Invariant measures exist without a growth condition

Henk Bruin[*], Weixiao Shen[†], and Sebastian van Strien

September 1, 2003

Abstract

Given a non-flat S-unimodal interval map f, we show that there exists C which only depends on the order of the critical point c such that if $|Df^n(f(c))| \geq C$ for all n sufficiently large, then f admits an absolutely continuous invariant probability measure (acip). As part of the proof we show that if the quotients of successive intervals of the principal nest of f are sufficiently small, then f admits an acip. As a special case, any S-unimodal map with critical order $\ell < 2 + \varepsilon$ having no central returns possesses an acip. These results imply that the summability assumptions in the theorems of Nowicki & van Strien [21] and Martens & Nowicki [17] can be weakened considerably.

1 Introduction

In this paper we consider S-unimodal C^3 maps $f\colon [0,1] \to [0,1]$. We assume the unique critical point c has order $\ell > 1$, *i.e.*, for x near c, there exists a C^2 diffeomorphism φ such that $f(x) = \varphi(|x-c|^\ell)$.

Theorem 1. *There exists $C = C(\ell)$ so that provided $|Df^n(f(c))| \geq C$ for all n sufficiently large, f admits an absolutely continuous invariant probability measure (acip).*

[*]HB was supported by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW)

[†]WS was supported by EPSRC grant GR/R73171/01

The problem dealt with in Theorem 1 has a long history, with contributions by amongst others [1], [22], [8], [5], [19], [20], [21]. In particular Theorem 1 shows that the well-known Collet-Eckmann condition ($|Df^n(f(c))| \leq C\gamma^n$ for some $\gamma \in (0,1)$, see [5]) or the more recent summability condition ($\sum_n |Df^n(f(c))|^{-1/\ell} < \infty$, see Nowicki & van Strien [21]) are far too restrictive. No growth is needed. Recently, many people are considering weakly hyperbolic systems (in particular in dimensions 2 and larger). Perhaps our techniques indicate that one might not always need to look for growth conditions.

A key idea in our proof is to construct an induced Markov map, and analyse the non-linearities and transition probabilities of the resulting random walk. This Markov map has branches with arbitrarily small ranges. The Markov map we construct is based on the so-called *principal nest*, and the estimates for the transition probabilities come from a careful analysis of the geometry of this principal nest. So let us define this nested sequence of neighbourhoods of the critical point c starting with $I_0 = (\hat{q}, q)$, where $q \in (0,1)$ is the orientation reversing fixed point of f and $f(\hat{q}) = f(q)$. Then define inductively I_{n+1} to be the central domain of the first return map to I_n. To continue the induction, we need to assume that c is recurrent, *i.e.,* $\omega(c) \ni c$. Without this assumption, f is a Misiurewicz map, and the conclusions of this paper then follow easily (or from well-known results). Write

$$\mu_n = |I_{n+1}|/|I_n|.$$

Our paper deals with the case that μ_n is small for all large n.

Before stating our result second theorem, let us first discuss μ_n. Estimating the μ_n has been an eminent problem in one-dimensional dynamics, cf. [6, 7, 9, 12]. More precisely, it has been asked if the *starting condition* [9]

$$\forall \varepsilon > 0 \; \exists n_0 > 0 \; \mu_{n_0} < \varepsilon. \tag{1}$$

holds. We speak of a *central return* of c to I_n if the first return $f^s(c)$ of c into I_n belongs also to I_{n+1}. If $\ell \leq 2$ and there are no central returns, an inductive argument ([9], [12]) shows that (1) implies

$$\forall \varepsilon > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 \; \mu_n < \varepsilon; \tag{2}$$

(if there are central returns at times $n(k)$ then in (2) then this only holds at all 'non-central' times. Lyubich [12] and Graczyk & Świątek [6], using complex methods, have established the starting conditions for quadratic maps.

Note that prior to the results [6, 12], the starting condition was verified for quadratic maps with so-called Fibonacci combinatorics [13, 11]. For this map, it is crucial that the critical order is $\ell = 2$, because for $\ell > 2$, (1) fails: μ_n does **not** tend to zero. More precisely, as was shown in [11],

$$\exists \varepsilon = \varepsilon(\ell) > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 \; \mu_n \leq \varepsilon \text{ and } \varepsilon(\ell) \searrow 0 \text{ as } \ell \searrow 2. \tag{3}$$

In fact, when ℓ is large then μ_n is close to 1 for all n (for the Fibonacci map); this implies that a Fibonacci map with large critical order possesses a Cantor attractor, see [4].

Recently, Shen [23] showed, by purely real methods, that for all C^3 S-unimodal maps without central returns that

- (1) holds for $\ell \in (1, 2]$,
- (3) holds for $\ell > 2$ close to 2.

In this paper we will show that (3), *i.e.*, large values of $|I_n|/|I_{n+1}|$ when n is large, guarantee the existence of an f-invariant measure μ that is absolutely continuous with respect to Lebesgue (acip).

Theorem 2. *There exists $\varepsilon = \varepsilon(\ell)$ such that if $|I_{n+1}| \leq \varepsilon |I_n|$ for all n sufficiently large, then f admits an acip.*

Remark 1. *We do not need to assume that f has no central returns for this theorem to hold.*

Theorem 2 extends a theorem of Martens & Nowicki [17] stating that $\sum_n \mu_n^{1/\ell} < \infty$ implies the existence of an acip. In fact, as they show, $\sum_n \mu_n^{1/\ell} < \infty$ implies the Nowicki-van Strien summability condition. Theorem 2 is strictly stronger: for example for the Fibonacci map with critical order $2 + \varepsilon$ the summability conditions fail, but our assumption holds. Theorem 2 also extends the result of Keller & Nowicki [11] for Fibonacci maps of order $2 + \eta$ to more general maps:

Corollary 1. *There exists $\eta > 0$ such that for every C^3 S-unimodal map f with critical order $\ell < 2 + \eta$, and with a finite number of central returns holds: If f has no periodic attractor, then f has an acip.*

Proof of Corollary 1. This follows from Shen's result [23] that under the above conditions, there exists $\varepsilon = \varepsilon(\ell)$ such that $|I_{n+1}| \leq \varepsilon |I_n|$ for n sufficiently large and that $\varepsilon \to 0$ as $\eta \to 0$. □

In [3], conditions (reminiscent of Fibonacci combinatorics) are given under which f has an acip, irrespective the critical order as long as $\ell < \infty$. One can interpret Corollary 1 as a proof that the only mechanism for unimodal maps with critical order $\ell < 2 + \eta$ not to have an acip, is by (deep) central returns, either of *almost restrictive interval* type (cf. [10]) or of *almost saddle node* type (cf. [2]).

2 Preliminaries and structure of the proof

Let us start making precise the condition on f. It is a C^3 unimodal map with negative Schwarzian derivative such that $f^2(c) < c < f(c)$ and $f^3(c) \geq f^2(c)$. Hence we can rescale f such that $f^2(c) = 0$ and $f(c) = 1$. The critical order $\ell \in (1, \infty)$, the critical point is recurrent but not periodic.

Let us first show that Theorem 2 implies our first theorem:

Proof of Theorem 1. Let $k(n)$ be the minimal integer for which $f^{k(n)}(c) \in I_n$. Then I_{n+1} is the pullback of I_n by $f^{k(n)}$. By real bounds, [18], there exists $\delta > 0$ (which does not depend on n) and a neighbourhood T of $f(I_{n+1})$, such that $f^{k(n)-1}$ maps T diffeomorphically onto a δ-scaled neighbourhood of I_n. Hence

$$\begin{aligned} |Df^{k(n)}(f(c))| &= |Df(f^{k(n)}(c))| \cdot |Df^{k(n)-1}(f(c))| \\ &\leq \ell |I_n|^{\ell-1} \cdot K \frac{|f^{k(n)}(I_{n+1})|}{|f(I_{n+1})|} \\ &\leq \ell |I_n|^{\ell-1} \cdot K \frac{|I_n|}{|I_{n+1}|^{\ell}} \leq \ell K \frac{|I_n|^{\ell}}{|I_{n+1}|^{\ell}}, \end{aligned}$$

where we have used the non-flatness of f and Koebe. Therefore, one obtains that $|I_{n+1}|/|I_n|$ is small provided $|Df^{k(n)}(f(c))|$ is large.

