Drift bifurcations of relative equilibria and transitions of spiral waves *

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Abstract

We consider dynamical systems that are equivariant under a noncompact Lie group of symmetries and the drift of relative equilibria in such systems. In particular, we investigate how the drift for a parametrized family of normally hyperbolic relative equilibria can change character at what we call a 'drift bifurcation'. To do this, we use results of Arnold to analyze parametrized families of elements in the Lie algebra of the symmetry group.

We examine effects in physical space of such drift bifurcations for planar reaction-diffusion systems and note that these effects can explain certain aspects of the transition from rigidly rotating spirals to rigidly propagating 'retracting waves'. This is a bifurcation observed in numerical simulations of excitable media where the rotation rate of a family of spirals slows down and gives way to a semi-infinite translating wavefront.

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1 Introduction

Let Γ be a finite-dimensional Lie group (not necessarily compact) and suppose that $u_t = F(u)$ is an evolution equation equivariant with respect to an action of Γ. A dynamically invariant subset X in phase space is called a *relative equilibrium* if X consists of a single group orbit under the action of Γ. Equivalently, X reduces to an ordinary equilibrium for the dynamics induced on the orbit space. The notion of relative equilibrium includes the case of a group orbit of equilibria and also includes *rotating waves.* A rotating wave is a flow-invariant group orbit on which the flow is periodic with time evolution corresponding to drift along the group orbit.

A relative equilibrium that is normally hyperbolic persists under small perturbations of the evolution operator and hence it is the dynamics on the relative equilibrium X itself that is of interest. We suppose (for simplicity of exposition) that X consists of points of trivial isotropy, and also that the group orbit X is diffeomorphic to the group Γ. (This last assumption is automatically satisfied for smooth actions of compact Lie groups Γ.) For compact groups, the typical dynamics on X has been classified by Field [5] and Krupa [10]. The relative equilibrium is foliated by closed flow-invariant subsets that are copies of a torus $K \subset \Gamma$ and the dynamics on these subsets consists of a transitive (irrational) linear flow. From their work, one finds that generically, K is a maximal torus in Γ.

In recent work [2], we obtained analogous results for Γ noncompact. There is now the possibility that the closed subgroup K is isomorphic to a copy of $\mathbb{R}$. Indeed, generically K is a maximal torus or $K \cong \mathbb{R}$. Further results are group-dependent. For the Euclidean groups $\mathbf{E}(n)$ with n even, generically K is a maximal torus but $K \cong \mathbb{R}$ occurs as a codimension one phenomenon. In contrast, when $\Gamma = \mathbf{E}(n)$, n odd, generically $K \cong \mathbb{R}$ and with codimension one K is a maximal torus. (A third possibility, which is realized for the symplectic groups, is that maximal tori and copies of $\mathbb{R}$ are both codimension zero.)

The above discussion suggests the notion of *drift bifurcation* whereby the subgroup K determining the drift on the relative equilibrium X varies as a parameter is varied. We assume normal hyperbolicity throughout, so the only bifurcation that occurs is in the drift on X. A particularly intriguing example, which motivated this work, occurs when $\Gamma = \mathbf{E}(2)$. By [2], generically we have a rotating wave (the maximal torus here is a circle) and atypically we have linear translation drift (corresponding to a copy of $\mathbb{R}$). As a parameter is varied, the speed of rotation may pass through zero leading to a change from counterclockwise to clockwise rotation. A simple calculation, reproduced in Section 6(a), shows that at the bifurcation point there is linear translation drift with nonzero speed. Moreover as the bifurcation point is approached, the center of rotation diverges to infinity. This behavior is strongly reminiscent of a bifurcation observed in numerical simulations of excitable media by Jahnke and Winfree [9], see also Mikhailov and Zykov [12] and Barkley and Kevrekidis [3].

In this paper, we classify drift bifurcations for noncompact symmetry groups and we explore the implications for applications such as the phenomena described in [12]. We consider the bifurcations both for relative equilibria and for periodic orbits. Our work should be contrasted with the recent work of [4, 8, 14, 16] which focuses on bifurcations *from* relative equilibria (and relative periodic orbits [15]) where the context is loss of normal hyperbolicity of the underlying relative equilibrium.

With regard to the example described above, a natural question is how significance can be attached to the center of rotation diverging to infinity. By choosing a different symmetrically placed initial condition, the center of rotation could be normalized 'without loss of generality' to the origin. The answer is that this normalization would be singular at the bifurcation point. The situation is completely analogous to the one described in Arnold [1]: in a parametrized family of matrices, it is not appropriate to suppose that each member of the family is in Jordan normal form without taking into account the dependence of the similarity transformations on parameters. Instead, it is shown in [1] how to construct a single normal form for the entire matrix family under smoothly varying similarity transformations.

In Section 2, we show how drift bifurcation for relative equilibria (and relative periodic orbits) fits into the context of [1]. In particular, we show how such bifurcations are governed by bifurcations of parametrized families in the Lie algebra $\mathrm{L}\Gamma$ of the symmetry group Γ. In Section 3 we extend the theory of Arnold [1] to the classification of bifurcations in $\mathrm{L}\Gamma$ and the computation of their versal unfoldings. Section 4 considers the case $\Gamma = \mathbf{O}(n)$. Our main purpose for doing this is that the results are required for understanding the case $\Gamma = \mathbf{E}(n)$. We discuss codimension one and two drift bifurcations for Euclidean symmetry in Section 5. Section 6 applies this to reaction diffusion systems on the plane and in particular the spiral wave/retracting wave transition.

2 Drift bifurcations and families in the Lie algebra

Let Γ be a finite dimensional Lie group acting linearly on a Banach space $\mathcal{B}$ and suppose that $u_t = F_\lambda(u)$ is a smoothly parametrized family of Γ-equivariant evolution equations, with $u \in \mathcal{B}$, $\lambda \in \mathbb{R}^k$. We suppose that when $\lambda = 0$, X is a relative equilibrium for F, equivalently for $u_0 \in X$, $F(u_0) = \xi u_0$ for some $\xi \in \mathrm{L}\Gamma$.

We make the following standing hypotheses.

(H1) The relative equilibrium X consists of points of trivial isotropy (that is, $\Sigma_{u_0} = \mathbf{1}$).

(H2) The group orbit $X = \Gamma u_0$ is an embedded submanifold of $\mathcal{B}$ (hence diffeomorphic to the group Γ).

(H3) The relative equilibrium X is normally hyperbolic.

As discussed at the end of this section, hypotheses (H1) and (H2) are easily relaxed, whereas hypothesis (H3) is somewhat problematic.

The time-evolution of u_0 is given by $u(t) = \exp(t\xi)u_0$ where ξ is an element in $\mathrm{L}\Gamma$. The subgroup $K \subset \Gamma$ mentioned in the introduction is the closure of this one-parameter subgroup: $K = \overline{\{\exp(t\xi) : t \in \mathbb{R}\}}$.

