Bifurcation from periodic solutions with spatiotemporal symmetry, including resonances and mode interactions

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Abstract

We study local bifurcation in equivariant dynamical systems from periodic solutions with a mixture of spatial and spatiotemporal symmetries.

In previous work, we focused primarily on codimension one bifurcations. In this paper, we show that the techniques used in the codimension one analysis can be extended to understand also higher codimension bifurcations, including resonant bifurcations and mode interactions. In particular, we present a general reduction scheme by which we relate bifurcations from periodic solutions to bifurcations from fixed points of twisted equivariant diffeomorphisms, which in turn are linked via normal form theory to bifurcations from equilibria of equivariant vector fields.

We also obtain a general theory for bifurcation from relative periodic solutions and we show how to incorporate time-reversal symmetries into our framework.

Key words: Equivariant bifurcation theory, Periodic and relative periodic solutions, Spatiotemporal symmetry, Mode interactions
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1 Introduction

Equivariant bifurcation theory is concerned, to a large extent, with local bifurcation theory for vector fields that are equivariant with respect to the action of a compact Lie group Γ, see Golubitsky, Stewart and Schaeffer [16]. In particular, a systematic approach to bifurcation from equilibria is laid out in [16]. This approach has been generalised to include bifurcation from relative equilibria (where a single group orbit is flow invariant), see Krupa [17], and also situations where Γ is noncompact but acts properly on a finite dimensional manifold, see Fiedler, Sandstede, Scheel and Wulff [12].

Recently, the corresponding theory for bifurcation from periodic solutions has been in development. The simplest case is when the periodic solution has only *spatial* symmetries (symmetries that fix the periodic solution pointwise in phase space). In that case the bifurcation analysis reduces to bifurcation from a fixed point for an equivariant diffeomorphism and has been studied by Chossat and Golubitsky [8] (see also Ruelle [28]).

The more complicated case in which a periodic solution has not only spatial but also *spatiotemporal* symmetries has been studied by Fiedler [11] (cyclic spatiotemporal symmetry), Buono [6] (abelian spatiotemporal symmetry), and Lamb and Melbourne [21] (general spatiotemporal symmetry). Further examples and applications involving bifurcation from periodic solutions with spatiotemporal symmetry can be found in [5,27].

In particular, in [21] we studied codimension one bifurcation from a periodic solution. This reduces to bifurcation from a fixed point for a twisted equivariant diffeomorphism, which reduces in turn to bifurcation from a fixed point for a diffeomorphism that is equivariant (through arbitrarily high order) with respect to an enlarged compact symmetry group. (See also [22] for a less technical discussion.) The novel ingredient in this reduction scheme is the systematic treatment of twisted equivariant diffeomorphisms.

In the present paper, we improve the approach of [21] in two ways. First, we consider general bifurcations of arbitrary codimension (including resonances and mode-interactions). Second, following in spirit the approach of Takens [31] and Lamb [18,19], instead of reducing to bifurcation from a fixed point for an equivariant diffeomorphism, we reduce to bifurcation from an equilibrium for an equivariant vector field (again, through arbitrarily high order).

In Wulff, Lamb and Melbourne [33], we considered bifurcation from *relative periodic solutions*. These are flow-invariant sets that reduce to periodic solutions at the Γ orbit space level. (Here, Γ is possibly noncompact but is assumed to act properly on a finite dimensional manifold.) It is shown in [33] how to reduce the problem to bifurcation from a periodic solution for an equivariant vector field with a compact symmetry group. This reduces in turn, by the theory outlined above, to bifurcation from an equilibrium for an equivariant vector field with an enlarged compact symmetry group. Hence, our present results when combined with [33] yield a systematic theory for bifurcation from relative periodic solutions.

Summarising, we have the following hierarchy of reductions:

$$\begin{array}{c}\text{relative periodic solution}\\ \downarrow\\ \text{periodic solution}\\ \downarrow\\ \text{fixed point for twisted equivariant diffeomorphism}\\ \downarrow\\ \text{equilibrium for equivariant vector field}\end{array}$$

We note that whereas the first two reductions involve no loss of information, the final step is only valid through arbitrarily high order. Using finite determinacy results of Field [14], we obtain persistence for certain solutions, such as (relative) periodic solutions and invariant tori.

This paper is organised as follows. In Section 2, we summarise the main results of our paper concerning bifurcation from periodic solutions with spatiotemporal symmetry. Illustrative examples are given in Section 3. Section 4 contains results about linear twisted equivariant maps, and Section 5 establishes a normal form theorem for nonlinear twisted equivariant diffeomorphisms. The results in Section 2 are proved in Section 6.

In Section 7, we extend our approach to bifurcations from periodic solutions for reversible equivariant vector fields (where in addition to equivariance the vector field possesses time-reversal symmetry).

Finally, the generalisation to bifurcations from relative periodic solutions is described in Section 8.

2 Statement of the main results

Let Γ be a compact Lie group acting orthogonally on $\mathbb{R}^n$. We consider the dynamics for a Γ-equivariant flow on $\mathbb{R}^n$. Suppose that $P = \{x(t), 0 \le t \le T\}$ is a periodic solution of minimal period T with initial condition $x_0 = x(0)$. The symmetries that leave P invariant come in two forms. There is the group of *spatial symmetries*

$$\Delta = \{\gamma \in \Gamma : \gamma x_0 = x_0\},$$

which is by definition the isotropy subgroup of x_0. It follows from equivariance that Δ is the isotropy subgroup of each point $x(t) \in P$. There is also the group of *spatiotemporal symmetries*

$$\Sigma = \{\gamma \in \Gamma : \gamma P = P\}.$$

For each $\sigma \in \Sigma$, there is a unique *time-shift* $T_\sigma \in [0, T)$ such that $\sigma x(t) = x(t + T_\sigma)$ for all t. It is easily verified that Δ is a normal subgroup of Σ and that either $\Sigma/\Delta \cong S^1$ or $\Sigma/\Delta \cong \mathbb{Z}_m$ for some $m \geq 1$.

When $\Sigma/\Delta \cong S^1$, the periodic solution is a *rotating wave* which is a special case of a relative equilibrium. Hence, we focus on the case $\Sigma/\Delta \cong \mathbb{Z}_m$, where P is called a *discrete rotating wave* [11]. (The case $\Sigma = \Delta$ was studied in [8] so our interest lies primarily in the case $m > 1$.) We assume that $\dim \Gamma = \dim \Sigma$. Then without loss we may suppose that $\Gamma = \Sigma$. (The case $\dim \Gamma > \dim \Sigma$ is addressed in [33], and again in Section 8.)

Now choose $\sigma \in \Sigma$ such that Σ is generated by Δ and σ. This induces an automorphism $\phi \in \mathrm{Aut}(\Delta)$ given by

$$\phi(\delta) = \sigma^{-1}\delta\sigma. \tag{2.1}$$

The element σ can be chosen [21, Lemma 2.1] so that the automorphism ϕ has finite order k, for some $k \geq 1$ [2]. Following [21], for each $N \geq 1$ we form the semidirect product $\Delta \rtimes \mathbb{Z}_N$ by adjoining to the group Δ an element τ satisfying the relations

$$\tau^N = 1, \quad \tau^{-1}\delta\tau = \phi(\delta). \tag{2.2}$$

2.1 Codimension one bifurcation

Choose k as above and form the semidirect product $\Delta \rtimes \mathbb{Z}_{2k}$. The main results in Lamb and Melbourne [21] can be encapsulated as follows:

Theorem 2.1 ([21] Codimension one bifurcation) *Codimension one bifurcations from a discrete rotating wave with spatial symmetry Δ and spatiotemporal symmetry Σ are in one-to-one "correspondence" with codimension one bifurcations from a fully symmetric equilibrium of a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field.*

[2] The results described below are valid for any choice of k. In practice, it is natural to choose σ so that k is as small as possible. There is no relationship between k and $m = |\Sigma/\Delta|$ except for the obvious one (when $\Sigma = \Delta$) that $k = 1$ if $m = 1$.

Remark 2.2 *(a) We write "correspondence" in quotes since the proof of the theorem relies on normal from theory for twisted equivariant maps [18], and aspects of the bifurcations that are beyond all orders are not necessarily preserved by the correspondence. However, a finite determinacy result of Field [14] ensures that many important features of the bifurcations are determined at finite order and hence are preserved by the correspondence. These features include the existence, symmetry and stability of branches of periodic solutions, and in the case of Hopf bifurcation branches of invariant tori.*

A precise description of the correspondence in Theorem 2.1 is deferred until Subsection 2.3. This result is a special case of Theorem 2.5 which is stated below and proved in Section 6.

(b) Theorem 2.1 is a reformulation of results in [21]. In particular, nonHopf and Hopf bifurcation in [21] correspond to steady state and Hopf bifurcation of $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector fields. (In the case of Hopf bifurcation, there is a further simplification whereby it is possible to reduce to Hopf bifurcation of a $\Delta \rtimes \mathbb{Z}_k$-equivariant vector field (see also Remark 2.4(b)). This fact is useful in doing computations, but for the sake of simplicity in the statement of our results, we have chosen to retain the factor 2.*)*

(c) The analysis of codimension one steady state and Hopf bifurcation for $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector fields involves the computation of irreducible representations for $\Delta \rtimes \mathbb{Z}_{2k}$. In [21], it is shown how to obtain these representations from the representations of Δ using induced representation theory.

2.2 Resonance

The generalisation of Theorem 2.1 to higher codimension bifurcations is complicated by the possibility of *resonances.*

First, we recall some key concepts from [21]. Suppose that $\Sigma/\Delta = \mathbb{Z}_m$ and that Σ is generated by Δ together with an element $\sigma \in \Sigma$. Suppose also that σ induces an automorphism $\phi \in \mathrm{Aut}(\Delta)$ of finite order k as discussed earlier. Let X be a Δ-invariant local cross-section to the discrete rotating wave P such that $P \cap X = \{x_0\}$. Then $\{\sigma^j X : \; j = 0, \ldots, m-1\}$ is a sequence of Δ-invariant local cross-sections spaced at time intervals of $1/m$'th of the period of P. Define $G^{(j)} : X \to \sigma^j X$ to be the first hit map for the flow and define the *first hit pullback* map $f : X \to X$ by $f = \sigma^{-1} G^{(1)}$. Observe that

(a) x_0 is a fixed point for f.
(b) f is *twisted equivariant*. That is,

$$f(\delta x) = \phi(\delta) f(x) \quad \text{for all}\, x \in X \,\text{and}\, \delta \in \Delta. \tag{2.3}$$

Here, $\phi \in \mathrm{Aut}(\Delta)$ is the automorphism $\phi(\delta) = \sigma^{-1}\delta\sigma$ introduced in equation (2.1).

(c) $G^{(j)} = \sigma^j f^j$ for $j \geq 1$.

(d) The Poincaré map $G : X \to X$ for the periodic solution P is given by $G = G^{(m)} = \sigma^m f^m$, has x_0 as a fixed point, and is Δ-equivariant.

We say that an eigenvalue μ of a matrix A is *rational* if $\mu = \exp(2\pi ic/d)$ where $c, d \in \mathbb{N}$. Otherwise μ is *irrational*.

