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Review of AdS/CFT Integrability, Chapter VI.2: Yangian Algebra

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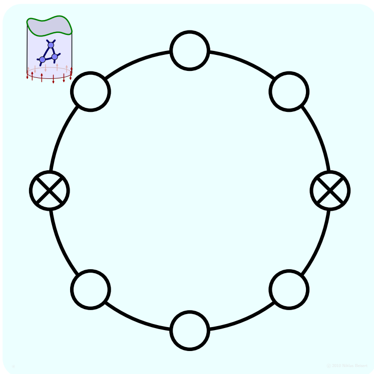

Abstract: We review the study of Hopf algebras, classical and quantum R-matrices, infinite-dimensional Yangian symmetries and their representations in the context of integrability for the $\mathcal{N}=4$ *vs* $AdS_5 \times S^5$ correspondence.

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1 Introduction

Despite the success obtained so far by the integrability program, many questions are left unanswered. Most notably, the problem remains of understanding what is the non-perturbative definition of the model that seems to reproduce so well all the available data [1]. Answering this question may also be important for a deeper understanding of the finite-size problem and its solution. An essential role in this respect is played by the symmetries of the factorized S-matrix. A clear sign is the presence of a Hopf algebra [2–4], then promoted to a Yangian [5]. In relativistic integrable quantum field theories, symmetries like the Yangian or quantum affine algebras completely determine the tensorial part of the S-matrix, up to an overall scalar factor. They also entail important consequences for the transfer matrices and for the Bethe equations [6]. This happens also in the AdS/CFT case [7, 8]. However, the AdS/CFT Yangian has very distinctive features still preventing a full mathematical understanding. For instance, there exists an additional Yangian symmetry of the S-matrix [9, 10] with properties not yet entirely understood, pointing to a new type of quantum group[1]. In order to give an ultimate solution of the AdS/CFT integrable system, one needs to understand the features of this novel quantum group, and of the associated quantum integrable model. The scope of this review is illustrating such group-theory aspects.

2 Hopf Algebras

Let us begin by recalling a few concepts in the theory of Hopf algebras, as these are very important algebraic structures appearing in the context of integrable models. We will attempt to motivate these concepts mostly from the physical viewpoint, and refer the reader to standard textbooks, such as [12], for a thorough treatment.

The starting point is the algebra of symmetries of a system. Let us consider the case when this algebra is a Lie (super)algebra $\mathfrak{g}$, and let us also consider its universal enveloping algebra $A \equiv U(\mathfrak{g})$. This step allows us to 'multiply' generators, besides taking the Lie bracket. In such universal enveloping algebra there is a *unit element* $\mathbb{1}$ with respect to the *multiplication map* μ. We think about multiplication as $\mu : A \otimes A \to A$, and we introduce a *unit map* $\eta : \mathbb{C} \to A$. A few compatibility conditions on these maps guarantee that we are dealing with the physical symmetries of, say, a single-particle system.

In order to treat multiparticle states, we equip our algebra with two more maps, and obtain a *bialgebra* structure. One map is the *coproduct* $\Delta : A \to A \otimes A$, which tells us how symmetry generators act on two-particle states. The other map is the *counit* $\varepsilon : A \to \mathbb{C}$. A list of compatibility axioms ensures that these maps are consistent with the (Lie) (super)algebra structure, so we can safely think of them as the symmetries we

[1]The relation with Yangian symmetry in n-p.t amplitudes [11] is also a fascinating problem.

started with, just acting on a Fock space. In fact, for a generic n-particle state, we can generalize the action of the coproduct as the composition $\Delta^n = ...(\Delta\otimes\mathbb{1}\otimes\mathbb{1})(\Delta\otimes\mathbb{1})\Delta$. The *coassociativity* axiom

$$(\Delta\otimes\mathbb{1})\Delta = (\mathbb{1}\otimes\Delta)\Delta \tag{2.1}$$

guarantees that a change in the positions of the Δ's in the sequence Δ^n is immaterial.

One more map turns our structure into a *Hopf algebra*. This map is the *antipode* $\Sigma : A \to A$, which is needed to define antiparticles (conjugated representations of the symmetry algebra). Therefore, the antipode should also be consistent with the (Lie) (super)algebra structure[2], and be compatible with the coproduct action. If a bialgebra admits an antipode, it is unique.

In the scattering theory of integrable models, the fundamental object encoding the dynamics is the two-particle S-matrix, which exchanges the momenta of the two particles, and reshuffles their colors. One has therefore the possibility of defining the coproduct action as acting on, say, *in* states. Likewise, the composed map $P\Delta \equiv \Delta^{op}$, with P the permutation map, will act on *out* states. The discovery of quantum groups revealed that these two actions need not be the same. They are the same only for *cocommutative* Hopf algebras, one example being the Leibniz rule $\Delta(a) = a\otimes\mathbb{1} + \mathbb{1}\otimes a$ one normally associates with local actions. In general, coproducts can be more complicated, as we will amply see in what follows[3].

However, as Δ and Δ^{op} produce tensor product representations of the same dimensions, they may be related by conjugation *via* an invertible element (the S-matrix itself). The Hopf-algebra is then said to be *quasi-cocommutative*, and, if the S-matrix satisfies an additional condition ('bootstrap' [13, 14]), it is called *quasi-triangular*. The S-matrix must also be compatible with the antipode map, a condition that in physical terms goes under the name of *crossing* symmetry. One can prove that bootstrap implies that the S-matrix satisfies the Yang-Baxter equation and the crossing condition.