First, we make explicit the dependence of the element $\xi \in \mathrm{L}\Gamma$ on the initial condition $u_0 \in X$. We recall the usual notation $\mathrm{Ad} : \Gamma \to \mathrm{Aut}(\mathrm{L}\Gamma)$ for the adjoint action of Γ on $\mathrm{L}\Gamma$.

Proposition 2.1 *Suppose that X is a relative equilibrium and $u_0, u_1 \in X$, so $u_1 = \gamma u_0$ for some $\gamma \in \Gamma$. If $F(u_0) = \xi u_0$, then $F(u_1) = (\mathrm{Ad}_\gamma \xi)u_1$.*

Proof Compute that

$$F(u_1) = F(\gamma u_0) = \gamma F(u_0) = \gamma \xi u_0 = \gamma \xi \gamma^{-1} \gamma u_0 = (\mathrm{Ad}_\gamma \xi)u_1. \qquad \blacksquare$$

By Proposition 2.1, the adjoint orbit of ξ under Γ is independent of the choice of initial condition $u_0 \in X$ and the dynamics on X is classified by the adjoint orbits for the action of Γ on $\mathrm{L}\Gamma$. For example, to compute the time evolution of trajectories on X, that is to exponentiate ξ, we can suppose without loss that ξ is a particularly simple representative of its adjoint orbit. Such a representative ξ is called a 'normal form'.

It follows from normal hyperbolicity (H3) that X extends to a smooth family of relative equilibria $X_\lambda = \Gamma u_0(\lambda)$ for F_λ, giving rise to a smooth family $\xi(\lambda) \in \mathrm{L}\Gamma$, $\xi(0) = \xi$, defined by $F(u_0(\lambda)) = \xi(\lambda)u_0(\lambda)$. Although F, X, u_0 and ξ depend smoothly on λ, the adjoint orbit of ξ and hence the dynamics on X may undergo bifurcations.

The above discussion indicates that such bifurcations in the dynamics on X, or *drift bifurcations*, are understood as bifurcations in the Lie algebra. The ideas of Arnold [1] can be used to compute normal forms for families of Lie algebra elements, to classify families by codimension, and to compute versal unfoldings. These ideas are recalled in Section 3. In particular, we require that the simplifying transformations via the adjoint action on the family $\xi(\lambda)$ depend smoothly on parameters.

Relative periodic orbits Recall that a flow-invariant Γ-invariant set P is called a *relative periodic orbit* if the orbit space P/Γ is an ordinary periodic orbit. As in the case of relative equilibria, we assume that P is a normally hyperbolic embedded submanifold of $\mathcal{B}$ consisting of points of trivial isotropy.

The flow on relative periodic orbits is classified for Γ compact by Krupa [10] and Field [6] and for Γ noncompact by Ashwin and Melbourne [2]. Let T be the period of the periodic solution on the orbit space P/Γ. If $u(0) = u_0 \in P$, then $u(T) = \gamma u_0$ for some $\gamma \in \Gamma$. Let H be the closed subgroup generated by γ. Generically, H is either a Cartan subgroup or a copy of $\mathbb{R}$ and the relative periodic orbit P is foliated either by irrational torus flows of dimension $\dim H + 1$ or by copies of $\mathbb{R}$ with unbounded linear flow. See [2] for details.

The element $\gamma \in \Gamma$ is well-defined, independent of the choice of $u_0 \in P$, up to conjugacy in Γ. Hence the dynamics on P is classified by conjugacy classes in Γ. Drift bifurcations for relative periodic orbits are governed by bifurcations of parametrized families of Lie group elements and the corresponding normal form theory requires that the families of conjugacies depend smoothly on parameters.

For groups where the exponential map $\mathrm{L}\Gamma \to \Gamma$ is surjective (necessarily meaning that Γ is connected), as is the case when $\Gamma = \mathbf{SE}(n)$, the classifications for parametrized families of Lie algebra elements and Lie group elements are identical. For groups where the exponential map is not surjective (e.g. $\mathbf{SL}(n)$) the classification for relative periodic orbits could in principle include more cases than one can find for relative equilibria. It follows that for such groups, the drift bifurcations associated with relative periodic orbits are different from those associated with relative equilibria.

Discussion of the hypotheses Hypothesis (H1) is unnecessary and can be lifted using the following standard argument. Suppose that $u_0 \in X$ has isotropy subgroup $\Sigma = \Sigma_{u_0}$. This subgroup is well-defined up to conjugacy since $\Sigma_{\gamma u_0} = \gamma \Sigma_{u_0} \gamma^{-1}$. Hence we can speak of the isotropy subgroup Σ of the relative equilibrium X. Let $N(\Sigma)$ denote the normalizer of Σ in Γ The quotient $G = N(\Sigma)/\Sigma$ governs the drifts on X. ((H2) is altered slightly: Γu_0 is now diffeomorphic to Γ/Σ). Hence the generic drift corresponds to either a maximal torus in G or a copy of $\mathbb{R}$ [5, 10, 2]. Similarly, the results on drift bifurcations described in this paper go through by replacing Γ with $G = N(\Sigma)/\Sigma$.

Next, we consider hypothesis (H2). This is satisfied for finite-dimensional compact Lie group actions and for many settings involving infinite-dimensional actions and noncompact groups. Provided Γ acts smoothly on u_0, the group orbit X is an immersed submanifold of $\mathcal{B}$. To ensure that X is embedded, we must exclude the presence of 'approximate symmetries' [2]. (A sequence $\{\gamma_n\} \in \Gamma/\Sigma$ is an approximate symmetry if there are no convergent subsequences and yet $\gamma_n u_0 \to u_0$.) The condition that Γ acts smoothly on u_0, and hence X, is very natural since drift on a relative equilibrium corresponds to time evolution and hence is smooth. If Γ does not act smoothly on u_0, we can replace $G = N(\Sigma)/\Sigma$ by a closed subgroup $H \subset G$ that does act smoothly. The condition on approximate symmetries is not so easily dealt with but will not cause any problems in the applications considered in this paper.

Finally, we consider hypothesis (H3). Of course, this hypothesis is generic for finite-dimensional actions of compact Lie groups and for many infinite-dimensional actions. The generalization to noncompactness and infinite-dimensionality lead to two different issues.

The first issue, which is unimportant for our purposes (though significant for bifurcations from relative equilibria [15]) arises from noncompactness of Γ and concerns the possibility of nonneutral eigenvalues along the group directions. Recall that for a compact Lie group, the spectrum of the linearized vector field in the directions along the group orbit consists of purely imaginary eigenvalues. As pointed out in [8, Appendix], this is no longer automatically true for noncompact Lie groups, though it is true for the Euclidean groups (and for any group with an invariant metric [15]). If there are such nonneutral eigenvalues, it is immediate that normal hyperbolicity is not a generic condition (though it may be an open condition).