Definition 2.3 *Let $L = (df)_0$ be the linearisation of the first hit pullback map f arising in a bifurcation problem for a discrete rotating wave.*

(a) We say that the bifurcation problem is nonresonant *if L^k has no rational eigenvalues other than ± 1. Otherwise the bifurcation problem is* resonant*. (Similarly, we speak of the twisted equivariant linear map L being* nonresonant *and* resonant*.)*

(b) Let $\mu_j = \exp(2\pi ic_j/d_j)$ be the rational eigenvalues of L^{2k} and suppose that c_j and d_j are in their lowest terms for each j. Define $\ell = \mathrm{lcm}\{d_j\}$ to be the least common multiple of the denominators d_j (if there are no rational eigenvalues, set $\ell = 1$). Then the bifurcation problem (and the twisted equivariant linear map L) is resonant of order ℓ.

Note that the nonresonant bifurcation problems are precisely those with order $\ell = 1$.

Remark 2.4 *(a) In systems without symmetry, resonances arise in Hopf bifurcation from periodic solutions for flows (or Hopf bifurcation from fixed points for diffeomorphisms), see Ruelle [28]. This phenomenon is identical to the issue that we are addressing in Definition 2.3, but the details are not identical. In particular, our order of resonance ℓ need not match up with the 'standard' order of resonance in nonequivariant systems, and some cases that one might normally think of as being resonant are swallowed up in the integer $2k$. (See the example in Subsection 3.3.)*

(b) The factor of 2 in the integer $2k\ell$ is sometimes unnecessary. In fact, when (the centre subspace component of) L^k is Δ-equivariantly isotopic to the identity (which is certainly the case if L^k has no eigenvalues at -1) it is possible to redefine ℓ in terms of the eigenvalues of L^k (instead of L^{2k}) and to replace $\Delta \rtimes \mathbb{Z}_{2k\ell}$ by $\Delta \rtimes \mathbb{Z}_{k\ell}$. We have chosen not to take this approach in developing the general theory, since the statement of our results would become considerably more complicated. However, in applying the theory, for example when studying Hopf bifurcation as in Section 3(b,c), it is often more convenient to remove the factor of 2 when possible.

(c) Our definition of nonresonance permits resonances between irrational eigenvalues of L^k (or L^{2k}) that lie on the unit circle. (For example, it is permissible to have eigenvalues $e^{i\omega}$, $e^{2i\omega}$ where $\omega \notin \pi\mathbb{Q}$.)

(d) Preliminary versions of these results were announced in [23]. Although our approach has not changed, some of the details have been improved. In particular, the definition of the order of resonance ℓ has been altered.

2.3 Higher codimension bifurcation

The main result in the paper is the following theorem, where we relate bifurcations from a discrete rotating wave to bifurcations from an equilibrium for a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field.

Theorem 2.5 *(a) Nonresonant bifurcations from a discrete rotating wave with spatial symmetry Δ and spatiotemporal symmetry Σ are in one-to-one "correspondence" with bifurcations from a fully symmetric equilibrium of a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field.*

(b) In the case of resonance of order ℓ, bifurcations from a discrete rotating wave "reduce" to bifurcations from a fully symmetric equilibrium of a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field.

The proof of this result can be found in Section 6.

The precise meaning (and usage) of this theorem requires some explanation. Let $\mathbb{R}^n$ be a representation of the compact Lie group $\Delta \rtimes \mathbb{Z}_{2k\ell}$ (in practice, this representation is the cross section X to the periodic orbit). In particular, the element τ (which was introduced in (2.2) and is a generator of $\mathbb{Z}_{2k\ell}$) acts on $\mathbb{R}^n$ and can be viewed as a linear map $\tau : \mathbb{R}^n \to \mathbb{R}^n$. (In fact, the conditions in (2.2) imply that τ^{-1} is a linear twisted equivariant map. The relationship between τ^{-1} and L will become clear in Sections 5 and 6.)

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field with an equilibrium at 0. After centre manifold reduction, we may suppose without loss that the spectrum of $(dh)_0$ lies entirely on the imaginary axis. Provided $n \geq 1$, there is a local bifurcation from 0 for the vector field h and the analysis proceeds as in standard equivariant bifurcation theory [16]. Theorem 2.5 states that bifurcations from the periodic solution for the underlying flow can be read off from the (to a large extent known) bifurcations from 0 for h. In standard equivariant bifurcation theory, it is necessary to take into account of the representation theory of the symmetry group, and similarly here we must take account of the representation theory of the enlarged symmetry group $\Delta \rtimes \mathbb{Z}_{2k\ell}$. In particular, it is necessary to take account not only the action of Δ but also the action of τ. Just as in the standard theory, all representations are considered up to isomorphism. However, when $\ell > 1$, there are some redundancies. In fact, it is only necessary to consider those representations in which τ^{2k} acts as an element of order precisely ℓ. There are additional redundancies when $\ell > 1$

which we do not make precise here. Nevertheless, it is certainly the case that any resonant bifurcation of order ℓ reduces to a bifurcation for h in which τ^{2k} acts as an element of order precisely ℓ.

There is still the question of how to translate the bifurcation results for h into bifurcation results for the underlying flow. In Section 5, it is shown that the twisted equivariant first hit pullback map f can be transformed by Δ-equivariant near-identity polynomial changes of coordinates into the form

$$f \sim \tau^{-1} \exp h \tag{2.4}$$

where h is a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field (and $\exp h$ is the time-one map). Here, $\sim$ denotes equality through arbitrarily high, but finite, order. In particular, h commutes with τ so that the dynamics of f can be recovered modulo flat terms (cf. Remark 2.2(a)). Again, the results of Field [14] guarantee that many important features are finitely determined and hence preserved by the correspondence.

Finally, we refer to Subsection 2.4 for details relating to the interpretation of bifurcating equilibria and periodic solutions for h in terms of the underlying flow.

Remark 2.6 *Resonant bifurcations have codimension at least two, so codimension one bifurcations have $\ell = 1$. Hence, Theorem 2.1 is a special case of Theorem 2.5(a), which is in turn a special case of Theorem 2.5(b).*

Remark 2.7 *Resonant bifurcations in the nonequivariant context ($\Delta = \mathbf{1}$) lead to $\mathbb{Z}_\ell$-equivariant vector fields. See for example Arnold [2] or Arrowsmith and Place [3] for discussions of the use of equivariant vector fields in the study of resonant bifurcations.*

There is the usual distinction between strong *and* weak *resonances [2,3]. Roughly speaking, the resonance corresponding to an eigenvalue $\mu = e^{2\pi ic/d}$ is strong if d is small and weak if d is large. In general, this distinction depends on the number of nonresonant and resonant eigenvalues (taking into account algebraic and geometric multiplicities) and also depends on the desired completeness of the analysis of the dynamics.*

The simplest solutions that occur in Hopf bifurcation are branches of invariant two-tori. It follows from Field [14] (building upon work of Ruelle [28]) that in most cases the branching and stability of two-tori are the same for bifurcations with resonance as for nonresonant bifurcations — only resonances of a specified low order are "strong" in this context. In particular, if one focuses on such aspects of the theory, it often suffices to consider bifurcation of $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector fields.

More delicate dynamics such as phaselocking on the invariant tori is influenced by resonances of all orders.

2.4 Existence and symmetry of bifurcating solutions

Let h denote the $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field in Theorem 2.5 (and Theorem 2.1). In this subsection, we describe the precise correspondence between certain solutions of h and the corresponding solutions for the underlying Γ-equivariant flow.

Branches of periodic solutions for the underlying flow

The fully symmetric equilibrium for the $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field h in Theorem 2.5 corresponds to the original periodic solution P for the underlying flow.

Similarly, bifurcating equilibria $x^{\rm bif}$ for the vector field h correspond to bifurcating periodic solutions $P^{\rm bif}$ for the underlying flow. As in [21], we define $J \subset \Delta \rtimes \mathbb{Z}_{2k\ell}$ to be the isotropy subgroup of $x^{\rm bif}$:

$$J = \{\gamma \in \Delta \rtimes \mathbb{Z}_{2k\ell} : \gamma x^{\rm bif} = x^{\rm bif}\}.$$

Note that elements of J can be written in the form $\tau^j \delta$ where $j = 0, 1, \ldots, 2k\ell - 1$ and $\delta \in \Delta$.

Proposition 2.8 *Suppose that $x^{\rm bif}$ is a bifurcating equilibrium with isotropy $J \subset \Delta \rtimes \mathbb{Z}_{2k\ell}$ for the vector field h. Let $p \geq 1$ be least such that $\tau^p \delta_0 \in J$ for some $\delta_0 \in \Delta$ and define $\sigma^{\rm bif} = \sigma^p \delta_0$.*

Then there is a bifurcating periodic solution $P^{\rm bif}$ for the underlying flow, and $P^{\rm bif}$ has spatial symmetry

$$\Delta^{\rm bif} = J \cap \Delta = \{\delta \in \Delta : \delta x^{\rm bif} = x^{\rm bif}\}.$$

and spatiotemporal symmetry $\Sigma^{\rm bif}$ generated by $\Delta^{\rm bif}$ and $\sigma^{\rm bif}$.

Proof. We make use of the relation (2.4). By Remark 2.6, detailing Field's determinacy result, we may suppose throughout that $f = \tau^{-1} g = \tau^{-1} \exp h$ where h is $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant and g is the time-one map of h. In particular, h commutes with τ.

Since $x^{\rm bif}$ is a point of intersection of $P^{\rm bif}$ with the Poincaré cross-section, it is immediate that $\Delta^{\rm bif} = J \cap \Delta$.

Next, we note that the equilibrium $x^{\rm bif}$ for h corresponds to a fixed point $x^{\rm bif}$ for the time-one map g and hence a periodic point for f. Indeed, $f^j(x^{\rm bif}) = \tau^{-j}x^{\rm bif}$.

The spatiotemporal symmetry group $\Sigma^{\rm bif}$ is a cyclic extension of $\Delta^{\rm bif}$, and so we search for the least $p \geq 1$ such that the first hit map $G^{(p)} : X \to \sigma^p X$ satisfies $G^{(p)}(x^{\rm bif}) = \sigma^{\rm bif}x^{\rm bif}$ where $\sigma^{\rm bif} \in \Sigma$. It then follows that $\Sigma^{\rm bif}$ is generated by $\Delta^{\rm bif}$ and $\sigma^{\rm bif}$.

By definition, $\sigma^{\rm bif}$ maps X into $\sigma^p X$ and hence $\sigma^{\rm bif} \in \sigma^p\Delta$. That is, $\sigma^{\rm bif} = \sigma^p\delta_0$ for some $\delta_0 \in \Delta$. In addition,

$$G^{(j)}(x^{\rm bif}) = \sigma^j f^j(x^{\rm bif}) = \sigma^j\tau^{-j}x^{\rm bif}.$$

Hence $G^{(p)}(x^{\rm bif}) = \sigma^p\delta_0 x^{\rm bif}$ if and only if $\tau^p\delta_0 x^{\rm bif} = x^{\rm bif}$. □

Remark 2.9 *We have the following correspondence between the symmetry J of the equilibrium $x^{\rm bif}$ and the spatiotemporal symmetry $\Sigma^{\rm bif}$ of the periodic solution $P^{\rm bif}$:*

$$\tau^j\delta \in J \subset \Delta \rtimes \mathbb{Z}_{2k\ell} \longleftrightarrow \sigma^j\delta \in \Sigma^{\rm bif}.$$

Absolute and relative periods of periodic solutions

A periodic trajectory $x(t)$ has *(absolute) period* T if $T > 0$ is least such that $x(T) = x(0)$. In a dynamical system with symmetry group Γ, the periodic trajectory has *relative period* T if $T > 0$ is least such that $x(T) \in \Gamma x(0)$. Write $T_{\rm abs}$ and $T_{\rm rel}$ to denote these periods. Note that $T_{\rm abs}$ is an integer multiple of $T_{\rm rel}$. Moreover, $T_{\rm abs}/T_{\rm rel} = |\Sigma/\Delta|$.

