Invariant measures exist without a growth condition

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Abstract

Given a non-flat S-unimodal interval map \( f \), we show that there exists \( C \) which only depends on the order of the critical point \( c \) such that if \( |Df^n(f(c))| \geq C \) for all \( n \) sufficiently large, then \( f \) admits an absolutely continuous invariant probability measure (acip). As part of the proof we show that if the quotients of successive intervals of the principal nest of \( f \) are sufficiently small, then \( f \) admits an acip. As a special case, any S-unimodal map with critical order \( \ell < 2 + \epsilon \) having no central returns possesses an acip. These results imply that the summability assumptions in the theorems of Nowicki & van Strien [21] and Martens & Nowicki [17] can be weakened considerably.

1 Introduction

In this paper we consider S-unimodal \( C^3 \) maps \( f : [0, 1] \to [0, 1] \). We assume the unique critical point \( c \) has order \( \ell > 1 \), i.e., for \( x \) near \( c \), there exists a \( C^2 \) diffeomorphism \( \varphi \) such that \( f(x) = \varphi(|x - c|^{\ell}) \).

Theorem 1. There exists \( C = C(\ell) \) so that provided \( |Df^n(f(c))| \geq C \) for all \( n \) sufficiently large, \( f \) admits an absolutely continuous invariant probability measure (acip).

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The problem dealt with in Theorem 1 has a long history, with contributions by amongst others [1], [22], [8], [5], [19], [20], [21]. In particular Theorem 1 shows that the well-known Collet-Eckmann condition \(|Df^n(f(c))| \leq C \gamma^n\) for some \(\gamma \in (0, 1)\), see [5]) or the more recent summability condition \((\sum_n |Df^n(f(c))|^{-1/\ell} < \infty, \text{see } \text{Nowicki & van Strien [21]}\) are far too restrictive. No growth is needed. Recently, many people are considering weakly hyperbolic systems (in particular in dimensions 2 and larger). Perhaps our techniques indicate that one might not always need to look for growth conditions.

A key idea in our proof is to construct an induced Markov map, and analyse the non-linearities and transition probabilities of the resulting random walk. This Markov map has branches with arbitrarily small ranges. The Markov map we construct is based on the so-called principal nest, and the estimates for the transition probabilities come from a careful analysis of the geometry of this principal nest. So let us define this nested sequence of neighbourhoods of the critical point \(c\) starting with \(I_0 = (\hat{q}, q)\), where \(q \in (0, 1)\) is the orientation reversing fixed point of \(f\) and \(f(\hat{q}) = f(q)\). Then define inductively \(I_{n+1}\) to be the central domain of the first return map to \(I_n\). To continue the induction, we need to assume that \(c\) is recurrent, i.e., \(\omega(c) \ni c\). Without this assumption, \(f\) is a Misiurewicz map, and the conclusions of this paper then follow easily (or from well-known results). Write

\[
\mu_n = |I_{n+1}|/|I_n|.
\]

Our paper deals with the case that \(\mu_n\) is small for all large \(n\).

Before stating our result second theorem, let us first discuss \(\mu_n\). Estimating the \(\mu_n\) has been an eminent problem in one-dimensional dynamics, cf. [6, 7, 9, 12]. More precisely, it has been asked if the starting condition [9]

\[
\forall \varepsilon > 0 \exists n_0 > 0 \mu_{n_0} < \varepsilon. \tag{1}
\]

holds. We speak of a central return of \(c\) to \(I_n\) if the first return \(f^*(c)\) of \(c\) into \(I_n\) belongs also to \(I_{n+1}\). If \(\ell \leq 2\) and there are no central returns, an inductive argument ([9], [12]) shows that (1) implies

\[
\forall \varepsilon > 0 \exists n_0 > 0 \forall n \geq n_0 \mu_n < \varepsilon; \tag{2}
\]

(if there are central returns at times \(n(k)\) then in (2) then this only holds at all ‘non-central’ times. Lyubich [12] and Graczyk & Świațek [6], using complex methods, have established the starting conditions for quadratic maps.

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Note that prior to the results [6, 12], the starting condition was verified for quadratic maps with so-called Fibonacci combinatorics [13, 11]. For this map, it is crucial that the critical order is \( \ell = 2 \), because for \( \ell > 2 \), (1) fails: \( \mu_n \) does not tend to zero. More precisely, as was shown in [11],

\[
\exists \varepsilon = \varepsilon(\ell) > 0 \exists n_0 > 0 \ \forall n \geq n_0 \ \mu_n \leq \varepsilon \text{ and } \varepsilon(\ell) \searrow 0 \text{ as } \ell \searrow 2.
\]  

(3)

In fact, when \( \ell \) is large then \( \mu_n \) is close to 1 for all \( n \) (for the Fibonacci map); this implies that a Fibonacci map with large critical order possesses a Cantor attractor, see [4].

Recently, Shen [23] showed, by purely real methods, that for all \( C^3 \) S-unimodal maps without central returns that

- (1) holds for \( \ell \in (1, 2] \),
- (3) holds for \( \ell > 2 \) close to 2.

In this paper we will show that (3), i.e., large values of \( |I_n|/|I_{n+1}| \) when \( n \) is large, guarantee the existence of an \( f \)-invariant measure \( \mu \) that is absolutely continuous with respect to Lebesgue (acip).

**Theorem 2.** There exists \( \varepsilon = \varepsilon(\ell) \) such that if \( |I_{n+1}| \leq \varepsilon |I_n| \) for all \( n \) sufficiently large, then \( f \) admits an acip.

**Remark 1.** We do not need to assume that \( f \) has no central returns for this theorem to hold.

Theorem 2 extends a theorem of Martens & Nowicki [17] stating that \( \sum_n \mu_n^{1/\ell} < \infty \) implies the existence of an acip. In fact, as they show, \( \sum_n \mu_n^{1/\ell} < \infty \) implies the Nowicki-van Strien summability condition. Theorem 2 is strictly stronger: for example for the Fibonacci map with critical order \( 2 + \varepsilon \) the summability conditions fail, but our assumption holds. Theorem 2 also extends the result of Keller & Nowicki [11] for Fibonacci maps of order \( 2 + \eta \) to more general maps:

**Corollary 1.** There exists \( \eta > 0 \) such that for every \( C^3 \) S-unimodal map \( f \) with critical order \( \ell < 2 + \eta \), and with a finite number of central returns holds: If \( f \) has no periodic attractor, then \( f \) has an acip.
Proof of Corollary 1. This follows from Shen’s result [23] that under the above conditions, there exists \( \varepsilon = \varepsilon(\ell) \) such that \( |I_{n+1}| \leq \varepsilon |I_n| \) for \( n \) sufficiently large and that \( \varepsilon \to 0 \) as \( \eta \to 0 \).

In [3], conditions (reminiscent of Fibonacci combinatorics) are given under which \( f \) has an acip, irrespective the critical order as long as \( \ell < \infty \). One can interpret Corollary 1 as a proof that the only mechanism for unimodal maps with critical order \( \ell < 2 + \eta \) not to have an acip, is by (deep) central returns, either of almost restrictive interval type (cf. [10]) or of almost saddle node type (cf. [2]).

2 Preliminaries and structure of the proof

Let us start making precise the condition on \( f \). It is a \( C^3 \) unimodal map with negative Schwarzian derivative such that \( f^2(c) < c < f(c) \) and \( f^3(c) \geq f^2(c) \). Hence we can rescale \( f \) such that \( f^2(c) = 0 \) and \( f(c) = 1 \). The critical order \( \ell \in (1, \infty) \), the critical point is recurrent but not periodic.

Let us first show that Theorem 2 implies our first theorem:

Proof of Theorem 1. Let \( k(n) \) be the minimal integer for which \( f^{k(n)}(c) \in I_n \). Then \( I_{n+1} \) is the pullback of \( I_n \) by \( f^{k(n)} \). By real bounds, [18], there exists \( \delta > 0 \) (which does not depend on \( n \)) and a neighbourhood \( T \) of \( f(I_{n+1}) \), such that \( f^{k(n)-1} \) maps \( T \) diffeomorphically onto a \( \delta \)-scaled neighbourhood of \( I_n \). Hence

\[
|D f^{k(n)}(f(c))| = |D f(f^{k(n)}(c))| \cdot |D f^{k(n)-1}(f(c))| \leq \ell |I_n|^{\ell-1} \cdot K \frac{|f^{k(n)}(I_{n+1})|}{|f(I_{n+1})|} \leq \ell |I_n|^{\ell-1} \cdot K \frac{|I_n|}{|I_{n+1}|} \leq \ell K \frac{|I_n|}{|I_{n+1}|} \ell,
\]

where we have used the non-flatness of \( f \) and Koebe. Therefore, one obtains that \( |I_{n+1}|/|I_n| \) is small provided \( |D f^{k(n)}(f(c))| \) is large.