The second issue arises for spatially-extended systems of PDEs. An in-depth discussion can be found in Sandstede, Scheel and Wulff [15]. For reaction-diffusion equations in $\mathbb{R}^n$, it follows from [15, Lemma 6.2] that 'localized solutions' that decay at infinity are generically normally hyperbolic. Unfortunately, solutions that do not decay at infinity to some constant are never normally hyperbolic [15, Lemma 6.3] due to the presence of essential spectrum (the complement in the spectrum of the set of isolated eigenvalues of finite multiplicity) intersecting the imaginary axis.

This discussion indicates that hypothesis (H3) is justified for localized solutions and is unjustified for nonlocalized solutions. As far as we know, the variation of relative equilibria that are nonhyperbolic due to the essential spectrum is not understood even away from bifurcation points. Nevertheless, the apparent robustness of spiral waves in excitable media suggests that with certain modifications (that we have not determined) the predictions obtained by assuming (H3) should still be meaningful.

3 Normal forms and versal unfoldings in $\mathrm{L}\Gamma$

In this section, we recall the ideas of Arnold [1] on parametrized families of Lie algebra elements. We recall the usual notation $\mathrm{Ad} : \Gamma \to \mathrm{Aut}(\mathrm{L}\Gamma)$ and $\mathrm{ad} : \mathrm{L}\Gamma \to \mathrm{End}(\mathrm{L}\Gamma)$ for the adjoint actions of Γ and $\mathrm{L}\Gamma$ on $\mathrm{L}\Gamma$. Also the Lie bracket of elements $A, B \in \mathrm{L}\Gamma$ is given by $[A,B] = \mathrm{ad}_A(B)$ and the centralizer of $A \in \mathrm{L}\Gamma$ is defined to be

$$Z(A) = \{B \in \mathrm{L}\Gamma : [A,B] = 0\}.$$

Let $A_0 \in \mathrm{L}\Gamma$ and consider the adjoint group orbit $\mathrm{Ad}_\Gamma A_0 \subset \mathrm{L}\Gamma$. We define the *codimension* of A_0 to be the codimension of the group orbit, so

$$\operatorname{codim} A_0 = \operatorname{codim} \mathrm{Ad}_\Gamma A_0 = \dim \mathrm{L}\Gamma - \dim \mathrm{Ad}_\Gamma A_0 = \dim Z(A_0).$$

The codimension of A_0 is equal to the minimum number of unfolding parameters required in a versal unfolding of A_0 [1].

Let $<\,,\,>$ be an inner product on $\mathrm{L}\Gamma$ and define

$$\mathcal{B}(A_0) = (\mathrm{ad}_\Gamma A_0)^\perp = \{B \in \mathrm{L}\Gamma :< B, [A_0, C] >= 0 \text{ for all } C \in \mathrm{L}\Gamma\}.$$

If $\{B_1, \cdots, B_k\}$ is a basis for $\mathcal{B}(A_0)$, then a versal unfolding of A_0 is given by

$$A_0 + \lambda_1 B_1 + \cdots + \lambda_k B_k.$$

In computing $Z(A_0)$ and $\mathcal{B}(A_0)$, we can of course first apply transformations in Ad_Γ to reduce A_0 to a simpler (normal) form. A further simplification is possible in certain circumstances: namely when the Lie algebra $\mathrm{L}\Gamma$ can be embedded in the Lie algebra M_n of real matrices in such a way that A_0^T (which is defined in M_n) lies in $\mathrm{L}\Gamma$. We take the inner product on $\mathrm{L}\Gamma$ to be the one induced by the inner product $\ll A, B \gg= \operatorname{tr} AB^T$ on M_n. In this case, we recover the following result of [1].

Proposition 3.1 *Suppose that* $\mathrm{L}\Gamma$ *is identified with a subspace of* M_n *as above. If* $A_0^T \in \mathrm{L}\Gamma$, *then* $\mathcal{B}(A_0) = Z(A_0^T)$.

Proof Let $B, C \in \mathrm{L}\Gamma$. We compute that

$$\ll B, [A_0, C] \gg= \operatorname{tr} B(A_0 C - C A_0)^T = \operatorname{tr}(A_0^T B - B A_0^T) C^T =\ll A_0^T B - B A_0^T, C \gg .$$

Hence, $B \in \mathcal{B}(A_0)$ precisely when $A_0^T B - B A_0^T$ is orthogonal to $\mathrm{L}\Gamma$. However, $A_0^T \in \mathrm{L}\Gamma$ implies that $A_0^T B - B A_0^T = [A_0^T, B] \in \mathrm{L}\Gamma$. It follows that $B \in \mathcal{B}(A_0)$ if and only if $[A_0^T, B] = 0$. ∎

Remark 3.2 Arnold [1] concentrates on the case $\Gamma = \mathbf{GL}(n)$ where the hypotheses of Proposition 3.1 are satisfied for all A_0. This is true also for any compact Lie group, for the symplectic group $\mathbf{Sp}_{2n}$ [7, 11], for the special linear group $\mathbf{SL}(n)$, and for the real classical Lie groups [13]. However, the hypotheses of the proposition are not satisfied for the Euclidean group $\mathbf{E}(n)$ considered in Section 5. Instead, we are forced to work directly with the definition of $\mathcal{B}(A_0)$. Of course, $\operatorname{codim} A_0$ can still be computed using centralizers.

There is a parallel theory for parametrized families of Lie group elements. Here, we consider group orbits $\Phi_\Gamma \gamma_0$ where Φ_δ denotes conjugation by δ. Then

$$\operatorname{codim} \gamma_0 = \dim Z_\Gamma(\gamma_0),$$

where $Z_\Gamma(\gamma_0) = \{\delta \in \Gamma : \delta\gamma_0 = \gamma_0\delta\}$ denotes the centralizer of γ_0 in Γ.

Let $T(\gamma_0) = (T_{\gamma_0}\Phi_\Gamma\gamma_0)\gamma_0^{-1}$ denote the tangent space at γ_0 transported by right multiplication to e. In particular, $T(\gamma_0) \subset \mathrm{L}\Gamma$. Again, we choose an inner product on $\mathrm{L}\Gamma$ and define

$$\mathcal{B}(\gamma_0) = T(\gamma_0)^\perp = \{B \in \mathrm{L}\Gamma : < B, C - \mathrm{Ad}_{\gamma_0} C >= 0 \text{ for all } C \in \mathrm{L}\Gamma\}.$$

If $\{B_1, \cdots, B_k\}$ is a basis for $\mathcal{B}(\gamma_0)$, then a versal unfolding of γ_0 is given by

$$\exp(\lambda_1 B_1 + \cdots + \lambda_k B_k)\gamma_0.$$

Finally, suppose that $\Gamma \subset \mathbf{GL}(n)$ is a matrix group (so that γ_0^T is defined as an invertible matrix). Then $\mathrm{L}\Gamma$ is identified with a subspace of M_n with the trace inner product and we have the characterization

$$\mathcal{B}(\gamma_0) = \{B \in \mathrm{L}\Gamma : B - \mathrm{Ad}_{\gamma_0^T} B \in \mathrm{L}\Gamma^\perp\}.$$

In particular, if $\gamma_0^T \in \Gamma$, then $\mathcal{B}(\gamma_0)$ consists of those matrices in $\mathrm{L}\Gamma$ that commute with the matrix γ_0^T.