In our set up, the underlying periodic solution P has spatiotemporal symmetry $\Sigma = \Gamma$ and we defined the positive integer $m = |\Sigma/\Delta|$. So

$$m = |\Sigma/\Delta| = T_{\rm abs}/T_{\rm rel}. \tag{2.5}$$

Let $T^{\rm bif}_{\rm abs}$ and $T^{\rm bif}_{\rm rel}$ denote the relative and absolute periods of the bifurcating periodic solution $P^{\rm bif}$. Corresponding to m, we define the positive integer $m^{\rm bif}$ to be

$$m^{\rm bif} = |\Sigma^{\rm bif}/\Delta^{\rm bif}| = T^{\rm bif}_{\rm abs}/T^{\rm bif}_{\rm rel}. \tag{2.6}$$

Proposition 2.10 *As the bifurcation point is approached,*

$$T^{\rm bif}_{\rm rel}/T_{\rm rel} \to p, \qquad T^{\rm bif}_{\rm abs}/T_{\rm abs} \to q,$$

where p is as in Proposition 2.8 and $q = pm^{\rm bif}/m$.

Proof. It is clear that $T_{\rm abs}^{\rm bif}$ is approximately an integer multiple of $T_{\rm abs}$ and similarly for the relative periods. Moreover, it follows from the proof of Proposition 2.8 that $T_{\rm rel}^{\rm bif}/T_{\rm rel}$ converges to the integer p. The expression for q follows from (2.5) and (2.6). $\square$

The integer q corresponds to the occurrence of a period q-tupling bifurcation. In certain simple situations (codimension one nonHopf bifurcations considered in [21, Proposition 4.5]) the only possibilities are $q = 1$ (period-preserving bifurcation) and $q = 2$ (period-doubling bifurcation). This corresponds to the case where the $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant normal form vector field h undergoes a codimension one steady-state bifurcation — $(dh)_0$ has no eigenvalues on the imaginary axis except at 0, and the kernel of $(dh)_0$ is absolutely irreducible under the action of $\Delta \rtimes \mathbb{Z}_{2k}$.

Branches of invariant 2-tori for the underlying flow

Let C be a bifurcating periodic solution for the normal form vector field h. Then C corresponds to an invariant 2-torus T for the underlying flow. Associated to the 2-torus, we define the symmetry groups

$$\Delta_T \subset \Sigma_T \subset \Gamma,$$

where Δ_T fixes the individual points in T and Σ_T fixes T as a set. These can be read off from the corresponding symmetry groups for C as follows (cf. [21]).

Define $J \subset J_C \subset \Delta \rtimes \mathbb{Z}_{2k\ell}$ to be the spatial and spatiotemporal symmetry groups of the periodic solution. Then $\Delta_T = J \cap \Delta$. Define $p \geq 1$ to be least such that $\tau^p \delta_0 \in J_C$ for some $\delta_0 \in \Delta$. Then Σ_T is generated by Δ_C and $\sigma^p \delta_0$, where $\Delta_C = J_C \cap \Delta$. (For an interpretation of the subgroup $\Delta_C = J_C \cap \Delta$, we refer to [21].)

The periodic solution C for the normal form vector field h either has discrete spatiotemporal symmetry ($\dim J_C = \dim J$) or is a rotating wave ($\dim J_C = \dim J + 1$). It follows that the invariant torus T for the underlying flow either has discrete spatiotemporal symmetry ($\dim \Sigma_T = \dim \Delta_T$) or is a modulated rotating wave ($\dim \Sigma_T = \dim \Delta_T + 1$).

When there is phase-locking to a periodic solution $P^{\rm bif} \subset T$, then $\Delta^{\rm bif} = \Delta_T$ and $\Sigma^{\rm bif}$ can be computed as in Proposition 2.8 and satisfies $\Delta_T \subset \Sigma^{\rm bif} \subset \Sigma_T$.

3 Examples

To illustrate our results, we discuss some examples of local bifurcation from a periodic solution $x(t)$ in a system with $\mathbb{D}_4$ symmetry. By scaling time, we may suppose that $x(t)$ has minimal period one. Suppose that $x(t)$ has spatiotemporal symmetry $\Sigma = \mathbb{D}_4 = \langle \kappa, R_{\pi/2} \rangle$ and spatial symmetry $\Delta = \mathbb{D}_2 = \langle \kappa, R_\pi \rangle$, where R_φ denotes rotation by angle φ. We can write $\Sigma = \langle \Delta, \sigma \rangle$, where $\sigma = R_{\pi/2}\kappa$. The periodic solution $x(t)$ is pointwise invariant under Δ, $\delta x(t) = x(t)$ for all t and $\delta \in \Delta$, and σ is a spatiotemporal symmetry, $\sigma x(t) = x(t + 1/2)$. From the group structure it follows that $k = 2$. This example is one of the simplest in which $k > 1$ arises.

In Subsections 3.1 and 3.2, we recall how our approach applies to the analysis of generic codimension one *nonHopf* bifurcation (that reduce to generic codimension one steady-state bifurcations of the normal form vector field) and generic codimension one nonresonant Hopf bifurcation (cf. Lamb and Melbourne [21]). In Subsections 3.3 and 3.4, we show that our approach extends to codimension two resonant Hopf bifurcation and nonHopf/nonHopf mode interactions.

3.1 Codimension one nonHopf bifurcation

NonHopf bifurcations correspond to steady-state bifurcations of a $\mathbb{D}_2 \rtimes \mathbb{Z}_4$-equivariant vector field. Generically, the group $\mathbb{D}_2 \rtimes \mathbb{Z}_4$ acts absolutely irreducibly on the centre subspace of the linearised vector field [16]. For the steady-state bifurcation problem we may invoke the equivariant branching lemma, asserting the existence of branches of equilibria with axial isotropy subgroups within $\mathbb{D}_2 \rtimes \mathbb{Z}_4$ (isotropy subgroups whose fixed point subspaces are one-dimensional). These equilibria correspond to periodic solutions for the underlying flow, and the symmetry properties of these periodic solutions can be computed as in Section 2(d).

We concentrate on the case where $\mathbb{D}_2 \rtimes \mathbb{Z}_4$ acts as the standard two-dimensional absolutely irreducible representation of $\mathbb{D}_4$. In particular, we suppose that the action of $\Delta = \mathbb{D}_2$ is given by

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

There are two possibilities for the action of the element τ generating $\mathbb{Z}_4$:

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Corresponding to these two choices, the kernel of the action is given by $\mathbb{Z}_2(\tau^2)$ and $\mathbb{Z}_2(\tau^2 R_\pi)$ respectively.

The results corresponding to the two choices of τ are summarised in Tables 1 and 2 respectively. We verify the entries of Table 1 where $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For the standard action of $\mathbb{D}_4$, generated by the fourfold rotation $R_{\frac{\pi}{2}}$ and reflection κ, the axial isotropy subgroups are $A = \mathbb{D}_1(\kappa)$ and $A = \mathbb{D}_1(R_{\frac{\pi}{2}}\kappa)$. Corresponding to $A = \mathbb{D}_1(\kappa)$ we have the isotropy subgroup $J \subset \mathbb{D}_2 \rtimes \mathbb{Z}_4$ generated by κ and τ^2 (including the kernel). Corresponding to $A = \mathbb{D}_1(R_{\frac{\pi}{2}}\kappa)$ we have the subgroup J generated by τ (since τ is the element of $\mathbb{D}_2 \rtimes \mathbb{Z}_4$ playing the role of $R_{\frac{\pi}{2}}\kappa \in \mathbb{D}_4$) and τ^2 (the kernel). The remaining entries of Table 1 are now easily read off as in Section 2(d).

Table 1
Symmetry types of bifurcating solutions in nonHopf bifurcation from a periodic solution with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$. The case where $\tau^2 = I$

A	J	Δ^{bif}	Σ^{bif}	σ^{bif}	m	m^{bif}	p	q
$\mathbb{D}_1(\kappa)$	$\mathbb{D}_2(\kappa, \tau^2)$	$\mathbb{D}_1(\kappa)$	$\mathbb{D}_1(\kappa)$	1	2	1	2	1
$\mathbb{D}_1(R_{\frac{\pi}{2}}\kappa)$	$\mathbb{Z}_4(\tau)$	1	$\mathbb{D}_1(R_{\frac{\pi}{2}}\kappa)$	$R_{\frac{\pi}{2}}\kappa$	2	2	1	1

Table 2
Symmetry types of bifurcating solutions in nonHopf bifurcation from a periodic solution with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$. The case where $\tau^2 = -I$

A	J	Δ^{bif}	Σ^{bif}	σ^{bif}	m	m^{bif}	p	q
$\mathbb{D}_1(\kappa)$	$\mathbb{D}_2(\kappa, \tau^2 R_\pi)$	$\mathbb{D}_1(\kappa)$	$\mathbb{D}_2(\kappa, R_\pi)$	R_π	2	2	2	2
$\mathbb{D}_1(R_{\frac{\pi}{2}}\kappa)$	$\mathbb{D}_2(\tau\kappa, \tau^2 R_\pi)$	1	$\mathbb{Z}_4(R_{\frac{\pi}{2}})$	$R_{\frac{\pi}{2}}$	2	4	1	2

Along the same lines, codimension one Hopf bifurcation of the periodic solution reduces to codimension one Hopf bifurcation of a symmetric equilibrium of a $\mathbb{D}_2 \rtimes \mathbb{Z}_4$ equivariant vector field. (Recall that $\ell = 1$ for all codimension one bifurcations.) By Remark 2.4(b), we can remove a factor of 2 and reduce to a $\mathbb{D}_2 \rtimes \mathbb{Z}_2$-equivariant vector field. Note that $\mathbb{D}_2 \rtimes \mathbb{Z}_2$ is isomorphic to $\mathbb{D}_4$. We deduce the existence of branches of invariant tori as in Section 2(d).

