INTERPOLATED ALLPASS FRACTIONAL-DELAY FILTERS USING ROOT DISPLACEMENT

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ABSTRACT

Fractional-delay filter is the general name given to filters modelling non-integer delays. Such filters have a flat phase delay for a wide frequency band, with the value of the phase delay approximating the fractional delay. A maximally-flat delay IIR fractional-delay filter can be obtained by the Thiran approximation. A simple and efficient method for obtaining filters modelling intermediate fractional delays from two Thiran fractional-delay filters is proposed. The proposed method allows continuously modifying the fractional delay. Computational complexity of the proposed method is discussed. A practical application of the method in model-based sound synthesis is given as an example.

1. INTRODUCTION

In many different areas of signal processing, the fixed sampling period becomes an obstacle for a variety of tasks. If one is to synchronize a digital signal to a received analog signal, the problems may arise. The solution to this and similar problems is to design and use filters that have a flat phase-delay for a desired frequency range. Such filters are known as the fractional-delay filters [1], and are used in diverse set of contexts in discrete-time signal processing, such as spatial audio synthesis [2], model-based sound synthesis [3], frequency synchronization in wireless telecommunications [4], sample-rate conversion [5], and array processing [6].

There exist different varieties of fractional-delay filters. Farrow structure [4], and Lagrange interpolators [7] are two simple examples of FIR-based fractional-delay filters. The popular Farrow structure allows continuous modification of the modelled fractional delay using a single parameter. However, if the magnitude response of the output signal is of concern, the Farrow structure is far from perfect, as it essentially has a low-pass magnitude response. This becomes a significant problem especially for audio, and acoustics applications where one needs to maintain the full bandwidth of the input signal. A feasible solution is to use allpass fractional-delay filters.

There are a number of different allpass fractional-delay design methods [8]. Major drawback in the great majority of those is the computational complexity required to obtain the filter coefficients. A simple allpass fractional-delay filter can be obtained by the Thiran approximation [9, 10]. However, it is not as straightforward with Thiran fractional-delay filters as the Farrow structure to continuously modify the fractional-delay. This paper presents a simple and effective method to obtain the coefficients of an intermediate fractional-delay filter given the poles of two Thiran fractional-delay filters. The method was previously proposed by the authors for the interpolation of arbitrary-shape minimum-phase FIR filters [11].

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Section 2 briefly summarizes fractional-delay filters with maximally-flat phase delay, and the Thiran approximation for obtaining the filter coefficients. Previously proposed variable fractional-delay filter structures are mentioned. Section 3 explains the root-displacement method, and its application to the interpolation of allpass fractional-delay filters. Accuracy and the computational complexity of the interpolation is discussed in Section 4. A practical example application of the method to model-based sound synthesis is given in Section 5, where the frequency of a digital waveguide string is shifted by varying the length of the feedback path. Section 6 concludes the paper.

2. MAXIMALLY-FLAT ALLPASS FRACTIONAL-DELAY FILTERS

An allpass fractional-delay filter with a maximally-flat phase delay models the non-integer delay, D. While the magnitude response of the filter is unity for each frequency, the phase-delay of the filter approximates the fractional delay over a suitable bandwidth. The ideal fractional delay can be represented as:

$$H_{ideal}(z) = z^{-D} \tag{1}$$

Therefore, a maximally-flat allpass fractional-delay filter will approximate this arbitrary delay element for a wide bandwidth, and have a phase delay exactly equal to D samples at DC. An Nth-order allpass IIR filter has the following transfer function:

$$H(z) = \frac{B(z)}{A(z)} = \sum_{k=0}^{N} a_{N-k} z^{-k} \Big/ \sum_{k=0}^{N} a_k z^{-k}, \tag{2}$$

where $B(z)$ and $A(z)$ are the numerator and denominator polynomials respectively. The coefficients of the allpass fractional delay filter can be calculated approximately using the following formula:

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{D-N+n}{D-N+n+k}, \tag{3}$$

where D is the total fractional delay, and k is the coefficient index. The filter with the given coefficients is guaranteed to be stable, and to have maximally-flat phase delay at low frequencies corresponding to the desired fractional delay, D, at the zero frequency. The Thiran approximation can be used to obtain a fractional-delay filter for $D > N-1$ [10].

