Liberal Egalitarianism and the Harm Principle*

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September 2, 2010

Abstract

This paper provides a novel analysis of liberal egalitarian principles stemming from John Rawls’ seminal work, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise Rawlsian egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed by Mariotti and Veneziani [17]. This is quite surprising, because the Harm principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the difference principle and of its lexicographic refinement are derived, including in the intergenerational context with an infinite number of agents.

JEL classification: D63 (Equity, Justice, Inequality, and Other Normative Criteria and Measurement); D70 (Analysis of Collective Decision-Making); Q01 (Sustainable development).

Keywords: Difference principle, leximin, harm principle, infinite utility streams.

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*We are grateful to José-Carlos Alcantud, Geir Asheim, Nick Baigent, Kaushik Basu, Andrés Carvajal, Bhaskar Dutta, Marc Fleurbaey, Peter Hammond, Koichi Tadenuma, Naoki Yoshihara, and audiences at the University of Warwick (CRETA), the London School of Economics, the University of Maastricht, K.U. Leuven, Hitotsubashi University (Kunitachi), Waseda University (Tokyo), the University of Massachusetts (Amherst), the Midwest Political Science Association conference (Chicago), the New Directions in Welfare Conference (Oxford), the Royal Economic Society Conference (Surrey), the Logic, Game Theory and Social Choice conference (Tsukuba) and the Social Choice and Welfare Conference (Moscow) for useful comments and suggestions. Special thanks go to François Maniquet and Marco Mariotti, whose comments have led to substantial improvements in the paper. The usual disclaimer applies.

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1 Introduction

Almost four decades after its publication, *A Theory of Justice* ([19]) maintains a prominent role in political philosophy, economics, and social choice. As Konow ([14], p. 1195) has recently noted, “the authors of nearly every subsequent normative treatment of justice have felt obliged to formulate their theories within Rawls’s framework, or at least to define their positions with reference to his contribution”. Among the most influential contributions of the book is the difference principle contained in Rawls’s second principle of justice, according to which inequalities should be allowed only insofar as they benefit the worst-off members of society. Both the difference principle, formally captured by the well-known maximin social welfare relation (henceforth, swr), and especially its lexicographic extension, the leximin swr, have generated a vast literature across disciplinary borders.

The Rawlsian difference principle and its extension are usually considered to have a strong egalitarian bias and are taken to represent the main alternative to libertarian (and utilitarian) approaches (see, e.g., [21], [24], [29]). Indeed, they play a prominent role in many different egalitarian theories, including in the modern theory of equality of opportunity (see, among the many recent contributions, [22], [23], [10]), in the normative analysis of international relations, intergenerational issues, and climate change (see, e.g., [16], [24], [25]), and in experimental approaches on egalitarian notions of fairness (for a survey, see [14]).

The classic characterisation of leximin, in fact, is due to Hammond ([11]) and it requires an axiom (the so-called Hammond Equity axiom) with a marked egalitarian content: in a welfaristic framework, Hammond Equity asserts that if \( x_i < y_i < y_j < x_j \) for two utility profiles \( x \) and \( y \), for which \( x_h = y_h \) for all agents \( h \neq i, j \), then \( y \) should be (weakly) socially preferred to \( x \). In a recent contribution, however, Mariotti and Veneziani ([17]) show that the leximin can be characterised using an axiom - the Harm Principle - that incorporates a liberal view of noninterference, without any explicit egalitarian content. This result is surprising and it raises a number of interesting issues for liberal approaches emphasising notions of individual autonomy or freedom in political philosophy and social choice, but it also sheds new light on the normative foundations of standard egalitarian principles.

This paper extends the analysis of the implications of liberal views of noninterference, as expressed in the Harm Principle, and it generalises ([17]) in a number of directions. Formally, it is shown that a weaker version of the Harm Principle, together with standard axioms in social choice, provides a unified axiomatic framework to analyse a set of swrs originating from the difference principle in a welfaristic framework. Theoretically, the analysis provides a novel statement, based on liberal principles, of the ethical intuitions behind a family of normative principles stemming from Rawls’s seminal work. On the one hand, the Harm Principle is formally different from standard informational invariance or separability axioms (see, e.g., [9], [21], [27]) and, unlike the latter, it has a clear normative content. On the other hand, unlike the Hammond Equity ax-
ion, the Harm principle does not incorporate an egalitarian intuition. Therefore, quite surprisingly, the ethical foundations of two SWRs traditionally considered as rather egalitarian - the difference principle and its lexicographic extension - rest only on the two standard axioms of Anonymity and Pareto efficiency, and on a liberal principle incorporating a noninterfering view. No axiom with a clear egalitarian content is necessary, and indeed the analysis in this paper provides a new meaning to the label ‘liberal egalitarianism’ usually associated with Rawls’s approach. Actually, this paper sheds new light on the importance of the notion of justice as impartiality incorporated in the Anonymity axiom in egalitarian approaches. This is particularly clear in societies with a finite number of agents: the Harm principle and the Pareto principle are consistent with some of the least egalitarian SWRs (e.g. the lexicographic dictatorships), and the Anonymity axiom plays a pivotal role in determining the egalitarian outcome. Our analysis also raises some interesting issues concerning the implications of liberal approaches emphasising a notion of individual autonomy, or freedom: if one endorses some standard axioms - such as Anonymity and the Pareto principle - the adoption of an arguably weak liberal view of noninterference leads straight to welfare egalitarianism. As noted by Mariotti and Veneziani ([17]), liberal noninterference implies equality, an insight that is proved to be robust in this paper.

To be specific, first of all, in economies with a finite number of agents, it is shown that a weaker version of the Harm Principle analysed in ([17]) is sufficient to characterise the lexicin social welfare ordering (henceforth, swo). This result is interesting because the weak Harm Principle captures liberal, noninterfering views even more clearly than the original Harm Principle. Further, based on the weak Harm Principle, a new characterisation of the maximin swo is provided.

Second, this paper analyses the maximin and the lexicin in the context of societies with an infinite number of agents. This is arguably a crucial task for egalitarians. In fact, despite Rawls’s claims to the contrary, there is no compelling reason to restrict the application of the difference principle to intragenerational justice. In the intergenerational context, a basic concern for impartiality arguably implies that principles of justice be applied to all present and future generations. The extension to the case with an infinite number of generations, however, is problematic for all of the main approaches, and indeed impossibility results easily obtain (see [15], [31], [13], [1]). A number of recent contributions have provided characterisation results for SWRs by dropping either completeness (see [5], [3], [6], [8], [4]) or transitivity (see [26]). But the definition of suitable anonymous and paretian SWRS is still an open question in the infinite context (for a thorough discussion, see [1]).

In this tradition, this paper provides various new characterisations of the maximin and the lexicin SWRS, based on the Harm Principle, in economies with an infinite number of agents. Although various formal frameworks and definitions have been proposed to analyse infinitely-lived societies, it is shown that the Harm Principle can be used to derive interesting results in all of the main approaches.

The rest of the paper is structured as follows. Section 2 lays out the ba-
sic framework. Section 3 characterises the leximin and the maximin swos in economies with a finite number of agents. Section 4 provides several characterisations of leximin and maximin swrs in societies with an infinite number of agents, in different frameworks. Section 5 concludes.

2 The framework

Let $X \equiv \mathbb{R}^N$ be the set of countably infinite utility streams, where $\mathbb{R}$ is the set of real numbers and $N$ is the set of natural numbers. An element of $X$ is $1u = (u_1, u_2, ...)$ and $u_t$ is the welfare level of agent $t$, or in the intergenerational context - of a representative member of generation $t \in \mathbb{N}$. For $T \in \mathbb{N}$, $1u_T = (u_1, ..., u_T)$ denotes the $T$-head of $1u$ and $T+1u = (u_{T+1}, u_{T+2}, ...)$ denotes its $T$-tail, so that $1u = (1u_T, T+1u)$; $1u_T$ denotes the welfare level of the worst-off generation of $1u$, and $\min(1u) = \min \{u_1, u_2, ...\}$ denotes the welfare level of the worst-off generation of $1u$ whenever it exists. $\epsilon$ denotes the stream of constant level of well-being equal to $\epsilon \in \mathbb{R}$, and, for the sake of notational simplicity, the $T$-head of $\epsilon$ is denoted as $\epsilon_T$ and the $T$-tail of $\epsilon$ is denoted as $\epsilon_T$. A permutation $\pi$ is a bijective mapping of $\mathbb{N}$ onto itself. A permutation $\pi$ of $\mathbb{N}$ is finite if there is $T \in \mathbb{N}$ such that $\pi(t) = t$, $\forall t > T$, and $\Pi$ is the set of all finite permutations of $\mathbb{N}$. For any $1u \in X$ and any $\pi \in \Pi$, let $\pi(1u) = (u_{\pi(t)})_{t\in\mathbb{N}}$ be a permutation of $1u$. For any $T \in \mathbb{N}$ and $1u \in X$, $1u_T$ is a permutation of $1u_T$ such that the components are ranked in ascending order.

For any $1u, v \in X$, $1u \geq v$ if and only if $u_t \geq v_t$, $\forall t \in \mathbb{N}$; $1u > v$ if and only if $1u \geq v$ and $1u \neq v$; and $1u \gg v$ if and only if $u_t > v_t$, $\forall t \in \mathbb{N}$.