As one can easily realize, the framework of Hopf algebras is particularly suitable for dealing with integrable scattering. Integrability reduces the scattering problem to an algebraic procedure, and the axioms we have been discussing just formalize that procedure. However, instead of being a mere translation, the mathematical framework of Hopf algebras provides a set of powerful theorems that unify the treatment of arbitrary representations. To this purpose, the notion of *universal R-matrix* is very important. This is an abstract solution to the quasi-cocommutativity condition, purely expressed in terms of algebra generators. This solution gives an expression for the S-matrix which is therefore free from a particular representation, at the same time being valid in any of them upon plug-in. As we will explicitly see in what follows, the

[2] Being the antipode connected to conjugation, one imposes $\Sigma(ab) = (-)^{ab}\Sigma(b)\Sigma(a)$, where multiplication is *via* the map μ.

[3] This is another reason why the Coleman-Mandula theorem does not apply to the S-matrices we will be discussing (besides being in $1+1$ dimensions).

study of the properties of the universal R-matrix reveals a big deal about the structure of the (hidden) symmetry algebra of the integrable system.

3 Yangians

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with generators $\mathfrak{J}^A$, structure constants f^{AB}_C defined by $[\mathfrak{J}^A,\mathfrak{J}^B] = f^{AB}_C \mathfrak{J}^C$ and a non-degenerate invariant bilinear form κ^{AB}. The Yangian $\mathscr{Y}(\mathfrak{g})$ of $\mathfrak{g}$ is a deformation of the universal enveloping algebra of half of the loop algebra of $\mathfrak{g}$. The loop algebra is defined by (4.5), "half" meaning non-negative indices m,n. Drinfeld gave two isomorphic realizations of the Yangian[4]. The first realization [25] is as follows. $\mathscr{Y}(g)$ is defined by relations between level zero generators $\mathfrak{J}^A$ and level one generators $\widehat{\mathfrak{J}}^A$:

$$[\mathfrak{J}^A,\mathfrak{J}^B] = f^{AB}_C \mathfrak{J}^C, \qquad [\mathfrak{J}^A,\widehat{\mathfrak{J}}^B] = f^{AB}_C \widehat{\mathfrak{J}}^C. \tag{3.1}$$

The generators of higher levels are derived recursively by computing the commutant, subject to the following Serre relations (for $\mathfrak{g} \neq \mathfrak{su}(2)$):

$$[\widehat{\mathfrak{J}}^A,[\widehat{\mathfrak{J}}^B,\mathfrak{J}^C]] + [\widehat{\mathfrak{J}}^B,[\widehat{\mathfrak{J}}^C,\mathfrak{J}^A]] + [\widehat{\mathfrak{J}}^C,[\widehat{\mathfrak{J}}^A,\mathfrak{J}^B]] = \frac{1}{4} f^{AG}_D f^{BH}_E f^{CK}_F f_{GHK} \mathfrak{J}^{\{D}\mathfrak{J}^E\mathfrak{J}^{F\}} \tag{3.2}$$

Indices are raised (lowered) with κ^{AB} (its inverse). The Yangian is equipped with a Hopf algebra structure. The coproduct is uniquely determined for all generators by specifying it on the level zero and one generators as follows:

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{J}^A, \qquad \Delta(\widehat{\mathfrak{J}}^A) = \widehat{\mathfrak{J}}^A \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathfrak{J}}^A + \frac{1}{2} f^A_{BC} \mathfrak{J}^B \otimes \mathfrak{J}^C. \tag{3.3}$$

Antipode and counit are easily obtained from the Hopf algebra definitions[5]. We will not present here Drinfeld's second realization of the Yangian [26], which is suitable for constructing the universal R-matrix [27]. It suffices to say that it explicitly solves the recursion implicit in the first realization.

3.1 The $\mathfrak{psu}(2,2|4)$ Yangian

Generically, the level zero local generators are realized on spin-chains as

$$\mathfrak{J}^A = \sum_k \mathfrak{J}^A(k), \qquad k \in \{spin-chain\ \ sites\}. \tag{3.4}$$

[4] The reader is referred to *e.g.* [12, 15–17] for a thorough treatment. We will not discuss the 'RTT' realization, see *e.g* [18, 17]. For generalizations to Lie superalgebras, see *e.g.* [19–24].

[5] *Via* a rescaling of the algebra generators, one can make a parameter (say, $\hbar$) appear in front of the mixed term $\frac{1}{2} f^A_{BC} \mathfrak{J}^B \otimes \mathfrak{J}^C$ in the Yangian coproduct (3.3). This parameter is sometimes useful as it can be made small, as in the classical limit, cf. section 3.

For infinite length, the level one Yangian generators are bilocal combinations

$$\widehat{\mathfrak{J}}^A = \sum_{k<n} f^A_{BC}\, \mathfrak{J}^B(k)\, \mathfrak{J}^C(n). \tag{3.5}$$

The relationship with the coproduct (3.3) will be clear later when discussing the Principal Chiral Model. Level n generators are $n+1$-local expressions. At finite length, boundary effects usually prevent from having conserved charges such as (3.5), while Casimirs of the Yangian may still be well-defined. We refer to [28] for a review.

The $\mathcal{N}=4$ SYM spin-chain is based on the superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$. The Yangian charges for infinite length have been constructed, at leading order in the 't Hooft coupling, in [29]. The Serre relations for the relevant representations have been proved in [30]. In [31] the first two Casimirs of the Yangian are computed and identified with the first two local abelian Hamiltonians of the spin-chain with periodic boundary conditions.