It is possible that \( f \) is renormalizable. In that case \( k(n) \) is equal to the period \( p \) of this renormalization for all \( n \) large and \( I_n \) shrinks to the largest periodic renormalization interval \( J \) (and so \( |I_{n+1}|/|I_n| \to 1 \)). Then use the same argument for the renormalization: repeat the construction of
the principal nest for \( f^p | J \). Assume \( f \) is \( s \) times renormalizable and \( J_n \) is its \( s \)-th renormalization interval with period \( p_s \). Intervals \( I_n \) associated to its \((s-1)\)-th renormalization shrink to the \( s \)-th renormalization interval \( J_n \), and therefore \(|Df^{p_s}(f(c))| \leq \ell K \frac{\| f \|_{s-1}}{\| f \|_s} \leq 2\ell K \) for \( n \) sufficiently large. But since \( p_s \geq 2^s \), this and the assumption of Theorem 1 imply that \( s \) must be bounded, and so \( f \) can only be finitely often renormalizable. Then consider instead of \( f \) its last renormalization \( f^s | J_n \). Since the above inequality gives that \( |I_n| / |I_{n-1}| \) is large for all \( n \) large (and in particular \( |I_n| \to 0 \) as \( n \to \infty \)), we can apply Theorem 2 and obtain an invariant measure. 

So it suffices to prove Theorem 2. The boundary points of each \( I_n \) are \textit{nice} in the sense of Martens [16], which means that \( f^i (\partial I_n) \notin I_n \) for all \( i > 0 \). In fact, \( f^i (\partial I_n) \notin I_{n-1} \). This allows the following priori estimates:

**Lemma 1.** If \( J \subset I_n \) is a component of the domain of the first return map to \( I_n \) for some \( n > 0 \), say \( f^s | J \) is this return, then there exists an interval \( T \supset f(J) \) such that \( f^{-1}(T) \subset I_n \) and such that \( f^{s-1}|T \) is a diffeomorphism onto \( I_{n-1} \).

**Proof of Lemma 1.** See Martens [16] or Section V.1 in [18].

The idea is now to construct a Markov induced map \( G \) over \( f \) with the intervals \( I_n \) as countable set of ranges: \( G \) is defined on a countable collection of intervals \( J_i \), \( G|J_i = f^{s_i}|J_i \) is a diffeomorphism and \( G(J_i) = I_n \) for some \( n \). We then will construct a \( G \)-invariant measure \( \nu \ll \text{Leb} \), and estimate \( \nu(I_n) \):

**Proposition 1.** Assume that \( \mu_n \leq \varepsilon \) for all \( n \geq n_0 \). If \( \varepsilon \) is sufficiently small, then the induced transformation \( G \) admits a \( \text{acip} \) \( \nu \). Moreover, there exists \( C_0 = C_0(f) \) such that \( \nu(I_n) \leq C_0 \sqrt{|I_n|} \) for all \( n \).

**Corollary 2.** Under the above conditions, \( f \) admits no Cantor attractor.

**Proof of Corollary 2.** This follows easily, for example, from the observation that any Cantor attractor has zero Lebesgue measure (see [15]), and, disregarding \( c \), is invariant by \( G \). Hence \( G \) cannot carry an \( \text{acip} \) if a Cantor attractor is present.

It should be noted that the distortion of the branches of \( G \) is in general not bounded; this comes from the fact that if \( G|J = f^s | J \) is such a branch
and $G(J) = I_n$, then this branch need not be extendible, i.e., if $T \supset J$ is the maximal interval on which $f^s$ of monotone, then $f^s(T)$ need not contain a definite scaled neighbourhood of $I_n$. In particular, $d\nu(x)/dx$ can not be expected to be bounded on any of the sets $I_n \setminus I_{n+1}$. However, we will still be able to derive the following result:

**Theorem 3.** There exists $\varepsilon = \varepsilon(\ell)$ such that if $|I_{n+1}| \leq \varepsilon |I_n|$ for all $n \geq n_0$, then $\sum s_i \nu(J_i) < \infty$.

Once this is obtained, the proof of the main theorem is straightforward.

**Proof of Theorem 2.** This follows by a standard pull-back construction. Given the $G$-invariant measure $\nu$, define $\mu$ by

$$
\mu(A) = \sum_i \sum_{j=0}^{s_i-1} \nu(f^{-j}(A) \cap J_i).
$$

As $f$ is non-singular with respect to Lebesgue, $\mu$ is absolutely continuous, and the $f$-invariance of $\mu$ is a standard exercise. The finiteness of $\mu$ follows directly from Theorem 3. $\square$

**Comments on constants:** In the following, $\ell$ is fixed, $\varepsilon_i$ denotes constants depending only on $\varepsilon$ which are small provided that $\varepsilon$ is. Constants $\rho_i$ depend only on $\ell$. Constants $C_i$ depend only on $f$. The numbers $n_0 \in \mathbb{N}$ and $\lambda \in (0, 1)$, which are defined in Section 4, also depend on $f$. For local use (i.e., within a proof), $B$ and $C = C(f)$ will denote a constant, which might vary within equations.

## 3 Construction of induced maps $G_n$ and $G$

Let $G_0$ be the first return map to $I_0$. Then $G_0$ has a finite number of branches, the central branch is the branch with the largest return time, and each non-central branch maps diffeomorphically onto $I_0$.

In this section we shall construct a sequence of maps $G_n$: $\cup_i J_i^{n+1} \rightarrow I_0$ inductively such that

1. $\cup_i J_i^{n+1}$ is a finite union and for $n \geq 1$, $G_n = G_{n-1}$ outside $I_n$;
2. The central branch $J_0^{n+1} = I_{n+1}$ and $G_n|I_{n+1}$ is the first return map to $I_n$;
3. for each $i \neq 0$, there exists $b_i \leq n$ such that such that $G_n : J_i^{n+1} \to I_{b_i}$ is a diffeomorphism;

4. the outermost branch maps onto $I_0$; more precisely, $J_i^{n+1} \subset I_n$ and \( \partial J_i^{n+1} \cap \partial I_n \neq \emptyset \) imply $G_n(J_i^{n+1}) = I_0$ (and the external point of such an interval $J_i^{n+1}$ maps to the fixed point $q$);

5. $G_n(x) = f^s(x)$ implies that $f(x), \ldots, f^{s-1}(x) \notin I_n$;

By definition $G_0$ satisfies the above statements, so let us assume that by induction $G_n$ exists with the above properties, and construct $G_{n+1}$.

Set $G_{n+1}(x) = G_n(x)$ for $x \notin I_{n+1}$. Let $k_n \in \mathbb{N} := \{1, 2, 3, \ldots \}$ be minimal so that $G^{k_n}_n(c) \in I_{n+1}$. This means that $k_n = 1$ if the return to $I_n$ is central. Define $K^0 = I_{n+1}$, $K^{k_n} = I_{n+2}$ and, for $0 \leq j \leq k_n - 1$, let $K^j$ be the component of $\text{dom}(G_n^{j+1})$ which contains $c$. Next define on $K^j \setminus K^{j+1}$

$$G_{n+1}(x) = \begin{cases} 
G^{j+1}_n(x) & \text{if } G^{j+1}_n(x) \in I_{n+1} \\
G^{j+2}_n(x) & \text{otherwise.}
\end{cases}$$

$G_{n+1} | I_{n+2} = G^{k_n}_n | I_{n+2}$ is the first return map to $I_{n+1}$. Properties (1) and (2) hold by construction for $G_{n+1}$. Property (3) holds because if $G^{j+1}_n(x) \in I_{n+1}$ for some $x \in I_{n+1} \setminus I_{n+2}$ then $G_{n+1}(J_i^{n+1}) = I_{n+1}$ for the corresponding domain $J_i^{n+1} \ni x$ and if $G^{j+1}_n(x) \notin I_{n+1}$ then by the induction assumption $G_{n+1}(J_i^{n+1})$ is equal to some domain $I_b$, $b \leq n$, because then $G_{n+1}(x) = G^{j+2}_n(x)$. Property (4) holds immediately because $\partial I_n$ is mapped by $G_n$ into $\partial I_0$. In order to show Property (5) holds, take $x \in K^j \setminus K^{j+1}$ and let $y = G^j(x)$. Note that $G^j_n | K^j$ is inside a component of $\text{dom}(G_n)$ and that all iterates $f(K^j), \ldots, G^j_n(K^j) \ni y$ are outside $I_{n+1}$. Since $G^{j+1}_n(x) = G_n(y)$ we get by induction that (5) holds for $G_{n+1}$ (using that it holds for $G_n$ and $y$ instead of $x$).