It is possible that f is renormalizable. In that case $k(n)$ is equal to the period p of this renormalization for all n large and I_n shrinks to the largest periodic renormalization interval J (and so $|I_{n+1}|/|I_n| \to 1$). Then use the same argument for the renormalization: repeat the construction of the principal nest for $f^p|J$. Assume f is s times renormalizable and J_s is its s-th renormalization interval with period p_s. Intervals I_n associated to its $(s-1)$-th renormalization shrink to the s-th renormalization interval J_s, and therefore $|Df^{p_s}(f(c))| \leq \ell K \frac{|I_n|^\ell}{|I_{n+1}|^\ell} \leq 2\ell K$ for n sufficiently large. But since $p_s \geq 2^s$, this and the assumption of Theorem 1 imply that s must be bounded, and so f can only be finitely often renormalizable. Then consider instead of f its last renormalization $f^s|J_s$. Since the above inequality gives that $|I_n|/|I_{n+1}|$ is large for all n large (and in particular $|I_n| \to 0$ as $n \to \infty$), we can apply Theorem 2 and obtain an invariant measure. □

So it suffices to prove Theorem 2. The boundary points of each I_n are *nice* in the sense of Martens [16], which means that $f^i(\partial I_n) \notin I_n$ for all $i > 0$. In fact, $f^i(\partial I_n) \notin I_{n-1}$. This allows the following priori estimates:

Lemma 1. *If $J \subset I_n$ is a component of the domain of the first return map to I_n for some $n > 0$, say $f^s|J$ is this return, then there exists an interval $T \supset f(J)$ such that $f^{-1}(T) \subset I_n$ and such that $f^{s-1}|T$ is a diffeomorphism onto I_{n-1}.*

Proof of Lemma 1. See Martens [16] or Section V.1 in [18]. □

The idea is now to construct a Markov induced map G over f with the intervals I_n as countable set of ranges: G is defined on a countable collection of intervals J_i, $G|J_i = f^{s_i}|J_i$ is a diffeomorphism and $G(J_i) = I_n$ for some n. We then will construct a G-invariant measure $\nu \ll \text{Leb}$, and estimate $\nu(I_n)$:

Proposition 1. *Assume that $\mu_n \leq \varepsilon$ for all $n \geq n_0$. If ε is sufficiently small, then the induced transformation G admits an acip ν. Moreover, there exists $C_0 = C_0(f)$ such that $\nu(I_n) \leq C_0\sqrt{|I_n|}$ for all n.*

Corollary 2. *Under the above conditions, f admits no Cantor attractor.*

Proof of Corollary 2. This follows easily, for example, from the observation that any Cantor attractor has zero Lebesgue measure (see [15]), and, disregarding c, is invariant by G. Hence G cannot carry an acip if a Cantor attractor is present. □

It should be noted that the distortion of the branches of G is in general not bounded; this comes from the fact that if $G|J = f^s|J$ is such a branch and $G(J) = I_n$, then this branch need not be extendible, *i.e.,* if $T \supset J$ is the maximal interval on which f^s of monotone, then $f^s(T)$ need not contain a definite scaled neighbourhood of I_n. In particular, $d\nu(x)/dx$ can not be expected to be bounded on any of the sets $I_n \setminus I_{n+1}$. However, we will still be able to derive the following result:

Theorem 3. *There exists $\varepsilon = \varepsilon(\ell)$ such that if $|I_{n+1}| \leq \varepsilon |I_n|$ for all $n \geq n_0$, then $\sum s_i \nu(J_i) < \infty$.*

Once this is obtained, the proof of the main theorem is straight forward.

Proof of Theorem 2. This follows by a standard pull-back construction. Given the G-invariant measure ν, define μ by

$$\mu(A) = \sum_i \sum_{j=0}^{s_i-1} \nu(f^{-j}(A) \cap J_i).$$

As f is non-singular with respect to Lebesgue, μ is absolutely continuous, and the f-invariance of μ is a standard exercise. The finiteness of μ follows directly from Theorem 3. □

Comments on constants: In the following, ℓ is fixed, ε_i denotes constants depending only on ε which are small provided that ε is. Constants ρ_i depend only on ℓ. Constants C_i depend only on f. The numbers $n_0 \in \mathbb{N}$ and $\lambda \in (0,1)$, which are defined in Section 4, also depend on f. For local use (*i.e.,* within a proof), B and $C = C(f)$ will denote a constant, which might vary within equations.

3 Construction of induced maps G_n and G

Let G_0 be the first return map to I_0. Then G_0 has a finite number of branches, the central branch is the branch with the largest return time, and each non-central branch maps diffeomorphically onto I_0.

In this section we shall construct a sequence of maps $G_n \colon \cup_i J_i^{n+1} \to I_0$ inductively such that

1. $\cup_i J_i^{n+1}$ is a finite union and for $n \geq 1$, $G_n = G_{n-1}$ outside I_n;
2. The central branch $J_0^{n+1} = I_{n+1}$ and $G_n | I_{n+1}$ is the first return map to I_n;

3. for each $i \neq 0$, there exists $b_i \leq n$ such that such that $G_n \colon J_i^{n+1} \to I_{b_i}$ is a diffeomorphism;

4. the outermost branch maps onto I_0; more precisely, $J_i^{n+1} \subset I_n$ and $\partial J_i^{n+1} \cap \partial I_n \neq \emptyset$ imply $G_n(J_i^{n+1}) = I_0$ (and the external point of such an interval J_i^{n+1} maps to the fixed point q);

5. $G_n(x) = f^s(x)$ implies that $f(x), \dots, f^{s-1}(x) \notin I_n$;

By definition G_0 satisfies the above statements, so let us assume that by induction G_n exists with the above properties, and construct G_{n+1}.

Set $G_{n+1}(x) = G_n(x)$ for $x \notin I_{n+1}$. Let $k_n \in \mathbb{N} := \{1, 2, 3, \dots\}$ be minimal so that $G_n^{k_n}(c) \in I_{n+1}$. This means that $k_n = 1$ if the return to I_n is central. Define $K^0 = I_{n+1}$, $K^{k_n} = I_{n+2}$ and, for $0 \leq j \leq k_n - 1$, let K^j be the component of $\mathrm{dom}(G_n^{j+1})$ which contains c. Next define on $K^j \setminus K^{j+1}$

$$G_{n+1}(x) = \begin{cases} G_n^{j+1}(x) & \text{if } G_n^{j+1}(x) \in I_{n+1} \\ \\ G_n^{j+2}(x) & \text{otherwise.} \end{cases}$$

$G_{n+1}|I_{n+2} = G_n^{k_n}|I_{n+2}$ is the first return map to I_{n+1}. Properties (1) and (2) hold by construction for G_{n+1}. Property (3) holds because if $G_n^{j+1}(x) \in I_{n+1}$ for some $x \in I_{n+1} \setminus I_{n+2}$ then $G_{n+1}(J_i^{n+1}) = I_{n+1}$ for the corresponding domain $J_i^{n+1} \ni x$ and if $G_n^{j+1}(x) \notin I_{n+1}$ then by the induction assumption $G_{n+1}(J_i^{n+1})$ is equal to some domain I_b, $b \leq n$, because then $G_{n+1}(x) = G_n^{j+2}(x)$. Property (4) holds immediately because ∂I_n is mapped by G_n into ∂I_0. In order to show Property (5) holds, take $x \in K^j \setminus K^{j+1}$ and let $y = G^j(x)$. Note that $G_n^j|K^j$ is inside a component of $\mathrm{dom}(G_n)$ and that all iterates $f(K^j), \dots, G_n^j(K^j) \ni y$ are outside I_{n+1}. Since $G_n^{j+1}(x) = G_n(y)$ we get by induction that (5) holds for G_{n+1} (using that it holds for G_n and y instead of x).

The induced map G is defined as follows: for each $n \geq 0$, each component of the domain J of G_n other than the central one I_{n+1} becomes a component of the domain of G, and $G|J = G_n|J$.

For later use, we compute by induction that if $x \in I_n \setminus I_{n+1}$, and $G(x) = f^s(x)$, then

$$s \leq t_0 \cdot (k_0 + 1) \cdots (k_{n-2} + 1) \cdot (k_{n-1} + 1), \tag{4}$$

where $t_0 = \min\{i > 0 \quad ; \quad f^i(c) \in I_0\}$.