Perturbative corrections to the Yangian charges in subsectors have been studied in [32–36]. The integrable structure of spin-chains with long-range (LR) interactions, like the one emerging from gauge perturbation theory, lies outside the established picture [37], but a large class of LR spin-chains has been shown to display Yangian symmetries, see also [38–41]. In absence of other standard tools, Yangian symmetry provides a formal proof of integrability order by order in perturbation theory. The two-loop expression of the Yangian (3.5) for the $\mathfrak{su}(2|1)$ sector has been derived in [35]. In [36], a large degeneracy of states in the $\mathfrak{psu}(1,1|2)$ sector is explained *via* nonlocal charges related to the loop-algebra of the $\mathfrak{su}(2)$ automorphism of $\mathfrak{psu}(1,1|2)$. Further references include [42, 43]. For a recent review we recommend [44].

Higher non-local charges analogous to (3.5) emerge in 2D classically integrable field theories [45,46]. If not anomalous, their quantum versions [47] form a Yangian. *E.g.*, for the Principal Chiral Model

$$\frac{d}{dt}\widehat{\mathfrak{J}}^A = \frac{d}{dt}\int_{-\infty}^{\infty} dx \left[\varepsilon_{\mu\nu} J^{\nu,A} + \frac{1}{2} f^A_{BC} J^B_\mu \int_{-\infty}^{x} dx' J^C_0(x')\right] = 0, \tag{3.6}$$

where $\mathfrak{J}^A$ are Noether currents for the global (left or right) group multiplication.

The classical integrability of the Green-Schwarz superstring sigma model in the $AdS_5 \times S^5$ background has been established in [48]. The corresponding infinite set of nonlocal classically-conserved charges is found according to a logic very close to the one described above (similar observations for the bosonic part of the action were made in [49]). Further work can be found in [50–57].

We conclude with a remark on the Hopf algebra structure of the nonlocal charges. How charges (3.6) can give rise to the coproduct (3.3) is shown in [58]. A semiclassical treatment [59, 60] is as follows. One imagines two well-separated solitonic excitations as the classical version of a scattering state. Soliton 1 is localized in the region $(-\infty, 0)$, soliton 2 in $(0, \infty)$. Defining the *semiclassical action* of a charge on such solution as evaluation on the profile, one splits the current-integration in individual domains relevant for each of the two solitons, respectively:

$$\begin{aligned}
\mathfrak{J}^A_{|profile} &= \int_{-\infty}^{\infty} dx J^A_{0\,|profile} = \int_{-\infty}^{0} dx J^A_0 + \int_0^{\infty} dx J^A_0 \longrightarrow \Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{J}^A, \\
\widehat{\mathfrak{J}}^A_{|profile} &= \Big[\int_{-\infty}^{0} dx J^A_1 + \frac{1}{2} f^A_{BC} \int_{-\infty}^{0} dx J^B_0(x) \int_{-\infty}^{x} dy J^C_0(y)\Big] \\
&\quad + \Big[\int_0^{\infty} dx J^A_1 + \frac{1}{2} f^A_{BC} \int_0^{\infty} dx J^B_0(x) \int_0^{x} dy J^C_0(y)\Big] \\
&\quad + \frac{1}{2} f^A_{BC} \int_0^{\infty} dx J^B_0(x) \int_{-\infty}^{0} dy J^C_0(y).
\end{aligned} \tag{3.7}$$

Upon quantization in absence of anomalies this gives (3.3) on the Hilbert space.

3.2 The centrally-extended $\mathfrak{psu}(2|2)$ Yangian

In the previous section, we have described how algebraic structures related to integrability arise at the two perturbative ends of the AdS/CFT correspondence. To fully exploit these powerful symmetries one needs to take a further step, which allows to go beyond the perturbative regimes. One introduces the choice of a vacuum state, and considers excitations upon this vacuum. This choice breaks the full $\mathfrak{psu}(2,2|4)$ symmetry down to a subalgebra. The excitations carry the quantum numbers of the unbroken symmetry, and they scatter *via* an integrable S-matrix.

The choice that is normally made is, for instance, to consider a string (composite operator) of Z fields (one of the three complex combinations of the six scalar fields of $\mathcal{N} = 4$ SYM) as the vacuum state. The unbroken symmetry consists then of two copies of the $\mathfrak{psu}(2|2)$ Lie superalgebra, which receive central extensions through quantum corrections. The same algebra appears on the string theory side. The excitations carrying the unbroken quantum numbers are called *magnons*, in analogy to the theory of spin-chains and magnetism.

3.2.1 The Hopf algebra of the S-matrix

Upon choosing a vacuum, the residual symmetry carried by the magnon excitations is (two copies of) the centrally extended $\mathfrak{psu}(2|2)$ Lie superalgebra (or $\mathfrak{psu}(2|2)_c$):

$$\begin{aligned}
&[\mathbb{L}_a{}^b, \mathbb{J}_c] = \delta^b_c \mathbb{J}_a - \tfrac{1}{2}\delta^b_a \mathbb{J}_c, && [\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma] = \delta^\beta_\gamma \mathbb{J}_\alpha - \tfrac{1}{2}\delta^\beta_\alpha \mathbb{J}_\gamma, \\
&[\mathbb{L}_a{}^b, \mathbb{J}^c] = -\delta^c_a \mathbb{J}^b + \tfrac{1}{2}\delta^b_a \mathbb{J}^c, && [\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma] = -\delta^\gamma_\alpha \mathbb{J}^\beta + \tfrac{1}{2}\delta^\beta_\alpha \mathbb{J}^\gamma, \\
&\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} = \varepsilon_{\alpha\beta}\varepsilon^{ab}\mathbb{C}, && \{\mathbb{S}_a{}^\alpha, \mathbb{S}_b{}^\beta\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathbb{C}^\dagger, \\
&\{\mathbb{Q}^a_\alpha, \mathbb{S}^\beta_b\} = \delta^a_b \mathbb{R}_\alpha{}^\beta + \delta^\beta_\alpha \mathbb{L}_b{}^a + \tfrac{1}{2}\delta^a_b \delta^\beta_\alpha \mathbb{H}.
\end{aligned} \tag{3.8}$$