The induced map $G$ is defined as follows: for each $n \geq 0$, each component of the domain $J$ of $G_n$ other than the central one $I_{n+1}$ becomes a component of the domain of $G$, and $G|J = G_n|J$.

For later use, we compute by induction that if $x \in I_n \setminus I_{n+1}$, and $G(x) = f^s(x)$, then

$$s \leq t_0 \cdot (k_0 + 1) \cdots (k_{n-2} + 1) \cdot (k_{n-1} + 1),$$

where $t_0 = \min \{ i > 0 ; f^i(c) \in I_0 \}$.
4 Distortion properties of the induced map

Suppose $\varphi : T \to \varphi(T)$ is a $C^1$ map. Let us define

$$\text{Dist}(\varphi) := \text{Dist}(\varphi, T) := \sup_{x,y \in T} \log \frac{\varphi'(x)}{\varphi'(y)}.$$ 

Let us say a diffeomorphism $h : J \to h(J)$ belongs to the distortion class $\mathcal{F}_p^C$ if it can be written as

$$Q \circ \varphi_q \circ \cdots \circ Q \circ \varphi_1,$$

with $q \leq p$, where $Q(x) = |x|^{\ell}$ and $\text{Dist}(\varphi, j) \leq C$ for all $1 \leq j \leq q$.

Let us fix a large positive integer $n_0$ such that $|I_n| \leq \varepsilon |I_{n-1}|$ for all $n \geq n_0$, and such that $f|_{I_{n_0}}$ can be written as $x \mapsto \varphi(|x|^\ell)$ with $\text{Dist}(\varphi) \leq 1/4$. By Lemma 1, it follows that for each $n \geq n_0$, if $J$ is a return domain to $I_n$, and $f^n|J$ is the return, then $f^n|J$ can be written as $x \mapsto \varphi(|x|^\ell)$ with $\text{Dist}(\varphi) \leq 1/2$ provided $\varepsilon$ is sufficiently small.

According to Mañé [14], the map $G$, restricted to the set of points which stay outside $I_{n_0+1}$ is a hyperbolic (uniformly expanding) system. Thus, there exists $C_1 = C_1(f) > 0$ and $\lambda = \lambda(f) \in (0,1)$ with the following property. For any $k \in \mathbb{N}$

1. if $x$ is a point such that $G^i(x)$ are defined and $G^i(x) \notin I_{n_0+1}$ for any $0 \leq i \leq k-1$, then

$$|(G^k)'(x)| \geq \frac{1}{C_1 \lambda^k};$$

2. if $J$ is an interval such that $G^k|J$ is defined, and $G^i(J) \cap I_{n_0+1} = \emptyset$ for all $0 \leq i \leq k-1$, then

$$\text{Dist}(G^k|J) \leq \log C_1.$$

We will use the notation $\alpha(y) = n$ if $y \in I_n \setminus I_{n+1}$.

**Proposition 2.** Let $m \geq 1$, and let $G^i : J \to I_m$ be an onto branch of $G^i$. There exists $C_2 = C_2(f)$ such that the following hold:

- Suppose that $\alpha(G^{i-1}J) > m$. Let $n > m$ and $1 \leq k \leq i$ be maximal such that

$$n = \alpha(G^{i-k}J) > \alpha(G^{i-k+1}J) > \cdots > \alpha(G^{i-1}J) > m.$$
Then $G^i|J$ can be written as $\psi \circ \varphi$ such that
\[
\text{Dist}(\psi) \leq \log C_2 \text{ and } \varphi \in \mathcal{F}_{2[n-m+1]}^1.
\]

- If $\alpha(G^{i-1}(J)) \leq m$ then $G^i|J$ can be written as $\psi \circ \varphi$ such that
  \[
  \text{Dist}(\psi) \leq \log C_2 \text{ and } \varphi \in \mathcal{F}_2^1.
  \]

**Proof.** Let $r$ denote the maximum of $\alpha(G^j(J))$ for $0 \leq j \leq i - 1$. Let $C = C(f)$ be a big constant. We shall prove by induction on $r$ the following stronger statement: $G^i|J$ can be written as $\psi \circ H \circ Q \circ \varphi_1$ with
\[
\text{Dist}(\psi) \leq \log C, \quad H \in \mathcal{F}_{2[n-m]+1}^1 \text{ and } \text{Dist}(\varphi_1) < 1/2.
\]

If $r \leq n_0$, then the distortion of $G^i|J$ is bounded by $\log C_1(f)$ as we remarked above. Hence the statement is true for $C > C_1$. So let us consider the case $r > n_0$.

For $0 \leq j \leq i - 1$, let $T_j$ denote the domain of $G$ which contains $G^j(J)$. For simplicity of notation, write $\alpha_j = \alpha(G^j(J))$. By definition of $n$, we have $\alpha_{i-k-1} \leq \alpha_{i-k} = n$. Note that $G^j|J$ extends to a diffeomorphism onto $I_{\alpha_j}$ for all $1 \leq j \leq i$.

**Case 1.** $n \leq n_0$. Then $\alpha_j \leq n_0$ for all $i - k \leq j \leq i - 1$, and so $\text{Dist}(G^k|G^{i-k}(J)) \leq \log C_1$. If $G(T_{i-k-1}) \supset I_{n-1}$, then $\text{Dist}(G^i|J)$ is bounded by the Koebe principle, and thus we are done. If $G(T_{i-k-1}) \subset I_n$, then $T_{i-k-1}$ is a return domain to $I_n$. Since $n \geq m \geq 1$, this return domain is well inside $I_n$, which implies that $G^{i-k-1}|J$ has bounded distortion. Since $n \leq n_0$, the distortion of $G|T_{i-k-1}$ has bounded distortion as well, and so the proposition is true for some universal constant $C$ (which depending on the a priori real bounds).

**Case 2.** $n > n_0$. Then similarly as above, we can show that $G^{i-k}|J$ can be written as $\varphi_2 \circ h_1$, with $\text{Dist}(\varphi_2) \leq 1/2$ and $h_1 \in \mathcal{F}_{1/2}^1$. If $k = 0$, then the proposition follows. Assume $k \geq 1$. Let $J' = G_{n-1}(G^{i-k}(J))$, and let $s \in \mathbb{N}$ be such that $G = G_n = G_{n-1}$ on $G^{i-k}(J)$. Since $\alpha_{i-k+1} \leq \alpha_{i-k}$, it follows from our construction that $G^j(J') \cap I_n = \emptyset$ for all $0 \leq j < s$. The same is true for $s \leq j \leq s - 1 + k - 1$ by definition of $n$. Thus
\[
\max_{0 \leq j \leq s-1+k-1} \alpha(G^j(J')) \leq n - 1 \leq r - 1.
\]
Applying the induction hypothesis to the map \( G^{k-1} \circ G^{i-1}|J' = G^{k-1} \circ G_{n-1}^{i-1}|J' \), we see that the map can be written as \( \psi \circ h \circ Q \circ \varphi \) with \( \text{Dist}(\psi) < C \), and \( \text{Dist}(\varphi) \leq 1/2 \), and \( h \in \mathcal{F}_{2(n-m)-1}^1 \). The map \( G_{n-1}|G^{i-k}(J) \) is a restriction of the first return map to \( I_{n-1} \), which is of the form \( \varphi_3 \circ Q \) with \( \text{Dist}(\varphi_3) \leq 1/2 \). Therefore

\[
G^i|J = G^{i-1+k-1}|J' \circ G_{n-1}|G^{i-k}(J) \circ G^{i-k}|J \\
= \psi \circ h \circ Q \circ (\varphi \circ \varphi_3) \circ Q \circ \varphi_2 \circ h_1.
\]