4 Distortion properties of the induced map

Suppose $\varphi : T \to \varphi(T)$ is a C^1 map. Let us define

$$\mathrm{Dist}(\varphi) := \mathrm{Dist}(\varphi, T) := \sup_{x,y \in T} \log \frac{\varphi'(x)}{\varphi'(y)}.$$

Let us say a diffeomorphism $h : J \to h(J)$ belongs to the distortion class $\mathcal{F}_p^C$ if it can be written as

$$Q \circ \varphi_q \circ Q \circ \varphi_{q-1} \circ \cdots \circ Q \circ \varphi_1,$$

with $q \leq p$, where $Q(x) = |x|^\ell$ and $\mathrm{Dist}(\varphi_j) \leq C$ for all $1 \leq j \leq q$.

Let us fix a large positive integer n_0 such that $|I_n| \leq \varepsilon |I_{n-1}|$ for all $n \geq n_0$, and such that $f|I_{n_0}$ can be written as $x \mapsto \varphi(|x|^l)$ with $\mathrm{Dist}(\varphi) \leq 1/4$. By Lemma 1, it follows that for each $n \geq n_0$, if J is a return domain to I_n, and $f^s|J$ is the return, then $f^s|J$ can be written as $x \mapsto \varphi(|x|^l)$ with $\mathrm{Dist}(\varphi) \leq 1/2$ provided ε is sufficiently small.

According to Mañé [14], the map G, restricted to the set of points which stay outside I_{n_0+1} is a hyperbolic (uniformly expanding) system. Thus, there exists $C_1 = C_1(f) > 0$ and $\lambda = \lambda(f) \in (0,1)$ with the following property. For any $k \in \mathbb{N}$

1. if x is a point such that $G^i(x)$ are defined and $G^i(x) \notin I_{n_0+1}$ for any $0 \leq i \leq k-1$, then
$$|(G^k)'(x)| \geq \frac{1}{C_1 \lambda^k};$$

2. if J is an interval such that $G^k|J$ is defined, and $G^i(J) \cap I_{n_0+1} = \emptyset$ for all $0 \leq i \leq k-1$, then
$$\mathrm{Dist}(G^k|J) \leq \log C_1.$$

We will use the notation $\alpha(y) = n$ if $y \in I_n \setminus I_{n+1}$.

Proposition 2. *Let $m \geq 1$, and let $G^i : J \to I_m$ be an onto branch of G^i. There exists $C_2 = C_2(f)$ such that the following hold:*

- *Suppose that $\alpha(G^{i-1}J) > m$. Let $n > m$ and $1 \leq k \leq i$ be maximal such that*
$$n = \alpha(G^{i-k}J) > \alpha(G^{i-k+1}J) > \cdots > \alpha(G^{i-1}J) > m.$$

Then $G^i|J$ can be written as $\psi \circ \varphi$ such that

$$\mathit{Dist}(\psi) \le \log C_2 \text{ and } \varphi \in \mathcal{F}^1_{2(n-m+1)}.$$

- *If $\alpha(G^{i-1}(J)) \le m$ then $G^i|J$ can be written as $\psi \circ \varphi$ such that*

$$\mathit{Dist}(\psi) \le \log C_2 \text{ and } \varphi \in \mathcal{F}^1_2.$$

Proof. Let r denote the maximum of $\alpha(G^j(J))$ for $0 \le j \le i-1$. Let $C = C(f)$ be a big constant. We shall prove by induction on r the following stronger statement: $G^i|J$ can be written as $\psi \circ H \circ Q \circ \varphi_1$ with

$$\mathrm{Dist}(\psi) \le \log C, \ H \in \mathcal{F}^1_{2(n-m)+1} \text{ and } \mathrm{Dist}(\varphi_1) < 1/2.$$

If $r \le n_0$, then the distortion of $G^i|J$ is bounded by $\log C_1(f)$ as we remarked above. Hence the statement is true for $C > C_1$. So let us consider the case $r > n_0$.

For $0 \le j \le i-1$, let T_j denote the domain of G which contains $G^j(J)$. For simplicity of notation, write $\alpha_j = \alpha(G^j(J))$. By definition of n, we have $\alpha_{i-k-1} \le \alpha_{i-k} = n$. Note that $G^j|J$ extends to a diffeomorphism onto I_{α_j} for all $1 \le j \le i$.

Case 1. $n \le n_0$. Then $\alpha_j \le n_0$ for all $i-k \le j \le i-1$, and so $\mathrm{Dist}(G^k|G^{i-k}(J)) \le \log C_1$. If $G(T_{i-k-1}) \supset I_{n-1}$, then $\mathrm{Dist}(G^{i-k}|J)$ is bounded by the Koebe principle, and thus we are done. If $G(T_{i-k-1}) \subset I_n$, then T_{i-k-1} is a return domain to I_n. Since $n \ge m \ge 1$, this return domain is well inside I_n, which implies that $G^{i-k-1}|J$ has bounded distortion. Since $n \le n_0$, the distortion of $G|T_{i-k-1}$ has bounded distortion as well, and so the proposition is true for some universal constant C (which depending on the a priori real bounds).

Case 2. $n > n_0$. Then similarly as above, we can show that $G^{i-k}|J$ can be written as $\varphi_2 \circ h_1$, with $\mathrm{Dist}(\varphi_2) \le 1/2$ and $h_1 \in \mathcal{F}^{1/2}_1$. If $k = 0$, then the proposition follows. Assume $k \ge 1$. Let $J' = G_{n-1}(G^{i-k}(J))$, and let $s \in \mathbb{N}$ be such that $G = G_n = G^s_{n-1}$ on $G^{i-k}(J)$. Since $\alpha_{i-k+1} < \alpha_{i-k}$, it follows from our construction that $G^j(J') \cap I_n = \emptyset$ for all $0 \le j < s$. The same is true for $s \le j \le s-1+k-1$ by definition of n. Thus

$$\max_{j=0}^{s-1+k-1} \alpha(G^j(J')) \le n-1 \le r-1.$$

Applying the induction hypothesis to the map $G^{k-1} \circ G^{s-1}|J' = G^{k-1} \circ G_{n-1}^{s-1}|J'$, we see that the map can be written as $\psi \circ h \circ Q \circ \varphi$ with $\mathrm{Dist}(\psi) < C$, and $\mathrm{Dist}(\varphi) \le 1/2$, and $h \in \mathcal{F}^1_{2(n-m)-1}$. The map $G_{n-1}|G^{i-k}(J)$ is a restriction of the first return map to I_{n-1}, which is of the form $\varphi_3 \circ Q$ with $\mathrm{Dist}(\varphi_3) \le 1/2$. Therefore

$$\begin{aligned} G^i|J &= G^{s-1+k-1}|J' \circ G_{n-1}|G^{i-k}(J) \circ G^{i-k}|J \\ &= \psi \circ h \circ Q \circ (\varphi \circ \varphi_3) \circ Q \circ \varphi_2 \circ h_1. \end{aligned}$$

Note that $\mathrm{Dist}(\varphi \circ \varphi_3) < 1$, and the induction step is completed. □

We will need another proposition to treat the case $m = 0$. By taking C_2 larger if necessary, we prove:

Proposition 3. *Consider any branch $G^i|J$. Let $n = \max_{j=0}^{i-1} \alpha(G^j J)$. Then $G^i|J$ can be written $\psi \circ H$ with*

$$\mathit{Dist}(\psi) \le \log C_2 \text{ and } H \in \mathcal{F}^1_{2n}.$$

Proof. First note that if $G^i(J) \subset I_1$, then the assertion follows immediately from the previous proposition. So we shall assume $G^i(J) = I_0$. Let us prove by induction that $G^i|J$ can be written as $\psi \circ H \circ Q \circ \varphi$, where ψ is an iterate of $G|(I_0 \setminus I_{n_0+1})$, and $H \in \mathcal{F}^1_{n-1}$, and $\mathrm{Dist}(\varphi) < 1/2$.

If $n \le n_0$, then the claim is clearly true. Assume $n > n_0$. Let $0 \le p < i$ be the largest such that $\alpha_p = n$. Using similar argument as in the proof of the previous proposition, the map $G^p|J$ can be written as $\varphi_0 \circ h$, where $\mathrm{Dist}(\varphi_0) < 1/2$, and $h \in \mathcal{F}_1^{1/2}$. Note that $\alpha(G^{p-1}J) \le \alpha(G^p J)$ by the maximality of $\alpha(G^p J)$. Let s be the positive integer such that

$$G|G^p J = G_{\alpha_p}|G^p J = G^s_{\alpha_p - 1}|G^p J,$$

and let $J' = G_{\alpha_p - 1}(G^p J)$. It follows from the construction of G and the maximality of α_p that $\alpha(G^j(J')) \le n-1$ for all $0 \le j \le s-2+(i-p)$. By the induction hypothesis, we can decompose the map $G^{i-p+s-1}|J'$ as $\psi_1 \circ H_1 \circ Q \circ \varphi_1$ such that ψ_1 is an iterate of $G|I_0 \setminus I_{n_0+1}$, and $H_1 \in \mathcal{F}^1_{n-2}$. The map $G_{n-1}|G^p J$ is a restriction of the first return map to I_{n-1}, and thus it can be written as $\varphi \circ Q$ with $\mathrm{Dist}(\varphi) < 1/2$. Combining all these facts, we decompose

$$G^i|J = \psi_1 \circ \{H_1 \circ [Q \circ (\varphi_1 \circ \varphi_0)]\} \circ h,$$

as required. This completes the proof of the induction step. $\square$

We are going to use the following lemma many times.