The generators $\mathbb{R}_\alpha{}^\beta$ and $\mathbb{L}_a{}^b$ form the two $\mathfrak{su}(2)$ subalgebras which, together with the central elements $\{\mathbb{H}, \mathbb{C}, \mathbb{C}^\dagger\}$, form the bosonic part of $\mathfrak{psu}(2|2)_c$. The names are reminiscent of the unbroken R- and Lorentz symmetry of the model. The fermionic part is generated by the supercharges $\mathbb{Q}_\alpha^a$ and $\mathbb{S}_b^\beta$. The 'dagger' symbol is to remember that, in unitary representations, the two charges are indeed conjugate of each other, and a similar conjugation condition holds for the supercharges.

The representation of [61] gives a *dynamical* spin-chain, *i.e.* sites can be created/destroyed by the action of the generators. The central charges act as

$$\mathbb{H}\,|p\rangle \,=\, \varepsilon(p)\,|p\rangle,\;\; \mathbb{C}\,|p\rangle \,=\, c(p)\,|pZ^-\rangle,\;\; \mathbb{C}^\dagger\,|p\rangle \,=\, \bar{c}(p)\,|pZ^+\rangle, \tag{3.9}$$

where $Z^{+(-)}$ adds (removes) one 'site' (*i.e.*, one of the scalar fields Z in the infinite string that constitutes the vacuum state) to (from) the chain. We denote as $|p\rangle$ the one-magnon state of momentum p. This state is given by $|p\rangle = \sum_n e^{ipn}\,|\cdots ZZ\,\phi(n)\,Z\cdots\rangle$, ϕ being one of the 4 possible orientations of the 'spin' in the fundamental representation of $\mathfrak{psu}(2|2)_c$. The eigenvalue $\varepsilon(p)$ is the energy (dispersion relation) of the magnon excitation. As we will shortly see, $c(p)$ contains the exponential of the momentum p itself. So does $\bar{c}(p)$, which in unitary (*alias*, real-momentum) representations is just the conjugate of $c(p)$.

The length-changing property can be interpreted, at the Hopf algebra level, as a nonlocal modification of the (otherwise trivial) coproduct [3,4]. Let us spell out the case of the central charges. When acting on a two-particle state, one computes

$$\begin{aligned}
&\mathbb{C}\otimes\mathbb{1}\,|p_1\rangle\otimes|p_2\rangle = \\
&\mathbb{C}\otimes\mathbb{1}\sum_{n_1<<n_2} e^{i\,p_1\,n_1+i\,p_2\,n_2}\,|\cdots ZZ\phi_1\,\underbrace{Z\cdots Z}_{n_2-n_1-1}\,\phi_2 Z\cdots\rangle \,= \\
&\qquad (\text{rescaling}\,n_2) \;=\; c(p_1)\,e^{ip_2}\,|p_1\rangle\otimes|p_2\rangle.
\end{aligned} \tag{3.10}$$

This action is non-local, since acting on the first magnon (with momentum p_1) produces a result which also depends on the momentum p_2 of the second magnon.

We must now impose compatibility of the S-matrix with the symmetry algebra carried by the excitations. Imposing such S-matrix invariance condition $\Delta(\mathbb{C})\mathrm{S} = \mathrm{S}\Delta(\mathbb{C})$ implies computing

$$\mathrm{S}\,\Delta(\mathbb{C}) \,=\mathrm{S}\,[\mathbb{C}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbb{C}] \,=\mathrm{S}\,[e^{ip_2}\mathbb{C}_{local}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbb{C}_{local}], \tag{3.11}$$

where $\mathbb{C}_{local}$ is the *local* part of $\mathbb{C}$, acting as $\mathbb{C}_{local}|p\rangle = c(p)|p\rangle$. An analogous argument works for $\Delta(\mathbb{C})\mathrm{S}$. One can rewrite (3.11) as

$$\Delta(\mathbb{C}_{local}) = \mathbb{C}_{local}\otimes e^{ip} + 1\otimes\mathbb{C}_{local}. \tag{3.12}$$

Formula (3.12) is the manifestation of a non-trivial Hopf-algebra coproduct[6]. Similarly, to all (super)charges of $\mathfrak{psu}(2|2)_c$, one assigns an additive quantum number $[[A]]$ s.t.

