
<td>2.391 -9.6</td>
<td>2.414 4.7</td>
</tr>
</tbody>
</table>

\[ \lambda_{nm} \text{ = frequency parameter } = \rho R^2 \omega^2_{nm} / E \]

\[ \omega^2_{nm} \text{ = natural frequency corresponding to mode (n, m)} \]
TABLE 6.4. Variation of the natural frequencies of the uniform cylindrical shell blade with degrees of freedom.

<table>
<thead>
<tr>
<th>Freq. No.</th>
<th>Finite element mesh, degrees of freedom and primary coordinates</th>
<th>Experimental values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2 x 2, 80, 16)</td>
<td>(3 x 2, 120, 24)</td>
</tr>
<tr>
<td></td>
<td>Result Hz</td>
<td>% Error</td>
</tr>
<tr>
<td>1</td>
<td>92.9</td>
<td>8.5</td>
</tr>
<tr>
<td>2</td>
<td>143.2</td>
<td>6.5</td>
</tr>
<tr>
<td>3</td>
<td>248.5</td>
<td>4.8</td>
</tr>
<tr>
<td>4</td>
<td>397.2</td>
<td>13.1</td>
</tr>
<tr>
<td>5</td>
<td>411.6</td>
<td>4.0</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Experimental values obtained by Lindberg, Olson and Sarazin (44, 1970).
TABLE 6.5. Comparison of the natural frequencies of uniform cylindrical shell blades with other solutions.

<table>
<thead>
<tr>
<th>Experimental values (Hz)</th>
<th>Finite Element Solutions (Hz)</th>
<th>Present Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>86.6</td>
<td>85.6</td>
<td>93.5</td>
</tr>
<tr>
<td>135.5</td>
<td>134.5</td>
<td>147.6</td>
</tr>
<tr>
<td>258.9</td>
<td>259.0</td>
<td>255.1</td>
</tr>
<tr>
<td>350.6</td>
<td>351.0</td>
<td>393.1</td>
</tr>
<tr>
<td>395.2</td>
<td>395.0</td>
<td>423.5</td>
</tr>
<tr>
<td>Average % Error</td>
<td>7.9</td>
<td>2.7</td>
</tr>
</tbody>
</table>
TABLE 6.6. Comparison of the natural frequencies of the uniform cylindrical shell blade calculated by a super-parametric and a shallow thin shell element representation.

<table>
<thead>
<tr>
<th>Experimental value - Lindberg, Olson and Sarazin (44, 1970)</th>
<th>Finite element solutions. (Number of degrees of freedom and primary coordinates)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Super-parametric shell element (80, 16) (165, 33) (280, 56) Thin shallow shell element (44)</td>
</tr>
<tr>
<td></td>
<td>Result Hz % Error Result Hz % Error Result Hz % Error Result Hz % Error Result Hz % Error</td>
</tr>
<tr>
<td>85.6</td>
<td>92.9 8.5</td>
</tr>
<tr>
<td>134.5</td>
<td>143.2 6.5</td>
</tr>
<tr>
<td>259</td>
<td>248.5 4.8</td>
</tr>
<tr>
<td>351</td>
<td>397.2 13.1</td>
</tr>
<tr>
<td>395</td>
<td>411.6 4.0</td>
</tr>
<tr>
<td>531</td>
<td>576.8 8.6</td>
</tr>
<tr>
<td>751</td>
<td>745.6 -0.7</td>
</tr>
<tr>
<td>743</td>
<td>823.7 10.9</td>
</tr>
<tr>
<td>790</td>
<td>853.6 8.0</td>
</tr>
<tr>
<td>809</td>
<td>959.7 18.6</td>
</tr>
<tr>
<td>Average error of first five natural frequencies</td>
<td>7.4    3.1    2.1   11.3    2.7    1.8</td>
</tr>
<tr>
<td>Average error of first ten natural frequencies</td>
<td>6.2    2.9    2.9   2.9    2.9    1.2</td>
</tr>
</tbody>
</table>
TABLE 6.7. Variation of the natural frequencies of the pre-stressed, uniform cylindrical shell blade with stress ratio.

<table>
<thead>
<tr>
<th>Stress ratio $\tau_{xx}/\tau_{xx}^c$</th>
<th>Natural frequencies (Hz)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>-2.00</td>
<td>142.0</td>
<td>179.3</td>
<td>277.9</td>
<td>412.3</td>
<td>474.7</td>
<td>587.6</td>
<td>791.7</td>
<td>939.3</td>
</tr>
<tr>
<td>-1.50</td>
<td>131.2</td>
<td>170.9</td>
<td>271.9</td>
<td>400.1</td>
<td>464.3</td>
<td>584.4</td>
<td>782.2</td>
<td>913.1</td>
</tr>
<tr>
<td>-1.00</td>
<td>118.6</td>
<td>161.5</td>
<td>265.6</td>
<td>386.5</td>
<td>452.6</td>
<td>581.2</td>
<td>771.5</td>
<td>884.5</td>
</tr>
<tr>
<td>-0.50</td>
<td>104.0</td>
<td>151.4</td>
<td>259.1</td>
<td>371.4</td>
<td>439.4</td>
<td>577.8</td>
<td>759.4</td>
<td>854.7</td>
</tr>
<tr>
<td>0.00</td>
<td>86.2</td>
<td>139.8</td>
<td>252.8</td>
<td>354.3</td>
<td>424.8</td>
<td>576.8</td>
<td>745.6</td>
<td>823.7</td>
</tr>
<tr>
<td>0.25</td>
<td>75.2</td>
<td>133.6</td>
<td>248.7</td>
<td>344.7</td>
<td>415.6</td>
<td>572.5</td>
<td>737.4</td>
<td>807.0</td>
</tr>
<tr>
<td>0.50</td>
<td>61.9</td>
<td>126.8</td>
<td>245.1</td>
<td>334.4</td>
<td>406.2</td>
<td>570.7</td>
<td>728.7</td>
<td>790.5</td>
</tr>
<tr>
<td>0.75</td>
<td>44.3</td>
<td>119.4</td>
<td>241.4</td>
<td>323.3</td>
<td>396.1</td>
<td>568.8</td>
<td>719.2</td>
<td>773.6</td>
</tr>
<tr>
<td>0.90</td>
<td>28.3</td>
<td>114.6</td>
<td>239.1</td>
<td>316.0</td>
<td>389.4</td>
<td>567.6</td>
<td>713.0</td>
<td>763.3</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0</td>
<td>111.1</td>
<td>237.5</td>
<td>311.0</td>
<td>384.7</td>
<td>566.8</td>
<td>708.7</td>
<td>756.4</td>
</tr>
<tr>
<td>% Decrease</td>
<td>-</td>
<td>61.3</td>
<td>17.0</td>
<td>32.6</td>
<td>23.4</td>
<td>3.6</td>
<td>11.7</td>
<td>24.2</td>
</tr>
</tbody>
</table>

The minimum critical stress $\tau_{xx}^c$ is $-8.83 \times 10^7$ N/m²

All values are calculated using a 3 x 3 finite element representation with 165 degrees of freedom and 33 primary coordinates. The primary coordinates are the transverse displacements.
TABLE 6.8. Variation of the natural frequencies of the tapered, cylindrical shell blade with degrees of freedom.

<table>
<thead>
<tr>
<th>Freq. no.</th>
<th>Finite element mesh, degrees of freedom and primary coordinates</th>
<th>Experimental values Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2 x 2, 80, 16)</td>
<td>(3 x 2, 120, 24)</td>
</tr>
<tr>
<td></td>
<td>Result</td>
<td>% Error</td>
</tr>
<tr>
<td>1</td>
<td>78.1</td>
<td>2.2</td>
</tr>
<tr>
<td>2</td>
<td>115.2</td>
<td>6.4</td>
</tr>
<tr>
<td>3</td>
<td>202.4</td>
<td>4.2</td>
</tr>
<tr>
<td>4</td>
<td>284.3</td>
<td>12.4</td>
</tr>
<tr>
<td>5</td>
<td>412.8</td>
<td>-3.2</td>
</tr>
<tr>
<td>6</td>
<td>480.5</td>
<td>12.6</td>
</tr>
<tr>
<td>7</td>
<td>525.8</td>
<td>12.9</td>
</tr>
<tr>
<td>8</td>
<td>599.5</td>
<td>14.7</td>
</tr>
<tr>
<td>9</td>
<td>855.8</td>
<td>26.3</td>
</tr>
<tr>
<td>10</td>
<td>880.9</td>
<td>27.1</td>
</tr>
<tr>
<td>11</td>
<td>824.6</td>
<td>4.7</td>
</tr>
<tr>
<td>12</td>
<td>891.3</td>
<td>10.2</td>
</tr>
<tr>
<td>13</td>
<td>979.3</td>
<td>8.7</td>
</tr>
<tr>
<td>14</td>
<td>1266.7</td>
<td>25.9</td>
</tr>
</tbody>
</table>

Experimental values obtained by Lindberg, Olson and Sarazin (44, 1970).
TABLE 6.9. Comparison of the natural frequencies of the tapered cylindrical shell blade calculated by a super-parametric and a shallow thin shell element representation.

| Experimental value - Lindberg, Olson and Sarazin (44, 1970) Hz | Finite element solutions (number of degrees of freedom and primary coordinates) | Thin shallow shell element Lindberg, Olson and Sarazin (44, 1970) |
|---|---|---|---|---|---|---|---|---|---|
| | Super parametric shell element | | | | | | | | |
| | (80, 16) | (165, 33) | (280, 56) | (72, 39) | (144, 76) | (240, 125) |
| | Result Hz | % Error | Result Hz | % Error | Result Hz | % Error | Result Hz | % Error | Result Hz | % Error |
| 76.4 | 78.1 | 2.2 | 73.5 | -3.9 | 72.0 | -5.5 | 83.6 | 9.4 | 79.6 | 4.2 |
| 108 | 115.2 | 6.4 | 113.3 | 4.6 | 112.0 | 3.7 | 117.5 | 8.8 | 114.6 | 6.1 |
| 202 | 202.4 | 0.2 | 205.2 | 1.5 | 202.6 | 0.5 | 223.5 | 10.6 | 213.9 | 5.9 |
| 253 | 284.3 | 12.4 | 265.6 | 4.7 | 249.8 | 1.2 | 321.1 | 26.9 | 270.5 | 6.9 |
| 364 | 412.8 | -3.2 | 389.5 | 6.8 | 377.2 | 3.5 | 411.1 | 12.9 | 377.3 | 3.7 |
| 426 | 480.5 | 12.6 | 451.3 | 5.8 | | | 457.8 | 7.5 | 452.3 | 6.2 |
| 465 | 525.8 | 12.9 | 503.6 | 8.1 | | | 519.2 | 11.7 | 480.5 | 3.3 |
| 572 | 599.5 | 4.7 | 579.5 | 1.2 | | | 589.5 | 3.1 | 581.3 | 1.6 |
| 677 | 855.8 | 26.3 | 756.7 | 11.6 | | | 707.0 | 4.4 | 690.1 | 1.9 |
| 692 | 880.9 | 27.1 | 811.7 | 17.1 | | | 824.8 | 19.1 | 755.0 | 9.1 |
| Average error of first five natural frequencies | | | | 4.9 | 4.3 | 3.1 | 13.7 | 5.4 | 3.5 |
| Average error of first ten natural frequencies | | | | 10.5 | 5.9 | | 7.3 | 4.0 |
TABLE 6.10. Variation of the natural frequencies of the pre-stressed, tapered cylindrical shell blade with stress ratio.

<table>
<thead>
<tr>
<th>Stress ratio ( \tau_{xx} / \tau_{xx}^c )</th>
<th>Natural frequencies (Hz)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.00</td>
<td></td>
<td>118.8</td>
<td>147.6</td>
<td>227.5</td>
<td>312.7</td>
<td>431.3</td>
<td>489.9</td>
<td>617.8</td>
<td>683.4</td>
</tr>
<tr>
<td>-1.50</td>
<td></td>
<td>110.0</td>
<td>140.3</td>
<td>222.3</td>
<td>303.4</td>
<td>422.4</td>
<td>487.3</td>
<td>606.2</td>
<td>654.6</td>
</tr>
<tr>
<td>-1.00</td>
<td></td>
<td>100.0</td>
<td>132.2</td>
<td>216.8</td>
<td>293.2</td>
<td>412.6</td>
<td>484.7</td>
<td>588.2</td>
<td>629.9</td>
</tr>
<tr>
<td>-0.50</td>
<td></td>
<td>88.1</td>
<td>123.4</td>
<td>211.1</td>
<td>280.9</td>
<td>401.7</td>
<td>482.1</td>
<td>560.2</td>
<td>612.4</td>
</tr>
<tr>
<td>0.00</td>
<td></td>
<td>73.5</td>
<td>113.3</td>
<td>205.2</td>
<td>265.6</td>
<td>389.5</td>
<td>480.5</td>
<td>525.8</td>
<td>599.5</td>
</tr>
<tr>
<td>0.25</td>
<td></td>
<td>64.4</td>
<td>107.9</td>
<td>201.7</td>
<td>256.0</td>
<td>382.3</td>
<td>477.4</td>
<td>507.4</td>
<td>593.4</td>
</tr>
<tr>
<td>0.50</td>
<td></td>
<td>53.3</td>
<td>101.9</td>
<td>198.2</td>
<td>244.8</td>
<td>374.6</td>
<td>474.5</td>
<td>490.0</td>
<td>587.5</td>
</tr>
<tr>
<td>0.75</td>
<td></td>
<td>38.3</td>
<td>95.2</td>
<td>194.6</td>
<td>231.2</td>
<td>366.1</td>
<td>464.9</td>
<td>479.3</td>
<td>581.6</td>
</tr>
<tr>
<td>0.90</td>
<td></td>
<td>24.2</td>
<td>90.8</td>
<td>192.1</td>
<td>221.6</td>
<td>360.5</td>
<td>455.3</td>
<td>477.1</td>
<td>577.9</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td>0.0</td>
<td>87.7</td>
<td>190.2</td>
<td>214.7</td>
<td>356.5</td>
<td>448.5</td>
<td>476.2</td>
<td>575.5</td>
</tr>
<tr>
<td>% Decrease</td>
<td></td>
<td>-</td>
<td>68.3</td>
<td>19.6</td>
<td>45.6</td>
<td>21.0</td>
<td>9.2</td>
<td>29.7</td>
<td>18.7</td>
</tr>
</tbody>
</table>

The minimum critical stress \( \tau_{xx}^c \) is \(-6.28 \times 10^7\) N/m².

All values are obtained using a 3 x 3 finite element representation with 165 degrees of freedom and 33 primary coordinates. The primary coordinates are the transverse displacements.
The simply supported boundary conditions are:

- At $x = 0.0$ or $x = a$
  \[ w = v = \frac{\partial w}{\partial y} = 0.0 \]

- At $y = 0.0$ or $y = a$
  \[ w = u = \frac{\partial w}{\partial x} = 0.0 \]

a) Spherical shell geometry

b) A $4 \times 3$ mesh of finite elements

FIG. 6.1. SPHERICAL SHELL GEOMETRY AND FINITE ELEMENT MESH - SIMPLY SUPPORTED BOUNDARY CONDITIONS.
FIG. 6.2. CONVERGENCE CURVES FOR THE FINITE ELEMENT MODELS OF THE SPHERICAL SHELL. (R/a = 1.0, Rh/a² = 0.02).
Computed values

Theoretical values

Reduced integration

\[ \lambda_{21} = 1.08 \quad (1.09) \]

\[ \lambda_{22} = 1.22 \quad (1.23) \]

Finite element model (6 × 6, 513,85)

Finite element mesh

Primary coordinates

Degrees of freedom

FIG. 6.3 VIBRATION MODES OF A SPHERICAL SHELL (R/a = 1.0, Rh/a^2 = 0.02)

- 124 -
The primary coordinates are the vertical deflections.

FIG. 6.4. LOCATION OF PRIMARY COORDINATES OF THE SPHERICAL SHELL.
\((R/a = 1.0, \, Rh/a^2 = 0.02)\)
Fig. 6.5. Variation of frequency parameter of spherical shell with primary coordinates. (R/a = 1.0, R/h/a^2 = 0.02)
FIG. 6.6. VARIATION OF THE FREQUENCY PARAMETER OF SPHERICAL SHELLS WITH THICKNESS RATIO. (R/a = 1.0)
\[ \beta = \frac{\text{Frequency parameter using corrected integration}}{\text{Frequency parameter using reduced integration}} \]

FIG. 6.7. RELATIVE VALUE OF FREQUENCY PARAMETER OF SPHERICAL SHELLS WITH THICKNESS RATIO \( R/a = 1.0 \)
FIG. 6.8. VARIATIONS OF FREQUENCY PARAMETER OF SPHERICAL SHELLS WITH RADIUS/LENGTH RATIO. (h/R = 0.02).
Poisson ratio = 0.3
Length of blade = a
Uniform thickness = h
Uniform rectangular cross section.

FIG. 6.9. SIDE VIEW OF PRE-TWISTED BLADE.
FIG. 6.10. CONVERGENCE CURVE FOR THE FINITE ELEMENT MODEL OF A PRE-TWISTED BLADE - MODE 1 TO 7.

