IMPROVED SINGLE FREQUENCY ESTIMATION WITH WIDE ACQUISITION RANGE

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An improved method for estimating the frequency of a single complex sinusoid in complex additive white Gaussian noise is proposed. The method uses a modified version of the weighted linear predictor to achieve optimal accuracy at low/moderate SNR while retaining its speed and wide acquisition range. Consequently, it has an advantage over known methods that use the weighted phase averager since they suffer from an increased threshold effect at frequencies approaching the full estimation range.

Introduction

Frequency estimation of a complex sinusoid in complex additive white Gaussian noise is a common problem in several applications [1] and many techniques have been proposed over the years [2-8]. It is well known that the optimal maximum likelihood (ML) estimator [2] is specified by the location of the peak of a periodogram and it achieves the Cramer-Rao lower bound (CRLB) [2] at low/moderate signal-to-noise ratios (SNR). However, the approach is considered to be too computationally intensive for real-time implementation, even with the use of FFT techniques.

Computationally efficient estimators include that of Kay [3], who proposed the weighted phase averager and a corresponding weighted linear predictor.

Mengali and Morelli (M&M) [4] used a similar approach to Kay's method to derive the weighted autocorrelation phase averager. M&M's method achieves the CRLB at low/moderate SNR while Kay's method achieves the bound at moderate/high SNR.

We present a modification to both methods based on weighted linear prediction. It is shown by theoretical analysis and computer simulations that the proposed method achieves the same accuracy as the earlier methods with similar complexity; but unlike the others, it retains its full estimation range. Also, it does not involve any latency arising from a stepped frequency search.

Consider the received signal to be a noisy exponential signal represented as

$$x_t = Ae^{j(2\pi f_0 t+\theta)} + n_t \qquad t = 0,1,2,\ldots,N-1 \tag{1}$$

where the amplitude *A*, frequency f_0 and phase θ are deterministic but unknown constants and *N* is the number of samples. The value of f_0 is assumed to be in the interval (-1/2,1/2) while noise $n_t = \alpha_t + j\beta_t$ is assumed to be a zero-mean complex white Gaussian process having variance σ^2, where α_t and β_t are real uncorrelated zero-mean Gaussian random variables having variance $\sigma^2/2$.

Frequency Estimation

Kay's estimator [3] was derived by replacing (1) with an approximate model under an assumption that the SNR $\left(A^2/\sigma^2\right)$ is large.

$$x_t \approx Ae^{j\left(2\pi f_0 t+\theta+\tilde{\beta}_t\right)} \qquad t = 0,1,2,\ldots,N-1 \tag{2}$$

where $\tilde{\beta}_t$ is a zero-mean white Gaussian noise [3] with a variance of $\sigma^2/2A^2$.

An observation of (2) shows that the differenced phase data of x_t gives

$$\angle x_t x_{t-1}^* = 2\pi f_0 + \tilde{\beta}_t - \tilde{\beta}_{t-1} \qquad t = 1,2,3,\ldots,N-1 \tag{3}$$

The ML estimate of f_0 based on the linear model of (3) was derived by Kay [3] as a weighted phase averager (WPA) shown as:

$$2\pi \hat{f}_0 = \sum_{t=1}^{N-1} w_t \angle x_t x_{t-1}^* \tag{4}$$

where w_t is a smoothing function given by

$$w_t = \frac{3N}{2(N^2-1)}\left\{1-\left(\frac{2t-N}{N}\right)^2\right\}, \qquad t = 1,2,3,\ldots\ldots\ldots N-1 \tag{5}$$

Note that the sum of w_t over all values of *t* in (5) is equal to 1. This means that the ML weights do not introduce bias into the estimator.

It was also proved in [3] that when $(\tilde{\beta}_t - \tilde{\beta}_{t-1}) << 1$ (i.e. at large enough SNR), a similarly weighted linear predictor (WLP) shown in (6) and formed by interchanging the angle and summation operations in (4) is identical to the WPA estimator of (4).