Given that the coefficients in Equation 3 are to be obtained, the calculation of the k^{th} coefficient requires (N+2) multiplications and

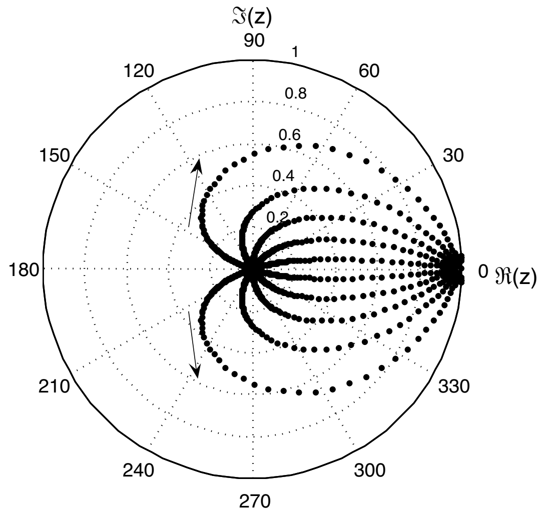

Fig. 1. The loci of the poles of 10^{th}-order Thiran filters. The fractional part of the delay is exponentially increased from 10^{-50} to 10^{50} samples.

(N+1) divisions if the complexity associated with the calculation of the initial combinatorial constant which can be obtained from a look-up table is ignored. For an Nth-order Thiran filter this amounts to $(N^2 + 2N)$ multiplications and $(N^2 + N)$ divisions. This presents an obstacle if one needs to continuously vary the fractional-delay modelled by the filter. Therefore, several different variable structures have been proposed [12, 13] which are derived from the ideas presented by Farrow for FIR filters. Although quite accurate, these filter structures are computationally more complex than a simple IIR filter of the same order.

3. ROOT DISPLACEMENT METHOD

Assume that we have two N^{th}-order Thiran fractional-delay filters, $H_1(z)$ and $H_2(z)$ modelling two different fractional delays D_1 and D_2. The aim of interpolation is to obtain a new allpass fractional-delay filter which models an intermediate delay, D_i, such that $D_1 < D_i < D_2$.

Poles and zeros of a real-valued allpass IIR filter are found at conjugate reciprocal positions. Therefore, we can limit the interpolation operation to the roots of the denominator polynomial, as the stability conditions are imposed on the magnitudes of the filter poles. The denominator of an N^{th}-order IIR filter is itself an Nth-order polynomial and the roots of a Thiran filter do not reveal themselves to a simple closed-form representation for $N \geq 5$ as stated by the Abel-Ruffini theorem. However, the roots can be obtained numerically by using the iterative polynomial root-finding techniques such as calculating the eigenvalues of the companion matrix by QR decomposition, or by utilizing the Newton-Raphson method [14]. Figure 1 shows the loci of the poles of 10^{th}-order Thiran fractional-delay filters modelling a fractional delay of 10+d, where d is varied logarithmically from 10^{-50} to 10^{50} samples.

It should be noted for Thiran filters that, only the odd-order filters have one and only one positive real pole, and the corresponding reciprocal real zero. It is possible to express the denominator polynomial, $A(z)$ in Equation 2 in the following way:

$$A_i(z) = \begin{cases} \left[1 - r_i z^{-1}\right] \times \prod_{k=1}^{(N-1)/2} \left[1 - c_{i,k}^2 z^{-2}\right], & \text{N odd} \\ \prod_{k=1}^{N/2} \left[1 - c_{i,k}^2 z^{-2}\right], & \text{N even} \end{cases} \quad (4)$$

where $\{c_{i,k}, c_{i,k}^*\}$ is the k^{th} complex conjugate pole pair and r_i is the real pole of the filter H_i.