\( (a/b = 2.0, \ h/b = 10.0, \ \varphi = 30.0) \)
FIG. 6.11. CONVERGENCE CURVE FOR THE FINITE ELEMENT MODEL OF A PRE-TWISTED BLADE - MODE 8 TO 13.
Finite element model (8 x 4, 560, 112)

\[
\lambda_1 = 3.343
\]

\[
\begin{align*}
\lambda_2 &= 15.77 \\
\lambda_3 &= 19.17 \\
\lambda_4 &= 48.45 \\
\lambda_5 &= 55.53 \\
\lambda_6 &= 87.72 \\
\lambda_7 &= 8442 \\
\lambda_8 &= 109.2 \\
\lambda_9 &= 117.8
\end{align*}
\]

\((a/b = 2.0, h/b = 10.0, \gamma = 30.0)\).
FIG. 6.13. VIBRATION MODES OF A PRE-TWISTED BLADE - MODE 10 to 15.
FIG. 6.14. VARIATION OF THE FREQUENCY PARAMETERS OF PRE-TWISTED BLADES WITH THICKNESS RATIO- FIRST AND THIRD MODE.
FIG. 6.15. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADES WITH THICKNESS RATIO - SECOND MODE.
FIG. 6.16. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADES WITH THICKNESS RATIO - FOURTH MODE.
FIG. 6.17. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADES WITH THICKNESS RATIO - FIFTH MODE.
FIG. 6.18. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADES WITH THICKNESS RATIO - SIXTH MODE.

- 139 -
FIG. 6.19. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADES WITH THICKNESS RATIO - SEVENTH MODE.
FIG. 6.20. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADES WITH THICKNESS RATIO - EIGHTH MODE.
FIG. 6.21. VARIATION OF FREQUENCY PARAMETERS OF PRE-TWISTED BLADES WITH ASPECT RATIO. (b/h = 15.0)
FIG. 6.22. VARIATION OF FREQUENCY PARAMETERS OF CANTILEVER PLATE ($c/h = 1.0$)
FIG. 6.23. VARIATION OF FREQUENCY PARAMETER OF CANTILEVER PLATES WITH THICKNESS RATIO (a/b = 2.0)
FIG. 6.24. SIMPLY SUPPORTED PRE-STRESSED CYLINDRICAL SHELL

\( a/b = 1.0, \, \frac{Rh}{a^2} = 0.02. \)

\[ \varepsilon_{xx} = \text{Stress} \]

\[ \nu = 0.3 \]
Finite element model \((5 \times 5, 352, 56)\)

\[
\lambda_{mn} = \frac{c R^2 \omega_{mn}^2}{E}
\]

**Mode (1, 2)**

\[
\lambda_{12} = 0.126 \\
(0.129)
\]

**Mode (1, 1)**

\[
\lambda_{11} = 0.264 \\
(0.264)
\]

**Mode (1, 3)**

\[
\lambda_{13} = 0.356 \\
(0.367)
\]

**Mode (2, 2)**

\[
\lambda_{22} = 0.492 \\
(0.478)
\]

**Mode (2, 3)**

\[
\lambda_{23} = 0.714 \\
(0.698)
\]

**Mode (2, 1)**

\[
\lambda_{21} = 0.742 \\
(0.729)
\]

**FIG. 6.25. VIBRATION MODES OF A CYLINDRICAL SHELL - MODES 1 to 6.**

\((a/b = 1.0, \, Rh/a^2 = 0.02)\)
FIG. 6.26. VIBRATION MODES OF CYLINDRICAL SHELL - MODES 7 to 12.
Finite element model (5 × 5, 52, 56)

First mode

\[ \lambda_{12} = 0.126 \]

(0.129)

Second mode

\[ \lambda_{11} = 0.264 \]

(0.264)

FIG. 6.27. ACCURACY OF VIBRATION MODES OF CYLINDRICAL SHELL

- FIRST AND SECOND MODE.
Finite element model (5×5, 352, 56)

Fourth mode
\[ \lambda_{22} = 0.492 \quad (0.478) \]

Sixth mode
\[ \lambda_{21} = 0.742 \quad (0.729) \]

FIG. 6.28. ACCURACY OF VIBRATION MODES OF CYLINDRICAL SHELL - FOURTH AND FIFTH MODE.
Finite element model \(4 \times 4, 223, 33\)

\(\sigma_{xx}^{c}\) - minimum buckling stress

- Analytical solution
- \(\bullet \bullet \bullet \bullet \) Computed solution
- Computed values corresponding to mode \((2, 2)\)

**FIG. 6.29. VARIATION OF FREQUENCY PARAMETER OF PRE-STRESSED CYLINDRICAL SHELL WITH STRESS RATIO.**

\((a/b = 1.0, \, R_h/a^2 = 0.02)\).
FIG. 6.30. CYLINDRICAL SHELL BLADES WITH UNIFORM AND TAPERED THICKNESS.

Uniform blade
\[ a = 0.305 \text{ m} \]
\[ b = 0.302 \text{ m} \]
\[ h_2 = h_1 = 0.00305 \text{ m} \]
\[ R = 0.610 \text{ m} \]

Tapered blade
\[ a = 0.305 \text{ m} \]
\[ b = 0.303 \text{ m} \]
\[ h_1 = 0.00122 \text{ m} \]
\[ h_2 = 0.00419 \text{ m} \]
\[ R = 0.762 \text{ m} \]

\[ \psi = 0.3 \]
\[ \varrho = 7800.0 \text{ Kg/m}^3 \]
\[ E = 2.0 \times 10^{11} \text{ N/m}^2 \]
The experimental vibration modes were obtained by Linberg, Olson and Sarazin (44, 1970).

FIG. 6.31. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF UNIFORM CYLINDRICAL SHELL BLADE - MODES 1 to 4.
FIG. 6.32. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF UNIFORM CYLINDRICAL SHELL BLADE - MODES 5 to 7.
FIG. 6.33. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF UNIFORM CYLINDRICAL SHELL BLADE—MODES 8 to 10.
FIG. 6.34. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF UNIFORM CYLINDRICAL SHELL BLADE - MODES 11 to 13.
FIG. 6.35. VARIATION OF NATURAL FREQUENCIES OF UNIFORM CYLINDRICAL SHELL BLADE WITH STRESS RATIO - MODES 1 to 3.
The experimental vibration modes were obtained by Lindberg, Olson and Sarazin (44, 1970).

FIG. 6.37. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF TAPERED CYLINDRICAL SHELL BLADE - MODES 1 to 4.
FIG. 6.38. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF TAPERED CYLINDRICAL SHELL BLADE - MODES 5 to 7.
FIG. 6.39. COMPUTED AND EXPERIMENTAL VIBRATION modes of TAPERED CYLINDRICAL SHELL BLADE - MODES 8 to 10.

Computed vibration modes
$\omega_8 = 569$ Hz

Experimental vibration modes
$\omega_8 = 572$ Hz

$\omega_9 = 706$ Hz

$\omega_9 = 677$ Hz

$\omega_{10} = 775$ Hz

$\omega_{10} = 692$ Hz
Computed vibration modes

$\omega_{11} = 805 \text{ Hz}$

Experimental vibration modes

$\omega_{11} = 787 \text{ Hz}$

$\omega_{12} = 857 \text{ Hz}$

$\omega_{12} = 808 \text{ Hz}$

$\omega_{13} = 951 \text{ Hz}$

$\omega_{13} = 900 \text{ Hz}$

FIG. 6.40. COMPUTED AND EXPERIMENTAL VIBRATION MODES OF TAPERED CYLINDRICAL SHELL BLADE - MODES 11 to 13.
Finite element model (3 x 3, 120, 16)

Experimental values.

Minimum critical stress $= \sigma_{xx}^c$
$= -6.28 \times 10^7$ N/m$^2$

FIG. 6.41. VARIATION OF NATURAL FREQUENCIES OF TAPERED CYLINDRICAL SHELL BLADE WITH STRESS RATIO-MODES 1 to 3.
FIG. 6.42 VARIATION OF NATURAL FREQUENCIES OF TAPERED CYLINDRICAL SHELL BLADE WITH STRESS RATIO MODE 4 to 6.
Stress ratio = actual stress/minimum buckling stress
Finite element model (3 * 3, 120, 160)

Stress ratio = -2.0
\[ \omega = 312 \text{ Hz} \]

Stress ratio = 0.0
\[ \omega = 266 \text{ Hz} \]

Stress ratio = 0.5
\[ \omega = 244 \text{ Hz} \]

Stress ratio = 1.0
\[ \omega = 217 \text{ Hz} \]

FIG. 6.43. EFFECT OF THE STRESS RATIO ON THE VIBRATION MODES OF TAPERED CYLINDRICAL SHELL BLADE - FOURTH MODE.
CHAPTER 7

DYNAMIC ANALYSIS OF ROTATING SHELLS

7.1 Introduction

It is only recently that the Finite Element Method has been developed and applied to the dynamic analysis of rotating structures. Rawtani (133, 1970) applied the Finite Element Method to the dynamic analysis of rotating, pre-twisted blades. The blade was idealized as an assembly of incompatible, constant thickness, triangular plate elements with 9 bending and 6 axial degrees of freedom, respectively. The influence of the steady state deformation on the dynamic characteristics of rotating, pre-twisted blades were investigated. However, the effect of the gyroscopic coupling is neglected. Rawtani's research is partially published by Rawtani and Dokainish (138, 1970; 18, 1971; 139, 1971; 140, 1972).

Henry (141, 1973) and Henry and Lallane (142, 1973; 143, 1974) represented a compressor blade as an assembly of triangular, plate elements. These elements are compatible, with 18 degrees of freedom and variable thickness. The computed natural frequencies and mode shapes of the rotating blade are compared with the dynamic characteristics of the non-rotating blade. The steady state deformation of the blade is calculated by an Iterative Method (57) and the equations of motion are obtained with reference to the deformed blade. The gyroscopic term was neglected in the analysis.

The equations of motion of any Finite Element Model of a rotating structure were developed by Mota Soares and Thomas (68, 1973; 144, 1974). These equations of motion, including the gyroscopic term, are derived with reference to any displacement finite element. The Model is further developed for a Super-parametric shell element representation of a rotating shell structure. A Reduced Integration Technique is used to evaluate the strain.
energy of the element. The dynamic characteristics of rotating, pre-twisted blades were investigated.

Zienkiewicz and Bossak (69, 1973) also derived the equation of motion of a rotating structure idealized as an assembly of finite elements. The structure is represented by three-dimensional Isoparametric elements and a Reduced Integration Technique is used to evaluate the strain energy of the element. In all applications, the rotating structure is considered to be a pre-stressed system, neglecting all other factors.

An important contribution to the development of the finite element dynamic analysis of rotating structures has been reported by Likins (145, 1972). This author developed a Finite Element Model of a structure spinning in space.

In this chapter, the equations of motion of a Finite Element Model of a rotating structure are derived. The Model is further developed for a super-parametric shell element representation of a rotating shell structure. The Reduced Integration Technique is used to evaluate the strain energy of the structure. This Model is applicable to thin or thick, deep or shallow, rotating shell structures. The dynamic characteristics of rotating, pre-twisted and cylindrical, shell blades are investigated.

7.2 Theoretical Analysis

Consider a rotating structure of volume $\Omega$ and density $\rho$, as shown in Figure 7.1, with reference to a fixed Cartesian set of coordinates $x_1, x_2, x_3$. The structure is also referred to a rotating set of coordinates $x, y, z$. The structure is rotating at a constant angular velocity $\omega$ and the axis of rotation is defined by the unit vector $\mathbf{a}$. The position vector of the origin of the rotating frame of reference is $\mathbf{r}_1$. Let $\mathbf{u}$ and $\mathbf{u}$ be the displacement and velocity vectors of a particle $P$ of position vector $\mathbf{r}_0(x_0, y_0, z_0)$ with reference to the rotating axes. Also, let $u, v, w$ be the displacement of a particle in the $x, y, z$ directions, respectively.
With reference to Figure 7.1, the instantaneous position and velocity vectors of a particle are,

\[
\mathbf{r}_1 = \mathbf{r}_0 + \mathbf{u} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\]  

(7.01)

and

\[
\dot{\mathbf{r}}_1 = \dot{\mathbf{u}} + \dot{\mathbf{r}}_0 + \mathbf{a} - \mathbf{u} = \dot{\mathbf{u}} + \mathbf{a} - \mathbf{r}_1
\]

where

\[
\mathbf{a} = \dot{\mathbf{u}}.
\]

The Kinetic Energy \( T \) of the rotating system is given by,

\[
T = \frac{1}{2} \int \rho \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 \, d\Omega = \frac{1}{2} \int \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, d\Omega + \frac{1}{2} \int \rho \mathbf{a} \cdot \mathbf{a} - \mathbf{r}_1 \cdot \mathbf{a} - \mathbf{r}_1 \, d\Omega
\]

(7.02)

It should be noticed that,

\[
\mathbf{a} - \mathbf{r}_1 = \mathbf{a} \begin{bmatrix} 0 \\ -a_x \\ a_y \end{bmatrix} \text{ and } \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}
\]

(7.03)

where

\[
[A] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}
\]

and consequently the Kinetic energy becomes,

\[
T = \frac{1}{2} \int \rho \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}^T \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} d\Omega + \frac{1}{2} \beta^2 \int \rho \begin{bmatrix} x_0 + u \\ y_0 + v \\ z_0 + w \end{bmatrix}^T \begin{bmatrix} x_0 + u \\ y_0 + v \\ z_0 + w \end{bmatrix} d\Omega
\]

167.
Let the rotating system be represented by finite elements of volume $\Omega_e$, shape function matrix and degrees of freedom are $[N]$ and $\{r\}$, respectively. Also $\{q\}$ and $[N]$ represent the global degrees of freedom and corresponding shape function matrix. Let $\{\dot{q}\}$ and $\{\dot{r}\}$ be the velocity vector of $\{q\}$ and $\{r\}$, respectively. Thus, the kinetic energy of the discretized system becomes,

$$
T = \frac{1}{2} \{\dot{q}\}^t \int_{\Omega} \rho \left[ [N] \right]^t [N] \{q\} \ d\Omega + \frac{1}{2} \beta^2 \int_{\Omega} \left\{ \begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right\}^t [A]^t [A] \left\{ \begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right\} d\Omega \\
+ \beta^2 \{q\}^t \int_{\Omega} \rho \left[ [N] \right]^t [A] \left[ [N] \right] d\Omega \{q\} \\
+ \beta \{\dot{q}\}^t \int_{\Omega} \rho \left[ [N] \right]^t [A] \left[ [N] \right] d\Omega \{q\} \\
+ \beta \{\dot{\dot{q}}\}^t \int_{\Omega} \rho \left[ [N] \right]^t [A] d\Omega \{q\} \\
+ \beta \{\dot{q}\}^t \int_{\Omega} \rho \left[ [N] \right]^t [A] d\Omega \{q\}
$$

(7.05)

The rotating structure is subjected to a large steady state displacement induced by the angular velocity. Therefore, it cannot be analysed by the linear theory of elasticity. However, as a First Approximation, the rotating structure can be considered to have large steady state elastic
displacements with small deformations. Also, it can be assumed that the oscillatory displacements are small. With these assumptions, the stress and strain tensor can be referred to the original undeformed geometry. Thus a Lagrangian Formulation of the problem is assumed.

The potential energy $V$ of a large displacement, small deformation structural system idealized as an assembly of finite elements is derived in chapter 6 to be,

$$ V = \frac{1}{2}(q)^{t}[K](q) + \frac{1}{2}(q)^{t}[KG](q) \quad (7.06) $$

where $[K]$ and $[KG]$ are the Stiffness and Geometric Matrices of the system.

Let $\{q_0\}$ and $\{q_1\}$ be the steady state and oscillatory components of the displacement vector. Let $\{q_1\}$ be the velocity vector of $\{q_1\}$. Also let $\{Q_1\}$ be the force vector acting on coordinates $\{q_1\}$. Then,

$$ \{q\} = \{q_0\} + \{q_1\} \quad \dot{\{q\}} = \{q_1\} \quad \{q_0\} \gg \{q_1\} \quad (7.07) $$

It should be noticed that the Lagrangian $(L = T - V)$ is a function of $\{q_0\}$, $\{q_1\}$ and $\{q_1\}$. Also the Geometric Matrix is only a function of $\{q_0\}$.

Applying Lagrange equations to the kinetic and potential energy of the system and neglecting second order terms, the equations of motion of the discretized rotating system become,

$$ \begin{bmatrix} [K] + [KG] - [MG] \end{bmatrix} \{q_0\} = \{Q_1\} \quad (7.08) $$

and

$$ \begin{bmatrix} M \end{bmatrix} \ddot{\{q_1\}} + \begin{bmatrix} MG \end{bmatrix} \dot{\{q_1\}} + \begin{bmatrix} [K] + [KG] - [MC] \end{bmatrix} \{q_1\} = \{Q_1\} \quad (7.09) $$

where

$$ [MG] = 2\beta \int_\Omega \rho [N]^{t}[A][N] d\Omega \quad - \quad \text{Gyroscopic matrix} \quad (7.10) $$

$$ [MC] = \beta^2 \int_\Omega \rho [N]^{t}[A][N] d\Omega \quad - \quad \text{Centrifugal Mass matrix} \quad (7.11) $$

$$ [QC] = \beta^2 \int_\Omega \rho [N]^{t}[A][A] \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} d\Omega \quad - \quad \text{Centrifugal Force vector} \quad (7.12) $$

169.
and $[M]$, $[K]$ and $[KG]$ are the Mass, Stiffness and Geometric Matrices already defined in Chapters 2 and 6.

Equations (7.08) and (7.09) represent a Nonlinear Static System and a Linear Dynamic System, respectively. At low angular velocities, the Nonlinear Static System can be approximated by a linear static system, since the stiffness matrix is much bigger than the other two matrices. Thus, at low angular velocities, equation (7.08) becomes,

$$[K]\{q_0\} = \{QC\} \quad (7.13)$$

At high angular velocities, the Nonlinear Static System can be solved by an Iterative Method (57). Also, the matrices of the linear dynamic system should be evaluated with reference to the deformed geometry.

Equation (7.09) represents a conservative system, since the Gyroscopic Matrix is skew-symmetric and, therefore, the energy dissipated during motion is zero.

The solution of equation (7.09) is discussed by Simpson (146, 1973) and Gupta (147, 1973).