4 Versal unfoldings with orthogonal symmetry

In this section, we apply the methods of Section 3 to the group $\Gamma = \mathbf{O}(n)$ of $n \times n$ orthogonal matrices. The Lie algebra $\mathrm{L}\mathbf{O}(n)$ consists of $n \times n$ skew-symmetric matrices. Obviously, the transpose of a skew-symmetric matrix is skew-symmetric, so Proposition 3.1 applies. Moreover, we have the simplification that $\mathcal{B}(A_0) = Z(A_0)$.

For $\omega > 0$, we define the 2×2 matrix $R_\omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. Let $R_{\omega,s}$ denote the direct sum of s copies of R_ω. Also 0_m denotes the $m \times m$ zero matrix. It is a standard result in linear algebra that every skew-symmetric matrix can be transformed by an orthogonal change of coordinates into a matrix of the form

$$A_0 = R_{\omega_1,s_1} \oplus R_{\omega_2,s_2} \oplus \cdots \oplus R_{\omega_r,s_r} \oplus 0_m, \tag{4.1}$$

where the $\omega_j > 0$ are distinct and $2(s_1 + \cdots + s_r) + m = n$. Moreover, this 'normal form' is unique up to ordering of the ω_j.

Proposition 4.1 *The codimension of the matrix A_0 in (4.1) is given by*

$$\begin{aligned}\operatorname{codim} A_0 = \dim Z(A_0) &= \dim Z(R_{\omega_1,s_1}) + \cdots + \dim Z(R_{\omega_r,s_r}) + \dim Z(0_m) \\ &= s_1^2 + \cdots + s_r^2 + m(m-1)/2.\end{aligned}$$

Proof Commuting matrices preserve the eigenspaces of A_0 so we have the direct sum

$$Z(A_0) = Z(R_{\omega_1,s_1}) \oplus \cdots \oplus Z(R_{\omega,s_r}) \oplus Z(0_m).$$

Clearly, $Z(0_m)$ consists of all $m \times m$ skew-symmetric matrices and hence has dimension $m(m-1)/2$. It remains to show that $\dim Z(R_{\omega,s}) = s^2$. Observe that any matrix that commutes with $R_{\omega,s}$ can be written as an $s \times s$ matrix of 2×2 blocks each of which commutes with R_ω. Such blocks have the form $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$. So far, we have computed the dimension to be $2s^2$, but it is easily seen that the skew-symmetry condition results in the required dimension s^2. (The diagonal blocks have $\alpha = 0$ and the nondiagonal blocks are related in pairs.) ∎

The eigenvalues $\pm i\omega_j$ of A_0 are moduli preserved by the adjoint action of $\mathbf{O}(n)$ on $\mathrm{L}\mathbf{O}(n)$. It is desirable to suppress their contribution to the codimension of A_0. Following Arnold [1], we consider the totality of normal forms that have the same structure as A_0 (same values for $s_1, \ldots, s_r$) but with different values for the ω_j. The corresponding set of adjoint orbits forms a 'bundle' in $\mathrm{L}\mathbf{O}(n)$ and the *bundle codimension* $\operatorname{codim}_b A_0$ of A_0 is defined to be the codimension of this bundle in $\mathrm{L}\mathbf{O}(n)$. Thus we obtain

$$\operatorname{codim}_b A_0 = (s_1^2 - 1) + \cdots + (s_r^2 - 1) + m(m-1)/2. \tag{4.2}$$

We use formula (4.2) to compute bundles of low codimension. Note that for $\Gamma = \mathbf{O}(n)$, $\operatorname{codim}_b A_0$ is additive over the summands of A_0 and moreover that the versal unfolding of A_0 is the direct sum of the versal unfoldings of the summands. Hence, we can restrict our computations to the cases $A_0 = R_{\omega,s}$ and $A_0 = 0_m$ which have codimension $s^2 - 1$ and $m(m-1)/2$ respectively. The summands of codimension zero are $R_{\omega,1}$ and 0_1. We obtain the result that $\operatorname{codim}_b A_0 = 0$ if and only if all eigenvalues of A_0 are simple. (In particular, A_0 is invertible when n is even and has a single zero eigenvalue when n is odd.)

There is one summand of codimension one, namely 0_2. Hence bundles of codimension one occur only when n is even and have the form $A_0 = R_{\omega_1} \oplus \cdots \oplus R_{\omega_{(n-2)/2}} \oplus 0_2$ with ω_j distinct. The versal unfolding is of course given by $R_{\omega_1} \oplus \cdots \oplus R_{\omega_{(n-2)/2}} \oplus R_\lambda$.

There are no summands of codimension two, but there are two summands of codimension three: $R_{\omega,2}$ (which can occur for $n \geq 4$) and 0_3 (which can occur for $n \geq 3$ odd). Versal unfoldings are given by

$$\begin{pmatrix} 0 & -\omega & -\lambda_2 & -\lambda_3 \\ \omega & 0 & \lambda_3 & -\lambda_2 \\ \lambda_2 & -\lambda_3 & 0 & -(\omega+\lambda_1) \\ \lambda_3 & \lambda_2 & \omega+\lambda_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\lambda_1 & -\lambda_2 \\ \lambda_1 & 0 & -\lambda_3 \\ \lambda_2 & \lambda_3 & 0 \end{pmatrix}.$$

5 Versal unfoldings with Euclidean symmetry

In this section, we consider the Euclidean group $\Gamma = \mathbf{E}(n)$. Recall the standard identification of $\mathbf{E}(n)$ with the subgroup of $\mathbf{GL}(n+1)$ consisting of matrices $\begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} = (R, v)$ where $R \in \mathbf{O}(n)$ and $v \in \mathbb{R}^n$. The Lie algebra $\mathrm{L}\mathbf{E}(n)$ then consists of $(n+1)\times(n+1)$ matrices $\widetilde{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} = (A, a)$ where $A \in \mathrm{L}\mathbf{O}(n)$ and $a \in \mathbb{R}^n$. Note that this representation of $\mathrm{L}\mathbf{E}(n)$ does not satisfy the transpose hypothesis of Proposition 3.1.

We proceed to compute the codimensions of the elements $\widetilde{A}_0 \in \mathrm{L}\mathbf{E}(n)$. The following proposition lists convenient normal forms for these elements. Recall from Section 4, the definitions of $R_{\omega,s}$, $\omega > 0$, and 0_m. Also, let $e_n = (0, \ldots, 0, 1)^T \in \mathbb{R}^n$.