The centre subspace consists of the direct sum of two isomorphic absolutely irreducible representations of $\mathbb{D}_2 \rtimes \mathbb{Z}_2 \cong \mathbb{D}_4$ [16]. We consider the standard representation of $\mathbb{D}_4$, so $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The analysis of Hopf bifurcation leads to further phase shift normal form symmetry so that h can be assumed to be $(\mathbb{D}_2 \rtimes \mathbb{Z}_2) \times S^1$-equivariant through arbitrarily high order. The equivariant Hopf theorem [16] guarantees the existence of branches of periodic solutions with $\mathbb{C}$-axial isotropy subgroups of $(\mathbb{D}_2 \rtimes \mathbb{Z}_2) \times S^1$ (isotropy subgroups whose fixed point subspaces are two-dimensional). It is well known [16] that the $\mathbb{C}$-axial subgroups (A) of the $\mathbb{D}_4 \times S^1$ action on the centre manifold are $\mathbb{D}_1((\kappa, 0)) \times \mathbb{Z}_2((R_\pi, \frac{1}{2}))$, $\mathbb{D}_1((R_{\frac{\pi}{2}}\kappa, 0)) \times \mathbb{Z}_2((R_\pi, \frac{1}{2}))$, and $\mathbb{Z}_4((R_{\frac{\pi}{2}}, \frac{1}{4}))$. From these subgroups we obtain J and J_C, the pointwise and setwise symmetry groups of the periodic solution in the normal form vector field. From there, we proceed as in Section 2(d) to we obtain the pointwise and setwise symmetries Δ_T and Σ_T, of the bifurcating tori. [3]

Swift [30] has carried out a more detailed analysis of Hopf bifurcation with $\mathbb{D}_4$ symmetry, in particular establishing the existence and asymptotic stability of primary branches of periodic solutions and invariant two-tori with minimal isotropy type (trivial spatial symmetry and $\mathbb{Z}_2(R_\pi)$ spatiotemporal symmetry). These results carry over directly to our situation. In particular, Swift's solutions correspond to (asymptotically stable) branches of invariant 2-tori and invariant 3-tori with spatial symmetry $\Delta_T = 1$ and setwise symmetry $\Sigma_T = \mathbb{Z}_2(R_\pi)$ as shown in Table 3. The actual dynamics on the invariant 3-tori may be complicated [25] and are beyond the scope of this paper.

[3] In [21] it was erroneously claimed that in case $A = \mathbb{Z}_4((R_{\frac{\pi}{2}}, \frac{1}{4}))$ we have $\Sigma_T = \mathbb{D}_4$. This has been corrected here.

Table 3
Symmetry types of bifurcating solutions in Hopf bifurcation from a periodic solution with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$

A	J	J_C	Δ_T	Σ_T
$\mathbb{D}_1((\kappa, 0)) \times \mathbb{Z}_2((R_\pi, \frac{1}{2}))$	$\langle\kappa\rangle$	$\langle\kappa, R_\pi\rangle$	$\mathbb{D}_1(\kappa)$	$\mathbb{D}_2(\kappa, R_\pi)$
$\mathbb{D}_1((R_{\frac{\pi}{2}}\kappa, 0)) \times \mathbb{Z}_2((R_\pi, \frac{1}{2}))$	$\langle\tau\rangle$	$\langle\tau, R_\pi\rangle$	1	$\mathbb{D}_2(R_{\frac{\pi}{2}}\kappa, R_\pi)$
$\mathbb{Z}_4((R_{\frac{\pi}{2}}, \frac{1}{4}))$	1	$\langle\tau\kappa\rangle$	1	$\mathbb{Z}_4(R_{\frac{\pi}{2}})$
$\mathbb{Z}_2((R_\pi, \frac{1}{2}))$	1	$\langle R_\pi\rangle$	1	$\mathbb{Z}_2(R_\pi)$

3.3 Codimension two resonant Hopf bifurcation

We consider the same situation as in Subsection 3.2, but now include the possibility of resonance. This means that $\tau^k = \tau^2$ acts as an element of order ℓ and the normal form vector field h is $\mathbb{D}_2 \rtimes \mathbb{Z}_{2\ell}$-equivariant. As described in Remark 2.7, the analysis for the nonresonant case still applies provided ℓ is large enough – so that h is S^1-equivariant through sufficiently high order. This situation is called *weak resonance.*

In particular, it can be shown that if $\ell \geq 5$ or if $\ell = 3$, then the results on existence and stability of the flow-invariant 2-tori and 3-tori computed in Subsection 3.2 for nonresonant Hopf bifurcation are retained by the resonant Hopf bifurcation. Hence, the only *strongly* resonant case is $\ell = 4$ (corresponding to the situation when τ^2 has eigenvalues $\pm i$).

To verify these values of ℓ, we note first that the dynamics computed by Swift [30] are 3-determined. Since the $\mathbb{D}_2 \rtimes \mathbb{Z}_2 \cong \mathbb{D}_4$-equivariance precludes even terms, it is sufficient to check that there is sufficient normal form symmetry to avoid cubic order resonance terms. In the context of $\mathbb{D}_4$-equivariant Hopf bifurcation it is straightforward to check that this means avoiding a resonance of order 4 (where the eigenvalues lie at $\pm i$).

The detailed analysis of resonant Hopf bifurcation (weak or strong resonance) leads to issues such as phase-locking, Arnold tongues and prevalence of irrational toral flows, which are outside the scope of this paper. Some of these issues are touched upon in Swift [30, Section 4] and the calculations there apply equally to our context.

3.4 Codimension two nonHopf/nonHopf mode interaction

NonHopf/nonHopf-mode interactions reduce to steady-state/steady-state mode interactions of a $\mathbb{D}_2 \rtimes \mathbb{Z}_4$-equivariant normal form vector field. Generically, the centre subspace is the direct sum of two (not necessarily isomorphic) absolutely irreducible representations. Again, we consider the case where each representation is effectively the standard two-dimensional representation of $\mathbb{D}_4$. It is convenient to use complex coordinates writing the centre subspace as $\mathbb{C}^2$ so that the action of $\mathbb{D}_2$ is given by

$$\kappa(z_1, z_2) = (\bar{z}_1, \bar{z}_2), \qquad R_\pi(z_1, z_2) = (-z_1, -z_2).$$

There are three cases corresponding to the possibilities

$$\tau(z_1, z_2) = (i\bar{z}_1, i\bar{z}_2), \qquad \tau(z_1, z_2) = (iz_1, iz_2), \qquad \tau(z_1, z_2) = (i\bar{z}_1, iz_2).$$

The first two possibilities for τ lead to an action of $\mathbb{D}_2 \rtimes \mathbb{Z}_4$ on $\mathbb{C}^2$ that is the direct sum of two isomorphic representations of the type in Subsection 3.1. After factoring out the two element kernel, we are left with the direct sum of two copies of the standard action of $\mathbb{D}_4$. Hence, modulo terms in the tail that break the normal form symmetry, the bifurcation analysis reduces to Takens-Bogdanov bifurcation with $\mathbb{D}_4$ symmetry. This bifurcation has been studied previously by Armbruster *et al.* [1] (see also [26] and references therein). The details of the bifurcation are rather complicated, and we content ourselves here with the observation that the isotropy subgroups are the same as in Subsection 3.1, but the fixed-point subspaces are two-dimensional. Although the equivariant branching lemma does not apply, branches of equilibria with these symmetry types exist and correspond to periodic solutions with the same spatiotemporal symmetry properties as in Subsection 3.1.

We now turn to the third possibility

$$\tau(z_1, z_2) = (i\bar{z}_1, iz_2).$$

There are four axial isotropy subgroups as shown in Table 4 (with fixed-point subspaces $\{(x_1, 0)\}$, $\{(e^{i\pi/4}x_1, 0)\}$, $\{(0, x_2)\}$, $\{(0, e^{i\pi/4}x_2)\}$ respectively, where $x_j \in \mathbb{R}$, $j = 1, 2$). The equivariant branching lemma guarantees existence of branches of solutions for these isotropy subgroups. There are also four submaximal isotropy subgroups with two-dimensional fixed-point subspaces. The remaining entries of Table 4 are computed as before. It turns out that the axial solutions are the union of the axial solutions in the individual representations (but it should be clear that this is not a general principle; this could not have been deduced without carrying out the computations).

Table 4
Symmetry types of bifurcating solutions in nonHopf/nonHopf bifurcation from a periodic solution with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$

J	$\Delta^{\rm bif}$	$\Sigma^{\rm bif}$	$\sigma^{\rm bif}$	m	$m^{\rm bif}$	p	q
$\langle \tau^2, \kappa \rangle$	$\mathbb{D}_1(\kappa)$	$\mathbb{D}_1(\kappa)$	1	2	1	2	1
$\langle \tau \rangle$	1	$\mathbb{D}_1(R_{\frac{\pi}{2}}\kappa)$	$R_{\frac{\pi}{2}}\kappa$	2	2	1	1
$\langle \tau^2 R_\pi, \kappa \rangle$	$\mathbb{D}_1(\kappa)$	$\mathbb{D}_2(\kappa, R_\pi)$	R_π	2	2	2	2
$\langle \tau^2 R_\pi, \tau\kappa \rangle$	1	$\mathbb{Z}_4(R_{\frac{\pi}{2}})$	$R_{\frac{\pi}{2}}$	2	4	1	2
$\langle \kappa \rangle$	$\mathbb{D}_1(\kappa)$	$\mathbb{D}_1(\kappa)$	1	2	1	4	2
$\langle \tau^2\kappa \rangle$	1	$\mathbb{D}_1(\kappa)$	κ	2	2	2	2
$\langle \tau^2 R_\pi \rangle$	1	$\mathbb{Z}_2(R_\pi)$	R_π	2	2	2	2
$\langle \tau^2 \rangle$	1	1	1	2	1	2	1

4 Linear twisted equivariant maps

We present some elementary, but crucial, results about twisted equivariant linear maps, which rely on ideas in Lamb [18, Lemma 2.2] and Wulff *et al.* [33, Lemma 2.2]. Our main result is the following decomposition theorem.

Theorem 4.1 *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a twisted equivariant linear map with eigenvalues on the unit circle. Suppose that L is resonant of order ℓ as in Definition 2.3. Then*

$$L = L_0 e^{B+N}$$

where the linear maps L_0, B and N have the following properties:

(a) L_0, B and N all commute with each other,
(b) L_0 is twisted equivariant, and B, N are equivariant,
(c) L_0, B are semisimple, and N is nilpotent,
(d) $L_0^{2k\ell} = I$; moreover L_0^{2k} has order precisely ℓ,
(e) B has no eigenvalues in $\pi i\mathbb{Q} - \{0\}$.

Corollary 4.2 *Let $S = L_0 e^B$. Then*

$$\overline{\langle S \rangle} \cong K \times \mathbb{Z}_{2k\ell},$$

where $\mathbb{Z}_{2k\ell}$ is generated by L_0 and K is the torus $K = \overline{\{e^{tB} : t \in \mathbb{R}\}}$.

Proof. It is standard that $\overline{\langle e^B \rangle} = K$ thanks to Theorem 4.1(e). Hence it follows from Theorem 4.1(a,d) that L_0 and e^B generate the group $K \times \mathbb{Z}_{2k\ell}$. Moreover, it is immediate that $S = L_0 e^B$ lies in this group and hence that $\overline{\langle S \rangle} \subset K \times \mathbb{Z}_{2k\ell}$.

To prove the reverse inclusion, it suffices to verify that $e^B \subset \overline{\langle S \rangle}$. But, $e^{2k\ell B} = (L_0 e^B)^{2k\ell} \subset \overline{\langle S \rangle}$, and again by Theorem 4.1(e), $\overline{\langle e^{2k\ell B} \rangle} = K$, so it follows that $e^B \subset K \subset \overline{\langle S \rangle}$. $\square$

In the remainder of this section, we prove Theorem 4.1.