It should be noticed that the gyroscopic matrix never becomes predominant. This matrix is proportional to the angular velocity, while the other matrices are either independent of the angular velocity or proportional to the square of the angular velocity. Thus, in general, the Gyroscopic Matrix can be neglected.

At low angular velocities, the First Approximation of the equation of motion of discretized rotating systems becomes,

$$[K]\{q_0\} = \{QC\}$$

$$[M]\{\ddot{q}_1\} + ([K] + [KG] - [MC])\{q_1\} = \{Q_1\} \quad (7.14)$$

which are a Linear Static and Linear Dynamic System, respectively.
The Finite Element Model developed in this section is applicable to any rotating structure. For example, this Model can be applicable to the dynamic analysis of rotating bladed discs. It is interesting to note that Gupta (147, 1973) developed the eigenvalue procedure to solve equation (7.09) to predict the dynamic characteristics of the SKYLAB Space Station.

In the following section the Finite Element Model is further developed for a super-parametric shell element representation of a rotating shell structure.

7.3 Super-parametric Shell Element

Details of the super-parametric shell element are presented in Chapter 4. The Geometric Mass Matrix of the element is developed in Chapter 6. In this section, the Centrifugal Mass and Gyroscopic Matrices and the Centrifugal Force Vector for the super-parametric shell element are developed.

7.3.1 Centrifugal mass matrix

With reference to section 7.2, the Element Centrifugal Mass Matrix $[M_{CE}]$ is given by,

$$ [M_{CE}] = \beta^2 \int_{\Omega_e} \varepsilon [N]^t [A]^t [A] [N] \ d\Omega $$

(7.15)

The sub-matrix of the shape function matrix for the super-parametric shell element is defined in Chapter 4 to be,

$$ [N_j] = N_j \begin{bmatrix} 1 & 0 & 0 & t_j \xi [\phi_j] \\ 0 & 1 & 0 & \\ 0 & 0 & 1 \end{bmatrix} $$

(7.16)

Then it can be demonstrated that the sub-matrix $[M_{CE_{ij}}]$ of the Centrifugal Mass Matrix is given by,
\[
[M_{C_{Eij}}] = 2B^2 \left[ [A]^t [A] \right] \left[ \begin{array}{c} [A] \end{array} \right] \left[ \begin{array}{c} [0] \\
[0]^t \\
\frac{1}{12} t_i t_j \left[ \phi_i \right]^t [A]^t [A] \left[ \phi_j \right] \\
\end{array} \right] \left\{ \begin{array}{c} \rho N_i N_j |J| d\xi d\eta \\
\end{array} \right\} 
\]

(7.17)

This equation can be integrated numerically using a 3 \times 3 Gaussian mesh. It should be noticed that the similarity of this equation with the mass matrix can save computer time.

7.3.2 Gyroscopic matrix

With reference to section 7.2, the Element Gyroscopic Matrix [MGE] is,

\[
[MGE] = 2B \int_{\Omega_e} \rho [N]^t [A] [N] d\Omega 
\]

(7.18)

The sub-matrix \([MGE_{ij}]\) for the super-parametric shell element can be proved to be,

\[
[MGE_{ij}] = 4B \left[ [A] \right] \left[ \begin{array}{c} [A] \\
[0]^t \\
\frac{1}{12} t_i t_j \left[ \phi_i \right]^t [A]^t [A] \left[ \phi_j \right] \\
\end{array} \right] \left\{ \begin{array}{c} \rho N_i N_j |J| d\xi d\eta \\
\end{array} \right\} 
\]

(7.19)

This equation can be integrated numerically using a 3 \times 3 Gaussian mesh. Again, the similarity of this equation with the mass matrix can save computer time.

7.3.3 Centrifugal force vector

With reference to section 7.2, the Element Centrifugal Force Vector \([QCE]\) is given by,

\[
{QCE} = B^2 \int_{\Omega_e} \rho [N]^t [A]^t [A] \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} d\Omega 
\]

(7.20)

172.
For the super-parametric shell element, the sub-matrix \{QCE_j\} of the Centrifugal Force Vector can be proved to be,

\[
\{QCE_j\} = 2\beta^2 \int \int \rho N_j \left[ \begin{array}{c}
[A]^t [A] \\
1/2 [\phi_j]^t [A]^t [A] \\
\end{array} \right] \left[ \begin{array}{c}
\Sigma N_i x_i \\
\Sigma N_i y_i \\
\Sigma N_i z_i \\
\end{array} \right] |J|d\xi d\eta (7.21)
\]

where \(x_i, y_i, z_i\) and \(\delta x_i, \delta y_i, \delta z_i\) are defined in Chapter 4.

This equation can be integrated numerically using a 3 * 3 Gaussian mesh.

7.4 Applications

The computer program developed, based on equation (7.14) and the corresponding Element Matrices for the super-parametric shell element, is applied to the pre-twisted blades of Figure 6.9 and to the cylindrical shell blades of Figure 6.30.

In all the applications, the Reduced Integration Technique is used to evaluate the strain energy of the shells. Also the Eigenvalue Economizer is used to reduce the number of degrees of freedom of the system. The primary coordinates are all the transverse displacements. The Poisson ratio is 0.3. Unless otherwise mentioned, the steady state deformation is neglected.

The effect of the angular velocity, disc radius, pre-twist angle, setting angle, thickness ratio, aspect ratio and steady state deformation on the dynamic characteristics of the blades are investigated. The natural frequencies of the rotating, pre-twisted blade are compared with the results of Rawtani (133, 1970).
7.5 **General Discussion**

A Finite Element Model for the dynamic analysis of rotating structures is presented. The matrices derived are further developed for the super-parametric shell element.

In the derivation of the Model, it is assumed that the structure deformations are small, although the displacements are large. With this assumption, which is true at low angular velocities \( \alpha < 1.0 \), the stress and strain tensors are referred to a Lagrangian Formulation of the problem. At high angular velocities \( \alpha > 1.0 \), the deformations are large and, consequently, the stress and strain tensors must be referred to an Eulerian Formulation of the problem. With small oscillations, the Eulerian Formulation of the problem is only required to calculate the steady state displacement of the structure.

The Finite Element Model of a rotating structure is equivalent to a Non-linear Static System and a Linear Dynamic System. Only at low angular velocities is the Model equivalent to a Linear Static and a Linear Dynamic System.

The dynamic convergence properties of a super-parametric shell element representation of rotating, pre-twisted and cylindrical shell blades are presented in Figures 7.3, 7.4 and 7.16. A 3 * 3 finite element mesh representation, with 165 degrees of freedom and 33 primary coordinates, is capable of predicting accurately the first eight and six natural frequencies of the pre-twisted and cylindrical shell blades, respectively. The corresponding modes of vibration can also be predicted accurately and are presented in Figures 7.5 and 7.17.

The variation of the frequency parameter of a pre-twisted blade with angular velocity and the corresponding modes of vibration are presented in Figures 7.6 and 7.7. It is clear that the angular velocity significantly changes the frequency parameters and mode shapes of the bending modes of...
the blade. The frequency parameters of the higher bending and torsion modes and the corresponding nodal lines are insignificantly affected by the angular velocity. For example, in this particular blade, the frequency parameter of the first two bending modes of the blade rotating at the angular velocity \( \alpha = 1.0 \) are increased by 94.4 and 28.0% respectively. The corresponding increase in the frequency parameter of the torsion modes is only 4.1 and 2.8%.

The effect of the steady state deformation on the frequency parameters of a pre-twisted blade is also presented in Figure 7.6. The effect is to increase the frequency parameter of the bending modes and to decrease the frequency parameters of the torsion modes. Also, the change is more marked in the higher bending modes and lower torsion modes. At low angular velocities \((\alpha < 1.0)\), the effect of the steady state deformation on the frequency parameters can be neglected, without major errors; in this particular blade this error is less than 5.1%.

The variation of the frequency parameters of two pre-twisted blades with angular velocity is presented in Figure 7.8. The aspect ratio of these blades are \( a/b = 1.0 \) and \( a/b = 2.0 \). It is clear that the frequency parameter of the third mode of the wider blade \((a/b = 1.0)\) does not change significantly. However, the nodal lines are significantly changed. At low angular velocities, this mode is the second bending mode, while at high angular velocities it is a completely different bending mode. The corresponding change in the nodal lines of the fourth mode are exactly the opposite. Although the corresponding transformation of these two modes of vibration is not shown in Figure 7.8 to the longer blade \((a/b = 2.0)\), the same phenomenon occurs but at much higher angular velocities. This transformation of the vibration modes becomes more complex with large aspect ratio thin blades.

The variation of the frequency parameters of a rotating thin plate
(a/b = 1.0) is ideal to demonstrate the transformation of the nodal lines with angular velocity. The frequency parameters as functions of the angular velocity are presented in Figure 7.9 and the corresponding mode shapes in Figure 7.10. In this case, the coupling is between the third and fourth modes. This coupling occurs at low angular velocities, since the corresponding frequency parameters of the non-rotating blade are only 31.5% apart. Figure 7.10 shows that the transformation of the nodal lines is a continuous process, therefore only certain modes can be transformed into other modes. Figure 7.9 also shows that the frequency parameter of the third mode of the blade with high angular velocity is almost identical to the frequency parameter of the non-rotating fourth mode. Figure 7.10 shows that the corresponding mode shapes are also similar.

It can be concluded that the effect of the angular velocity on the natural frequency of certain modes can be predicted by the analysis of the non-rotating blade. The closeness of certain non-rotating frequency parameters determines the variation of the frequency parameter with angular velocity. Also, the rate of increase of the frequency parameters with angular velocity is dependent on the mode shapes at the particular angular velocity.

Figure 7.11 presents the variation of the frequency parameters of a rotating pre-twisted blade with the radius ratio. Only the frequency parameters of the lower bending modes are dependent on the radius ratio. In this particular blade a variation of the radius ratio from 0.0 to 2.0 increases the frequency parameters of the first two bending modes by 63.6 and 14.5%, respectively.

The frequency parameters of the lower modes of a rotating, pretwisted blade are slightly dependent on the setting angle, as shown in Figure 7.12. In this particular blade an increase of the setting angle from 0.0 to 90.0
decreases the frequency parameter of the first two modes by 11.1 and 7.0%, respectively. There is no apparent change in the value of the frequency parameter of the higher modes.

The variation of the frequency parameters of a rotating and non-rotating pre-twisted blade with pre-twist angle and thickness ratio are presented in Figure 7.13 and 7.14, respectively. It is clear that the effect of the pre-twist angle and thickness ratio on the frequency parameters of the rotating and non-rotating blades is almost identical. The difference between the absolute values of the frequency parameters of the rotating and non-rotating blades are caused by the angular velocity.

Figure 7.15 presents the variation of the frequency parameters of a rotating and non-rotating pre-twisted blade with aspect ratio. With the exception of the torsional modes of the low aspect ratio blades, this variation is almost identical. Also all the frequency parameters of the bending modes are increased with angular velocity, while the frequency parameters of the torsional modes of high aspect ratio blades are almost independent of angular velocity.

The influence of the angular velocity on the natural frequencies of the uniform and tapered cylindrical shells are presented in Figures 7.18 and 7.24, respectively. This relationship between natural frequencies and angular velocity is much more complex than with the pre-twisted blades. This is a consequence of the closeness of the natural frequencies of the non-rotating cylindrical shells, which permits the transformation of several modes, even at low angular velocities.

The complexity of the transformation of the modes of vibration with angular velocity are presented in Figures 7.19, 7.20, 7.25 and 7.26. For example, the variation of the natural frequency of the fifth mode is very complex, as a consequence of the significant change in the nodal lines. At certain values of the angular velocity, these nodal lines are similar to
the fourth or sixth mode. Thus the gradient of the curve representing the fifth mode takes the same gradient as the fourth and sixth mode curves and vice versa, thus giving a very complex curve to represent the variation of the natural frequencies with angular velocity.

It can be concluded that the angular velocity increases the natural frequencies of all modes. The rate of increase is dependent on each mode shape at the particular angular velocity. The rate of increase of the natural frequency of the torsional bending modes is small compared with the other modes. Although the relationship between the natural frequencies of the cylindrical shells and angular velocity is very complex, it is still possible to conclude the basic shape of the curve by an analysis of the non-rotating system.

Figure 7.18 also shows that a change in the radius of the disc does not change the basic geometry of the curve representing the natural frequencies as functions of the angular velocity.

The influence of the steady state deformation on the natural frequencies of the rotating, uniform cylindrical shell is presented in Figure 7.21. The effect is to increase the natural frequencies of the torsional bending modes and to decrease the natural frequencies of the other modes. With some torsional bending modes, the steady state deformation increases the natural frequencies at low angular velocities and decreases the natural frequencies at high angular velocities. The opposite also occurs in other modes. However, at low angular velocities ($\alpha < 1.0$) the effect of the steady state deformation can be neglected without major errors.

The variation of the natural frequencies of the rotating, cylindrical shell with radius ratio is presented in Figure 7.22. An increase in the radius of the disc also increases the natural frequencies of all modes. However, the influence of the radius of disc on the natural frequencies of the torsional bending and of the higher modes can be neglected without
any major error.

Figure 7.23 shows the variation of the natural frequencies of the rotating cylindrical shell with setting angle. Only the natural frequencies of the lower modes are affected by a change in the setting angle.
FIG. 7.1. ROTATING STRUCTURE.

$\Omega$ = Domain
$\beta$ = Angular velocity

- Undefomed structure.
- Deformed structure.
FIG. 7.2. ROTATING BLADE DIVIDED INTO 3 x 2 MESH OF FINITE ELEMENTS.

R = radius of disc
θ = setting angle
y and y', z and z' are parallel axis.
FIG. 7.3. CONVERGENCE CURVE FOR THE FINITE ELEMENT MODEL OF THE ROTATING PRE-TWISTED BLADE-MODE 1 to 5.
FIG. 7.4. CONVERGENCE CURVE OF FINITE ELEMENT MODEL OF PRE-TWISTED BLADE MODES 6 to 9.

Frequency parameter \( \nu = \frac{\rho L^4}{E I_2/(1 - \nu^2)} \)

-183 -
FIG. 7.5. VIBRATION MODES OF ROTATING PRE-TWISTED BLADE (α/b = 2.0, b/h = 16.0, θ = 60.0, ν = 30.0, R = 2.0, θ = 1.0).
FIG. 7.6. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH ANGULAR VELOCITY. (a/b = 2.0, b/h = 16.0, \( \theta = 60.0 \), \( \phi = 30.0 \), \( \bar{R} = 2.0 \)).
Insignificant change in others vibration modes.

**FIG. 7.7. INFLUENCE OF THE ANGULAR VELOCITY ON THE VIBRATION MODES OF PRE-TWISTED BLADE.** (α/b = 2.0, b/h = 16.0, γ = 30.0, θ = 60.0, R = 2.0).
Finite element model (3 x 3, 165, 33)

FIG. 7.8. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH ANGULAR VELOCITY RATIO. ($\theta = 60.0$, $\phi = 30.0$, $b/h = 16.0$, $\bar{R} = 2.0$).
FIG. 7.9. VARIATION OF FREQUENCY PARAMETERS OF PRE-TWISTED BLADE WITH ANGULAR VELOCITY RATIO. \((a/b = 1.0, b/h = 100.0, \theta = 90.0, \gamma = 0.0, \bar{R} = 1.0)\)
FIG. 7.10. VARIATION OF THE VIBRATION MODES OF A PRE-TWISTED BLADE WITH ANGULAR VELOCITY. (α/β = 1.0, R = 1.0, b/h = 1.00.0, ψ = 0.0, θ = 90.0).
FIG. 7.11. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH RADIUS RATIO. \((a/b = 2.0, b/h = 16.0, \theta = 60.0, \psi = 30.0)\)
FIG. 7.12 VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH SETTING ANGLE. \((\alpha/b = 2.0, b/h = 16.0, \gamma' = 30.0, \bar{R} = 2.0)\)
FIG. 7.13. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH PRETWIST ANGLE. ($a/b = 2.0$, $b/h = 16.0$, $\theta = 60.0$, $\bar{R} = 2.0$)
FIG. 7.14. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH THICKNESS RATIO. (a/b = 2.0, \( \theta = 60.0 \), \( \psi = 30.0 \), \( \bar{R} = 2.0 \))
FIG. 7.15. VARIATION OF FREQUENCY PARAMETER OF PRE-TWISTED BLADE WITH ASPECT RATIO (b/h = 15.0, 0.0 - 60.0, $\phi = 30.0$, $R = 2.0$)
FIG. 7.16. CONVERGENCE CURVE FOR THE FINITE ELEMENT MODEL OF THE ROTATING UNIFORM AND TAPERED CYLINDRICAL SHELL BLADES. ($\theta = 90.0$, $R = 1.0$).
FIG. 7.17. VIBRATION Modes OF UNIFORM CYLINDRICAL SHELL BLADE
(α= 1.0, θ = 90.0, R = 1.0).
FIG. 7.18. VARIATION OF THE NATURAL FREQUENCIES OF THE UNIFORM CYLINDRICAL SHELL BLADE WITH ANGULAR VELOCITY.
FIG. 7.19. VARIATION OF THE VIBRATION MODES OF THE UNIFORM CYLINDRICAL SHELL BLADE WITH ANGULAR VELOCITY. FIRST AND THIRD MODE.

Finite element model (3 x 3, 165,33)
FIG. 7.20. VARIATION OF THE VIBRATION MODES OF THE UNIFORM CYLINDRICAL SHELL BLADE WITH ANGULAR VELOCITY. FOURTH AND SIXTH MODE.
FIG. 7.21. INFLUENCE OF THE STEADY STATE DEFORMATION ON THE NATURAL FREQUENCIES OF THE UNIFORM CYLINDRICAL SHELL BLADE.
FIG. 7.22. VARIATION OF THE NATURAL FREQUENCIES OF THE ROTATING, UNIFORM CYLINDRICAL SHELL BLADE WITH RADIUS RATIO.
FIG. 7.23. VARIATION OF THE NATURAL FREQUENCIES OF ROTATING UNIFORM CYLINDRICAL SHELL BLADE WITH SETTING ANGLE.