Lemma 2. *If $h : J \to I$ is a diffeomorphism in $\mathcal{F}_p^1$, and $A \subset J$ is a measurable set, then*

$$\frac{1}{(\ell e)^p} \frac{\mathrm{Leb}(h(A))}{|I|} \leq \frac{\mathrm{Leb}(A)}{|J|} \leq e^p \Big(\frac{\mathrm{Leb}(h(A))}{|I|} \Big)^{1/\ell^p}. \tag{5}$$

Proof. First we note that for any interval $T \subset \mathbb{R} \setminus \{0\}$ and any measurable set $A \subset T$, we have

$$\frac{\mathrm{Leb}(A)}{|T|} \leq \Big(\frac{\mathrm{Leb}(Q(A))}{|Q(T)|} \Big)^{1/\ell}.$$

To see this, note that for a fixed $\mathrm{Leb}(Q(A))$, the left hand side takes its maximum in the case that A is an interval adjacent to the endpoint of ∂T which is closer to 0.

It suffices to prove the two inequalities in case $p = 1$. So let us consider the case $h = Q \circ \varphi$ with $\mathrm{Dist}(\varphi) \leq 1$. For any $A \subset J$, we have

$$\frac{\mathrm{Leb}(A)}{|J|} \leq e \frac{\mathrm{Leb}(\varphi(A))}{|\varphi(J)|} \leq e \Big(\frac{\mathrm{Leb}(h(A))}{|h(I)|} \Big)^{1/\ell}.$$

This proves the second inequality of (5). On the other hand,

$$\begin{aligned} \frac{\mathrm{Leb}(\varphi(A))}{|\varphi(J)|} &= 1 - \frac{\mathrm{Leb}(\varphi(J \setminus A))}{|\varphi(J)|} \\ &\geq 1 - \Big(\frac{\mathrm{Leb}(h(J) \setminus h(A))}{|h(J)|} \Big)^{1/\ell} \\ &= 1 - \Big(1 - \frac{\mathrm{Leb}(h(A))}{|I|} \Big)^{1/\ell} \\ &\geq \frac{1}{\ell} \frac{\mathrm{Leb}(h(A))}{|I|}, \end{aligned}$$

and thus

$$\frac{\mathrm{Leb}(A)}{|J|} \geq \frac{1}{e} \frac{\mathrm{Leb}(\varphi(A))}{|\varphi(J)|} \geq \frac{1}{e\ell} \frac{\mathrm{Leb}(h(A))}{|I|},$$

proving the first inequality. $\square$

5 Outermost branches

Within I_n, there are two special branches which have common endpoints with I_n. These branches always mapped onto I_0 by the map G, and need special care in our argument. In this section, we shall prove that these branches can not be too small.

Proposition 4. *There exist a constant $\rho_1 = \rho_1(\ell) > 0$ and a constant $C_3 = C_3(f) > 0$, such that if J_n is one of the two outermost branches of G in I_n, then*

$$\frac{|J_n|}{|I_n|} \geq \frac{\rho_1^n}{C_3}.$$

Proof. Let $\delta_n := |J_n|/|I_n|$ and $\hat{J}_{n-1}$ the outer-most branch of $I_{n-1} \setminus I_n$ for which $\hat{J}_{n-1} \supset G_{n-1}(J_n)$. Write $G_{n-1}|I_n = f^{t_n}$. Since this is a first return, one has $\mathrm{Dist}(f^{t_n-1}|f(I_n)) \leq 1$ for all n sufficiently big.
Case 1. $G_{n-1}(c) \notin \hat{J}_{n-1}$. Then by the distortion bound for $f^{t_n-1}|f(I_n)$,

$$\frac{|f(a)-f(c)|}{|f(b)-f(c)|} = 1 + \frac{|f(a)-f(b)|}{|f(b)-f(c)|} \geq 1 + C\delta_{n-1},$$

where a and b are the end points of J_n with b between a and c. Hence, using that c is a critical point of order ℓ,

$$\frac{|a-c|}{|b-c|} \geq (1 + C\delta_{n-1})^{1/\ell} \geq 1 + \frac{C\delta_{n-1}}{\ell}.$$

Hence

$$\delta_n = \frac{|J_n|}{|I_n|} = \frac{1}{2}\frac{|a-b|}{|a-c|} \geq \frac{1}{2}\left(1 - \frac{1}{1 + C\delta_{n-1}/\ell}\right) \geq C\delta_{n-1}/\ell.$$

By induction, $|J_n|/|I_n| \geq \rho_1^n/C_3$ for $\rho_1 = \rho_1(\ell) \asymp 1/\ell$.
Case 2. $G_{n-1}(c) \in \hat{J}_{n-1}$. Note that $G_{n-1}(\hat{J}_{n-1}) = I_0$ and that $G_{n-1}^2 J_n$ intersects an outermost branch $\hat{J}_0$ of I_0. Let $p \geq 0$ be minimal so that $G_{n-1}^{p+2}(c) \notin \hat{J}_0$. Then $|\hat{J}_0|/|G_{n-1}^{p+2} I_n|$ is bounded from below (by a bound which depends only on f), and since $G_{n-1}^{p+2}(J_n) = \hat{J}_0$, and $f|(I_0 \setminus I_1)$ is hyperbolic this implies

$$|G_{n-1}^2 J_n|/|G_{n-1}^2 I_n| \geq C > 0.$$

According to the distortion control on $G_{n-1}|\hat{J}_{n-1}$ given by Proposition 3, this implies

$$|G_{n-1}J_n|/|G_{n-1}I_n| \geq C\rho^n > 0.$$

Since I_n is a first return domain of G_{n-1}, by Lemma 1, this implies

$$|J_n|/|I_n| \geq \rho_1^n/C_3,$$

with $\rho_1 = \rho_1(\ell) \asymp 1/\ell$ and $C_3 = C_3(f) \asymp 1/C$. □

6 Improved decay for deep returns

Let x and m be so that $G_n^m(x)$ is well-defined and $G_n^i(x) \notin I_{n+1}$ for $0 \leq i < m$. Let $T_i = T_i(x)$ be the component of $\mathrm{dom}(G_n)$ which contains $G_n^i(x)$. Define $\alpha(y) = j$ if $y \in I_j \setminus I_{j+1}$ and $s(y) = s$ if $G(y) = f^s(y) = G_{\alpha(y)}(y)$. Let t_n be the return time of c to I_n under f. Define

$$\Lambda = \{0 \leq i \leq m-2 \quad ; \quad \alpha(T_{i+1}) \geq \alpha(T_i)\},$$

$$N = \sum_{i \in \Lambda} [\alpha(T_{i+1}) - \alpha(T_i) + 1] \text{ and } r = \#\Lambda.$$

Moreover, define

$$T_0' = \{y \in T_0 \quad ; \quad G_n^i(y) \in T_i \text{ for all } i \leq m-1\}.$$

If $\varphi : T \to \varphi(T)$ is a homeomorphism and $J \subset T$ is a subinterval of T, we denote the components of $T \setminus J$ by L and R, and write

$$Cr(T, J) := \frac{|T| \cdot |J|}{|L| \cdot |R|}$$

for the *cross-ratio* of J in T.

Lemma 3. *Assume that $\alpha(T_i) \geq n_0$ for all $i = 0, \ldots, m-2$, then for $\varepsilon_1 \asymp \varepsilon^{1/\ell}$*

- *$Cr(T_0, T_0') \leq \varepsilon_1^N$ if $r \geq 1$;*
- *for each interval $J \subset G_n(T_{m-1})$ with $J \ni G_n^m(x)$, and $J' := \{y \in T_0' \quad ; \quad G_n^m(y) \in J\}$ we have*

  $$Cr(T_0, J') \leq \varepsilon_1^N \cdot Cr(G_n(T_{m-1}), J)$$

  (even if $r = 0$).

