Analysis of Mobile Terminal Diversity Antennas

T. W. C. Brown, S. R. Saunders, B. G. Evans

Abstract
Antenna diversity has for many years been deployed at the base station to provide significant diversity gain. More recently diversity has also been implemented at the mobile and it has been observed through measurement that antennas still give low correlation even when closely spaced. This paper analyses why de-correlation exists mainly due to angular diversity effects from the two antennas and mutual coupling between them. Although there is low correlation there is still often a reduction in efficiency when antennas are close together so this is also investigated in order to see what scope there is for diversity at the mobile. The impact on efficiency will then determine what effect is made on the diversity system as a whole.

1. Introduction
Spatial Diversity at the base station has been considered the most effective form of diversity to use in mobile communications in order to mitigate multipath fading [1] since it is practical to space the antennas out as far as necessary. In recent days, however, there has been increasing use of antenna diversity in mobile terminals and possibly even in smaller indoor base stations. With physical size restrictions, the antennas are required to be closer together and measurements in the past have shown that closely spaced omni-directional antennas (0.1 wavelengths apart) can still give low correlation [2] even though they should theoretically be at least 0.5 wavelengths apart. Given this low correlation there is possibly scope for effective antenna diversity at the mobile terminal.

Although low correlation is seen at the mobile in some cases, it has still not been fully resolved on a theoretical level as to why this is the case. Further to this, it is not straightforward to evaluate mathematically. It is evident that the antenna effects play a part as well as the propagation environment, which will both be investigated. The two key considerations regarding the antennas are the effects of the mutual coupling between them and also how they change the field patterns of each antenna while in the presence of the other. Such antenna interactions show that coupling will change the spatial correlation to a small extent and also the field pattern interactions will cause some angular diversity and possibly polarization diversity.

Even though antenna coupling may be beneficial to diversity in terms of reducing correlation, it may also have disadvantages as far as their respective mean effective gains are concerned. To begin with, the antennas are likely to be dissimilar resulting in their mean effective gains being too different (more than 3dB) giving no significant diversity gain. Also the efficiency of an antenna may reduce considerably while in the presence of another which will have a dramatic effect on the gain in signal to noise ratio at the output. Therefore it is important to consider all these factors to determine whether a second antenna in a mobile terminal will actually improve the output compared to a single antenna.

This paper first outlines the background theory for spatial diversity in an urban fading environment in section II and then it will show how coupling of antennas can affect
the spatial characteristics in section III. A framework for theoretical analysis of angular and polarization diversity of closely spaced antennas is given in section IV and section V. Finally section VI and the remaining sections show how the spatial and angular diversity characteristics are combined using two dipoles as an example and also a mobile handset with a planar and meandered monopole antenna.

2. Correlation between two points in space

Before considering the effects of antennas, it is important to establish how the spatial correlation is evaluated in an urban fading environment at the mobile. This depends on the three-dimensional angle of arrival characteristics and the distance between the two points which will cause a phase delay. It is assumed that, since the antennas are close together, the local mean power levels available at the two branches are the same. Hence, in order to evaluate the spatial diversity gain only correlation has to be considered. Complex correlation can be evaluated using the closed form expression [3]:

\[
\rho_{12} = \frac{\int \left[ XPR.E_{\theta}(\theta, \phi)E_{\phi}(\theta, \phi) + E_{\phi}(\theta, \phi)E_{\theta}(\theta, \phi) \right] \sin \theta \mathrm{d}\theta \mathrm{d}l}{\sqrt{\sigma_1^2 \sigma_2^2}}
\]

where:

\[
\sigma_n^2 = \int_0^{2\pi} \int_0^{2\pi} \left[ XPR.E_{\theta}(\theta, \phi)E_{\phi}(\theta, \phi) + E_{\phi}(\theta, \phi)E_{\theta}(\theta, \phi) \right] \sin \theta \mathrm{d}\theta \mathrm{d}l \mathrm{d}\phi
\]

In an urban Rayleigh fading environment, it is assumed that there are a large number of uniformly distributed scatterers in azimuth which allows the principle found by Clarke [4] to be applied to find the envelope correlation:

\[
\rho_e = \left| \rho_{12} \right|^2
\]