Finite element model

(3 * 3, 165, 33)

$\alpha = 1.0, \bar{R} = 1.0$
FIG. 7.24. VARIATION OF THE NATURAL FREQUENCIES OF THE ROTATING TAPERED CYLINDRICAL SHELL BLADE WITH ANGULAR VELOCITY.
FIG. 7.25. VARIATION OF THE VIBRATION MODES OF THE TAPERED CYLINDRICAL SHELL BLADE WITH ANGULAR VELOCITY. ($\alpha = 0.0$ to $\alpha = 1.0$)
FIG. 7.26 VARIATION OF THE VIBRATION MODES OF THE TAPERED CYLINDRICAL SHELL BLADE WITH ANGULAR VELOCITY. ($\alpha$ = 1.1 to 2.0)
8.1 Introduction

Although the Finite Element Method has been extensively applied to solid mechanics problems, it has a very short history in fluid mechanics applications. At first, the Finite Element Method was associated with the variational statement of the problem. Thus, it is natural that potential flow problems, which are governed by the Laplace equation, were the first to be solved by finite elements. The application of the Finite Element Method to the solution of field problems, including Laplace and Poisson equations, was first proposed by Zienkiewicz and Cheung (148, 1965).

Martin (149, 1969) developed a triangular fluid element and applied it to two dimensional ideal flow problems. Other triangular elements have also been developed by Doctors (150, 1970), Chan (151, 1971), Aral (152, 1971) and Norrie and De Vries (153, 1972) and applied to the same problem.

A triangular fluid element has been developed and applied to two-dimensional viscous fluid flow problems by Oden and Somogyi (154, 1969). The same problem has also been solved and other fluid elements developed by Tong (155, 1971), Cheng (156, 1972) and Olson (157, 1973).

Extensive literature surveys on the development and application of the Finite Element Method to fluid mechanics problems have been presented by Oden (158, 1973) and Norrie and De Vries (159, 1973).

The application of the Finite Element Method to the dynamic analysis of structures submerged in a fluid medium was first proposed by Zienkiewicz, Irons and Nath (160, 1966) and Zienkiewicz (161, 1967). Later, Holland (162, 1969) developed several rectangular and triangular fluid elements to calculate the hydrodynamic mass of structures submerged in a fluid medium.
Zienkiewicz and Newton (19, 1969) developed the theory of the application of the Finite Element Method to the dynamic analysis of structures submerged in an incompressible or compressible fluid. The structure was represented by beam elements and the fluid domain by rectangular elements. Using a different approach, Matsumoto (163, 1972) also developed the theory of the problem.

The Finite Element Method has been applied to the dynamic analysis of plates submerged in an incompressible fluid by Chowdhury (164, 1972) and Selby and Severn (165, 1972). The fluid domain was represented by hexahedron elements.

Newton, Chenault and Smith (166, 1974) used finite elements to predict the dynamic characteristics of partially submerged cylinders.

In this chapter, a Finite Element Model is developed for a shell structure submerged in an incompressible or a compressible ideal fluid. The structure is represented by super-parametric shell elements and the fluid domain by three-dimensional isoparametric elements. The model is applied to the dynamic analysis of plate structures submerged in water. The computed natural frequencies are compared with the experimental values obtained by Lindholm, Kana, Chu and Abramson (167, 1965).

In all the applications, the plate is represented by super-parametric shell elements with 80 structural degrees of freedom. The Eigenvalue Economizer is used and the transverse displacements are the primary coordinates. Generally, the fluid domain is represented by 80 isoparametric fluid elements with 177 fluid degrees of freedom. Figure 8.1 shows the isoparametric fluid element and Figure 8.2 shows the representation of the water by fluid elements. The Reduced Integration Technique is used to evaluate the strain energy of the shell structure.

As a consequence of the storage available in the computers used (ICL 1905 and 1906), it was not possible to predict the dynamic characteristics of submerged shell structures has just been presented by Blaker (202, 1975).
of shell structures submerged in a fluid medium.

8.2 Finite Element Method in Fluid Mechanics

The basic equations of fluid mechanics are the Continuity, Momentum and Energy Balance equations. In the case of an isentropic flow, the Energy equation may be discarded and the equations describing the flow become,

\[ \frac{dp}{dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad \text{(Continuity Equation)} \quad (8.01) \]

and

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \quad \text{(Momentum Equation)} \quad (8.02) \]

where \( \rho \) is the density of the fluid, \( p \) is the pressure of the fluid, \( \mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k} \) is the velocity vector of a fluid particle with reference to an \( x, y, z \) Cartesian system and \( \nabla = \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \) is the Gradient Operator.

Noting that the Bulk Modulus of a fluid is defined by,

\[ B = \rho \frac{dp}{dp} \quad (8.03) \]

then, the Continuity equation becomes,

\[ \nabla \cdot \mathbf{u} + \frac{1}{B} \frac{\partial p}{\partial t} = 0. \quad (8.04) \]

Consequently, the equation of motion of the fluid becomes,

\[ \nabla^2 p - \frac{\partial^2 p}{\partial t^2} = 0 \quad (8.05) \]

where \( \nabla^2 = \nabla \cdot \nabla \) is the Laplacian Operator and \( c = \sqrt{B/\rho} \) is the fluid velocity of sound.

For an incompressible fluid, the equation of motion becomes the Laplace equation,

\[ \nabla^2 p = 0. \quad (8.06) \]

The Boundary Conditions of equation (8.05) are given by,

\[ \nabla p = -\rho \mathbf{u} \frac{\partial \mathbf{u}}{\partial t} \quad \text{or} \quad \frac{\partial p}{\partial n} = -\rho \mathbf{u} \cdot \mathbf{n} \frac{\partial \mathbf{u}}{\partial t} \quad (8.07) \]

208.
where \( \mathbf{n} \) is the unit vector normal to the boundary and \( u_n \) is the fluid velocity in the \( \mathbf{n} \) direction. Equations (8.05) and (8.07) define the pressure of a compressible non-viscous fluid.

The boundary condition equation (8.07) will be extended to the particular cases of fixed, moving and free surfaces. At a fixed surface, the velocity component normal to the boundary must be zero,

\[
\mathbf{n} \cdot \mathbf{u} = 0.
\]  
(8.08)

Multiplying equation (8.07) by \( \mathbf{n} \) and imposing the constraint defined by equation (8.08), then the boundary conditions at a fixed surface become,

\[
\mathbf{n} \cdot \mathbf{v}_p = 0 \quad \text{or} \quad \frac{\partial p}{\partial \mathbf{n}} = 0
\]  
(8.09)

At a moving boundary, where the surface point has a velocity \( \mathbf{v} \), the fluid velocity component normal to the surface must equal the boundary velocity normal to the surface,

\[
\mathbf{n} \cdot (\mathbf{u} - \mathbf{v}) = 0.
\]  
(8.10)

In this case, the boundary conditions become,

\[
\mathbf{n} \cdot \mathbf{v}_p = -\rho \mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial t} \quad \text{or} \quad \frac{\partial p}{\partial \mathbf{n}} = -\rho \frac{\partial \mathbf{v}_n}{\partial t}
\]  
(8.11)

where \( \mathbf{v}_n \) is the surface velocity in the \( \mathbf{n} \) direction.

At a free surface, in the absence of surface waves, the boundary conditions become,

\[
p = 0
\]  
(8.12)

This boundary condition is generally adequate. However, if surface waves are generated, then it is important to take the gravity into consideration. In this case the boundary condition at the free surface becomes,

\[
\frac{\partial^2 p}{\partial t^2} + g \frac{\partial p}{\partial \mathbf{n}} = 0
\]  
(8.13)

where \( g \) is the gravity acceleration.

Consider a fluid domain represented by finite elements. Vector \( \{\mathbf{p}\} \)
represents the unknown pressures within the fluid domain. Also \( \Omega_e \) and \( S_e \) are the volume and surface of a fluid finite element. Within each element, the pressure is defined by,

\[
p = [N^*]\{f\} = \sum_{j=1}^{m} N_j^* f_j
\]  

(8.14)

where \( j = 1, 2, ..., m \)

\( m \) = number of degrees of freedom of fluid element

\([N^*]\) is the Shape Function Matrix

\( \{f\} \) is the vector of unknown pressures.

With reference to Chapter 2, a Galerkin Finite Element Model of equation (8.05) is given by,

\[
\int_{\Omega_e} \nabla N_i^* \cdot \nabla N_j^* \, d\Omega f_j - \frac{1}{c^2} \int_{\Omega_e} N_i^* N_j^* \, d\Omega \frac{d^2 f_j}{dt^2} = \int_{S_e} N_i^* \frac{\partial p}{\partial n} \, dS
\]  

(8.15)

where \( \frac{d^2 f_j}{dt^2} \) = \( \dot{f}_j \) \( i, j = 1, 2, ..., m \).

or

\[
[HE]\{f\} + [AE]\{\dot{f}\} = \{BE\}
\]  

(8.16)

where

\[
HE_{ij} = \int_{\Omega_e} \nabla N_i^* \cdot \nabla N_j^* \, d\Omega \quad - \text{Element Fluid Stiffness Matrix}
\]  

(8.17)

\[
AE_{ij} = \frac{1}{c^2} \int_{\Omega_e} N_i^* N_j^* \, d\Omega \quad - \text{Element Compressibility Matrix}
\]  

(8.18)

and

\[
BE_j = \int_{S_e} N_j^* \frac{\partial p}{\partial n} \, dS \quad - \text{Element Fluid Vector}
\]  

(8.19)

\( j = 1, 2, ..., m \)
After assembly of all fluid finite elements and imposing the boundary conditions, then equation (8.16) becomes,

\[ [A] \ddot{p} + [H](p) = \{B\} \]  \hspace{1cm} (8.20)

This equation is similar to the structural dynamic equation; thus matrix \([A]\) is equivalent to the mass matrix, while matrix \([H]\) corresponds to the stiffness matrix and \(\{B\}\) is equivalent to the applied force vector.

8.3 Finite Element Dynamic Analysis of Submerged Shells

Consider a structure defined by the displacement vector \(\{q\}\), submerged in a fluid defined by the pressure vector \(\{p\}\). The structure is represented by finite elements, whose shape function matrix is \([N]\) and displacement vector \(\{r\}\). The fluid domain is represented by finite elements, whose shape function matrix is \([N^*]\) and pressure vector \(\{f\}\).

The equation of motion of the structure is given by,

\[ [M] \ddot{q} + [K] q = \{Q\} \]  \hspace{1cm} (8.21)

where \([M]\) and \([K]\) are the Mass and Stiffness Matrices and \(\{Q\}\) is the Force Vector due to the fluid pressure.

With reference to Chapter 6, the Element Force Vector is defined by,

\[ \{Q_E\} = \int_{S_e} [N]^t p \, dS \]  \hspace{1cm} (8.22)

where \(dS\) is a vector defining the area \(dS = n dS\),

\(n\) is the vector normal to the boundary and \(dS\) is an infinitesimal area;

or

\[ \{Q_E\} = [L_E]\{f\} \]  \hspace{1cm} (8.23)
where

\[ \begin{bmatrix} \text{LE} \end{bmatrix} = \int_{S_e} [N]^t [N^*] d\mathbf{s} = \text{Element fluid/structure matrix} \quad (8.24) \]

After assembly of all the structural finite elements, the structure equation of motion becomes,

\[ [M]\{\ddot{q}\} + [K]\{q\} = [L]\{p\} \quad (8.25) \]

At the interface, the structure behaves as a moving boundary. This boundary condition is defined by equation (8.11). Consequently, the Element Fluid Vector becomes,

\[ \{BE\} = -\rho [LE]^t \{\mathbf{r}\} \quad (8.26) \]

Thus, the pressure equation becomes,

\[ [A]\{\ddot{p}\} + [H]\{p\} = -\rho [L]^t \{q\}. \quad (8.28) \]

For Incompressible Fluids this equation becomes,

\[ \{p\} = -\rho [H]^{-1} [L]^t \{\mathbf{q}\}, \quad (8.29) \]

and, consequently, the equation of motion of a submerged structure becomes,

\[ [[M] + [MA]]\{\ddot{q}\} + [K]\{q\} = \{0\} \quad (8.30) \]

where

\[ [MA] = \rho [L][H]^{-1}[L]^t \quad (8.31) \]

is the Added Mass Matrix.

The equation of motion of a structure submerged in a compressible fluid is given by equations (8.25) and (8.28). In matrix form these equations become,
Although all the system matrices are symmetric, equation (8.32) leads to an unsymmetric eigenvalue problem. However, this equation can be transformed into a symmetric matrix, as follows:

\[
\begin{bmatrix}
[M] + [MA] \\
\rho [L]^t [A] \\
\rho [A] [H]^{-1} [L]^t \\
\end{bmatrix}
\begin{bmatrix}
\{q\} \\
\{\dot{p}\} \\
\{\ddot{p}\} \\
\end{bmatrix}
+ \begin{bmatrix}
[K] \\
[0] \\
[0] \\
\end{bmatrix}
\begin{bmatrix}
\{q\} \\
\{p\} \\
\{\ddot{p}\} \\
\end{bmatrix}
= \begin{bmatrix}
\{0\} \\
\{0\} \\
\{0\} \\
\end{bmatrix}
\] (8.33)

This equation is symmetric, therefore the eigenvalue problem is of the standard form.

8.4 Isoparametric Fluid and Super-parametric Shell Elements

Consider the three-dimensional isoparametric element, shown in Figure 8.1 with 20 nodes and 20 degrees of freedom. The element is referred to a set of local non-dimensional curvilinear coordinates \(\xi, \eta, \zeta\) and to a set of global Cartesian coordinates \(x, y, z\).

The Cartesian coordinates are related to the curvilinear coordinates by,

\[
x = [N^*]\{x_i\}; \quad y = [N^*]\{y_i\}; \quad z = [N^*]\{z_i\},
\] (8.34)

where \(\{x_i\}\) are the \(x\) coordinates of the nodal points and \([N^*(\xi, \eta, \zeta)]\) is the Shape Function Matrix for the isoparametric element.

The shape functions for the corner nodes are,

\[
N_i^* = \frac{1}{2}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta)(\xi \xi_i + \eta \eta_i + \zeta \zeta_i - 2)
\] (8.35)

and for a typical mid-side node (\(\xi_i = 0\)) are,

\[
N_i^* = \frac{1}{2}(1 - \xi^2)(1 + \eta_i \eta)(1 + \zeta_i \zeta)
\] (8.36)

where \(\xi_i\) are the value of \(\xi\) at the nodal points.
Similarly the pressure \( p \) is described as,

\[
p = [N^*] \{f\} \quad \text{or} \quad p = \sum_{j=1}^{20} N_j f_j \tag{8.37}
\]

where \( \{f\} \) is the element pressure vector.

### 8.4.1 Fluid stiffness matrix

The coefficients of the Element Fluid Stiffness Matrix are presented in section (8.2) to be,

\[
\begin{align*}
\mathbf{H}_{ij} &= \int \left( \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}
\end{array} \right) \left( \begin{array}{c}
\frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial y} \\
\frac{\partial N_j}{\partial z}
\end{array} \right) \, d\Omega \\
&= \int \left( \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}
\end{array} \right)^T \left[ \mathbf{J} \right]^{-1} \left[ \mathbf{J} \right]^{-1} \left( \begin{array}{c}
\frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial y} \\
\frac{\partial N_j}{\partial z}
\end{array} \right) \, |\mathbf{J}| \, d\xi d\eta d\zeta \\
&= \left( \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}
\end{array} \right)^T \left( \begin{array}{c}
\frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial y} \\
\frac{\partial N_j}{\partial z}
\end{array} \right) \, |\mathbf{J}| \, d\xi d\eta d\zeta 
\end{align*}
\tag{8.38}
\]

With reference to Chapter 4, this equation is transformed to the curvilinear coordinates and becomes,

\[
\begin{align*}
\mathbf{H}_{ij} &= \int \left( \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}
\end{array} \right) \left( \begin{array}{c}
\frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial y} \\
\frac{\partial N_j}{\partial z}
\end{array} \right) \, d\Omega \\
&= \int \left( \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}
\end{array} \right)^T \left[ \mathbf{J} \right]^{-1} \left[ \mathbf{J} \right]^{-1} \left( \begin{array}{c}
\frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial y} \\
\frac{\partial N_j}{\partial z}
\end{array} \right) \, |\mathbf{J}| \, d\xi d\eta d\zeta \\
&= \left( \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}
\end{array} \right)^T \left( \begin{array}{c}
\frac{\partial N_j}{\partial x} \\
\frac{\partial N_j}{\partial y} \\
\frac{\partial N_j}{\partial z}
\end{array} \right) \, |\mathbf{J}| \, d\xi d\eta d\zeta 
\end{align*}
\tag{8.39}
\]

where \( |\mathbf{J}| \) is the Jacobian Matrix relating the Cartesian and curvilinear coordinates.

This equation can be integrated numerically using a 3 * 3 * 3 Gaussian mesh.

### 8.4.2 Fluid compressibility matrix

The coefficients of the Fluid Compressibility Matrix are presented in section 8.2. After transformation into the curvilinear coordinates, the Element Fluid Compressibility Matrix becomes,

\[
\mathbf{[AE]} = \frac{1}{c^2} \int \int \int \left( \begin{array}{c}
[N^*]^T [N^*] \left| \mathbf{J} \right| \, d\xi d\eta d\zeta
\end{array} \right) \\
\tag{8.40}
\]

This equation is integrated numerically using a 3 * 3 * 3 Gaussian mesh.