Proof of Lemma 3. For $0 \le j \le m-2$, write

$$\begin{aligned} Cr(I_{\alpha(T_j)}, G_n^j T_0') &\le Cr(T_j, G_n^j T_0') \\ &\le Cr(G_n T_j, G_n^{j+1} T_0') \\ &\le Cr(I_{\alpha(T_{j+1})}, G_n^{j+1} T_0'). \end{aligned}$$

Here the first and third inequality hold by inclusion of intervals, and the second inequality because f has negative Schwarzian derivative. Note that $G_n T_j \supset I_{\alpha(T_j)}$. If $j \in \Lambda$ then one gets improved inequalities: if

$$G_n T_j \supset I_{\alpha(T_j)-1} \supset I_{\alpha(T_{j+1})-1},$$

then in the third inequality one gets an additional factor $\varepsilon_1^{[\alpha(T_{j+1})-\alpha(T_j)+1]}$, while if $G_n T_j = I_{\alpha(T_j)} \supset I_{\alpha(T_{j+1})}$ then in the first inequality one gets an factor ε_1 (because then G_n is a first return and so a composition of x^ℓ and a map which extends diffeomorphically to $I_{\alpha(T_j)-1}$) and in the third we get an additional factor

$$\varepsilon_1^{[\alpha(T_{j+1})-\alpha(T_j)]}.$$

To prove the second assertion of the lemma one proceeds in the same way. Note that all this holds, provided $\alpha(T_j) \ge n_0$ for each $j \in \Lambda$ where n_0 is chosen so that $|I_{n+1}|/|I_n| < \varepsilon$ for $n \ge n_0$. □

Let k_n be as in Section 3.

Corollary 3. *There exists $C_4 = C_4(f) > 1$ and $\varepsilon_2 \asymp \varepsilon_1^{1/\ell}$ with the following property.*

(1) If $\alpha(G_n^i(I_{n+2})) \ge n_0$ for all $0 \le i \le k_n$, then

$$\frac{|I_{n+2}|}{|I_{n+1}|} \le C_4 \varepsilon_2^{k_n}.$$

(2) If $\alpha(G_n^i(I_{n+2})) \le n_0$ for some $1 \le i \le k_n$, then

$$\frac{|I_{n+2}|}{|I_{n+1}|} \le C_4 \varepsilon_2^{n-n_0}.$$

Proof. (1) Let $x = G_n(c)$ and $m = k_n - 1$, and let T_i, Λ, N be defined as above. Write $n' = \alpha(G_n^{k_n-1}(c))$. Note that $\alpha(G_n(c)) = n$. Then

$$\sum_{i=0}^{m-1} [\alpha(T_{i+1}) - \alpha(T_i)] = n' - n.$$

Thus

$$\begin{aligned} N - r &= \sum_{i\in\Lambda}[\alpha(T_{i+1}) - \alpha(T_i)] \\ &= n' - n + \sum_{i\notin\Lambda}[\alpha(T_i) - \alpha(T_{i+1})] \\ &\geq n' - n + m - r, \end{aligned}$$

which implies

$$N \geq n' - n + m. \tag{6}$$

Let $J = G_n^{k_n}(I_{n+2})$. Then

$$Cr(G_n(T_{m-1}), J) \leq Cr(I_{n'}, I_{n+1}) \leq 3\varepsilon_1^{n+1-n'}.$$

Applying the last part of the previous lemma, we obtain

$$Cr(T_0, G_n(I_{n+2})) \leq 3\varepsilon_1^N \varepsilon_1^{n+1-n'} \leq 3\varepsilon_1^{m+1} = 3\varepsilon_1^{k_n},$$

which implies this corollary.

(2) Let $p < k_n$ be the largest integer for which $\alpha(G^p(I_{n+2})) \leq n_0$. Let $\tilde{\Lambda} = \{p \leq i \leq k_n - 2 : i \in \Lambda\}$, and let $\tilde{N} = \sum_{i\in\tilde{\Lambda}'}[\alpha(T_{i+1}) - \alpha(T_i) + 1]$. Then we can show similarly

$$\tilde{N} \geq n' - n_0 + k_n - p \geq n' - n_0,$$

and

$$Cr(T_p, G_n^p(I_{n+2})) \leq \varepsilon_1^{\tilde{N}} Cr(I_{n'}, I_{n+1}) \leq \varepsilon_1^{n-n_0},$$

which implies the statement. $\square$

7 Improved decay in general

Let $I_{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I_{n+2}$ be the domains of G_n^j as in Section 3.

Lemma 4. *Assume*

$$K^i \Supset K^{i+1} = K^{i+2} = \cdots = K^{i+m} \Supset K^{i+m+1}.$$

Then there exists $C_5 = C_5(f) > 0$ *and* $\rho_2 = \rho_2(\ell) \in (0,1)$ *such that (provided* n *is sufficiently large)*

$$\frac{|K^{i+1}|}{|K^i|} \leq (1 - \rho_2^n/C_5)^m. \tag{7}$$

Proof of Lemma 4. By construction, $G^{i+1}K^i$ contains the outermost domain of some interval I_j, with $j \leq n$, while $G^{i+1}K^{i+1} \subset I_j$ is not contained in that outermost domain. By Proposition 4, this outermost domain is at least $\rho_1^j/C_3 (\geq \rho_1^n/C_3)$ times as long as $|I_j|$. By Propositions 2 and 3, the map $G^i|G_nK^i = G_n^i|G_nK^i$ can be written as $\psi \circ H$ with

$$\mathrm{Dist}(\psi) \leq \log C_3 \text{ and } H \in \mathcal{F}^1_{2n}.$$

By the left inequality of (5), this implies that

$$\frac{|G_n(K^i \setminus K^{i+1})|}{|G_nK^i|} \geq \rho^n/C,$$

for some $\rho = \rho(\ell) \in (0,1)$. Since $G_n|K^i$ is a restriction of the first return map to I_n, it follows that

$$\frac{|K^{i+1}|}{|K^i|} \leq 1 - \rho^n/C.$$

for n large. Hence, at least provided $\frac{\log m}{n}$ is not too large, i.e., bounded by a universal constant, (7) holds (taking $\rho_2 > 0$ small). So we need to consider the case that $\frac{\log m}{n}$ is large. Then $K^{i+1} = \cdots = K^{i+m}$, $G_n^{i+2}K^{i+1}$ is contained in an outermost domain, and so one of the endpoints of $G_n^{i+3}K^{i+1}$ is a boundary point of I_0. Using that $K^{i+1} = \cdots = K^{i+m}$,

$$\frac{|G_n^{i+3}K^{i+1}|}{|\hat{J}_0|} \leq C\lambda^m,$$

where $\hat{J}_0$ is the outermost branch of I_0, $C = C(f)$, and $\lambda \in (0,1)$ comes from the beginning of Section 4. The distortion control given by Proposition 3 gives

$$\frac{|G_n^{i+1}K^{i+1}|}{|T_{i+1}|} \leq C\lambda^{m/\ell^{2n}},$$

where T_{i+1} is the domain of G_n^2 containing $G_n^{i+1}K^{i+1}$. Since $|G_n^{i+1}(K^i \setminus K^{i+1})| \geq \rho_1^n |T_{i+1}|/C$, it follows

$$\frac{|G_n^{i+1}K^{i+1}|}{|G_n^{i+1}K^i|} \leq C\frac{\lambda^{m/\ell^{2n}}}{\rho_1^n}.$$

Using the distortion control given by Proposition 2 or 3, and equation (5), we obtain

$$\frac{|G_n(K^{i+1})|}{|G_n(K^i)|} \leq Ce^n\lambda^{m/\ell^{4n}}/\rho_1^{n/\ell^{2n}}.$$

Pulling back by the first return map $G_n|K^i$, we obtain

$$\frac{|K^{i+1}|}{|K^i|} \leq Ce^{n/\ell}\lambda^{m/\ell^{4n+1}}/\rho_1^{n/\ell^{2n+1}},$$

which clearly implies (7) when $\frac{\log m}{n} \gg 4\log\ell$ and $\rho_2 \ll \ell^{-4}$. □

Lemma 5. *Let $\lambda \in (0,1)$ be as in the beginning of Section 4. Let m be so that $I_{n+1} = K^0 = \cdots = K^m \neq K^{m+1} \supset I_{n+2}$. Assume $m \geq 1$. Then*

$$\frac{|I_{n+1}|}{|I_n|} \leq C_6\lambda^{m/\ell^{n+1}},$$

where $C_6 = C_6(f)$ is a constant.