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A\otimes e^{i[[A]]p} + \mathbb{1}\otimes\mathfrak{J}^A, \tag{3.13}$$

[6] We remark that a (nonlocal) basis change for spin-chain states can produce e^{ip} factors in different places in the coproduct (possibly with a different power), with no deep consequences.

which gives a (Lie) superalgebra homomorphism. Counit and antipode are derived from the Hopf algebra axioms, and the whole structure defines a consistent Hopf algebra. The S-matrix invariance should be written as

$$\Delta^{op} R = R\Delta \tag{3.14}$$

(quasi-cocommutativity), where the invertible R-matrix is defined as $R = P\mathrm{S}$, P being the graded permutation. There is a consistency requirement: since $\Delta(\mathbb{C})$ is central,

$$\Delta^{op}(\mathbb{C})R = R\Delta(\mathbb{C}) = \Delta(\mathbb{C})R \qquad \Longrightarrow \qquad \Delta^{op}(\mathbb{C}) = \Delta(\mathbb{C}). \tag{3.15}$$

This is guaranteed by interpreting as algebraic condition the physical requirement

$$U \equiv e^{ip}\,\mathbb{1} = \kappa\mathbb{C} + \mathbb{1} \tag{3.16}$$

for a constant κ related to the coupling g [61].

A version of the coproduct (3.13) was shown to emerge from the dual worldsheet string-theory. In [62], the result was reproduced by applying the standard Bernard-LeClair procedure [58] to the light-cone worldsheet Noether charges obtained in [63].

A semi-classical argument, based on the same reasoning presented at the end of section 3, is as follows. The light-cone worldsheet Noether supercharges have nonlocal contributions in the physical fields:

$$\mathfrak{J}^A = \int_{-\infty}^{\infty} d\sigma\, J_0^A(\sigma)\, e^{i[[A]]\int_{-\infty}^{\sigma} d\sigma'\,\partial x^-(\sigma')}. \tag{3.17}$$

If we consider, as before, two well-separated soliton excitations, the *semiclassical action* of these charges on such a scattering state is again obtained by splitting the integrals:

$$\begin{aligned} \mathfrak{J}^A{}_{|profile} &= \int_{-\infty}^{\infty} d\sigma\, J_0^A(\sigma)_{|profile}\, e^{i[[A]]\int_{-\infty}^{\sigma} d\sigma'\,\partial x^-(\sigma')_{|profile}} \\ &= \int_{-\infty}^{0} d\sigma\, J_0^A(\sigma) e^{i[[A]]\int_{-\infty}^{\sigma} d\sigma'\,\partial x^-(\sigma')} + \int_{0}^{\infty} d\sigma\, J_0^A(\sigma)\, e^{i[[A]]\int_{-\infty}^{0} d\sigma'\,\partial x^-(\sigma')}\, e^{i[[A]]\int_{0}^{\sigma} d\sigma'\,\partial x^-(\sigma')} \\ &\sim \mathfrak{J}_1^A + e^{i[[A]]p_1}\mathfrak{J}_2^A \;\longrightarrow\; \Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes \mathbb{1} + e^{i[[A]]p} \otimes \mathfrak{J}^A, \end{aligned} \tag{3.18}$$

where one has used the definition of the worldsheet momentum for the first excitation.

From the Hopf-algebra antipode Σ one derives derive the so-called 'antiparticle' representation $\tilde{\mathfrak{J}}^A$ and the corresponding charge-conjugation matrix C:

$$\Sigma(\mathfrak{J}^A) = C^{-1}\,[\tilde{\mathfrak{J}}^A]^{st}\,C, \tag{3.19}$$

where M^{st} is the supertranspose of M. These are the ingredients entering the crossing-symmetry relations originally written down in [2], where the existence of an underlying Hopf-algebra of the S-matrix was conjectured. The antiparticle representation and the constraints on the overall scalar factor of the S-matrix as found in [2], naturally follow from (3.19) combined with the general formulas

$$(\Sigma \otimes \mathbb{1})R = (\mathbb{1} \otimes \Sigma^{-1})R = R^{-1}, \tag{3.20}$$

where the antipode is derived from the coproduct (3.13).

A reformulation in terms of a Zamolodchikov-Faddeev (ZF) algebra has been given in [64]. There, the basic objects are creation and annihilation operators, with commutation relations given in terms of the S-matrix. Also, a q-deformation of this structure and of the one-dimensional Hubbard model is studied in [65, 66].

3.2.2 The Yangian of the S-matrix

The S-matrix in the fundamental representation has been shown to possess $\mathfrak{psu}(2|2)_c$ Yangian symmetry [5]. In order to be a Lie superalgebra homomorphism, the coproduct should respect (3.1). Therefore, the structure of the Yangian coproduct has to take into account the deformation in (3.13):

$$\Delta(\widehat{\mathfrak{J}}^A) = \widehat{\mathfrak{J}}^A \otimes \mathbb{1} + U^{[[A]]} \otimes \widehat{\mathfrak{J}}^A + \frac{1}{2} f^A_{BC} \mathfrak{J}^B U^{[[C]]} \otimes \mathfrak{J}^C. \tag{3.21}$$

The representation for $\widehat{\mathfrak{J}}^A$ is the so-called *evaluation* representation, typically obtained by multiplying level-zero generators by a 'spectral' parameter. Here

$$\widehat{\mathfrak{J}}^A = u \mathfrak{J}^A = ig\,(x^+ + \frac{1}{x^+} - \frac{i}{2g})\,\mathfrak{J}^A. \tag{3.22}$$

The variables $x^\pm$ parameterize the fundamental representation (conventions as in [8]).

A special remark concerns the dual structure constants f^A_{BC}. They should reproduce the general form (3.3), and analogous ones with all indices lowered should be used to prove the Serre relations (3.2). However, since the Killing form of $\mathfrak{psu}(2|2)_c$ is zero, one has a problem in defining these structure constants. In [5], the quantities f^A_{BC} are explicitly given as a list of numbers, without necessarily referring to an index-lowering procedure[7]. The table of coproducts is in this way fully determined.