Note that \( \text{Dist}(\varphi \circ \varphi_3) < 1 \), and the induction step is completed. \( \square \)

We will need another proposition to treat the case \( m = 0 \). By taking \( C_2 \) larger if necessary, we prove:

**Proposition 3.** Consider any branch \( G^i|J \). Let \( n = \max_{j=0}^{i-1} \alpha(G^jJ) \). Then \( G^i|J \) can be written \( \psi \circ H \) with

\[
\text{Dist}(\psi) \leq \log C_2 \quad \text{and} \quad H \in \mathcal{F}_{2n}^1.
\]

**Proof.** First note that if \( G^i(J) \subset I_1 \), then the assertion follows immediately from the previous proposition. So we shall assume \( G^i(J) = I_0 \). Let us prove by induction that \( G^i|J \) can be written as \( \psi \circ H \circ Q \circ \varphi \), where \( \psi \) is an iterate of \( G|(I_0 \setminus I_{n_0+1}) \), and \( H \in \mathcal{F}_{n_1}^1 \), and \( \text{Dist}(\varphi) < 1/2 \).

If \( n \leq n_0 \), then the claim is clearly true. Assume \( n > n_0 \). Let \( 0 \leq p < i \) be the largest such that \( \alpha_p = n \). Using similar argument as in the proof of the previous proposition, the map \( G^p|J \) can be written as \( \varphi_0 \circ h \), where \( \text{Dist}(\varphi_0) < 1/2 \), and \( h \in \mathcal{F}_{1/2}^1 \). Note that \( \alpha(G^p|J) \leq \alpha(G^pJ) \) by the maximality of \( \alpha(G^pJ) \). Let \( s \) be the positive integer such that

\[
G|G^pJ = G_{\alpha_p}|G^pJ = G_{\alpha_{p-1}}|G^pJ,
\]

and let \( J' = G_{\alpha_{p-1}}(G^pJ) \). It follows from the construction of \( G \) and the maximality of \( \alpha_p \) that \( \alpha(G^j(J')) \leq n-1 \) for all \( 0 \leq j \leq s - 2 + (i-p) \). By the induction hypothesis, we can decompose the map \( G^{i-p+s-1}|J' \) as \( \psi_1 \circ H_1 \circ Q \circ \varphi_1 \) such that \( \psi_1 \) is an iterate of \( G|(I_0 \setminus I_{n_0+1}) \), and \( H_1 \in \mathcal{F}_{n_2}^1 \). The map \( G_{n-1}|G^pJ \) is a restriction of the first return map to \( I_{n-1} \), and thus it can be written as \( \varphi \circ Q \) with \( \text{Dist}(\varphi) < 1/2 \). Combining all these facts, we decompose

\[
G^i|J = \psi_1 \circ \{ H_1 \circ [Q \circ (\varphi_1 \circ \varphi_0)] \} \circ h,
\]

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as required. This completes the proof of the induction step. \qed

We are going to use the following lemma many times.

**Lemma 2.** If \( h : J \to I \) is a diffeomorphism in \( \mathcal{F}_p \), and \( A \subset J \) is a measurable set, then

\[
\frac{1}{(\ell e)^p} \frac{\text{Leb}(h(A))}{|I|} \leq \frac{\text{Leb}(A)}{|J|} \leq e^p \left( \frac{\text{Leb}(h(A))}{|I|} \right)^{1/p}.
\]

**Proof.** First we note that for any interval \( T \subset \mathbb{R} \setminus \{0\} \) and any measurable set \( A \subset T \), we have

\[
\frac{\text{Leb}(A)}{|T|} \leq \left( \frac{\text{Leb}(Q(A))}{|Q(T)|} \right)^{1/t}.
\]

To see this, note that for a fixed \( \text{Leb}(Q(A)) \), the left hand side takes its maximum in the case that \( A \) is an interval adjacent to the endpoint of \( \partial T \) which is closer to 0.

It suffices to prove the two inequalities in case \( p = 1 \). So let us consider the case \( h = Q \circ \varphi \) with \( \text{Dist}(\varphi) \leq 1 \). For any \( A \subset J \), we have

\[
\frac{\text{Leb}(A)}{|J|} \leq e \frac{\text{Leb}(\varphi(A))}{|\varphi(J)|} \leq e \left( \frac{\text{Leb}(h(A))}{|h(I)|} \right)^{1/t}.
\]

This proves the second inequality of (5). On the other hand,

\[
\frac{\text{Leb}(\varphi(A))}{|\varphi(J)|} = 1 - \frac{\text{Leb}(\varphi(J \setminus A))}{|\varphi(J)|}
\]

\[
\geq 1 - \left( \frac{\text{Leb}(h(J \setminus h(A)))}{|h(J)|} \right)^{1/t}
\]

\[
= 1 - (1 - \frac{\text{Leb}(h(A))}{|I|})^{1/t}
\]

\[
\geq \frac{1}{e} \frac{\text{Leb}(h(A))}{|I|} - \frac{1}{e \ell} \frac{\text{Leb}(h(A))}{|I|},
\]

and thus

\[
\frac{\text{Leb}(A)}{|J|} \geq \frac{1}{e} \frac{\text{Leb}(\varphi(A))}{|\varphi(J)|} \geq \frac{1}{e \ell} \frac{\text{Leb}(h(A))}{|I|},
\]

proving the first inequality. \qed

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5 Outermost branches

Within $I_n$, there are two special branches which have common endpoints with $I_n$. These branches always mapped onto $I_0$ by the map $G$, and need special care in our argument. In this section, we shall prove that these branches can not be too small.

**Proposition 4.** There exist a constant $\rho_1 = \rho_1(\ell) > 0$ and a constant $C_3 = C_3(f) > 0$, such that if $J_n$ is one of the two outermost branches of $G$ in $I_n$, then

$$\frac{|J_n|}{|I_n|} \geq \frac{\rho_1}{C_3}.$$ 

**Proof.** Let $\delta_n := |J_n|/|I_n|$ and $\hat{J}_{n-1}$ the outer-most branch of $I_{n-1} \setminus I_n$ for which $\hat{J}_{n-1} \supset G_{n-1}(J_n)$. Write $G_{n-1}|I_n = f^{t_n}$. Since this is a first return, one has $\text{Dist} (f^{t_n-1}|f(I_n)) \leq 1$ for all $n$ sufficiently big.

**Case 1.** $G_{n-1}(c) \notin \hat{J}_{n-1}$. Then by the distortion bound for $f^{t_n-1}|f(I_n)$,

$$\frac{|f(a) - f(c)|}{|f(b) - f(c)|} = 1 + \frac{|f(a) - f(b)|}{|f(b) - f(c)|} \geq 1 + C\delta_{n-1},$$

where $a$ and $b$ are the end points of $J_n$ with $b$ between $a$ and $c$. Hence, using that $c$ is a critical point of order $\ell$,

$$\frac{|a - c|}{|b - c|} \geq (1 + C\delta_{n-1})^{1/\ell} \geq 1 + \frac{C\delta_{n-1}}{\ell}.$$

Hence

$$\delta_n = \frac{|J_n|}{|I_n|} = \frac{1}{2} \frac{|a - b|}{|a - c|} \geq \frac{1}{2} \left( 1 - \frac{1}{1 + C\delta_{n-1}/\ell} \right) \geq C\delta_{n-1}/\ell.$$

By induction, $|J_n|/|I_n| \geq \rho_1^2/C_3$ for $\rho_1 = \rho_1(\ell) \asymp 1/\ell$.