Proposition 5.1 *Every element of* $\mathrm{L}\mathbf{E}(n)$ *can be transformed under the adjoint action of* $\mathbf{E}(n)$ *into one of the following normal forms* $\widetilde{A}_0 = (A_0, a_0)$*:*

$$A_0 = R_{\omega_1,s_1} \oplus \cdots \oplus R_{\omega_r,s_r} \oplus 0_m$$

for distinct $\omega_j > 0$ *and* $2(s_1 + \cdots + s_r) + m = n$*; and*

$$a_0 = 0 \quad \text{if } m = 0, \qquad a_0 = \alpha e_n,\ \alpha \geq 0, \quad \text{if } m > 0.$$

The normal forms are unique up to ordering of the ω_j*.*

Proof Conjugating by pure rotations and reflections $(R, 0) \in \mathbf{E}(n)$, we can arrange that $A_0 \in \mathrm{L}\mathbf{O}(n)$ is in the normal form described in Section 4. Conjugating by pure translations (I, v) we can assume that $a_0 \in \ker A_0$. In particular, if $m = 0$ so A_0 is invertible, we have $a_0 = 0$. Finally, if $m > 0$, we consider a pure rotation $(R, 0)$ where R is the identity on the range of A_0. Such a rotation preserves A_0, and restricts to an arbitrary rotation on $\ker A_0$. Hence, we can rotate a_0 onto the final coordinate axis. Also, we can reflect a_0 if necessary so that $\alpha \geq 0$. ∎

Proposition 5.2 *Suppose that $\widetilde{A}_0 = (A_0, a_0) \in \mathrm{L}\mathbf{E}(n)$ and let $Z_{\mathrm{L}\mathbf{O}(n)}(A_0)$ denote the centralizer of A_0 in $\mathrm{L}\mathbf{O}(n)$. Then $\widetilde{B} = (B, b) \in Z(\widetilde{A}_0)$ if and only if*

$$B \in Z_{\mathrm{L}\mathbf{O}(n)}(A_0) \quad \textit{and} \quad A_0 b = B a_0.$$

Moreover, if $\widetilde{A}_0$ is in normal form as in Proposition 5.1, then $A_0 b = B a_0 = 0$.

Proof The first statement of the proposition is a direct calculation. Suppose that $\widetilde{A}_0$ is in normal form. Then, in particular, $a_0 \in \ker A_0$. The condition that $B \in Z_{\mathrm{L}\mathbf{O}(n)}(A_0)$ guarantees that B preserves $\ker A_0$. Hence, $B a_0 \in \ker A_0$. At the same time, $B a_0 = A_0 b \in \operatorname{range} A_0$. Since A_0 is skew-symmetric, we have $B a_0 = 0$. ∎

In the next result, we proceed directly to the computation of $\mathrm{codim}_b \widetilde{A}_0$. We define two normal forms $\widetilde{A}_0$ and $\widetilde{A}_0'$ to be bundle equivalent if A_0 and A_0' are equivalent as in Section 4 (so the values of ω_j may vary but $s_1, \ldots, s_r$ are fixed) and in addition a_0, a_0' either both vanish or are both nonzero. (Thus we allow scalings of a_0 by a positive scalar.)

Corollary 5.3 *Suppose that $\widetilde{A}_0 = (A_0, a_0)$ is in normal form as in Proposition 5.1. Then*

$$\mathrm{codim}_b \widetilde{A}_0 = (s_1^2 - 1) + \cdots + (s_r^2 - 1) + m(m+1)/2 \quad \textit{if } a_0 = 0,$$

and

$$\mathrm{codim}_b \widetilde{A}_0 = (s_1^2 - 1) + \cdots + (s_r^2 - 1) + m(m-1)/2 \quad \textit{if } a_0 \neq 0.$$

Proof We compute $\dim Z(\widetilde{A}_0)$ using Proposition 5.2. The conditions $B \in Z_{\mathrm{L}\mathbf{O}(n)}(A_0)$ and $A_0 b = 0$ yield the dimension $(s_1^2 + \cdots + s_r^2 + m(m-1)/2) + m$. When $a_0 = 0$, there are no further constraints and we obtain the dimension $s_1^2 + \cdots + s_r^2 + m(m+1)/2$. The formula for the bundle codimension follows. When $a_0 \neq 0$, the additional constraint $B a_0 = 0$ forces $m-1$ independent coefficients in B to vanish hence reducing the dimension by $m-1$. The bundle codimension is reduced further by one corresponding to the scaling of a_0. ∎

Proposition 5.4 *Suppose that $\widetilde{A}_0 = (A_0, a_0)$ is in normal form as in Proposition 5.1. When $a_0 = 0$, we have that $\widetilde{B} = (B, b) \in \mathcal{B}(\widetilde{A}_0)$ if and only if $B \in Z_{\mathrm{L}\mathbf{O}(n)}(A_0)$ and $b \in \ker A_0$. When $a_0 \neq 0$, we have that $\widetilde{B} = (B, b) \in \mathcal{B}(\widetilde{A}_0)$ if and only if $B \in Z_{\mathrm{L}\mathbf{O}(n)}(A_0)$ and $b \in \mathbb{R}\{e_n\}$.*

Proof The case $a_0 = 0$ is straightforward: $\widetilde{A}_0^T \in \mathbf{LE}(n)$ and hence by Proposition 3.1 we have $\mathcal{B}(\widetilde{A}_0) = Z(\widetilde{A}_0^T)$ leading to the required characterization of $\mathcal{B}(\widetilde{A}_0)$.

The case $a_0 \neq 0$ is more difficult since Proposition 3.1 does not apply. However, a calculation starting from the definition of $\mathcal{B}(\widetilde{A}_0)$ shows that $\widetilde{B} = (B, b) \in \mathcal{B}(\widetilde{A}_0)$ if and only if

$$\mathrm{tr}(A_0 B - B A_0) C + (A_0 c - C a_0) \cdot b = 0,$$

for all $\widetilde{C} = (C, c) \in \mathbf{LE}(n)$. It is clear that this condition is satisfied when $B \in Z_{\mathbf{LO}(n)}(A_0)$ and $b \in \mathbb{R}\{e_n\}$. Moreover, a dimension count using the proof of Corollary 5.3 shows that we have accounted for the whole of $\mathcal{B}(\widetilde{A}_0)$. ∎

Corollary 5.5 *The classification of low codimension normal forms is as follows.*

(a) For each n there is a unique normal form of codimension zero:

$$A_0 = R_{\omega_1} \oplus \cdots \oplus R_{\omega_{n/2}}, \quad a_0 = 0, \qquad (n \text{ even})$$

$$A_0 = R_{\omega_1} \oplus \cdots \oplus R_{\omega_{(n-1)/2}} \oplus 0_1, \quad a_0 = \alpha e_n,\ \alpha > 0, \qquad (n \text{ odd})$$

(b) For each n there is a unique normal form of codimension one with versal unfolding:

$$A_0 = R_{\omega_1} \oplus \cdots \oplus R_{\omega_{(n-2)/2}} \oplus R_\lambda, \quad a_0 = \alpha e_n,\ \alpha > 0, \qquad (n \text{ even})$$

$$A_0 = R_{\omega_1} \oplus \cdots \oplus R_{\omega_{(n-1)/2}} \oplus 0_1, \quad a_0 = \lambda e_n, \qquad (n \text{ odd})$$

(c) There are no normal forms of codimension two.