Proof of Lemma 5. Note that $G_n|I_{n+1}$ is a first return map to I_n, and so there exists a neighbourhood $T \ni f(c)$ such that $f^{t_n-1}\colon T \to I_{n-1}$ is a diffeomorphism and $f^{-1}(T) \subset I_n$. Therefore

$$\frac{|I_{n+1}|}{|I_n|} \leq \left(\frac{|G_nI_{n+1}|}{|I_{n-1}|}\right)^{1/\ell}.$$

If $m \geq 1$, then $G_n(I_{n+1})$ is contained in an outermost branch J_n in I_n. Similarly as before

$$\frac{|G_nI_{n+1}|}{|J_n|} \leq C\lambda^{m/\ell^n} \text{ and so } \frac{|I_{n+1}|}{|I_n|} \leq C\lambda^{m/\ell^{n+1}}.$$

□

Lemma 6. *There exists $\varepsilon(\ell)$ so that if $|I_{n+1}| \le \varepsilon |I_n|$ for all n sufficiently large, then for all n sufficiently large,*

$$\frac{|I_{n+2}|}{|I_{n+1}|} \le \frac{1}{(k_n+1)^4}.$$

Proof of Lemma 6. Consider $\alpha(G_n^i c)$ for $1 \le i < k_n$. If all these are larger than n_0 then by Corollary 3

$$\frac{|I_{n+2}|}{|I_{n+1}|} \le C_4 \varepsilon_2^{k_n} < \frac{1}{(k_n+1)^4}.$$

So assume that there exists $1 \le i < k_n$ such that $\alpha(G_n^i c) \le n_0$. Then at least we have

$$\frac{|I_{n+2}|}{|I_{n+1}|} \le C_4 \varepsilon_2^{(n-n_0)/\ell},$$

by the second statement of Corollary 3. This implies the lemma, unless $k_n \ge \varepsilon_2^{-(n-n_0)/(4\ell)}/C_4$. Let m as before be so that $I_{n+1} = K^0 = K^1 = \cdots = K^m \ne K^{m+1} \supset I_{n+2}$. Then respectively by the previous lemma and by Lemma 4,

$$\frac{|I_{n+1}|}{|I_n|} \le C\lambda^{m/\ell^{n+1}} \text{ and } \frac{|I_{n+2}|}{|I_{n+1}|} \le (1-\rho_2^n/C_5)^{k_n-m}.$$

Case 1. $m < k_n/2$. According to the second inequality, we have

$$\frac{|I_{n+2}|}{|I_{n+1}|} \le (1-\rho_2^n/C_5)^{k_n/2} \le \frac{1}{(k_n+1)^4},$$

provided we choose $\varepsilon(\ell)$ so small that for ε_2 from Corollary 3, $\varepsilon_2 < \rho_2^{4\ell}$ and we take n sufficiently large. Here we have used the assumption that $k_n \ge \varepsilon_2^{-(n-n_0)/(4\ell)}/C_4$.
Case 2. $m \ge k_n/2$. Then by the first inequality,

$$Cr(I_n, I_{n+1}) \asymp \frac{|I_{n+1}|}{|I_n|} \le \lambda^{k_n/2\ell^{n+1}}.$$

By Lemma 1, there is an interval $T \ni f(c)$ such that $f^{-1}(T) \subset I_{n+1}$ and such that $f^{t_{n+1}-1} : T \to I_n$ is a diffeomorphism, where t_{n+1} is the first return time of c to I_{n+1}. Since also $f^{t_{n+1}}(I_{n+2}) \subset I_{n+1}$, we obtain

$$\begin{aligned} Cr(T, f(I_{n+2})) &\leq Cr(f^{t_{n+1}-1}(T), f^{t_{n+1}}(I_{n+2})) \\ &\leq Cr(I_n, I_{n+1}) \\ &\asymp \lambda^{k_n/2\ell^{n+1}}. \end{aligned}$$

Since $f^{-1}(T) \subset I_{n+1}$, $f(I_{n+1})$ contains a component of $T \setminus f(I_{n+2})$. Thus

$$\frac{|f(I_{n+2})|}{|f(I_{n+1})|} \leq Cr(T, f(I_{n+2})) \leq C\lambda^{k_n/2\ell^{n+1}}.$$

Finally, the non-flatness of the critical point gives

$$\frac{|I_{n+2}|}{|I_{n+1}|} \leq C\lambda^{k_n/2\ell^{n+2}} \leq \frac{1}{(k_n+1)^4},$$

provided that $\varepsilon_2 < \ell^{-4\ell}$ and n is sufficiently large. □

8 The measure for the induced map

In this section we prove the existence of an acip for the induced map G.

Proof of Proposition 1. We will use the result by Straube [24] claiming that G has an acip if (and only if) there exists some $\eta \in (0,1)$ and $\delta > 0$ such that for every measurable set A of measure $\text{Leb}(A) < \delta$ holds $\text{Leb}(G^{-k}(A)) < \eta|I_0|$.

The assumptions give that there exists a constant B with the following property: If J is any branch of G^k and $G^k(J) = I_n$, then

$$\frac{\text{Leb}(\{x \in J; G^k(x) \in I_{n+m}\})}{|J|} \leq B\frac{|I_{n+m}|}{|I_n|}. \tag{8}$$

This includes trivially the branch of G^0, that is the identity. Note that B is a distortion constant, and $B \leq 2$ for $\varepsilon \approx 0$ and $n \geq n_0$. So we can assume that $B\sqrt{\varepsilon}/(1-\sqrt{\varepsilon}) < 1/3$. Moreover, $|I_n| \leq \varepsilon^{n-m}|I_m|$ for all $n \geq m \geq n_0$.

Lemma 7. *If J is a branch of G^{k-1} such that $G^{k-1}(J) = I_{n+1}$, then*

$$\text{Leb}(\{x \in J; \alpha(G^k(x)) \geq n+1\}) \leq \frac{1}{6}|J|, \tag{9}$$

provided $n \geq n_0$.

Proof. Let $I_{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I_{n+2}$ be as in Section 3. For each $0 \leq i \leq k_n - 1$ with $K^i \neq K^{i+1}$, there can be at most two branches of G, symmetric w.r.t. the critical point, which map onto I_{n+1}. We claim that each of these branches P lies deep inside K^i (if they exist). To see this, let $s \in \mathbb{N}$ be such that $G|P = f^s|P$. Then by our construction, f^{s-1} maps an interval $T \ni f(c)$ onto some interval I_j with $j \leq n$, and $f^{-1}(T) = K_i$. Since $f^{s-1}(f(P)) = I_{n+1}$ lies deep inside I_j, it follows from the Koebe principle that $f(P)$ lies deep inside T. The claim follows from the non-flatness of the critical point.

Let U_{n+1} be the union of those domains of G inside $I_{n+1} \setminus I_{n+2}$ which are mapped onto I_{n+1} by G. Then it follows from the Koebe principle

$$\mathrm{Leb}(\{x \in J : G^{k-1}(x) \in U_{n+1}\}) \leq \frac{1}{10}|J|.$$

It remains to consider branches of J' of $G^k|J$ for which $G^k(J') = I_{n'}$ with $n' \leq n$. But using the remark before this lemma, we obtain an estimate for this part also, and thus we conclude the proof. $\square$

Write $y_{n,k} = \mathrm{Leb}(\{x \in I_0; \alpha(G^k(x)) = n\})$. Take $C_0 > 6B/|I_{n_0}|$. We will show by induction that $y_{n,k} \leq C_0\sqrt{|I_n|}$ for all $n, k \geq 0$. For $k = 0$, this is obvious, and the choice of C_0 assures that $y_{n,k} \leq C_0\sqrt{|I_n|}$ for all $n < n_0$.

Now for the inductive step, assume that $y_{n,k-1} \leq C_0\sqrt{|I_n|}$ for all n. Pick n such that (9) holds (*i.e.*, $n \geq n_0 + 1$), and write $y^{n'}_{n,k}$ for the measure of the set x such that $\alpha(G^{k-1}x) = n'$ and $\alpha(G^k x) = n$.