Let $\succ$ be a (binary) relation over $X$. For any $1u, v \in X$, $1u \succ v$ stands for $(1u, v) \in \succ$ and $1u \not\succ v$ for $(1u, v) \not\in \succ$; $\succ$ stands for “at least as good as”. The asymmetric factor $\succ$ of $\succ$ is defined by $1u \succ v$ if and only if $1u > v$ and $v \not\succ 1u$, and the symmetric part $\sim$ of $\succ$ is defined by $1u \sim v$ if and only if $1u \succ v$ and $v \succ 1u$. They stand, respectively, for “strictly better than” and “indifferent to”. A relation $\succ$ on $X$ is said to be: reflexive if, for any $1u \in X$, $1u \succ 1u$; complete if, for any $1u, v \in X$, $1u \neq v$ implies $1u \succ v$ or $v \succ 1u$; transitive if, for any $1u, v, w \in X$, $1u \succ v \succ w$ implies $1u \succ w$. $\succ$ is a quasi-ordering if it is reflexive and transitive, while $\succ$ is an ordering if it is a complete quasi-ordering. Let $\succ$ and $\succ'$ be relations on $X$: $\succ'$ is an extension of $\succ$ if $\succ \subseteq \succ'$ and $\succ' \subseteq \succ$.

If there are only a finite set $\{1, ..., T\} = N \subset \mathbb{N}$ of agents, or generations, $X_T$ denotes the set of utility streams of $X$ truncated at $T = |N|$, where $|N|$ is the cardinality of $N$. In order to simplify the notation, in economies with a finite number of agents the symbol $u$ is used instead of $1u_T$. With the obvious adaptations, the notation spelled out above is carried over utility streams in $X_T$. 


3 Egalitarian Principles in Societies with a Finite Number of Agents

This section analyses liberal egalitarianism in societies with a finite number of agents. First, the characterisation of the leximin swo derived by Mariotti and Veneziani ([17], Theorem 1, p. 126) is strengthened by weakening the main axiom incorporating a liberal view of noninterference, the Harm Principle. Then, based on the weak Harm Principle, a novel characterisation of Rawls’s difference principle, as formalised in the maximin swo, is provided.

3.1 The Leximin

According to the leximin, that society is best which lexicographically maximises the welfare of its worst-off members. Formally, the leximin relation \( \succ_{LM} \) on \( X_T \) is defined as follows. The asymmetric factor \( \succ_{LM} \) of \( \succ_{LM} \) is defined by:

\[
u \succ_{LM} v \iff \bar{u}_i > \bar{v}_i \text{ or } \exists i \in N \setminus \{1\} : \bar{u}_j = \bar{v}_j \left( \forall j \in N : j < i \right) \text{ and } \bar{u}_i > \bar{v}_i.\]

The symmetric factor \( \sim_{LM} \) of \( \succ_{LM} \) is defined by:

\[
u \sim_{LM} v \iff \bar{u}_i = \bar{v}_i, \forall i \in N.\]

\( \succ_{LM} \) is easily shown to be an ordering. Classic analyses of the leximin swo typically involve the following three axioms (see [11]).

**Strong Pareto Optimality, SPO:** \( \forall u, v \in X_T : u \succ v \iff u \succ v. \)

**Anonymity, A:** \( \forall u \in X_T \) and \( \forall \pi \text{ of } N, u \sim \pi(u). \)

**Hammond Equity, HE:** \( \forall u, v \in X_T : u_i < v_i < v_j \left( \forall j \in N \right) \exists i, j \in N, u_k = v_k \forall k \in N \setminus \{i, j\} \Rightarrow v \succ u. \)

The first two axioms are standard in social choice theory and need no further comment. It is important to note, instead, that HE expresses a clear concern for equality in welfare distributions, for it asserts that among any two welfare allocations which are not Pareto-ranked and differ only in two components, society should prefer the more egalitarian one. The classic characterisation by Hammond ([11]) states that a swo is the leximin ordering if and only if it satisfies SPO, A, and HE.\(^1\)

In a recent contribution, Mariotti and Veneziani ([17]) drop HE and introduce a new axiom, called the Harm Principle (HP), which is meant to capture a liberal view of noninterference whenever individual choices have no effect on others. To be precise, starting from two welfare allocations \( u \) and \( v \) such that \( u \) is socially preferred to \( v \), consider two different allocations \( u' \) and \( v' \) such that agent \( i \) is worse off at these than at the corresponding starting allocations, the other agents are equally well off, and agent \( i \) prefers \( u' \) to \( v' \). The decrease

\(^1\)See also the related Hammond ([12]) and the generalisation by Tungodden ([28], [29]).
in agent $i$’s welfare may be due to her negligence or her bad luck, but in any case HP states that society’s preference over $u'$ and $v'$ should coincide with $i$’s preferences. In this sense, HP requires that, having already suffered a welfare loss in both allocations, agent $i$ should not be punished in the swo by changing social preferences against her. This seems a rather intuitive way of capturing a liberal view of noninterference, and the name of the axiom is meant to echo John Stuart Mill’s famous formulation in his essay *On Liberty* (see [30], and the discussion in [17] and [18]). Yet, although it has no explicit egalitarian content, quite surprisingly, Mariotti and Veneziani ([17], Theorem 1, p. 126) prove that, jointly with SPO and A, HP characterises the leximin swo.

In this paper, the implications of liberal, noninterfering views in social choice are explored further. As a first step, a weaker version of HP is formally stated as follows.

**Weak Harm Principle, WHP:** $\forall u, v, u', v' \in X_T : u \succ v$ and $u', v'$ are such that, $\exists i \in N$,

\[
\begin{align*}
& u'_i < u_i \\
& v'_i < v_i \\
& u'_j = u_j \forall j \neq i \\
& v'_j = v_j \forall j \neq i
\end{align*}
\]

implies $v' \neq u'$ whenever $u'_i > v'_i$.

WHP weakens the axiom proposed by Mariotti and Veneziani ([17]) in that it does not require that society’s preferences over $u'$ and $v'$ be identical with agent $i$’s, but only that society should *not reverse* the strict preference between $u$ and $v$ to a strict preference for $v'$ over $u'$ (possibly except when $i$ prefers otherwise). In this sense, the liberal content of WHP, and the requirement that agent $i$ should not be punished in the swo by changing social preferences against her, is even clearer, and WHP strongly emphasises the negative prescription of the Harm Principle ensuring individual protection from unjustified interference. The surprising characterisation result provided in ([17]) can then be strengthened.

**Theorem 1.** A swo $\succ$ on $X_T$ is the leximin ordering if and only if it satisfies Anonymity (A), Strong Pareto Optimality (SPO), and the Weak Harm Principle (WHP).

**Proof.** ($\Rightarrow$) Let $\succ$ on $X_T$ be the leximin ordering, i.e., $\succ \rightarrow \succ^{LM}$. Since WHP is weaker than HP, the proof that $\succ^{LM}$ on $X_T$ meets SPO, A, and WHP follows from the proof of necessity in ([17], Theorem 1, p. 126).

($\Leftarrow$) Let $\succ$ on $X_T$ be a swo satisfying SPO, A, and WHP. We show that $\succ$ on $X_T$ is the leximin swo. Thus, we should prove that, $\forall u, v \in X_T$,

\[
\begin{align*}
& u \sim^{LM} v \iff u \sim v \tag{1} \\
& u \succ^{LM} v \iff u \succ v \tag{2}
\end{align*}
\]
First, we prove the implication \((\Rightarrow)\) of (1). If \(u \sim^{LM} v\), then \(\bar{u} = \bar{v}\), and so \(u \sim v\), by A.

Next, we prove the implication \((\Rightarrow)\) of (2). Suppose that \(u >^{LM} v\), and so, by definition \(\bar{u}_1 > \bar{v}_1\) or \(\exists t \in \{2, \ldots, T\}\) such that \(\bar{u}_s = \bar{v}_s\ \forall 1 \leq s < t\) and \(\bar{u}_t > \bar{v}_t\). Suppose, by contradiction, that \(v > u\). Note that since \(\succ\) satisfies A, in what follows we can focus, without loss of generality, either on \(u\) and \(v\), or on the ranked vectors \(\bar{u}\) and \(\bar{v}\). Therefore, suppose \(\bar{v} > \bar{u}\). As SPO holds it must be the case that \(\bar{v}_l > \bar{u}_l\) for some \(l > t\). Let

\[
    k = \min\{t < l \leq T|\bar{v}_l > \bar{u}_l\}
\]

By A, let \(v_i = \bar{v}_k\) and let \(u_i = \bar{u}_{k-g}\), for some \(1 \leq g < k\), where \(\bar{u}_{k-g} > \bar{v}_{k-g}\). Then, let two real numbers \(d_1, d_2 > 0\), and consider vectors \(u', v'\) and the corresponding ranked vectors \(\bar{u}', \bar{v}'\) formed from \(\bar{u}, \bar{v}\) as follows: first, \(\bar{u}_{k-g}\) is lowered to \(\bar{u}_{k-g} - d_1\) such that \(\bar{u}_{k-g} - d_1 > \bar{v}_{k-g}\); next, \(\bar{v}_k\) is lowered to \(\bar{v}_k - d_2\) such that \(\bar{u}_k > \bar{v}_k - d_2 > \bar{u}_{k-g} - d_1\); finally, all other entries of \(\bar{u}\) and \(\bar{v}\) are unchanged. By construction \(\bar{u}'_j \geq \bar{v}'_j\) for all \(j \leq k\), with \(\bar{u}'_{k-g} > \bar{v}'_{k-g}\), whereas WHP, combined with A, implies \(v' \succ u'\). By SPO, \(d_1, d_2 > 0\) can be chosen so that \(\bar{v}' > \bar{u}'\), without loss of generality. Consider two cases:

a) Suppose that \(\bar{u}_k > \bar{u}_l\), but \(\bar{u}_l \geq \bar{v}_l\) for all \(l > k\). It follows that \(\bar{u}' \succ \bar{v}'\), and so SPO implies that \(u' \succ v'\), a contradiction.