$$2\pi \hat{f}_0 = \angle \sum_{t=1}^{N-1} w_t x_t x_{t-1}^* \tag{6}$$

However, the WLP estimator is sub-optimal at moderately high SNR in contrast to the WPA [3]. In order to rectify this problem, we propose a more accurate approximation for (1). We re-write (1) as:

$$x_t = Ae^{j(2\pi f_0 t+\theta)} + n_t = Ae^{j(2\pi f_0 t+\theta)}\left[1+\tilde{n}_t\right] \qquad (7)$$

where $\tilde{n}_t \triangleq n_t e^{-j(2\pi f_0 t+\theta)} / A$

Note that $A\tilde{n}_t = A(\tilde{\alpha}_t + j\tilde{\beta}_t)$ has the same statistics with n_t. Consequently, $\tilde{\alpha}_t$ and $\tilde{\beta}_t$ are real uncorrelated zero-mean Gaussian random variables having a variance of $\sigma^2/2A^2$. Equation (7) can be expanded as:

$$x_t = A\sqrt{(1+\tilde{\alpha}_t)^2 + \tilde{\beta}_t^{\,2}}\, e^{j(2\pi f_0 t+\theta)} e^{j\left(\tan^{-1}\{\tilde{\beta}_t/(1+\tilde{\alpha}_t)\}\right)} \qquad (8)$$

A comparison between (8) and (6) shows that at low/moderate SNR, the profile of the optimal ML weights used for the WLP can be significantly distorted by the noisy amplitude of the differenced phase data, thus degrading its performance. This can be easily seen when $w_t\left|x_t x^*_{t-1}\right|$ is simulated/plotted for different SNR values, with A=1. The WPA does not encounter this problem as the amplitude information is already eliminated in (3) before the optimal weighting process in (4) is applied. Therefore in the first proposed method, also referred to as the weighted 'normalized' linear predictor (WNLP), the observed data is normalized by its amplitude as $\tilde{x}_t = x_t / |x_t|$. Under the assumption that the $\tilde{n}_t$<<1 (i.e. at large enough SNR), we have:

$$\tilde{x}_t \approx e^{j\left(2\pi f_0 t+\theta+\tilde{\beta}_t\right)} \qquad (9)$$

It should be noted that (2) and (9) are essentially the same equation since A is a constant. As such, all other derivations made by using (2) in [3] remain valid for (9). The first proposed estimator (WNLP) is thus given as:

$$2\pi \hat{f}_0 = \angle \sum_{t=1}^{N-1} w_t \tilde{x}_t \tilde{x}_{t-1}^* \tag{10}$$

It is known that Kay's estimator exhibits a threshold effect at low SNR [3]. To improve on this, Mengali and Morelli [4] proposed a frequency estimator using the autocorrelation of the received signal given by:

$$R(m) \triangleq \frac{1}{N-m} \sum_{t=m}^{N-1} x_t x_{t-m}^*, \qquad 0 \le m \le M \tag{11}$$

where m is the autocorrelation lag and M is a design parameter.

Substituting (7) into (11), as done in [4], we have the autocorrelation expressed as:

$$R(m) = e^{j2\pi m f_0} \left[1 + \xi(m)\right] \tag{12}$$

where

$$\xi(m) \triangleq \frac{1}{N-m} \sum_{k=m}^{N-1} \left[\tilde{n}_t + \tilde{n}_{t-m} + \tilde{n}_t \tilde{n}^*{}_{t-m} \right] \tag{13}$$

An inspection of (13) shows that $\xi(m)$ is a zero-mean noise term and has a reduced variance when compared to $\tilde{n}_t$. Using the same approach as in (7)-(10) and with the assumption that the SNR>>1, we have

$$R(m) \approx B_m e^{j\left(2\pi m f_0 + \xi_I(m)\right)} \tag{14}$$

$$\angle R(m) R^*(m-1) = 2\pi f_0 + \xi_I(m) + \xi_I(m-1) \qquad 0 \le m \le M \tag{15}$$

where B_m is a noisy amplitude and $\xi_I(m)$ is the imaginary part of $\xi(m)$.

M&M uses a similar approach to Kay's [3] to determine optimal weights w_m for (15) and the resulting estimator is given as:

$$2\pi\hat{f}_0 = \sum_{m=1}^{M} w_m \angle R(m) R^*(m-1) \tag{16}$$

where $M=N/2$ and w_m is the smoothing function shown in [4] as:

$$w(m) \triangleq \frac{3[(N-m)(N-m+1) - M(N-M)]}{M(4M^2 - 6MN + 3N^2 - 1)} \tag{17}$$

Based on the proof provided in [3] that shows the equivalence of the WPA and the WLP, and further analysis in (7)-(10), we propose a second frequency estimator, also referred to as the weighted 'normalized autocorrelation' linear predictor (WNALP) and shown as:

$$2\pi\hat{f}_0 = \angle \sum_{m=1}^{M} w_m \tilde{R}(m)\, \tilde{R}^*(m-1) \tag{18}$$

where $\tilde{R}(m) = R(m)/|R(m)|$ is the autocorrelation normalized by its amplitude.