Then by equations (8), (9) and induction,

$$\begin{aligned} y_{n,k} &= \sum_{n'<n_0} y^{n'}_{n,k-1} + \sum_{n_0 \leq n' < n} y^{n'}_{n,k-1} + y^{n}_{n,k-1} + \sum_{n'>n} y^{n'}_{n,k-1} \\ &\leq B\frac{|I_n|}{|I_{n_0}|} + \sum_{n'<n} C_0 B \frac{|I_n|}{|I_{n'}|}\sqrt{|I_{n'}|} + \frac{C_0}{6}\sqrt{|I_n|} + \sum_{n'>n} C_0\sqrt{|I_{n'}|} \\ &\leq C_0\sqrt{|I_n|}\left(\frac{1}{6} + \sum_{n'<n} B(\sqrt{\varepsilon})^{n-n'} + \frac{1}{6} + \sum_{n'>n}(\sqrt{\varepsilon})^{n'-n}\right) \\ &< (\frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3})C_0\sqrt{|I^n|} = C_0\sqrt{|I_n|}. \end{aligned}$$

If an acip ν exists, then it can be written as $\nu(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{Leb}(G^{-i}A)$.

Therefore,

$$\nu(I_n) \le C_0\sqrt{|I_n|}. \tag{10}$$

Take $\eta \in (0,1)$. Fix n_1 such that $\sum_{n\ge n_1} y_{n,k} < \eta/2$ for all $k \ge 0$. We need to show that we can choose $\delta > 0$ so that if $A \subset I_0$ is a set of measure $\mathrm{Leb}(A) < \delta$, then $\mathrm{Leb}(G^{-k}(A)) < \eta$ for all $k \ge 0$. By the choice of n_1, it suffices to show that $\mathrm{Leb}(G^{-k}(A)) < \eta/2$, $k \ge 0$, for any $A \subset I_0 \setminus I_{n_1}$.

Assume that $A \subset I_n \setminus I_{n+1}$ for some $n < n_1$. Proposition 2 shows that any onto branch $G^k : J \to I_n$ can be written as $\psi \circ \varphi$ with

$$\mathrm{Dist}(\psi) \le \log C_2 \text{ and } \varphi \in \mathcal{F}^1_{3(m-n+1)},$$

where

$$m = \alpha(G^i J) > \alpha(G^{i+1}J) > \cdots > \alpha(G^{k-1}J) > n$$

for some $i < k$. Clearly $i \ge k - m + n - 1$. For such a branch we have

$$\mathrm{Leb}(G^{-k}A \cap J) \le C_2 B\big(\frac{|A|}{|I_n|}\big)^{1/\ell^{3(m-n+1)}}|J|.$$

For fixed m, the total measure of the set of points arriving to I_n in this fashion is bounded by $\sum_{i=k-m+n-1}^{k-1} y_{m,i} \le (m-n+1)\cdot C_0 \cdot \sqrt{|I_m|}$. Summing over all branches J (including the ones that do have extensions and hence distortion bounded by C_1), and all $m \ge n$, we find

$$\mathrm{Leb}(G^{-k}A) \le C_1\frac{|A|}{|I_n|} + \sum_{m\ge n}(m-n+1)C_0\sqrt{|I_m|}C_2B\big(\frac{|A|}{|I_n|}\big)^{1/\ell^{3(m-n+1)}}.$$

Thus $\mathrm{Leb}(G^{-k}A) \le \eta/2n_1$ for any integer k and any $A \subset I_n \setminus I_{n+1}$, $n < n_1$, with $|A| \le \delta$, provided δ is sufficiently small. It follows that if $A \subset I_0 \setminus I_{n_1}$ has sufficiently small measure, then $\mathrm{Leb}(G^{-k}A) < n_1\eta/(2n_1) = \eta/2$. This concludes the verification of Straube's condition. □

9 Summability

We finish by proving Theorem 3. This theorem follows immediately from the next lemma.

Lemma 8. *The partial sum* $\sum_{J_j \subset I_{n+1} \setminus I_{n+2}} s_j \nu(J_j)$ *is exponentially small in* n.

Proof of Lemma 8. Let $I_{n+1} = K^0 \supset \cdots \supset K^{k_n} = I_{n+2}$ be as in Section 3, and let $m \geq 0$ be minimal such that $K^m \supsetneq K^{m+1}$. Let us first comment on the induce times s_j. If $J_j \subset K^i \setminus K^{i+1}$, then $G|J_j$ corresponds to at most $(i+2)$ iterates of G_n, and thus $s_j \leq (i+2)t_0(k_0+1)\cdots(k_{n-1}+1)$ according to (4). For $J_j \subset K^m \setminus K^{m+1}$, we need a better estimate than (4). Note that if $m \geq 2$, then $G_n^p(J_j)$ is contained in one of the outermost branches in I_0 for all $2 \leq p \leq m-1$, where iterates of G corresponds to f^2, and thus we have in this case that

$$s_j \leq 2t_0(k_0+1)\cdots(k_{n-1}+1) + 2m. \tag{11}$$

By Lemma 6, we have

$$\sqrt{|I_{n+1}|}(k_0+1)(k_1+1)\cdots(k_{n-1}+1) \leq C|I_{n+1}|^{1/4} \leq C\varepsilon^{n/4}. \tag{12}$$

A direct computation shows that $m\lambda^{m/\ell^n}/R^n \leq C = C(\lambda)$ for $\lambda \in (0,1)$ and all $m, n \geq 0$, provided $R > \ell$. So, by Lemma 5, we have

$$m|I_{n+1}|^{1/2} \leq Cm\lambda^{m/2\ell^{n+1}}|I_n|^{1/2} \leq C\ell^{2n}|I_n|^{1/2} \leq C(\sqrt{\varepsilon}\ell^2)^n. \tag{13}$$

Sum over outermost branches: Note that if J_i is an outermost branch in I_{n+1}, then $s_i \leq 2t_0(k_0+1)\cdots(k_{n-1}+1) + 2m$ by (11). Using also the obvious estimate $\nu(J_i) \leq \nu(I_{n+1}) \leq C_0\sqrt{I_{n+1}}$, we obtain

$$\begin{aligned} s_i\nu(J_i) &\leq 4C_0(t_0(k_0+1)\cdots(k_{n-1}+1) + m)\sqrt{|I_{n+1}|} \\ &\leq C(\varepsilon^{n/4} + (\sqrt{\varepsilon}\ell^2)^n) \end{aligned}$$

according to (12) and (13). Since there are only two outermost branches, the term over these branches is exponentially small in n (provided that ε is sufficiently small).

Sum over all other branches: Note that if A is a subset of a component of $I_{n+1} \setminus I_{n+2}$, and the distance $d(A, \partial I_{n+1}) \geq \delta \cdot \mathrm{diam}(A)$, then the Koebe distortion lemma gives that for every $i \geq 0$ and every onto branch $G^i : J \to I_{n+1}$, we have

$$\frac{\mathrm{Leb}(G^{-i}(A) \cap J)}{|J|} \leq K(\delta)\frac{\mathrm{Leb}(A)}{|I_{n+1}|},$$

where $K(\delta) = 2(1+\delta)^2/\delta^2$. Hence

$$\mathrm{Leb}(G^{-i}A) \le \sum_{G^i J = I_{n+1}} K(\delta) \frac{\mathrm{Leb}(A)}{|I_{n+1}|} |J| + \sum_{G^i J \supsetneq I_{n+1}} K(1/\varepsilon) \frac{\mathrm{Leb}(A)}{|I_{n+1}|} \mathrm{Leb}(G^{-i}(I_{n+1}) \cap J),$$

so that $\mathrm{Leb}(G^{-i}A)/\mathrm{Leb}(G^{-i}I_{n+1}) \le K(\delta)\mathrm{Leb}(A)/|I_{n+1}|$. In particular, this implies that

$$\nu(A) \le K(\delta)\nu(I_{n+1}) \frac{\mathrm{Leb}(A)}{|I_{n+1}|}.$$

By Proposition 4, the length of each of the outermost branches is as least ρ_1^n/C_3, and thus for any other branch $J_j \subset I_{n+1} \setminus I_{n+2}$,

$$d(J_j, \partial I_{n+1}) \ge \rho_1^n |J_j|/C_3,$$

which implies

$$\frac{\nu(J_j)}{|J_j|} \le \frac{C}{\rho_1^{2n}} \frac{\nu(I_{n+1})}{|I_{n+1}|} \le \frac{C}{\rho_1^{2n}} \frac{C_0}{\sqrt{|I_{n+1}|}}.$$