b) Suppose that \(\bar{v}_l > \bar{u}_l\) for some \(l > k\). Note that by construction \(\bar{v}'_l = \bar{v}_l\) and \(\bar{u}'_l = \bar{u}_l\) for all \(l > k\). Then, let

\[
    k' = \min\{k < l \leq T|\bar{v}'_l > \bar{u}'_l\}
\]

where \(k' > k\). The above argument can be applied to \(u', v'\) to derive vectors \(\bar{u}'', \bar{v}''\) such that \(\bar{u}''_j \geq \bar{v}''_j\) for all \(j \leq k'\), whereas WHP, combined with A and SPO, implies \(v'' \succ u''\). And so on. After a finite number of iterations \(s\), two vectors \(\bar{u}^s, \bar{v}^s\) can be derived such that, by WHP, combined with A and SPO, we have that \(\bar{v}^s \succ \bar{u}^s\), but \(\bar{u}^s > \bar{v}^s\) so that SPO implies \(\bar{u}^s \succ \bar{v}^s\), yielding a contradiction.

We have proved that if \(u \succ^{LM} v\) then \(u \succeq v\). Suppose now, by contradiction, that \(v \sim u\), or equivalently \(\bar{v} \sim \bar{u}\). Since, by our supposition, \(\bar{v}_t < \bar{u}_t\), there exists \(\epsilon > 0\) such that \(\bar{v}_t < \bar{u}_t - \epsilon < \bar{u}_t\). Let \(\bar{u}^\epsilon \in X_T\) be a vector such that \(\bar{u}^\epsilon_t = \bar{u}_t - \epsilon\) and \(\bar{u}^\epsilon_j = \bar{u}_j\) for all \(j \neq t\). It follows that \(\bar{u}^\epsilon \succ^{LM} \bar{v}\) but \(\bar{v} \succ \bar{u}^\epsilon\) by SPO and the transitivity of \(\succeq\). Hence, the above argument can be applied to \(\bar{v}\) and \(\bar{u}^\epsilon\), yielding the desired contradiction. 

The properties in Theorem 1 are clearly independent.

Theorem 1 has a number of interesting theoretical implications. Firstly, it implies that HE and WHP are equivalent in the presence of A and SPO, even though they are logically independent. However, it can be proved that if \(N = \{1, 2\}\), then in the presence of SPO, HE implies WHP, but the converse is not true (see [18]). This implies that the above characterisation is far from trivial, given that under SPO, HE is actually stronger than WHP, at least in some cases. Secondly, and perhaps more interestingly, Theorem 1 puts the
normative foundations of leximin under a rather different light. For, unlike in standard results, the egalitarian swo is characterised without appealing to any axioms with a clear egalitarian content. Actually, it is easily shown that SPO and WHP alone are compatible with some of the least egalitarian swos, namely the lexicographic dictatorships, which proves that WHP imposes no significant egalitarian restriction. As a result, Theorem 1 highlights the normative strength of the Anonymity axiom in determining the egalitarian outcome, an important insight which is not obvious in standard characterisations based on HE.

The main implication of Theorem 1, however, is that it proves that the core intuition of Mariotti and Veneziani ([17]) concerning the implications of liberal noninterfering views is robust: a strongly egalitarian swo can be characterised with an even weaker axiom that only incorporates a liberal view of noninterference. In the next sections, this intuition is extended further and it is shown that the counterintuitive implications of liberal noninterfering principles in terms of egalitarian orderings are quite general and robust. Analogous characterisations of a whole family of principles inspired by Rawls’s theory are obtained in societies with both finite and infinite populations, based on the Harm Principle.

3.2 The Difference Principle

The maximin relation $\succeq^M = \succ^M \cup \sim^M$ on $X_T$ is defined as follows. The asymmetric factor $\succ^M$ of $\succeq^M$ is defined by:

$$u \succ^M v \iff \tilde{u}_1 > \tilde{v}_1.$$  

The symmetric factor $\sim^M$ of $\succeq^M$ is defined by:

$$u \sim^M v \iff \tilde{u}_1 = \tilde{v}_1.$$  

$\succeq^M$ is easily shown to be an ordering. The maximin swo formalises Rawls’s difference principle. As is well-known, the maximin does not satisfy SPO, and therefore the following, weaker axiom is imposed.

**Weak Pareto Optimality, WPO:** $\forall u, v \in X_T : u \succeq v \Rightarrow u \succ v$.

Second, a continuity axiom is imposed, which represents a standard inter-profile condition requiring the swo to vary continuously with changes in utility streams. This axiom is common in characterisations of the maximin swo (see, e.g., [7]).

**Continuity, C:** $\forall u \in X_T, \{v \in X_T | v \succ u\}$ is closed and $\{v \in X_T | u \succ v\}$ is closed.

The next Theorem shows that the combination of Anonymity (A), Weak Pareto Optimality (WPO), Continuity (C), and the Weak Harm Principle (WHP) characterises the maximin swo.

**Theorem 2.** A swo $\succeq$ on $X_T$ is the maximin ordering if and only if it satisfies Anonymity (A), Weak Pareto Optimality (WPO), Continuity (C), and the Weak Harm Principle (WHP).
Proof. ($\Rightarrow$) Let $\succ$ on $X_T$ be the maximin ordering, i.e., $\succ = \succ^M$. It can be easily verified that $\succ^M$ on $X_T$ satisfies WPO, A, C, and WHP.

($\Leftarrow$) Let $\succ$ on $X_T$ be a swo satisfying A, WPO, WHP, and C. We show that $\succ$ is the maximin swo. We prove that, $\forall u, v \in X_T,\n\n u \succ^M v \Leftrightarrow u \succ v \tag{3}$

and

$u \sim^M v \Leftrightarrow u \sim v. \tag{4}$

Note that as $\succ$ on $X_T$ satisfies A, in what follows we can focus either on $u$ and $v$, or on the ranked vectors $\bar{u}$ and $\bar{v}$, without loss of generality.

First, we show that the implication ($\Rightarrow$) of (3) is satisfied. Take any $u, v \in X_T$. Suppose that $u \succ^M v \Leftrightarrow \bar{u}_1 > \bar{v}_1$ and assume, by contradiction, that $v \succ u$, or equivalently, $\bar{v} \succ \bar{u}$. As WPO holds, $\bar{v}_j \geq \bar{u}_j$ for some $j \in N$, otherwise a contradiction immediately obtains. We proceed according to the following steps.

Step 1. Let

$$k = \min \{l \in N| \bar{v}_l \geq \bar{u}_l\}.$$ 

By A, let $v_i = \bar{v}_k$ and let $u_i = \bar{u}_1$. Then, consider two real numbers $d_1, d_2 > 0$, and two vectors $u', v' -$ together with the corresponding ranked vectors $\bar{u}', \bar{v}' \in X_T$ - formed from $\bar{u}, \bar{v}$ as follows: $\bar{u}_1$ is lowered to $\bar{u}_1 - d_1 > \bar{v}_1$; $\bar{v}_k$ is lowered to $\bar{v}_k > \bar{v}_k - d_2 > \bar{u}_1 - d_1$; and all other entries of $\bar{u}$ and $\bar{v}$ are unchanged. By construction $\bar{u}'_j > \bar{v}'_j$ for all $j \leq k$, whereas by WHP and A, we have $\bar{v}' \succ \bar{u}'$.

Step 2. Let

$$0 < \epsilon < \min\{\bar{u}'_j - \bar{v}'_j| \forall j \leq k\}$$

and define $\bar{v}' = \bar{v}' + \epsilon \cdot \bar{u}'$. By construction, $\bar{v}' \ll \bar{v}'$, and $\bar{v}'_j < \bar{u}'_j$ for all $j \leq k$. WPO implies $\bar{v}' \succ \bar{v}'$. As $\bar{v}' \succ \bar{u}'$, by step 1, the transitivity of $\succ$ implies $\bar{v}' \succ \bar{u}'$.

If $\bar{u}'_j > \bar{v}'_j$ for all $j \in N$, WPO implies $\bar{u}' \succ \bar{v}'$, a contradiction. Otherwise, let $\bar{v}'_l \geq \bar{u}'_l$ for some $l > k$. Then, let

$$k' = \min \{l \in N| \bar{v}'_l \geq \bar{u}'_l\}$$

where $k' > k$.