214.
8.4.3 Fluid/structure matrix

The Fluid/Structure Matrix relates the fluid finite element with the structure finite element. This matrix is given by,

$$[LE] = \int_{Se} [N]^t [N^*] dS$$  \hspace{1cm} (8.41)

With a super-parametric shell element representation, the fluid pressure is mainly acting on the surface defined by $\zeta = 1$ or $\zeta = -1$. In this case,

$$dS = \{J1\}d\xi d\eta$$

where

$$\{J1\} = \begin{bmatrix} \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \eta} & -\frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} & \frac{\partial x}{\partial \eta} & -\frac{\partial z}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \zeta} \end{bmatrix}$$  \hspace{1cm} (8.42)

With reference to Chapter 4, the shape function sub-matrix for the super-parametric shell element is given by,

$$[N_j] = N_j \begin{bmatrix} 1 & 0 & 0 & t_j \zeta [\phi_j] \\ 0 & 1 & 0 & \end{bmatrix}$$  \hspace{1cm} (8.43)

Consequently, the sub-vector of the Fluid/Structure Matrix becomes,

$$\{LE_{ij}\} = \begin{bmatrix} 1 & 0 & 0 & t_j \zeta [\phi_j] \\ 0 & 1 & 0 & \end{bmatrix} \begin{bmatrix} 1 & \end{bmatrix} N_i^* N_j \{J1\} d\xi d\eta$$  \hspace{1cm} (8.44)

This equation can be numerically integrated using a 3 * 3 Gaussian mesh.

Similarly, if the pressure is acting on other surfaces, corresponding Fluid/Structure Matrices can be developed.
8.4.4 Boundary conditions

There is no difficulty in imposing the boundary conditions of a free surface. However, the boundary conditions of a fixed surface cannot be easily satisfied.

Let the pressure vector representing the Finite Element Model of a fluid domain be defined as,

\[ \{p\} = \begin{cases} \{p_b\} \\ \{p_a\} \end{cases} \tag{8.45} \]

where \( \{p_b\} \) = sub-vector representing the nodal pressures subjected to fixed surface boundary conditions

\( \{p_a\} \) = sub-vector representing all other nodal pressures.

The boundary conditions of a fixed surface is given by equation (8.09).

Then,

\[ \frac{\partial p}{\partial n} = \left( \begin{array}{c} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{array} \right) \left( \begin{array}{c} \frac{\partial x}{\partial n} \\ \frac{\partial y}{\partial n} \\ \frac{\partial z}{\partial n} \end{array} \right) = 0 \tag{8.46} \]

At a nodal point \( s \) of coordinates \( \xi_s, \eta_s, \zeta_s \) or \( x_s, y_s, z_s \), this equation becomes,

\[ \frac{\partial p_s}{\partial n} = \{E_s\}^t \{p\} = 0 \tag{8.47} \]

where

\[ \{E_s\}^t = \begin{bmatrix} \frac{\partial x_s}{\partial n} \\ \frac{\partial y_s}{\partial n} \\ \frac{\partial z_s}{\partial n} \end{bmatrix}^t \left[ J \right]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \left[ N^*(\xi_s, \eta_s, \zeta_s) \right] \\ \frac{\partial}{\partial \eta} \left[ N^*(\xi_s, \eta_s, \zeta_s) \right] \\ \frac{\partial}{\partial \zeta} \left[ N^*(\xi_s, \eta_s, \zeta_s) \right] \end{bmatrix} \]

Imposing the boundary conditions at the \( b \) nodal points, then equation (8.47) becomes,

\[ \frac{\partial}{\partial n} \{p_b\} = [E]\{p\} = 0 \tag{8.48} \]
where

\[
[E] = \begin{bmatrix}
{E_1}^t \\
\vdots \\
{E_s}^t \\
{E_b}^t
\end{bmatrix}
\]

It is convenient to express this equation as,

\[
[[EB] \; [EA]] \begin{bmatrix} \{p_b\} \\ \{p_a\} \end{bmatrix} = \{0\} \tag{8.49}
\]

Consequently,

\[
\{p\} = [T_b] \{p_a\} \tag{8.50}
\]

where

\[
[T_b] = \begin{bmatrix}
-[EB]^{-1}[EA] \\
[I]
\end{bmatrix}
\]

\([I] = \text{Unit Matrix.}
\]

After imposing the boundary conditions, the Fluid Stiffness Matrix becomes,

\[
[H_{B}] = [T_b]^t[H][T_b]. \tag{8.51}
\]

\section{8.5 General Discussion}

A Finite Element Model for the dynamic analysis of shell structures submerged in an incompressible or compressible fluid is developed. The Model is based on a super-parametric shell element representation of the structure and on an isoparametric element representation of the fluid.

The computed dynamic characteristics of submerged structures are dependent on the number of structural and fluid elements. An increase in the number of structural degrees of freedom demands a significant increase in the number of fluid degrees of freedom, therefore it is necessary to have the minimum possible structural degrees of freedom. However, the number of structural degrees of freedom must be capable of predicting the lower natural frequencies and mode shapes of the structure in the vacuum.

217.
Also, the distribution of the fluid degrees of freedom was found to be very important. The majority of fluid elements should be near the structure/fluid boundaries. In all the applications the discretized fluid domain is as shown in Figure 8.2.

The convergence properties of the Finite Element Model is presented in Table 8.1. It is seen that the natural frequency converges slowly with the number of degrees of freedom. This can be explained, since it was not possible to have more fluid elements in the \( y \) direction and near to the structure. For this reason, the higher natural frequencies could not be predicted accurately. However, the computed fundamental natural frequency is only 1.7% in error. Chowdhury (164, 1972) with almost the same number of fluid and structural degrees of freedom, obtained the fundamental natural frequency with a 5.6% error.

The accuracy of the Finite Element Model in predicting the dynamic characteristics of submerged structures is clearly seen with reference to Table 8.2. In all the applications the computed fundamental natural frequency is less than 6.8% in error. The average error, which is more representative of the accuracy, is only 2.0%.

The effect on the fundamental frequency parameter of a submerged plate with aspect ratio is presented in Figure 8.3. The results are compared with the corresponding values of the plate in vacuum. It can be seen that the aspect ratio has a significant influence on the fundamental frequency parameter of the submerged plate. For example, the decrease in the fundamental frequency of submerged plates (\( h/b = 76.5 \)) with aspect ratio varying from \( a/b = 0.5 \) to 5.0 is 41.0%. The variation of the fundamental frequency of the same plates in vacuum is only 3.1%.

Also, contrary to the theory of plates in vacuum, the effect of the variation of the thickness ratio on the fundamental frequency parameters of submerged plates is significant, as shown in Figure 8.4. For example, the
decrease in the fundamental frequency parameter of a submerged plate \((a/b = 2.0)\) with thickness ratio varying from \(b/h = 10.0\) to \(100.0\) is 124%.

The height of fluid above the structure has a significant effect on the natural frequencies of submerged structures, as shown in Figure 8.5. However, for deeply submerged shells \((d/a > 1)\), the natural frequencies are independent of the height of fluid above the structure. This is the main reason for representing the fluid domain as shown in Figure 8.2.

The natural frequencies of a submerged structure are almost independent of the depth of fluid below the structure, as shown in Figure 8.6.
TABLE 8.1. Variation of the fundamental natural frequency of a submerged plate with degrees of freedom of the fluid domain. 
(a/b = 1.0, h/b = 0.0131, d/a = 1.0)

<table>
<thead>
<tr>
<th>Number of structural elements</th>
<th>Number of fluid elements</th>
<th>Number of structural degrees of freedom</th>
<th>Number of fluid degrees of freedom</th>
<th>Fundamental natural frequency (Hz)</th>
<th>Percentage Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>48</td>
<td>80</td>
<td>93</td>
<td>29.1</td>
<td>24.9</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
<td>80</td>
<td>122</td>
<td>27.9</td>
<td>19.7</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>80</td>
<td>177</td>
<td>23.7</td>
<td>1.7</td>
</tr>
</tbody>
</table>

Finite element solution by Choudhury (164, 1972)

|                      |                           |                                        | 198                                | 24.6                              | 5.6             |

Experimental value by Lindholm, Kana, Chu and Abranson (167, 1965)
### TABLE 8.2. Comparison of experimental and computed fundamental natural frequency of submerged plates. (In all applications d/b = 1.0 and b = 0.203m)

<table>
<thead>
<tr>
<th>Plate Number</th>
<th>a/b</th>
<th>h/b x 10²</th>
<th>Exact vacuum</th>
<th>Computed water</th>
<th>Experimental water</th>
<th>Difference</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0</td>
<td>12.40</td>
<td>20.9</td>
<td>15.6</td>
<td>14.6</td>
<td>6.8</td>
<td>40.0</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>6.11</td>
<td>65.7</td>
<td>42.3</td>
<td>40.3</td>
<td>4.9</td>
<td>55.3</td>
</tr>
<tr>
<td>3</td>
<td>3.0</td>
<td>6.11</td>
<td>29.1</td>
<td>18.1</td>
<td>17.8</td>
<td>1.7</td>
<td>60.8</td>
</tr>
<tr>
<td>4</td>
<td>5.0</td>
<td>6.11</td>
<td>10.4</td>
<td>6.4</td>
<td>6.3</td>
<td>1.6</td>
<td>62.5</td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
<td>2.38</td>
<td>99.5</td>
<td>54.3</td>
<td>51.4</td>
<td>5.6</td>
<td>83.2</td>
</tr>
<tr>
<td>6</td>
<td>2.0</td>
<td>2.38</td>
<td>24.7</td>
<td>12.4</td>
<td>12.1</td>
<td>2.5</td>
<td>99.2</td>
</tr>
<tr>
<td>7</td>
<td>3.0</td>
<td>2.38</td>
<td>10.9</td>
<td>5.2</td>
<td>5.1</td>
<td>2.0</td>
<td>109.6</td>
</tr>
<tr>
<td>8</td>
<td>5.0</td>
<td>2.38</td>
<td>3.9</td>
<td>1.8</td>
<td>1.8</td>
<td>0.0</td>
<td>116.7</td>
</tr>
<tr>
<td>9</td>
<td>0.5</td>
<td>1.31</td>
<td>223.0</td>
<td>108.2</td>
<td>106.0</td>
<td>2.1</td>
<td>106.1</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>1.31</td>
<td>55.6</td>
<td>23.7</td>
<td>23.3</td>
<td>1.7</td>
<td>134.6</td>
</tr>
<tr>
<td>11</td>
<td>2.0</td>
<td>1.31</td>
<td>13.8</td>
<td>5.3</td>
<td>5.1</td>
<td>3.9</td>
<td>160.4</td>
</tr>
<tr>
<td>12</td>
<td>3.0</td>
<td>1.31</td>
<td>6.1</td>
<td>2.3</td>
<td>2.3</td>
<td>0.0</td>
<td>165.2</td>
</tr>
<tr>
<td>13</td>
<td>0.5</td>
<td>0.90</td>
<td>159.0</td>
<td>64.1</td>
<td>63.5</td>
<td>0.9</td>
<td>148.0</td>
</tr>
<tr>
<td>14</td>
<td>1.0</td>
<td>0.90</td>
<td>39.7</td>
<td>14.0</td>
<td>14.6</td>
<td>-4.3</td>
<td>183.6</td>
</tr>
<tr>
<td>15</td>
<td>2.0</td>
<td>0.90</td>
<td>9.9</td>
<td>3.1</td>
<td>3.1</td>
<td>0.0</td>
<td>219.4</td>
</tr>
</tbody>
</table>

Average difference: 2.0%

Experimental values obtained by Lindholm, Kana, Chu and Abranson (167, 1965).
Structural finite element model (2x2, 80, 16)
Fluid finite element model (80, 177)

FIG. 8.2. REPRESENTATION OF THE SUBMERGED PLATE AND FLUID DOMAIN BY FINITE ELEMENTS.
Plate theory in vacuum, Leissa (137, 1969)

FIG. 8.3. VARIATION OF FREQUENCY PARAMETER OF SUBMERGED PLATE WITH ASPECT RATIO.
FIG. 8.4. VARIATION OF THE FREQUENCY PARAMETER OF SUBMERGED PLATE WITH THICKNESS RATIO.
FIG. 8.5. VARIATION OF THE FREQUENCY PARAMETER OF SUBMERGED PLATE WITH HEIGHT RATIO.
FIG. 8.6. VARIATION OF THE FREQUENCY PARAMETER OF SUBMERGED PLATE WITH DEPTH RATIO.
9.1 Introduction

The phenomenon of dynamic stability or instability combines aspects of dynamics and stability. It is basically a dynamic process but it occurs in structures under loading conditions typical to stability problems. Dynamic Instability or more correctly, Parametric Resonance, is the excitation of transverse oscillations due to periodic longitudinal forces.

The dynamic instability problem was first recognised by Rayleigh (168, 1887), who investigated the stability of strings under variable tension. The dynamic stability of a uniform beam subjected to periodical axial forces was first studied by Beliaev (1069, 1924). Utida and Sezawa (170, 1940) reported on the dynamic stability of the same problem and also verified experimentally the existence of second regions of instability. Lubkin and Stoker (171, 1943) demonstrated that the problem reduces to the evaluation of the instability of the Mathieu Equation and determined the regions of instability of this equation.

A review of the literature on the dynamic stability problem up to 1951, is presented by Beilin and Dzhanelidge (172, 1953). The most comprehensive and complete work on the theory of dynamic stability is presented by Bolotin (173, 1964). In his book, Bolotin determines the regions of dynamic instabilities for several structural systems, including multi-degree of freedom systems, beams, frames, plates and shells. The influence of damping and non-linear terms on the dynamic stability of structures are determined. The book is also an excellent source of further references.

Within the past few years, interest in the dynamic stability of elastic systems has increased. This interest is reflected in the appearance
of a large number of publications on the subject. Unfortunately, only a few of these publications are in English, the majority being written in Russian. Recently, several international conferences related totally or partially to the dynamic stability of structural systems have occurred and have been edited by Herrman (174, 1967) and Leipholz (175, 1971; 176, 1972).

The Finite Element Method was first applied to the dynamic stability of structural systems by Brown, Hutt and Salama (177, 1968). Using the simple beam element, these authors solved the problem of the dynamic stability of beams subjected to several boundary conditions.

Using an incompatible, rectangular plate element, Hutt (178, 1968) and Hutt and Salama (20, 1971) solved the problem of the dynamic stability of plates subjected to several boundary conditions. The effect of the viscous damping on the dynamic stability of the plates was investigated.

Roberts (179, 1971) and Burney (180, 1971) applied the Finite Element Method to the dynamic stability of beams and two dimensional frames, with several boundary conditions. Roberts (179) introduced the concept of sub-structures and component mode synthesis to the dynamic stability problem. Burney (180) also investigated experimentally the dynamic stability of beams and two dimensional frames.

The development of the Finite Element Method to the static, linear and non-linear, stability of structural systems is at a very advanced stage. Martin (181, 1971) presents a review of the majority of publications, up to 1969. Gallagher (182, 1973) has extended this review.

In this chapter, the Finite Element Method is applied to the dynamic instability analysis of shells. The structure is represented by super-parametric shell elements. The theory of the dynamic instability analysis of multi-degree of freedom systems is introduced.
9.2 Dynamic Instability of Multi-degree of Freedom Systems

The equations of motion of a system subjected to static and dynamic stresses is given by,

\[
[M] \ddot{q} + \left( [K] - [KG] - [KT] \right) q = 0 \tag{9.01}
\]

where \( [KT] \) is the Geometric Matrix or Initial Stress Matrix due to the dynamic forces. All other matrices have been defined in Chapter 6.

These \( n \) Lagrange equations of motion can be replaced by \( 2n \) Hamilton Canonical equations by defining a new variable as follows:

\[
\{p\} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \tag{9.02}
\]

Then, equation (9.01) becomes,

\[
\{p\} + [A(t)]\{p\} = 0 \tag{9.03}
\]

where

\[
[A] = \begin{bmatrix}
[0] & [-1] \\
[M]^{-1} & [K] - [KG] - [KT] & [0]
\end{bmatrix}
\]

\( t = \text{time} \)

When the dynamic forces acting on the system are periodic, with period \( T \), it is clear that equation (9.03) is periodic with period \( T \). It is not necessary to solve the periodic equation to determine the stability or instability of its solution, but to predict that the solution is bounded or unbounded. The theory to predict the stability or instability of this periodic equation is frequently referred to as the Floquet Theory.

Assuming that the solution of the periodic equation is known in the interval \( t = 0 \) to \( t = T \), then,
\[ \{p\} = [H(t)]\{d\} = \sum_{i=1}^{2n} d_i \{H_i(t)\} \quad (9.04) \]

where \([H]\) is the Fundamental Matrix, \(\{H_i\}\) is a vector of this Fundamental Matrix, \(\{d\}\) is a vector of constants and \(d_i\) is an element of this vector.

Let \(\{p_0\}\) be the initial conditions at the time \(t = 0\) of the problem. Consequently,

\[ \{p\} = [H(t)][H(0)]^{-1}\{p_0\} \quad (9.05) \]

Meirovitch (183, 1970) has demonstrated that if \([H(t)]\) is a Fundamental Matrix of the periodic equation, then \([H(t + T)]\) is also a Fundamental Matrix of the same equation. Consequently,

\[ \{P\} = [H(t + T)][H(T)]^{-1}\{p_0\} \quad (9.06) \]

\[ [H(t + T)] = [H(t)][C] \quad [C] = [H(0)]^{-1}[H(T)] \]

Matrix \([C]\) is a constant matrix, known as the Monodromy Matrix of the Fundamental Matrix.