Proposition 4.3 ([18]) *Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a nonsingular linear map. Then*

(a) L can be written uniquely in the form $L = Se^N$ where S is semisimple, N is nilpotent, and $SN = NS$.
(b) L is equivariant if and only if S and N are equivariant.
(c) L is twisted equivariant if and only if S is twisted equivariant and N is equivariant.

Proof. The Jordan-Chevalley decomposition $L = Se^N$ in (a) is standard. Moreover, one direction in each of (b) and (c) is trivial. We prove the nontrivial directions in (b) and (c).

Suppose L is Δ-equivariant, and let $\delta \in \Delta$. Then

$$L = \delta^{-1} L \delta = \delta^{-1} S \delta \, e^{\delta^{-1} N \delta}.$$

Conjugation by δ is a similarity transformation, and hence $\delta^{-1} S \delta$ is semisimple, $\delta^{-1} N \delta$ is nilpotent, and these two linear maps commute. It follows from the uniqueness of the decomposition $L = Se^N$ that $\delta^{-1} S \delta = S$ and $\delta^{-1} N \delta = N$. Since $\delta \in \Delta$ is arbitrary, S and N are equivariant matrices, proving (b).

Finally, suppose that L is twisted equivariant. Note that $L^k = S^k e^{kN}$ where S^k is semisimple, kN is nilpotent and S^k commutes with kN. Since L^k is equivariant, it follows from part (b) that S^k and kN are equivariant. In particular, N is equivariant. Hence e^N is equivariant, and $S = Le^{-N}$ is twisted equivariant, proving (c). $\square$

Proof of Theorem 4.1. By Proposition 4.3, we have the decomposition $L = Se^N = e^N S$. In particular, S is semisimple and twisted equivariant. Let μ denote an eigenvalue of S^k with eigenspace E_μ. Since the matrices L, S and e^N commute with S^k, they restrict to linear maps on E_μ. Moreover, it follows from equivariance of S^k that Δ acts on E_μ. Hence, we may reduce without loss to E_μ.

We divide into the cases where μ is rational or irrational on the unit circle. If μ is rational (so $\mu^2 = \exp(2\pi ic/d)$ where d divides ℓ) then we simply take $L_0 = S$ and $B = 0$.

If μ is irrational, choose ν so that $\nu^k = \mu$ and take $L_0 = (1/\nu)S$, $e^B = \nu I$. (In fact, $B = i\theta I$ where $e^{i\theta} = \nu$.) $\square$

5 Normal forms for twisted equivariant maps

Theorem 4.1 gives a decomposition of nonsingular twisted equivariant linear maps. This decomposition property forms the basis of the following nonlinear normal form theorem which extends a classical result of Takens [31] and its twisted equivariant version [18].

We restrict attention to twisted equivariant diffeomorphisms f satisfying $f(0) = 0$, whose eigenvalues of $(df)_0$ all lie on the complex unit circle. The order of resonance ℓ of $(df)_0$ is defined as in Definition 2.3.

Theorem 5.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Δ twisted equivariant diffeomorphism satisfying $f(0) = 0$ and set $L = (df)_0$. Suppose that all eigenvalues of L lie on the complex unit circle and write $L = L_0 e^{B+N}$ as in Theorem 4.1 (so in particular $L_0^{2k\ell} = I$).*

Then there exists a Δ-equivariant coordinate transformation T such that the Taylor expansions of

$$TfT^{-1} \quad \text{and} \quad L_0 g$$

agree through arbitrarily high order where g is the time-one map of a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant autonomous vector field $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $h(0) = 0$ and $(dh)_0 = B + N$. Here, $\mathbb{Z}_{2k\ell}$ is generated by $\tau = L_0^{-1}$.

Lemma 5.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Δ twisted equivariant diffeomorphism satisfying $f(0) = 0$ and set $L = (df)_0$. Suppose that all eigenvalues of L lie on the complex unit circle and write $L = Se^N$ as in Proposition 4.3.*

Then there exists a Δ-equivariant coordinate transformation T such that the Taylor expansions of

$$TfT^{-1} \quad \text{and} \quad Sg'$$

agree through arbitrarily high order where g' is the time-one map of a Δ-equivariant and S-equivariant autonomous vector field h' in $\mathbb{R}^n$ satisfying $h'(0) = 0$ and $(dh')_0 = N$.

Proof. We follow Takens [31], who proved this result in the case $\Delta = \mathbf{1}$.

Given a smooth map f, we let f_m denote the Taylor expansion of f up to order m. Then, we can write (modulo terms of order $> m$)

$$f_m = Lg_m = L(\exp \hat{h})_m = L\Big(\exp(\hat{h}_m)\Big)_m$$

where $\hat{h}_m(0) = 0$ and $(d\hat{h}_m)_0 = I$. Here $\hat{h}_m$ is the unique [31] polynomial vector field whose time-one map agrees with g through order m. It follows from uniqueness of $\hat{h}_m$, twisted equivariance of f and L, and naturality of exp, that $\hat{h}_m$ is Δ-equivariant.

Takens [31] showed how to choose T so that $\hat{h}_m$ is transformed into an S-equivariant vector field. (So $Tf_mT^{-1} = L(\exp(\hat{h}_m))_m$ where $\hat{h}_m$ commutes with S.) It was pointed out in Lamb [18] that T could be chosen to be Δ-equivariant. Then $\hat{h}_m$ is Δ-equivariant as well as S-equivariant. For completeness, we provide here the details of this argument.

We suppose inductively that $\hat{h}_{m-1}$ is Δ- and S-equivariant and derive the same result for $\hat{h}_m$. The result is trivially true when $m = 1$, so we suppose $m \geq 2$. Following [31], we consider near-identity coordinate transformations of the form $T = \exp P$ where P is a homogeneous polynomial vector field P of degree m. Ignoring terms of order higher than m, we obtain

$$Le^{\hat{h}} \mapsto e^P Le^{\hat{h}} e^{-P} = L(L^{-1}e^P L)e^{\hat{h}}e^{-P} = L\exp\{L^{-1}PL + \hat{h} - P\}.$$

Hence,

$$\hat{h}_m \;\mapsto\; \hat{h}_m + \mathrm{Ad}_{L^{-1}}(P) - P, \tag{5.1}$$

where $\mathrm{Ad}_{L^{-1}}(P) = L^{-1}PL$. Since P is homogeneous of degree m, this transformation does not affect $\hat{h}_{m-1}$.

It is well known [31], that the complement of the image of $(\mathrm{Ad}_{L^{-1}} - I)$ in the vector space $\mathcal{P}_m$ of homogeneous polynomial vector fields of degree m can be chosen to be S-equivariant. Hence, we can choose $P \in \mathcal{P}_m$ so that $\hat{h}_m$ is S-equivariant.

Next we average the right-hand-side of (5.1) over the compact Lie group Δ to obtain

$$\hat{h}_m + \mathrm{Ad}_{L^{-1}}(Q) - Q, \tag{5.2}$$

where Q is the polynomial P averaged over Δ. In particular, Q (and hence T) is now a Δ-equivariant change of coordinates. Moreover, since S is twisted equivariant, it follows that S-equivariance of (5.2) is preserved in the averaging process. Hence the transformed and averaged vector field $\hat{h}_m$ is both Δ- and S-equivariant.

To arrive at the normal form proposed by Takens in [31], we merge N with the vector field expansion, writing (modulo higher order terms)

$$\exp(N)\exp(\hat{h}_m) = \exp(h'),$$

where h' is Δ- and S-equivariant (because N and $\hat{h}_m$ are), and $(dh')_0 = (d\hat{h}_m)_0 + N = N$. $\square$

Remark 5.3 *Having established the normal form $\tau^{-1}\exp h$ for f, one can now apply standard procedures for transforming h itself further into normal form. For instance, h can be normalised also with respect to the nilpotent part of the linear part of $(dh)_0$, as in [10].*

Proof of Theorem 5.1. By Lemma 5.2, we can transform f through arbitrarily high order into the normal form Sg' where $g' = \exp h'$. By Theorem 4.1, $S = L_0 e^B$ and so $Sg' = L_0 e^B e^{h'}$. Since h' is S-equivariant, it follows from Corollary 4.2 that h' commutes with e^{tB} for all $t \in \mathbb{R}$ and hence h' commutes with B. In particular, we can write $e^B e^{h'} = e^{B+h'}$. This means that we can replace the normal form Sg' by the normal form $L_0 g$ where g is the time-one map of the vector field $h = B + h'$. By Theorem 4.1 and Lemma 5.2, B and h' are Δ- and S-equivariant, so h is Δ- and S-equivariant. Finally, $\mathbb{Z}_{2k\ell}$-equivariance follows from Corollary 4.2. $\square$

Remark 5.4 *Let K be the torus $K = T^j = \overline{\{e^{tB} : t \in \mathbb{R}\}}$ defined in Corollary 4.2. The proof of Theorem 5.1 shows that the normal form vector field h is actually S-equivariant and hence $T^j \times \mathbb{Z}_{2k\ell}$-equivariant by Corollary 4.2. Altogether, h is $(\Delta \rtimes \mathbb{Z}_{2k\ell}) \times T^j$-equivariant.*

The T^j-equivariance is precisely the normal form symmetry expected for a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field h with linearisation $(dh)_0 = B + N$.

Remark 5.5 *As a consequence of merging B into the normal vector field h, the linear part of h has the same codimension within the class of $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant linear vector fields, as the linear part of f has in the class of linear twisted equivariant diffeomorphisms.*

6 Proof of the results in Section 2

Suppose that $f : X \to X$ is a Δ-twisted equivariant map with a fixed point (which we normalise so that $f(0) = 0$). Then the centre manifold of f is Δ-invariant. By centre manifold reduction, we may reduce by restricting f to the centre manifold in which case the eigenvalues of $(df)_0$ all lie on the complex unit circle. Locally, we may identify the centre manifold with $\mathbb{R}^n$, where n is the dimension of the centre manifold.

Recall that $L = (df)_0$ has the decomposition $L = L_0 e^{B+N}$ described in Theorem 4.1. Since L_0 is twisted equivariant,

$$L_0 \delta L_0^{-1} = \phi(\delta) = \tau^{-1} \delta \tau.$$

Moreover, by Theorem 4.1(iv), L_0^{2k} has order precisely ℓ. Hence, the linear map L_0 may be identified with τ^{-1} where τ is the generator of $\mathbb{Z}_{2k\ell}$ in Section 2. We rewrite the decomposition of L as

$$L = \tau^{-1} e^{B+N},$$

In the nonresonant case, we have $\ell = 1$ and hence $\tau^{2k} = I$. Theorem 5.1 states that through arbitrarily high order we can write

$$f = \tau^{-1} g = \tau^{-1} e^h,$$

where g is the time-one map of a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field h and $(dh)_0 = B + N$. Moreover, $(dh)_0$ has no eigenvalues in $\pi i \mathbb{Q} - \{0\}$ so that h is a nonresonant vector field.