It can be seen from equations (1) and (2) that correlation depends on the cross polar ratio, XPR, the vertical and horizontal E-fields, \( E_{\theta} \) and \( E_{\phi} \), and the angle of arrival (AOA) statistics, \( p(\theta, \phi) \). In the case of spatial diversity only vertical E-fields have to be considered which causes the correlation to become independent of XPR. When evaluating spatial correlation, it has often been the case that only the azimuth angle of arrival has been taken into account, which is assumed to be uniform [5]. In this case equation (1) can be simplified to the following equation as shown by Vaughan and Bach Andersen [6 eq. (35)]:

\[
\rho_{12} = \frac{1}{2\pi} \int_0^{2\pi} e^{i\beta l \cos \phi} \mathrm{d}\phi = J_0(\beta l)
\]

where \( \beta \) is the phase constant and \( J_0(x) \) is a zero order Bessel function. In an urban fading environment, measurements have shown that there is an elevation angle of arrival distribution [5], \( p(\theta, \phi) \), which is usually considered to be Gaussian so it can be applied to mathematical models. Therefore the correlation can be calculated using equation (1) to now give:
\[
\rho_{12} = \int_0^{2\pi} \int_0^\pi p_\theta(\theta, \phi) e^{j\beta \cos \theta} \sin \theta d\phi d\theta
\]  
(5)

From measurements undertaken by Taga [5] it is assumed that the standard deviation of this Gaussian distribution is 20° to 40° and that the mean is around 20° above the horizontal. Figure 1 shows that the correlation versus distance (when the standard deviation is 20° and when it is 40°) has a similar output to that of the zero order Bessel function. Therefore the zero order Bessel function is a good approximation for the correlation between two horizontally separated points in space. As a point to note, the curve will be the same as the zero order Bessel function when the mean and standard deviation are both 0°.

In figure 1, the points could also have a vertical spacing, \( h \), as well as horizontal spacing, \( d \). Hence they could be diagonally spaced. Therefore the trigonometry can be expanded further to include angle \( \theta \) as well as \( \phi \) shown in figure 2. The two points are shown as horizontally separated and also have a vertical separation, \( h \).

![Figure 1 - Graph showing the spatial complex correlation between two points in space due to the three dimensional urban angle of arrival distribution compared with the zero order Bessel function](image)

![Figure 2 - Diagram showing the respective angles to two diagonally spaced points](image)

Using figure 2, the phase difference, \( \Delta \phi \), between the two points, if \( x >> d \) and \( x >> h \), can be derived by simple trigonometry [1], using the phase constant, \( \beta \), as:

\[
\Delta \phi = \beta \sqrt{(d^2 + h^2)} \cos \zeta
\]  
(6)

Where the angle \( \zeta \) is derived in Appendix 1 as:
\[
\cos \zeta = \sin(\theta + \delta \theta \text{sgn } \phi)\sin \phi
\]  
(7)

Where the “\text{sgn}” function refers to the polarity of a value. The change in angle, \(\theta\), given as \(\delta \theta\) is resolved as:

\[
\tan \delta \theta = \frac{h}{d}
\]  
(8)

Note \(\delta \theta \neq 90^\circ\) as explained in Appendix 1, where in this case \(\cos \zeta\) is simply \(\cos \theta\). Therefore this can be applied in the same way as for equations (4) and (5) so that:

\[
\rho_{12} = \int \int e^{j/h}(d^2 + h^2 \cos \zeta) p_\rho(\theta, \phi) \sin \theta d\phi d\theta
\]  
(9)

Equation (9) is plotted in figure 3 with 20° mean and standard deviation to show how increasing \(h\) changes the correlation versus \(d\). It is interesting to note that there are some peak points in correlation greater than when the points have non-zero values for \(d\). Consequently, the zero order Bessel function cannot be applied as an approximation when there is vertical separation. This concludes the spatial diversity available due to the mobile fading environment although coupling effects have not been considered, which are inherent in the practical case.

![Figure 3](image)

**Figure 3** - Graph showing the effects on complex correlation when the value of \(h\) is increased

### 3. Effects of coupling on spatial correlation

As far as spatial correlation is concerned, there are two antenna effects that need to be modeled theoretically. One is that of possible differences in impedance (which may have some mismatch) and the mutual impedance between them when the antennas are in the presence of each other. In both cases, the overall signal at the output of the two antennas will be changed and will have some effect on correlation. If the theory used by Vaughan and Bach Andersen [6] is applied to derive the load voltage \(V_L\) from the output voltage at the antenna, \(V_E\), then it can be derived using a transfer matrix \(S_T\):

\[
V_L = Z_L (Z_A + Z_L)^{-1} V_E = S_T V_E
\]  
(10)