Proof Restricting to codimension less than three, we see immediately that $s_1 = \ldots = s_r = 1$. Suppose that $a_0 = 0$ so that $\mathrm{codim}_b \widetilde{A}_0 = m(m+1)/2$. The values $m = 0$ and $m = 1$ give low codimension normal forms for n even and n odd respectively. When $a_0 \neq 0$, we have $\mathrm{codim}_b \widetilde{A}_0 = m(m-1)/2$ where $m \geq 1$ (since $\ker A_0 \ni a_0$). The values $m = 1$ and $m = 2$ give low codimension normal forms for n odd and n even respectively. The versal unfoldings are easily computed by Proposition 5.4. ∎

The normal forms and versal unfoldings for $\mathbf{E}(2)$ and $\mathbf{E}(3)$ are shown in Tables 1 and 2 respectively. Note that the low codimension normal forms of Corollary 5.5 occur as well as two normal forms of codimension three and one of codimension six. Further normal forms of codimension three occur for $n \geq 4$ with purely imaginary eigenvalues of multiplicity two.

$\widetilde{A}_0 = (A_0, a_0)$	$\mathrm{codim}_b \widetilde{A}_0$	versal unfolding
$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	0	$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$	1	$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	3	$\begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}$

Table 1: Versal unfoldings in $\mathrm{L}\mathbf{E}(2)$

$\widetilde{A}_0 = (A_0, a_0)$	$\mathrm{codim}_b \widetilde{A}_0$	versal unfolding
$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}$	0	$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}$
$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	1	$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}$	3	$\begin{pmatrix} 0 & -\lambda_1 & -\lambda_2 \\ \lambda_1 & 0 & -\lambda_3 \\ \lambda_2 & \lambda_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	6	$\begin{pmatrix} 0 & -\lambda_1 & -\lambda_2 \\ \lambda_1 & 0 & -\lambda_3 \\ \lambda_2 & \lambda_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix}$

Table 2: Versal unfoldings in $\mathrm{L}\mathbf{E}(3)$

6 Drift bifurcations in planar evolution equations

We now investigate the appearance of drift bifurcations for Euclidean-equivariant evolution equations on the plane and their connections with planar pattern formation. The bifurcation itself is described in Subsection (a). In Subsection (b), we make the connection with the spiral wave-retracting wave transition mentioned in the introduction. It turns out that many, though not all, of the features of this transition can be explained by the drift bifurcation of Subsection (a). However, the discrepancies between our theory and numerical simulations suggest that Hypothesis (H3) of Section 2 breaks down. Alternative, but related, scenarios are described in Subsections (c) and (d). It appears that, though of interest in their own right, the latter scenarios do not explain the spiral wave-retracting wave transition.

(a) A drift bifurcation in the plane

In Section 5, we showed that there was a single codimension one drift bifurcation of relative equilibria in systems with $\mathbf{E}(2)$ symmetry. Using complex notation, we identify $\xi \in \mathbf{LSE}(2)$ with the matrix

$$\xi = \begin{pmatrix} i\omega & \beta \\ 0 & 0 \end{pmatrix},$$

where $\omega \in \mathbb{R}$ and $\beta \in \mathbb{C}$. For the codimension one bifurcation, we require that $\omega(0) = 0$, $\omega'(0) \neq 0$, $\beta(0) \neq 0$. A versal unfolding is given in Table 1:

$$\xi(\lambda) = \begin{pmatrix} i\lambda & i\alpha \\ 0 & 0 \end{pmatrix},$$

for some fixed $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Exponentiating, we obtain

$$\exp t\xi(\lambda) = \begin{pmatrix} e^{i\lambda t} & \dfrac{\alpha}{\lambda}(e^{i\lambda t} - 1) \\ 0 & 1 \end{pmatrix}.$$

We deduce that this bifurcation occurs for rotating waves with slow speed of rotation λ. In the limit of zero rotation $\lambda = 0$, the rotating wave is replaced by a translating wave translating with nonzero speed α. On the other side of the bifurcation point, we have a wave rotating slowly in the opposite direction.

At first sight, it is not clear how such a transition could be continuous in a system of PDEs. We now show that at least in principle, there is no obstruction to such a transition in planar PDEs.

We consider relative equilibria for systems of Euclidean equivariant PDEs in the plane, for instance reaction diffusion equations. Suppose that $X = \mathbf{E}(2) \cdot u_0$ is a relative equilibrium satisfying hypotheses (H1)–(H3) from Section 2. By [2] we can say that the state u_0 generically rotates rigidly and exceptionally (codimension one) translates in some fixed direction.

Now suppose that $u_0 = u_0(\lambda)$ depends on a parameter $\lambda \in \mathbb{R}$. As λ varies, the drift on the relative equilibrium varies. The time evolution is given by

$$u(\lambda, t) = \exp(t\xi(\lambda))u_0(\lambda).$$

where $\xi(\lambda)$ is as given above. By normal hyperbolicity (H3) of the relative equilibrium, both ξ and the shape u_0 depend smoothly on λ.

At this point, we introduce the spatial dependence $u_0 = u_0(x, \lambda)$ where $x \in \mathbb{R}^2 \cong \mathbb{C}$. When $\lambda \neq 0$, the solution is rotating with slow speed λ. The center of rotation $c(\lambda) \in \mathbb{C}$ is given by the solution to the equation

$$\exp(t\xi)c = c,$$

that is,

$$e^{i\lambda t}c + \frac{\alpha}{\lambda}(e^{i\lambda t} - 1) = c.$$

Solving this equation, we obtain

$$c(\lambda) = -\frac{\alpha}{\lambda}.$$

We conclude that as the speed of rotation approaches zero, the center of rotation of the rigidly rotating solution diverges to infinity. In the limit, there is pure translation with finite speed of propagation and the center of rotation then comes back in from infinity from the opposite direction.