Conversely, if h is a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field satisfying $(dh)_0 = B + N$, then f is a twisted equivariant vector field with linear part $L_0 e^{B+N}$. If h is nonresonant, then so is f. Hence, h is a general nonresonant $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field. Theorem 5.1 extends to diffeomorphisms with parameters and we obtain Theorem 2.5(a). (Actually, it follows from our construction that $(dh)_0$ has no eigenvalues on $i\pi\mathbb{Q}$ other than at 0, but this is not important for the bifurcations of h.)

Next, we consider the resonant case. Since Δ and τ generate $\Delta \rtimes \mathbb{Z}_{2k\ell}$ we obtain Theorem 2.5(b). As discussed in the paragraphs after Theorem 2.5(b), there are further restrictions on the $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant normal form vector field h. By Theorem 4.1(d), we may suppose that $\tau = L_0^{-1}$ has the property that τ^{2k} has order precisely ℓ. Because the resonant part of $(df)_0$ has been absorbed inside L_0, it follows that B has zero eigenvalues corresponding to the resonances. There are no other restrictions on the linear and nonlinear terms.

7 Reversible equivariant systems

In this section, we show how the normal form approach applies not only to equivariant systems, but also to reversible equivariant systems.

We say that the differential equation $\dot{x} = F(x)$ is Γ-*reversible equivariant* if there exists a group homomorphism

$$\chi : \Gamma \to \{\pm 1\},$$

so that whenever $x(t)$ is a solution, then so is $\gamma x(\chi(\gamma)t)$. In case Γ acts as a linear representation on $\mathbb{R}^n$, the vector field F is Γ-reversible equivariant if

$$F\gamma = \chi(\gamma)\gamma F, \quad \forall \gamma \in \Gamma.$$

The elements γ of Γ for which $\chi(\gamma) = -1$ are called *time-reversal* symmetries of f. A dynamical system possessing a time-reversal symmetry is usually called *reversible*. Note that the notions of reversibility for vector fields and diffeomorphisms are different: a diffeomorphism f is called Γ-reversible equivariant if $\gamma f^{\chi(\gamma)} = f\gamma$, $\gamma \in \Gamma$.

Now suppose we have a periodic orbit $x(t)$ that is setwise invariant with respect to a group $\Sigma \subset \Gamma$. Let $\Sigma^+ = \{\gamma \in \Sigma \mid \chi(\gamma) = +1\}$ and let $\Delta \subset \Sigma$ be the subgroup of Σ that fixes $x(t)$ pointwise. Assume furthermore that the periodic orbit is reversible, so $\Sigma \neq \Sigma^+$, and is not a rotating wave (so not contained entirely in a Σ^+ group orbit). Then it is easily verified that Σ has the following structure:

$$\Delta \trianglelefteq \Sigma^+ \trianglelefteq \Sigma, \quad \Sigma/\Sigma^+ \cong \mathbb{Z}_2, \quad \Sigma/\Delta \cong \mathbb{D}_m, \quad \Sigma^+/\Delta \cong \mathbb{Z}_m.$$

As the equivariant context, we set up a cross-section X to the periodic orbit, invariant under Δ and at least one time-reversal symmetry ρ (this can be obtained without loss of generality). Let $\Sigma_X = \langle \Delta, \rho \rangle = \Delta \cup \rho\Delta$. The first hit pullback map $f = \sigma^{-1}G^{(1)} : X \to X$ is Σ_X *twisted reversible equivariant*:

$$f\delta = \phi(\delta)f, \quad \forall \delta \in \Delta$$

where $\phi : \Delta \to \Delta$ is defined as in (2.1), and

$$f\gamma = \phi(\gamma)f^{-1}, \quad \forall \gamma \in \rho\Delta$$

where $\phi : \rho\Delta \to \rho\Delta$ is defined by

$$\phi(\gamma) = \sigma^{-1}\gamma\sigma^{-1}.$$

Thus the automorphism $\phi : \Delta \to \Delta$ is extended to a map on Σ_X. In general, $\phi : \Sigma_X \to \Sigma_X$ is no longer an automorphism, and we refer to it as the *twist morphism*.

We may assume without loss that the order of σ is finite [21] and hence the twist morphism ϕ has finite order k. We now define the abstract group Ξ generated by Σ_X and an element τ such that

$$\tau^{-1}\delta\tau = \phi(\delta)\ \forall\delta \in \Delta, \quad \tau^{-1}\gamma\tau^{-1} = \phi(\gamma)\ \forall\gamma \in \rho\Delta, \text{ and } \tau^{2k\ell} = 1.$$

We note that τ and Δ generate the group $\Xi^+ = \Delta \rtimes \mathbb{Z}_{2k\ell}$. We have $\Delta \trianglelefteq \Xi^+ \trianglelefteq \Xi$, with $\Xi/\Delta \cong \mathbb{D}_{2k\ell}$ and $\Xi^+/\Delta \cong \mathbb{Z}_{2k\ell}$.

Given this structure, Theorem 5.1 extends naturally to the twisted reversible equivariant context. Define the order of resonance ℓ of $(df)_0$ as in Definition 2.3.

Theorem 7.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Σ_X twisted reversible equivariant diffeomorphism satisfying $f(0) = 0$ and set $L = (df)_0$. Suppose that all eigenvalues of L lie on the complex unit circle and write $L = L_0 e^{B+N}$ where $L_0^{2k\ell} = I$.*

Then there exists a Σ_X-equivariant coordinate transformation T such that the Taylor expansions of

$$TfT^{-1} \quad \text{and} \quad L_0 g$$

agree through arbitrarily high order where g is the time-one map of a Ξ-reversible equivariant autonomous vector field $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $h(0) = 0$ and $(dh)_0 = B + N$.

Proof. The strategy of the proof is essentially similar to that of the proof of Theorem 5.1. We provide a sketch.

First, there exists a decomposition property of the linear part analogous to the one given in Theorem 4.1:

$$L = (df)_0 = L_0 e^{B+N},$$

where L_0 is a Σ_X-twisted reversible equivariant map, $L_0^{2k\ell} = I$, B is semisimple and generates a torus T^j, and N is nilpotent. Moreover, B and N commute and are Ξ-reversible equivariant (as linear vector fields), where $\chi : \Xi \to \{\pm 1\}$ so that $\chi(\xi) = 1$ if and only if $\xi \in \Xi^+$. Here $\Xi = \langle \Xi^+, \rho \rangle \cong \langle L_0, \Sigma_X \rangle$, where $\Xi^+ = \Delta \rtimes \mathbb{Z}_{2k\ell} \cong \langle L_0, \Delta \rangle$.

As before, we write $f = Lg$, where $L = (df)_0$. Importantly [20], because L is Σ_X-twisted reversible equivariant, the diffeomorphism g is $\langle \Delta, \rho L \rangle$-reversible equivariant, where $\langle \Delta, \rho L \rangle$ is the group generated by Δ and ρL. That is,

$\delta g = g\delta$ for all $\delta \in \Delta$, and $\rho L g = g^{-1}\rho L$. Writing $g = \exp(\hat{h})$, it follows by uniqueness that the Taylor expansion of $\hat{h}$ is $\langle \Delta, \rho L\rangle$-reversible equivariant.

We now note that TfT^{-1} is Σ_X-twisted reversible equivariant provided that T is Σ_X-equivariant. Writing $T = \exp(P)$, $\hat{h}$ transforms according to (5.2). One readily verifies that $\mathrm{Ad}_{L^{-1}}(P) - P$ is indeed $\langle \Delta, \rho L\rangle$-reversible equivariant if P is Σ_X-equivariant.

Let $S = L_0 e^B$ denote the semisimple part of $(df)_0$. As before, the complement to $\mathrm{Im}(\mathrm{Ad}_{L^{-1}} - I)$ can be chosen to consist of S-equivariant homogeneous polynomial vector fields, and averaging over Σ_X shows that this complement can be chosen to consist of S-equivariant vector fields when the domain is restricted to Σ_X-equivariant homogeneous polynomial vector fields P.

Hence, after an appropriate coordinate transformation, the normal form vector field $\hat{h}$ can be made $\langle S, \Delta\rangle$-equivariant and ρL-reversible. Again we write - modulo higher order terms - $\exp(N)\exp(\hat{h}) = \exp(h')$ where h' is $\langle S, \Delta\rangle$-equivariant. By well-known composition properties of reversible maps (see for example [20]), as $\exp(N)$ and $\exp(\hat{h})$ are both $\overline{\langle \rho L, S, \Delta\rangle}$-reversible equivariant, $\exp(h')$ and h' are $\overline{\langle \rho L e^{-N}, S, \Delta\rangle}$-reversible equivariant, [4] where

$$\overline{\langle \rho L e^{-N}, S, \Delta\rangle} = \overline{\langle \rho, S, \Delta\rangle} = \langle \rho, L_0, \Delta\rangle \times T^j \cong \Xi \times T^j.$$

Finally, we merge e^B with $e^{h'}$. As B commutes with h', this yields e^h which is Σ_X-reversible equivariant, where $h = B + h'$ and $(dh)_0 = B + N$. $\square$

Branches of periodic solutions for the underlying flow

Bifurcating equilibria x^{bif} for the vector field h correspond to bifurcating periodic solutions P^{bif} for the underlying flow. As in Section 2(d), we define $J \subset \Xi$ to be the isotropy subgroup of x^{bif}:

$$J = \{\gamma \in \Xi : \gamma x^{\mathrm{bif}} = x^{\mathrm{bif}}\}.$$

Note that elements of J can be written in the form $\tau^j \rho^\varepsilon \delta$ where $j = 0, 1, \dots, 2k\ell - 1$, $\varepsilon \in \{0, 1\}$ and $\delta \in \Delta$.

Proposition 7.2 *Suppose that x^{bif} is a bifurcating equilibrium with isotropy $J \subset \Xi$ for the vector field h. Let $p \geq 1$ be least such that $\tau^p \delta_0 \in J$ for some $\delta_0 \in \Delta$ and define $\sigma^{\mathrm{bif}} = \sigma^p \delta_0$.*

[4] Note the different notions of reversibility for vector fields and diffeomorphisms: a vector field F is R-reversible if $RF = -FR$ whereas a diffeomorphism f is called R-reversible if $Rf = f^{-1}R$.

Then there is a bifurcating periodic solution P^{bif} for the underlying flow, and P^{bif} has spatial symmetry

$$\Delta^{\mathrm{bif}} = J \cap \Delta = \{\delta \in \Delta : \delta x^{\mathrm{bif}} = x^{\mathrm{bif}}\}.$$

If $\tau^j \rho\delta \in J$ for some j and $\delta \in \Delta$, then let $\rho^{\mathrm{bif}} = \tau^j \rho\delta$ (not unique), and P^{bif} has spatiotemporal symmetry Σ^{bif} generated by Δ^{bif} and σ^{bif} and ρ^{bif}.

Otherwise P^{bif} has spatiotemporal symmetry Σ^{bif} generated by Δ^{bif} and σ^{bif}.