Another remark concerns the dependence of the spectral parameter u on the representation variables $x^\pm$, or, equivalently, on the eigenvalues of the central charges of $\mathfrak{psu}(2|2)_c$. For simple Lie algebras, the spectral parameter is typically an additional variable attached to the evaluation representation. Together with the existence of a *shift*-automorphism $u \to u + const$ of the Yangian in evaluation representations, this implies that the Yangian-invariant S-matrix is of difference-form $\mathrm{S} = \mathrm{S}(u_1 - u_2)$. The dependence of u on the central charges alters this property, and one does not have a difference form in the fundamental S-matrix (see [72] and section 3.1.1).

The full quantum S-matrix is also invariant under the following exact symmetry, found in [9] and shortly afterwards confirmed in [10]:

$$\begin{aligned}
\Delta(\widehat{\mathbb{B}}') &= \widehat{\mathbb{B}}' \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathbb{B}}' + \frac{i}{2g}(\mathbb{S}^\alpha_a \otimes \mathbb{Q}^a_\alpha + \mathbb{Q}^a_\alpha \otimes \mathbb{S}^\alpha_a),\\
\Sigma(\widehat{\mathbb{B}}') &= -\widehat{\mathbb{B}}' + \frac{2i}{g}\mathbb{H},\\
\widehat{\mathbb{B}}' &= \frac{1}{4}(x^+ + x^- - 1/x^+ - 1/x^-)\,diag(1,1,-1,-1).
\end{aligned} \tag{3.23}$$

[7] An argument in [5] suggests interpreting these quantities as dual structure constants in an enlarged algebra with invertible Killing form, see also [67, 68]. This algebra is obtained by adjoining the $\mathfrak{sl}(2)$ automorphism of $\mathfrak{psu}(2|2)_c$ [69, 70]. Apart from allowing inversion of the Killing form and determination of f^A_{BC}, these extra generators would drop out of the final form of the Yangian coproduct (3.21). We also refer to [71] for a derivation of the Yangian coproducts using the exceptional Lie superalgebra $\mathfrak{D}(2,1;\alpha)$.

This coproduct is reminiscent of a level one Yangian symmetry (cf. (3.3)). We will see in the next section the relevance of this generator for the classical r-matrix. Commuting this symmetry with the (level zero) generators, one obtains novel exact Yangian (super)symmetries of S [9]. The latter act on bosons and fermions with two *different* spectral parameters, reducing in the classical limit to the supercharges of [73].

4 The classical r-matrix

The form of the Yangian we discussed resembles the standard one while simultaneously showing some unexpected features. In order to gain a deeper understanding it is commonly advantageous to study certain limits. One important instance is the *classical* limit, *i.e.* one studies perturbations of the R-matrix around the identity:

$$R = \mathbb{1} \otimes \mathbb{1} + \hbar r + \mathscr{O}(\hbar^2), \tag{4.1}$$

$\hbar$ being a small parameter. The first-order term r is called the *classical* r-matrix[8]. One can easily prove that, if R satisfies the Yang-Baxter equation (YBE), r satisfies the *classical* YBE (CYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{4.2}$$

In known cases, studying (4.2) one can classify the solutions of the YBE itself, and the possible quantum group structures underlying such solutions (Belavin-Drinfeld theorem [74,75]). We will not reproduce here the details. Knowing the r-matrix, there is a standard procedure for constructing an associate Lie bialgebra, and *quantizing* it[9] in terms of so-called 'Manin triples' (see *e.g.* [15]). The quantum structures for simple Lie algebras are elliptic quantum groups, (trigonometric) quantum groups and Yangians. Analogous theorems for superalgebras are investigated in [76–78]. An illuminating example is Yang's r-matrix (C_2 is the quadratic Casimir)

$$r = \frac{C_2}{u_2 - u_1} \ . \tag{4.3}$$

This is the prototypical rational solution of the CYBE[10]. The geometric series gives

$$r = \frac{C_2}{u_2 - u_1} = \frac{\mathfrak{J}^A \otimes \mathfrak{J}_A}{u_2 - u_1} = \sum_{n \geq 0} \mathfrak{J}^A u_1^n \otimes \mathfrak{J}_A u_2^{-n-1} = \sum_{n \geq 0} \mathfrak{J}_n^A \otimes \mathfrak{J}_{A,-n-1}, \tag{4.4}$$

for $|u_1/u_2| < 1$). Such rewriting attributes dependence on the u_1 (u_2) to operators in the first (second) space (*factorization*). This gives r the form of tensor product of algebra representations. Assigning $\mathfrak{J}_n^A = u^n \mathfrak{J}^A$ in (4.4) gives loop-algebra relations

$$[\mathfrak{J}_m^A, \mathfrak{J}_n^B] = f_C^{AB} \mathfrak{J}_{m+n}^C. \tag{4.5}$$

[8] r lives in $\mathfrak{g} \otimes \mathfrak{g}$, for $\mathfrak{g}$ an algebra, R in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

[9] Meaning completing the Lie bialgebra to a quantum group (classical r- to quantum R-matrix).

[10] Since by definition $[C_2, \mathfrak{J}^A \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{J}^A] = 0 \ \forall A$, the CYBE is easily proven for (4.3).