In the case of two diversity antennas the network is two port so the transfer matrix can be resolved when the \(Z\) parameters are converted to \(S\) parameters [12] giving more clarity compared to the equations used by Vaughan and Scott [8]:

\[
\begin{align*}
V_L &= Z_L (Z_A + Z_L)^{-1} V_E \\
&= \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\end{align*}
\]
The transfer matrix clearly outlines the power transfer components, \( s_{fn} \), and the isolation components \( s_r \). It is now possible to compare the equivalent E-field voltages so that:

\[
E_{\theta_1} \equiv V_{E1} = s_{f1}V_{E1} + s_rV_{E2} \quad (12)
\]

\[
E_{\theta_2} \equiv V_{E2} = s_{f2}V_{E2} + s_rV_{E1} \quad (13)
\]

such that:

\[
V_{E1} = r e^{j(\alpha - \beta x)} \quad (14)
\]

\[
V_{E2} = r e^{j\left(\alpha - \beta \left[x + \sqrt{d^2 + h^2} \sin(\theta - \theta_0) \sin \phi\right]\right)} \quad (15)
\]

where \( r \) is a wave signal and \( x \) is a variable phase distance for the carrier wave at time \( t \). The application of this to equation (8) gives a new correlation function when the antenna effects are considered:

\[
\rho_{12} = \frac{s_{f1}s_r^* + s_{f2}s_r + (s_{f1}s_{f2} + |s_{f1}|^2)^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} e^{j\beta \sqrt{d^2 + h^2} \cos \zeta} p_\theta(\theta,\phi) \sin \theta \sin \phi d\theta}{\left( |s_{f1}|^2 + |s_{f2}|^2 \right) + 2\Re \left\{ s_{f1}s_{f2}^* \int_0^{\frac{\pi}{2}} e^{j\beta \sqrt{d^2 + h^2} \cos \zeta} p_\theta(\theta,\phi) \sin \theta \sin \phi d\theta \right\}}
\]

(16)

As can be seen, equation (16) reduces to equation (9) when there is no isolation (i.e. \( s_r = 0 \)) and the input ports are matched so \( s_{f1} = s_{f2} = 0.5 \). It must be noted at this point therefore that both the transmission, \( s_{fn} \), and isolation, \( s_i \), change as the distance between two closely spaced antennas changes. Likewise, \( z_{11}, z_{22} \) and \( z_{12} \) will all change with distance considerably when the antennas are less than 0.5 wavelengths apart. An example is given in figure 4 that shows measurements conducted with two half-wavelength dipoles, which are shortened to remove their self reactance, \( x_{11} \). The mutual impedance, \( z_{12} \), is in agreement with theoretical analysis [7]. As can be seen, \( z_{11} \) (equal to \( z_{22} \)) is by no means constant when \( d \) is less than 0.5 wavelengths so therefore the antenna’s impedance will be changing as well.
4. Angular diversity effects

Angular correlation requires careful measurement or simulation of the two antennas to extract all spatial aspects of the diversity antennas and concentrate only on the angular effects. Figure 5 helps to explain how this is done with two horizontally separated dipoles. First of all, antenna 1 is excited with the antenna feed at the centre of rotation to produce its spherical field pattern. Having measured or simulated antenna 1, antenna 2 can now be excited, while the two antennas are re-positioned so that feed 2 is at the centre of rotation as indicated by the axis in dotted lines. Again the spherical field pattern will be taken which is a mirror image of the first field pattern in this situation. In both cases the non-excited antenna should have a terminated load.

Although the antennas have been measured at their relative phase centers, the effects of possible mismatch at the output ports need to be removed from any measurements undertaken. This can be carried out by taking the division of the antenna and load matrix, $Z_A Z_L^{-1}$, which can be defined as follows if the system is reciprocal (i.e. $z_{12} = z_{21}$):
\[ Z_A Z_L^{-1} = \frac{1}{Z_0} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \]  

(17)

so that:

\[ V_A = Z_A Z_L^{-1} V_L \]  

(18)

Applying this to the complex antenna field amplitude patterns, \( A_{\theta}(\theta, \phi) \) and \( A_{\phi}(\theta, \phi) \), the voltages at the antenna outputs can be derived for correlation from vectors in the two polarizations, \( V_A^{\theta} \) and \( V_A^{\phi} \). It is important to note, however that for each antenna, the other is a parasitic (i.e. it is not driven) so this causes \( z_{12} \) and \( z_{21} \) to be zero. Therefore:

\[ E_{\theta 1} \equiv V_{A\theta 1} = \frac{z_{11}}{Z_0} A_{\theta 1} \]  