Remark 7.3 *We have the following correspondence between the symmetry J of the equilibrium x^{bif} and the spatiotemporal symmetry Σ^{bif} of the periodic solution P^{bif}:*

$$\tau^j \rho^\varepsilon \delta \in J \subset \Xi \longleftrightarrow \sigma^j \rho^\varepsilon \delta \in \Sigma^{\mathrm{bif}}.$$

We adopt the definition of the relative period T_{rel} of a periodic solution as the smallest $T > 0$ so that $x(T) = \Sigma^+ x(0)$. Then we have $m = |\Sigma^+/\Delta| = T_{\mathrm{abs}}/T_{\mathrm{rel}}$, and m^{bif} defined analogously. As the bifurcation point is approached,

$$T^{\mathrm{bif}}_{\mathrm{abs}}/T_{\mathrm{abs}} \to p, \qquad T^{\mathrm{bif}}_{\mathrm{rel}}/T_{\mathrm{rel}} \to q,$$

where p is as in Proposition 7.2 and $q = pm^{\mathrm{bif}}/m$. Again the integer q corresponds to the occurrence of a period q-tupling bifurcation.

The above normal form result may be used to study nonHopf bifurcation from isolated periodic solutions, reducing to steady-state bifurcation of a reversible equivariant normal form vector field. A detailed discussion of steady-state bifurcation in reversible equivariant vector fields can be found in Buono *et al.* [7]. As an illustration of the general procedure, we discuss how codimension one bifurcation reduces to codimension one reversible equivariant (and in fact, equivariant) steady-state bifurcations. We consider two elementary cases: $\mathbb{Z}_2$-reversible and $\mathbb{D}_2$-reversible equivariant systems. Note that they strongly resemble the well-studied subject of subharmonic branching of (nonisolated) periodic solutions in $\mathbb{Z}_2$-reversible systems, cf. [29,32]. See also Ciocci and Vanderbauwhede [9], for a related study of subharmonic bifurcation of $\mathbb{Z}_2$-reversible diffeomorphisms.

Example 1 (Bifurcation from $\mathbb{Z}_2$-symmetric periodic solutions) *We consider a $\mathbb{Z}_2(R)$-reversible vector field in $\mathbb{R}^{2n+1}$, where the fixed point subspace of R has dimension n. We consider the case that this vector field has an R-reversible symmetric periodic solution $x(t)$, satisfying $x(t) = x(t+1)$ and $Rx(t) = x(-t)$. It is not difficult to verify that such solutions generically arise persistently and are isolated [29]. The return map f for this periodic solution* *may be chosen to be a diffeomorphism of a $2n$-dimensional section X, satisfying $R(X) = X$. Then f is R-reversible: $fR = Rf^{-1}$ (where we consider, naturally, the action of R restricted to X). Moreover, f has an R-invariant fixed point in X, corresponding to the periodic solution, which we choose to have coordinates* 0.

By reversibility, after restriction to the centre manifold, the eigenvalues of $(df)_0$ arise in pairs $\{\lambda, \lambda^{-1}\}$ with λ on the unit circle in $\mathbb{C}$, and isolated pairs of such eigenvalues (if not equal to ± 1) are forced to move on the unit circle if parameters are varied. For simplicity, we assume that the centre manifold is two-dimensional and hence admits one pair of eigenvalues on the complex unit circle.

We now consider a bifurcation point, at which $L_0^{2\ell} = 1$ for some ℓ. It is clear that this is a codimension one phenomenon. By Theorem 7.1, the corresponding normal form vector field is $\mathbb{D}_{2\ell}$-reversible equivariant.

In particular, if $\ell = 1$, the representation of $\mathbb{D}_2$ is the direct sum of two nonisomorphic absolutely irreducible representations, either acting faithfully or acting unfaithfully as $\mathbb{D}_1$. The latter situation corresponds to the case when $\lambda = 1$, with R acting as $\mathrm{diag}(1, -1)$ *and τ acting trivially. The former situation refers to the case when $\lambda = -1$, with R acting in the same way, but now with τ acting as $-I$.*

If $\ell \geq 2$, we need to consider the absolutely irreducible representations of $\mathbb{D}_{2\ell}$ on X that act faithfully, or unfaithfully as $\mathbb{D}_\ell$: all other representation reduce to cases covered in the analysis with lower values of ℓ Namely, $\mathbb{D}_{2\ell}$ acts faithfully if and only if $\lambda = e^{2\pi i/2\ell}$ and it acts like $\mathbb{D}_\ell$ if $\lambda = e^{2\pi i/\ell}$.

Conveniently, the $\mathbb{D}_{2\ell}$-reversible equivariant normal form vector field h is related to a $\mathbb{D}_{2\ell}$-equivariant vector field g [7]:

$$h = Ag, \quad \textit{where } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, since A is invertible, the equilibrium solutions of the reversible equivariant vector field h coincide exactly with those of the equivariant vector field g. This means that we can infer the steady-state bifurcation theory directly from that of the equivariant vector field g.

We first consider the case $\ell > 1$. Let the action of $\mathbb{D}_{2\ell}$ on the centre manifold be generated by the reflection R and rotation $R_{\pi/\ell}$ or $R_{2\pi/\ell}$, the latter depending on the representation of $\mathbb{D}_{2\ell}$. In order to avoid duplication of results, we note that in case of the representation with generator $R_{2\pi/\ell}$, it suffices to consider the case that ℓ is odd. Namely, the corresponding case with ℓ even is in fact a *resonance of order $\ell/2$.*

As usual, by the equivariant branching lemma we have branches of equilibria with axial isotropy $\mathbb{D}_1(R)$ and $\mathbb{D}_1(RR_{\pi/\ell})$ or $\mathbb{D}_1(RR_{2\pi/\ell})$. To obtain J, we identify $R_{\pi/\ell} \to \tau$ or $R_{2\pi/\ell} \to \tau$. Finally, in order to obtain $\Sigma^{\rm bif}$, we identify $\tau \to I$. The results are summarised in Table 5. Note that the branches of

Table 5
Symmetry types of bifurcating solutions in codimension one nonHopf (subharmonic) bifurcation from a reversible periodic solution with spatiotemporal symmetry $\Sigma = \mathbb{Z}_2(R)$, spatial symmetry $\Delta = 1$, and normal form symmetry $\mathbb{D}_{2\ell}(R,\tau) = \mathbb{Z}_{2\ell}(\tau) \rtimes \mathbb{Z}_2(R)$ with $\ell \geq 2$.

ℓ	τ^ℓ	J	$\Delta^{\rm bif}$	$\Sigma^{\rm bif}$	$\sigma^{\rm bif}$	$\rho^{\rm bif}$	m	$m^{\rm bif}$	p	q
odd/even	-1	$\mathbb{D}_1(R)$	1	$\mathbb{D}_1(R)$	1	R	1	1	2ℓ	2ℓ
odd/even	-1	$\mathbb{D}_1(R\tau)$	1	$\mathbb{D}_1(R)$	1	R	1	1	2ℓ	2ℓ
odd	$+1$	$\mathbb{D}_1(R,\tau^\ell)$	1	$\mathbb{D}_1(R)$	1	R	1	1	ℓ	ℓ
odd	$+1$	$\mathbb{D}_1(R\tau,\tau^\ell)$	1	$\mathbb{D}_1(R)$	1	R	1	1	ℓ	ℓ

new solutions correspond to periodic solutions with approximately q-times the period of the reference solution. Such bifurcations are usually referred to as q-tupling when $q \geq 3$. Note that in contrast to periodic solutions in equivariant vector fields [21], reversible periodic solutions can display generic codimension one q-tupling, with $q \geq 3$.

If $\ell = 1$, we have to consider two cases: $\mathbb{D}_2$ acting as $\mathbb{Z}_2$ ($\tau = +1$), or $\mathbb{D}_2$ acting faithfully ($\tau = -1$). In both cases, X is the direct sum of two nonisomorphic irreducible representations. In case $\tau = +1$, one of the bifurcations that arises involves a turning point bifurcation in which the periodic solution collides with another one (with the same symmetry properties) after which both cease to exist. The results are summarised in Table 6. Note that of the solutions listed for $\tau = +1$ and $\tau = -1$, only one of the two branches appears at a codimension one bifurcation. It depends on the nilpotent part of $(df)_0$ which branch arises. Namely, in this case, $(df)_0$ is not semi-simple and the symmetry property of the proper eigenvector determines which case arises.

Example 2 (Bifurcation from $\mathbb{D}_2$-symmetric periodic solutions) *We consider a $\mathbb{D}_2(R,S)$-reversible equivariant vector field in $\mathbb{R}^{2n+1}$, with R and RS having n-dimensional fixed point subspaces, the latter to guarantee the typically persistent existence of isolated $\mathbb{D}_2$-symmetric periodic solutions [19]. We consider such an isolated periodic solution with spatiotemporal symmetry $\Sigma = \mathbb{D}_2$, so that $Rx(t) = x(-t)$ and $Sx(t) = x(t + \frac{1}{2})$, and consequently $\Delta = 1$. Also, since S commutes with R, the twist morphism has order $k = 1$.*

Table 6
Symmetry types of bifurcating solutions in codimension one nonHopf bifurcation from a reversible periodic solution with spatiotemporal symmetry $\Sigma = \mathbb{Z}_2(R)$, spatial symmetry $\Delta = 1$, and normal form symmetry $\mathbb{Z}_2(R) \times \mathbb{Z}_2(\tau)$ $(\ell = 1)$. Of each of the two possible branches listed for $\tau = \pm 1$, only one appears at the bifurcation point. The nilpotent part of the linear part selects which branch appears.

τ	J	Δ^{bif}	Σ^{bif}	σ^{bif}	ρ^{bif}	m	m^{bif}	p	q	comment
$+1$	$\mathbb{D}_2(R,\tau)$	1	$\mathbb{D}_1(R)$	1	R	1	1	1	1	turning point
$+1$	$\mathbb{D}_1(\tau)$	1	1	1	1	1	1	1	1	
-1	$\mathbb{D}_1(R)$	1	$\mathbb{D}_1(R)$	1	R	1	1	2	2	
-1	$\mathbb{D}_1(R\tau)$	1	$\mathbb{D}_1(R)$	1	R	1	1	2	2	

Table 7
Symmetry types of bifurcating solutions in codimension one nonHopf (subharmonic) bifurcation from a reversible periodic solution with spatiotemporal symmetry $\Sigma = \mathbb{D}_2(R,S) = \mathbb{Z}_2(R) \times \mathbb{Z}_2(S)$, spatial symmetry $\Delta = 1$, and normal form symmetry $\mathbb{D}_{2\ell}(R,\tau) = \mathbb{Z}_{2\ell}(\tau) \rtimes \mathbb{Z}_2(R)$ with $\ell \geq 2$.

ℓ	τ^ℓ	J	Δ^{bif}	Σ^{bif}	σ^{bif}	ρ^{bif}	m	m^{bif}	p	q
odd/even	-1	$\mathbb{D}_1(R)$	1	$\mathbb{D}_1(R)$	1	R	2	1	2ℓ	ℓ
odd/even	-1	$\mathbb{D}_1(R\tau)$	1	$\mathbb{D}_1(RS)$	1	RS	2	1	2ℓ	ℓ
odd	$+1$	$\mathbb{D}_2(R,\tau^\ell)$	1	$\mathbb{D}_2(R,S)$	S	R	2	2	ℓ	ℓ
odd	$+1$	$\mathbb{D}_2(R\tau,\tau^\ell)$	1	$\mathbb{D}_2(R,S)$	S	R	2	2	ℓ	ℓ

We are led to study a first hit pullback map f that is R-reversible with $f(0) = 0$. As in Example 1, we consider the fixed point 0 with $(df)_0$ being such that ℓ is least such that $L_0^{2k\ell} = I$, which is a codimension one phenomenon.