Therefore the sum of $s_j \nu(J_j)$ over all branches other than the outermost ones is bounded from above by the following

$$\frac{C}{\rho_1^{2n}} \frac{C_0}{\sqrt{|I_{n+1}|}} \sum_{J_j} s_j |J_j| = \frac{C}{\rho_1^{2n}} \frac{C_0}{\sqrt{|I_{n+1}|}} \Big(\sum_{J_j \subset K^m \setminus K^{m+1}} + \sum_{J_j \subset K^{m+1} \setminus I_{n+2}} s_j |J_j| \Big)$$

(note that $K^m = I_{n+1}$). Let us first estimate the first part of this sum. Using (11), Corollary 3, Lemma 5 and (13), we obtain

$$\begin{aligned}
&\frac{1}{\rho_1^{2n} \sqrt{|I_{n+1}|}} \sum_{J_j \subset K^m \setminus K^{m+1}} s_j |J_j| \le \\
&\le \frac{2}{\rho_1^{2n}} \left(t_0(k_0+1) \cdots (k_{n-1}+1) + m \right) \frac{|K^m \setminus K^{m+1}|}{\sqrt{|I_{n+1}|}} \\
&\le \frac{2}{\rho_1^{2n}} \left(t_0(k_0+1) \cdots (k_{n-1}+1) + m \right) \sqrt{|I_{n+1}|} \\
&\le \frac{2}{\rho_1^{2n}} \left(t_0(k_0+1) \cdots (k_{n-1}+1) \right) C_4^n \varepsilon_2^{k_0+\cdots+k_{n-1}} C_6 \lambda^{m/\ell^{n+1}} + \\
&\quad + \quad 2C \left(\frac{\sqrt{\varepsilon}\ell^2}{\rho_1^2} \right)^n,
\end{aligned}$$

is exponentially small provided that ε is sufficiently small.

For each domain $J_j \subset K^i \setminus K^{i+1}$ with $i \geq m+1$, we have

$$s_j \leq (i+2)t_0(k_0+1)\cdots(k_{n-1}+1) \leq C(i+2)\Big(\frac{1}{|I_{n+1}|}\Big)^{1/4}.$$

Therefore,

$$\begin{aligned}
\frac{1}{\sqrt{|I_{n+1}|}} \sum_{J_j \subset K^{m+1}\setminus I_{n+2}} s_j|J_j| &= \frac{1}{\sqrt{|I_{n+1}|}} \sum_{i=m+1}^{k_n-1} \sum_{J_j \subset K^i \setminus K^{i+1}} s_j|J_j| \\
&\leq \frac{C}{|I_{n+1}|^{3/4}} \sum_{i=m+1}^{k_n-1} (i+2)|K^i| \\
&= C \cdot |I_{n+1}|^{1/4} \sum_{i=m+1}^{k_n-1} (i+2)\frac{|K^i|}{|I_{n+1}|}.
\end{aligned}$$

By Lemma 4 (applied repeatedly),

$$\frac{|K^i|}{|I_{n+1}|} = \frac{|K^i|}{|K^m|} \leq \Big(1 - \frac{\rho_2^n}{C_5}\Big)^{i-m},$$

which implies

$$\begin{aligned}
\sum_{i=m+1}^{k_n-1} (i+2)\frac{|K^i|}{|I_{n+1}|} &\leq \sum_{i>m} (i+2)\left(1-\rho_2^n/C_5\right)^{i-m} \\
&\leq (m+2)\frac{C_5}{\rho_2^n} + \Big(\frac{C_5}{\rho_2^n}\Big)^2 \\
&\leq 2C(m+2)\frac{1}{\rho_2^{2n}}.
\end{aligned}$$

Thus, using again Lemma 5,

$$\begin{aligned}
\frac{1}{\sqrt{|I_{n+1}|}} \sum_{J_j \subset K^{m+1}\setminus I_{n+2}} s_j|J_j| &\leq C|I_{n+1}|^{1/4}(m+2)\frac{1}{\rho_2^{2n}} \\
&\leq C\Big(\frac{\varepsilon^{1/4}}{\rho_2^2}\ell^3\Big)^n C_6(m+2)\lambda^{m/\ell^{n+1}},
\end{aligned}$$

and so

$$\frac{1}{\rho_1^{2n}}\frac{1}{\sqrt{|I_{n+1}|}}\sum_{J_i\subset K^{m+1}\setminus I_{n+2}} s_i|J_i| \le C(m+2)\lambda^{m/\ell^{n+1}}\Big(\frac{\varepsilon}{\rho_1^8\rho_2^8}\Big)^{n/4},$$

which is again exponentially small in n provided that ε is sufficiently small. This completes the proof. □

References

[1] R. Bowen, *Invariant measures for Markov maps of the interval*, Commun. Math. Phys. **69** (1979) 1–17

[2] H. Bruin, *Topological conditions for the existence of invariant measures for unimodal maps,* Ergod. Th. & Dynam. Sys. **14** (1994) 433-451.

[3] H. Bruin, *Topological conditions for the existence of Cantor attractors,* Trans. Amer. Math. Soc. **350** (1998) 2229-2263.

[4] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Wild Cantor attractors exist,* Annals of Math. **143** (1996) 97-130.

[5] P. Collet and J.-P. Eckmann, *Positive Liapunov exponents and absolute continuity for maps of the interval*, Ergod. Th. & Dyn. Sys. **3** (1983) 13–46

[6] J. Graczyk, G. Świątek, *Induced expansion for quadratic polynomials,* Ann. Sci Éc. Norm. Súp. **29** (1996) 399-482.

[7] J. Graczyk, D. Sands, G. Świątek, *Decay of geometry for unimodal maps: Negative Schwarzian case,* Preprint (2000).

[8] M.V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Commun. Math. Phys. **81** (1981) 39–88

[9] M. Jakobson, G. Świątek, *Metric properties of non-renormalizable S-unimodal maps. I. Induced expansion and invariant measures,* Ergod. Th. & Dynam. Sys. **14** (1994) 721–755.

[10] S. Johnson, *Singular measures without restrictive intervals,* Commun. Math. Phys. **110** (1987) 185–190.

[11] G. Keller, T. Nowicki, *Fibonacci maps re(aℓ)visited,* Ergod. Th. & Dyn. Sys. **15** (1995) 99-120.

[12] M. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps,* Ann. of Math. **140** (1994) 347–404 and *Erratum* Manuscript (2000).

[13] M. Lyubich, J. Milnor, *The Fibonacci unimodal map,* J. Amer. Math. Soc. **6** (1993) 425-457.

[14] R. Mañé, *Hyperbolicity, sinks and measure in one-dimensional dynamics,* Comm. Math. Phys. **100** (1985), no. 4, 495–524.

[15] M. Martens, *Interval dynamics,* Ph.D. Thesis, Delft (1990).

[16] M. Martens, *Distortion results and invariant Cantor sets of unimodal maps,* Ergod. Th. & Dynam. Sys. **14** (1994) 331–349.

[17] M. Martens, T. Nowicki, *Invariant measures for typical quadratic maps,* Géométrie complexe et systèmes dynamiques (Orsay, 1995). Astérisque **261** (2000) 239–252.

[18] W. de Melo, S. van Strien, *One-dimensional dynamics,* Springer (1993).

[19] M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Publ. Math. I.H.E.S. **53** (1981) 17–51

[20] T. Nowicki, *A positive Liapunov exponent for the critical value of an S-unimodal mapping implies uniform hyperbolicity*, Ergod. Th. & Dynam. Sys. **8** (1988) 425–435

[21] T. Nowicki, S. van Strien, *Invariant measures exist under a summability condition,* Invent. Math. **105** (1991) 123–136.

[22] G. Pianigiani, *Absolutely continuous invariant measures on the interval for the process $x_{n+1} = Ax_n(1 - x_n)$*, Boll. Un. Mat. Ital. **16** (1979) 364–378

[23] W. Shen, *Decay geometry for unimodal maps: an elementary proof,* Preprint Warwick (2002).

[24] E. Straube, *On the existence of invariant absolutely continuous measures,* Commun. Math. Phys. **81** (1981) 27–30.

Department of Mathematics
University of Groningen
P.O. Box 800, 9700 AV Groningen
The Netherlands
bruin@math.rug.nl
http://www.math.rug.nl/~bruin

Department of Mathematics
University of Warwick
Coventry CV4 7AL
UK
wxshen@maths.warwick.ac.uk
http://www.maths.warwick.ac.uk/~wxshen

Department of Mathematics
University of Warwick
Coventry CV4 7AL
UK
strien@maths.warwick.ac.uk
http://www.maths.warwick.ac.uk/~strien