The above steps 1-2 can be applied to $\bar{u}', \bar{v}'$ to derive vectors $\bar{u}'', \bar{v}''$ such that $\bar{u}''_j > \bar{v}''_j$ for all $j \leq k'$, whereas $\bar{v}'' \succ \bar{u}''$. By WPO, a contradiction is obtained whenever $\bar{u}''_j > \bar{v}''_j$ for all $j \in N$. Otherwise, let $\bar{v}''_l \geq \bar{u}''_l$ for some $l > k'$. And so on. After a finite number $s$ of iterations, two vectors $\bar{u}'', \bar{v}''$ can be derived such that $\bar{v}'' \succ \bar{u}''$, by steps 1-2, but $\bar{u}'' \succ \bar{v}''$, by WPO, a contradiction. Therefore, it must be $\bar{u} \succ \bar{v}$ whenever $\bar{u} \succ^M \bar{v}$. We have to rule out the possibility that $\bar{u} \sim \bar{v}$. We proceed by contradiction. Suppose that $\bar{u} \sim \bar{v}$. Since $\bar{v}_1 < \bar{u}_1$, there exists $\epsilon > 0$ such that $\bar{v}' = \bar{v} + \epsilon \cdot \bar{u}'$ and $\bar{v}'_1 < \bar{u}_1$ so that $\bar{u} \succ^M \bar{v}'$. However, by WPO and transitivity of $\succ$ it follows that $\bar{v}' \succ \bar{u}$. Then the above reasoning can be applied to $\bar{v}'$ and $\bar{u}$ to obtain the desired contradiction.
Now, we show that the implication (\(\Rightarrow\)) of (4) is met as well. Suppose that \(\bar{u} \sim_{M} \bar{v} \iff \bar{u}_1 = \bar{v}_1\). Assume, to the contrary, that \(\bar{u} \not\sim \bar{v}\). Without loss of generality, let \(\bar{u} \succ \bar{v}\). By A, it must be \(\bar{u} \not\succeq \bar{v}\). As \(\bar{u} \succ \bar{v}\), it follows from C that there exist neighbourhoods \(S(\bar{u})\) and \(S(\bar{v})\) of \(\bar{u}\) and \(\bar{v}\) such that \(u' \succ v'\) for all \(u' \in S(\bar{u})\) and for all \(v' \in S(\bar{v})\). Then, there exists \(v' \in S(\bar{v})\) such that \(v' \gg \bar{v}\) and \(\bar{u} \succ v' \sim \bar{v}'\), so that \(\bar{u} \succ \bar{v}'\) but \(\bar{v}' \succ_{M} \bar{u}\). By the implication (3) proved above, it follows that \(\bar{v}' \succ \bar{u}\), a contradiction.

The properties in Theorem 2 are clearly independent.\(^2\)

The theoretical relevance of the latter result can be appreciated if Theorem 2 is compared with alternative characterisations. Unlike informational invariance axioms often used in the literature (see, for example, [20], [21]), WHP has a clear ethical foundation, but, as noted above, no obvious egalitarian implication. In a recent contribution, Bosmans and Ooghe ([7]) characterise the maximin SWO using Anonymity (A), Weak Pareto Optimality (WPO), Continuity (C), and Hammond Equity (HE). Instead, as in the case of the lexicmin ordering analysed above, Theorem 2 characterises the maximin without appealing to an axiom like HE, which arguably has a marked egalitarian content, and using instead WHP, which only incorporates a liberal, noninterfering view of society.

4 Egalitarian Principles in the Infinitely-Lived Society

In this section, the axiomatic analysis of the difference principle and of its lexicographic refinement is extended to infinitely-lived economies, focusing on the role of liberal views of noninterference as formulated in the Harm Principle. As already noted, the case with an infinite number of agents raises a number of issues concerning the existence and the characterisation of SWOS, and different definitions of the lexicmin SWR can be provided in order to compare (countably) infinite utility streams. In this section, first, the framework proposed by Asheim and Tungodden ([3]) is adopted, and an alternative characterisation of their lexicmin SWRS is provided. Then, a new characterisation of an infinite-horizon ordering extension of a lexicmin SWR recently proposed by Bossert, Sprumont and Suzumura ([8]) is provided. Finally, the framework of Asheim and Tungodden ([3]) is extended to analyse the maximin SWR, and a new characterisation of the difference principle is proposed in the context of infinitely-lived economies.

4.1 The Leximin SWR

Following Asheim and Tungodden, there are two different ways of formally defining the lexicmin SWR. The first one is the so-called ‘weak lexicmin’, or W-Leximin, and can be formalised as follows.

\(^2\)It is worth noting that a stronger characterisation result of the maximin SWO can be provided by replacing the standard continuity property C with the following weaker property:

**Weak Continuity, WC:** \(\forall u, v \in X_T, u \succ v \Rightarrow \exists z \in \mathbb{R}_{++}^T : v \not\succ (u - z)\).
**Definition 1.** (Definition 2, [3], p. 224) For all \(1 u, 1 v \in X\), \(1 u \sim_{LM} 1 v \Leftrightarrow \exists T \geq 1\) such that \(\forall T \geq T\); \(1 u_T = 1 v_T\); and \(1 u \succ_{LM} 1 v \Leftrightarrow \exists T \geq 1\) such that \(\forall T \geq T\); \(\exists t \in \{1, ..., T\}: \tilde{u}_s = \tilde{v}_s\) \(\forall 1 \leq s < t\) and \(\tilde{u}_t > \tilde{v}_t\).

The characterisation results below are based on some standard axioms. The first three axioms are similar to those used in the finite case, and need no further comment, except possibly noting that in this context, WHP is weaker than the version in Section 2 above, since it only holds for vectors with the same tail.

**Finite Anonymity, FA:** \(\forall 1 u \in X\) and \(\forall \pi \in \Pi\), \(\pi(1 u) \sim 1 u\).

**Strong Pareto Optimality, SPO:** \(\forall 1 u, 1 v \in X\); \(1 u > 1 v\) \(\Rightarrow 1 u \succ 1 v\).

**Weak Harm Principle, WHP:** \(\forall 1 u, 1 v, 1 u' \in X\): \(\exists T \geq 1\) \(1 u = (1 u_{T,T+1} v) \succ 1 v\), and \(1 u' \prec 1 v \prec 1 u\) are such that, \(\exists t \leq T\),

\[
\begin{align*}
  u'_{i} &< u_{i} \\
v'_{i} &< v_{i} \\
u'_{j} &= u_{j} \forall j \neq i \\
v'_{j} &= v_{j} \forall j \neq i
\end{align*}
\]

implies \(1 v' \prec 1 u'\) whenever \(u'_{i} > v'_{i}\).

Next, following Asheim and Tungodden ([3], p. 223), an axiom is imposed, which represents a mainly technical requirement to deal with infinite-dimensional vectors.

**Weak Preference Continuity, WPC:** \(\forall 1 u, 1 v \in X\): \(\exists T \geq 1\) \(1 u = (1 u_{T,T+1} v) \succ 1 v\), \(\forall T \geq T\); \(\Rightarrow 1 u \succ 1 v\).

WPC (and the same is true for the stronger SPC discussed below) establishes “a link to the standard finite setting of distributive justice, by transforming the comparison of any two infinite utility paths to an infinite number of comparisons of utility paths each containing a finite number of generations” ([3], p. 223). In the same vein, the next axiom states that the swr should at least be able to compare (infinite-dimensional) vectors with the same tail. This seems an obviously desirable property which imposes a minimum requirement of completeness on the swr.

**Minimal Completeness, MC:** \(\forall 1 u, 1 v \in X\), \(\exists T \geq 1\) \((1 u_{T,T+1} v) \succ 1 v\) \(\forall T \geq T\); \(\Rightarrow 1 u \succ 1 v\).

The next Theorem proves that the combination of Finite Anonymity (FA), Strong Pareto Optimality (SPO), Weak Harm Principle (WHP*), Weak Preference Continuity (WPC), and Minimal Completeness (MC), characterises the leximin swr.

**Theorem 3.** \(\succ\) is an extension of \(\succ_{LM}^{*}\) if and only if \(\succ\) satisfies Finite Anonymity (FA), Strong Pareto Optimality (SPO), Weak Harm Principle (WHP*), Weak Preference Continuity (WPC), and Minimal Completeness (MC).
Proof. (**⇒**) Let \( \succ^{LM^*} \subseteq \succ \). It is easy to see that \( \succ \) meets FA and SPO.
By observing that \( \succ^{LM^*} \) is complete for comparisons between utility streams having the same tail it is also easy to see that \( \succ \) satisfies WPC and MC. We show that \( \succ \) meets WHP*.
Take any \( u, v \in X \) such that \( \exists T \geq 1 \), \( u \succ v \) and \( u' \succ v' \) are such that, \( \exists i \leq T, u_i' < u_i \), \( v_i' < v_i \), \( u_j' = u_j \forall j \neq i \), \( v_j' = v_j \forall j \neq i \). It must be true that \( u_i' > v_i' \). As \( \succ^{LM^*} \) is complete for comparisons between utility streams having the same tail, it must be true that \( 1u \succ^{LM^*} 1v \). Therefore, by definition, \( \exists T \geq 1 \) such that \( \forall T' \geq T \), \( \exists t \in \{1, ..., T\} \), \( u_t = \tilde{u}, \forall 1 \leq s < t \) and \( u_s > \tilde{u} \). Take any \( T' \geq T \). As \( T' < \infty \) it follows from Theorem 1 in [17], p. 126 that there exists \( t < t' \leq T' \) such that \( s \prec t \). We show that \( 1u \succ^{LM^*} 1v \) follows that \( 1u \succ 1v' \) as \( \succ^{LM^*} \subseteq \succ \).