Let us consider only the positive conjugate poles of the fractional-delay filter. We first sort the positive conjugate poles according to their angles, and pair them with respect to their angular proximity. From the paired poles, root displacement vectors, $\{\overrightarrow{\nu_k}\}$, are calculated such that:

$$\overrightarrow{\nu_k} = c_{2,k} - c_{1,k} \quad (5)$$

The interpolated complex poles can be calculated using the root displacement vectors as:

$$c_{int,k} = c_{1,k} + \rho\overrightarrow{\nu_k} = [1 - \rho]\, c_{1,k} + \rho\, c_{2,k}, \quad (6)$$

where ρ is a constant between 0 and 1. The magnitude of the interpolated pole can be calculated as:

$$\begin{aligned} |c_{int,k}| &= |(1-\rho)c_{1,k} + \rho\, c_{2,k}| \\ &= \left[(1-\rho)^2 |c_{1,k}|^2 + \rho^2 |c_{2,k}|^2 + \right. \\ &\quad \left. (2\rho - 2\rho^2) |c_{1,k}| |c_{2,k}| \cos\xi \right]^{\frac{1}{2}}, \end{aligned} \quad (7)$$

where ξ is the angular separation between the poles. Therefore, it can be shown for $0 \leq \rho \leq 1$ that the interpolated poles are inside the unit circle if the original poles are inside the unit circle (i.e. $|c_{1,k}|, |c_{2,k}| < 1$). Thus, the proposed interpolation method retains the stability of the original filters.

Consider two poles, $c_{1,k} = K_1 + jL_1$ and $c_{2,k} = K_2 + jL_2$, with their complex conjugate pairs. The interpolated complex conjugate pole pair can be calculated as:

$$\begin{aligned} \{c_{int,k}\} &= K_{int,k} \pm jL_{int,k} \\ &= [(1-\rho)K_1 + \rho K_2] \pm j\,[(1-\rho)L_1 + \rho L_2]\,. \end{aligned} \quad (8)$$

The interpolated poles constitute the roots of the denominator polynomial, $A_{int,k}(z)$, of a second-order section of the allpass fractional delay filter, $H_{int,k}(z) = B_{int,k}(z)/A_{int,k}(z)$, such that:

$$\begin{aligned} A_{int,k}(z) &= 1 + a_{k,1}z^{-1} + a_{k,2}z^{-2}, \\ &= 1 - 2K_{int,k}z^{-1} + (K_{int,k}^2 + L_{int,k}^2)z^{-2}. \end{aligned} \quad (9)$$

The denominator coefficients can thus be obtained easily from the real and complex parts of the interpolated pole (see Figure 2). The numerator coefficients are simply obtained by "mirroring" the denominator polynomial. The fractional-delay filter can be implemented as a cascade of tuneable second-order allpass IIR sections. If the filter order, N, is odd, the remaining real pole can be interpolated in the same way. The corresponding pole/zero pair can be appended to the cascade as a first-order block.

Figure 3 shows the phase delays of interpolated filters between two 10^{th}-order Thiran fractional-delay filters modelling 10.1 and 10.3 samples delay. The figure depicts phase delays of the interpolated filters with 0.05 sample intervals, as well as phase delays of the calculated Thiran fractional delay filters. It may be observed that in comparison with the original filters representing the same fractional delay, the interpolated filters have a shorter bandwidth of flat phase delay, although the maximal-flatness condition is not fully satisfied.

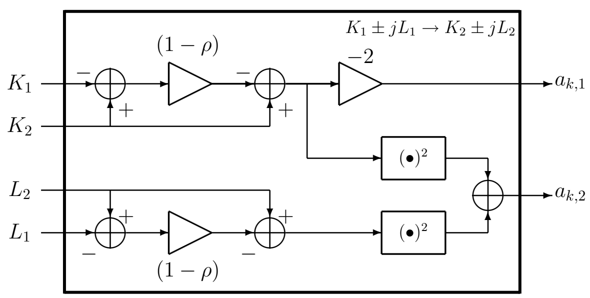

Fig. 2. Obtaining the filter coefficients of the interpolated second-order IIR section from original filter poles.