Unlike our proposed methods (WNLP and WNALP) where the noise has been averaged out by the optimal weights before taking the angle function, there is no noise averaging performed as yet when the angle function is used in Kay's and M&M's methods. This makes them more susceptible to large estimation errors as $|f_0|$ approaches 0.5 since the output of the angle (arg) function is always in the interval $[-\pi, \pi]$.

3. Computer simulations

Computer simulations (10,000 runs) were performed to verify the performance of the proposed methods in comparison to Kay's and M&M's methods. A data record of N=24 was used in all cases. Figure 1 and 2 show the mean square error (MSE) of the frequency estimate. It can be seen that our proposed methods (WNLP and WNALP) attain the CRLB at low/moderate SNR (unlike the WLP) and maintain such performance even when the frequency is increased from 0.05 to 0.45 in contrast to Kay's and M&M's estimator.

4. Conclusions

In this letter, we have proposed two new techniques for single frequency estimation by improving upon the weighted linear predictor suggested by Kay. Results show that the estimators achieve similar accuracy with that of Kay's method and M&M's method respectively while maintaining a full acquisition range of ~0.5 which is not achievable by their counterparts.

References

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Figure/Table captions

Fig. 1 Frequency MSE for the different estimators f_0=0.05, N=24

Fig. 2 Frequency MSE for the different estimators f_0=0.45, N=24

Figure 1

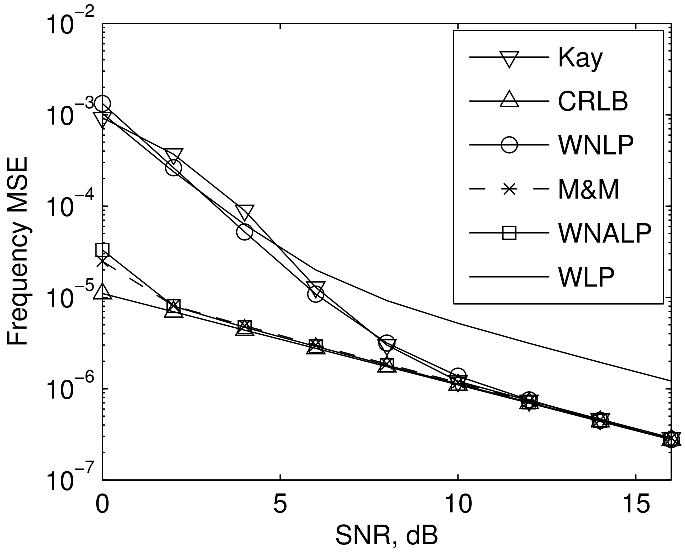
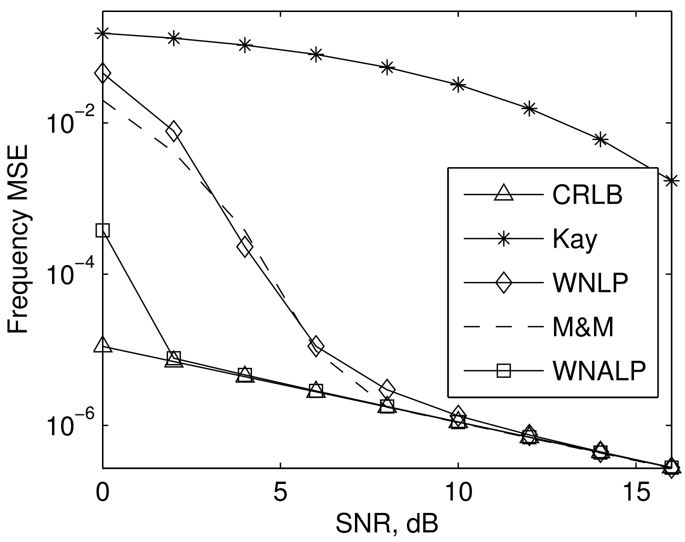

Frequency MSE
SNR, dB
Kay
CRLB
WNLP
M&M
WNALP
WLP
10^−2
10^−3
10^−4
10^−5
10^−6
10^−7
0
5
10
15

Figure 2


Frequency MSE
SNR, dB
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Kay
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WNALP
10^{-2}
10^{-4}
10^{-6}
0
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15