(**⇐**) Suppose that \( \succ \) satisfies FA, SPO, WHP*, WPC, and MC. We show that \( \sim^{LM^*} \subseteq \sim \) and \( \succ^{LM^*} \subseteq \succ \). Take any \( u, v \in X \).

Assume that \( u \sim^{LM^*} v \). By definition, \( \exists T \geq 1 \) such that \( \forall T \geq T \), \( 1u \succ \tilde{u} \), and \( 1v \succ \tilde{v} \), for any \( T \geq T \). It follows that \( 1u \sim 1v \), by FA.

Next, suppose that \( u \succ^{LM^*} v \). By definition, \( \exists T \geq 1 \) such that \( \forall T \geq T \), \( \exists t \in \{1, ..., T\} \) such that \( u_t = \tilde{u}, \forall 1 \leq s < t \) and \( u_s > \tilde{u} \). Take any such \( T \) and consider the vector \( w \equiv (1u, T_{T+1}) \). Note that \( w \succ^{LM^*} 1v \). We want to show that \( 1w \succ 1v \). By FA and transitivity, we can consider \( 1\tilde{w} \equiv (1\tilde{u}, T_{T+1}) \) and \( 1\tilde{v} \equiv (1\tilde{v}, T_{T+1}) \). Suppose that \( 1\tilde{v} \succ 1\tilde{w} \). We distinguish two cases.

Case 1. \( 1\tilde{v} \succ 1\tilde{w} \)
As SPO holds it must be the case that \( \tilde{v}_l > \tilde{w}_l \) for some \( l < t \). Let \( k = \min\{t < l \leq T | \tilde{v}_l > \tilde{w}_l\} \).

By FA, let \( v_l = \tilde{v}_k \) and let \( w_l = \tilde{w}_{k-g} \), for some \( 1 \leq g < k \), where \( \tilde{w}_{k-g} > \tilde{v}_{k-g} \). Then, let two real numbers \( d_1, d_2 > 0 \), and consider vectors \( 1w, 1v \) formed from \( 1\tilde{w}, 1\tilde{v} \) as follows: \( \tilde{w}_{k-g} \) is lowered to \( \tilde{w}_{k-g} - d_1 \), such that \( \tilde{w}_{k-g} - d_1 > \tilde{v}_{k-g} \); \( \tilde{v}_k \) is lowered to \( \tilde{v}_k - d_2 \), such that \( \tilde{w}_k > \tilde{v}_k - d_2 > \tilde{w}_{k-g} - d_1 \); and all other entries of \( 1\tilde{w} \) and \( 1\tilde{v} \) are unchanged. By FA, consider \( 1\tilde{w}' = (1\tilde{w}', T_{T+1}) \) and \( 1\tilde{v}' = (1\tilde{v}', T_{T+1}) \). By construction \( \tilde{w}'_j \geq \tilde{v}'_j \) for all \( j \leq k \), with \( \tilde{w}'_{k-g} > \tilde{v}'_{k-g} \), whereas WHP*, combined with MC and FA, implies \( \tilde{v}' \succ \tilde{w}' \). Furthermore, by SPO, it is possible to choose \( d_1, d_2 > 0 \), such that \( \tilde{v}' \succ \tilde{w}' \), without loss of generality. Consider two cases:

a) Suppose that \( \tilde{v}_k > \tilde{w}_k \), but \( \tilde{v}_t \geq \tilde{w}_t \) for all \( l < k \). It follows that \( 1\tilde{w}' \succ 1\tilde{v}' \), and so SPO implies that \( 1\tilde{w}' > 1\tilde{v}' \), a contradiction.

b) Suppose that \( \tilde{v}_l > \tilde{w}_l \) for some \( l > k \). Note that by construction \( \tilde{v}'_l = \tilde{v}_l \) and \( \tilde{w}'_l = \tilde{w}_l \) for all \( l > k \). Then, let \( k' = \min\{k < l \leq T | \tilde{v}'_l > \tilde{w}'_l\} \).

where \( k' > k \). The above argument can be applied to \( 1\tilde{w}', 1\tilde{v}' \) to derive vectors \( 1\tilde{w}''', 1\tilde{v}''' \) such that \( \tilde{w}'''_j \geq \tilde{v}'''_j \) for all \( j \leq k' \), whereas WHP*, combined with MC, FA, and SPO, implies \( 1\tilde{v}''' > 1\tilde{w}''' \). And so on. After a finite number of iterations \( s \), two vectors \( 1\tilde{w}^s, 1\tilde{v}^s \) can be derived such that, by WHP*, combined
with \( MC \), \( FA \), and \( SPO \), we have that \( _1\tilde{v}^s \succ 1\tilde{w}^s \), but \( SPO \) implies \( 1\tilde{w}^s \succ 1\tilde{v}^s \), yielding a contradiction.

**Case 2.** \( 1\tilde{v} \sim 1\tilde{w} \)

Since, by our supposition, \( \tilde{v}_t < \tilde{u}_t \equiv \tilde{w}_t \), there exists \( \epsilon > 0 \) such that \( \tilde{v}_t < \tilde{w}_t - \epsilon < \tilde{w}_t \). Let \( 1\tilde{w}^c \in X \) be a vector such that \( \tilde{w}^c_t = \tilde{w}_t - \epsilon \) and \( \tilde{w}^c_j = \tilde{w}_j \) for all \( j \neq t \). It follows that \( 1\tilde{w}^c \succ^{LM^*} 1\tilde{v} \) but \( 1\tilde{v} \succ 1\tilde{w}^s \) by \( SPO \) and the transitivity of \( \succ \). Hence, the argument of **Case 1** above can be applied to \( 1\tilde{v} \) and \( 1\tilde{w}^c \), yielding the desired contradiction.

It follows from \( MC \) that \( 1\tilde{w} \succ 1\tilde{v} \). \( FA \), combined with the transitivity of \( \succ \), implies that \((1u_T, t+1v) \succ 1v \). Since it holds true for any \( T \geq \tilde{T} \), \( WPC \) implies \( 1\tilde{v} \succ 1v \), as desired.

The properties in Theorem 3 are independent (see Annex).

It is worth stressing again that in societies with an infinite number of agents, or generations, there is no obvious, and unanimously accepted, definition of the leximin - the S-Leximin - that can be formalised as follows.

**Definition 2.** (Definition 1, [3], p. 224) For all \( 1u_1, v \in X \), \( 1u \succ^{LM^*} \exists v \Leftrightarrow \exists \tilde{T} \geq 1 \) such that \( \forall T \geq \tilde{T} \): either \( 1\tilde{u}_T = 1\tilde{v}_T \), or \( \exists t \in \{1, ..., T\} : \tilde{u}_s = \tilde{v}_s \forall 1 \leq s < t \) and \( \tilde{u}_t > \tilde{v}_t \).

The above analysis focuses on the W-Leximin because the continuity axiom \( WPC \) is much weaker - and thus possibly more appealing (as argued by Basu and Mitra [6], p. 358) - than the Strong Preference Continuity property adopted by Asheim and Tungodden ([3], p. 223) to characterise the S-leximin. Strong Preference Continuity can be formalised as follows.

**Strong Preference Continuity**, \( SPC \): \( \forall 1u_1, v \in X : \exists \tilde{T} \geq 1 \) such that \((1u_T, t+1v) \succ 1v \forall T \geq \tilde{T} \), and \( \forall \tilde{T} \geq 1 \exists T \geq \tilde{T} \) such that \((1u_{T+1}, v) \succ 1v \Rightarrow 1v \succ 1v \).

A result analogous to Theorem 3 can be established for the stronger definition 2 by replacing \( WPC \) with \( SPC \). It can be easily obtained through a trivial modification of the parts of the proof of Theorem 3 that involve \( WPC \), and by observing that the necessity of \( WHP^* \) can be easily established along the same lines as in Theorem 3.

**Theorem 4.** \( \succ \) is an extension of \( \succ_S^{LM^*} \) if and only if \( \succ \) satisfies Finite Anonymity (\( FA \)), Strong Pareto Optimality (\( SPO \)), Weak Harm Principle (\( WHP^* \)), Strong Preference Continuity (\( SPC \)), and Minimal Completeness (\( MC \)).

The properties in Theorem 4 are independent (see Annex).

Theorems 3 and 4 identify the relevant class of leximin \( SWR^* \)s by postulating a continuity property on the quasi-ordering (respectively, \( WPC \) and \( SPC \)), which represents a mainly technical requirement in ranking infinite utility streams. As axioms such as \( SPO \) and \( FA \) may be considered ethically more defensible than continuity axioms, Bossert, Sprumont and Suzumura ([8]) have not postulated
any continuity property on the quasi-ordering and have provided a characterisation of a subclass of the class of orderings satisfying SPO, FA, and an infinite version of HE. Formally, the relationship between the characterisation of the leximin by Bossert et al. ([8]) and that by Asheim and Tungodden ([3]) is analogous to the relationship between the characterisation of the utilitarian SWR by Basu and Mitra ([6]) and the characterisations of the more restrictive utilitarian SWR induced by the overtaking criterion (see the discussion in [8], p. 580). This relationship is explored below by extending the analysis of WHP* to the framework developed by Bossert et al. ([8]).