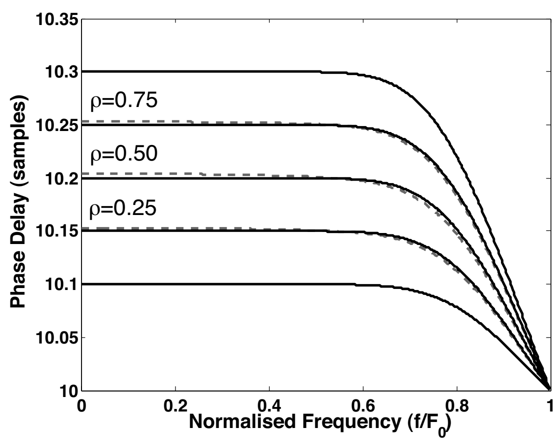

Fig. 3. Phase delays of original (solid line) vs. interpolated (dashed line) fractional delay filters.

4. ACCURACY AND COMPUTATIONAL COMPLEXITY

The accuracy of interpolation may be examined from the frequency response error (FRE), which is the absolute deviation of the interpolated filter from the ideal fractional delay. It should be observed that, as the interpolated filters are also allpass, the FRE represents the error in the modelled fractional-delay.

$$\mathrm{FRE}(e^{j\omega}) = 20\log_{10}\left|e^{-j\omega D_i} - H_{int}(e^{j\omega})\right| \qquad (10)$$

where D_i is the desired intermediate delay, and $H_{int}(e^{j\omega})$ is the frequency response of the interpolated allpass fractional-delay filter. Figure 4 shows FRE functions for an interpolated fractional-delay value of 8.3 samples using different original filters ($8.25-8.35$, $8.2-8.4$, $8.15-8.45$, $8.1-8.5$, $8.05-8.55$, and $8.0-8.6$) as well as the FRE for an original Thiran allpass filter modelling the fractional delay of 8.3 samples. It may be observed from the figure that the FRE is generally lower than around -30dB in the frequency band of interest regardless of the original filters used in the given range. However, it may also be observed that the difference in the delay modelled by each original filter also affects the accuracy of the interpolation. A smaller difference results in a lower FRE. It should be noted that the proposed interpolation provides an approximation

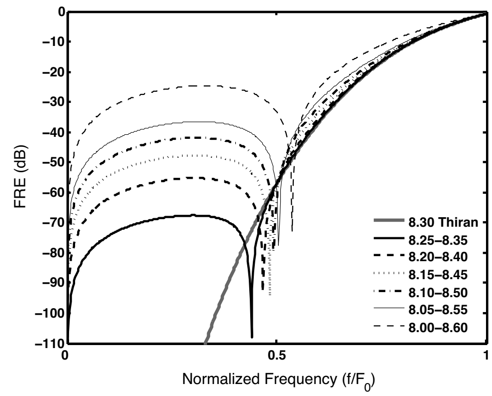

Fig. 4. The frequency response error (FRE) for interpolated fractional-delay filters.

to a Thiran allpass filter which already is a bandlimited approximation to an ideal transfer function modelling a fractional delay. Thus the proposed interpolation is more suitable for applications which can tolerate a slight error of approximation. It should be emphasized that the interpolated filters are not maximally-flat. However, they still approximate the desired phase delay successfully.

As for the computational complexity, the proposed method is quite efficient. Given that the poles of the coefficients are not calculated during runtime and stored in the memory instead, the calculation of an intermediate filter from the roots of two Nth-order Thiran filters requires $4N$ multiplications, and no divisions are necessary. Therefore, the interpolation approach is more efficient than calculating the filter coefficients for $N > 1$. This provides an efficient way of approximately obtaining intermediate delays at low computational cost.