(19)

\[ E_{\theta 2} \equiv V_{A\theta 2} = \frac{z_{22}}{Z_0} A_{\theta 2} \]  

(20)

\[ E_{\phi 1} \equiv V_{A\phi 1} = \frac{z_{11}}{Z_0} A_{\phi 1} \]  

(21)

\[ E_{\phi 2} \equiv V_{A\phi 2} = \frac{z_{22}}{Z_0} A_{\phi 2} \]  

(22)

By substituting equations (19) to (22) into equation (1), the resultant angular correlation is as follows:

\[ \rho_{12} = \frac{\int_0^{2\pi} \int_0^{2\pi} \left( XPR A_{\phi 1} A_{\phi 2}^* p_\theta(\theta, \phi) \right) \sin \theta d\phi d\theta}{\int_0^{2\pi} \int_0^{2\pi} \left( XPR |A_{\phi 1}|^2 p_\theta(\theta, \phi) \right) \sin \theta d\phi d\theta} \]

\[ - \frac{\int_0^{2\pi} \int_{-\pi}^{\pi} \left( XPR |A_{\phi 2}|^2 p_\phi(\theta, \phi) \right) \sin \phi d\theta d\phi}{\int_0^{2\pi} \int_{-\pi}^{\pi} \left( XPR |A_{\phi 2}|^2 p_\phi(\theta, \phi) \right) \sin \phi d\theta d\phi} \]

\[ \left| z_{11} \right| \left| z_{22} \right| \]

(23)

Therefore, it can be concluded from equation (23) that the only effects of impedance are that the phase of the complex correlation will change, and not the magnitude. With matched antennas, there will be no change. To obtain the correct angular correlation it needs to be performed in continuous closed integral form as shown above. For practical measurements and simulations, the correlation has to be evaluated in discrete form with a sufficient number of measurement samples. Taking azimuth or elevation cuts with 1° spacing in 10° steps has proved sufficient for mobile terminal antenna applications. It has also been found in certain cases that the complex angular correlation can have an imaginary part, which indicates that there is correlation due to the phase differences between the antennas as well as the differences in magnitude.
As can be seen from equation (23) the angular correlation is now dependent on $XPR$. It will be assumed in this paper that this is 6dB [9] for typical urban environments.

5. Polarization diversity effects
Angular correlation may indicate the level of angular contribution that mobile terminal antennas may be able to provide although it does not give any indication as to what polarizations the antennas have and what polarization contribution there is. Therefore there is a case to resolve the polarization correlation from the relative $E_\theta$ and $E_\phi$ polarizations. The relative polarizations are given in equations (24) to (27) representing each polarization as a proportion of the total field magnitude.

\[
E_{\theta 1} = \frac{|A_{\theta 1}|}{\sqrt{|A_{\theta 1}|^2 + |A_{\phi 1}|^2}} \quad (24)
\]

\[
E_{\theta 2} = \frac{|A_{\theta 2}|}{\sqrt{|A_{\theta 2}|^2 + |A_{\phi 2}|^2}} \quad (25)
\]

\[
E_{\phi 1} = \frac{|A_{\phi 1}|}{\sqrt{|A_{\theta 1}|^2 + |A_{\phi 1}|^2}} \quad (26)
\]

\[
E_{\phi 2} = \frac{|A_{\phi 2}|}{\sqrt{|A_{\theta 2}|^2 + |A_{\phi 2}|^2}} \quad (27)
\]

Therefore the polarization correlation is derived as follows when substituting into equation (1):