For each $T \in \mathbb{N}$, let the leximin ordering on $X_T$ be denoted as $\succsim^L_T$. The definition of the leximin SWR proposed by Bossert et al. ([8]) can be formulated as follows. Define a relation $\succsim^L_T \subseteq X \times X$ by letting, for all $1u, 1v \in X$,

$$1u \succsim^L_T 1v \iff 1uT \succsim^L_T 1vT \text{ and } T_{+1}u \succeq T_{+1}v.$$  \hspace{1cm} (5)

The relation $\succsim^L_T$ can be shown to be reflexive and transitive for all $T \in \mathbb{N}$. Then the leximin SWR on $X$ is $\succsim = \bigcup_{T \in \mathbb{N}} \succsim^L_T$ ([8], p. 586): it is reflexive and transitive, but not necessarily complete. Moreover, Bossert et al. show that $\succsim^L$ satisfies the following property ([8], p. 586, equation (14)):

$$\forall 1u, 1v \in X : \exists T \in \mathbb{N} \text{ such that } 1u \succsim^L_T 1v \iff 1u \succsim^L 1v.$$  \hspace{1cm} (6)

The next Theorem shows that the set of ordering extensions of $\succsim^L$ characterised by Finite Anonymity (FA), Strong Pareto Optimality (SPO), and the weak Harm Principle (WHP*), is non-empty.

**Theorem 5.** $\succsim$ is an ordering extension of $\succsim^L$ if and only if $\succsim$ satisfies Finite Anonymity (FA), Strong Pareto Optimality (SPO), and the weak Harm Principle (WHP*).

**Proof.** ($\Rightarrow$) The proof that any ordering extension of $\succsim^L$ satisfies FA and SPO is as in ([8], Theorem 2, p. 586). We only need to prove that any ordering extension $\succsim$ of $\succsim^L$ satisfies WHP*. Consider any $1u, 1v, 1u', 1v' \in X$ such that $\exists T \geq 1 1u = (1uT, T_{+1}u) \succsim 1v$, and $1u', 1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j \forall j \neq i$, $v'_j = v_j \forall j \neq i$. We show that $1u' \succsim 1v'$ whenever $u'_i > v'_i$. Since $\succsim^L_T$ is complete and $T_{+1}v = T_{+1}u$ it cannot be $1uT \succsim^L_T 1vT$, otherwise $(1v, 1u) \in \succsim^L \subseteq \succsim$ which contradicts $1u \succsim 1v$. Thus, we have that $1uT \succsim^L_T 1vT$, $1vT \not\succsim^L_T 1uT$, and $T_{+1}v = T_{+1}u$, so that $(1u, 1v) \in \succsim^L_T$ by (5). It follows from (6) that $(1u, 1v) \in \succsim^L$. As $1u'$ and $1v'$ are such that, $\exists i \leq T$, $u'_i < u_i$, $v'_i < v_i$, $u'_j = u_j \forall j \neq i$, $v'_j = v_j \forall j \neq i$, it can easily be shown, as in ([17]), that $1u'T \succsim^L_T 1v'T$ whenever $u'_i > v'_i$. As $T_{+1}v' = T_{+1}u'$ and $1u'T \succsim^L_T 1v'T$ it follows from (5) that $1u' \succsim^L_T 1v'$, and therefore $1u' \succsim^L 1v'$ by (6). But since $\succsim$ is an ordering extension of $\succsim^L$ it follows that $1u' \succsim 1v'$.

($\Leftarrow$) The proof is as in ([8], Theorem 2, p. 587), using the characterisation of the $T$-person leximin in Theorem 1.

Theorem 5 characterises a larger class of orderings than that identified by Theorems 3 or 4, because in the latter results an additional continuity axiom is employed, but it is strikingly similar to the characterisation in the finite context.
Finally, it is worth noting that the Weak Harm Principle (WHP*) can also be used to characterise the intergenerational version of the lexmin SWO recently proposed by Sakai ([26]), which drops transitivity but retains completeness. In particular, if one replaces Hammond Equity with WHP*, a modified version of his characterisation results ([26], Lemma 6, p. 17; and Theorem 5, p. 18) can easily be proved.

4.2 The Maximin SWR

In this subsection, Rawls’s difference principle is analysed in the context of economies with an infinite number of agents. First of all, the analysis focuses on the subset of utility streams that reach a minimum in a finite period. Formally, define the following subset $Y$ of $X$:

$$Y = \{ u \in X | \exists T : 1u_T = 1u_T \forall T \geq T' \}.$$ 

The maximin SWR can be formally defined as follows.

**Definition 3.** For all $1u, 1v \in Y$, $1u \sim^M 1v \iff \min(1u) = \min(1v)$, and $1u \succ^M 1v \iff \min(1u) > \min(1v)$.

Let $\succ^M = \succ^M \cup \sim^M$. It is easy to show that $\succ^M$ is a complete quasi-ordering on $Y$. In the framework proposed by Asheim and Tungodden ([3]), Definition 3 has equivalent reformulations.

**Theorem 6.** For all $1u, 1v \in Y$, the following statements are equivalent:

(a) $\min(1u) = \min(1v) ; \min(1u) > \min(1v)$;

(b) $\exists T \geq 1 : 1u_T = 1u_T \forall T \geq T'$; $\exists T \geq 1 : 1u_T > 1u_T \forall T \geq T'$;

(c) $\exists T \geq 1 : 1u_T = 1u_T$ and $[1u_T = 1u_T = 1u_T \forall T \geq T]$; $\exists T \geq 1 : 1u_T > 1u_T$ and $[1u_T = 1u_T$ and $1u_T = 1u_T \forall T \geq T]$.

**Proof.** Obvious, so omitted.

It is worth noting that the relevant $\hat{T}$ in part (b) may be different from that in part (c) of the latter proposition.

In order to prove the main characterisation result, the following four standard axioms are imposed, which are analogous to those used in the finite setting, and need no further comment, except noting that WC* is a significant weakening of standard continuity axioms. Continuity requires that if $1u$ is strictly better than $1v$, then any vector sufficient close to $1u$ should be strictly better than any vector sufficient close to $1v$, and, as is well-known, in its full strength continuity is problematic in the infinite setting (see [13], [31]). WC* only requires the existence of some vector close to $1u$ such that the strict preference is not reversed. Instead, the stronger formulation of the Harm Principle originally proposed by Mariotti and Veneziani ([17]) is used.

**Finite Anonymity, FA*:** $\forall 1u \in Y$ and $\forall \pi \in \Pi \Rightarrow \pi(1u) \sim 1u$.

**Weak Pareto Optimality, WPO*:** $\forall 1u, 1v \in Y, 1u \succ 1v \Rightarrow 1u \succ 1v$. 


**Lemma 1**. Let the premises of the statement hold. Assume, to the contrary, that

\[ u_i' < u_i, \quad v_i' < v_i, \quad u_j' = u_j \quad \forall j \neq i \quad v_j' = v_j \quad \forall j \neq i. \]

implies \( u_i' > v_i' \) whenever \( u_i' > v_i' \).

**Weak Continuity**, **WC**: \( \forall u, v \in Y, \; 1u = (1u_{T+1} + v) > v \Rightarrow \exists z \in \mathbb{R}^N_{++} : 1u - vz \in Y. \)

In addition to the above requirements, a weak consistency requirement is imposed.

**Weak Dominance Consistency**, **WDC**: \( \forall u, v, w \in Y, \; 1w = (1w_{T+1} + w) > v \Rightarrow \exists T \geq 1 \; (1w_{T+1} + w) > (1v_{T+1} + w) \forall u' \geq T \Rightarrow 1u' \geq 1v. \)

In analogy with **WPC**, **WDC** is mainly a technical requirement that provides a link to the finite setting by transforming the comparison of two infinite utility paths to an infinite number of comparisons of utility paths each containing a finite number of generations. Axioms similar to **WDC** are common in the literature (see, e.g., [6], [1], [2]).

Finally, the next axiom requires that \( \succ \) be complete at least when comparing elements of \( Y \) with the same tail. This requirement is weak and it seems uncontroversial, for it is obviously desirable to be able to rank as many vectors as possible.

**Minimal Completeness**, **MC**: \( \forall u, v \in Y, \; 1u \not= 1v, \; \exists T \geq 1 : 1u_{T+1} = 1v_{T+1} \Rightarrow 1u \not= 1v \) or \( 1v \not= 1u. \)

In order to derive the main characterisation result concerning the maximin **swr**, it is first proved that any **swr** satisfying **WC**, **MC**, and **WPO** also satisfies monotonicity.

**Lemma 1**. Let \( \succ \) on \( X \) be a **swr** satisfying **WPO**, **WC**, and **MC**. Then, \( \forall u, v \in Y, \; 1u \succ (1v_{T+1} + u) \) for some \( T \geq 1 \Rightarrow 1u \succ (1v_{T+1} + u). \)

**Proof**. Let the premises of the statement hold. Assume, to the contrary, that \( 1u \not< (1v_{T+1} + u). \) **MC** implies \( (1v_{T+1} + u) \succ 1u. \) It follows from **WC** that \( 1u \not< (1v_{T+1} + u) - 1z \) for some \( z \in \mathbb{R}^N_{++}, \) with \( (1v_{T+1} + u) - 1z \equiv 1w \in Y. \) It follows from \( 1u \succ (1v_{T+1} + u) \) that \( 1u \succ 1w. \) **WPO** implies \( 1u \succ 1w, \) a contradiction.