Another important property of the Fundamental Matrix can be proved by assuming,

\[ [Z(t)] = [H(t)][e^{-\frac{t}{T}\log[C]}] \quad (9.07) \]

Then it is clear that \([Z(t)]\) is also a periodic matrix with period \(T\).

The Monodromy Matrix is not unique, but depends on the particular Fundamental Matrix. However, the eigenvalues associated with the Monodromy Matrix are unique because any other Fundamental Matrix possesses a similar Monodromy Matrix. To prove this statement, consider another Fundamental Matrix related to the original matrix by,

\[ [\tilde{H}(t)] = [H(t)][B] \quad (9.08) \]
where $[B]$ is a constant, non-singular matrix. Consequently,

$$[A(t + T)] = [A(t)][B]^{-1}[C][B]. \quad (9.09)$$

Therefore, any Fundamental Matrix has a Similar Monodromy Matrix. It follows that all the Fundamental Matrices of the periodic equation determine uniquely all the quantities associated with the Monodromy Matrix, which are invariant under Similarity Transformations, such as the eigenvalues.

The transformation matrix $[B]$ can be selected arbitrarily. It is desirable to choose this matrix such that $[B]^{-1}[C][B]$ is a diagonal matrix. In this case, the Characteristic Equation is given by,

$$| [C] - \lambda [I] | = 0 \quad (9.10)$$

where $\lambda$ is the eigenvalue parameter.

This equation has $2n$ eigenvalues. It is assumed that there are not multiple eigenvalues. Let $[\lambda]$ be a diagonal matrix of all the eigenvalues $\lambda_i$, then

$$[H(t)] = [Z(t)] \left[ e^{\frac{t}{T} \log [\lambda]} \right]$$

and

$$\{p\} = [Z(t)] \left[ e^{\frac{t}{T} \log[\lambda]} \right] \{d\} \quad (9.11)$$

OR

$$\{H_i(t)\} = \{Z_i(t)\} e^{\frac{t}{T} \log \lambda_i} \quad (9.12)$$

and

$$\{p\} = \sum_{i=1}^{2n} d_i \{Z_i(t)\} e^{\frac{t}{T} \log \lambda_i}$$

where $\{Z_i\}$ is a vector corresponding to a column of the periodic matrix $[Z]$.

In general, the eigenvalues are complex. Let,

$$\lambda_i = a_i + jb_i \quad (9.13)$$

232.
where \( a_i \) and \( b_i \) are the real and complex part of the eigenvalue and \( j = \sqrt{-1} \). Then equation (9.12) becomes,

\[
\{p\} = \sum_{i=1}^{2n} d_i \{X_i(t)\} e^{-\frac{t}{T} \log \sqrt{a_i^2 + b_i^2}}
\]  
\[ (9.14) \]

where

\[
\{X_i(t)\} = \{Z_i(t)\} e^{\frac{t}{T} \tan^{-1}(b_i/a_i)}
\]

Therefore, if any eigenvalue has an absolute value greater than unity then instability will occur.

To proceed further with the analysis of the stability of the periodic equation, it is necessary to investigate the properties of the eigenvalues of the Characteristic Equation. It can be proved that if \( \lambda_i \) is an eigenvalue of the Characteristic Equation, then \( 1/\lambda_i \) is also another eigenvalue of the same equation. The proof of this statement will be presented for the case when \([A(t)]\) is an Even Period Function of time. This is the case with most practical applications. Then,

\[
[A(t)] = [A(-t)]
\]  
\[ (9.15) \]

and, consequently,

\[
\{H_i(-t)\} = \{Z_i(-t)\} e^{\frac{t}{T} \log 1/\lambda_i}
\]  
\[ (9.16) \]

Therefore, \( 1/\lambda_i \) is also an eigenvalue. Bolotin (173, 1964) proves that this property is not restricted to Even Functions.

Another restriction on the eigenvalues of the Characteristic Equation is that the eigenvalues must occur in complex conjugate pairs. It may be useful to consider a graphical representation, in the Complex Plane, of the various possible cases of the eigenvalues subjected to its constraints.
This is presented in Figure 9.1. If \( \lambda_i \) is an eigenvalue, then \( \lambda_{i+1} \), \( \lambda_{i+n} \) and \( \lambda_{i+n+1} \) are also eigenvalues and defined as,

\[
\begin{align*}
\lambda_i &= a_i + jb_i \\
\lambda_{i+1} &= a_i - jb_i \\
\lambda_{i+n} &= a_i - jb_i/a_i^2 + b_i^2 \\
\lambda_{i+n+1} &= a_i + jb_i/a_i^2 + b_i^2
\end{align*}
\]  

(9.17)

It can be concluded that the area inside the unit circle, Figure 9.1, represents stable solutions, while the area outside this circle represents unstable solutions. The Reciprocity Property of the eigenvalues demands that the only possible stable solutions must lie on the Unit Circle.

Therefore, for stable solutions,

\[
\{p\} = \sum_{i=1}^{n} d_i (Z_i(t)) e^{jT \tan^{-1}(b_i/a_i)}
\]  

(9.18)

Since the eigenvalues occur in complex conjugates, the limiting values of \( \tan^{-1}(b_i/a_i) \) are zero and \( \pi \). Except at these two limiting values, the vector \( \{p\} \) is Almost Periodic. It is, in fact, a summation of products of two periodic functions with different periods.

At \( \tan^{-1}(b_i/a_i) = 0 \), the solution becomes,

\[
\{p\} = \sum_{i=1}^{2n} d_i (Z_i(t))
\]  

(9.19)

and, therefore, the solution is Periodic with period \( T \).

At \( \tan^{-1}(b_i/a_i) = \pi \), the solution becomes,

\[
\{p\} = \sum_{i=1}^{2n} d_i (Z_i(t)) e^{j \pi T}
\]  

(9.20)

which is also a Period Function with period \( 2T \).

It can be concluded that at the boundaries of the regions of instability,
the periodic system governed by the differential equation (9.01) has periodic solutions with period T and 2T. Also the Regions of Dynamic Instabilities are separated from the Regions of Dynamic Stability by periodic solutions with period T and 2T.

Since on the stability/instability boundaries the motion is periodic, the solution can be represented by a Fourier Series of period T or 2T. It is assumed that the dynamic stresses are sinusoidal with frequency v. Then the equation of motion becomes,

\[ [M]\{\ddot{q}\} + [(K) - (KG) - \cos \beta t[KT]]\{q\} = \{0\}. \quad (9.21) \]

When the solution of this equation with period 2T exists, it may be represented by the following Fourier Series:

\[ \{q\} = \sum_{r=1,3,5,...}^{\infty} \{a_r\} \sin \frac{r\pi}{2} + \{b_r\} \cos \frac{r\pi}{2} \quad (9.22) \]

where \(\{a_r\}\) and \(\{b_r\}\) are time independent vectors. Consequently,

\[
\begin{bmatrix}
    [K] & -(KG) + \frac{1}{4}[KT] - \frac{9\beta^2}{4}[M] & -\frac{1}{2}[KT] & {[0]} & \{a_1\} \\
    -\frac{1}{2}[KT] & [K] & -(9\beta^2/4)[M] & -\frac{1}{2}[KT] & \{a_2\} \\
    [0] & -\frac{1}{2}[KT] & [K] & -(9\beta^2/4)[M] & \{a_3\} \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} = 0
\]

and

235.
A solution of period $2T$ exists, if the determinants of these matrices are zero. Combining both determinants,

$$\begin{bmatrix}
[K] - [KG] - \frac{1}{2}[KT] - \frac{\beta^4}{16}[M] & -\frac{1}{2}[KT] & 0 & \cdots \\
-\frac{1}{2}[KT] & [K] - [KG] - \frac{9\beta^2}{4}[M] & -\frac{1}{2}[KT] & \cdots \\
0 & -\frac{1}{2}[KT] & [K] - [KG] - \frac{25\beta^2}{4}[M] & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}\begin{bmatrix}b_1 \\ b_2 \\ b_3 \\ \vdots\end{bmatrix} = 0$$

Similarly, the solution with period $T$ may be represented by the following Fourier Series:

$$\{q\} = \frac{1}{2}\{b_0\} + \sum_{r=2,4,6,\ldots}^{\infty} \{a_r\}\sin r\beta \frac{t}{2} + \{b_r\}\cos r\beta \frac{t}{2} \quad (9.25)$$

In this case a solution exists if the following two determinants are zero,

$$\begin{bmatrix}
[K] - [KG] - \beta^2[M] & -\frac{1}{2}[KT] & 0 & \cdots \\
-\frac{1}{2}[KT] & [K] - [KG] - 4\beta^2[M] & -\frac{1}{2}[KT] & \cdots \\
0 & -\frac{1}{2}[KT] & [K] - [KG] - 9\beta^2[M] & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix} = 0$$

and
These three infinite determinants can be reasonably approximated to,

\[
\begin{vmatrix}
[K] - [KG] + \frac{1}{2}[KT] - \frac{\beta^2}{4} [M]
\end{vmatrix} = 0
\]  
(9.27)

and

\[
\begin{vmatrix}
[K] - [KG] - \frac{\beta^2}{4} [M]
\end{vmatrix} = 0
\]  
(9.28)

Equation (9.27) defines the Principal Region of Dynamic Instability. Equation (9.28) defines the Second Region of Dynamic Instability. Both these Regions of Instabilities can be improved by taking higher order determinants. Also, the frequencies corresponding to the boundaries of the Principal Region of Dynamic Instabilities are approximately twice the natural frequencies of a system, which is subjected to the static forces plus or minus one half of the dynamic forces.

Consequently, the dynamic instability problem can be solved by analysing two pre-stressed systems. Thus, the pre-stressed Finite Element Model of a shell structure developed in Chapter 6 can be used to predict the Instability Regions of shells subjected to periodic longitudinal forces.
9.3 Applications

The pre-stressed shell Finite Element Model of Chapter 6 is applied to the dynamic stability of simply supported cylindrical shells and cylindrical shell blades. These shells are shown in Figures (6.24) and (6.30), respectively. The ratio of the dynamic to static stress is 0.3.

The analytical solution of the natural frequencies of a simply supported, pre-stressed, cylindrical shell is presented in Chapter 6. This solution is used to derive the theoretical Principal Instability Regions of the shell subjected to longitudinal periodic forces.

In all the applications the Reduced Integration Technique is used to evaluate the strain energy of the structure. Also the Eigenvalue Economizer is used to reduce the number of degrees of freedom of the problem. The primary degrees of freedom are the vertical displacements. Poisson ratio is 0.3.

9.4 General Discussion

A general method of predicting the Principal Instability Regions of shells with arbitrary geometry, thickness and boundary conditions is presented. It is found that a 3 * 3 mesh of super-parametric elements representation of shell structures is capable of determining the Principal Instability Regions with acceptable accuracy.

Figure 9.2 and 9.3 compares the computed results with the analytical Principal Instability Regions of a simply supported, cylindrical shell with axial periodic forces.

Figures 9.4 and 9.5 show the Principal Instability Regions of the uniform and tapered cylindrical shell blades, respectively. These shells are subjected to axial periodic forces. With high frequency forces, or low stress ratio, the Instability Regions become almost lines having values twice the natural frequencies of the pre-stressed system. These figures demonstrate clearly the link between Statics, Dynamics and Stability.
FIG. 9.1. UNIT CIRCLE IN THE COMPLEX PLANE.
\[ \omega_{mn} = \text{Natural frequency of the shell without pre-stress.} \]

\[ \varepsilon_{xx}^0 + \varepsilon_{xx}^+ \cos \beta t \]

\[ \varepsilon_{xx}^0 = \text{static stress} \]

\[ \varepsilon_{xx}^+ = \text{Dynamic stress} \]

\[ \varepsilon_{xx}^+ / \varepsilon_{xx}^0 = 0.3 \]

\[ \beta = \text{Frequency of the dynamic stress.} \]

\[ (\varepsilon_{xx}^c)_{mn} = \text{Critical stress corresponding to the buckling mode (m, n)} \]

**Finite element model** (3×3, 120, 16)

**FIG. 9.2. PRINCIPAL REGIONS OF DYNAMIC INSTABILITY OF SIMPLY SUPPORTED CYLINDRICAL SHELL.**
$\omega_{mn}$ = Natural frequency of pre-stressed shell

$\tau_{xx}^t / \tau_{xx}^c = 0.3$

Finite element model $(3 \times 3, 120, 16)$

FIG. 9.3. PRINCIPAL REGIONS OF DYNAMIC INSTABILITY OF SIMPLY SUPPORTED CYLINDRICAL SHELL.
\( \omega_i \) = Natural frequency of blade.
\( \sigma_{xx}^{\infty} \) = Minimum dynamic buckling stress.

\[
\tau_{xx} = \tau_{xx}^0 + \tau_{xx}^t \cos \beta t \\
\tau_{xx}^e / \tau_{xx}^0 = 0.3
\]

Finite element model (3 x 3, 165, 33)

**FIG. 9.4.** PRINCIPAL REGION OF DYNAMIC INSTABILITY OF THE UNIFORM CYLINDRICAL SHELL BLADE.
Finite element model \((3 \times 3, 165, 33)\)

\[
\frac{\tau_{xx}^t}{\tau_{xx}^o} = 0.3
\]

**Fig. 9.5. Principal regions of dynamic instability of the tapered cylindrical shell blade.**

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**243**
CHAPTER 10

TRANSIENT ANALYSIS OF SHELLS

10.1 Introduction

The Finite Element Method was introduced to the transient analysis of structural systems by Clough and Wilson (184, 1962). Also, Carr (91, 1967; 92, 1967) predicted the transient response of cylindrical shells using finite elements.

It is only recently that the Finite Element Method has been applied to predict the transient response of arbitrary shell structures. This is a direct consequence of the late development of an efficient shell element. The flat shell element or the simple curved shell elements were incapable of predicting the natural frequencies of the shell with acceptable accuracy, such that a transient analysis can be accurate. The extensive literature surveys of Stricklin (185, 1971) and Clough and Wilson (186, 1971; 187, 1971) demonstrate that until 1971, the Finite Element Method had only been applied to the transient analysis of cylindrical shells or shells of revolution.

Recently, Key and Beisinger (188, 1971) and Key and Krieg (21, 1972) used a finite difference method to integrate the equation of motion of arbitrary shell structures. The structure was represented by an assembly of quadrilateral, curved shell elements. These authors developed a diagonal mass matrix from the consistent mass matrix. This development simplified the numerical integration of the equation of motion.

Progress has also been made in the nonlinear dynamic analysis of shells. Yeh (189, 1970) introduced the Finite Element Method to the analysis of nonlinear transient response of cylindrical shells. An extensive literature survey on the subject is presented by Bathe, Ramm and Wilson (190, 1975).
The dynamic response of undamped, linear, multi-degree of freedom structural systems can be calculated by solving the following equation:

\[
[M]\{\ddot{q}(t)\} + [K]\{q(t)\} = \{Q(t)\}
\]  

(10.01)

In the dynamic analysis of pre-stressed, rotating or submerged structures, \([M]\) and \([K]\) represent the total Mass and Stiffness Matrices of the system.

Equation (10.01) represents a linear system of ordinary differential equations. Its solution can be achieved by direct integration of the simultaneous differential equations or by uncoupling the equations and solving independently each differential equation. These two techniques of solution are known as Direct Integration and Mode Superposition (or Modal Analysis) Methods.

In this chapter the Mode Superposition Method is introduced. The accuracy, efficiency and stability of both methods are discussed. Also, the Modal Analysis Method is used to calculate the transient response of shell structures subjected to arbitrary dynamic forces. The structure is represented by an assembly of super-parametric shell elements. The Reduced Integration Technique is used to evaluate the strain energy of the structure. The Eigenvalue Economizer is used to reduce the number of degrees of freedom of the discretized system. The uncoupled equations of motion of the Finite Element Model are integrated by the Simpson's Rule. The super-parametric shell element modal analysis model is applied to the transient response of shell structures, including pre-stressed and rotating systems, due to arbitrary forcing functions. The computed transient responses are compared with analytical solutions.
10.2 Mode Superposition Method

The basic concept of the Mode Superposition Method is that the displacement vector of equation (10.01) can be expressed as a linear combination of the eigenvectors of this system, thus,

\[ \{q\} = [U]\{p\} \quad (10.02) \]

where \([U]\) is the normalized Modal Matrix formed by the eigenvectors of the system and \(\{p\}\) is the vector of modal amplitudes.

Equation (10.02) can also be considered as a transformation of coordinates, from the original \(\{q\}\) set of coordinates to a new set \(\{p\}\) coordinates known as the Principal Coordinates. Transforming the coordinates of the equation of motion and pre-multiplying this equation by the transpose of the Modal Matrix, then equation (10.02) becomes,

\[ [U]^t[M][U]\{\ddot{p}\} + [U]^t[K][U]\{p\} = [U]^t\{Q\} \quad (10.03) \]

Applying the Orthogonality properties of the eigenvectors, which are presented by Meirovitch (191, 1967), then it can be demonstrated that this equation becomes,

\[ \{\ddot{p}\} + \left[ \omega^2 \right]\{p\} = [U]^t\{Q\} = \{P\} \quad (10.04) \]

where \([\omega^2]\) is a diagonal matrix of the eigenvalues of the system.