**Case 2.** $G_{n-1}(c) \in \hat{J}_{n-1}$. Note that $G_{n-1}(\hat{J}_{n-1}) = I_0$ and that $G_{n-1}^2J_n$ intersects an outermost branch $\hat{J}_0$ of $I_0$. Let $p \geq 0$ be minimal so that $G_{n-1}^{p+2}(c) \notin \hat{J}_0$. Then $|\hat{J}_0|/|G_{n-1}^{p+2}I_n|$ is bounded from below (by a bound which depends only on $f$), and since $G_{n-1}^{p+2}(J_n) = \hat{J}_0$, and $f|/(I_0 \setminus I_1)$ is hyperbolic this implies

$$|G_{n-1}^2J_n|/|G_{n-1}^2I_n| \geq C > 0.$$ 

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According to the distortion control on $G_{n-1}|\hat{J}_{n-1}$ given by Proposition 3, this implies

$$|G_{n-1}J_n|/|G_{n-1}I_n| \geq C\rho^n > 0.$$ 

Since $I_n$ is a first return domain of $G_{n-1}$, by Lemma 1, this implies

$$|J_n|/|I_n| \geq \rho_1^n/C_3,$$

with $\rho_1 = \rho_1(\ell) \times 1/\ell$ and $C_3 = C_3(f) \times 1/C$. 

\[\square\]

6 Improved decay for deep returns

Let $x$ and $m$ be so that $G^n(x)$ is well-defined and $G^n_i(x) \notin I_{n+1}$ for $0 \leq i < m$. Let $T_i = T_i(x)$ be the component of $\text{dom}(G_n)$ which contains $G^n_i(x)$. Define $\alpha(y) = j$ if $y \in I_j \setminus I_{j+1}$ and $s(y) = s$ if $G(y) = f^s(y) = G_{\alpha(y)}(y)$. Let $t_n$ be the return time of $c$ to $I_n$ under $f$. Define

$$\Lambda = \{0 \leq i \leq m-2 ; \; \alpha(T_{i+1}) \geq \alpha(T_i)\},$$

$$N = \sum_{i \in \Lambda} [\alpha(T_{i+1}) - \alpha(T_i) + 1] \text{ and } r = \#\Lambda.$$

Moreover, define

$$T'_0 = \{y \in T_0 ; \; G^n_i(y) \in T_i \text{ for all } i \leq m-1\}.$$

If $\varphi : T \to \varphi(T)$ is a homeomorphism and $J \subset T$ is a subinterval of $T$, we denote the components of $T \setminus J$ by $L$ and $R$, and write

$$Cr(T, J) := \frac{|T| \cdot |J|}{|L| \cdot |R|}$$

for the cross-ratio of $J$ in $T$.

**Lemma 3.** Assume that $\alpha(T_i) \geq n_0$ for all $i = 0, \ldots, m-2$, then for $\varepsilon_1 \propto \varepsilon^{1/r}$

- $Cr(T_0, T'_0) \leq \varepsilon_1^N$ if $r \geq 1$;

- for each interval $J \subset G_n(T_{m-1})$ with $J \ni G^n_m(x)$, and $J' := \{y \in T'_0 ; \; G^n_m(y) \in J\}$ we have

$$Cr(T_0, J') \leq \varepsilon_1^N \cdot Cr(G_n(T_{m-1}), J)$$

(even if $r = 0$).
Proof of Lemma 3. For \(0 \leq j \leq m - 2\), write
\[
Cr(I_{a(T_j)}, G_n^{j}T_0^j) \leq Cr(T_j, G_n^jT_0^j) \\
\leq Cr(G_nT_j, G_n^{j+1}T_0^j) \\
\leq Cr(I_{a(T_{j+1})}, G_n^{j+1}T_0^j).
\]
Here the first and third inequality hold by inclusion of intervals, and the second inequality because \(f\) has negative Schwarzian derivative. Note that \(G_nT_j \supset I_{a(T_j)}\). If \(j \in \Lambda\) then one gets improved inequalities: if
\[
G_nT_j \supset I_{a(T_j)-1} \supset I_{a(T_{j+1})-1},
\]
then in the third inequality one gets an additional factor \(\varepsilon_1^{[\alpha(T_{j+1})-\alpha(T_j)+1]}\), while if \(G_nT_j = I_{a(T_j)} \supset I_{a(T_{j+1})}\) then in the first inequality one gets a factor \(\varepsilon_1\) (because then \(G_n\) is a first return and so a composition of \(x^i\) and a map which extends diffeomorphically to \(I_{a(T_j)-1}\)) and in the third we get an additional factor
\[
\varepsilon_1^{[\alpha(T_{j+1})-\alpha(T_j)]}.
\]
To prove the second assertion of the lemma one proceeds in the same way. Note that all this holds, provided \(\alpha(T_j) \geq n_0\) for each \(j \in \Lambda\) where \(n_0\) is chosen so that \(|I_{n+1}|/|I_n| < \varepsilon\) for \(n \geq n_0\). \(\square\)

Let \(k_n\) be as in Section 3.

Corollary 3. There exists \(C_4 = C_4(f) > 1\) and \(\varepsilon_2 \asymp \varepsilon_1^{1/\ell}\) with the following property.

1. If \(\alpha(G_n^i(I_{n+2})) \geq n_0\) for all \(0 \leq i \leq k_n\), then
\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4\varepsilon_2^{k_n}.
\]

2. If \(\alpha(G_n^i(I_{n+2})) \leq n_0\) for some \(1 \leq i \leq k_n\), then
\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4\varepsilon_2^{n-n_0}.
\]

Proof. (1) Let \(x = G_n(c)\) and \(m = k_n - 1\), and let \(T_i, \Lambda, N\) be defined as above. Write \(n' = \alpha(G_n^{k_n-1}(c))\). Note that \(\alpha(G_n(c)) = n\). Then
\[
\sum_{i=0}^{m-1} |\alpha(T_{i+1}) - \alpha(T_i)| = n' - n.
\]
Thus

\[ N - r = \sum_{i \in \Lambda} [\alpha(T_{i+1}) - \alpha(T_i)] \]
\[ = n' - n + \sum_{i \notin \Lambda} [\alpha(T_i) - \alpha(T_{i+1})] \]
\[ \geq n' - n + m - r, \]

which implies

\[ N \geq n' - n + m. \] (6)

Let \( J = G_{n}^{k_n} (I_{n+2}) \). Then

\[ Cr(G_n(T_{m-1}), J) \leq Cr(I_{n'}, I_{n+1}) \leq 3\varepsilon_1^{n+1-n'}. \]

Applying the last part of the previous lemma, we obtain

\[ Cr(T_0, G_n(I_{n+2})) \leq 3\varepsilon_1^{N_n} \leq 3\varepsilon_1^{m+1} = 3\varepsilon_1^{k_n}, \]

which implies this corollary.

(2) Let \( p < k_n \) be the largest integer for which \( \alpha(G_n(I_{n+2})) \leq n_0 \). Let \( \tilde{\Lambda} = \{ p \leq i \leq k_n - 2 : i \in \Lambda \} \), and let \( \tilde{N} = \sum_{i \in \tilde{\Lambda}} [\alpha(T_{i+1}) - \alpha(T_i) + 1] \). Then we can show similarly

\[ \tilde{N} \geq n' - n_0 + k_n - p \geq n' - n_0, \]

and

\[ Cr(T_p, G_n(I_{n+2})) \leq \tilde{\varepsilon}^N Cr(I_{n'}, I_{n+1}) \leq \varepsilon_1^{n-n_0}, \]

which implies the statement. \( \square \)

7 Improved decay in general

Let \( I_{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I_{n+2} \) be the domains of \( G_n^i \) as in Section 3.
Lemma 4. Assume
\[ K^i \supseteq K^{i+1} = K^{i+2} = \cdots = K^{i+m} \supseteq K^{i+m+1}. \]

Then there exists \( C_5 = C_5(f) > 0 \) and \( \rho_2 = \rho_2(\ell) \in (0, 1) \) such that (provided \( n \) is sufficiently large)
\[
\frac{|K^{i+1}|}{|K^i|} \leq (1 - \rho_2^n/C_5)^m. \tag{7}
\]

Proof of Lemma 4. By construction, \( G^{i+1}K^i \) contains the outermost domain of some interval \( I_j \), with \( j \leq n \), while \( G^{i+1}K^{i+1} \subset I_j \) is not contained in that outermost domain. By Proposition 4, this outermost domain is at least \( \rho_2^j/C_3(\geq \rho_2^i/C_3) \) times as large as \( |I_j| \). By Propositions 2 and 3, the map \( G^i|G_nK^i = G^i_n|G_nK^i \) can be written as \( \psi \circ H \) with
\[
\text{Dist}(\psi) \leq \log C_3 \text{ and } H \in \mathcal{F}_2^1.
\]