Given Lemma 1, the next Theorem proves that the combination of the above axioms characterises the maximin **swr**.

**Theorem 7**. \( \succ \) on \( X \) is an extension of \( \succ^M \) on \( Y \) if and only if \( \succ \) satisfies Finite Anonymity (**FA**), Weak Pareto Optimality (**WPO**), Harm Principle (**HP**), Weak Continuity (**WC**), Weak Dominance Consistency (**WDC**), and Minimal Completeness (**MC**).
Proof. \((\Rightarrow)\) It is easy to see that \(\succcurlyeq\) meets \(\text{FA}^*, \text{WPO}^*, \text{HP}, \text{WC}, \text{WDC},\) and \(\text{MC}^*\) whenever \(\succcurlyeq\) is an extension of \(\succeq^M\).

\((\Leftarrow)\) Suppose that \(\succcurlyeq\) meets \(\text{FA}^*, \text{WPO}^*, \text{HP}, \text{WC}, \text{WDC},\) and \(\text{MC}^*\). We show that \(\succeq^M \subseteq \succcurlyeq\), that is, \(\forall 1 u, 1 v \in Y,\)

\[ 1 u \succcurlyeq^M 1 v \Rightarrow 1 u \succcurlyeq 1 v \]

and

\[ 1 u \sim^M 1 v \Rightarrow 1 u \sim 1 v. \]

First we show that the implication \((\Rightarrow)\) of (7) is met. For take any \(1 u, 1 v \in Y\) such that \(\min(1 u) > \min(1 v)\). By Theorem 6 it follows from \(\min(1 u) > \min(1 v)\) that there exists \(\bar{T} \geq 1\) such that, for all \(T \geq \bar{T}, 1 u_T > 1 v_T, 1 w_T = 1 u_T\) and \(1 w_T = 1 v_T\). Let \(a_t = \max \{u_t, v_t\}\), all \(t\), and take any \(w_T > a_t + \varepsilon\) for some \(\varepsilon > 0\), all \(t \leq \bar{T}\), and \(w_T > \max \{a_t, 1 w_T\}\), all \(t > \bar{T}\). Clearly, \(1 w \notin Y\) and \(1 w > 1 v\) and \(1 w \gg 1 u\). Then, take any \(T \geq \bar{T}\) and consider the following vectors: \((1 u_T, 1 T - 1 u)\) and \((1 T, 1 T - 1 u)\). Clearly, \((1 u_T, 1 T - 1 u)\) and \((1 T, 1 T - 1 u)\) are in \(Y\) and \((1 u_T, 1 T - 1 u) >^M (1 T, 1 T - 1 u)\). We show that \((1 u_T, 1 T - 1 u) > (1 T, 1 T - 1 u)\). Assume, to the contrary, that \((1 u_T, 1 T - 1 u) \not\succcurlyeq (1 T, 1 T - 1 u)\). \(\text{MC}^*\) implies that \((1 T, 1 T - 1 u) \succcurlyeq (1 T, 1 T - 1 u)\). Let \(1 x = (1 u_T, 1 T - 1 u)\) and \(1 y = (1 u_T, 1 T - 1 u)\), so that \(1 x > 1 y\).

As \(\text{FA}^*\) holds, let \(1 x_1, 1 y_1\) be such that \(T = T + 1 x = T + 1 y = T + 1 \bar{y}\), and \(1 x_T, 1 y_T\) are such that \(x_1 \leq \ldots \leq x_T\) and \(y_1 \leq \ldots \leq y_T\), so that \(1 x \sim 1 x\) and \(1 y \sim 1 y\). If \(1 x_T < 1 y_T\) then Lemma 1 and transitivity of \(\succcurlyeq\) imply that \(1 y \sim 1 x\), a contradiction. Otherwise, let \(\bar{x}_i > \bar{y}_i\) for some \(t \leq T\). Let \(k = \min \{l \leq T \mid \bar{x}_l > \bar{y}_l\}\).

Let \(1 x\) and \(1 y\) be two finite permutations of \(\mathbb{N}\) such that \(T + 1 x = T + 1 y = T + 1 \bar{y}\) and, for some \(i \leq T\), \(x_i = \bar{x}_k\) and \(y_i = \bar{y}_i\). By \(\text{FA}^*\), \(1 x \sim 1 \bar{x}\) and \(1 y \sim 1 \bar{y}\), so that by transitivity \(1 x \succcurlyeq 1 \bar{y}\). Then, let two real numbers \(d_1, d_2 > 0\), and consider vectors \(1 y', 1 x'\) formed from \(1 y, 1 x\) as follows: first, \(1 y_i\) is lowered to \(y_i - d_1 > x_i\); next, \(x_i\) is lowered to \(x_i - d_2 > y_i - d_1\); finally, all other entries of \(1 y\) and \(1 x\) are unchanged. Clearly \(1 y', 1 x' \in Y\) and it follows from \(\text{HP}\) that \(1 x' \succcurlyeq 1 y'\). Let \(1 x''\) and \(1 y''\) be two finite permutations of \(\mathbb{N}\) such that \(T + 1 x'' = T + 1 y'' = T + 1 \bar{y}\) and \(1 x'', 1 y''\) are such that \(x_i' \leq \ldots \leq x_T'\) and \(y_i' \leq \ldots \leq y_T'\). By construction, \(y_i' \geq y_i''\) for all \(j \leq k,\) with strict inequality holding for at least some \(j\). By \(\text{FA}^*\) and transitivity, \(1 x'' \succcurlyeq 1 y''\). If \(y_j' > y_j''\) for all \(j \leq T\), Lemma 1 implies \(1 y' \succcurlyeq 1 x''\), a contradiction. Otherwise, let \(x_i'' > y_i''\) for some \(T \geq l > k\). Then, let

\[ k' = \min \{l \leq T \mid x_i'' > y_i''\} \]

where \(k' > k\). The above reasoning can be applied to \(1 y''\) and \(1 x''\) to derive vectors \(1 y''\) and \(1 x''\) such that \(y_j'' > x_j''\) for all \(j \leq k'\), with strict inequality holding for at least some \(j,\) whereas \(1 x'' \succcurlyeq 1 y''\). By Lemma 1, a contradiction is obtained whenever \(y_j'' > x_j''\) for all \(j \leq T\). Otherwise, let \(x_i'' > y_i''\)
for some $T \geq l > k'$. And so on. After a finite number $s$ of iterations, two vectors $\hat{y}^s, \hat{x}^s$ can be derived such that $\hat{x}^s \succ \hat{y}^s$, but $\tilde{y}^s \succ \tilde{x}^s$, by Lemma 1, a contradiction. We conclude that $1y \equiv (1w_{T,T+1} w) \succ 1x \equiv (1w_{T,T+1} w)$.

Since $(1w_{T,T+1} w) \succ (1w_{T,T+1} w)$ for any $T \geq \tilde{T}$ and WPO* implies that $1w \succ 1u$ and $1w \succ 1v$, it follows from WDC that $1u \sim 1v$. We show that $1u \sim 1v$. Assume that $1v \succ 1u$ so that $1v \sim 1u$. Take any $0 < \epsilon < (\min (1u) - \min (1v))$ and consider $1v + \epsilon \equiv 1v' \in Y$. The transitivity of $\succ$ and WPO* imply that $1v' \succ 1u$ but $\min (1u) > \min (1v')$. The above reasoning can be applied to $1u$ and $1v'$ by taking a vector $1w' \in Y$ such that $1w' \succ 1u$ and $1w' \succ 1v'$ to arrive to the conclusion that $1u \sim 1v'$ which yields the desired contradiction.

Next, we show that the implication $(\Rightarrow)$ of (8) is met as well. Take any $1u, 1v \in Y$ such that $\min (1u) = \min (1v)$. By Theorem 6 it follows that $\exists T \geq 1$ such that $1u_T = 1v_T$ and $1w_T = 1w_{T} = 1w_T \forall T \geq \tilde{T}$. If $1v = \pi (1u)$ for some $\pi \in \Pi$, FA* implies $1u \sim 1v$. Otherwise, let $1v \neq \pi (1u)$ for all $\pi \in \Pi$. Let $a_t \equiv \max \{u_t, v_t\}$, all $t$, and take any $1w$ such that $w_t = a_t + \epsilon$ for some $\epsilon > 0$, all $t \leq \tilde{T}$, and $w_t > \max \{a_t, 1w_T\}$, all $t > \tilde{T}$. Clearly, $1w \in Y$ and $1w \sim 1v$ and $1w \sim 1u$. Then, take any $T \geq \tilde{T}$ and consider the vectors $(1w_{T,T+1} w)$ and $(1w_{T,T+1} w)$. Clearly, $(1w_{T,T+1} w)$ and $(1w_{T,T+1} w)$ are in $Y$. Let $1x \equiv (1w_{T,T+1} w)$ and $1y \equiv (1w_{T,T+1} w)$, so that $\min (1x) = \min (1y)$. We show that $1x \sim 1y$. Assume, to the contrary, that $1x \neq 1y$, so that either $1x \succ 1y$ or $1y \succ 1x$ holds by MC*. Without loss of generality, suppose $1y \succ 1x$. As $\succ$ meets WC* it follows that, for some $1z \in \mathbb{N}$ such that $(1y - 1z) \in Y$, $1x \neq 1y - 1z$. Since $\min (1x) > \min (1y - 1z)$ it follows from the implication $(\Rightarrow)$ of (7) proved above that $1x \succ (1y - 1z)$, a contradiction. Therefore $1u \sim 1v$.