(b) Transition from spiral waves to retracting waves

In numerical simulations [9], a transition from a slowly rotating spiral wave to a retracting wave has been observed on reducing excitability. This transition is illustrated in Mikhailov and Zykov [12, Figure 6] and is reproduced in Figure 1. As a parameter is varied, the spiral slows down, the curvature of the wave fronts becomes small as the spiral unwinds, the core of the spiral becomes unboundedly large, and the center of rotation goes to infinity. In the limit, there is a traveling pulse translating with finite nonzero speed. As the bifurcation parameter is varied further, the traveling pulse appears to continue to translate linearly with nonzero speed. The sequence of apparently stable asymptotic states is shown in Figure 1. It is convenient to consider the 'spiral tip' as a feature of the pattern that stays approximately unchanged

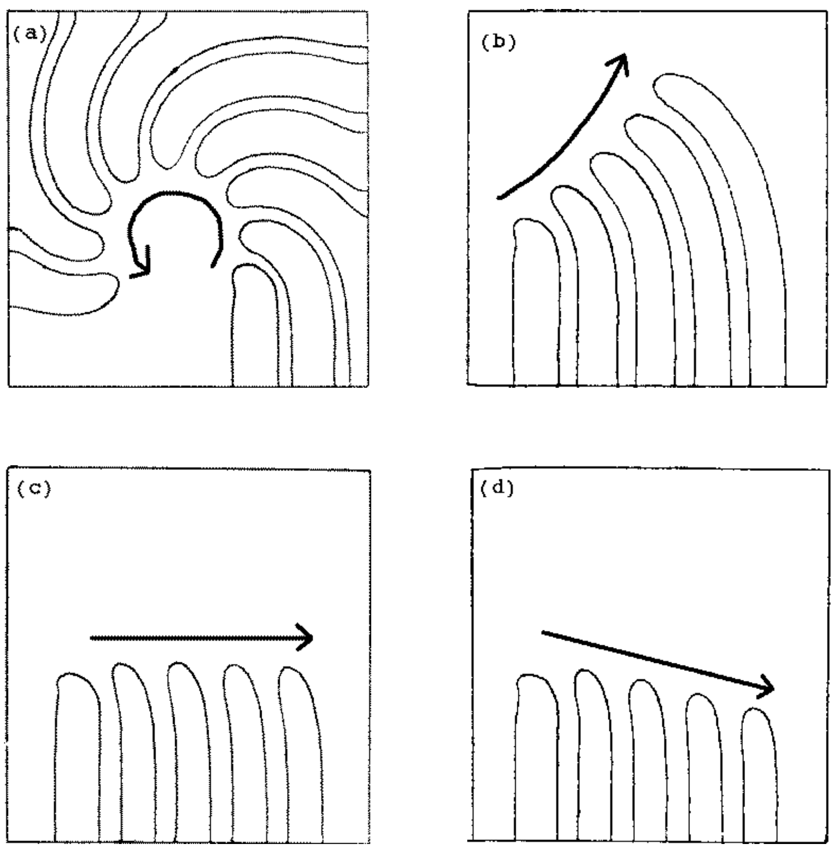

Figure 1: Observed bifurcation of a spiral wave to a retracting wave on reducing the excitability of the medium. The four diagrams depict several snapshots of a single spiral wave at different instants in time for four different values of the bifurcation parameter. The spiral wave in (a) has a core size that grows (b) on approaching the bifurcation point. In the far field it is a source of outwardly propagating waves. At the bifurcation point (c), and beyond (d), it takes the form of a semi-infinite translating wave. Adapted with permission from [12, Figure 6].

throughout the bifurcation and to interpret the bifurcation in terms of the motion of this tip.

Certain features of this transition are captured by our results in Subsection (a). In particular, we obtain as a codimension one phenomenon that the center of rotation goes to infinity and that the limiting motion is translation with finite nonzero speed.

The vanishing curvature and the infinite core are easily accounted for by combining our results with the kinematic theory of excitable media. In excitable media, it is assumed that wave fronts propagate in the normal direction to the front with magnitude determined by the curvature of the front. Often the kinematic theory is used to determine the motion of wave fronts given their curvature. For relative equilibria, the motion is determined by an element in the three-dimensional Lie algebra $L\mathbf{SE}(2)$ and it seems fruitful to apply the kinematic theory in reverse — regarding the curvature of the wave fronts as determined by the motion of the fronts. It is now an easy kinematic-style argument to see that the drift bifurcation in the motion drives the vanishing curvature and the growth of the core.

Unfortunately, our theory breaks down on the other side of the bifurcation point. Under our assumptions, we predict that the spiral will begin to rotate slowly in the opposite direction and hence by the kinematic theory, we obtain a reverse-wound spiral, as shown in Figure 2. This does not appear to be what is observed in the numerical simulations and leads us to conclude that (H3) is not satisfied at the bifurcation. In other words, a complete description of the transition must take into account the non-localized nature of the spiral and retracting waves.

Multi-armed spirals Consider a multi-armed spiral with $\mathbb{Z}_\ell$ spatial symmetry, $\ell \geq 2$. Hypothesis (H1) is no longer satisfied and hence the effective symmetry group is not the whole of $\mathbf{SE}(2)$. As explained at the end of Section 2, we replace the group $\mathbf{SE}(2)$ by $N(\mathbb{Z}_\ell)/\mathbb{Z}_\ell \cong \mathbf{SO}(2)$. The drift bifurcations are then determined by the results in Section 4. In particular, assuming Hypothesis (H3) to be valid, the center of rotation is fixed and the limiting state is stationary. It then follows from the reverse kinematic argument that the wave fronts straighten out, just as for a one-armed spiral. However, we predict that in the case of a multi-armed spiral, the core remains of finite size and the center of the core remains stationary throughout the transition. After the transition, the spiral rotates in the opposite direction.

Since Hypothesis (H3) is problematic, it is unclear what will transpire in practice after the bifurcation point. However, since the center of rotation is fixed, the core remains finite as long as we remain in relative equilibrium. Moreover, with a finite core, it is difficult to see how there can be a transition to retracting waves. Hence, the case of a multi-armed spiral is quite different from the case of a single armed spiral.

We note that the kinematic theory alone does not clearly distinguish between

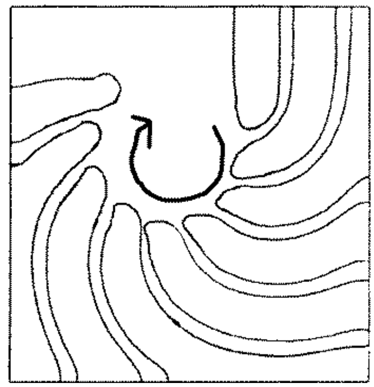

Figure 2: A reverse-wound spiral. In a drift bifurcation, we predict the sequence of diagrams (a), (b) and (c) in Figure 1. However, instead of diagram (d), we infer the existence of a reverse-wound spiral wave as shown here.

multi-armed spirals and one-armed spirals, whereas these cases are clearly distinguished (in terms of the movement of the center of rotation) on grounds of symmetry. This adds strength to our argument for applying the kinematic theory in reverse; using symmetry to predict motion and then kinematics to predict curvature of fronts.

(c) A codimension two bifurcation: simultaneous drift bifurcation and transcritical bifurcation

In Subsection (b), we demonstrated that many features of the spiral wave-retracting wave transition could be explained in terms of a codimension one drift bifurcation. In particular, this explanation accounts for parts (a), (b) and (c) of Figure 1 (up to and including the bifurcation point) but not part (d) (after the bifurcation point). In this subsection, we present an alternative scenario that completely reproduces Figure 1.