\[
\rho_2 = -\frac{\left( \int_0^{2\pi} \int_0^{2\pi} \left( XPR \cdot \frac{|A_{\theta 1}|^2}{\sqrt{|A_{\theta 1}|^2 + |A_{\phi 1}|^2} |A_{\phi 2}|^2} + \frac{|A_{\phi 1}|^2}{\sqrt{|A_{\theta 1}|^2 + |A_{\phi 1}|^2} |A_{\theta 2}|^2} \right) \rho_x(\theta, \phi) \sin \theta d\theta d\phi \int_0^{2\pi} \int_0^{2\pi} \left( XPR \cdot \frac{|A_{\theta 1}|^2}{|A_{\theta 1}|^2 + |A_{\phi 1}|^2} \rho_x(\theta, \phi) \sin \theta d\theta d\phi \right) \times \left( \int_0^{2\pi} \int_0^{2\pi} \left( XPR \cdot \frac{|A_{\phi 1}|^2}{|A_{\phi 1}|^2 + |A_{\phi 2}|^2} \rho_x(\theta, \phi) \sin \theta d\theta d\phi \right) \right) \right)}{\left( \int_0^{2\pi} \int_0^{2\pi} \left( XPR \cdot \frac{|A_{\theta 1}|^2}{|A_{\theta 1}|^2 + |A_{\phi 1}|^2} \rho_x(\theta, \phi) \sin \theta d\theta d\phi \right) \right) \times \left( \int_0^{2\pi} \int_0^{2\pi} \left( XPR \cdot \frac{|A_{\phi 1}|^2}{|A_{\phi 1}|^2 + |A_{\phi 2}|^2} \rho_x(\theta, \phi) \sin \theta d\theta d\phi \right) \right)}
\]

Polarization correlation is therefore inherent within angular correlation and is a measure of the polarization contribution within an angular diversity system. Cross
polarized antennas will show a lower correlation, although not necessarily zero since many horizontal linear polarized antennas have $E_\theta$ as well as $E_\phi$ components.

### 6. Overall correlation

Combining the spatial and angular diversity the overall correlation is shown in equation (24) as if the antenna patterns were correlated normally. Having established the three types of correlation, the spatial and angular correlation can be compared in the case of closely spaced monopole measurements carried out in an anechoic chamber (polarization correlation is always unity here).

$$
\rho_{12} = \frac{\int_0^{2\pi} \int_0^{2\pi} \left[ XPR|A_{\theta 1}|^2 p_x(\theta, \phi) + |A_{\theta 2}|^2 p_x(\theta, \phi) \right] \sin \theta d\theta d\phi}{\int_0^{2\pi} \int_0^{2\pi} \left[ XPR|A_{\theta 1}|^2 p_x(\theta, \phi) + |A_{\theta 2}|^2 p_x(\theta, \phi) \right] \sin \theta d\theta d\phi}
$$

Equation (29)

![Graph showing the spatial, angular and overall correlation for two closely spaced dipoles](image)

**Figure 6** - Graph showing the spatial, angular and overall correlation for two closely spaced dipoles

It is clearly shown in figure 6 that as the dipoles become closer than 0.5 wavelengths, the angular de-correlation is more predominant as would be expected. Beyond 0.5 wavelengths, the spatial diversity takes over. The correlation was evaluated by taking elevation cuts of the antenna patterns at $10^0$ steps to provide sufficient sampling.

### 7. Efficiency of closely spaced antennas

Although correlation may be reduced due to the antennas interacting with each other when close together, the antenna efficiency is also reduced. If the output fading signals were to be compared between a single antenna and two antennas, in many cases the mean power level will therefore be reduced even though the fading on the signal may be also reduced. Consequently the mean signal to noise ratio at the output is not necessarily improved. To find how the efficiency is reduced, the mean effective gain (MEG) of the dipoles need to be evaluated [5]:

$$
MEG = \frac{\int_0^{2\pi} \int_0^{2\pi} XPR G_{\theta}(\theta, \phi) p_x(\theta, \phi) + \frac{1}{1 + XPR} G_{\phi}(\theta, \phi) p_x(\theta, \phi) \sin \theta d\theta d\phi}{\int_0^{2\pi} \int_0^{2\pi} XPR G_{\theta}(\theta, \phi) p_x(\theta, \phi) + \frac{1}{1 + XPR} G_{\phi}(\theta, \phi) p_x(\theta, \phi) \sin \theta d\theta d\phi}
$$

Equation (30)
where $G_{\theta}$ and $G_{\phi}$ are the vertical and horizontal polarization gains respectively. An analysis of the MEG versus spatial distance is shown in figure 7.

![Figure 7](image)

**Figure 7 - Graph showing the mean effective gain versus horizontal separation of two half-wavelength dipoles**

Figure 7 shows that, in the case of two monopoles, the reduction in the mean effective gain, as also investigated by James and Fujimoto [11], reduces when the antennas are closer than 0.5 wavelengths. Beyond 0.5 wavelengths the mean effective gain is the same as that of a single monopole based on the angle of arrival model and cross-polar ratio applied here. Bringing the antennas close therefore reduces the efficiency as well as correlation and so a compromise needs to be made. Figure 7 indicates that for a reduction in mean effective gain of less than 1dB, the antennas must be at least 0.15 wavelengths apart. It is possible that some degree of diversity gain will still be achievable at 0.1 wavelengths since less than 3dB reduction is met there.