The properties in Theorem 7 are independent (see Annex).

Theorem 7 provides an original characterisation of the maximin SWR in societies with an infinite number of agents. This result is interesting per se, as compared to alternative characterisations of the maximin. For example, Lauerw (15) characterises the maximin SWO by an anonymous social welfare function (SWF) defined over the set of bounded infinite utility streams, by imposing a strong version of HE according to which for any two bounded infinite vectors $1u, 1v$ such that $u_i \geq v_i \geq u_j$ for some $i, j \in \mathbb{N}$ and $u_k = v_k \forall k \in \mathbb{N} \backslash \{i, j\}$, then $1v \succ 1u$. The main focus of this paper is different and so the question of the characterisation of the maximin SWO by an anonymous and liberal SWF remains open. It is worth noting, however, that Theorem 7 does characterise the maximin SWR on a different set of infinite utility streams, which can be unbounded above, and to this aim neither the continuity condition, nor to the so-called “repetition approximation principle” imposed by Lauerw (15), p. 146) are necessary. Indeed, subject to the domain restriction, and except for the rather mild condition WDC, the axioms are strikingly similar to those used to characterise the maximin SWO in finite economies.3

3This is even more evident in the light of the discussion in footnote 2 above.
Perhaps more importantly, Theorem 7 provides further support to the main theoretical arguments of this paper. For it confirms that the main intuitions concerning the role of the liberal notion of noninterference embodied in the Harm Principle are robust and they do not depend on the specific definition of the maximin and leximin SWRs adopted to rank infinite utility streams.

5 Conclusions

This paper provides a novel analysis of liberal egalitarian principles stemming from John Rawls’s seminal work, in societies with a finite and an infinite number of agents. A unified framework of analysis is set up, which allows one to characterise a family of egalitarian principles by means of a weaker version of a new axiom - the Harm Principle - recently proposed in [17]. This is quite surprising, because the Harm Principle is meant to capture a liberal requirement of noninterference and it incorporates no obvious egalitarian content. A set of new characterisations of the maximin and of its lexicographic refinement are derived, including in the intergenerational context with an infinite number of agents and using different definitions of the relevant SWRs proposed in the literature.

The results presented in this paper have two main sets of implications from a theoretical viewpoint. First, they shed new light on the ethical foundations of the egalitarian approaches stemming from Rawls’s difference principle. In fact, both the leximin and the maximin are characterised by some standard axioms (such as Anonymity and the Pareto Principle) together with a liberal principle incorporating only a noninterfering view of society. No axiom with an explicitly egalitarian content is necessary in order to derive the main liberal egalitarian principles. Second, from the viewpoint of liberal approaches emphasising a notion of individual autonomy, or freedom, they have a rather counterintuitive implication. For they prove that, in a number of different contexts, liberal noninterfering principles lead straight to welfare egalitarianism.

References


Annex: Independence of Axioms

The proofs of the independence of the axioms used to characterise the maximin and leximin SWO are obvious and therefore they are omitted. It is worth noting, however, that some of the examples below can be easily adapted to apply to the finite context.

Independence of axioms used in Theorem 3

In order to complete the proof of Theorem 3, we show that the axioms are tight.

For an example violating only FA, define $\succ$ on $X$ in the following way:

\[
1u = 1v \Rightarrow 1u \sim 1v
\]

\[
\exists T \in \mathbb{N} : u_t = v_t \forall t < T \text{ and } u_T > v_T \Rightarrow 1u \succ 1v
\]

The SWR $\succ$ on $X$ is not an extension of the leximin SWR $\succeq LM^*$. The SWR $\succ$ on $X$ satisfies all properties except FA.

For an example violating only SPO, define $\succ$ on $X$ in the following way:

\[
\forall 1u, 1v \in X, 1u \sim 1v. \text{ The SWR } \succ \text{ on } X \text{ is not an extension of the leximin SWR } \succeq LM^*. \text{ Clearly, the SWR } \succ \text{ on } X \text{ satisfies all properties except SPO.}
\]

For an example violating only WHP, define $\succ$ on $X$ to be the leximax SWR, i.e.,

\[
1u \sim LX 1v \Leftrightarrow \exists T \geq 1, \forall T \geq \hat{T}, \exists \{1, \ldots, T\} : u_s = v_s (\forall t < s \leq T) \text{ and } u_t > v_t.
\]

The SWR $\succ$ on $X$ is not an extension of the leximin SWR. The SWR $\succ$ on $X$ satisfies all properties except WHP.

For an example violating only MC, define $\succ$ on $X$ in the following way:

\[
\exists \pi \in \Pi : 1u = \pi(1v) \Rightarrow 1u \sim 1v
\]

\[
1u > 1v \Rightarrow 1u \succ 1v
\]

The SWR $\succ$ on $X$ is not an extension of the leximin SWR. The SWR $\succ$ on $X$ satisfies all properties except MC.

For an example violating only WPC, $\forall T \in \mathbb{N}$, let the leximin ordering on $X_T$ be denoted as $\preceq_T$. Define $\succ_T^L$ on $X$ as in (5). Then, let $\succ = \bigcup_{T \in \mathbb{N}} \succ_T^L$. By definition, this relation is reflexive and transitive. The SWR $\succ$ on $X$ is not an extension of the leximin SWR. The SWR $\succ$ on $X$ satisfies all properties but WPC. [To see that WPC is violated consider the following vectors, $1v = (3, \text{con}0)$ and $1u = (2, \text{con}1)$. Then, $(1u, 1v) \not\succ$ and $(1u_{T,T+1}^T, 1v) \in \succ_T^L \forall T \geq 2].$
Independence of axioms used in Theorem 4

In order to complete the proof of Theorem 4, we show that the axioms are tight.

As Strong Preference Continuity (SPC) implies Weak Preference Continuity (WPC), the above examples show that the axioms used in Theorem 4 are tight as well.

Independence of axioms used in Theorem 7

In order to complete the proof of Theorem 7, we show that the axioms are tight.

For an example violating only \( FA^* \), define \( \succ \) on \( X \) in the following way:

\[ \forall u, v \in X, \; i) \; u = v \iff u \sim v; \; ii) \; u \succ v \iff u \sim v \]. The \( SWR \succ \) on \( X \) is not an extension of the maximin \( SWR \succ M^* \). The \( SWR \succ \) on \( X \) satisfies all properties except \( FA^* \).

For an example violating only \( WPO^* \), define \( \succ \) on \( X \) in the following way:

\[ \forall u, v \in X, \; u \succ v \iff \exists T \geq 1 \text{ such that } \forall T \geq T : \sum_{t=1}^{T} u_t \geq \sum_{t=1}^{T} v_t \]. The \( SWR \succ \) on \( X \) is not an extension of the maximin \( SWR \succ M^* \). The \( SWR \succ \) on \( X \) satisfies all properties except \( WPO^* \).

For an example violating only \( HP \), define \( \succ \) on \( X \) in the following way: for all \( u, v \in X, u \succ v \iff \exists T \geq 1 \text{ such that } \forall T \geq T : u_T = T v_T \), or \( \exists t \in \{1, \ldots, T\} : u_t = T v_t \forall 1 \leq s \neq t \text{ and } u_t > v_t \). The \( SWR \succ \) on \( X \) is not an extension of the maximin \( SWR \succ M^* \). The \( SWR \succ \) on \( X \) satisfies all properties except \( HP \).

For an example violating only \( WC^* \), define \( \succ \) on \( X \) in the following way:

\[ \forall u, v \in X, \; u \succ v \iff \exists T \geq 1 \text{ such that } \forall T \geq T : u_T = T v_T \]. The \( SWR \succ \) on \( X \) is not an extension of the maximin \( SWR \succ M^* \). The \( SWR \succ \) on \( X \) satisfies all properties except \( WC^* \).

For an example violating only \( WDC \), define \( \succ \) on \( X \) in the following way:

\[ \forall u, v \in Y, \]
\[ u \sim M^* v, \exists T \geq 1 : u_T = T v \implies u \sim v \]
\[ u \succ M^* v \implies u \succ v \]
\[ u \sim M^* v, \exists T \geq 1 : u_T = T v \implies u \neq v \text{ and } v \neq u. \]

Furthermore, \( \forall u, v \in X \setminus Y \) and \( u \neq v \). The \( SWR \succ \) on \( X \) is not an extension of the maximin \( SWR \succ M^* \). The \( SWR \succ \) on \( X \) satisfies all properties except \( WDC \).

For an example violating only \( MC^* \), let \( \sigma \) be a permutation of \( N \). Let \( \Sigma \) be the set of all permutations of \( N \). Define \( \succ \) on \( X \) in the following way:

\[ \forall u, v \in X, \]
\[ u = \pi(1)v \implies u \sim v \]
\[ u \succ \sigma(1)v \implies u \succ v \]

The \( SWR \succ \) on \( X \) is not an extension of the maximin \( SWR \succ M^* \). The \( SWR \succ \) on \( X \) satisfies all properties except \( MC^* \).