Wulff [16] introduced the space C_{eucl} of uniformly continuous functions on which $\mathbf{E}(2)$ acts as a strongly continuous group. We suppose that the shape u_0 in the previous section lies in C_{eucl} for all values of λ and slows down as before as λ approaches zero. We suppose in addition that this family of relative equilibrium consists of sinks inside of C_{eucl} for all λ. However, we suppose that as λ passes through zero, there is a loss of stability (with zero eigenvalue) in directions outside C_{eucl}. Thus there is a transcritical bifurcation out of C_{eucl}. Translations act continuously on the whole of C_{unif} but rotations act continuously only on C_{eucl}. Hence the bifurcating states cannot rotate, but generically translate with nonzero speed.

The prediction is that there are unstable traveling pulses before the bifurcation and unstable rotating spirals after the bifurcation. The unstable spirals are stable within C_{eucl}.

Unfortunately, the experimental behavior appears to be codimension one, whereas our scenario has codimension two. We do not know of a mechanism whereby bifurcation out of C_{eucl} should occur precisely when the speed of rotation goes to zero.

(d) Pitchfork bifurcation from a reflection-symmetric pulse

We consider a possible codimension one bifurcation from a symmetric state that causes a bifurcation to generic drift of the bifurcating symmetry broken solutions. For this, we consider evolution on a space where rotations act continuously, for example $L^2(\mathbb{R}^2)$, $C_0(\mathbb{R}^2)$ or the space C_{eucl} considered in Wulff [16].

Suppose that we have a family of (localized) reflection symmetric relative equilibria $u_0(\lambda)$ that undergo a reflection symmetry breaking steady-state bifurcation at $\lambda = 0$.

It follows from [2] that the pulses undergo translation drift parallel to the axis of reflection with generically nonzero speed. In contrast, the branching asymmetric states generically rotate with nonzero speed.

More information on the drifting of pulses and spirals near the bifurcation can be obtained by performing a center bundle reduction [14, 8]. Let $\alpha \in \mathbb{C}$ denote the translation speed of the pulses at the bifurcation point. Generically, $\alpha \neq 0$. If we choose coordinates, so that the reflection fixing the pulse state acts on $\mathbb{C}$ as $p \mapsto \bar{p}$, then $\alpha \in \mathbb{R}$.

Proposition 6.1 *There is a reduction to an $\mathbf{E}(2)$-equivariant vector field on a four-dimensional center bundle $Y = S^1 \times \mathbb{C} \times \mathbb{R}$, where the action of $\mathbf{E}(2)$ is given by*

$$(\theta, v) \cdot (\phi, p, x) = (\phi + \theta, e^{i\theta}p + v, x), \quad \kappa \cdot (\phi, p, x) = (-\phi, \bar{p}, -x),$$

for $(\theta, v) \in \mathbf{SE}(2)$, $\kappa \in \mathbb{D}_1$, and $(\phi, p, x) \in Y$.

Proof Since the pulse solution u_0 has isotropy $\mathbb{D}_1$, the group orbit $\mathbf{E}(2)u_0$ is diffeomorphic to $\mathbf{E}(2)/\mathbb{D}_1 \cong \mathbf{SE}(2) \cong S^1 \times \mathbb{C}$. The normal vector field is $\mathbb{D}_1$-equivariant and has a one-dimensional center manifold $\mathbb{R}$. Since the steady-state bifurcation is assumed to be symmetry-breaking, the action of $\mathbb{D}_1$ on $\mathbb{R}$ is given by $x \mapsto -x$.

Center bundle reduction leads to a four-dimensional center bundle with base space $S^1 \times \mathbb{C}$ and fiber $\mathbb{R}$. The action of $\mathbb{D}_1$ on $\mathbb{R}$ extends to an action of $\mathbf{E}(2)$ on $\mathbb{R}$ (where $\gamma x = x$ for $x \in \mathbf{SE}(2)$ and $\gamma x = -x$ for $x \in \mathbf{E}(2) - \mathbf{SE}(2)$). Hence, it follows from [8] that the center bundle is a trivial bundle $S^1 \times \mathbb{C} \times \mathbb{R}$. Moreover, the action of $\mathbf{E}(2)$ on $\mathbb{R}$ is as given.

The action of $\mathbf{SE}(2)$ on $S^1 \times \mathbb{C}$ is given by group multiplication. Finally, observe that $\kappa \cdot (\phi, p) = (-\phi, \kappa p) \cdot \kappa = (-\phi, \bar{p}) \cdot \kappa$. Hence $\kappa(\phi, p)u_0 = (-\phi, \bar{p})u_0$. This gives the action of $\mathbb{D}_1$ on $S^1 \times \mathbb{C}$. ∎

Proposition 6.2 *The equations on the center bundle have the form*

$$\begin{aligned}\dot{\phi} &= xf(x^2,\lambda)\\ \dot{p} &= e^{i\phi}\{g_1(x^2,\lambda)+ixg_2(x^2,\lambda)\}\\ \dot{x} &= xh(x^2,\lambda)\end{aligned}$$

where $f, g_1, g_2, h : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $g_1(0,0)=\alpha$.

Proof Write the vector field in the form (F^ϕ, F^p, F^x). It follows from $\mathbf{SE}(2)$-equivariance that $F^\phi = F^\phi(x)$, $F^p(x) = e^{i\phi}g(x)$, $F^x = h(x)$ (this is the same calculation as in [4] or [8]). Finally, the action of $\mathbb{D}_1$ forces F^ϕ and h to be odd in x, and in addition $g(-x) = \overline{g(x)}$. ∎

We suppose also that $h_\lambda(0,0) > 0$ and $h_{x^2}(0,0) < 0$, thus ensuring that the pulse state is asymptotically stable for $\lambda < 0$ and that there is a loss of stability at $\lambda = 0$ resulting in a supercritical pitchfork bifurcation of asymmetric states.

The nontrivial zeroes of the $\dot{x}$ equation are given by $x(\lambda) = \pm k\sqrt{\lambda}+O(\lambda^{3/2})$ where k is a positive constant. Substituting into the $\dot{\phi}$ equation, and integrating, we obtain

$$\phi(t) = \pm kf(0,0)\sqrt{\lambda}t + O(\lambda^{3/2}).$$

Finally, we have

$$\dot{p} = g_1(0,0)e^{\pm ikf(0,0)\sqrt{\lambda}t} + O(\lambda),$$

so that

$$p(t) = \frac{\alpha}{\pm ikf(0,0)\sqrt{\lambda}}e^{\pm ikf(0,0)\sqrt{\lambda}t} + O(\lambda).$$

It follows that the rotation frequency of the spiral state decreases to 0 at the bifurcation point and is of order $\sqrt{\lambda}$. In addition, the radius of rotation goes to infinity.

The analysis above explains calculations of Barkley and Kevrekidis [3] but suffers from the difficulty, in common with [3], that the introduction of reflection symmetry is artificial (since the retracting waves are asymmetric). We note that the scenario in this subsection leads to quite different predictions to the scenarios in the previous two subsections. In particular, the speed of rotation scales as $\sqrt{\lambda}$ (just as in [3]) whereas in Subsections (b) and (c) the speed of rotation scales linearly with the bifurcation parameter. In all cases, the rate of growth of the center of rotation is inversely proportional to the speed of rotation.

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