The loop algebra is precisely the 'classical' limit of the Yangian $\mathscr{Y}(\mathfrak{g})$ (see section 3). With this example one realizes how *rational* solutions of the CYBE, such as (4.3), starting as not-better specified elements of $\mathfrak{g}\otimes\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$, give rise to Yangians upon quantization (namely, their quantized version takes values in $\mathscr{Y}(\mathfrak{g})\otimes\mathscr{Y}(\mathfrak{g})$). For related aspects concerning the classical r-matrix, see [45, 46].

4.1 $\mathfrak{psu}(2|2)_c$

In the case of the S-matrix found in [61], the parameter controlling the classical expansion is naturally the inverse of the coupling constant g (near-BMN limit [79]):

$$R = \mathbb{1}\otimes\mathbb{1} + \frac{1}{g}r + \mathscr{O}(\frac{1}{g^2}). \tag{4.6}$$

The classical r-matrix r is identified with the tree-level string scattering matrix computed in [62]. In the parameterization of [80] one has

$$x^{\pm}(x) = x\sqrt{1-\frac{1}{g^2(x-\frac{1}{x})^2}} \pm \frac{ix}{g(x-\frac{1}{x})} \;\rightarrow\; x. \tag{4.7}$$

One sends g to ∞ with x fixed. x is interpreted as an unconstrained 'classical' variable. This classical limit was studied in [81]. The target is finding the complete algebra the r-matrix takes values in, whose quantization can reveal the full quantum symmetry of the S-matrix. The fundamental representation tends to a limiting centrally-extended $\mathfrak{psu}(2|2)$, with generators parameterized by x. The classical r-matrix $r = r(x_1,x_2)$ is not of difference form. The Lie superalgebra is not simple and has zero dual Coxeter number. This prevents applying Belavin-Drinfeld type of theorems. Nevertheless, r has a simple pole at $x_1 - x_2 = 0$ with residue[11] the Casimir C_2 of $\mathfrak{gl}(2|2)$:

$$C_2 = \sum_{i,j=1}^{4} (-)^{[j]} E_{ij}\otimes E_{ji}, \tag{4.8}$$

with E_{ij} matrices with all zeros but 1 in position (i,j), and $[j]$ the fermionic grading of the index j. In the absence of a quadratic Casimir for $\mathfrak{psu}(2|2)_c$, the classical r-matrix displays on the pole (it 'borrows') the Casimir of a bigger algebra[12] for which a non-degenerate form exists and the quadratic Casimir can be constructed. This 'borrowing' reminds a mathematical prescription due to Khoroshkin and Tolstoy [82, 27]. One expects that, if a universal R-matrix exists and if it has to be of Khoroshkin-Tolstoy type, an additional Cartan element of type $\mathbb{B}$ has to appear.

Type-$\mathbb{B}$ generators play an important role in factorizing r. The present r is more complicated than Yang's one, and it is harder to find a suitable geometric-like series

[11] As a consequence of the CYBE, such residue must be a Casimir.

[12] $\mathfrak{gl}(2|2)$ is obtained by adjoining to $\mathfrak{su}(2|2)$ the non-supertraceless element $\mathbb{B} = diag(1,1,-1,-1)$.

expansion. A first proposal for the fundamental representation was given [73], with a Yangian tower of $\mathbb{B}$'s coupled to a tower of $\mathbb{H}$'s to achieve factorization. This proposal fails to reproduce the bound-state classical r-matrix [83].

A universal formula was advanced in [10]. It has been shown to reproduce also the classical limit of the bound-state S-matrix [84, 8], and it reads

$$r = \frac{\mathscr{T} - \tilde{\mathbb{B}} \otimes \mathbb{H} - \mathbb{H} \otimes \tilde{\mathbb{B}}}{i(u_1 - u_2)} - \frac{\tilde{\mathbb{B}} \otimes \mathbb{H}}{iu_2} + \frac{\mathbb{H} \otimes \tilde{\mathbb{B}}}{iu_1} - \frac{\mathbb{H} \otimes \mathbb{H}}{\frac{2iu_1u_2}{u_1-u_2}}, \tag{4.9}$$

$$\mathscr{T} = 2\left(\mathbb{R}_\beta^{\ \alpha} \otimes \mathbb{R}_\alpha^{\ \beta} - \mathbb{L}_b^{\ a} \otimes \mathbb{L}_a^{\ b} + \mathbb{S}_a^{\ \alpha} \otimes \mathbb{Q}_\alpha^{\ a} - \mathbb{Q}_\alpha^{\ a} \otimes \mathbb{S}_a^{\ \alpha}\right),$$
$$\tilde{\mathbb{B}} = \frac{1}{4\,\varepsilon(p)}\, diag(1,1,-1,-1). \tag{4.10}$$

In this formula, the generators are in their classical limit, the variable u is the classical limit of (3.22), and $\varepsilon(p)$ is the classical energy (cf. section 4.1). All classical Yangian generators are obtained as $\mathfrak{J}_n = u^n \mathfrak{J}$ after factorizing *via* the geometric series expansion. Quantization of this formula is an open problem. The classical analysis seems to suggest that the triple central extension may have to merge into some sort of deformation of the loop algebra of $\mathfrak{gl}(2|2)$, where the additional generator $\mathbb{B}$ is sitting. Another open question is how to relate the results described here to the r, s non-ultralocal structure of the $\mathfrak{psu}(2,2|4)$ sigma-model [45, 46, 85].