8. Analysis of spatial and angular diversity in a mobile handset

To see an example case of mobile diversity in a handset, the following example was taken shown in figures 8 (a) and (b) where a dual band GSM planar inverted-F (PIFA) antenna and a tapered meander monopole based on [13] were measured as appropriate. Different monopole wire lengths were used for the two frequencies measured at being 920MHz and 1800MHz.

![Figure 8](image)

**Figure 8 - Photographs of the handset antenna used to analyze spatial and angular diversity (a) the dual band planar antenna on one side (b) the meander monopole on the other side**

The antennas are spaced $0.23\lambda$ apart at 920MHz and $0.6\lambda$ apart at 1800MHz with differing field patterns that are co-polarized. The return loss for the PIFA at both frequencies is at least 10dB and for the meander it is around 9dB. Therefore some minor impedance effects were present. Coupling between the antennas was -8dB at 920MHz and $-12$dB at 1800MHz.
<table>
<thead>
<tr>
<th>Type of Correlation</th>
<th>( \rho_{12} ) at 920MHz</th>
<th>( \rho_{12} ) at 1800MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>-0.40 - j0.71</td>
<td>-0.29 – j0.75</td>
</tr>
<tr>
<td>Spatial</td>
<td>0.96 + j0.34</td>
<td>0.17 - j0.28</td>
</tr>
<tr>
<td>Angular</td>
<td>-0.69 - j0.40</td>
<td>-0.44 – j0.40</td>
</tr>
<tr>
<td>Polarization</td>
<td>0.98</td>
<td>0.89</td>
</tr>
<tr>
<td>PIFA MEG/dB</td>
<td>-3.54</td>
<td>-4.58</td>
</tr>
<tr>
<td>Meander MEG/dB</td>
<td>-4.59</td>
<td>0.77</td>
</tr>
</tbody>
</table>

The correlation results are presented in figure 9 from three dimensional antenna elevation plots at 10° steps. The mean effective gain of the antennas at their respective frequencies is also given so that the diversity potential of the antennas can be determined. As can be seen, the spatial correlation and polarization correlation are high at 920MHz, so the system is predominantly angular here. Angular correlation is still high, however so the diversity would be ineffective. For 1800MHz there is a lower spatial correlation due to increased distance and differing azimuth patterns as shown in figure 10. Therefore a better overall de-correlation is achieved although there is a significant loss in efficiency such that the mean effective gains have a difference beyond 3dB unlike 920MHz. Therefore neither frequency provides suitable diversity. However, effects of the human head have not been considered here, which could have a significant role in changing these results.

9. Conclusion

A theory to deduce spatial, angular and polarization correlation between two antennas with impedance and coupling effects has been presented. The application of this has been shown in the case of dipoles and a mobile handset with a planar and meandered monopole antenna. Results illustrate how spatial and angular diversity contribute to the overall diversity. Further to this it has been shown that the angular aspects of diversity come at the expense of efficiency loss.
10. Appendix 1 – Derivation of the three dimensional angle

To derive the angle $\zeta$ in section II, a closer look at figure 2 will show the following in figure 11 where the relation of $\zeta$ to $\theta$ and $\phi$ can be seen.

![Diagram showing a closer view of figure 2 analyzing the three dimensional angle](image)

The difference in length, $\Delta l$, can be derived by simple trigonometry as follows:

$$\Delta l = \sqrt{d^2 + h^2} \cos(\theta - (90 - \theta))\cos(90 - \phi)$$

(31)

where $\theta$ is defined in equation (8). This can be simplified and used to derive the angle, $\zeta$, as follows:

$$\cos \zeta = \frac{\Delta l}{\sqrt{d^2 + h^2}} = \sin(\theta + \theta)\sin \phi$$

(32)

This is only valid for positive values of $\phi$ due to laws of Pythagoras’ Theorem. Therefore to account for negative values of $\phi$ the function is better defined as:

$$\cos \zeta = \sin(\theta + \theta \text{sgn} \phi)\sin \phi$$

(33)

Where $\text{sgn} \phi$ returns $-1$ for negative values and $+1$ for values greater than or equal to zero. However, when $\theta = 90^\circ$, there is an exception to the rule and $\cos \zeta$ is $\cos \theta$.

References