Accordingly, the normal form symmetry is $\mathbb{D}_{2\ell}(R,\tau)$, and as in Example 1 the steady-state bifurcation problem reduces to that of a $\mathbb{D}_{2\ell}$-equivariant vector field (with the same map A). The bifurcation analysis runs parallel to that of Example 1, the only difference being the symmetry properties of the periodic solution branches. The analysis of the cases $\ell \geq 2$ and $\ell = 1$ is summarised in Table 7 and Table 8.

Hamiltonian systems

In the case of Hamiltonian flows, supposing that Γ acts (anti)symplectically, one obtains an iso-energetic first hit pullback map f that is a diffeomorphism which preserves or reverses the symplectic form (symplectomorphism or anti-

Table 8
Symmetry types of bifurcating solutions in codimension one nonHopf bifurcation from a reversible periodic solution with spatiotemporal symmetry $\Sigma = \mathbb{D}_2(R, S) = \mathbb{Z}_2(R) \times \mathbb{Z}_2(S)$, spatial symmetry $\Delta = 1$, and normal form symmetry $\mathbb{Z}_2(R) \times \mathbb{Z}_2(\tau)$ ($\ell = 1$). Of each of the two possible branches listed for $\tau = \pm 1$, only one appears at the bifurcation point. The nilpotent part of the linear part selects which branch appears.

τ	J	Δ^{bif}	Σ^{bif}	σ^{bif}	ρ^{bif}	m	m^{bif}	p	q	comment
$+1$	$\mathbb{D}_2(R, \tau)$	1	$\mathbb{D}_2(R, S)$	S	R	2	2	1	1	turning point
$+1$	$\mathbb{D}_1(\tau)$	1	$\mathbb{D}_1(S)$	S	1	2	2	1	1	
-1	$\mathbb{D}_1(R)$	1	$\mathbb{D}_1(R)$	1	R	2	1	2	1	
-1	$\mathbb{D}_1(R\tau)$	1	$\mathbb{D}_1(RS)$	1	RS	2	1	2	1	

symplectomorphism). The above normal form result extends with the normal form vector field now being Hamiltonian, as Hamiltonian vector fields form a Lie subalgebra of vector fields on $\mathbb{R}^{2n}$.

Theorem 7.4 *Let $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a Σ twisted reversible equivariant (anti)symplectic diffeomorphism satisfying $f(0) = 0$, and set $L = (df)_0$. Suppose that all eigenvalues of L lie on the complex unit circle and write $L = L_0 e^{B+N}$ where $L_0^{2k\ell} = I$.*

Then there exists a Σ-equivariant symplectic coordinate transformation T such that the Taylor expansions of

$$TfT^{-1} \quad \text{and} \quad L_0 g$$

agree through arbitrarily high order where g is the time-one map of a Ξ-reversible equivariant autonomous Hamiltonian vector field $h : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying $h(0) = 0$, and $(dh)_0 = B + N$.

Proof. If f is (anti)symplectic twisted reversible equivariant, then L and L_0 are too, and $g = L^{-1} f$ is symplectic reversible equivariant. If we have an (anti)symplectic diffeomorphism that is (twisted) equivariant but not reversible, the result follows directly from Takens [31], using the Lie algebra structure of equivariant Hamiltonian vector fields. The reversible Hamiltonian vector fields do not form a Lie algebra, however, so we need to verify whether the argument in the proof of Theorem 7.1 carries over. It is straightforward to verify that it does, with $\hat{h}$, B and N now being reversible equivariant Hamiltonian vector fields. $\square$

Analogous results can be obtained for first hit pullback maps with other structures, such as for instance volume preserving diffeomorphisms yielding divergence free normal form vector fields.

8 Relative periodic solutions

In this section, we recall (certain aspects of) the framework established in [33] for studying bifurcation from relative periodic solutions and we show how to incorporate our main results of this paper.

We continue to assume (for the moment) that Γ is a compact Lie group acting orthogonally on $\mathbb{R}^n$. However, we drop the assumption that $\dim \Gamma = \dim \Sigma$. As before, we consider the dynamics for a smooth Γ-equivariant flow. Instead of restricting attention to periodic solutions, we consider general solutions that are periodic modulo the group action.

Recall that a trajectory $u(t)$ is said to be relatively periodic if there exists a $T > 0$ least such that $u(T) \in \Gamma u(0)$. By scaling time, we may suppose that u_0 has relative period $T = 1$. The corresponding *relative periodic solution* $\mathcal{P}$ is defined to be

$$\mathcal{P} = \{\gamma u(t) : \gamma \in \Gamma,\ t \in [0, 1)\}.$$

By definition, $\mathcal{P}$ is both flow-invariant and Γ-invariant. Under the flow induced on the orbit space, $\mathcal{P}/\Gamma \cong S^1$ is an ordinary periodic solution of period 1.

Again, we define the group of spatial symmetries

$$\Delta = \{\gamma \in \Gamma : \gamma u(0) = u(0)\}.$$

By construction, there is an element $\sigma \in \Gamma$ such that $u(1) = \sigma u(0)$. The spatiotemporal symmetry group Σ corresponding to the trajectory $u(t)$ is defined to be the closed subgroup of Γ generated by Δ and σ (so Σ/Δ is a topologically cyclic group of the form $T^p \times \mathbb{Z}_q$). The element σ generates an automorphism $\phi \in \mathrm{Aut}(\Delta)$ and can be chosen so that ϕ has finite order k [33].

We form the semidirect product $\Delta \rtimes \mathbb{Z}_{2k}$ as before, adjoining to Δ an element τ satisfying the relations

$$\tau^{2k} = 1, \quad \tau^{-1}\delta\tau = \phi(\delta).$$

Theorem 8.1 ([33]) *The dynamics in a neighbourhood of $\mathcal{P}$ is equivalent, modulo slow drifts along group orbits, to the dynamics in the neighbourhood of an isolated discrete rotating wave $y(t)$ with spatial symmetry Δ and spatiotemporal symmetry $\Delta \rtimes \mathbb{Z}_{2k}$.*

More precisely, there is a Γ-invariant neighbourhood U of $\mathcal{P}$ and a $\Delta \rtimes \mathbb{Z}_{2k}$-invariant neighbourhood V of $y(t)$ such that the dynamics on the orbit spaces U/Γ and $V/(\Delta \rtimes \mathbb{Z}_{2k})$ are topologically conjugate.

Moreover, each symmetry $\tau^p\delta \in \Delta \rtimes \mathbb{Z}_{2k}$ acting on X corresponds to a symmetry of the form $\gamma\sigma^p\delta \in \Gamma$ acting on U, where γ is near identity.

In particular, there is a one-to-one correspondence between (group orbits of) periodic solutions lying close to $y(t)$, and relative periodic solutions lying close to $\mathcal{P}$.

At an abstract level, Theorem 8.1 reduces bifurcation from a relative periodic solution for a Γ-equivariant flow to bifurcation from an isolated discrete rotating wave for a $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant flow. This in turn reduces (modulo terms of arbitrarily high order) to bifurcation from an equilibrium for a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field.

For a more precise statement of Theorem 8.1, we refer to [33]. However, the following discussion illustrates what the theorem is saying. Suppose that $y^{\mathrm{bif}}(t)$ is a periodic solution bifurcating from $y(t)$. Suppose that $y^{\mathrm{bif}}(t)$ has spatial symmetry Δ^{bif}, and let $p \geq 1$ be least such that $\tau^p\delta_0$ is a spatiotemporal symmetry of $y^{\mathrm{bif}}(t)$ for some $\delta_0 \in \Delta$. Then there is a relative periodic solution $\mathcal{P}^{\mathrm{bif}}$ bifurcating from $\mathcal{P}$ with spatial symmetry Δ^{bif} and spatiotemporal symmetry Σ^{bif} where Σ^{bif} is the closed group generated by Δ^{bif} and $\gamma\sigma^p\delta_0$ where γ is near identity. Moreover, γ lies in the centraliser of Δ^{bif}.

The periodic solution $y^{\mathrm{bif}}(t)$ and $\mathcal{P}^{\mathrm{bif}}$ have the same relative period T (which is close to the integer p) and satisfy $y^{\mathrm{bif}}(T) = \tau^p\delta_0 y^{\mathrm{bif}}(0)$ and $u^{\mathrm{bif}}(T) = \gamma\sigma^p\delta_0 u^{\mathrm{bif}}(0)$ respectively.

Remark 8.2 *Combining the results of Section 2 with Theorem 8.1, we have shown that relative periodic solutions $\mathcal{P}^{\mathrm{bif}}$ that arise through bifurcation from a relative periodic solution for a Γ-equivariant flow correspond to equilibria x^{bif} bifurcating from an equilibrium for a $\Delta \rtimes \mathbb{Z}_{2k\ell}$-equivariant vector field h.*

Moreover, if $J \subset \Delta \rtimes \mathbb{Z}_{2k\ell}$ is the isotropy subgroup of x^{bif} and $p \geq 1$ is least such that $\tau^p\delta_0 \in J$ for some $\delta_0 \in \Delta$, then $\mathcal{P}^{\mathrm{bif}}$ has spatial symmetry $\Delta^{\mathrm{bif}} = J \cap \Delta$ and spatiotemporal symmetry Σ^{bif} generated by Δ^{bif} and $\gamma\sigma^p\delta_0$ where γ is near identity. More generally, we have the correspondence

$$\tau^j\delta \in J \subset \Delta \rtimes \mathbb{Z}_{2k\ell} \longleftrightarrow \gamma_{j,\delta}\sigma^j\delta \in \Gamma,$$

where $\gamma_{j,\delta}$ is a near identity element in $Z(\Delta^{\mathrm{bif}})$.

Remark 8.3 *Theorem 8.1 represents only part of the theory given in Wulff* et al. *[33]. For example, Theorem 8.1 does not provide a means of computing* *the element γ in the spatiotemporal symmetry $\gamma\sigma^p\delta_0$. We note that γ is a general near identity element in $Z(\Delta^{\mathrm{bif}})$, and hence the results of Krupa and Field [17,13] can be used to determine the expected drift on the bifurcating relative periodic solutions.*

In addition, the implications of such drifts in phase space for phenomena viewed in physical space has been studied recently [15] in the context of bifurcation from relative equilibria. The corresponding analysis for bifurcation from relative periodic solutions requires the full strength of the results in [33].

More generally, the results in [33] hold for many noncompact Lie groups Γ, including the Euclidean group. Hence, Theorem 8.1 is valid for such symmetry groups. We note that the computation of the slow drift γ is particularly important in this context, since the value of γ determines whether the drift is compact or unbounded. (The genericity results of [17,13] generalise to the noncompact group setting, see [4] and [33, Section 4.2], but do not predict the actual *value of γ.) Again, γ may be determined using the results in [33].*

As in the equivariant case, the bifurcation theory for reversible periodic solutions can also be applied to understand bifurcation from reversible relative periodic solutions. However, we will not expand on any details here. For discussions on the bundle structures in the reversible equivariant, and reversible equivariant Hamiltonian contexts, see [24,34].

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