5. AN EXAMPLE: VARIABLE DIGITAL WAVEGUIDE STRING

Digital waveguides present a flexible and efficient tool for synthesizing musical instrument sounds, modelling the vocal tract, or simulating the room acoustics. A very simple example is the digital waveguide string model [15] which is used for modelling plucked string instruments like the guitar. The digital waveguide string is essentially a damped digital oscillator. The fundamental frequency of the digital string depends on the length of the digital string, or in other words the total delay in the feedback path. It is possible to use variable fractional delay filters to change the total delay in the feedback path to model *true glissando* (i.e. continuous pitch shift).

Figure 5 shows the digital waveguide string whose feedback path is implemented with a fixed delay of 10 samples, and five 4^{th}-order Thiran fractional delay filters in series. The filter $W_d(z) = 0.965/(1-0.03z^{-1})$ is a single pole loss filter for exponentially damping the output [16].

For demonstrating the effect of interpolation of the fractional delay filters, we excite the digital waveguide string with a Hamming pulse of 10 samples. The proposed method is used to vary the fractional delay modelled by each fractional-delay between 4.0 and 8.0 samples with steps of 0.04 samples. Therefore, the poles of two Thiran filters modelling these delays are stored and the interpolation is made between those. Initially, the total delay of the digital waveg-

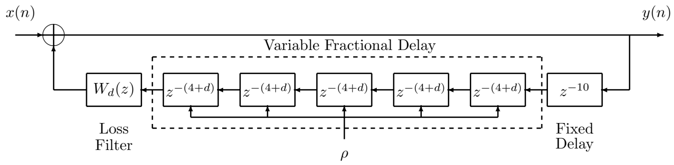

Fig. 5. Simple digital waveguide string as implemented with variable Thiran fractional delay filters in the feedback path.

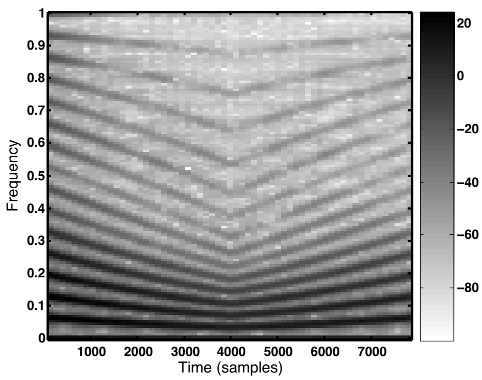

Fig. 6. Magnitude spectrogram of the glissando on the variable fractional-delay line digital waveguide string

uide string is 30 samples corresponding to a fundamental frequency of $f = 166$ Hz at a sampling frequency of $F_0 = 8$ kHz. Coefficient update is carried out at every 40 samples. At the 4000th sample, the total delay modelled by each fractional delay filter is 8 samples, increasing the effective length of the digital waveguide string to 50 samples. This corresponds to 80 Hz at the same sampling frequency. The length is reduced back to 30 samples in the following 4000 sample period at the same rate-of-change.

Figure 6 shows the magnitude spectrogram of this glissando effect. Frequency shift may be observed both with the fundamental frequency and with the higher harmonics. However, due to the bandwidth limitations of the Thiran filters, the change in the higher harmonics is not ideal.

6. CONCLUSIONS

A simple and effective interpolation strategy which utilizes the poles of an allpass fractional-delay filter was presented in this paper. The method allows continuous variation of the fractional-delay using a single parameter.

The major advantage of the method is the reduced computational complexity which requires $4N$ multiplications for an Nth order allpass fractional-delay filter. This is more efficient than obtaining the filter coefficients at runtime for $N > 1$.

The interpolated fractional-delay filters are stable allpass filters. The error introduced by the interpolation is not substantial when compared with original filters. However, the method is more suitable for applications where efficiency rather than accuracy is favoured. True glissando on a fractional-delay digital waveguide string is presented to demonstrate the usage of the method.

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