This equation is an uncoupled system of ordinary differential equations which can be solved by Laplace Transforms, Fourier Series, Direct Integration, Finite Differences or Runge-Kutta Methods. Alternatively, its solution is the Duhamel's or Convolution Integral,

\[ p_s(t) = \frac{1}{\omega_s} \int_0^t p_s(\alpha) \sin \omega_s(t - \alpha) d\alpha \quad (10.05) \]

where

\[ \omega_s \text{ - natural frequencies} \]
n = number of degrees of freedom
s = 1, 2, ..., n
Ps - components of vector \( \{P\} \)
\( \alpha \) - dummy variable
t - time

For general forcing functions, the Duhamel's Integral can be numerically integrated by Simpson's Rule. To improve the efficiency of a computer program, it is convenient to expand the Duhamel's Integral as,

\[
p_S = \frac{1}{\omega_s} \sin \omega_s t \int_0^t p_S(\alpha) \cos \omega_s \alpha d\alpha + \frac{1}{\omega_s} \cos \omega_s t \int_0^t p_S(\alpha) \sin \omega_s \alpha d\alpha
\]

(10.06)

The numerical integration of this equation by the Simpson's Rule is always stable provided that small steps are taken. The minimum magnitude of these steps are a function of the frequency of the external forces and of the natural frequencies of the system. Different steps should be used to evaluate the integral of each particular mode.

It is clear that higher modes require smaller time steps. Therefore, very large computer time is required to evaluate the contribution of these modes. Also, from equation (10.06), it can be generally concluded that the contribution of the higher modes becomes small and therefore can be neglected. Consequently, an approximated solution of the transient response of the system can be predicted by neglecting the contribution of the higher modes.

Modal Analysis is a general method of solution of linear, ordinary or partial, system of differential equations. It is an exact technique if the contribution of all modes are considered. In section (10.5) the Modal Analysis Method is used to transform the partial differential equation of motion of a thin shallow shell into a series of ordinary differential equations.
Another important application of the Mode Superposition concept has been the development of the Modal Synthesis Technique of Sub-structures. Full details of this technique are presented by Newbert and Raney (192, 1971).

10.3 Direct Integration Methods

In the most efficient Numerical Methods of solving the transient response of a dynamic system, the displacement, velocity and acceleration vectors at any time \( t \) are related to the corresponding vectors at time \( t + \delta t \), where \( \delta t \) is a small interval of time. The time domain is divided into a sequence of equal time sub-domains. The response at the end of a sub-domain is the initial condition of the next sub-domain. These methods can be divided into two categories: the Finite Difference and Numerical Integration Methods. Clough (193, 1973) briefly describes the most important of these methods.

In each category, the methods can be classified as Explicit and Implicit procedures. The Explicit formulation expresses the new vectors only in terms of vectors of preceding time which are already known. The Implicit formulation includes the new vectors in the expression for these vectors and, therefore, are solved by iteration.

Important criteria in selecting any of the numerical methods are accuracy, computational efficiency and stability of solution. A comparison of the efficiency of several methods is presented by Clough (194, 1971), Dunham, Nickell and Stricklin (195, 1972), Goudreau and Taylor (196, 1972), and Clough and Bathe (197, 1973). The stability of several methods is discussed by Nickell (198, 1971) and Fu (199, 1972).

From these publications it can be concluded that the Newmark, Houbolt and Wilson Methods are probably the most stable, efficient and accurate methods available. Details of these methods are presented by Clough (193, 1973).
10.4 Choice of Method

The two basic approaches to dynamic response analysis, Mode Superposition and Direct Integration, obviously differ greatly in their computational processes and each is best suitable to a certain class of problem. The Mode Superposition Method is most efficient if the essential dynamic response of the structure can be represented by the first few eigenvectors. This is the case if the applied forcing functions are not extremely complex and with low frequency. Also the distribution of the frequency spectrum of the system should not be compact.

In systems subjected to complex forcing functions or high frequency functions, or systems with a compact natural frequency distribution, a significant number of eigenvectors are required to represent the response. Consequently, a large eigenvalue problem must be solved. In these problems, the Direct Methods of Integration of the coupled equations are more efficient. These methods avoid the solution of large eigenvalue problems and automatically take account of all modes. Also, the Direct Methods can be used in non-linear dynamic response problems, while the Mode Superposition Method is only applicable to linear problems.

Recently, Bathe and Wilson (200, 1973) demonstrated that the Direct Integration Method is equivalent to a Mode Superposition Analysis in which all eigenvalues and eigenvectors have been calculated and the uncoupled equations are integrated with a common time step $\delta t$. The integration is accurate for those modes for which $\delta t \omega_i$ is small, where $\omega_i$ is the natural frequency of the particular mode. Also, the response in the modes for which $\delta t \omega_i$ is large is eliminated. Therefore, the direct integration is equivalent to a mode superposition analysis in which only the lowest modes of the system are considered. The exact number of modes effectively included in the analysis depend on the time step and the distribution of
the natural frequencies. Clearly, the Direct Integration Method is most efficient when the natural frequency spectrum of the system is compact, which is when the Mode Superposition Method is least efficient.

The natural frequency spectrum of shell structures are usually more compact than the natural frequencies of other structural systems. Thus, in general, the Mode Superposition Method of dynamic response is more efficient in analysing beam or plate structures than shell structures. However, in shells with cantilever type boundary conditions, as in turbine or compressor blades, the distribution of the lower natural frequencies are not compact. If the shell is not subjected to complex forcing functions, the Modal Analysis Method is more efficient than any of the Direct Methods. For this reason, the Modal Analysis Method is selected.

10.5 Applications

The Super-parametric Shell Element Modal Analysis Model developed in this chapter is applied to the transient response of spherical and cylindrical shells due to Sinusoidal and Impact forces. The Model is also applied to pre-stressed cylindrical shells and a rotating cylindrical shell blade. The computed results are compared with analytical solutions.

The Model is applied to the following shell structures. These problems are subjected to Sinusoidal and Impact Forces.

1) Spherical shell \( (E = 2.0 \times 10^{11} \text{ N/m}^2, \rho = 7800 \text{ Kg/m}^3, a = b = R = 0.1m, h = 0.002m) \) with simply supported boundary conditions and square base. The forces are acting at \( x = 0.6a \) and \( y = 0.4b \). This shell is shown in Figure 6.1.

2) Cylindrical shell \( (E = 2.0 \times 10^{11} \text{ N/m}^2, \rho = 7800 \text{ Kg/m}^3, a = b = R = 0.1m, h = 0.002m) \) with simply supported boundary conditions. The forces are acting at \( x = 0.6a \) and \( y = 0.4b \). This shell is shown in Figure 6.24.

250.
3) Pre-stressed cylindrical shell \((E = 2.0 \times 10^{11} \text{ N/m}^2, \rho = 7800 \text{ Kg/m}^3, a = b = R = 0.1m, h = 0.002m)\) with simply supported boundary conditions. The pre-stress is 0.25 of the minimum critical stress and the forces are acting at \(x = 0.75a\) and \(y = 0.25b\). This shell is shown in Figure 6.24.

4) Cylindrical shell blade, shown in Figure 6.30, with uniform thickness.

5) Pre-stressed, cylindrical shell blade.

6) Rotating, cylindrical shell blade.

In all the applications, the transverse displacements are the primary coordinates. The Poisson ratio is 0.3.

10.5.1 Analytical solutions

The analytical solution of the transient response of shells due to dynamic forces is only possible in a few special combinations of geometry, boundary conditions and forcing functions. One of these combinations is the spherical or cylindrical shell, with simply supported boundary conditions and subjected to sinusoidal or impact forces.

The equation of motion of a shallow, thin shell subjected to tangential stress is given by equation (6.33). Also, the natural frequencies of a simply supported shallow shell subjected to tangential stresses is given by equation (6.37).

With the nomenclature of section 6.4.1 and using the Modal Analysis Method, the transient response of the shell can be proved to be,

\[
w = v^4 \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(x, y) q_{mn}(t) \right)
\]

where

\[
q_{mn} = \frac{1}{\omega_{mn}} \int_0^t q_{mn}(\alpha) \sin \omega_{mn}(t - \alpha) d\alpha
\]

\(\alpha = \text{dummy variable}\)
\[ Q_{mn} = \int_a^b \int_0^b \phi_{mn}(x, y) P(x, y, t) \, dx \, dy \]

and \( \omega_{mn} \) are the natural frequencies of the system.

Consider a Sinusoidal Force, \( P = P_0 \sin \beta t \), applicable at point \( x = c \) and \( y = d \). The non-dimensional, vertical, transient displacement of the shallow shell can be proved to be,

\[
\rho a b^2 \frac{w'}{4P_0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi c/a) \sin(n\pi d/b) \sin(m\pi x/a) \sin(n\pi y/b) \times \frac{\omega_{mn}}{\beta} \sin \beta t - \sin \omega_{mn} t) / (\omega_{mn}/\beta)((\omega_{mn}/\beta)^2 - 1) \quad (10.08)
\]

Consider an Impact Force of magnitude \( P_0 \) and duration \( t_o \), acting at point \( x = c \) and \( y = d \). The non-dimensional, vertical, transient response can be proved to be,

\[
\rho a b^2 \frac{w'}{4P_0} = ( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi c/a) \sin(n\pi d/b) \sin(m\pi x/a) \sin(n\pi y/b) \times (1 - \cos \omega_{mn} t) \beta^2/\omega_{mn}^2 \quad t \leq t_o \quad (10.09)
\]

and

\[
\rho a b^2 \frac{w'}{4P_0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi c/a) \sin(n\pi d/b) \sin(m\pi x/a) \sin(n\pi y/b) \times (\cos \omega_{mn}(t - t_0) - \cos \omega_{mn} t) \beta^2/\omega_{mn}^2 \quad t > t_o
\]

where \( \beta = 2\pi/t_0 \).

10.6 General Discussion

A method is presented to calculate the transient response of shells with arbitrary geometry and subjected to any forcing functions. The shell is represented by super-parametric shell elements; the transient response is calculated by Modal Analysis and the integration of the Duhamel's Integral is by the Simpson's Rule. The method is also applicable to other linear
problems, such as pre-stressed, submerged or rotating shells. The model is applied to the transient analysis of shell structures, including pre-stressed and rotating shells.

Generally, the transient analysis of shell structures requires a larger discretized system than a free vibration analysis of the same structure. This increase in the degrees of freedom is dependent on the forcing functions, the geometry of the structure and its boundary conditions, but it is independent of the method of integration of the equations of motion.

The spherical shell with simply supported boundary conditions and subjected to simple forcing functions requires a large number of degrees of freedom to be analysed accurately. The computed and analytical transient response of a spherical shell to sinusoidal and impact forces is presented in Figures 10.1 and 10.3. The computed transient response, which has the contribution of the first 30 modes, has an average error of approximately 10%. Also, the convergence of the computed solution is very slow.

The convergence of the analytical solution is also very slow, as seen in Figures 10.2 and 10.4. The rate of convergence of the analytical solution is approximately dependent on the sequence $\sum_{i=1}^{\infty} \frac{1}{\omega_i^2}$. Since the natural frequency distribution is compact, this sequence converges very slowly. Also in the Finite Element Modal Analysis Model the same phenomenon occurs, therefore the necessity of a large number of degrees of freedom. The use of Direct Methods to integrate the equations of motion would not eliminate this problem.

The computed and analytical response due to sinusoidal or impact forces of a simply supported cylindrical shell is presented in Figures 10.5 and 10.7. These responses are on average more accurate than in the spherical shell case, which are calculated with the same number of degrees of freedom. The reason for the improvement can be seen in Figures 10.6 and 10.8, which
show the convergence of the analytical solutions. This improvement is a direct consequence of the fact that the distribution of the natural frequencies of the cylindrical shell is less compact than the distribution of the natural frequencies of the spherical shell.

Figure 10.9 shows the computed and analytical transient response due to a sinusoidal force, of a simply supported, pre-stressed, cylindrical shell. Although the errors are larger than in the previous applications, these errors are caused by the analysis of a smaller finite element representation.

The comparison of the transient response of the cylindrical shell, with and without pre-stress, demonstrates the influence of the pre-stress in the dynamic analysis of pre-stressed structures. This is shown in Figure 10.5 and 10.9.

The transient response of the cylindrical shell blade due to the sinusoidal and impact forces are presented in Figures 10.10 to 10.15. In these figures the influence of the pre-stress and angular velocity are considered. It is clear that a 3 * 3 mesh representation, with 165 degrees of freedom and 33 primary coordinates, of the blade is adequate to calculate the transient response. Also the transient response can be calculated, with acceptable accuracy, by considering only the first 5 modes. Thus, the transient response of this shell blade can be calculated very efficiently. This improvement is a direct consequence of the distribution of the natural frequencies of the cylindrical shell blade.

These figures also show the influence of the pre-stress and angular velocity on the transient response of the shells. In the case of the sinusoidal forces, the maximum transient response is decreased by 12.1% and 39.5% due to the pre-stress and angular velocity, respectively. The corresponding decrease with the impact forces are 5.8% and 21.6%, respectively. Although these decreases are relative to these particular problems, it demonstrates the importance of considering the pre-stress or angular velocity in the transient analysis of pre-stressed or rotating shells.
The superparametric shell element representation of shell structures and Modal Analysis are only partially successful. However, it is possible to study the efficiency of the method by analysing the free vibration problem, since its efficiency is primarily dependent on the distribution of the natural frequency spectrum of the system.

In general, thin shells tend to have compact distribution of natural frequencies, therefore, Modal Analysis is only partially successful to these problems. However, any shell with cantilever type boundary conditions does not have a compact distribution of the natural frequencies and, consequently, Modal Analysis can be very efficient.

It can be concluded that the Super-parametric Shell Element Modal Analysis Model is very efficient to calculate the transient response of blades, but the model is only partially successful to large numbers of shell problems.

The complexity of the forcing function has an influence in the efficiency of the Modal Analysis Method. With very complex forcing functions the Direct Integration Methods are superior to the Mode Superposition Method.

An increase in the angular velocity or pre-stress tend to give a more compact spectrum of natural frequencies, and consequently, decreases the efficiency of the Modal Analysis Method. At high angular velocity or pre-stress a Direct Method of integration is probably more efficient than the Mode Superposition Method.
FIG. 10.1. TRANSIENT RESPONSE OF THE SPHERICAL SHELL SUBJECTED TO A SINEOIDAL FORCE.
FIG. 10.2. ANALYTICAL TRANSIENT RESPONSE OF THE SPHERICAL SHELL SubjectED TO A SINUSOIDAL FORCE.
FIG. 10.3. TRANSIENT RESPONSE OF THE SPHERICAL SHELL SUBJECTED TO AN IMPACT FORCE.
FIG. 10.4. ANALYTICAL TRANSIENT RESPONSE OF THE SPHERICAL SHELL SUBJECTED TO AN IMPACT FORCE.
FIG. 10.5. TRANSIENT RESPONSE OF THE CYLINDRICAL SHELL SUBJECTED TO A SINUSOIDAL FORCE.
FIG. 10.6.  ANALYTICAL TRANSIENT RESPONSE OF THE CYLINDRICAL SHELL SUBJECTED TO A SINUSOIDAL FORCE.
FIG. 10.7. TRANSIENT RESPONSE OF THE CYLINDRICAL SHELL SUBJECT TO AN IMPACT FORCE.
FIG. 10.8. ANALYTICAL TRANSIENT RESPONSE OF THE CYLINDRICAL SHELL SUBJECTED TO AN IMPACT FORCE.
FIG. 10.9. TRANSIENT RESPONSE OF THE PRE-STRESSED CYLINDRICAL SHELL SUBMITTED TO A SINEOIDAL FORCE.
FIG. 10.10. TRANSIENT RESPONSE OF THE CYLINDRICAL SHELL BLADE SUBJECTED TO SINUSOIDAL FORCES.
FIG. 10.11. TRANSIENT RESPONSE OF THE PRE-STRESSED CYLINDRICAL SHELL SUBJECTED TO SINUSOIDAL FORCES.

\[ P = P_0 \sin \beta \cdot t \]

\[ P_0 = 100.0 \text{ N} \]

\[ \beta = 50.0 \text{ Hz} \]
Radius of disc/length of blade = 1.0
Angular velocity = 50.0 revolutions/second
$\beta$ = forcing frequency = 50.0 Hz.

Finite element model ($3 \times 3, 165, 33$)

Points of application of forces.
Points of which the transient response is calculated.

FIG. 10.12. TRANSIENT RESPONSE OF THE ROTATING CYLINDRICAL SHELL BLADE SUBJECTED TO SINEOIDAL FORCES.
FIG. 10.13. TRANSIENT RESPONSE OF THE CYLINDRICAL SHELL BLADE SUBJECTED TO IMPACT FORCES.
FIG. 10.14. TRANSIENT RESPONSE AT THE PRE-STRESSED CYLINDRICAL SHELL BLADE SUBJECTED TO IMPACT FORCES
Angular velocity = 50.0 revolutions/second
Disc radius / length of blade = 1.0

Finite element model (3, 165, 33)

FIG. 10.15. TRANSIENT RESPONSE OF THE ROTATING CYLINDRICAL SHELL BLADE SUBJECTED TO IMPACT FORCES.
CHAPTER 11
CONCLUSIONS

Finite Element Models to predict the natural frequencies, modes of vibration and transient response of shell structures, including pre-stressed, rotating and submerged systems, have been developed. These shell structures have arbitrary geometry, thickness and boundary conditions and are subjected to arbitrary forcing functions. Also, a Finite Element Model to predict the dynamic stability and instability regions of arbitrary shells subjected to axial periodic forces has been developed.

The Reduced Integration Technique transforms the super-parametric shell element into a very efficient, accurate and versatile shell element. It becomes applicable to the linear and nonlinear, static and dynamic analysis of thin or thick, shallow or deep, shell structures. In one particular example, the representation with reduced integration is 500% more efficient than the representation with correct integration. Also, the increase in efficiency is more significant with thin shells and at higher modes.

In general a finite element representation with less than 10 super-parametric shell elements, 200 degrees of freedom and 40 primary coordinates is capable of predicting accurately the lower natural frequencies and vibration modes of arbitrary shell structures. The actual number of accurate natural frequencies and vibration modes is dependent on the geometry and boundary conditions of the problem. This representation can predict accurately the first ten natural frequencies of shell blades. However, the same representation can predict only the first five natural frequencies of shell structures with simply supported boundary conditions.