4.1.1 Difference Form

Formula (4.9) displays an interesting structure where the dependence on the spectral parameter u is (almost purely) of difference form. The non-difference form is encoded in the representation labels $x^{\pm}(u)$ appearing in the symmetry generators, and in the last three terms of formula (4.9). Moreover, Drinfeld's second realization for the $\mathfrak{psu}(2|2)_c$ Yangian has been obtained in [86], together with the suitable evaluation representation. The Yangian Serre relations, which were left as an open question in [5], are proven to be satisfied in the second realization (see also [87].) The representation of [86] possesses a shift-automorphism $u \to u + const$, which normally guarantees the difference form of the S-matrix. All this suggests the following, provided an algebraic interpretation of the last three terms in formula (4.9) can be found that generalizes to the full quantum case (possibly along the case of the ideas reported in [10] in terms of twists). Modulo this interpretation, one might hope to achieve a rewriting of the quantum S-matrix such that the dependence on u_1 and u_2 is (almost purely) of difference form, the rest being taken care of by suitable combinations of algebra generators[13]. One would expect this as the result of evaluating a

[13]In the fundamental representation, such a rewriting has been shown to be possible in [88]. The resulting form is reminiscent of what a Khoroshkin-Tolstoy type of formula (or some natural quantization of the classical r-matrix (4.9)) would look like in this representation.

hypothetical Yangian universal R-matrix in this particular representation. This expectation seems to be consistent with recent studies of the exceptional Lie superalgebra $\mathfrak{D}(2,1;\alpha)$ [61,71,87][14], and with the explicit form of the bound state S-matrix (see next section).

5 The bound state S-matrix

The previous discussion highlights the importance of investigating the structure of the S-matrix for generic representations of $\mathfrak{psu}(2|2)_c$. One motivation is obtaining the universal R-matrix and understanding the role of the $\widehat{\mathbb{B}}'$ symmetry. There is also a more stringent need related to finite-size corrections to the energies according to the TBA approach [90]. According to this philosophy, it becomes crucial to have a concrete realization of the (mirror) bound state S-matrices. Usually, these can be *bootstrapped* once the S-matrix of fundamental constituents is known [13,14]. However, the present case is more complicated. The fundamental S-matrix does not reduce to a projector on the bound state pole, related to the fact that the tensor product of two short representations (generically irreducible) becomes reducible but indecomposable on the pole. The only way to construct the S-matrix for bound states seems to be a direct derivation from the Lie superalgebra invariance in each bound state representation. This becomes rapidly cumbersome [91]. Moreover, this does not uniquely fix the S-matrix when the bound state number increases, and one needs to resort to YBE, or, as shown in [84], to Yangian invariance. The Yangian eventually provides an efficient solution to this problem and it allows to uniquely determine the S-matrix for arbitrary bound state numbers [8].

The bound state representations are atypical (short) completely symmetric representations of dimension 4ℓ, $\ell = 1,2,\dots$. They all extend to evaluation representations of the Yangian, with appropriate evaluation parameter u [84]. A convenient realization is given in terms of differential operators acting on the space of degree M polynomials (superfields) in two bosonic (w_a, $a=1,2$) and two fermionic (θ_α, $\alpha=1,2$) variables. All details can be found in [8]. The essence of the construction consists in finding a closed subset of states $|x_i\rangle$ for which the S-matrix can be computed exactly in terms of a definite matrix M. One then generates all other states $|y_A\rangle$ by acting with (Yangian) coproducts on this closed subsector, and using quasi-cocommutativity:

$$R|y_A\rangle = R\,\Delta(\mathbb{J})^i_A\,|x_i\rangle = \Delta^{op}(\mathbb{J})^i_A\,R\,|x_i\rangle = \Delta^{op}(\mathbb{J})^i_A\,M^j_i\,|x_j\rangle. \tag{5.1}$$

On the other hand, $R|y_A\rangle = R^B_A\,|y_A\rangle = R^B_A\,\Delta(\mathbb{J})^i_B\,|x_i\rangle$. The task is to find as many states as needed to invert the above relation, namely $R^B_A = \Delta^{op}(\mathbb{J})^i_A\,M^j_i\,[\Delta(\mathbb{J})^{-1}]^B_j$.

The construction automatically provides a *factorizing twist* [92] for the R-matrix

[14] $\mathfrak{psu}(2|2)_c$ can be obtained by suitable contraction of $\mathfrak{D}(2,1;\alpha)$. See also [89].

in the bound state representations (hence also for the fundamental representation):

$$R = F_{21} \times F_{12}{}^{-1}. \tag{5.2}$$

However, we remark that the coproduct twisted with F_{12} is by construction cocommutative, but, as expected, not at all trivial. Furthermore, apart perhaps from the overall factor, the bound state S-matrix depends only on $u_1 - u_2$, on combinatorial factors involving the integer bound-state components, and on specific combination of algebra labels a_i, b_i, c_i, d_i. These combinations are the same noticed in [88]. It remains hard to figure out a universal formula reproducing this S-matrix. Nevertheless, it looks like such a universal object would treat the evaluation parameters of the Yangian as truly independent variables, the latter appearing only in difference-form due to the Yangian shift-automorphism. The rest of the labels would appear because of the presence in the universal R-matrix of the (super)charges in the typical 'positive $\otimes$ negative'-roots combinations, breaking the difference-form due to the constraint that links the evaluation parameter to the central charges. This is consistent with the findings of [93], where one of the blocks of the S-matrix has been related to the universal R-matrix of the Yangian of $\mathfrak{sl}(2)$ in arbitrary bound state representations.

The bound state S-matrix have been utilized in [94] to verify certain conjectures appeared in the literature, concerning the eigenvalues of the transfer matrix in specific short representations [70]. Long representations have been studied in [95].

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