By the left inequality of (5), this implies that
\[
\frac{|G_n(K^i \setminus K^{i+1})|}{|G_nK^i|} \geq \rho^n/C,
\]
for some \( \rho = \rho(\ell) \in (0, 1) \). Since \( G_n|K^i \) is a restriction of the first return map to \( I_n \), it follows that
\[
\frac{|K^{i+1}|}{|K^i|} \leq 1 - \rho^n/C.
\]
for \( n \) large. Hence, at least provided \( \frac{\log m}{n} \) is not too large, i.e., bounded by a universal constant, (7) holds (taking \( \rho_2 > 0 \) small). So we need to consider the case that \( \frac{\log m}{n} \) is large. Then \( K^{i+1} = \cdots = K^{i+m} \), \( G^{i+2}K^{i+1} \) is contained in an outermost domain, and so one of the endpoints of \( G^{i+3}K^{i+1} \) is a boundary point of \( I_0 \). Using that \( K^{i+1} = \cdots = K^{i+m} \),
\[
\frac{|G^{i+3}K^{i+1}|}{|\hat{J}_0|} \leq C\lambda^m,
\]
where \( \hat{J}_0 \) is the outermost branch of \( I_0 \), \( C = C(f) \), and \( \lambda \in (0, 1) \) comes from the beginning of Section 4. The distortion control given by Proposition 3 gives
\[
\frac{|G^{i+1}K^{i+1}|}{|T_{i+1}|} \leq C\lambda^{m/\ell^2n},
\]
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where $T_{i+1}$ is the domain of $G^2_n$ containing $G^{i+1}_n K^{i+1}$. Since $|G^{i+1}_n(K^i \setminus K^{i+1})| \geq \rho_1^n |T_{i+1}|/C$, it follows

$$\frac{|G^{i+1}_n K^{i+1}|}{|G^{i+1}_n K^i|} \leq C \frac{\lambda^{m/\ell^2}}{\rho_1^n}.$$  

Using the distortion control given by Proposition 2 or 3, and equation (5), we obtain

$$\frac{|G_n(K^{i+1})|}{|G_n(K^i)|} \leq C e^{n \lambda^m/\ell^{4n}} / \rho_1^n.$$ 

Pulling back by the first return map $G_n|K^i$, we obtain

$$\frac{|K^{i+1}|}{|K^i|} \leq C e^{n \lambda^m/\ell^{4n+1}} / \rho_1^n,$$

which clearly implies (7) when $\log \frac{m}{n} \gg 4 \log \ell$ and $\rho_2 \ll \ell^{-4}$. \hfill \Box

**Lemma 5.** Let $\lambda \in (0, 1)$ be as in the beginning of Section 4. Let $m$ be so that $I_{n+1} = K^0 = \cdots = K^m \neq K^{m+1} \supset I_{n+2}$. Assume $m \geq 1$. Then

$$\frac{|I_{n+1}|}{|I_n|} \leq C_6 \lambda^{m/\ell^{n+1}},$$

where $C_6 = C_6(f)$ is a constant.

**Proof of Lemma 5.** Note that $G_n|I_{n+1}$ is a first return map to $I_n$, and so there exists a neighbourhood $T \ni f(c)$ such that $f^{t_n - 1} : T \to I_{n-1}$ is a diffeomorphism and $f^{-1}(T) \subset I_n$. Therefore

$$\frac{|I_{n+1}|}{|I_n|} \leq \left( \frac{|G_n I_{n+1}|}{|I_{n-1}|} \right)^{1/\ell}.$$ 

If $m \geq 1$, then $G_n(I_{n+1})$ is contained in an outermost branch $J_n$ in $I_n$. Similarly as before

$$\frac{|G_n I_{n+1}|}{|J_n|} \leq C \lambda^{m/\ell^n}$$

and so

$$\frac{|I_{n+1}|}{|I_n|} \leq C \lambda^{m/\ell^{n+1}}.$$ 

\hfill \Box
Lemma 6. There exists \( \varepsilon(\ell) \) so that if

\[ |I_{n+1}| \leq \varepsilon |I_n| \]

for all \( n \) sufficiently large, then for all \( n \) sufficiently large,

\[ \frac{|I_{n+2}|}{|I_{n+1}|} \leq \frac{1}{(k_n + 1)^4}. \]

Proof of Lemma 6. Consider \( \alpha(G_n^i, c) \) for \( 1 \leq i < k_n \). If all these are larger than \( n_0 \) then by Corollary 3,

\[ \frac{|I_{n+2}|}{|I_{n+1}|} \leq C_4 \varepsilon_2^{k_n} \leq \frac{1}{(k_n + 1)^4}. \]

So assume that there exists \( 1 \leq i < k_n \) such that \( \alpha(G_n^i, c) \leq n_0 \). Then at least we have

\[ \frac{|I_{n+1}|}{|I_n|} \leq C_4 \varepsilon_2^{\frac{n-n_0}{\ell}} \]

by the second statement of Corollary 3. This implies the lemma, unless

\[ k_n \geq \varepsilon_2^{-\frac{n-n_0}{\ell}/C_4} \]

Let \( m \) as before be so that \( I_{n+1} = K^0 = K^1 = \cdots = K^m \neq K^{m+1} \supset I_{n+2} \). Then respectively by the previous lemma and by Lemma 4,

\[ \frac{|I_{n+1}|}{|I_n|} \leq C \lambda^{\ell_{n+1}} \text{ and } \frac{|I_{n+2}|}{|I_{n+1}|} \leq (1 - \rho_2^n/C_5)^{k_n-m}. \]

Case 1. \( m < k_n/2 \). According to the second inequality, we have

\[ \frac{|I_{n+2}|}{|I_{n+1}|} \leq (1 - \rho_2^n/C_5)^{k_n/2} \leq \frac{1}{(k_n + 1)^4}. \]

provided we choose \( \varepsilon(\ell) \) so small that for \( \varepsilon_2 \) from Corollary 3, \( \varepsilon_2 < \rho_2^\ell \) and we take \( n \) sufficiently large. Here we have used the assumption that

\[ k_n \geq \varepsilon_2^{-\frac{n-n_0}{\ell}/C_4} \]

Case 2. \( m \geq k_n/2 \). Then by the first inequality,

\[ Cr(I_n, I_{n+1}) \times \frac{|I_{n+1}|}{|I_n|} \leq \lambda^{k_n/2\ell^{n+1}}. \]

By Lemma 1, there is an interval \( T \ni f(c) \) such that \( f^{-1}(T) \cap I_{n+1} \) and such that \( f_{t_{n+1}}^{t_{n+2}} : T \to I_n \) is a diffeomorphism, where \( t_{n+1} \) is the first return time.
of $c$ to $I_{n+1}$. Since also $f^{t_{n}+1}(I_{n+2}) \subset I_{n+1}$, we obtain
\[
Cr(T, f(I_{n+2})) \leq Cr(f^{t_{n}+1}(T), f^{t_{n}+1}(I_{n+2})) \\
\leq Cr(I_{n}, I_{n+1}) \\
\asymp \lambda^{k_{n}/2 \ell^{n+1}}.
\]
Since $f^{-1}(T) \subset I_{n+1}$, $f(I_{n+1})$ contains a component of $T \setminus f(I_{n+2})$. Thus
\[
\frac{|f(I_{n+2})|}{|f(I_{n+1})|} \leq Cr(T, f(I_{n+2})) \leq C \lambda^{k_{n}/2 \ell^{n+1}}.
\]
Finally, the non-flatness of the critical point gives
\[
\frac{|I_{n+2}|}{|I_{n+1}|} \leq C \lambda^{k_{n}/2 \ell^{n+2}} \leq \frac{1}{(k_{n} + 1)^{4}},
\]
provided that $\varepsilon_{2} < \ell^{-4 \ell}$ and $n$ is sufficiently large. \hfill \qed

8 The measure for the induced map

In this section we prove the existence of an acip for the induced map $G$.

**Proof of Proposition 1.** We will use the result by Straube [24] claiming that $G$ has an acip if (and only if) there exists some $\eta \in (0, 1)$ and $\delta > 0$ such that for every measurable set $A$ of measure $\text{Leb}(A) < \delta$ holds $\text{Leb}(G^{-k}(A)) < \eta |I_{0}|$.