The super-parametric shell element with reduced integration is one of the most efficient thin shallow shell elements ever developed. In all the applications, it was concluded that for a thin shallow shell representation
with an equivalent number of degrees of freedom and primary coordinates, the super-parametric element with reduced integration is superior to the Lindberg element (118).

Besides, the super-parametric element offers other advantages over any of the thin shallow shell elements. Firstly, the element is also applicable to general shell analysis and not restricted to thin shallow shells.

Secondly, the element is developed with reference to curvilinear coordinates and not to the projected Cartesian coordinates as in the majority of shell elements. Therefore, the shallowness or deepness of the shell is almost irrelevant to the efficiency and accuracy of the element.

Finally, the element is based on the Theory of Elasticity and, therefore, the development to nonlinear analysis is simpler than the corresponding development of an element based on shell theories. Since it is necessary to evaluate the nonlinear strain energy of rotating or pre-stressed systems, the development of finite elements based on shell theories to represent these problems are extremely difficult, as a consequence of the complexity of the nonlinear shell theories.

A Finite Element Model for the dynamic analysis of pre-stressed structures is equivalent to a Nonlinear Static System and a Linear Dynamic System. Only at low stress levels can the static system be assumed to be linear.

A Lagrangian Model is not applicable to shell structures subjected to very high stress levels. In this case an Eulerian Model to represent the equivalent static system must be used. However, a Lagrange Model for the equivalent dynamic system is still valid.

The finite element dynamic analysis of pre-stressed structures usually requires the same number of finite elements as the analysis of the structures without pre-stress. However, when the pre-stress field has large stress gradients more finite elements are required to evaluate the nonlinear strain energy than the linear strain energy. Also with negative stress fields, the lower vibration modes are not necessarily the same vibration modes as
the structure without pre-stress and, consequently, more degrees of freedom are required to predict these modes.

The influence of the stress field on the natural frequencies, mode shapes and transient response of shell structures is extensively documented in this investigation. In general, a positive stress field increases the natural frequencies of the system, while a negative stress field decreases the natural frequencies. However, the transient response can increase or decrease depending on the distributions of the natural frequencies of the structures and harmonics of the forcing functions.

For constant stress fields, the natural frequencies of the pre-stressed system are proportional to the square root of the stress.

The vibration modes are also modified by the stress field. Thus, it is not absolutely correct to analyse pre-stressed structural systems by assuming identical eigenfunctions as the structure without pre-stress.

The limitations of the thin and shallow shell theories and the Theory of Thin Plates are extensively illustrated in this investigation. The shallow shell theories are only applicable to shell structures with shallowness ratio \( R/a \) less than 1.0. In other applications these theories predict higher natural frequencies. The thin shell theories are only applicable to shell structures with thickness ratio \( a/h \) less than 25. In other applications these theories predict higher natural frequencies. The theory of thin plates is only applicable to plates with thickness ratio \( b/h \) less than 15. It predicts higher natural frequencies when applied to thick plates.

The influence of the aspect ratio, thickness ratio and pre-twist angle on the dynamic characteristics of pre-twisted blades is extensively reported in this Thesis.

The frequency parameters of bending modes of thin, pre-twisted blades are not affected by the thickness ratio. However, for thick blades the frequency parameters of the higher modes are significantly changed with
thickness ratio. Also, the frequency parameters of torsion-bending modes of thin blades are significantly affected by a small variation in the thickness ratio.

A small variation in the aspect ratio of pre-twisted blades changes significantly the frequency parameters of torsion-bending modes. However, the frequency parameters of the bending modes are not significantly affected.

Variation of the pre-twisted angle changes significantly the frequency parameters of the torsion-bending modes of thin blades. The corresponding change in thick blades is small. Also, an increase of the pre-twist angle reduces the frequency parameters of the bending modes of thin and thick blades considered. The relative reductions are smaller with thick blades and at higher vibration modes.

The dynamic stability and instability regions of structures subjected to axial periodic forces can be predicted by a Finite Element Model for the dynamic analysis of pre-stressed structures. The regions can be predicted by analysing the structure subjected to two stress fields, which are equal to the static stress field plus or minus one half of the dynamic stress field.

A Finite Element Model of a rotating structure is equivalent to a Nonlinear Static System and a Linear Dynamic System. With the assumption of small oscillations, the Dynamic System can be always represented by a Lagrangian Model. However, only at low angular velocities can the Static System be represented by a Lagrangian Model. At high angular velocities, the Static System must be represented by an Eulerian Model. Alternatively, a Lagrangian Model can be used with an incremental procedure. Only at low angular velocity is the Finite Element Model of a rotating structure equivalent to a Lagrangian Linear Static System and a Lagrangian Linear Dynamic System.
The dynamic convergence properties of a super-parametric shell element representation of rotating shell blades is extensively illustrated. A representation of less than 10 elements, 200 degrees of freedom and 40 primary coordinates is capable of predicting accurately the lowest ten natural frequencies and vibration modes of rotating, arbitrary shell blades. This representation is also capable of predicting the transient response of rotating shell blades.

The angular velocity changes significantly the natural frequency spectrum and the corresponding vibration modes of a rotating blade. In general, the angular velocity increases the natural frequencies of the blade and the natural frequency spectrum becomes more compact. Also, the rate of increase is dependent on the vibration mode at the particular angular velocity. The rate of increase of the natural frequencies of the bending modes are much larger than the corresponding rate of increase of the torsional modes. Also, the rate of increase is greater in the lower vibration modes.

The vibration modes of a rotating blade are drastically modified with angular velocity. This modification of the modes can be such that exchange between modes occurs. This transformation is a continuous process and, therefore, only certain modes can be transformed into other modes. This phenomenon is extensively illustrated in this Thesis. With high aspect ratio blades this transformation does not occur at low angular velocities and, therefore, the relationship between natural frequencies and angular velocity is simple. However, with small aspect ratio thin shell blades, this transformation occurs at low angular velocities. Consequently, in these blades the relationship between natural frequencies and angular velocity is very complex.

The complexity of the relationship between natural frequencies of small aspect ratio thin shell blades with angular velocity is a consequence
of the compact distribution of the natural frequency spectrum of these blades. This compact distribution allows a transformation of modes even at low angular velocities.

The natural frequency distribution of large aspect ratio blades are not compact and, consequently, only at very high angular velocities the transformation of modes will occur.

Although the relationship between the natural frequencies of rotating blades with angular velocity can be very complex, a dynamic analysis of the non-rotating blade can provide the basic shape of this relationship. Unfortunately, this analysis is more qualitatively than quantitatively accurate.

The steady state deformation of rotating blades has the effect of increasing the natural frequencies of the torsional bending modes and decreasing the natural frequencies of the bending modes. At angular velocities less than the fundamental natural frequency of the non-rotating blade, this effect can be neglected without major errors.

With small aspect ratio thin shell blades, some modes can be a torsional bending mode at low angular velocity and a bending mode at higher angular velocity. Consequently, the effect of the steady state deformation is to increase the natural frequency at low angular velocity and to decrease the natural frequency at higher angular velocities. In other modes, the reverse occurs.

The influence of the disc radius, setting angle, thickness ratio, aspect ratio and pre-twist angle on the dynamic characteristics of rotating shell blades is extensively illustrated in this investigation.

An increase in the radius of the disc increases the natural frequencies of all modes. However, in the case of large aspect ratio blades, this increase is negligible for the torsional bending and higher modes and, therefore, only the natural frequencies of the lower bending modes are significantly affected.
With small aspect ratio thin shell blades, the relationship between the natural frequencies and disc radius can be very complex. This is a consequence of the possible transformation of several modes as the disc radius changes, even at constant angular velocity.

The dynamic characteristics of the lower modes of rotating blades are slightly dependent on the setting angle. The effect on the higher modes is negligible.

The curves defining the relationship between the natural frequencies of pre-twisted rotating blades and the pre-twist angle or thickness ratio are similar to the corresponding curves of the non-rotating blade. In fact these curves are parallel, indicating that only the absolute value of the natural frequencies are different and caused by the angular velocity.

With the exception of the lower torsional modes of low aspect ratio blades, the curves defining the relationship between natural frequencies of rotating pre-twisted blades and aspect ratio are also parallel to the corresponding curves of the non-rotating blades. This indicates that only the absolute value of the natural frequencies of the rotating and non-rotating blades are different and caused by the angular velocity.

The limitation of the computer available did not permit to analyse completely the convergence properties of the super-parametric shell/isoparametric fluid model when applied to the dynamic analysis of shell structures submerged in a fluid medium. However, in the applications to submerged plate structures, this model is more efficient and accurate than previously developed models.

The accuracy of the dynamic characteristics of submerged structures predicted by a Fluid/Structure Finite Element Model is significantly dependent on the number and its distribution of the fluid finite elements. It is imperative to have the majority of fluid elements to represent the fluid domain adjacent to the structure. In fact, the fluid domain must be repre
sented by a similar mesh as in a structural system subjected to a high stress concentration field.

The structure should be represented by the minimum number of elements capable of predicting the lower natural frequencies and vibration modes of the structure in vacuum. This is a consequence of the compatibility condition between structural and fluid degrees of freedom, since an increase in the structural degrees of freedom demands a much larger increase in the fluid degrees of freedom.

The natural frequencies of a submerged structure are significantly reduced by the height of fluid above the structure, if this height is less than the length of the structure. For deeply submerged structures, the natural frequencies are independent of further increases in the height of fluid above the structure and are much lower than in vacuum.

The natural frequencies of a submerged structure are almost independent of the depth of fluid below the structure.

Contrary to the Theory of Plates in vacuum, the fundamental natural frequency of a submerged plate is significantly dependent on the aspect and thickness ratios.

In general, it is possible to eliminate 80 to 95% of the degrees of freedom of a super-parametric shell element representation of a shell structure. The actual reduction is dependent on the selection of the primary coordinates and the number of degrees of freedom of the discretized system. In the finite element dynamic analysis of rotating or pre-stressed shell structures, the primary coordinates of the equivalent Static and Dynamic Systems are not the same coordinates.

The super-parametric shell element/modal analysis model for the transient response of general shell structures subjected to arbitrary forcing functions has some limitations. The model is very efficient in the prediction of the transient response of shell blades, but in other shell structures the model can be inefficient. The model is also applicable to pre-
stressed, submerged and rotating shell structures.

The efficiency of the model is dependent on the distribution of the natural frequency spectrum of the shell structure. When the natural frequency spectrum of the shell is compact, the model is not efficient. However the model is very efficient when the natural frequency spectrum is not compact.

In general, the model is not efficient since thin shells tend to have compact distribution of natural frequencies. However, the natural frequency spectrum of shell blades are not compact and, therefore, the model is very efficient.

In general, the finite element transient analysis of shell structures requires a larger discretized system than a free vibration analysis. This increase in degrees of freedom is dependent on the forcing functions and natural frequency distribution of the shell structure, but it is independent of the integration method (Direct Integration or Modal Analysis) of the equations of motion. However, in the particular case of shell blades subjected to simple forcing functions, the transient model does not necessarily need to have more degrees of freedom than a free vibration model.

The variation of the natural frequencies distribution of a shell structure with stress fields modifies the efficiency of the Modal Analysis or Direct Integration methods of transient response. With a positive stress field, the natural frequency spectrum becomes more compact and, therefore, the modal analysis method decreases its efficiency, while the direct integration method increases their efficiency. With negative stress fields, the natural frequency spectrum is less compact and, therefore, the modal analysis method increases its efficiency, while the direct integration methods decrease their efficiency. At very high positive stress fields, the direct integration methods are superior to the modal analysis method.
An increase in the angular velocity of blades transforms the natural frequency spectrum such that it becomes more compact. Thus, the angular velocity decreases the efficiency of the Modal Analysis Method of transient response of structures. At high angular velocities, the natural frequency spectrum of small aspect ratio thin shell blades are very compact and, therefore, a Direct Integration Method is more efficient than the Modal Analysis Method to calculate the transient response of these blades.

The complexity of the forcing functions have an influence on the efficiency of the finite element transient model. With very complex forcing functions, the Direct Integration Methods are superior to the Modal Analysis Method.

The effect of the angular velocity and stress field on the transient response of shell structures is extensively illustrated in this Thesis. The angular velocity and stress field can change significantly the transient response of the structure. The magnitude of the transient response variation is dependent on the distribution of the natural frequency spectrum of the structure and forcing function harmonics.
The development of a Finite Element Model capable of predicting the dynamic characteristics and transient response of shell structures, subjected to temperature fields, is of practical importance. This model can be further developed to the dynamic analysis of rotating shells subjected to thermal stresses.

The development of a Finite Element Model to predict the dynamic characteristics and transient response of shells subjected to aerodynamic forces is also of practical importance. This model can be further developed to predict the aerodynamic instability of shells, the dynamic analysis of rotating shells subjected to aerodynamic forces and the dynamic analysis of shells submerged in a fluid flow.

The development of Non-Linear Finite Element Models for the dynamic analysis of shells is also another possible development of this investigation. Lagrangian and Eulerian Models can be developed. These Models can further be developed for the dynamic analysis of pre-stressed and rotating shells.

The development of a Finite Element Model to predict the dynamic behaviour of turbomachines is of practical importance. The rotating and non-rotating components of the turbomachine can be represented by finite elements. The bearings, which couple the rotating and non-rotating components, can be represented by an equivalent system with stiffness and damping.
APPENDIX A - NUMERICAL INTEGRATION AND CONVERGENCE

In the evaluation of Element Matrices of a numerically integrated element, the following type of equation must be calculated,

\[
\int \int \int [-1 -1 -1] [C(\xi, \eta, \zeta)] d\xi d\eta d\zeta \quad \text{(A.01)}
\]

The most commonly used method of integrating this equation is the Gauss Quadrature. This method is introduced to one-dimensional functions and generalised to equation (A.01). The accuracy and convergence of the integration are discussed.

Consider the integral of a function, which is expanded in a polynomial form,

\[
\int_{-1}^{1} F(\xi) d\xi = \sum_{i=0}^{m} a_i \int_{-1}^{1} \xi^i d\xi = \sum_{i=0,2,4,\ldots,m}^{m} 2a_i/i + 1 \quad \text{(A.02)}
\]

It is assumed that the integral can be calculated by the following expression:

\[
\int_{-1}^{1} F(\xi) d\xi = \sum_{j=0}^{n-1} A_j F(\xi_j) \quad \text{(A.03)}
\]

where \( A_j \) and \( \xi_j \) are the Gaussian Weight and coordinates coefficients and \( n \) the number of Gaussian points.

Consequently, equation (A.02) becomes,

\[
\sum_{i=0,2,4,\ldots,m}^{m} 2a_i/i + 1 = \sum_{j=0}^{m} A_j \sum_{s=0}^{m} a_s \xi_j^s \quad \text{(A.04)}
\]

\[ m = 2n-1. \]
The solutions of this equation to 2 and 3 Gaussian points are,

\[
A_0 = A_1 = 1; \quad \xi_0 = -1/\sqrt{3}; \quad \xi_1 = 1/\sqrt{3} \tag{A.05}
\]

\[
A_0 = A_2 = 5/9; \quad A_1 = 8/9; \quad \xi_0 = -\sqrt{3}/5; \quad \xi_1 = 0; \quad \xi_2 = \sqrt{3}/5
\]

It is important to recognise that the coordinates of an \(n\) point Gauss Quadrature are the roots of the Legendre Polynomials of degree \(n\). Also, an \(n\) point Gauss Quadrature is capable of integrating exactly a polynomial of degree \(2n + 1\).

In the general case, equation (A.01) can be integrated using an \(n \times m \times p\) Gaussian Mesh,

\[
\int \int \int [C(\xi, n, \zeta)] d\xi d\eta d\zeta = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{s=0}^{p-1} A_i A_j A_s [C(\xi_i, n_j, \zeta_s)] \tag{A.06}
\]

Although this equation can be integrated exactly, the Correct Gaussian Mesh is not economic. It is necessary to use a lower order of integration. However, this order of integration must ensure convergence to the correct value.

In the context of solid mechanics, Convergence means that in the limit, as the finite element size decreases, the stresses according to the finite element solution are close to the true stress of the system. Further, as the element sizes are small, the variation in stress within an element or over a few neighbouring elements is also small. Consequently, if an arbitrary patch of elements can accept any state of constant stress or strain, then the convergence is established. This criterion was developed by Irons (201, 1970).

Also consider an Isoparametric element which is subjected to a constant stress field. The stress field induces certain nodal forces which are the only contact with adjacent elements. Even for an approximately
integrated element whose stiffness coefficients are not exact, these forces must be exactly integrated. The inter-element forces, due to internal stresses can be proved to be,

\[
\{QE\} = \int_{\Omega_e} [B_L]^t \{\tau\} d\Omega \tag{A.07}
\]

where \([B_L]\) and \(\{\tau\}\) are the linear strain matrix and stress vector, respectively. These forces must be integrated exactly, consequently, it can be concluded that the following equation,

\[
\left\{ \begin{array}{cc}
1 \\
J \\
-1 \\
\end{array} \right\} \left( \begin{array}{c}
d\xi \\
d\eta \\
d\zeta \\
\end{array} \right) d\xi d\eta d\zeta \tag{A.08}
\]

which evaluates the volume of the element, must be integrated exactly.

While this Minimum Order of Integration is sufficient to ensure that convergence is achieved, as the element size tends to zero, the Bound properties are no longer valid. Also, with only a few elements the Stiffness Matrix can become singular.

In the case of the super-parametric parabolic shell element, the Minimum order of integration of the strain energy is a 2 * 2 * 1 Gaussian mesh. The integrand of the kinetic energy is of higher order than the integrand of the strain energy. Consequently, the Mass Matrix of the element must be evaluated by at least a 3 * 3 * 2 Gaussian mesh.
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