The assumptions give that there exists a constant $B$ with the following property: If $J$ is any branch of $G^{k}$ and $G^{k}(J) = I_{n}$, then
\[
\text{Leb}(\{x \in J; G^{k}(x) \in I_{n+m}\}) \leq B \frac{|I_{n+m}|}{|I_{n}|}.
\]
(8)

This includes trivially the branch of $G^{0}$, that is the identity. Note that $B$ is a distortion constant, and $B \leq 2$ for $\varepsilon \approx 0$ and $n \geq n_{0}$. So we can assume that $B \sqrt{\varepsilon}/(1 - \sqrt{\varepsilon}) < 1/3$. Moreover, $|I_{n}| \leq \varepsilon^{n-m}|I_{m}|$ for all $n \geq m \geq n_{0}$.

**Lemma 7.** If $J$ is a branch of $G^{k-1}$ such that $G^{k-1}(J) = I_{n+1}$, then
\[
\text{Leb}(\{x \in J; \alpha(G^{k}(x)) \geq n + 1\}) \leq \frac{1}{6} |J|, \hfill (9)
\]
provided $n \geq n_{0}$.
Proof. Let $I_{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I_{n+2}$ be as in Section 3. For each $0 \leq i \leq k_n - 1$ with $K^i \neq K^{i+1}$, there can be at most two branches of $G$, symmetric w.r.t. the critical point, which map onto $I_{n+1}$. We claim that each of these branches $P$ lies deep inside $K^i$ (if they exist). To see this, let $s \in \mathbb{N}$ be such that $G^s|P = f^s|P$. Then by our construction, $f^{-1}$ maps an interval $T \ni f(e)$ onto some interval $I_j$ with $j \leq n$, and $f^{-1}(T) = K_i$. Since $f^{-1}(f(P)) = I_{n+1}$ lies deep inside $I_j$, it follows from the Koebe principle that $f(P)$ lies deep inside $T$. The claim follows from the non-flatness of the critical point.

Let $U_{n+1}$ be the union of those domains of $G$ inside $I_{n+1} \setminus I_{n+2}$ which are mapped onto $I_{n+1}$ by $G$. Then it follows from the Koebe principle

$$\text{Leb}(\{x \in J : G^{k-1}(x) \in U_{n+1}\}) \leq \frac{1}{10}|J|.$$

It remains to consider branches of $J'$ of $G^k|J$ for which $G^k(J') = I_{n'}$ with $n' \leq n$. But using the remark before this lemma, we obtain an estimate for this part also, and thus we conclude the proof. \(\square\)

Write $y_{n,k} = \text{Leb}(\{x \in I_0; \alpha(G^k(x)) = n\})$. Take $C_0 > 6B/|I_{n_0}|$. We will show by induction that $y_{n,k} \leq C_0 \sqrt{|I_n|}$ for all $n, k \geq 0$. For $k = 0$, this is obvious, and the choice of $C_0$ assures that $y_{n,k} \leq C_0 \sqrt{|I_n|}$ for all $n < n_0$.

Now for the inductive step, assume that $y_{n,k-1} \leq C_0 \sqrt{|I_n|}$ for all $n$. Pick $n$ such that (9) holds (i.e., $n \geq n_0 + 1$), and write $y_{n,k}$ for the measure of the set $x$ such that $\alpha(G^{k-1}x) = n' \text{ and } \alpha(G^kx) = n$.

Then by equations (8), (9) and induction,

$$y_{n,k} = \sum_{n' < n} y_{n,k-1} + \sum_{n_0 \leq n' < n} y_{n,k-1} + y_{n,k-1} + \sum_{n' > n} y_{n,k-1}$$

$$\leq B \frac{|I_n|}{|I_{n_0}|} + \sum_{n' < n} C_0 B \frac{|I_n|}{|I_{n'}|} \sqrt{|I_{n'}|} + \frac{C_0}{6} \sqrt{|I_n|} + \sum_{n' > n} C_0 \sqrt{|I_{n'}|}$$

$$\leq C_0 \sqrt{|I_n|} \left( \frac{1}{6} + \sum_{n' < n} B(\sqrt{\epsilon})^{n-n'} + \frac{1}{6} + \sum_{n' > n} (\sqrt{\epsilon})^{n'-n} \right)$$

$$< (\frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3})C_0 \sqrt{|I_n|} = C_0 \sqrt{|I_n|}.$$

If an acip $\nu$ exists, then it can be written as $\nu(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{Leb}(G^{-i}A)$. 20
Therefore,

\[ \nu(I_n) \leq C_0 \sqrt{|I_n|}. \tag{10} \]

Take \( \eta \in (0, 1) \). Fix \( n_1 \) such that \( \sum_{n \geq n_1} y_{n,k} < \eta/2 \) for all \( k \geq 0 \). We need to show that we can choose \( \delta > 0 \) so that if \( A \subset I_0 \) is a set of measure \( \text{Leb}(A) < \delta \), then \( \text{Leb}(G^{-k}(A)) < \eta \) for all \( k \geq 0 \). By the choice of \( n_1 \), it suffices to show that \( \text{Leb}(G^{-k}(A)) < \eta/2, k \geq 0 \), for any \( A \subset I_0 \setminus I_{n_1} \).

Assume that \( A \subset I_n \setminus I_{n+1} \) for some \( n < n_1 \). Proposition 2 shows that any onto branch \( G^k : J \to I_n \) can be written as \( \psi \circ \varphi \) with

\[ \text{Dist}(\psi) \leq \log C_2 \text{ and } \varphi \in \mathcal{F}_{\mathbb{R}^1}^{m-n+1}, \]

where

\[ m = \alpha(G^i J) > \alpha(G^{i+1} J) > \cdots > \alpha(G^{k-1} J) > n \]

for some \( i < k \). Clearly \( i \geq k - m + n - 1 \). For such a branch we have

\[ \text{Leb}(G^{-k} A \cap J) \leq C_2 B \left( \frac{|A|}{|I_n|} \right)^{1/\delta^{m-n+1}} |J|. \]

For fixed \( m \), the total measure of the set of points arriving to \( I_n \) in this fashion is bounded by \( \sum_{i=k-m+n-1}^{k-1} y_{m,i} \leq (m - n + 1) \cdot C_0 \cdot \sqrt{|I_m|} \). Summing over all branches \( J \) (including the ones that do have extensions and hence distortion bounded by \( C_1 \)), and all \( m \geq n \), we find

\[ \text{Leb}(G^{-k} A) \leq C_1 \frac{|A|}{|I_n|} + \sum_{m \geq n} (m - n + 1) C_0 \sqrt{|I_m|} C_2 B \left( \frac{|A|}{|I_n|} \right)^{1/\delta^{m-n+1}}. \]

Thus \( \text{Leb}(G^{-k} A) \leq \eta/2n_1 \) for any integer \( k \) and any \( A \subset I_n \setminus I_{n+1} \), \( n < n_1 \), with \( |A| \leq \delta \), provided \( \delta \) is sufficiently small. It follows that if \( A \subset I_0 \setminus I_{n_1} \) has sufficiently small measure, then \( \text{Leb}(G^{-k} A) < n_1 \eta/(2n_1) = \eta/2 \). This concludes the verification of Straube’s condition. \( \square \)

9 Summability

We finish by proving Theorem 3. This theorem follows immediately from the next lemma.
Lemma 8. The partial sum \( \sum_{J_j \subseteq I_{n+1} \setminus I_{n+2}} s_j \nu(J_j) \) is exponentially small in \( n \).

Proof of Lemma 8. Let \( I_{n+1} = K^0 \supseteq \cdots \supseteq K^{k_n} = I_{n+2} \) be as in Section 3, and let \( m \geq 0 \) be minimal such that \( K^m \supseteq K^{m+1} \). Let us first comment on the induce times \( s_j \). If \( J_j \subseteq K^i \setminus K^{i+1} \), then \( G|J_j \) corresponds to at most \( (i + 2) \) iterates of \( G_n \), and thus \( s_j \leq (i + 2)t_0(k_0 + 1) \cdots (k_{n-1} + 1) \) according to (4). For \( J_j \subseteq K^m \setminus K^{m+1} \), we need a better estimate than (4). Note that if \( m \geq 2 \), then \( G_n^J(J_j) \) is contained in one of the outermost branches in \( I_0 \) for all \( 2 \leq p \leq m - 1 \), where iterates of \( G \) corresponds to \( f^2 \), and thus we have in this case that

\[
s_j \leq 2t_0(k_0 + 1) \cdots (k_{n-1} + 1) + 2m. \tag{11}\]

By Lemma 6, we have

\[
\sqrt{|I_{n+1}|} (k_0 + 1)(k_1 + 1) \cdots (k_{n-1} + 1) \leq C|I_{n+1}|^{1/4} \leq C \varepsilon^{n/4}. \tag{12}\]

A direct computation shows that \( m\lambda^{m/\ell^m} / R^n \leq C = C(\lambda) \) for \( \lambda \in (0, 1) \) and all \( m, n \geq 0 \), provided \( R > \ell \). So, by Lemma 5, we have

\[
m|I_{n+1}|^{1/2} \leq C m \lambda^{m/2\ell^m} |I_n|^{1/2} \leq C(\lambda)^n |I_n|^{1/2} \leq C(\varepsilon \ell^2)^n. \tag{13}\]

**Sum over outermost branches:** Note that if \( J_i \) is an outermost branch in \( I_{n+1} \), then \( s_i \leq 2t_0(k_0 + 1) \cdots (k_{n-1} + 1) + 2m \) by (11). Using also the obvious estimate \( \nu(J_i) \leq \nu(I_{n+1}) \leq C_0 \sqrt{|I_{n+1}|} \), we obtain

\[
s_i \nu(J_i) \leq 4C_0 (t_0(k_0 + 1) \cdots (k_{n-1} + 1) + m) \sqrt{|I_{n+1}|} \\
\leq C(\varepsilon^{n/4} + (\varepsilon \ell^2)^n)\]

according to (12) and (13). Since there are only two outermost branches, the term over these branches is exponentially small in \( n \) (provided that \( \varepsilon \) is sufficiently small).

**Sum over all other branches:** Note that if \( A \) is a subset of a component of \( I_{n+1} \setminus I_{n+2} \), and the distance \( d(A, \partial I_{n+1}) \geq \delta \cdot \text{diam}(A) \), then the Koebe distortion lemma gives that for every \( i \geq 0 \) and every onto branch \( G^i: J \to I_{n+1} \), we have

\[
\frac{\text{Leb}(G^{-i}(A) \cap J)}{|J|} \leq K(\delta) \frac{\text{Leb}(A)}{|I_{n+1}|},
\]

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where $K(\delta) = 2(1 + \delta)^2/\delta^2$. Hence

$$\operatorname{Leb}(G^{-i}A) \leq \sum_{G^iJ = I_{n+1}} K(\delta) \frac{\operatorname{Leb}(A)}{|I_{n+1}|} |J|$$

$$+ \sum_{G^iJ \supset I_{n+1}} K(1/\varepsilon) \frac{\operatorname{Leb}(A)}{|I_{n+1}|} \operatorname{Leb}(G^{-i}(I_{n+1}) \cap J),$$

so that $\operatorname{Leb}(G^{-i}A)/\operatorname{Leb}(G^{-i}I_{n+1}) \leq K(\delta)\operatorname{Leb}(A)/|I_{n+1}|$. In particular, this implies that

$$\nu(A) \leq K(\delta)\nu(I_{n+1}) \frac{\operatorname{Leb}(A)}{|I_{n+1}|}.$$  

By Proposition 4, the length of each of the outermost branches is as least $\rho^n_1/C_3$, and thus for any other branch $J_j \subset I_{n+1} \setminus I_{n+2}$,

$$d(J_j, \partial I_{n+1}) \geq \rho^n_1 |J_j|/C_3,$$

which implies

$$\frac{\nu(J_j)}{|J_j|} \leq \frac{C \nu(I_{n+1})}{\rho^n_1 |I_{n+1}|} \leq \frac{C C_0}{\rho^n_1 \sqrt{|I_{n+1}|}}.$$  

Therefore the sum of $s_j \nu(J_j)$ over all branches other than the outermost ones is bounded from above by the following

$$\frac{C}{\rho^n_1 \sqrt{|I_{n+1}|}} \sum_{J_j} s_j |J_j| = \frac{C}{\rho^n_1 \sqrt{|I_{n+1}|}} \left( \sum_{J_j \subset K \setminus K^m} + \sum_{J_j \subset K^m \setminus I_{n+2}} \right) s_j |J_j|$$

(note that $K^m = I_{n+1}$). Let us first estimate the first part of this sum. Using (11), Corollary 3, Lemma 5 and (13), we obtain

$$\frac{1}{\rho^n_1 \sqrt{|I_{n+1}|}} \sum_{J_j \subset K \setminus K^m} s_j |J_j| \leq$$

$$\leq \frac{2}{\rho^n_1} (t_0(k_0 + 1) \cdots (k_{n-1} + 1) + m) \frac{|K \setminus K^m|}{\sqrt{|I_{n+1}|}}$$

$$\leq \frac{2}{\rho^n_1} (t_0(k_0 + 1) \cdots (k_{n-1} + 1) + m) \sqrt{|I_{n+1}|}$$

$$\leq \frac{2}{\rho^n_1} (t_0(k_0 + 1) \cdots (k_{n-1} + 1)) C_4 \varepsilon_{k_0+\cdots+k_{n-1}} C_6 \lambda^{m/n+1} +$$

$$+ 2C \left( \frac{\sqrt{\varepsilon} \rho_1^n}{\rho_1^n} \right)^n,$$
is exponentially small provided that $\varepsilon$ is sufficiently small.

For each domain $J_j \subset K^i \setminus K^{i+1}$ with $i \geq m + 1$, we have

$$s_j \leq (i + 2) t_0 (k_0 + 1) \cdots (k_{n-1} + 1) \leq C (i + 2) \left( \frac{1}{|I_{i+1}|} \right)^{1/4}.$$

Therefore,

$$\frac{1}{\sqrt{|I_{n+1}|}} \sum_{J_j \subset K^m \setminus I_{n+2}} s_j |J_j| = \frac{1}{\sqrt{|I_{n+1}|}} \sum_{i=m+1}^{k_{n-1}} \sum_{J_j \subset K^i \setminus K^{i+1}} s_j |J_j| \leq \frac{C}{|I_{n+1}|^{3/4}} \sum_{i=m+1}^{k_{n-1}} (i + 2) |K^i|$$

$$= C \cdot |I_{n+1}|^{1/4} \sum_{i=m+1}^{k_{n-1}} (i + 2) \frac{|K^i|}{|I_{n+1}|}.$$

By Lemma 4 (applied repeatedly),

$$\frac{|K^i|}{|I_{n+1}|} = \frac{|K^i|}{|K^m|} \leq \left(1 - \frac{\rho_n^2}{C_5}\right)^{i-m},$$

which implies

$$\sum_{i=m+1}^{k_{n-1}} (i + 2) \frac{|K^i|}{|I_{n+1}|} \leq \sum_{i=m}^{\infty} (i + 2) \left(1 - \frac{\rho_n^2}{C_5}\right)^{i-m} \leq (m + 2) \frac{C_5}{\rho_n^2} + \left(\frac{C_5}{\rho_n^2}\right)^2 \leq 2 C (m + 2) \frac{1}{\rho_n^2}.$$

Thus, using again Lemma 5,

$$\frac{1}{\sqrt{|I_{n+1}|}} \sum_{J_j \subset K^m \setminus I_{n+2}} s_j |J_j| \leq C |I_{n+1}|^{1/4} (m + 2) \frac{1}{\rho_n^2} \leq C \left(\frac{\varepsilon^{1/4}}{\rho_n^2}\right)^n C_6 (m + 2) \lambda^m |I_{n+1}|^{1/4},$$

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and so
\[
\frac{1}{\rho_1^n} \frac{1}{\sqrt{|I_{n+1}|}} \sum_{j \in K^{m+1} \setminus I_{n+2}} s_j |J_j| \leq C(m + 2)\lambda^{m/m+1} \left( \frac{\varepsilon}{\rho_1 \rho_2} \right)^{n/4},
\]
which is again exponentially small in \( n \) provided that \( \varepsilon \) is sufficiently small. This completes the proof. \( \square \)

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